MA/MSCMT-06

June - Examination 2019

M.A./ M.Sc. (Final) Mathematics Examination

Analysis and Advanced Calculus

Paper - MA/MSCMT-06

Time : 3 Hours] **[Max. Marks :- 80**

Note: The question paper is divided into three sections A, B and C. Use of non-programmable scientific calculator is allowed in this paper.

Section - A **8 × 2 = 16**

(Contain eight (08) Very Short Answer Type Questions)

Note: Examinees have to attempt all questions. Each question is of 02 marks and maximum word limit may be thirty words.

1) (i) Explain weak convergence.

(ii) State open mapping theorem.

(iii) Define continuous linear functional.

(iv) Define Hilbert space.

(v) Define Ortho-normal set.

(vi) Define adjoint operator.

(vii) Define derivative of a map.

(viii) Define step function.

Section - B **$4 \times 8 = 32$**

(contains Eight Short Answer Type Questions)

Note: Examinees will have to answer any four (4) question. Each question is of 08 marks. Examinees have to delimit each answer in maximum 200 words.

2) If $C(X)$ be a linear space of all bounded continuous scalar valued functions defined on a topological space X. Then show that $C(X)$ is a Banach space under the norm

$$\| f \| = \sup \left\{ | f(x) | : x \in X \right\}, f \in C(x)$$

3. If N and N' be normed linear spaces and $B(N, N')$ is the set of all bounded linear transformation from N into N' then prove that $B(N, N')$ is a normed linear space under the norm

$\| T \| = \sup \left\{ | T(x) | : \| x \| \leq 1 \right\} \forall x \in N$ with respect to pointwise linear operations $(T + S)(x) = T(x) + S(x)$ and $(\alpha T)(x) = \alpha T(x)$, for real α.

4. If M be a closed linear subspace of a Hilbert space H and x be a vector not in M if $d = d(x, M)$ then prove that there exists a unique vector y_0 in M such that $\| x - y_0 \| = d$

5. If $\{e_1, e_2, ...e_n\}$ be finite Ortho-normal set in a Hilbert space H and H and x be any vector in H, then prove that

$$\sum_{i=1}^{n} | (x, e_i) |^2 \leq \| x \|^2.$$

6. If P is a projection on a Hilbert space H with range M and null space N then prove that $M \perp N$ if and only if P is self adjoint and in this case $N = M^{\perp}$

7. Let X and Y be any two Banach spaces over the same field K of scalars and V be an open subset of X. Let $f : \mathrm{V} \to Y$ be continuous function. Let u, v be any two distinct points of V such that $[u, v] \subset V$ and f is differentiable in $[u, v]$. Then prove that
$\| f(v) - f(u) \| \leq \| v - u \| \sup \{ \| Df(x) \| : x \in [u, v] \}$

8. If f be a function defined on the interval $[a, b]$ of R into R such that f is m times differentiable in interval $[a, b]$ and (m+1) times differentiable in interval (a, b) then prove that

$$f(b) = f(a) + (b - a)Df(a) + \ldots.. + \frac{(b-a)^m}{m!}$$

$$D^m f(a) + \frac{(b-a)^m}{m!} D^m f(a) + \frac{(b-a)^{m+1}}{(m+1)!} D^{m+1} f(c)$$

Where $c \in (a, b)$

9. If f be a function on a compact interval $[a, b]$ of R into a Banach space X over K.
Then prove that f is regulated if and only if the following conditions are satisfied.
   (i) for each point $c \in [a, b)$
   $\lim\limits_{\substack{t \to c \\ t > c}} f(t)$ exists.
   (ii) for each point $c \in (a, b]$
   $\lim\limits_{\substack{t \to c \\ t > c}} f(t)$ exists.

Section - C **2 × 16 = 32**

(Contains 4 Long Answer Type Questions)

Note: Examinees will have to answer any two (02) questions. Each question is of 16 marks. Examinees have to answer in maximum 500 words.

10) State and prove Risez representation theorem.

11) If *B* and *B*' be Banach spaces and *T* is a continuous linear transformation of *B* onto *B*', then prove that the image of every open sphere centred at origin in B contains an open sphere centred at origin in *B*'.

12) 1f *f* be a functional defined on a linear subspace M of a normed linear space N, $x_0 \notin M$ and $M_0 = [M \cup \{x_0\}] = \{x + ax_0 : x \in M$, 'α' isreal$\}$ is the linear subspace spanned by M and x_0, then prove that *f* can be extended to a functional f_0 defined on M_0 such that $\| f_0 \| = \| f \|$.

13) State and prove spectral theorem.