MA/MSCMT-06

June - Examination 2018

M.A./M.Sc. (Final) Mathematics Examination

Analysis and Advanced Calculus

Paper - MA/MSCMT-06

Time : 3 Hours] **[Max. Marks :- 80**

Note: The question paper is divided into three sections A, B and C. Write answers as per the given instructions.

Section - A **$8 \times 2 = 16$**

(Very Short Answer Type Questions)

Note: Answer **all** questions. As per the nature of the question delimit your answer in one word, one sentence or maximum up to 30 words. Each question carries 2 marks.

1) (i) Define a function space.

(ii) What is multi linear mapping?

(iii) Define an inner product space.

(iv) Define an adjoint operator on a Hilbert space.

(v) Define derivatives on a Banach space.

(vi) Define regulated function for a Banach space.

(vii) What is $\in$ - approximate solution for a differential equation $\frac{dx}{dt} = g(t, x)$?

(viii)Define Eigen space of an operator T on a Hilbert space.

Section - B **4 × 8 = 32**

(Short Answer Questions)

Note: Answer **any four** questions. Each answer should not exceed 200 words. Each question carries 8 marks.

2) Show that every compact subset of a normed space is bounded but its converse is need not be true.

3) Prove that an inner product in a Hilbert space is jointly continuous.

4) If $\{e_i\}$ is an orthonormal set in a Hilbert space H, then prove that $\sum_{i=1}^{\infty} |(x, e_i)|^2 = \|x\|^2, \forall x \in H$

5) If P and Q are projection on closed linear subspace M and N of a Hilbert space H, then prove that $M \perp N \Leftrightarrow PQ = 0 \Leftrightarrow QP = 0$

6) Let X and Y be Banach space over the same field K and V be an open subset of X. Let $f: V \to Y$ be a $(n+1)$ times differentiable function. If the interval $[a, a+h]$ is contained in V and if $\| f^{n+1}(x) \| \leq M, \forall x \in V$. Then prove that

$$\left\| f(a+h) - f(a) - f'(a)h - \frac{f^n(a)}{2!}h^2 - \text{.........} - \frac{f^n(a)}{n!}h^n \right\| \leq \frac{M\| h \|^{n+1}}{(n+1)!}$$

7) Let u be a non-negative continuous function on an interval $[0, c]$, $c>0$ satisfying the inequality $u(t) \leq at + k\int_0^1 u(s)ds, \forall t \in [0, c]$

then prove that $u(t) \leq \frac{a}{k}(e^{kt} - 1)$ for $t \in [0, c]$

8) Prove that the limit of convergent sequence in a normed space is unique.

9) Prove that if M is a closed linear subspace of Hilbert space H then $H = M \oplus M^{\perp}$

Section - C **$2 \times 16 = 32$**

(Long Answer Questions)

Note: Answer **any two** questions. You have to delimit your each answer maximum up to 500 words. Each question carries 16 marks.

10) If C(X) be a linear space of all bounded continuous scalar valued function defined on a topological space X. Then show that C(X) is a Banach space under the norm $\| f \| = Sup\{| f(x): x \in X |\}, \forall f \in C(X)$

11) If B is a complex Banach space whose norm obeys the parallelogram law, and if an inner product is defined on B by $4(x,y) = \| x+y \|^2 - \| x-y \|^2 + i\| x+iy \|^2 - i\| x-iy \|^2$, then prove that B is a Hilbert space.

12) Let H be a Hilbert space and B(H) be the complex Banach space of all bounded linear transformation on H into H. Then prove that the adjoint operator T* of T∈B(H) has the following properties

(a) $(T+S)^* = T^* + S^*$ (b) $(\alpha T)^* = \bar{\alpha} T^*$

(c) $(TS)^* = S^* T^*$ (d) $T^{**} = T$

(e) $\| T^* \| = \| T \|$ (f) $\| T^* T \| = \| T \|^2$

(g) $(T^*)^{-1} = (T^{-1})^*$

13) Let f be a function on a compact interval $[a, b]$ of R into a Banach space X over K. Then prove that f is regulated iff the following conditions are satisfied

(i) $\forall c \in [a,b), \lim_{\substack{t \to c \\ t > c}} f(t)$ exists

(ii) $\forall c \in (a,b], \lim_{\substack{t \to c \\ t < c}} f(t)$ exists