M.A./ M.Sc. Final Examination, June 2015
MATHEMATICS
Paper: MT-06 (Analysis & Advanced Calculus)

Time Allowed: 3 Hours
Maximum Marks: 80

Note:- The Question paper is divided into three sections A, B, and C. Use of calculator is allowed in this paper.

Section- A

Section 'A' contain 08 Very Short Answer Type Questions. Examinees have to attempt all questions. Each question is of 02 marks and maximum word limit is thirty words.

1. (i). Define sequentially compact normed linear space.
(ii). Define the second dual space of N.
(iii). Define orthogonal complement for the Hilbert space H.
(iv). Define the eigenvalue and eigenvectors for Hilbert space H.
(v). Define Hahn-Banach Theorem.
(vi). State closed graph theorem.
(vii). What do you mean by weak convergence of a sequence?
(viii). Give an example of normed linear space.

Section- B

Section 'B' contain 08 Short Answer Type Questions. Examinees will have to answer any four (04) questions. Each question is of 08 marks. Examinees have to delimit each answer in maximum 200 words.

2. If T be a linear transformation from a normed linear space into normed space N', then show that T is continuous either at every point or at no point of N.
3. Prove that every compact subset of a normal space is bounded but the converse is not true.
4. Show that the space $l_2{}^n$ consisting of all n types $x = (x_1, \dots \dots . x_n)$ of complex numbers and the inner product on $l_2{}^n$ is defined as $(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i$, where $y = (y_1, \dots \dots \dots y_n)$ is an inner product space.
5. Show that an orthonormal set S in a Hilbert space H is complete iff for $x \perp S \Rightarrow x = 0 \; \forall \, x \in H$.
6. Show that a closed linear subspace M of a Hilbert space H reduces an operator $T \Leftrightarrow M$ is invariant under both T and T^*
7. If X and Y be Banach spaces over the same field K of scalars and V be an open subset of X. Let $f : V \to Y$ is differentiable at $x \in V$, then show that all the directional derivatives of f exists at x and $D_v f(x) = D\, f(x). v$, where $v \in V$ is a unit vector.
8. Let f be a regulated function on a compact interval [a, b] of R into a Banach space X, then show that for each $t \in (a, b)$, the function $F: [a, b] \to X, F(t) = \int_a^t f \;, \;\; t \in [a, b]$ is continuous.
9. State and prove the Schwarz inequality.

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Section- C

Section 'C' contain 04 Long Answer Type Questions. Examinees will have to answer any two (02) questions. Each question is of 16 marks. Examinees have to delimit each answer in maximum 500 words.

10. (i). Let M be a closed linear subspace of a Hilbert space H and x be a vector not in M. If $d = d\,(x, M)$ then show that ∃ a unique vector y_0 in M s.t. $\|x - y_0\| = d$.

    (ii). State and Prove the Bessel's inequality.
11. (i). Show that every Hilbert space is reflexive.

    (ii). If T is an operator on a Hilbert space H, then prove that $(T_x, x) = 0 \;\; \forall \, x \in H$ iff T = 0.
12. (i). Let X be a Banach space over the field K of scalars and let $f : [a, b] \to X \; and \; g: [a, b] \to R$ be continuous and differentiable functions s.t. $\|Df(t)\| \leq Dg(t)$ at each point $t \in (a, b)$ then prove:

    $\|f(b) - f(a)\| \leq g(b) - g(a)$

    (ii). Let f be a function defined on the interval [a, b] of R into R s.t. f is m times differentiable in [a, b] and {m+1} times differentiable in interval (a, b), then prove :

    $f(b) = f(a) + (b - a)Df(a) + \cdots + \frac{(b-a)^m}{m!} D^2 f(a) + \frac{(b-a)^{m+1}}{(m+1)!} D^{m+1} f(c)$

    Where $c \in (a, b)$.
13. (i). State and prove the uniform boundedness theorem.

    (ii). State and prove Reisz Lemma.