12. Let $\frac{K}{F}$ be a field extension. Then show that an element $a \in K$ is algebraic over F if and only if $F(a)$ is a finite extension of F, *i.e.* $[F(a) : F]$ is finite.

13. Let V be a finite dimensional inner product space. Let $t : V \to V$ be a linear transformation. Then show that their exists a unique linear transformation $t^* : V \to V$ such that :

$$\langle t(u), v \rangle = \langle u, t^*(v) \rangle$$

for all $u, v \in V$.

MAMT-01/MSCMT-01

June – Examination 2023

M.A./M.Sc. (Previous) Examination

MATHEMATICS

(Advanced Algebra)

Paper : MAMT-01/MSCMT-01

*Time : **3** Hours*] [*Maximum Marks : **80***

Note :– The question paper is divided into three Sections A, B and C. Write answers as per the given instructions. Use of non-programmable Scientific Calculator is allowed in this paper.

Section–A **8×2=16**

(Very Short Answer Type Questions)

Note :– Answer all questions. As per the nature of the question, delimit your answer in one word, one sentence or maximum up to **30** words. Each question carries 2 marks.

1. (i) Define Conjugate Class.

(ii) Define Euclidean Ring.

(iii) Define Kernel of linear transformation.

(iv) Define normal field extension.

(v) Define Galois field.

(vi) Define rank of matrix.

(vii) Define inner product space.

(viii) Define normal operator.

Section–B **4×8=32**

(Short Answer Type Questions)

Note :– Answer any *four* questions. Each answer should not exceed **200** words. Each question carries 8 marks.

2. State and prove class equation of a group.

3. If G is a solvable group, then show that every subgroup of G is also solvable.

4. Let R be a ring with unity and M be an R-module. Let N be finitely generated submodule of M generated by a subset $A = \{a_1, a_2, \ldots\ldots, a_n\}$ of M. Then show that :

$$N = RA = Ra_1 + Ra_2 + \ldots. + Ra_n$$

5. Show that a polynomial of degree n over a field F can have at most n roots in any extension field.

6. Let $t : V \to V'$ be a linear transformation and V is finite dimensional, then show that :

$$\dim V = \text{Rank}(t) + \text{Nullity}(t)$$

7. If a square matrix A of order n, over a field F has n distinct eigen values λ_i; $i = 1, 2, \ldots, n$, then show that there is an invertible matrix P such that :

$$P^{-1}AP = \text{diag}(\lambda_1\ \lambda_2, \ldots., \lambda_n)$$

8. Show that every orthonormal set of vectors is a linearly independent set in an inner product space.

9. Let V be an inner product space. Then show that for any arbitrary vectors $u, v \in V$, we have :

$$|<u, v>| \leq \|u\|\,\|v\|$$

Section–C **2×16=32**

(Long Answer Type Questions)

Note :– Answer any *two* questions. You have to delimit your each answer maximum up to **500** words. Each question carries 16 marks.

10. Show that a group G is an internal direct product of its subgroups $H_1, H_2, \ldots., H_n$ if and only if :

(i) $G = H_1H_2 \ldots. H_n$

(ii) $H_1, H_2, \ldots, H_n$ are all normal subgroups of G

(iii) $H_i \cap (H_1H_2 \ldots H_{i-1}\ H_{i+1} \ldots H_n) = \{e\}$

11. State and prove unique factorization theorem.