10. State and prove butterfly theorem.

11. (a) If K be a field extension of field F, then prove that an element $a \in K$ is algebraic over F if and only if $F(a)$ is finite extension of F.

(b) If Q be the field of rational numbers, then show that :

$$Q\left(\sqrt{2}, \sqrt{3}\right) = Q\left(\sqrt{2} + \sqrt{3}\right)$$

12. State and prove fundamental theorem of Galois theory.

13. If V is a finite dimensional vector space over a field F and $t : V \rightarrow V$ be a linear transformation, then prove that :

(a) The matrix A of t is a diagonal matrix having the eigenvalues of t as diagonal entries if and only if A is corresponding to a basis of V consisting of eigenvalues of linear transformation t.

(b) The eigenvalues of t are exactly the diagonal entries of A and each appearing on the diagonal as many times as the dimension of its eigenspace.

MAMT–01/MSCMT–01

December – Examination 2022

M.A./M.Sc. (Previous) Examination

MATHEMATICS

(Advanced Algebra)

Paper : MAMT-01/MSCMT-01

*Time : **3** Hours*] [*Maximum Marks :* ***80***

Note :– The question paper is divided into three Sections A, B and C. Write answers as per the given instructions. Use of non-programmable Scientific Calculator is allowed in this paper.

Section–A **8×2=16**

(Very Short Answer Type Questions)

Note :– Answer all questions. As per the nature of the question delimit your answer in one word, one sentence or maximum up to **30** words. Each question carries 2 marks.

1. (i) Define internal direct product.

   (ii) Define normalizer of an element in a group.

   (iii) Show that every homomorphic image of a solvable group is solvable.

   (iv) Define unique factorization domain.

   (v) Define image of a linear transformation.

   (vi) Define Galois extension.

   (vii) Define row rank of a matrix.

   (viii) Define orthogonal complement of a vector.

Section–B 4×8=32

(Short Answer Type Questions)

Note :– Answer any *four* questions. Each answer should not exceed **200** words. Each question carries 8 marks.

2. If $G_i\,(1 \le i \le n)$ be n groups and G is external direct product of these groups. If e_i be the identity element of the group G_i for each $i\,(1 \le i \le n)$, then prove that for each i $H_i = \{(e_1, e_2, \ldots\ldots, e_{i-1}, x_i, e_{i+1}, \ldots\ldots\ldots, e_n) \mid x_i \in G_i\}$ is a normal subgroup of G.

3. If G is a group and H be a subgroup of G, then prove that H is normal subgroup of G and G/H is abelian if and only if $G' \subset H$, where G' is derived subgroup of G.

4. If M and M′ be two R-modules, then prove that the set Hom_R (M, M′) is an abelian group under pointwise addition of morphism.

5. If K be extension of the field of rational numbers Q, then show that any automorphism of K must leave every element of Q fixed.

6. If $B = \{b_1 = (-1, 1, 1), b_2 = (1, -1, 1), b_3 = (1, 1, -1)\}$ is a basis of $V_3(R)$, then find the basis dual to B.

7. If $t : R^3 \to R^3$ be a linear transformation defined by $t\,(a, b, c) = (3a + c, -2a + b, -a + 2b + 4c)$, $\forall (a, b, c) \in R^3$, then what is the matrix of t in ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$, where $\alpha_1 = (1, 0, 1)$, $\alpha_2 = (-1, 2, 1)$, $\alpha_3 = (2, 1, 1)$.

8. Prove that a linear transformation t from a finite dimensional inner product space V into itself is symmetric if and only if its matrix relative to some ortho-normal basis of V is symmetric.

9. Prove that the eigenvalues of a self-adjoint linear transformation are real.

Section–C 2×16=32

(Long Answer Type Questions)

Note :– Answer any *two* questions. You have to delimit your each answer maximum up to **500** words. Each question carries 16 marks.