UNIT-1
Electrostatics

Structure of the Unit

1.0 Objectives

This unit constitutes the basic concepts of static (time invariant)electric field and potential. One can learn the Gauss's law and its applications. We learn the usefulness of spherical and cylindrical coordinates for solving the certain kinds of problems in electrostatics. The better physical insight of behaviour of electric field and potential across the interface can be got by studying the boundary conditions at the interface.

1.1 Introduction

This unit introduces Coulomb's law ,Gauss's Law and boundary conditions on electric fields and potentials. Gauss's law is developed and shown in both integral and differential form. The concept of circulation of the electric field is related to its conservative nature is discussed. The concept of boundary conditions for electric field and potential is introduced in this unit.

1.2 Electric Field

According to Coulomb's law , electrostatic force F between two point charges q and Q which are placed in free space at a distance $\mathscr{r}$ is expressed mathematically as

$$F = k\frac{Qq}{\mathscr{r}^2}$$

This force acts along the line joining the charges.

$$\boxed{k = \frac{1}{4\pi\varepsilon_0} \simeq 9\times 10^9\,\frac{N\,m^2}{C^2}}$$

$$\boxed{\varepsilon_0 = 8.854\times 10^{-12}\,\frac{C^2}{N\,m^2}}$$

The constant ε_0 is called permittivity of free space.

Force on q_2 due to q_1 can be written in vector form as

$$\boxed{\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\mathscr{r}^2}\hat{\mathscr{r}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\mathscr{r}^3}\vec{\mathscr{r}}} \qquad (1)$$

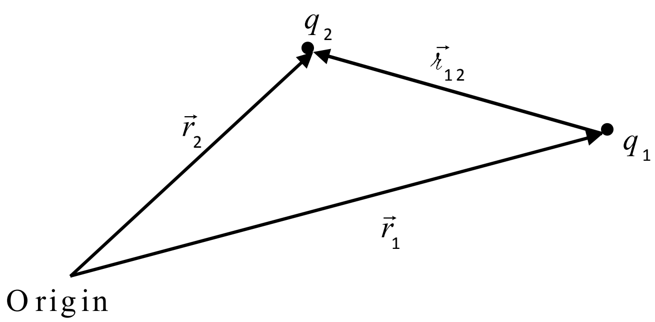


Figure1.1

$$\Rightarrow \vec{F}_{1\to 2} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\mathscr{r}_{1\to 2}^2}\hat{\mathscr{r}}_{1\to 2} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\mathscr{r}_{1\to 2}^3}\vec{\mathscr{r}}_{1\to 2}$$

$$\Rightarrow \vec{F}_{1\to 2} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{\left|\vec{r}_2 - \vec{r}_1\right|^3}\left(\vec{r}_2 - \vec{r}_1\right)$$

where $\vec{r}_{1\to 2} = \left(\vec{r}_2 - \vec{r}_1\right)$ =Position vector of q_2 −Position vector of q_1

$\left|\vec{r}_{1\to 2}\right| = \left|\vec{r}_2 - \vec{r}_1\right|$

From eq.(1) taking $q_1 = Q$ (Source charge) and $q_2 = q$ (test charge),we have

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}}$$

$$\vec{F} = q\left[\frac{1}{4\pi\varepsilon_0}\frac{Q}{\mathscr{r}^2}\hat{\mathscr{r}}\right]$$

$\boxed{\vec{F} = q\vec{E}}$ (Force on the test charge *q*)

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\mathscr{r}^2}\hat{\mathscr{r}}} \qquad (2)$$

and $\vec{E} = \dfrac{\vec{F}}{q}$

$\vec{E}$ is called ***electric field (or electric field intensity***) due to point charge Q.

SI unit of $\vec{E}$ is $\dfrac{N}{C}$.

Thus "Electric field $\vec{E}$ at a point is the force experienced per unit charge at rest state at that point of space." $\vec{E} = \vec{E}(\vec{r})$ has a value at each point in the space, so it is called vector point field.

For definition of electric field , we can write

$$\vec{E} = \lim_{q\to 0}\frac{\vec{F}}{q}$$

The test charge q should be infinitesimally small because large value of the test charge will disturb the original charge distribution of primary charges that produces $\vec{E}$.

If there are more than two charges, then we use principle of superposition for determination of the force on a particular charge. If there are N point charges $Q_1, Q_2, Q_3, ...Q_N$ (source charges) placed respectively at distances $\mathscr{r}_1, \mathscr{r}_2, \mathscr{r}_3, ...\mathscr{r}_N$ from charge q then by principle of superposition total force on the charge q(test charge)is given by

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ...\vec{F}_N$$

$$\vec{F} = k\frac{qQ_1}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 + k\frac{qQ_2}{\mathscr{r}_2^2}\hat{\mathscr{r}}_2 + ... k\frac{qQ_N}{\mathscr{r}_N^2}\hat{\mathscr{r}}_N$$

Here position vectors of $Q_1, Q_2, Q_3, ...Q_N$ are $\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_N$ respectively and test charge q is located at position $\vec{r}$, then

$$\vec{\mathscr{r}}_1 = \vec{r} - \vec{r}_1 , \vec{\mathscr{r}}_2 = \vec{r} - \vec{r}_2 ,....$$

$$\therefore \vec{F} = q\left(k\frac{Q_1}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 + k\frac{Q_2}{\mathscr{r}_2^2}\hat{\mathscr{r}}_2 + ... k\frac{Q_N}{\mathscr{r}_N^2}\hat{\mathscr{r}}_N \right)$$

$$\therefore \vec{F} = q\vec{E}$$

where $\vec{E} = k\frac{Q_1}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 + k\frac{Q_2}{\mathscr{r}_2^2}\hat{\mathscr{r}}_2 + ... k\frac{Q_N}{\mathscr{r}_N^2}\hat{\mathscr{r}}_N$

$$\boxed{\vec{E} = k\sum\frac{Q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i}$$

or $\quad \vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + ...\vec{E}_N$

Above expression represents the ***principle of superposition*** for electric field.

For continuous charge distribution

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{\mathscr{r}^2}\hat{\mathscr{r}}}$$

Now we consider the Coulomb's law for the general case of volume charge.

Volume charge density is $\rho = \frac{dq}{d\tau}$ (in C/m^3) ,where differential charge dq is present in a differential volume $d\tau$

The electric field at a point $\vec{r} = (x, y, z)$ in terms of integral over the volume charge distribution $\rho(x', y', z')$ is written as

$$\boxed{\vec{E}(\vec{r})=\frac{1}{4\pi\varepsilon_0}\int_{\tau'}\frac{\rho(\vec{r}')\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}d\tau'} \tag{3}$$

For given charge distribution, $\vec{E}(\vec{r})$ is a function of unprimed coordinates (x,y,z) $\rho(\vec{r}')$ is a function of primed coordinates (x',y',z')

$\vec{\mathscr{r}}=\vec{r}-\vec{r}'=(x-x')\hat{i}+(y-y')\hat{j}+(z-z')\hat{k}$,

Volume element $d\tau'=dx'dy'dz'$

For line charge distribution

$$\boxed{\vec{E}(\vec{r})=\frac{1}{4\pi\varepsilon_0}\int_{L'}\frac{\lambda(\vec{r}')\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dl'}$$

Line charge density is $\lambda=\dfrac{dq}{dl}$ (in C/m),where differential charge dq is present on a differential length dl .

For surface charge distribution

$$\boxed{\vec{E}(\vec{r})=\frac{1}{4\pi\varepsilon_0}\int_{S'}\frac{\sigma(\vec{r}')\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dS'}$$

Surface charge density is $\sigma=\dfrac{dq}{dS}$ (in C/m^2),where differential charge dq is present on a differential area dS .

1.3 Gauss's Law

The theorem is stated mathematically as follows –

$$\boxed{\oint\vec{E}.d\vec{S}=\frac{q_{enclosed}}{\varepsilon_0}} \tag{4}$$

where $q_{\text{enclosed}}=\Sigma\ q_{\text{inside}}$ = algebraic sum of charges inside the enclosed volume.

"***That net outward electric flux through a closed surface is equal to the sum of the charges inside the enclosed volume divided by the permittivity of free space***".

Here such a hypothetical closed surface is known as Gaussian Surface.

Gauss's law is the easiest way of calculating electric field in situations in which charge distribution has symmetry such as spherical distribution of charge, an infinite line charge etc. Gauss's law is also true in non symmetrical situations, but in that case it will not be so useful for the evaluation of electric fields.

Important Points:

1. Electric flux $\oint \vec{E}.d\vec{S}$ is independent of the size and shape of the Gaussian surface as long as q_{enclosed} is same.
2. Electric flux does not depend on the location of charge inside the closed surface *S*, whereas electric field at each point on surface *S*, is dependent on location of charge.
3. Electric flux is unaltered by the charge outside the closed surface *S*, but outside charge contributes in electric field at each point on the surface *S*.

The Concept of Solid Angle:

Solid angle is analogous in three dimensional of the ordinary two dimensional angle. Now we consider an infinitesimal small area *dS* which subtends an infinitesimal small solid angle $d\Omega$ at a point *O* (see Fig. 1.2).

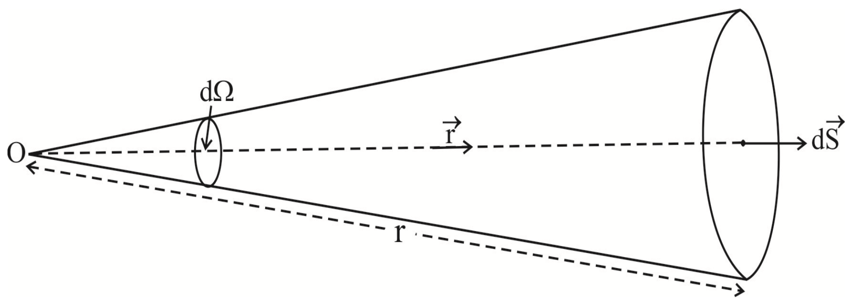


Figure1.2

Here *r* is the distance from the vertex *O* to the surface element *dS.* Solid angle $d\Omega$ is related to opening of cone around its vertex. Mathematically, we have

$$d\Omega = \frac{dS}{r^2}$$

Unit of solid angle is "Steradian" . From the definition of solid angle, it is obvious that the solid angle is dimensionless quantity.

(i) Suppose an elemental area vector vector $d\vec{S}$ makes an angle θ with radial vector $\vec{r}$ (see figure 1.3)

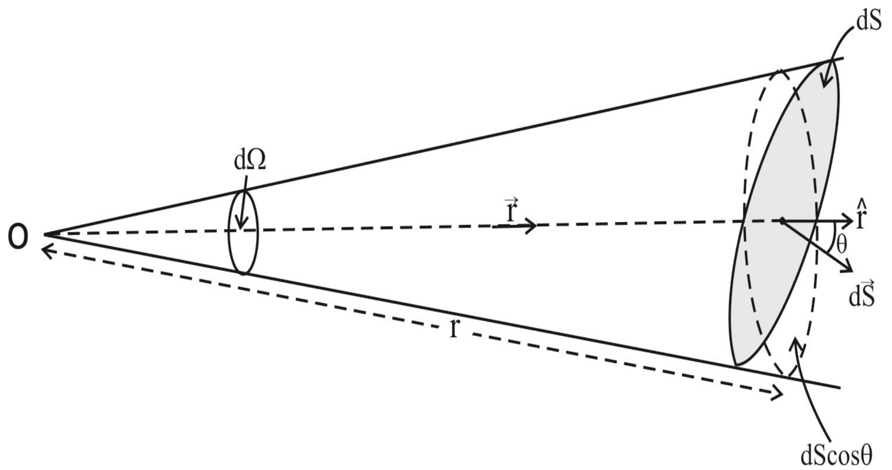


Figure1.3

Projection area perpendicular to $\vec{r}$ is $dS\cos\theta$. Now solid angle $d\Omega$ is the ratio of this projected area to r^2

i.e. $$\boxed{d\Omega = \frac{dS\cos\theta}{r^2} = \frac{d\vec{S}.\hat{r}}{r^2}} \qquad (5)$$

(ii) In figure 1.4, area elements dS_1, dS_2 and dS_3 subtend same solid angle at the point *O,* because for them, opening of cone around its vertex *O* is same.

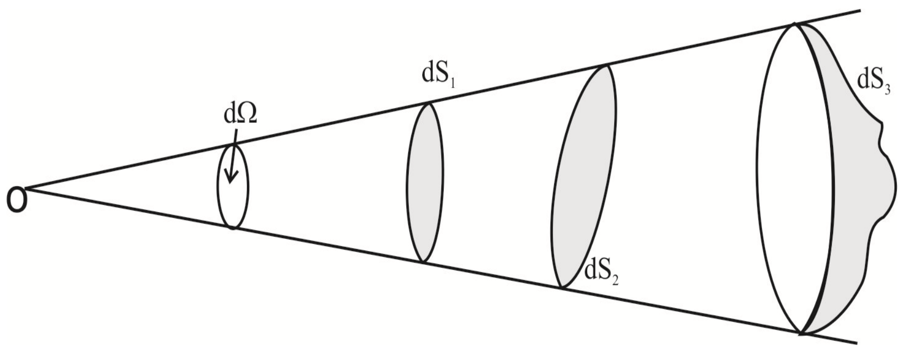


Figure1.4

(iii)Solid angle subtended by sphere at the centre is 4π

$$\therefore \oint d\Omega = \oint \frac{dS}{r^2}$$

$$\Rightarrow \Omega = \frac{1}{r^2}\oint dS \qquad \because r = \text{constant}$$

$$\Rightarrow \Omega = \frac{1}{r^2}(4\pi r^2) = 4\pi$$

4π = *complete (full) solid angle*

Whatever may be shape or size of a closed surface, above result holds at any internal point surrounded by the closed surface.

Therefore we have general result:

Entire closed surface subtends solid angle

$$\Omega = \begin{cases} 4\pi & \text{at an internal point} \\ 0 & \text{at an external point} \end{cases} \tag{6}$$

Proof of Gauss's Theorem:-

We consider a point charge q surrounded by a hypothetical closed surface S of an arbitrary shape as shown in Figure1.5

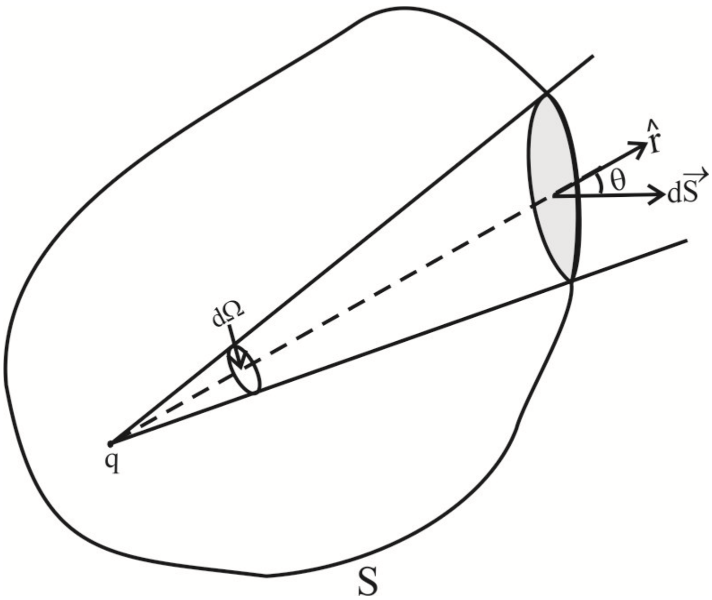


Figure1.5

Electric flux through an infinitesimal area element dS is $d\phi$, then

$$d\phi = \vec{E}.d\vec{S} = EdS\cos\theta$$

where θ is the angle between electric field $\vec{E}$ and area vector $d\vec{S}$

Due to the point charge q, electric field at a distance r is given by $\vec{E} = \frac{1}{4\pi \in_0} \frac{q}{r^2} \hat{r}$

$$\therefore d\phi = EdS\cos\theta = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r^2}\right)dS\cos\theta \tag{7}$$

Since electric field is directed along $\hat{r}$, therefore θ is the also angle between $\hat{r}$ and $d\vec{S}$. Here area element $d\vec{S}$ subtends an infinitesimal solid angle $d\Omega$ at the point charge q.

From eq.(5) ,we have $d\Omega = \frac{d\vec{S}.\hat{r}}{r^2}$

$$\Rightarrow \quad d\Omega = \frac{dS\cos\theta}{r^2} \tag{8}$$

From eq.(7) & (8)

$$d\phi = \frac{1}{4\pi\varepsilon_0} q d\Omega \tag{9}$$

Total flux through the closed surface S is

$$\phi = \oint_S d\phi = \oint_S \frac{1}{4\pi\varepsilon_0} q d\Omega$$

$$\Rightarrow \phi = \frac{q}{4\pi\varepsilon_0} \oint d\Omega$$

$$\Rightarrow \phi = \frac{q}{4\pi\varepsilon_0} \Omega \tag{10}$$

Here the hypothetical closed surface S subtends the total solid angle Ω at the point of location of the charge q.

From eq. (6) & (10)

$$\boxed{\phi = \begin{cases} \dfrac{q}{\varepsilon_0} & \text{if q lies inside the surface S} \\ 0 & \text{if q lies outside the surface S} \end{cases}} \tag{11}$$

Thus, in electrostatics, ***Gauss's Law is direct consequence of Coulomb's inverse square law***. Above expression shows Gauss's law for the single point charge.

If there are number of charges $q_1, q_2, q_3, ...q_N$ inside the closed surface and their contributions in electric field are $\vec{E}_1, \vec{E}_2, \vec{E}_3 \ldots\ldots\ldots \vec{E}_N$ respectively ,then by principle of superposition we have

$$\phi = \oint \vec{E}.d\vec{S}$$

$$= \oint (\vec{E}_1 + \vec{E}_2 + \vec{E}_3 + ...\vec{E}_N).d\vec{S}$$

$$\Rightarrow \phi = \oint \vec{E}_1.d\vec{S} + \oint \vec{E}_2.d\vec{S} + \oint \vec{E}_3.d\vec{S} + ...\oint \vec{E}_N.d\vec{S}$$

From eq.(11)

$$\phi = \frac{q_1}{\varepsilon_0} + \frac{q_2}{\varepsilon_0} + \frac{q_3}{\varepsilon_0} + ...\frac{q_N}{\varepsilon_0}$$

$$= \frac{\Sigma q_i}{\varepsilon_0}$$

$$\Rightarrow \phi = \frac{q_{enclosed}}{\varepsilon_0}$$

If the charge q lies outside the closed surface S, then it will not contribute in flux through surface S. We can verify this with help of Figure 1.6

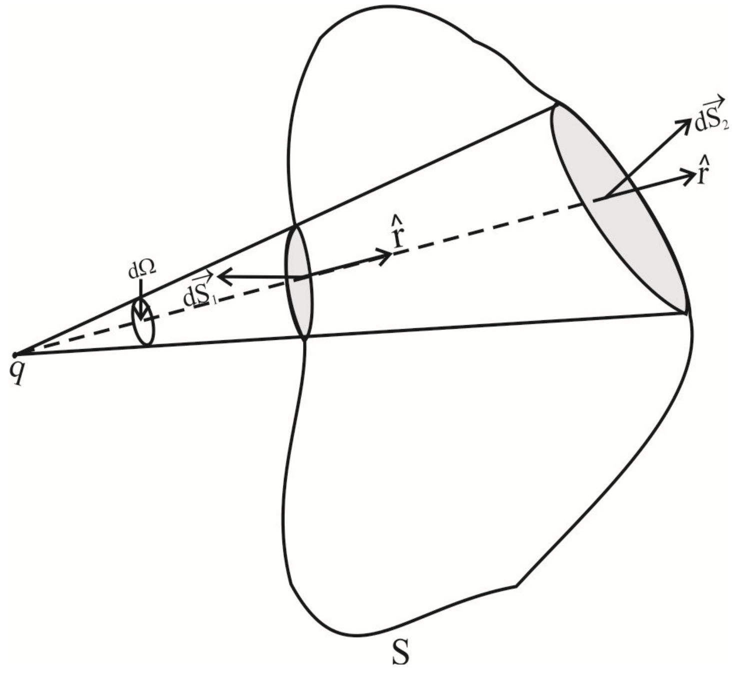


Figure1.6

A cone of solid angle $d\Omega$ is drawn with vertex at the point charge q. The cone cuts the surface S with intercepting area $d\vec{S}_1$ and $d\vec{S}_2$. Area elements $d\vec{S}_1$ and $d\vec{S}_2$ subtend solid angles of same magnetite $d\Omega$ at the point charge q.

From eq.(5) $d\Omega = \frac{d\vec{S}.\hat{r}}{r^2}$

For element $d\vec{S}_1$ $\Rightarrow d\Omega_1 = \frac{d\vec{S}_1.\hat{r}}{r^2} = \frac{dS_1 \cos\theta_1}{r^2}$

$\because \theta_1$ is obtuse angle, so $d\Omega_1$ is negative $\Rightarrow d\Omega_1 = -d\Omega$

For element $d\vec{S}_2$ $\Rightarrow d\Omega_2 = \frac{d\vec{S}_2.\hat{r}}{r^2} = \frac{dS_2 \cos\theta_2}{r^2}$

$\because \theta_2$ is acute angle, so $d\Omega_2$ is positive $d\Omega_2 = +\ d\Omega$

Flux contribution due to $d\vec{S}_1$ and $d\vec{S}_2$ are $d\phi_1$ and $d\phi_2$ respectively. Therefore net flux contribution due to both area elements is given by

$$d\phi = d\phi_1 + d\phi_2$$

From eq.(9)

$$d\phi = \frac{1}{4\pi\varepsilon_0} q d\Omega_1 + \frac{1}{4\pi\varepsilon_0} q d\Omega_2$$

$$= \frac{1}{4\pi\varepsilon_0} q(-d\Omega) + \frac{1}{4\pi\varepsilon_0} q d\Omega$$

$$= 0$$

If q is positive, then flux through dS_1 is an inward flux ,whereas flux through dS_2 is an outward flux, but algebraic sum of their contribution to net outward flux is zero.

This holds for all other elemental cones which cut the closed surface S and have vertices at the same outside position of the charge q.

We can say that contribution of outside charge in electric flux through closed surface is zero, because the incoming flux is equal to the outgoing flux. This verifies Gauss's Law.

Differential form of Gauss's Law of Electrostatics:

From Gauss's theorem

$$\oint_S \vec{E}.d\vec{S} = \frac{q_{enclosed}}{\varepsilon_0} \tag{12}$$

For continuous distribution of charge, we can write

$$q_{enclosed} = \int dq = \int_{\tau} \rho d\tau$$

where volume element $d\tau$ contains charge dq

$$\therefore \rho = \frac{dq}{d\tau} = \text{Volume charge density in } \frac{C}{m^3}$$

$$\Rightarrow q_{\ enclosed} = \int_{\tau} \rho d\tau \tag{13}$$

The charge density ρ may vary within the volume τ .

From eq.(12) &(13) $\boxed{\oint_S \vec{E}.d\vec{S} = \frac{1}{\varepsilon_0}\int_{\tau} \rho d\tau}$ (***Integral form of Gauss's law***)

Here surface S encloses the volume τ .By applying Gauss's divergence theorem

$$\int_{\tau}\left(\vec{\nabla}.\vec{E}\right)d\tau = \frac{1}{\varepsilon_0}\int_{\tau} \rho d\tau \qquad \left[\because \oint_S \vec{E}.d\vec{S} = \int_{\tau} \vec{\nabla}.\vec{E}\,d\tau\right]$$

Above equation holds for any arbitrary volume, therefore

$\boxed{\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}}$ (***Differential Form of Gauss's law***)

This equation is also known as ***Maxwell's first equation in electromagnetism***.

$$\Rightarrow \vec{\nabla}.\vec{E}\left(\vec{r}\right) = \frac{\rho\left(\vec{r}\right)}{\varepsilon_0} \tag{14}$$

Divergence of $\vec{E}$ at position $\vec{r}$ depends on the volume charge density ρ at that position $\vec{r}$ of the point i.e. div $\vec{E}$ is a function of co-ordinates and it may vary from point to point ,that's why differential form of Gauss's law is known as point form or local differential equation.

In Cartesian co-ordinates

$\boxed{\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\varepsilon_0}}$ where $\vec{E} = E_x \hat{i} + E_y \hat{j} + E_z \hat{k}$

In Spherical co-ordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2E_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,E_\theta\right)+\frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}=\frac{\rho}{\varepsilon_0}$$

where $\vec{E}=E_r\hat{r}+E_\theta\hat{\theta}+E_\phi\hat{\phi}$

In Cylindrical co-ordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rE_r\right)+\frac{1}{r}\frac{\partial E_\phi}{\partial\phi}+\frac{\partial E_z}{\partial z}=\frac{\rho}{\varepsilon_0}$$

where $\vec{E}=E_r\hat{r}+E_\phi\hat{\phi}+E_z\hat{z}$

We know that

$div\,\vec{E}=\underset{d\tau\to 0}{Lim}\dfrac{\oint\vec{E}.d\vec{S}}{d\tau}$ = Outflow of electric flux per unit volume at a given point in the limit of infinitesimal volume $d\tau$.

$div\,\vec{E}=\dfrac{\rho}{\varepsilon_0}$ means electric flux per unit volume at a point is equal to volume charge density ρ (charge per unit volume) at that point divided by ε_0 .

1.4 Illustrative Examples

Example1: An electric field is given(in spherical coordinates) by

$$\vec{E}=\frac{a}{r^2}\left(-\sin\theta\cos\phi\,\hat{r}+\cos\theta\cos\phi\hat{\theta}-\sin\phi\hat{\phi}\right)$$

Determine the volume charge density in the space(except the origin) .

Sol. Divergence of the electric field in spherical coordinates is given by

$$div\,\vec{E}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2E_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(E_\theta\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}$$

$$\Rightarrow div\,\vec{E}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(-r^2\frac{a}{r^2}\sin\theta\cos\phi\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{a}{r^2}\cos\theta\cos\phi\sin\theta\right)$$

$$+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(-\frac{a}{r^2}\sin\phi\right)$$

$$\Rightarrow div\,\vec{E}=0+\frac{1}{r\sin\theta}\frac{a}{r^2}\cos\phi\left(\cos^2\theta-\sin^2\theta\right)-\frac{1}{r\sin\theta}\frac{a}{r^2}\cos\phi$$

$$\Rightarrow div\vec{E} = \frac{a}{r^3}\frac{\cos\phi}{\sin\theta}\left(\cos^2\theta - \sin^2\theta\right) - \frac{a}{r^3}\frac{\cos\phi}{\sin\theta}$$

$$\Rightarrow div\vec{E} = \frac{a}{r^3}\frac{\cos\phi}{\sin\theta}\left(\cos^2\theta - \sin^2\theta - 1\right)$$

$$\Rightarrow div\vec{E} = \frac{a}{r^3}\frac{\cos\phi}{\sin\theta}\left(-2\sin^2\theta\right)$$

$$\Rightarrow div\vec{E} = -2\frac{a}{r^3}\sin\theta\cos\phi$$

$$\because div\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\therefore \rho = -2\frac{a\varepsilon_0}{r^3}\sin\theta\cos\phi$$

1.5 Self Learning Exercise-I

Q.1 Write the differential form of Gauss's law.

Q.2 Suppose Coulomb's law behaves as inverse cube instead of inverse square, then Gauss's law would not hold. Justify your answer.

Q.3 An infinitely long wire has linear charge density $10\frac{nC}{m}$ and this wire is located at $y = 3m,\ z = 4m$ .Find the electric field at the origin.

Q.4 A spherical shell carries volume charge density $\rho = \alpha r$ in the region $a \le r \le b$,where α is a constant . Find the electric field in the region

(i) $a \le r \le b$ (ii) $r > b$

1.6 Scalar Potential

Now we consider the Coulomb's law for the general case of volume charge.
From eq.(3),the electric field at a point $\vec{r} = (x, y, z)$ in terms of integral over the volume charge distribution $\rho(x', y', z')$ is written as

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\tau'}\frac{\rho(\vec{r}')\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau' \qquad (15)$$

Now we find the gradient of $\frac{1}{\mathscr{r}}$

$$\nabla\frac{1}{\mathscr{r}}=\nabla\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

$$=\nabla\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right)$$

$$=\sum\hat{i}\,\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right)$$

$$=\sum\hat{i}\left(-\frac{1}{2}\frac{2(x-x')+0+0}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{\frac{3}{2}}}\right)$$

$$=-\sum\frac{(x-x')\hat{i}}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{\frac{3}{2}}}$$

$$=-\frac{(x-x')\hat{i}+(y-y')\hat{j}+(z-z')\hat{k}}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{\frac{3}{2}}}$$

$$=-\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}=-\frac{\vec{\mathscr{r}}}{\mathscr{r}^3}$$

$$\therefore\quad \boxed{\nabla\frac{1}{\mathscr{r}}=-\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}} \tag{16}$$

From eq.(15)and(16)

$$\vec{E}(\vec{r})=-\frac{1}{4\pi\varepsilon_0}\int_{\tau'}\rho(\vec{r}')\nabla\left(\frac{1}{\mathscr{r}}\right)d\tau'$$

Here integration is over the primed coordinates (x',y',z') ,whereas gradient operation involves unprimed coordinates(x,y,z).So gradient operator can be taken outside the integral sign.

$$\vec{E}(\vec{r})=-\frac{1}{4\pi\varepsilon_0}\nabla\int_{\tau'}\rho(\vec{r}')\left(\frac{1}{\mathscr{r}}\right)d\tau'$$

$$\vec{E}(\vec{r})=-\nabla\left(\frac{1}{4\pi\varepsilon_0}\int_{\tau'}\frac{\rho(\vec{r}')}{\mathscr{r}}d\tau'\right)$$

$$\boxed{\vec{E} = -\nabla V} \tag{17}$$

where $$\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\tau'} \frac{\rho(\vec{r}')}{\mathscr{r}} d\tau' = \frac{1}{4\pi\varepsilon_0}\int_{\tau'} \frac{dq}{\mathscr{r}}} \tag{18}$$

Here V is known as electrostatic potential or scalar potential.

Here potential due to a point charge q is

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}}}$$

For collection of charges

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{\mathscr{r}_i}} \tag{19}$$

Therefore potential obeys the principle of superposition

$$V = V_1 + V_2 + V_3 + \ldots$$

For line charge distribution

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{L'} \frac{\lambda(\vec{r}')}{\mathscr{r}} dl'}$$

For surface charge distribution

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{S'} \frac{\sigma(\vec{r}')}{\mathscr{r}} dS'}$$

Symbol ϕ can also be used for potential.

Relation $\vec{E} = -\nabla V$ can be expressed in different-different coordinate systems as

In Cartesian co-ordinates

$$\boxed{\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right)}$$

In Spherical co-ordinates

$$\boxed{\vec{E} = -\left(\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}\right)}$$

In Cylindrical co-ordinates

$$\boxed{\vec{E} = -\left(\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{z}\right)}$$

Electric field $\vec{E}$ is a vector (three components)quantity, whereas potential V is a scalar(one component)quantity. Electric field can be derived from scalar function V by the gradient operation, so it is easy to deal with the function V .

Taking curl both sides of eq.(17)

$$\vec{\nabla}\times\vec{E}=\vec{\nabla}\times\left(-\nabla V\right)=0 \quad \left\{\because CurlGradV=0\right\}$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E}=0} \tag{20}$$

By Stoke's theorem

$$\int_S\left(\vec{\nabla}\times\vec{E}\right).d\vec{S}=\oint_C\vec{E}.d\vec{l} \tag{21}$$

Above expression holds for any arbitrary open area S enclosing the curve C .

By using eq.(20),(21) we can say that $\oint_C\vec{E}.d\vec{l}$ would vanish over any closed path.

Hence $\vec{\nabla}\times\vec{E}=0 \quad i.e. \boxed{\oint_C\vec{E}.d\vec{l}=0}$

$\vec{\nabla}\times\vec{E}=0$ means ***conservative field or irrotational field*** and line integral of electric field is independent of path.

We can get the potential difference between two points in following way

$$dV=\vec{\nabla}V.d\vec{l}$$

$$\Rightarrow dV=-\vec{E}.d\vec{l} \qquad \left\{\because \vec{E}=-\vec{\nabla}V\right\}$$

$$\Rightarrow \int_{initial}^{final}dV=-\int_{initial}^{final}\vec{E}.d\vec{l}$$

$$\Rightarrow \boxed{V_{final}-V_{initial}=-\int_{initial}^{final}\vec{E}.d\vec{l}}$$

We can take initial point as reference point $\vec{r}_{ref}$ and observation point $\vec{r}$, then we have

$$\boxed{V-V_{ref}=-\int_{\vec{r}_{ref}}^{\vec{r}}\vec{E}.d\vec{l}} \tag{22}$$

Here choice of reference point is arbitrary. Generally V_{ref} is taken zero at the reference point for convenience. Therefore

$$V=-\int_{\vec{r}_{ref}}^{\vec{r}}\vec{E}.d\vec{l} \tag{23}$$

Generally charge resides in a finite region of space ,so at infinite distance electric field vanishes .In such cases it is convenient to take infinity as reference point ,then eq.(23) becomes

$$V = -\int_{\infty}^{\vec{r}} \vec{E}.d\vec{l} \tag{24}$$

We know that the electric field is the force on unit test charge. Therefore from eq.(24) one can interpret the electrostatic potential at a point as the work done in bringing unit positive test charge from infinity to that observation point without changing the kinetic energy of this unit test charge.

1.7 Electrostatic Boundary Conditions

In electrostatics there are many problems in which surface charge density or potential is specified on boundary surfaces. Now we will discuss the behaviour of electric field and potential in crossing the boundary. The conditions that must be satisfied by fields or potentials at interface are known as boundary conditions.

(i) Boundary Condition on Normal Component of Electric Field

This boundary condition can be obtained by using Gauss's Law. We consider thin cylindrical Gaussian pillbox(Figure1.7) which intersects the interface. Interface has surface charge density $\sigma \frac{C}{m^2}$. Here σ may vary from point to point on the interface.

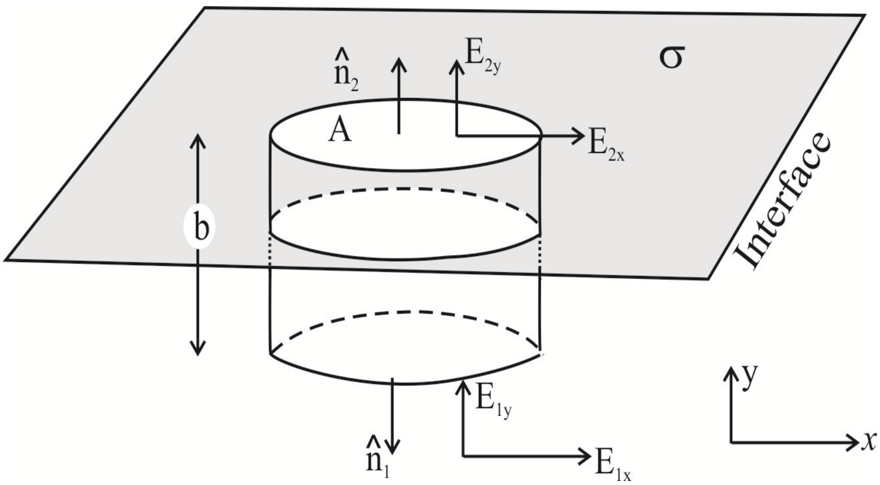


Figure1.7

Let the pillbox cuts out the area A on the interface and the height of the pillbox being negligibly small in comparison with diameter of its flat surface. Flux through the sides (curved surface) of the pillbox is negligible, because thickness b of the pillbox is taken to be arbitrarily small.

By applying Gauss's law to the pillbox

$$\oint \vec{E}.d\vec{S} = \frac{q_{enclosed}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

(Here we have assumed the area A to be extremely small so that σ remains constant on the area A)

We consider plane of the area A to be perpendicular to y axis(i.e. area vector $\vec{A}$ along the y axis).Here x component of electric field does not contribute in flux as it is parallel to the flat surface.

Thus $\vec{E}_1 . \vec{A}_1 + \vec{E}_2 . \vec{A}_2 = \dfrac{\sigma A}{\varepsilon_0}$

$$\Rightarrow \vec{E}_1 . \hat{n}_1 A + \vec{E}_2 . \hat{n}_2 A = \frac{\sigma A}{\varepsilon_0}$$

$$\Rightarrow (-E_{1y})A + (E_{2y}A) = \frac{\sigma A}{\varepsilon_0}$$

$$\Rightarrow E_{2y} - E_{1y} = \frac{\sigma}{\varepsilon_0}$$

$$\Rightarrow \boxed{E_{2n} - E_{1n} = \frac{\sigma}{\varepsilon_0}} \qquad (25)$$

"Here E_{2n} is the component of electric field which is normal to surface and just above it. Similarly normal component just below the surface is E_{1n}. Thus "***There is a discontinuity of*** $\dfrac{\sigma}{\varepsilon_0}$ ***in the normal component of electric field at the boundary***".

If surface charge density σ is zero then normal component of electric field is continuous across the boundary.

(ii) Boundary Condition on Tangential Component of Electric Field:

We know that ***electrostatic field*** $\vec{E}$ ***is conservative field , therefore circulation*** $\oint \vec{E}.d\vec{r}$ ***for any closed path is zero***. We use this property for determination of boundary condition on tangential component of electrostatic field $\vec{E}$.

Let us imagine a rectangular path *ABCD* as shown in diagram. Segments *BC* and *DA* are made extremely small and their contribution to the line integral is negligible.

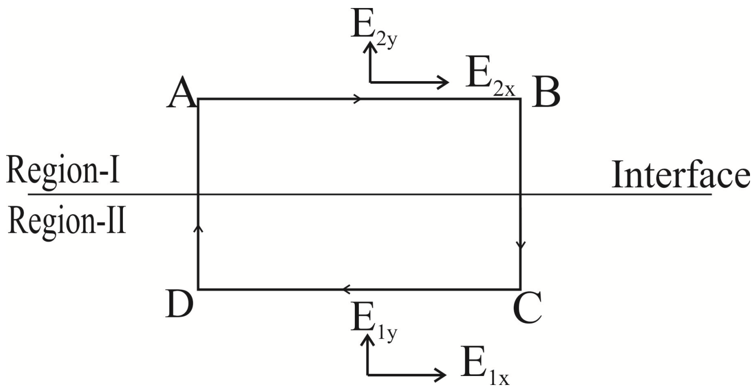


Figure1.8

Using $\oint \vec{E}.\vec{dl} = 0$

$$\vec{E}_2.\vec{AB} + \vec{E}_1.\vec{CD} = 0$$

$$\Rightarrow \; E_{2x} l + (-E_{1x}) l = 0 \qquad \{\because AB = CD = l\}$$

$$\Rightarrow \; E_{2x} = E_{1x}$$

$$\Rightarrow \; E_{2t} = E_{1t} \tag{26}$$

Here E_{1t} is the tangential component of electric field just below the interface and E_{2t} is the tangential component of electric field just above the interface. Thus "***The tangential component of electric field is always continuous across the interface***".

We can combine the boundary conditions into a single expression

From eq. (25)

$$E_{2n} - E_{1n} = \frac{\sigma}{\varepsilon_0}$$

$$\Rightarrow \; E_{2n}\hat{j} - E_{1n}\hat{j} = \frac{\sigma}{\varepsilon_0}\hat{j} \tag{27}$$

From eq. (26)

$$E_{2t} - E_{1t} = 0$$

$$\Rightarrow E_{2t}\hat{i} - E_{1t}\hat{i} = 0 \tag{28}$$

Adding eq. (27) & (28)

$$\left(E_{2t}\hat{i} + E_{2n}\hat{j}\right) - \left(E_{1t}\hat{i} + E_{1n}\hat{j}\right) = \frac{\sigma}{\in}\hat{j}$$

$$\Rightarrow \vec{E}_2 - \vec{E}_1 = \frac{\sigma}{\varepsilon_0}\hat{j}$$

$$\boxed{\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}} \tag{29}$$

where $\hat{n}$ is the unit vector normal to the surface and it is directed from below to above. Therefore by knowing the field on one side of the interface, we can find the field on the other side of the interface.

(iii) Boundary Condition on Potential :

We consider a segment *AB* of infinitesimal small length across the interface

Figure1.9

Potential difference

$$V_{above} - V_{below} = -\int_B^A \vec{E}.d\vec{r}$$

As the path length *AB* tends to zero its contribution to line integral can be neglected.

$$V_{above} - V_{below} \cong 0$$

$$\Rightarrow V_{above} = V_{below}$$

Therefore ***electrostatic potential is continuous across the interface***.

For example, we take uniformly charged thin spherical shell having total charge Q and radius R. Electric field just outside the shell is $E_2 = \dfrac{Q}{4\pi\varepsilon_0 R^2}$ or $E_2 = \dfrac{4\pi R^2\sigma}{4\pi\varepsilon_0 R^2} = \dfrac{\sigma}{\varepsilon_0}$ and electric field inside the shell is $E_1 = 0$. Therefore $E_2 - E_1 = \dfrac{\sigma}{\varepsilon_0} - 0 = \dfrac{\sigma}{\varepsilon_0}$ i.e. there is a discontinuity of $\dfrac{\sigma}{\varepsilon_0}$ in electric field in crossing the boundary. Electric potential just outside the shell is $V_2 = \dfrac{Q}{4\pi\varepsilon_0 R}$ and just inside the shell is also $V_1 = \dfrac{Q}{4\pi\varepsilon_0 R}$. Therefore $V_2 - V_1 = 0$. Hence there is no discontinuity in potential in crossing the boundary.

1.8 Discontinuity in Potential due to Dipole Layer

Consider a dipole layer which consists of closely spaced two surfaces S_1 and S_2. The surfaces S_1 and S_2 have equal and opposite surface charge densities. Suppose S_1 has surface charge density σ, then at just opposite points on the surface S_2 surface charge density is $-\sigma$ as shown in Figure1.10

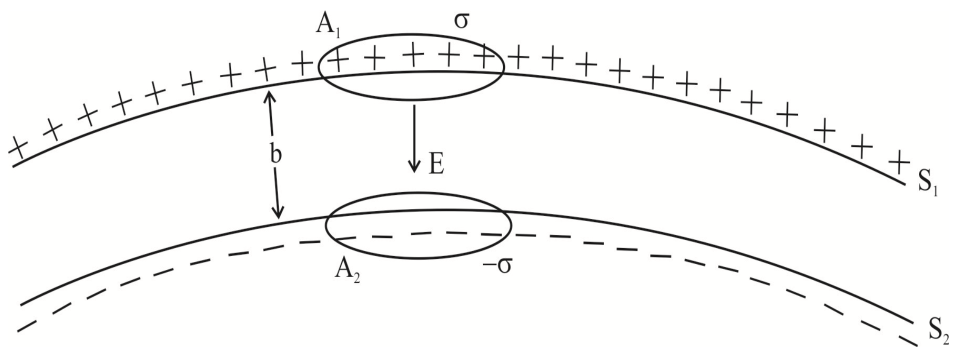


Figure1.10

Now we have to find out the change in potential in crossing the dipole layer.

$$V_2 - V_1 = -\int_{S_1}^{S_2} \vec{E}.d\vec{r}$$

Electric field inside the dipole is $\dfrac{\sigma}{\varepsilon_0}$. Since S_1 and S_2 are closely spaced, therefore a point between the dipole layer sees oppositely charged two sheets of infinite dimensions. Visualization of $E=\frac{\sigma}{\varepsilon_0}$ is analogous to electric field between the plates of parallel plate capacitor\. In case of dipolar layer, area elements A_1 and A_2 (see figure1.10) serve as parallel plates for observation point (which lies in between these area elements).

$$\therefore |V_2-V_1|=\left|\int \vec{E}.d\vec{r}\right|$$

$$=\left|\int \frac{\sigma}{\varepsilon_0}dr\right|=\left|\frac{\sigma}{\varepsilon_0}\int dr\right|$$

$$|V_2-V_1|=\left|\frac{\sigma b}{\varepsilon_0}\right| \tag{30}$$

where infinitesimal small distance between layers S_1 and S_2 is $b(\ i.e. b\to 0)$

Note that in case of ideal dipole layer surface charge density σ has infinitely high value i.e. $\sigma\to\infty$ therefore $(\sigma b)$is finite . By eq.(30) we can say that there is a discontinuity of $\frac{\sigma b}{\varepsilon_0}$ in potential in crossing the dipole layer.

1.9 Illustrative Examples

Example 2: An electric field is given(in spherical coordinates) by

$$\vec{E}=\frac{\alpha}{r^3}\cos\theta\hat{r}+\frac{\beta}{r^3}\sin\theta\hat{\theta}$$

Find the relation between α and β for this field to be an electrostatic field.

Sol. For conservative nature of electrostatic field $\vec{\nabla}\times\vec{E}=0$

In spherical coordinates , we have

$$\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ E_r & rE_\theta & r\sin\theta E_\phi\end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ \left(\dfrac{\alpha}{r^3}\cos\theta\right) & r\left(\dfrac{\beta}{r^3}\sin\theta\right) & r\sin\theta(0) \end{vmatrix} = 0$$

$$\Rightarrow \hat{r}(0) - r\hat{\theta}(0) + r\sin\theta\hat{\phi}\left[\frac{\partial}{\partial r}\left(\frac{\beta}{r^2}\sin\theta\right) - \frac{\partial}{\partial \theta}\left(\frac{\alpha}{r^3}\cos\theta\right)\right] = 0$$

$$\Rightarrow \left(\frac{-2\beta}{r^3}\sin\theta\right) - \frac{\partial}{\partial\theta}\left(-\frac{\alpha}{r^3}\sin\theta\right) = 0$$

$$\Rightarrow \alpha = 2\beta$$

Example 3: Electric potential is given(in spherical coordinates) by

$V = \dfrac{a}{r}\cos\theta\sin\phi$ where a is constant. Find the expression for electric field. Also calculate the electric filed at point $\left(1, \dfrac{\pi}{3}, \dfrac{\pi}{6}\right)$

Sol. Electric field in spherical coordinates is given by

$$\vec{E} = -\left(\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}\right)$$

$$\vec{E} = -\left(-\frac{a}{r^2}\cos\theta\sin\phi\hat{r} - \frac{1}{r}\frac{a}{r}\sin\theta\sin\phi\hat{\theta} + \frac{1}{r\sin\theta}\frac{a}{r}\cos\theta\cos\phi\hat{\phi}\right)$$

$$\Rightarrow \vec{E} = \frac{a}{r^2}\cos\theta\sin\phi\hat{r} + \frac{a}{r^2}\sin\theta\sin\phi\hat{\theta} - \frac{a}{r^2}\cot\theta\cos\phi\hat{\phi}$$

Electric field at point $\left(1, \dfrac{\pi}{3}, \dfrac{\pi}{6}\right)$ is

$$\vec{E} = \frac{a}{1^2}\cos\frac{\pi}{3}\sin\frac{\pi}{6}\hat{r} + \frac{a}{1^2}\sin\frac{\pi}{3}\sin\frac{\pi}{6}\hat{\theta} - \frac{a}{1^2}\cot\frac{\pi}{3}\cos\frac{\pi}{6}\hat{\phi}$$

$$\Rightarrow \vec{E} = a\left(\frac{1}{4}\hat{r} + \frac{\sqrt{3}}{4}\hat{\theta} - \frac{1}{2}\hat{\phi}\right)$$

Example 4: A potential in some region is given(in cylindrical coordinates) by $V = ar^2 z\cos\phi$

Find (i) Electric field (ii) charge density

Sol. Electric field

$$\vec{E} = -\left(\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{z}\right)$$

$$\Rightarrow \vec{E} = -\left\{\frac{\partial}{\partial r}\left(ar^2 z\cos\phi\right)\hat{r} + \frac{1}{r}\frac{\partial}{\partial \phi}\left(ar^2 z\cos\phi\right)\hat{\phi} + \frac{\partial}{\partial z}\left(ar^2 z\cos\phi\right)\hat{z}\right\}$$

$$\Rightarrow \vec{E} = -\left\{2arz\cos\phi\hat{r} - \frac{1}{r}ar^2 z\sin\phi\hat{\phi} + ar^2\cos\phi\hat{z}\right\}$$

$$\Rightarrow \vec{E} = -2arz\cos\phi\hat{r} + arz\sin\phi\hat{\phi} - ar^2\cos\phi\hat{z}$$

From Gauss's Divergence law $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$

$$\because \vec{\nabla}.\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}\left(rE_r\right) + \frac{1}{r}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(-r2arz\cos\phi\right) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(arz\sin\phi\right) + \frac{\partial}{\partial z}\left(-ar^2\cos\phi\right)$$

$$= \frac{1}{r}\left(-4arz\cos\phi\right) + \frac{1}{r}\left(arz\cos\phi\right) + 0$$

$$= -4az\cos\phi + az\cos\phi$$

$$= -3az\cos\phi$$

Charge density $\rho = -3\varepsilon_0 az\cos\phi \quad \because \vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$

1.10 Self Learning Exercise-II

Q.1 "The potential difference between any two points must be zero for the electrostatic field to be a conservative field". Is this statement true?

Q.2 Consider a uniformly charged sphere of radius R and charge density ρ.

Write the values of the divergence and curl of electrostatic field for given region

(i) r < R (ii) r > R

Q.3 The distance between the plates of parallel plate capacitor is 5cm and potential difference is 100Volt. Find the force experienced by an alpha particle entered into the field.

Q.4 An electric field in some region is given (in spherical coordinates) by

$\vec{E} = a\,r\cos\theta\,\hat{r} - a\,r\sin 2\theta\,\hat{\theta}$,where a is constant. Evaluate the volume charge density.

1.11 Summary

1. For continuous charge distribution electric field

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{\mathcal{r}^2}\hat{\mathcal{r}}$$

2. Gauss's Law

$$\oint \vec{E}.d\vec{S} = \frac{q_{enclosed}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_V \rho dV \qquad (\textit{Integral form of Gauss's law})$$

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0} \qquad (\textit{Differential form of Gauss's law})$$

3. Electric field $\vec{E} = -\nabla V$

where electrostatic potential $V(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_{\tau'} \frac{\rho(\vec{r}\,')}{\mathcal{r}}d\tau' = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_{\tau'} \frac{dq}{\mathcal{r}}$

Potential Difference $V_{final} - V_{initial} = -\displaystyle\int_{initial}^{final} \vec{E}.d\vec{l}$

4. Boundary condition on electric field at interface

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

1.12 Glossary

Boundary Conditions: The conditions that must be satisfied by fields or potentials at interface are known as boundary conditions.

Conservative Field: For vector field $\vec{A}$ to be conservative field, $\vec{\nabla}\times\vec{A}$ must be zero .

1.13 Answers to Self Learning Exercises

Answers to Self Learning Exercise-I

Ans.2: We consider a point charge q at the centre of the sphere (Gaussian surface).Now electric flux passing through the spherical surface of radius r is $\oint \vec{E}.d\vec{S} = E.4\pi r^2 = \frac{\alpha q}{r^3} 4\pi r^2 = 4\pi\alpha q \frac{1}{r} \neq$ constant .This electric flux depends on radius of sphere r. Therefore Gauss's law would not hold in this case.

Ans.3: $E = \frac{2k\lambda}{\imath}\hat{\imath} = \left(-21.6\hat{j} - 28.8\hat{k}\right)\frac{N}{C}$, Here $\hat{\imath} = \frac{-3\hat{j} - 4\hat{k}}{5}$

Ans.4:(i) $\vec{E} = \frac{\alpha}{4\varepsilon_0}\left(r^2 - \frac{a^4}{r^2}\right)\hat{r}$ **(ii)** $\vec{E} = \frac{\alpha}{4\varepsilon_0}\left(\frac{b^4 - a^4}{r^2}\right)\hat{r}$

Answers to Self Learning Exercise-II

Ans.1: False

Ans.2: (i) For r < R ; $div\vec{E} = \frac{\rho}{\varepsilon_0}, Curl\vec{E} = 0$

(ii) For r > R ; $div\vec{E} = 0, Curl\,\vec{E} = 0$

Ans.3: $E = -\frac{dV}{dx} = \frac{100}{5\times10^{-2}} = 2000\frac{N}{C}$

$F = qE = 2\times1.6\times10^{-19}\times2000 = 6.4\times10^{-16}N$

Ans.4: $a\varepsilon_0\left(3\cos\theta - 6\cos^2\theta + 2\right)$

1.14 Exercise

Section A:Very Short Answer Type Questions

Q.1 "A potential is given by $V = 2yz^3 + zx$,then $yz^3 = 2$ is the equipotential curve on the yz plane."Is this statement true?

Q.2 "The line integral $\int_A^B \vec{E}.d\vec{l}$ for any electrostatic field has the same value for all paths from the point A to the point B". Do you agree with this statement?

Q.3 An electric field in some region is given by $\vec{E} = ax\hat{i} + by\hat{j} + cz\hat{k}$,where a,b,c are constants. Find the volume charge density.

Q.4 Plane $x = 5m$ carries charge 10 $\frac{nC}{m^2}$ and plane $y = 15m$ carries charge -5 $\frac{nC}{m^2}$.

What is the electric field at the origin ? Here $\frac{1}{4\pi\varepsilon_0} = 9\times10^9 \frac{Nm^2}{C^2}$

Section B: Short Answer Type Questions

Q.5 Use Gauss's law to prove that the electric field at the surface of a conductor is normal to the surface and has magnitude $\frac{\sigma}{\varepsilon_0}$,where σ is the local surface charge density on that surface. Also check that the result is consistent with the boundary conditions.

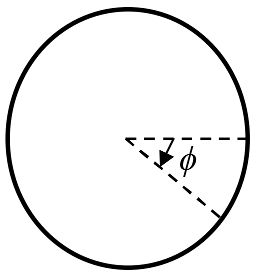
Q.6 Obtain the boundary conditions on normal and tangential components of electric field.

Q.7 An electric field in some region is given (in cylindrical coordinates) by

$\vec{E} = ar\sin\phi\hat{r} + br\cos\phi\hat{\phi}$,where a and b are constants. Evaluate the volume charge density.

Q.8 A ring of radius R has linear charge density density $\lambda = \lambda_0 \cos^2\phi$.

Find the electric potential at a point (0,0,z) on the axis of the ring. Consider origin at the centre of the ring.



Section C: Long Answer Type Questions

Q.9 A long cylinder has volume charge density $\rho = \alpha r^2$,where α is constant and r is the distance from the axis of the cylinder. Radius of the cylinder is R.

Use Gauss's law to find the electric field

(i) inside the cylinder (ii) outside the cylinder .

1.15 Answers to Exercise

Ans.1: Yes, on yz plane $x = 0$. Therefore V has value $V = 2.2 + 0 = 4$ which is constant.

Ans.2: Yes, because electrostatic field is conservative field and it is path independent .

Ans.3: $\varepsilon_0 \left(a + b + c \right)$

Ans.4: $\left(-180\pi\hat{i} + 90\pi\hat{j} \right) \frac{V}{m}$

Ans.7: $\varepsilon_0 \left(2a\sin\phi - b\cos\phi \right)$

Ans.8: $\frac{\lambda_0 R}{4\varepsilon_0 \sqrt{R^2 + z^2}}$

Ans.9: **(i)** $\vec{E} = \frac{\alpha r^3}{4\varepsilon_0}\hat{r}$ **(ii)** $\vec{E} = \frac{\alpha R^4}{4\varepsilon_0 r}\hat{r}$

References and Suggested Readings

1. John David Jackson ,Classical Electrodynamics ,Third Edition ,Wiley India Pvt.Ltd.,2010
2. David J. Griffiths ,Introduction to Electrodynamics, Third Edition ,Pearson Education ,Inc.,2009
3. Matthew N.O. Sadiku ,Elements of Electromagnetism, Third Edition ,Oxford University Press,2003
4. A.S.Mahajan & A.A.Rangwala ,Electricity and Magnetism, Tata McGraw Hill,2002
5. D.C.Chattopadhyay &P.C. Rakshit ,Electricity & Magnetism ,Second Edition ,New Central Book Agency P.Ltd 1993
6. John R. Reitz, Frederick J. Milford, Robert W. Christy , Foundations of Electromagnetic Theory (4th Edition) ,Addison-Wesley, 2008
7. Arthur Kip, Fundamental of Electricity &Magnetism(Second Edition),McGraw Hill International Editions,1969

UNIT-2
Poisson and Laplace Equations

Structure of the Unit

2.0 Objectives

Electrostatics deals with those phenomena that originate from time-independent distributions of charge and fields. The theory of the electrostatic field is based on Coulomb's law. The behaviour of an electrostatic field can be described by Poisson equation. In regions of space that lack charge density, the scalar potential satisfies the Laplace equation. In this chapter we shall focus upon the solution of Poisson and Laplace differential equations satisfying certain boundary conditions with the help of Green functions.

2.1 Introduction

Macroscopic electrodynamics is concerned with the study of electromagnetic fields in space that is occupied by matter. Like all macroscopic theories, electrodynamics deals with physical quantities averaged over elements of volume which are "Physically infinitesimal" ignoring the microscopic variations of the quantities which result from the molecular structure of matter.

Behaviours of an electrostatic field is given by Poisson and Laplace equations along with suitable boundary conditions. In this chapter we shall solve these equations with the help of Green's theorem.

2.2 Poisson and Laplace Equations

We know that an electrostatic field $\vec{E}$ can described by the two differential equations :

$$\boxed{\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}} \quad (\textbf{\textit{Gauss's law}}) \tag{1}$$

and

$$\boxed{\vec{\nabla} \times \vec{E} = 0} \quad (\rho \text{ is charge density}) \tag{2}$$

Equation (2) is equivalent to the statement that

$$\boxed{\vec{E} = -\vec{\nabla}\phi} \tag{3}$$

where ϕ is scalar potential.

From (1) an (3), we readily obtain

$$\vec{\nabla}.(-\vec{\nabla}\phi) = \frac{\rho}{\varepsilon_0}$$

or $$\boxed{\nabla^2\phi = -\frac{\rho}{\varepsilon_0}} \quad (\textbf{\textit{Poisson Equation}}) \tag{4}$$

Equation (4) is called the Poisson equation.

In regions of space where charge density $\rho = 0$, Eq (4)becomes

$$\boxed{\nabla^2\phi = 0} \quad (\textbf{\textit{Laplace}} \text{ equation}) \tag{5}$$

Equation (5) is called the Laplace equation.

We now determine the field produced by a point charge. From symmetry considerations, it is clear that it is directed along the radius-vector from the point at which the charge e is located.

From the same considerations it is clear that the value E of the field depends only on the distance R from the charge.

To find this absolute value, we make use of the Gauss's law (Eq.1) in the integral form :

Flux of electric field out of the closed surface $= \dfrac{charge\ enclosed}{\epsilon_0}$

$$\text{i.e. } E \times 4\pi R^2 = \frac{e}{\epsilon_0}$$

$$E = \frac{e}{4\pi \epsilon_0 R^2}$$

In vector notation

$$\vec{E} = \frac{e}{4\pi \epsilon_0 R^2}\hat{R} = \frac{e}{4\pi \epsilon_0 R^3}\hat{R} \tag{6}$$

This is Coulomb's law. The potential of this field is clearly

$$\phi = \frac{e}{4\pi \epsilon_0 R} \tag{7}$$

If we have a system of charges then the field produced by this system is equal, according to the principle of superposition, to the sum of fields produced by each of the particles individually. In particular the potential of such a field is

$$\boxed{\phi = \frac{1}{4\pi \epsilon_0}\sum_a \frac{e_a}{R_a}}$$

,where R_a is the distance from the charge e_a to the point at which we are determining the potential. If we introduce the charge density ρ, this formula takes on the form :

$$\boxed{\phi = \frac{1}{4\pi \epsilon_0}\int \frac{\rho dV}{R}} \tag{8}$$

where R is the distance from the volume element dV to the given point of the field . eq (8) is the solution of Poisson Equation (4).

We note a mathematical relation which is obtained from (4) by substituting the values of ρ and ϕ for a point charge, i.e. $\rho = e\delta(\vec{R})$ and $\phi = \dfrac{e}{4\pi \in_0 R}$ we then find

$$\nabla^2\left(\frac{e}{4\pi \in_0 R}\right) = -\frac{1}{\in_0} e\delta(\vec{R})$$

or $$\boxed{\nabla^2\left(\frac{1}{R}\right) = -4\pi\delta(\vec{R})} \qquad (9)$$

2.3 Green's Theorem

If electrostatic problems always involved localized discrete or continuous distributions of charge with no boundary surfaces, the general solution(8) could be the most convenient and straight forward solution to any problem. There would be no need of the Poisson or Laplace equation.

In actual fact, many of the problems of electrostatics involve finite regions of space, with or without charge inside, and with prescribed boundary conditions on the bounding surfaces.

To handle the boundary conditions it is necessary to develop some new mathematical tools, namely the identities or theorems due to George Green (1824). These follow as simple applications of the divergence theorem.

$$\int_V \vec{\nabla}.\vec{A}\, d^3x = \oint_S \vec{A}.\hat{n}\, da \qquad \text{(Integral form of Gauss's law)} \qquad (10)$$

Let $\vec{A} = \phi\vec{\nabla}\psi$

where ϕ and ψ are arbitrary scalar fields.

Using $\nabla.(\phi\nabla\psi) = \nabla\phi.\nabla\psi + \phi\nabla^2\psi$

and $\phi\nabla\psi.\hat{n} = \phi\dfrac{\partial\psi}{\partial n}$ in (10), we get

$$\int_V [\phi\nabla^2\psi + \nabla\phi.\nabla\psi]\mathrm{d}^3 x = \oint_S \phi\frac{\partial\psi}{\partial n} da \qquad (11)$$

This result is known as Green's first identity.

If we write down (11) again with ϕ and ψ interchanged and then subtract it form (11), the $\nabla\phi.\nabla\psi$ terms cancel, we obtains Green's second identity :

$$\int_V [\phi\nabla^2\psi - \psi\nabla^2\phi]\,\mathrm{d}^3 x = \oint_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] da \qquad (12)$$

The Poisson differential equation for the potential can be converted into an integral equation, if we choose

$$\psi = \frac{1}{R} = \frac{1}{\left|\vec{x} - \vec{x}'\right|} \;,$$

where $\vec{x}$ is the point of observation and $\vec{x}\,'$ is the integration variable.

Further we put $\phi = \Phi$ (the scalar potential) and use $\nabla^2\Phi = -\frac{\rho}{\varepsilon_0}$ from (9) ,we know

that $\nabla^2\left(\frac{1}{R}\right) = -4\pi\delta(\vec{x} - \vec{x}')$,so that (12) becomes :

$$\int_V \left[-4\pi\Phi(\vec{x}')\delta(\vec{x} - \vec{x}') + \frac{1}{\epsilon_0 R}\rho(\vec{x})\,\mathrm{d}^3 x'\right] = \oint_s \left[\Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial\Phi}{\partial n'}\right] da'$$

If the point $\vec{x}$ lies within the volume V, we obtain:

$$-4\pi\,\Phi(\vec{x}) + \frac{1}{\epsilon_0}\int_v \frac{\rho(\vec{x}')\,\mathrm{d}^3 x'}{R} = \oint_S \left[\Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial\Phi}{\partial n'}\right] da'$$

or $$\Phi(\vec{x}) = \frac{1}{4\pi\,\epsilon_0}\int_V \frac{\rho(\vec{x}')\,\mathrm{d}^3 x'}{R} + \frac{1}{4\pi}\oint_s \left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right] da' \qquad (13)$$

where we have used the well known property of δ-function viz.

$$\int_V \Phi(\vec{x}')\delta(\vec{x} - \vec{x}')\,\mathrm{d}^3 x' = \Phi(\vec{x})$$

If $\vec{x}$ lies outside the surface S, the left-hand-side of (13) is zero. This follows from the discontinuities in electric field and potential across the surface.

From the result (13) it may be noted that, if the surface S goes to infinity and the electric field on S falls faster than R^{-1},then the surface integral vanishes and (13) reduces to the familiar result :

$$\Phi(\vec{x}) = \frac{1}{4\pi \in_0} \int \frac{\rho(\vec{x'})}{|\vec{x} - \vec{x'}|} d^3x'$$

From (13) we also note that for a charge-free volume, the potential anywhere inside the volume (a solution of the Laplace equation) is expressed in terms of the potential and its normal derivative only on the surface of the volume.

This rather surprising result is not a solution to a boundary-value problem, but only an integral statement, since the arbitrary specification of both Φ and $\frac{\partial \Phi}{\partial n}$ is an over specifications of the problem.

We shall discuss in detail the techniques yielding solutions for appropriate boundary conditions using Green's theorem (Eq.12)

2.4 Dirichlet/Neumann Boundary Conditions

What boundary conditions are appropriate for the Poisson (or Laplace) equation to ensure that a unique and well-behaved (i.e. physically reasonable) solution will exist inside the bounded region?

From experience we are led to believe that the ***specification of the potential on closed surface (e.g. a system of conductors held at different potentials) defines a unique potential problem. This is called a Dirichlet problem or Dirichlet boundary conditions.***

Similarly it is ***plausible specifications of the electric field (normal derivative of the potential) everywhere on the surface (corresponding to a given surface charge density) also defines a unique problem. Specification of the normal derivative is known as the Neumann boundary conditions.***

2.5 Formal solution of Electrostatic Boundary-value Problem with Green Function

We can solve Poisson or Laplace equation in a finite volume V with either Dirichlet or Neumann boundary conditions on the bounding surface S by using Greens's theorem.

In obtaining the result (13) - not a solution -We chose the function to be $\frac{1}{|\vec{x}-\vec{x}'|}$,it being a potential of a unit point source, satisfying the equation.

$$\boxed{\nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\delta(\vec{x}-\vec{x}')} \tag{14}$$

The function $\frac{1}{|\vec{x}-\vec{x}'|}$ is only one of class of functions depending on the variables $\vec{x}$ and $\vec{x}'$ and called "***Green functions***", which satisfy (14).

In general

$$\boxed{\nabla^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')} \tag{15}$$

where $G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}')$ (16)

With the function F satisfying the Laplace equation inside the volume V :

$$\nabla'^2 F(\vec{x},\vec{x}') = 0 \tag{17}$$

Recalling Green's second identity of Green's theorem (Eq12) viz.

$$\int_V (\phi\nabla^2\Psi - \Psi\nabla^2\phi)\,\mathrm{d}^3 x = \oint_s \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] da$$

Now we substituting

$$\phi = \Phi, \psi = G(\vec{x},\vec{x}')$$

and using property of G viz.

$$\nabla'^2 G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$$

We obtain

$$\Phi(\vec{x}) = \frac{1}{4\pi\in_0}\int_V \rho(\vec{x}')G(\vec{x},\vec{x}')\,\mathrm{d}^3 x' + \frac{1}{4\pi}\int_S \left[G(\vec{x},\vec{x}')\frac{\partial\Phi}{\partial n'} - \Phi(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\right] da' \tag{18}$$

The freedom available in the definition of G (Eq.16) means that we can make the surface integral depend only on the chosen type of boundary conditions.

Thus for Dirichlet boundary conditions we demand

$$G_D(\vec{x},\vec{x}') = 0 \ \text{ for } \vec{x}' \, on \, S \tag{19}$$

Then the first term in the surface integral in (18) vanishes and the solution is

$$\Phi(\vec{x}) = \frac{1}{4\pi \in_0} \int_v \rho(\vec{x'}) G_D(\vec{x}, \vec{x'}) \mathrm{d}^3 x' - \frac{1}{4\pi} \oint_s \Phi(\vec{x'}) \frac{\partial G_D}{\partial n'} da' \tag{20}$$

For Neumann boundary conditions, the obvious choice of boundary condition on $G(\vec{x}, \vec{x'})$ seems to be

$$\frac{\partial G_N}{\partial n'}(\vec{x}, \vec{x'}) = 0 \quad \text{for } \vec{x'} \, on \, S$$

Since that makes the second term in the surface integral in (18) vanish, as desired

But an application of Gauss's theorem to (14) shows that

$$\oint \frac{\partial G}{\partial n'} da' = -4\pi$$

Consequently the simplest allowable boundary condition on G_N is

$$\frac{\partial G_N}{\partial n'}(\vec{x}, \vec{x'}) = -\frac{4\pi}{S} \quad \text{for } \vec{x}' \quad \text{on} \quad S \tag{21}$$

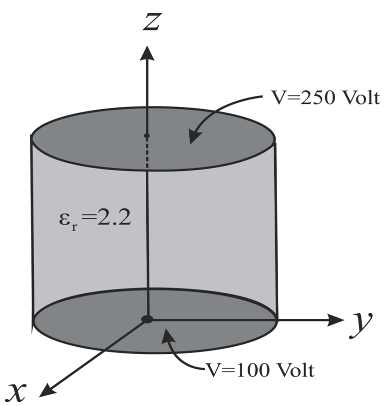
where S is the total area of the boundary surface. Then the solution is

$$\phi(\vec{x}) = \langle \Phi \rangle_S + \frac{1}{4\pi \in_0} \int_V \rho(\vec{x'}) \mathrm{G}_N(\vec{x}, \vec{x'}) \mathrm{d}^3 x' + \frac{1}{4\pi} \int_S \frac{\partial \phi}{\partial n'} G_N da'$$

Where $\langle \Phi \rangle_S$ is the average value of the potential over the whole surface.

2.6 Illustrative Examples

Example1 The parallel conducting disks are separated by 5 mm and contain a dielectric for which $\in_r = 2.2$. Determine the charge densities on the disks. See figure.



Sol. $V = Az + B$

$$A = \frac{\Delta V}{\Delta z} = \frac{250-100}{5\times10^{-3}}\frac{V}{m}$$

$$= 3\times10^{4}\frac{V}{m}$$

$$\because \vec{E} = -\vec{\nabla}V = -3\times10^{4}\hat{z}\frac{V}{m}$$

$$\vec{D} = \in_0 \in_r \vec{E}$$

$$= -5.84\times10^{-7}\hat{z}\frac{e}{m^2}$$

Since $\vec{D}$ is constant between disks and

$D_n = \rho_s$ at a conductor surface.

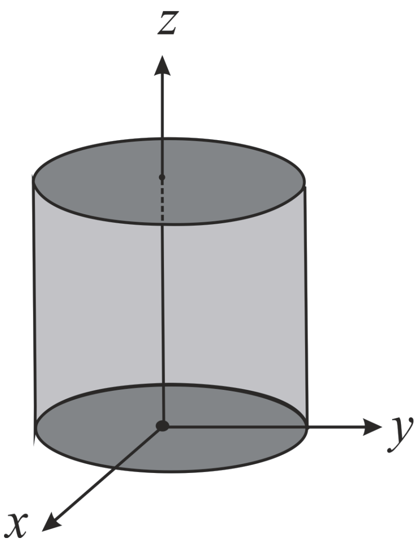
$$\rho_s = \pm 5.84\times10^{-7}\frac{C}{m^2}$$

+ on the upper plate and - on the lower plate.

2.7 Self Learning Exercise-I

Q.1 Write Laplace Equation in rectangular coordinates.

Q.2 Find the potential function V for the region between the parallel circular disks of Figure shown.



Q.3 What are Dirichlet and Neumann boundary conditions?

Q.4 From the Laplace equation $\Delta\Phi = 0$ (where Δ means Laplacian operator ∇^2), show that the potential of the electric field can nowhere have a maximum or a minimum.

2.8 Electrostatic Potential Energy and Energy Density

If a point charge is brought from infinity to a point $\vec{x}_i$ in a region of localized electric fields described by the scalar potential Φ (which vanishes at infinity), the work done on the charge (and hence the potential energy) is given by

$$\boxed{W_i = q_i.\Phi(\vec{x}_i)} \tag{1}$$

The potential Φ can be viewed as produced by an array of (n-1) charges q_j (j=1,2,……n-1) at positions $\vec{x}_j$. Then

$$\Phi(\vec{x}_i) = \frac{1}{4\pi \in_0}\sum_{j=1}^{n-1}\frac{q_j}{|\vec{x}_i - \vec{x}_j|} \tag{2}$$

So that the potential energy of the charge q_i is

$$W_i = \frac{q_i}{4\pi \in_0}\sum_{j=1}^{n-1}\frac{q_j}{|\vec{x}_i - \vec{x}_j|} \tag{3}$$

The total potential energy of all charges due to all the forces acting between them is

$$W = \frac{1}{4\pi \in_0}\sum_{i=1}^{n}\sum_{j<i}\frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} \tag{4}$$

A most symmetric form can be written by summing over *i* and *j* unrestricted and then dividing by 2 :

$$\boxed{W = \frac{1}{8\pi \in_0}\sum_{i}\sum_{j}\frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|}} \tag{5}$$

It is understood that $i = j$ (infinite self energy terms) are omitted in the double sum.

For a continuous charge distribution, the potential energy takes the form :

$$\boxed{W=\frac{1}{2}\int \rho(\vec{x})\Phi(\vec{x})\,\mathrm{d}^3 x} \tag{6}$$

An alternative and very useful approach is to emphasize the electric field and to interpret the energy as being stored in the electric field surrounding the charges. To obtain this form, we make use of the Poisson equation to eliminate the charge density ρ from (6) :

$$W=-\frac{\epsilon_0}{2}\int \Phi\nabla^2\Phi d^3x$$

$$\left(use\nabla^2\Phi=-\frac{\rho}{\epsilon_0}\right)$$

Integration by parts we get

$$W=-\frac{\epsilon_0}{2}\left\{\Phi(\nabla\phi)\Big|_{\text{surface at infinity}}-\int\left(\nabla\phi\right)^2 d^3x\right\}$$

The first term vanishes at infinity

$W=\frac{\epsilon_0}{2}\int\left|E\right|^2 d^3x$, where the integration is over all space. (7)

Capacitance :

Consider a system of n conductors, each with potential V_i and total charge Q_i (i=1, 2, ….n) in otherwise empty space, the electrostatic potential energy can be expressed in terms of the potentials alone and certain geometrical quantities called coefficients of capacity.

For a given configuration of the conductors, the potential of the conductor can be written as

$$V_i=\sum_{j=1}^{n} p_{ij}Q_j \qquad \text{(i=1,2, ……n)} \tag{1}$$

Where the p_{ij} depend on the geometry of the conductors.

These *n* equations can be inverted to yield the charge on the *i*th conductor in terms of all the potentials :

$$Q_i=\sum_{j=1}^{n} C_{ij}V_j \qquad \text{(i=1,2, ……n)} \tag{2}$$

The coefficients C_{ii} are called capacities or capacitances while the C_{ij} , $i \neq j$ are called coefficients of induction.

"The capacitance of a conductor is therefore the total charge on the conductor when it is maintained at unit potential, all other conductors being held at zero potential".

Sometimes the capacitance of two conductors carrying equal and opposite charges in the presence of other grounded conductors is defined as the ratio of the charge on one conductor to the potential difference between them.

The potential energy for the system of conductors is

$$W = \frac{1}{2}\sum_{i=1}^{n} Q_i V_i = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{b} C_{ij} V_i V_j$$

2.9 Illustrative Examples

Example 2: Calculate the force per unit area on the surface of a conductor with a surface charge density $\sigma(\vec{x})$.

Sol. In the immediate neighbourhood of the surface the ***energy density*** is

$$\boxed{w = \frac{\epsilon_0}{2}|E|^2}$$

But using Gauss's Law

$$EA = \frac{\sigma A}{\epsilon_0}$$

Therefore

$$\boxed{w = \frac{\sigma^2}{2\epsilon_0}}$$

If we now imagine a small outward displacement of an infinitesimal area Δa of the conducting surface ,the electrostatic energy decreases by an amount that is the product of energy density and the excluded volume $\Delta x . \Delta a$.

$$\Delta W = -\frac{\sigma^2}{2\epsilon_0}\Delta a . \Delta x \equiv F . \Delta x$$

This means that there is an ***outward force per unit area*** are equal to $\frac{\sigma^2}{2\in_0} = w$ at the surface of the conductor.

Example 3: Determine the density of the thermionic current between two infinite plate electrodes in a vacuum. This is an application of Poisson equation.

Sol. It is well known that heated metals emit from their surface into the surrounding space a beam of free electrons.

If we apply a definite potential difference to two metal electrodes and heat the negative electrode (Cathode), then the electrons continuously emitted by the glowing cathode will be attracted to the surface of the positive electrode (anode).

The beam of electrons travelling from the cathode to the anode is equivalent to an electronic current. This current is called "thermionic".

We choose the axes of Cartesian coordinates so that then origin is on the cathode and the x-axis is perpendicular to the plane of the electrodes and is directed toward the anode.

We assume that the cathode potential equals zero and anode potential equals ϕ_a.

It follows from considerations of symmetry that equipotential surface are parallel to the electrodes ; hence $\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial z} = 0$ and the Poisson equation or the space between the electrodes becomes

$$\frac{\partial^2 \phi}{\partial x^2} = -\frac{\rho}{\in_0} \tag{1}$$

If we denote $n(x)$ the number of electrons per unit of volume in the space between the electrodes at the distance x from the cathode, and by e the absolute value of the charge on an electron, then the charge density at this distance will be:

$\rho = -n(x)e$

Denoting by v(x) the velocity of an electron at the distance x from the cathode and by $\phi(x)$ the potential at the same distance, we get

$$\frac{mv^2(x)}{2} = e\phi(x) \tag{2}$$

Finally the density j of the electric current is

$$j = en(x)v(x) \tag{3}$$

Now using Poisson equation

$$\frac{\partial^2 \phi}{\partial x^2} = -\frac{\rho}{\epsilon_0} = \frac{ne}{\epsilon_0} = \frac{j}{\epsilon_0 v}$$

Substituting for v from (2) :

$$v = \sqrt{\frac{2e\phi}{m}}$$, we get

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{j}{\epsilon_0}\sqrt{\frac{m}{2e}}\phi^{-\frac{1}{2}} \tag{4}$$

Or $\frac{\partial^2 \phi}{\partial x^2} = Aj\phi^{-\frac{1}{2}}$ where $A = \sqrt{\frac{m}{2e}}.\frac{1}{\epsilon_0}$ (5)

Integrating the differential equation (5) with the condition of the problem viz.

$\frac{\partial^2 \phi}{\partial x^2} = 0$ at x = 0

and $\frac{\partial \phi}{\partial x} = 0$ at x = 0 we find

$$\phi = (Aj)^{2/3} x^{4/3}$$

If we denote the distance between the anode and cathode by L, then when $x=L$, $\phi = \phi_a$ Hence

$$\phi_a = (Aj)^{2/3} L^{4/3}$$

Hence $j = \frac{1}{AL^2}\phi_a^{3/2}$

i.e. $j \alpha \phi_a^{3/2}$

In other words the density of a thermionic current does not obey Ohm's law, but grows proportional to the power $\frac{3}{2}$ of the voltage ϕ_a .

Example 4: The region between two concentric right circular cylinders contains a uniform charge density ρ. Use Poisson's equation to find V.

Sol. Poisson equation reduces to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right)=-\frac{\rho}{\in}$$

$$\Rightarrow \frac{d}{dr}\left(r\frac{dV}{dr}\right)=-\frac{\rho r}{\in}$$

Integrating

$$\Rightarrow r\frac{dV}{dr}=-\frac{\rho r^2}{2\in}+A$$

$$\Rightarrow \frac{dV}{dr}=-\frac{\rho r}{2\in}+\frac{A}{r}$$

$$\Rightarrow \int dV=\int\left(-\frac{\rho r}{2\in}+\frac{A}{r}\right)dr$$

$$\Rightarrow V=-\frac{\rho r^2}{4\in}+A\ln r+B$$

2.10 Self Learning Exercise-II

Q.1 Define the capacitance of an isolated conductor i.e. a conductor infinitely removed from all other conductor.

Q.2 Let R stand for the distance from a given point of space to an arbitrarily chosen initial point P.

Show that the scalar $\psi=\frac{1}{R}$ complies with the Laplace equation.

$\nabla^2\left(\frac{1}{R}\right)=0$,the point $R=0$ is not considered.

2.11 Summary

In this chapter we have learnt that when the region of interest contains charges in a known distribution ρ, Poisson's equation can be used to determine the potential function. Very often the region is charge-free (as well as being of uniform permittivity) – Poisson equation then becomes $\nabla^2\phi=0$, which is Laplace's equation. Laplace's equation provides a method whereby the potential function f can be obtained subject to the conditions on the boundary conditions.

We have discussed the boundary conditions that are appropriate for the Poisson (or Laplace) equations to ensure that a unique and well-behaved solution will exist inside the bounded region.

2.12 Glossary

Energy density**:** Energy per unit volume

Thermionic emission**:** driving force of the thermionic emission is thermal energy which provides the thermal energy to electrons to overcome the barrier

2.13 Answers to Self Learning Exercises

Answers to Self Learning Exercise-I

Ans.1: $\boxed{\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0}$

Ans.2: Since V is not a function of r or ϕ ,so Laplace equation reduces to and the solution is $\frac{d^2V}{dz^2}=0$ and the solution is $V=Az+B$.

Ans.4: In order that Φ have an extreme value, it would be necessary that the first derivatives of $\Phi$with respect to the coordinates be zero, and that the second derivatives

$$\frac{\partial^2\Phi}{\partial x^2},\frac{\partial^2\Phi}{\partial y^2},\frac{\partial^2\Phi}{\partial z^2}$$

,all have the same sign. This last condition is impossible, since in that case $\Delta\Phi=0$could not be satisfied.

Answers to Self Learning Exercise-II

Ans.1: The capacitance of an isolated conductor is defined as the magnitude of the charge needed to impart a unit potential to this conductor. It is assumed that the additive constant in the expression for the potential is selected so that the potential equals zero at infinity.

2.14 Exercise

Q.1 Write Laplace equation in cylindrical and spherical coordinates.

Q.2 What is the capacitance of an isolated sphere?

Q.3 In Cartesian coordinates a potential is a function of x only. At x = -2.0 cm , V = 25.0 Volt and $E = 1.5\times10^{3}(-\hat{x})\frac{V}{m}$ throughout the region. Find V at x = 3.0 cm.

Q.4 Deduce Green's first and second theorems.

Q.5 Deduce an expression for the electrostatic potential energy and energy density.

Q.6 Prove $\nabla^2\left(\frac{1}{\left|\vec{x}-\vec{x}'\right|}\right) = -4\pi\delta(\vec{x}-\vec{x}')$

Q.7 Deduce Poisson's and Laplace's equation. Write Solution of Poisson equation in terms of Green's function.

2.15 Answers to Exercise

Ans.1: In cylindrical coordinates (ρ,ϕ,z) the Laplace equation is

$$\frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

In spherical coordinates (r,θ,ϕ) Laplace equation is

$$\boxed{\frac{1}{r}\frac{\partial^2}{\partial r^2}(\mathrm{r}\,\Phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0}$$

Ans.2: $C = \frac{Q}{V} = \frac{Q}{Q/4\pi\in_0 R} = 4\pi\in_0 R$

Ans.3: 100 V

References and Suggested Readings

1. John David Jackson ,Classical Electrodynamics ,Third Edition ,John-Wiley Sons,2010

2. L.D.Landau ,E.M.Lifshitz and L.P.Pitaevskii, Electrodynamics of Continuous Media (Second Edition)Butterworth-Heinemann,2010

UNIT- 3
Boundary Value Problems in Electrostatics : Methods of Images

Structure of the Unit

3.0 Objective

This chapter deals with the boundary value problems in electrostatics and their treatment to determine the electrostatics quantities like the potential, surface charge density, electric field etc. on the boundary surfaces.

3.1 Introduction

There are many problems in Electrostatics, which involves boundary surfaces on these surfaces either the potential or the surface charge density is specified. The formal solution of such problems using the method of Green's functions may be difficult in some cases as it is difficult to determine the correct Green's function.

Hence a number of approaches to solve electrostatic boundary value problems have been developed. Some of them are: (1) The method of images; (2) Expansion in orthogonal functions; (3) The finite element analysis (FEA), which is a numerical method and comprises of use of complex-variable techniques. In this section we are going to study the method of images technique.

3.2 The Method of Images

The method of images concerns with the problem of one or more point charges in the presence of boundary surfaces, some of the examples are: grounded conductors or conductors held at fixed potential. From the geometry of the problem we can infer a small number of suitably placed charges of appropriate magnitudes, which are external to our region of interest and can simulate the required boundary conditions. These charges are known as "image charges" and the replacement of the actual problem with image charges with boundaries by an enlarged region is called the "method of images".

To understand the method of images, let us take a simple example of a point charge located in front of an infinite plane conductor at zero potential as shown in the figure given below :

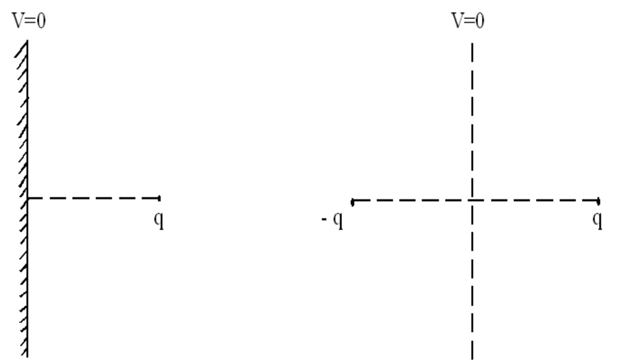


Figure 3.1 (a) The Original potential problem 3.1 (b) The equivalent image problem

We can show that the problem of original charge can be replaced by an equal and opposite charge located at the mirror-image point behind the plane defined by the position of the conductor and the original charge itself.

3.3 Point Charge in the presence of a Grounded Conducting Sphere

Any stationary charge distribution near a grounded conducting plane can be treated by the method of images technique, by introducing its mirror image charge at appropriate distance inside the conducting plane. Consider a point charge q located at $\overrightarrow{y}$ distance relative to the origin, on which a grounded conducting sphere of radius $'R'$ is centered.

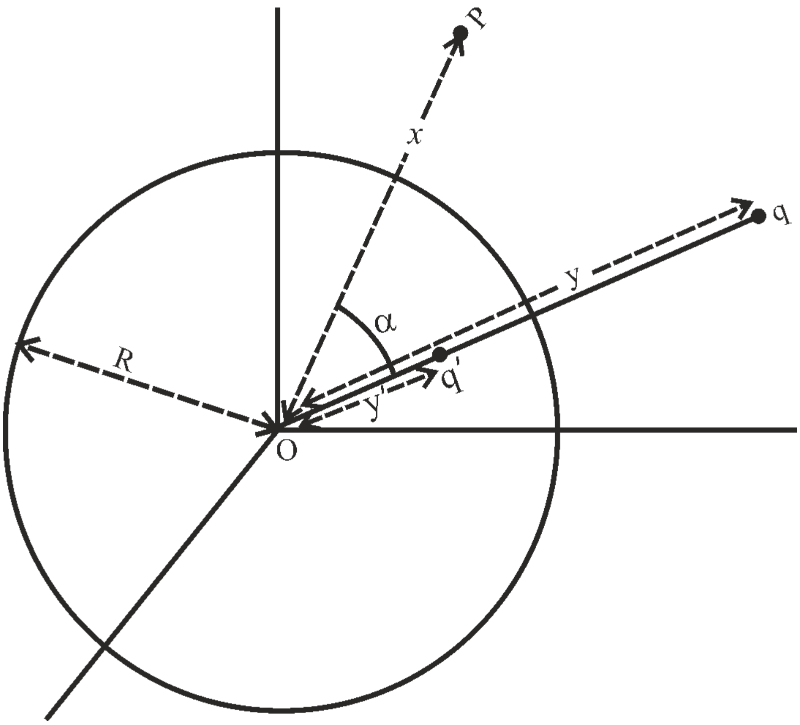


Figure 3.2 (a) Conducting Sphere of radius R and charge q, image charge q′

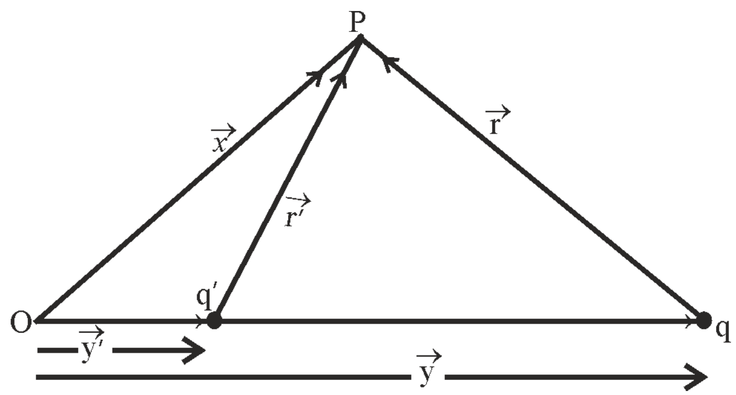


Figure 3.2(b) Vector diagram for the problem

By symmetry of the problem

we can see that the image charge q' will lie on the line connecting the origin O to the charge q.

If we consider the charge q outside the sphere, the image charge q' will lie inside the sphere such that $(V(x) = 0)_{x=R}$. We can write the potential due to charges q and q' at point P is :

$$V(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{q'}{r'}\right] \tag{3.1}$$

$$where\ r = (\vec{x} - \vec{y}), r' = (\vec{x} - \vec{y'})$$

We now try to choose q' and $|y'|$ such that this potential must vanish at $|x| = R$. If x is factored out of the first term and y' out of the second, the potential at $x = R$ becomes:

$$V\ (x = R) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{R\left[1-\frac{y}{R}\right]} + \frac{q'}{y'\left[1-\frac{R}{y'}\right]}\right] \tag{3.2}$$

From eqn (2) we can see that the choices

$$\frac{q}{R} = -\frac{q'}{y'}\ \ ,\ \frac{y}{R} = \frac{R}{y'} \tag{3.3}$$

to make $V(x = R) = 0$. Hence the magnitude and position of image charge are :

$$q' = -\frac{R}{y}q\ \ ;\ y' = \frac{R^2}{y} \tag{3.4}$$

to the right of the centre of sphere. We can see that as the charge q is brought closer to the sphere, the image charge grows in magnitude and moves out from the center of the sphere. When q is just outside the surface of the sphere, the image charge is equal and opposite in magnitude and lies just beneath the surface.

Now let us calculate the actual surface charge density induced on the surface of the sphere, which is give as :

$$\boxed{\sigma = - \in_0 \left.\frac{\partial V}{\partial x}\right|_{x=R}} \tag{3.5}$$

Where $\frac{\partial V}{\partial x}$ is the normal derivative of V at the surface. We know that

$$V(x) = \frac{1}{4\pi \in_0}\left[\frac{q}{[\vec{x}-\vec{y}]} + \frac{q'}{\left[\vec{x}-\vec{y'}\right]}\right]$$

Using $q'$and y' from eqn. (4), we get :

$$V(x) = \frac{1}{4\pi\in_0}\left[\frac{q}{[\vec{x}-\vec{y}]} + \frac{Rq}{y\left[x-\frac{R^2}{y}\right]}\right] \tag{3.6}$$

Differentiating and using $x = R$, we get

$$\sigma = - \in_0 \left.\frac{\partial V}{\partial x}\right|_{x=R}$$

$$\sigma = -\frac{q}{4\pi R^2}\left(\frac{R}{y}\right)\frac{\left(1-\frac{R^2}{y^2}\right)}{\left(1+\frac{R^2}{y^2}-2\frac{R}{y}\cos\alpha\right)^{3/2}} \tag{3.7}$$

Where α is the angle between $\vec{x}$ and $\vec{y}$. We can also show by direct integration that the total induced charge on the sphere is equal to the magnitude of the image charge as it must be according to Gauss's law.

The force acting on the charge q can be calculated as the force between the charge q and the image charge q'.

The distance between the two charges is

$$y - y' = y\left(1-\frac{R^2}{y^2}\right).$$

Hence the attractive force according to Coulomb's law is :

$$\left|\vec{f}\right| = \frac{1}{4\pi\epsilon_0}\frac{q^2}{R^2}\left(\frac{R}{y}\right)^3\ (1-\frac{R^2}{y^2})^{-2} \tag{3.8}$$

For large separation the force is an inverse curve law, but close to the sphere it is proportional to the inverse square of the distance away from the surface of the sphere.

3.4 Illustrative Examples

Example 3.1: Find the force on the charge $+q$ in the figure given below. The xy plane is a grounded conductor.

Solution : Consider two image charges $+2q$ at $z = -d$ and $-q$ at $z = -3d$ and then calculating the force on $+q$ is :

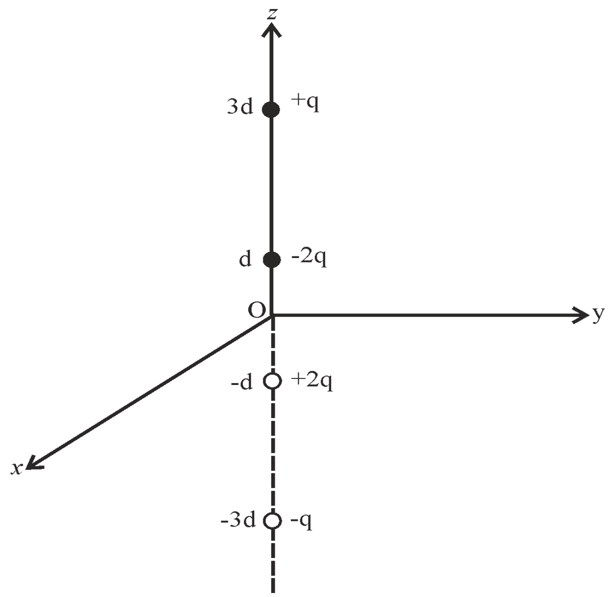


Figure 3.3

$$F = \frac{q}{4\pi\epsilon_0}\left[\frac{-2q}{(2d)^2}+\frac{2q}{(4d)^2}\frac{-q}{(6d)^2}\right]\hat{z}$$

$$= \frac{q^2}{4\pi\ \epsilon_0\ d^2}\left(-\frac{1}{2}+\frac{1}{8}-\frac{1}{36}\right)\hat{z}$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{29q^2}{72d^2}\right)\hat{z} \tag{3.9}$$

The expression gives the total force on the charge q. Similarly we can calculate the force on other charges also.

3.5 Self Learning Exercise – I

Section A :Very Short Answer type Questions

Q.1 Write down the magnitude and position of image charge for the point charge in the presence of grounded conducting sphere.

Q.2 Discuss the case when the point charge is situated just outside the surface of the grounded conducting sphere, specially the potential.

Q.3 What do you mean by image charges?

Section B : Short Answer type Questions

Q.4 Show that the total induced charge on the sphere is equal to the magnitude of the image charge.

Hint : You can use equation **3.7** and the total induced charge can be obtained by direct integration of equation **3.7**.

Q.5 For a point charge on a grounded conducting sphere calculate the total force acting on the surface of the sphere using the expression.

$$dF = \left(\frac{\sigma^2}{2\epsilon_0}\right) da \qquad (3.10)$$

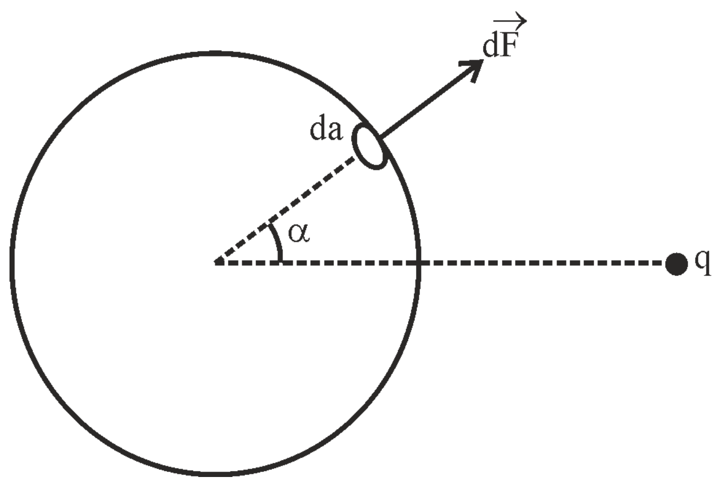


Figure 3.4

where da is the area element and show that the force will be equal to the force given in equation 3.8.

Q.6 Calculate the energy of the system of a point charge in the presence of a grounded conducting sphere.

3.6 Point Charge in the Presence of a Charged Insulated, Conducting Sphere

In last section we have seen that a surface charge density was induced on the sphere as the charge q is present near a grounded conducting sphere. The induced charge was of total amount q' and was distributed over the surface in such a way to be in equilibrium under all forces.

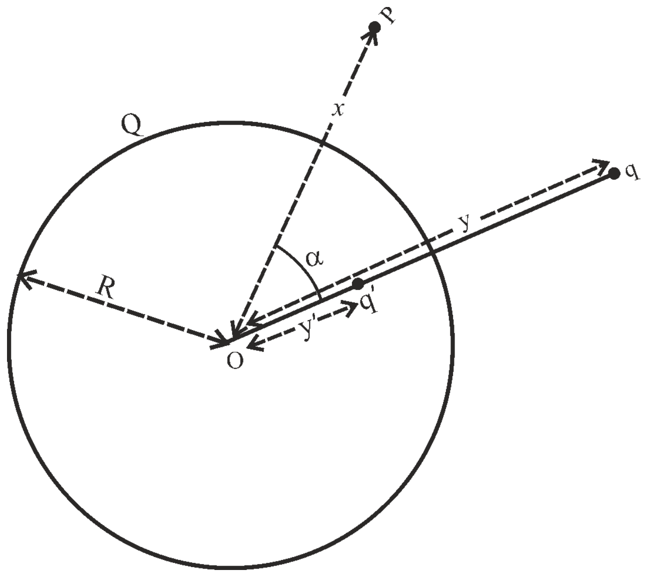


Figure 3.5 A charged, insulated conducting

Now we have an insulated conducting sphere with total charge Q in the presence of a point charge q, the solution for the potential can be taken by linear superposition. Consider that we have a grounded conducting sphere with its charge q' distributed over its surface.

We then disconnect the ground wire and an amount of charge $(Q - q')$ is added Q on the sphere. This added charge $(Q - q')$ will distribute uniformly over the surface of the sphere. Hence the potential due to the added charge $(Q - q')$ will be the same as if a point charge of some magnitude were at the origin.

The potential due to charges q, its image q' and the potential of a point charge $(Q - q')$ at the origin will be :

$$V(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\vec{x}-\vec{y}|} + \frac{q'}{[x-y']} + \frac{Q-q'}{|x|}\right] \quad (3.11)$$

Hence using q' and y' from equation 3.3 we can write :

$$V(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\vec{x}-\vec{y}|} + \frac{Rq}{y\left[\vec{x}-\frac{R^2}{y^2}\vec{y}\right]} + \frac{Q+\frac{R}{y}q}{|x|}\right] \quad (3.12)$$

The force acting on the charge q can be written down directly from Coulomb's law. It is along the radial direction of q and has the magnitude:

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\left[\frac{qq'}{(y-y')^2} + \frac{q(Q-q')}{y^2}\right]\hat{y} \quad (3.13)$$

Which can be written as :

$$\vec{F} = \frac{q}{4\pi \epsilon_0}\left[\frac{Q}{y^2} - \frac{q'}{y^2} + \frac{q'}{(y-y')^2}\right]\hat{y}$$

$$= \frac{q}{4\pi \epsilon_0}\left[\frac{Q}{y^2} - q'\left[\frac{1}{y^2} - \frac{1}{(y-y')^2}\right]\right]\hat{y}$$

$$= \frac{q}{4\pi \epsilon_0}\left[\frac{Q}{y^2} - q'\left[\frac{y'^2 - 2yy'}{y^2\,(y-y')^2}\right]\right]\hat{y}$$

$$= \frac{q}{4\pi \epsilon_0}\left[\frac{Q}{y^2} - q'\left[\frac{y'(y' - 2y}{y^2\,(y-y')^2}\right]\right]\hat{y}$$

Using q' and y' from equation 3.3 we get :

$$\vec{F} = \frac{q}{4\pi \epsilon_0\, y^2}\left[Q - \frac{Rq}{y}\left\{\frac{\left(\frac{R^2}{y}\right)\left(2y - \frac{R^2}{y}\right)}{y^2\left(y-\frac{R^2}{y}\right)^2}\right\}\right]\hat{y}$$

and finally the force

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q}{y^2}\left[Q - \frac{qR^3\,(2y^2 - R^2)}{y\,(y^2-R^2)^2}\right]\hat{y} \quad (3.14)$$

Using the limit $y >> a$, this force reduces to the coulomb's law for two small bodies. If the sphere is charged oppositely to q or is unchanged, the force is attractive at all distances. Even if the charge Q is of same sign as of q, the force becomes attractive at very close distances.

With this example we can explain why an excess charge on the surface does not immediately leave because of mutual repulsion of the individual charges. If the charge is removed from the surface, the image force tries to attract it back and with sufficient amount of work, charge can be removed from the surface (the work function).

3.7 Illustrative Examples

Example 3.2 A point charge q is situated a distance a from the center of a sphere at a potential V_0 (relative to infinity). Find the force of attraction between point charge q and a neutral conducting sphere.

Solution : Consider an image charge 'q'' at a distance 'b' from the centre of the sphere

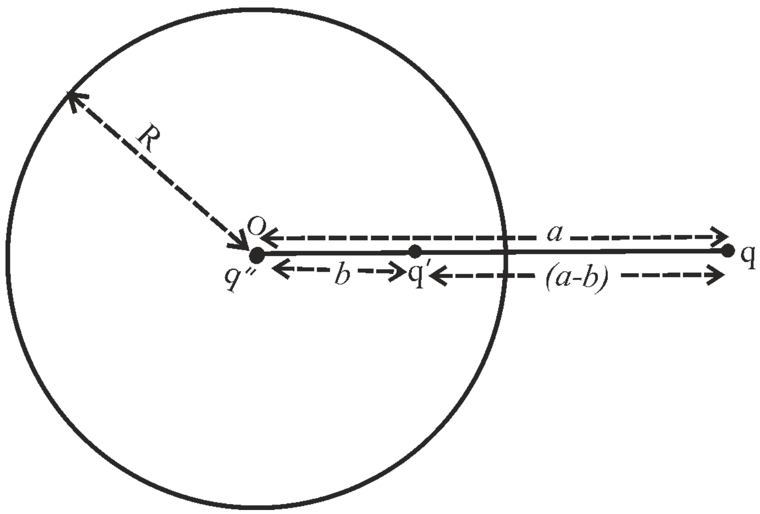


Figure 3.6

Now place a second image charge q'', at the centre of the sphere, this will use to make the sphere equipotential and increase that potential from zero to , $V_0 = \frac{1}{4\pi\epsilon_0}\frac{q''}{R}$,

hence our second image charge :

$$q'' = 4\pi \in_0 \; V_0 \, R \tag{3.15}$$

at the centre of sphere.

As per the problem, for a neutral sphere,

$$q' + q'' = 0 \tag{3.16}$$

Hence the force on the point charge can be calculated by coulomb's interaction between various charges:

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \, q \left(\frac{q''}{a^2} + \frac{q'}{(a-b)^2}\right) \tag{3.17}$$

$$= \frac{qq'}{4\pi\epsilon_0} \left(-\frac{1}{a^2} + \frac{1}{(a-b)^2}\right) \text{ using equation (3.16)}$$

$$= \frac{qq'}{4\pi\epsilon_0} \frac{b\,(2a-b)}{a^2\,(a-b)^2} \text{ using equation (3.4) we can write :}$$

$$F = \frac{q\left(-\frac{Rq}{a}\right)}{4\pi\epsilon_0} \frac{(R^2/a)\,(2a- R^2/a)}{a^2\,(a- R^2/a)^2}$$

and hence

$$F = -\frac{q^2}{4\pi\epsilon_0} \left(\frac{R}{a}\right)^3 \frac{(2a^2-R^2)}{(a^2-R^2)^2} \tag{3.18}$$

We can drop the minus sign as in the problem we have been asked for the force of attraction.

3.8 Point Charge near a Conducting Sphere at Fixed Potential

As we have discussed the case of a point charge near a sphere at fixed potential in example 3.2 and obtained the force for a neutral conducting sphere. Similarly considering a point charge near a conducting sphere held at a fixed potential V, the potential can be taken same as of the charged sphere, except that the charge $(Q - q')$ at the center is replaced by a charge $4\pi \in_0 V_R$, where R is the radius of the sphere.

We can see from **3.12**, since at $|\vec{x}| = R$ the first two terms cancel and last time will be equal to V as required. Thus for this problem we can write the potential as:

$$V(x) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{|\vec{x} - \vec{y}|} - \frac{Rq}{y\left|\vec{x} - \frac{R^2}{y^2}\vec{y}\right|}\right] + \frac{V_a}{|\vec{x}|} \tag{3.19}$$

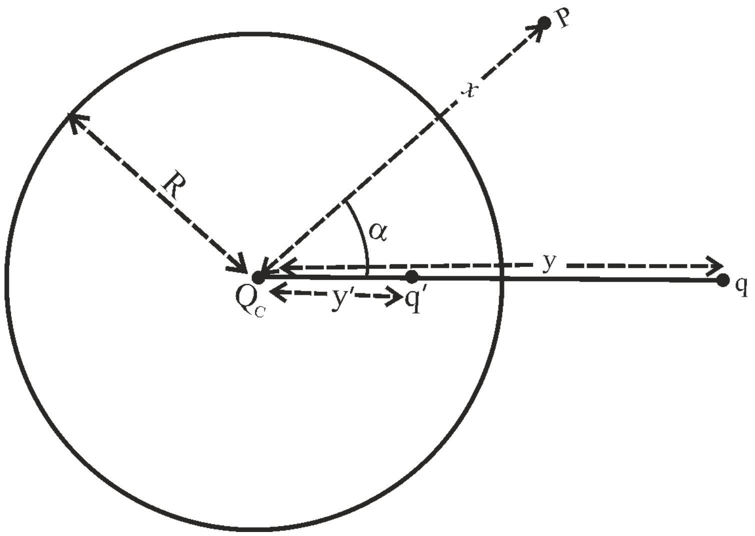


Figure 3.7 The conducting sphere at a fixed potential V

In 3.19, the first term is the potential term due to charge q, the second term is due to the image charge q' and the last term is due to the fixed potential V on the sphere.

The force on the charge q due to the sphere at fixed potential con be calculated by having coulomb's attraction force terms in between q and other charges. Hence :

$$\overrightarrow{F} = \frac{1}{4\pi\epsilon_0} q \left[\frac{q\prime}{(y-y\prime)^2} + \frac{4\pi\epsilon_0 VR}{y^2}\right]\hat{y} \tag{3.20}$$

Using values of q' and y' from **3.16** we can write

$$\overrightarrow{F} = \frac{1}{4\pi \epsilon_0} q \left[\frac{4\pi \epsilon_0 \ VR}{y^2} + \frac{(Rq/y)}{\left(y - \frac{R^2}{y}\right)^2}\right]\hat{y}$$

Solving this we can write:

$$\overrightarrow{F} = \frac{q}{y^2} \left[V_R - \frac{1}{4\pi\epsilon_0} \frac{q\,R\,y^2}{(y^2-R^2)^2}\right]\hat{y} \tag{3.21}$$

We can see that the force on a point charge q due to an insulated, conducting sphere at fixed potential can have positive values for repulsive forces and can have negative values for attractive forces. Regardless of the value of the potential charge $4\pi \epsilon_0 V_a$ the force is always attractive at close distances because of the induced surface charge.

Example 3.3 : A uniform line charge λ is placed on an infinite straight wire, a distance d above a grounded conducting plane. Then,

(a) find the potential in the region above the region above the plane and

(b) find the induced charge density σ on the conducting plane.

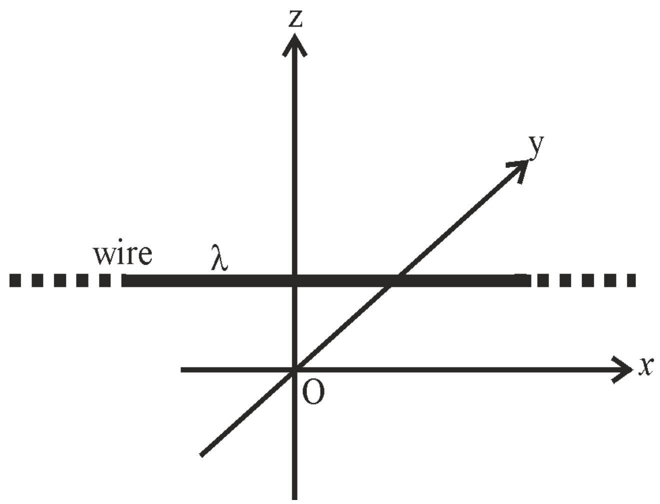


Figure 3.8

Solution (a) Let us consider the wire is parallel to the x-axis and it is directly above it and the conducting plane is in the xy-plane.

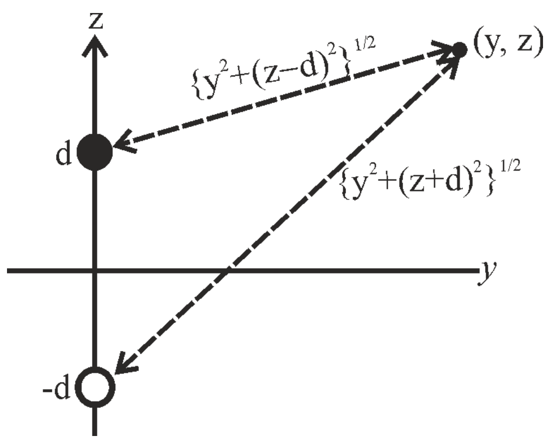


Figure 3.9

Considering an image line charge ($-\lambda$) just below d distance in grounded conducting plane. The potential of line charge ($+\lambda$) at d distance can be obtained as :

$$V_{+} = -\frac{\lambda}{2\pi\epsilon_0}\left(\frac{\{y^2+(z-d)^2\}^{1/2}}{d}\right) \tag{3.22}$$

and similarly the potential of line charge $(-\lambda)$ at $(-d)$ distance will be :

$$V_{-} = +\frac{\lambda}{2\pi\epsilon_0} log_e \left[\frac{\{y^2+(z+d)^2\}^{1/2}}{d}\right] \tag{3.23}$$

and the total potential for the line charge and its image line charge is given by :

$$V(y,z) = \frac{2\lambda}{4\pi\epsilon_0} log_e \left[\frac{\{y^2+(z+d)^2\}^{1/2}}{\{y^2+(z-d)^2\}^{1/2}}\right]$$

$$V(y,z) = \frac{\lambda}{4\pi\epsilon_0} log_e \left[\frac{y^2+(z+d)^2}{y^2+(z-d)^2}\right] \tag{3.24}$$

The expression 3.24 represents the potential in the region above the plane.

(b) For calculating induced charge density, we know that,

$$\sigma = -\epsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=R} \tag{3.25}$$

Hence

$$\sigma\,(y) = -\epsilon_0 \frac{\lambda}{4\pi\,\epsilon_0}\left\{\frac{1}{y^2+(z+d)^2}\,2(z+d) - \frac{1}{y^2+(z-d)^2}\,2(z-d)\right\}|z=0$$

$$\sigma\,(y) = -\frac{2\lambda}{4\pi}\left\{\frac{d}{y^2+d^2} + \frac{d}{y^2+d^2}\,2(z-d)\right\}$$

$$\sigma\,(y) = -\frac{\lambda d}{\pi(y^2+d^2)} \tag{3.26}$$

Equation 3.26 will give the induced charge density on the conducting plane and we can determine the induced charge on the conducting plane using the equation.

3.9 Self learning Exercise –II

Section A:Very Short Answer Type Questions

Q.1 What do you understand by work function? Discuss work function in electrostatic point of view.

Q.2 For the problem of a point charge near a conducting sphere at fixed potential V, show that the potential can be replaced by a charge at the centre of the sphere.

Q.3 Discuss the nature of force for a point charge near a conducting sphere at a fixed potential.

Section B: Short Answer Type Questions

Q.4 Show that the total charge induced in example 3.3 on the strip of line charge of width l parallel to the y axis, is given by $-\lambda l$.

Q.5 A point charge q of mass m is released from (rest at a distance d from) an infinite grounded conducting plane. How long will it take for the charge to hit the plane?

Q.6 Write down the potential for a point charge in the presence of a charged, insulated conduction sphere.

3.10 Summary

The unit starts with the introduction of boundary value problems in electrostatics. The problems are given with the charge and potential on the surfaces. The formal solution of such problems using method of images is developed in this unit. The solution using this method makes the problem easier to solve. The problems of point charge near a grounded conducting sphere, in the presence of a charged, insulated conducting sphere, near a conducting sphere at fixed potential have been discussed in this unit and solution in the form of potential, induced charge density and force are obtained.

3.11 Glossary

Induce : to cause something to happen

Grounding: To make potential to zero

***Insulate*:** Prevent the passage of electricity to or from (something) by covering it in non-conducting material

3.12 Answers to Self Learning Exercises

Answer to Self learning Exercise –I

Ans.1 : $q' = \frac{R}{y} q \, , y' = \frac{R^2}{y}$

Ans.2 : Approximately zero

Ans.3 : Definition (See 3.2)

Ans.6 : $-\frac{1}{4\pi\epsilon_0}\frac{q^2}{2}log_e\frac{(y-R)}{(y+R)}$

Answer to Self learning Exercise – II

Ans.3 : Repulsive and attractive at close distances

Ans.5 : $\frac{\pi d}{q}\sqrt{2\pi\,\epsilon_0\,md}$

Ans.6 : $V(x) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{(\vec{x}-\vec{y})} - \frac{Rq}{y\left|\vec{x}-\frac{R^2}{y^2}\vec{y}\right|} + \frac{\left(Q+\frac{Rq}{y}\right)}{|\vec{x}|}\right]$

3.13 Exercise

Section – A (Very short Answer type Questions)

Q.1 Whether image charges are always located outside of region where $V(x)$ and $E(x)$ are to be calculated :

Q.2 Whether image charges are always of opposite sign.

Q.3 Whether image charge is of same strength as of original charge.

Q.4 What is the total charge induced on surface of the conducting sphere?

Q.5 Write down the formula for face on a point clearing q dee to an insulated, conducting sphere at fixed potential.

Section – B (Short Answer type Questions)

Q.6 Calculate the potential for a point charge q held at a distance d above an infinite grounded conducting plane.

(Consider the grounded conducting plane as xy-plane and the charge is along z-axis)

Q.7 Find out the induced surface charge density for problem 6.

Q.8 Show that the total induced charge for problem 6 will be same as image charge.

Q.9 Determine the energy in problem 6 by calculating the work required to bring point charge q in from infinity.

Q.10 Two infinite parallel grounded conducting planes are held a distance 'a' a part. A point charge q is placed in the region between them, a distance x from plate. Find the force on q. Check your answer for the special cases $a \rightarrow \infty$ and $x = (a/2)$.

Section – C (Long Answer type questions)

Q.11 Using the method of images. Discuss the problem of a conducting sphere in a uniform potential and find :

(i) The potential inside the sphere of radius R.

(ii) The potential at far off point outside the sphere.

(iii) The induced surface charge density.

Q.12 A point charge q is situated a distance r from the center of a grounded conducting sphere of radius R. Find the potential outside the sphere and the force of attraction between the charge and sphere.

Q.13 A point charge q is brought to a position d away from an infinite plane conductor at zero potential. Using method of images, find:

(i) The surface-charge density induced on the plane.

(ii) The force between the plane and the charge and also the force between the charge and its image.

(iii) The work required to remove the charge from its position to infinity.

Q.14 Two long straight wires, carrying opposite uniform line charges $\pm\lambda$ are situated on either side of a long conducting cylinder of radius R. The wires are at a distance 'a' from the axis of the cylinder. Find the potential at point $(\vec{r}\,, \phi)$ from the centre of the cylinder.

Q.15 Find the potential, force and induced surface charge density for a point charge in the presence of a charged, insulated conducting sphere.

3.14 Answers to Exercise

Ans.1: Yes

Ans.2: May or may not be

Ans.3: May or may not be (depending on the nature of the problem).

Ans.4: Equal to image charge q'.

Ans.5: $\vec{F} = \frac{q}{y^2}\left[VR - \frac{1}{4\pi\epsilon_0}\frac{q\,R\,y^3}{(y^2-R^2)^2}\right]\hat{y}$

Ans.6: $V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{(x^2+y^2+(z-d)^2)^{1/2}} - \frac{q}{(x^2+y^2+(z+d)^2)^{1/2}}\right]$

Ans.7: $\sigma\,(x,y) = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}$

Ans.8: $Q = -q$

Ans.9: $W = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}$

Ans.10: $F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{4}\left\{\left[\frac{1}{(a-x)^2} + \frac{1}{(2a-x)^2} + \frac{1}{(3a-x)^2} + \cdots\right] - \left[\frac{1}{x^2} + \frac{1}{(a-x)^2} + \frac{1}{(2a-x)^2} + \cdots\right]\right\}$

When $a \to \infty, x = \frac{a}{2} = f = 0$

Ans.14:

$$V(r,\phi) = \frac{\lambda}{4\pi\epsilon_0}\,log_e\left[\frac{(r^2+a^2+2ra\cos\phi)\left[\left(\frac{ra}{R}\right)^2+R^2-2ra\cos\phi\right]}{(r^2+a^2-2ra\cos\phi)\left[\left(\frac{ra}{R}\right)^2+R^2+2ra\cos\phi\right]}\right]$$

References and Suggested Readings

1. David J. Griffiths, Introduction to Electrodynamics, 2nd Edition, Prentice-Hall,2000
2. J.D. Jackson, Classical Electrodynamics, Wiley Eastern Limited,2002.
3. W.K.H. Panofsky and M. Phillips, Classical Electricity and Magnetism, 2nd edition, Addison Wesley, 1962.
4. Mathew N.O. Sadiku, Elements of Electromagnetics, Oxford University Press, 2001.
5. Classical theory of Electrodynamics by Landau & Lifshitz. (Pergaman press, New York).

UNIT- 4
Conducting Sphere in a Uniform Electric Field by Method of Image

Structure of the Unit

4.0 Objectives

After interacting with the material presented here students will be able to understand

1. Method of image by an example of conducting sphere in an uniform electric field
2. Green function
3. Conducting sphere with Hemispheres at different potential
4. Orthogonal functions and its expansion

4.1 Introduction

There are four major analytical methods for solving boundary values problems:

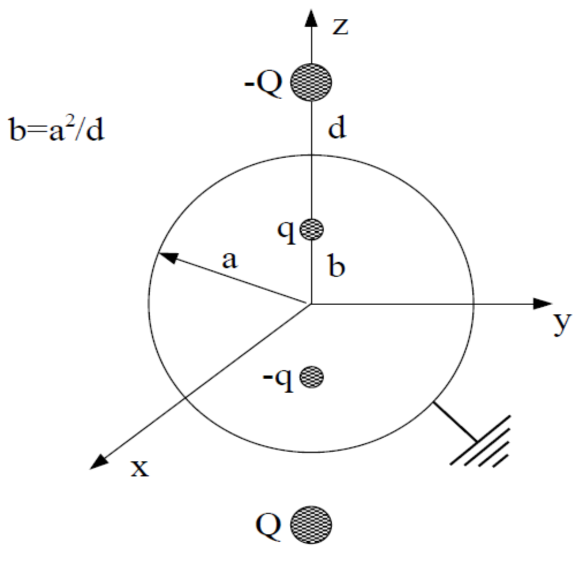
1. Image charges
2. Green functions
3. Expansion in orthogonal functions
4. Conformal mappings.

In this chapter we will only deal with the first three methods. The method of images and few problems we have discussed in the last chapter. Here we will discuss only conducting sphere in an uniform electric field by method of image.

4.2 Conducting sphere in an uniform electric field by method of image

Let us consider a grounded conducting sphere, which means $\Phi(x) = 0$ on the sphere, placed in a region where there was initially a uniform electric field $E_0 = E_0\hat{z}$ produced by some far away fixed charges. Here, $\hat{z}$ is a unit vector pointing in the z direction. We approach this problem by replacing it with another one which will become equivalent to the first one in some limit.

Let the sphere be centered at the origin and let there be not a uniform applied field but rather a charge Q placed at the point (0, 0, −d) and another charge −Q placed at the point (0, 0, d) in Cartesian coordinates.



The resulting potential configuration is easily solved by the image method; there are images of the charges $\pm$Q in the sphere at (0; 0;-a^2/d) and at (0; 0; a^2/d); they have size –Qa/d and Qa/d, respectively.

The potential produced by these four charges is zero on the surface of the sphere. Thus we have solved the problem of a grounded sphere in the presence of two symmetrically located equal and opposite charges. We could equally well think of the sphere as isolated (not electrically connected to anything) and neutral, because the total image charge is zero.

Now we want to think about what happens if we let Q become increasingly large and at the same time move the real charges farther and farther away from the sphere in such a way that the field they produce at the origin is constant. This field is E(x) = (2Q/d^2) $\hat{z}$, so if Q is increased at a rate proportional to d^2, the field at the origin is unaffected. As d becomes very large in comparison with the radius a of the sphere, not only will the applied field at the origin have this value, but it will have very nearly this value everywhere in the vicinity of the sphere. The difference becomes negligible in the limit $\frac{d}{a} \to \infty$ .Hence we recover the configuration presented in the original problem of a sphere placed in a uniform applied field. If we pick $E_0 = 2Q/d^2$, or, more appropriately, $Q = E_0 d^2/2$, we have the solution in the limit of $d \to \infty$:

$$\varphi(x) = \lim_{d\to\infty}\left[\frac{E_0 d^2}{2}\frac{1}{\left(d^2 + r^2 + 2rdcos\theta\right)^{\frac{1}{2}}} - \frac{E_0 d^2 a}{2d}\frac{1}{\left(\frac{a^4}{d^2} + r^2 + 2r\left(\frac{a^2}{d}\right)cos\theta\right)^{\frac{1}{2}}}\right]$$

$$+\lim_{d\to\infty}\left[-\frac{E_0 d^2}{2}\frac{1}{\left(d^2 + r^2 - 2rdcos\theta\right)^{\frac{1}{2}}} + \frac{E_0 d^2 a}{2d}\frac{1}{\left(\frac{a^4}{d^2} + r^2 - 2r\left(\frac{a^2}{d}\right)cos\theta\right)^{\frac{1}{2}}}\right]$$

$$= \lim_{d\to\infty}\left[\pm\frac{E_0 d}{2}\frac{1}{\left(1+\frac{r^2}{d^2}\pm 2\frac{r}{d}cos\theta\right)^{\frac{1}{2}}}\mp\frac{E_0 da}{2r}\frac{1}{\left(1+\frac{a^4}{d^2r^2}\pm\left(\frac{a^2}{rd}\right)cos\theta\right)^{\frac{1}{2}}}\right]$$

$$= -E_0 r\ cos\theta + \frac{E_0 a^3}{r^2 cos\theta}$$

The first term, $-E_0 r\ cos\theta$, is the potential of the applied constant field, E_0. The second is the potential produced by the induced surface charge density on the sphere. This has the characteristic form of an electric dipole field, of which we shall hear more presently. The dipole moment p associated with any charge distribution is defined by the equation

$$p = \int d^3x \times \rho(x)$$

in the present case the dipole moment of the sphere may be found either from the surface charge distribution or from the image charge distribution. Taking the latter tack, we find

$$p = \int d^3x \times \frac{E_0 da}{2}\left[-\delta\left(z+\frac{a^2}{d}\right)\delta(y)\delta(x)+\delta\left(z-\frac{a^2}{d}\right)\delta(y)\delta(x)\right]$$

$$= \frac{E_0 da}{2}\left[\left(\frac{a^2}{d}\right)\hat{z}+\left(\frac{a^2}{d}\right)\hat{z}\right]$$

$$= E_0 a^3 \hat{z}$$

Comparison with the expression for the potential shows that the dipolar part of the potential may be written as

$$\varphi(x) = p.\frac{x}{r^3}$$

The charge density on the surface of the sphere may be found in the usual way:

$$4\pi\sigma = E_r$$

$$\boxed{E_r = -\left[\frac{\partial\varphi}{\partial r}\right]_{r=a}}$$

$$= E_0 cos\theta + \frac{2E_0}{a^3} a^3 cos\theta$$
$$= 3E_0 cos\theta$$

Hence

$$\sigma(\theta) = \frac{3}{4\pi} E_0 cos\theta$$

4.3 Green's Function Method for the Sphere: general solution for potential

Next, let us consider an example of the use of the Green's function method by considering a Dirichlet potential problem inside of a sphere. The task is to calculate the potential distribution inside of an empty ($\rho(x) = 0, \in V$) spherical cavity of radius a, given some specified potential distribution V (θ,ϕ) on the surface of the sphere We can immediately invoke the Green's function expression

$$\varphi(x) = -\frac{1}{4\pi}\int d^2x'\varphi(x')\frac{\partial G(x,x')}{\partial n'}$$

and we already know that,

$$G(x,x') = \frac{1}{|x-x'|} - \frac{a}{r'}\frac{1}{\left|x - \left(\frac{a^2}{r'^2}\right)x'\right|}$$

since G(x,x′) is the potential at x due to a unit point charge at x′ (x,x′∈ V), and we have just solved this problem. If we let γ be the angle between x and x′,

$$G(x,x') = \frac{1}{(r^2 + r'^2 - 2rr'cos\gamma)^{\frac{1}{2}}} - \frac{a}{r'}\frac{1}{\left(r^2 + (a^4/r'^2) - 2r\left(\frac{a^2}{r'}\right)cos\gamma\right)^{\frac{1}{2}}}$$

Then

$$\left[\frac{\partial G\left(x,x'\right)}{\partial n'}\right]_s = \left[\frac{\partial G\left(x,x'\right)}{\partial r'}\right]_{r'=a}$$

$$= -\frac{1}{2}\left[\frac{2a - 2rcos\gamma}{\left(r^2 + a^2 - 2racos\right)^{\frac{3}{2}}} - \frac{2ar^2 - 2ra^2cos\gamma}{\left(r^2a^2 + a^4 - 2ra^3\cos\right)^{\frac{3}{2}}}\right]$$

$$= \frac{a\left(1-\frac{r^2}{a^2}\right)}{\left(r^2+a^2-2racos\right)^{\frac{3}{2}}}$$

$$= -\frac{1}{a^2}\frac{\left(1-\in^2\right)}{\left(1+\in^2-2\in\cos\right)^{\frac{3}{2}}}$$

where $\in = \frac{r}{a}$. For simplicity, let us suppose that ρ(x) = 0 inside of the sphere. Then

$$\varphi(X) = -\frac{1}{4\pi}\int_0^{2\pi} d\varphi' \int_0^{\pi} sin\theta' d\theta' V(\theta',\varphi') \frac{(1-\in^2)}{(1+\in^2-2\in cos\gamma)^{\frac{3}{2}}}$$

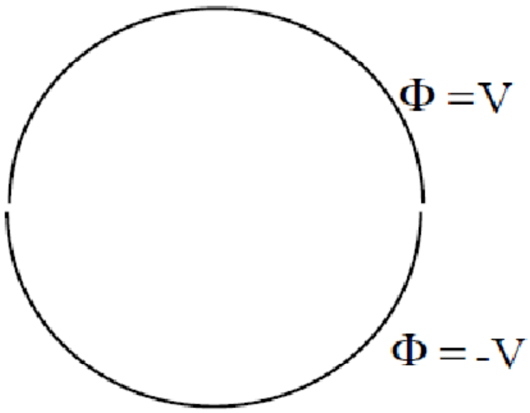
In terms of θ, φ and θ', φ'

$$cos\gamma = cos\theta cos\theta' + sin\theta sin\theta' \cos(\varphi-\varphi')$$

This integral can rarely be done in closed form in terms of simple functions; however, it is generally a simple matter to carry out the integrals numerically.

4.4 Conducting sphere with Hemispheres at different potential

Example



$$V(\theta,\varphi) = \begin{cases} V, & 0 \le \theta \le \frac{\pi}{2} \\ -V, & \frac{\pi}{2} \le \theta \le \pi \end{cases}$$

Then the answer will not depend on ϕ, so we may arbitrarily set ϕ equal to zero and proceed. Using $\in\equiv r/a$, we have

$$\varphi(\in,\theta)=\frac{V}{4\pi}(1-\in^2)\int_0^{2\pi}d\varphi'\left[\int_0^{\frac{\pi}{2}}\frac{\sin\theta' d\theta'}{(1+\in^2-2\in\cos\gamma)^{\frac{3}{2}}}-\int_{\pi/2}^{\pi}\frac{\sin\theta' d\theta'}{(1+\in^2-2\in\cos\gamma)^{\frac{3}{2}}}\right]$$

The integral is still difficult in the general case. For $\theta = 0$, it is easier:

$$\varphi(\in,0)=\frac{V}{4\pi}(1-\in^2)2\pi\left[\int_0^1\frac{du}{(1+\in^2-2\in u)^{\frac{3}{2}}}-\int_{-1}^0\frac{du}{(1+\in^2-2\in u)^{\frac{3}{2}}}\right]$$

These integrals are easily completed with the result that

$$\varphi(\in,0)=\frac{V}{\in}\left[1-\frac{(1-\in^2)}{\sqrt{(1+\in^2)}}\right]$$

An alternative approach, valid for $r/a \ll 1$, is to expand the integrand in powers of $\in$ and then to complete the integration term by term. This is straightforward with a symbolic manipulator but tedious by hand. Either way, a solution in powers of $\in$ is generated.

$$\varphi(\in,0)=\frac{3V}{2}\left[\in\cos\theta-\frac{7}{12}\epsilon^3\left(\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta\right)+O(\epsilon^5)\right]$$

Legendre polynomials

$$P_1(\cos\theta)=\cos\theta$$

$$P_3(\cos\theta)=\frac{5}{2}\cos^3\theta-\frac{3}{2}\cos\theta$$

etc. Note that only terms which are odd in $\cos\theta$ enter into the sum, due to the symmetry of the boundary conditions.

4.5 Self Learning Exercise -I

Short Answer Type Questions

Q.1 State the analytical methods for solving boundary values problems.

Q.2 Define method of image.

Q.3 Write down the charge density on the surface of the sphere

Q.4 Define Green's function and Green's function solution equation.

4.6 Orthogonal Functions and Expansions

Consider a set of functions $U_n(x)$ (real or complex) defined on the interval (a, b). Then the two functions $U_n(x)$ and $U_m(x)$ are called orthogonal if

$$\boxed{\int_a^b dx U_n^*(x) U_m(x) = 0, m \neq n}$$

the superscript * denotes complex conjugation. Further, the functions $U_n(x)$ are normalized on the interval,

$$\boxed{\int_a^b dx U_n^*(x) U_n(x) = \int_a^b dx |U_n|^2 = 1}$$

Combining these equations we have

$$\boxed{\int_a^b dx U_n^*(x) U_m(x) = \delta_{nm} = \begin{cases} 0, n \neq m \\ 1, n = m \end{cases}}$$

The functions $U_n(x)$ are said to be orthonormal; ***δ_{nm} is called a Kronecker delta function.***

Next, we attempt to expand, on the interval $a \leq x \leq b$, an arbitrary function f(x) as a linear combination of the functions $U_n(x)$, which are referred to as basis functions. Keeping just N terms in the expansion, one has

$$f(x) \approx \sum_{1}^{N} a_n U_n(x)$$

We need a criterion for choosing the coefficients in the expansion; a standard criterion is to minimize the mean square error E which may be defined as follows:

$$E = \int_a^b dx |f(x) - \sum_{n=1}^{N} a_n U_n(x)|^2$$

$$= \int_a^b dx (f^*(x) - \sum_{n=1}^{N} a_n^* U_n^*(x))(f(x) - \sum_{n=1}^{N} a_n U_n(x))$$

The conditions for an extremum are

$$\frac{\partial E}{\partial a_k}_{a_k^*} = 0 = \frac{\partial E}{\partial a_k^*}_{a_k}$$

where a_k and a^*_k have been treated as independent variables Application of these conditions leads to

$$0 = \int_a^b dx(f^*(x) - \sum_{n=1}^{N} a_n^* U_n^*(x))\, U_k(x)$$

$$= \int_a^b dx(f(x) - \sum_{n=1}^{N} a_n U_n(x))\, U_k^*(x)$$

or, making use of the orthonormality of the basis functions,

$$a_k = \int_a^b dx f(x) U_k^*(x)$$

with a^*_n given by the complex conjugate of this relation. If the basis

functions are orthogonal but not normalized, then one finds

$$\boxed{a_k = \frac{\int_a^b dx\, f(x) U_k^*(x)}{\int_a^b dx \left|U_k(x)\right|^2}}$$

The set of basis functions $U_n(x)$ is said to be complete if the mean square error can be made arbitrarily small by keeping a sufficiently large number of terms in the sum. Then one says that the sum converges in the mean to the given function. If we are a bit careless, we can then write

$$f(x) = \sum_n a_n\, U_n(x)$$

$$= \sum_n \int_a^b dx' f(x') U_n^*(x')\, U_n(x)$$

$$= \int_a^b dx' \sum U_n^*(x')\, U_n(x) f(x')$$

from which it is evident that

$$\boxed{\sum U_n^*(x') U_n(x) = \delta(x - x')}$$

for a complete set of functions. This equation is called the completeness or closure relation.

We may easily generalize to a space of arbitrary dimension. For example, in two dimensions we may have the space of x and y with $a \leq x \leq b$, and $c \leq y \leq d$ and complete sets of orthonormal functions $U_n(x)$ and $V_m(y)$ on the respective intervals. Then the arbitrary function f(x,y) has the expansion

$$f(x,y) = \sum_{nm} A_{nm} U_n(x) V_m(y)$$

Where

$$A_{nm} = \int_a^b dx \int_a^b dy\, f(x,y) U_n^*(x) V_m^*(y)$$

Returning to the one-dimensional case, suppose that the interval is infinite, $-\infty < x < \infty$. Then the index n of the functions $U_n(x)$ may become a continuous index, $U_n(x) \rightarrow U(x;\rho)$.

A familiar example of this is the Fourier integral which is the limit of a Fourier series when the interval on which functions are expanded becomes infinite. Consider that we have the interval $-a/2 < x < a/2$. Then the Fourier series may be built from the basis functions

$$U_m(x) = \frac{1}{\sqrt{a}} e^{-\frac{i2\pi m x}{a}}$$

With $m=0,\pm1,\pm2,\ldots$ these functions form a complete orthonormal set. The expansion of f(x) is

$$f(x) = \frac{1}{\sqrt{a}} \sum_{m=0}^{\infty} A_m\, e^{-\frac{i2\pi m x}{a}}$$

With

$$A_m = \frac{1}{\sqrt{a}} \int_{-a/2}^{a/2} dx f(x)\, e^{-\frac{i2\pi m x}{a}}$$

The closure relation is

$$\frac{1}{a}\sum_{m=0}^{\infty} e^{-\frac{i2\pi m(x-x')}{a}} = \delta(x-x')$$

Now define k ≡ 2πm/a or m = ka/2π. Also, define

$$A_m = \sqrt{\frac{2\pi}{a}}A(k).$$

Note that for a →∞, k takes on a set of values that approach a continuum. Thus

$$f(x) = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty}\frac{a}{2\pi}dke^{ikx}\sqrt{\frac{2\pi}{a}}A(k)$$
$$= \frac{1}{\sqrt{2\pi}}\int dke^{ikx}A(k)$$

While

$$\sqrt{\frac{2\pi}{a}}A(k) = \frac{1}{\sqrt{a}}\int dxf(x)e^{-ikx}$$

Or

$$\boxed{A(k) = \frac{1}{\sqrt{2\pi}}\int dx\, f(x)e^{-ikx}}$$

while the closure relation now reads

$$\boxed{\frac{1}{2\pi}\int dk\, e^{ik(x-x')} = \delta(x-x')}$$

thus e^{ikx} form a complete set (this is also a useful representation of the Dirac delta function).

Note that we can also write this equation as

$$\frac{1}{2\pi}\int dxe^{ik(x-x')} = \delta(k-k')$$

which is the orthonormalization expression of the complete set of functions U(x,k) on the infinite x interval. These functions are

$$U(x,k) = \frac{1}{\sqrt{2\pi}} e^{ixk}$$

4.7 Self Learning Exercise- II

Very Short Answer Type Questions

Q.1 Define orthogonal functions.

Short Answer Type Questions

Q.2 Explain normal functions.

Q.3 Write down the orthonormal condition of function.

Long Answer Type Questions

Q.4 Two concentric spheres have radii a,b (b>a) and each s divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemispheres of the outer sphere are maintained at potential V The other hemispheres are at zero potential.

Determine the potential in the region $a \leq r \leq b$ as a series of Legendre polynomials. Include terms at least upto l=4. Check your solution against known results in the limiting cases $b \to \infty$ and $a \to 0$.

4.8 Summary

In this chapter we firstly introduce method of image and understand conducting sphere in an uniform electric field by method of image followed by Green function for the sphere and understand conducting sphere with Hemispheres at different potential and at last we discussed orthogonal functions and its expansion.

4.9 Glossary

Uniform field :

A uniform field is one in which the electric field is constant at every point. It can be approximated by placing two conducting plates parallel to each other and maintaining a voltage (potential difference) between them; it is only an approximation because of boundary effects (near the edge of the planes, electric field is distorted because the plane does not continue). Assuming infinite planes, the magnitude of the electric field E is:

$$E = -\frac{\Delta\varphi}{d}$$

where $\Delta\varphi$ is the potential difference between the plates and d is the distance separating the plates. The negative sign arises as positive charges repel, so a positive charge will experience a force away from the positively charged plate, in the opposite direction to that in which the voltage increases.

Method of image: The method of image charges (also known as the method of images and method of mirror charges) is a basic problem-solving tool in electrostatics. The name originates from the replacement of certain elements in the original layout with imaginary charges, which replicates the boundary conditions of the problem (see Dirichlet boundary conditions or Neumann boundary conditions).

Green's function: A fundamental solution of a linear differential equation satisfying homogeneous boundary conditions. (other names include influence function, impulse response, source solution).

Green's function solution equation: Formal solution to a boundary value problem in the form of one or more integrals, each of which contains a Green's function and a nonhomogeneous term (``driving term''). The non-homogeneous terms may be boundary conditions, initial conditions, or volume energy generation.

Orthogonal function:

A set of functions, any two of which, by analogy to orthogonal vectors, vanish if their product is summed by integration over a specified interval.

For example, $f(x)$ and $g(x)$ are orthogonal in the interval $x = a$ to $x = b$ if

$$\int_a^b f(x)g(x)dx = 0$$

The functions are also said to be normal if

$$\int_a^b |f(x)|^2 = 1$$

$$\int_a^b |g(x)|^2 = 1$$

The most familiar examples of such functions, many of which have great importance in mathematical physics, are the sine and cosine functions between zero and 2π.

4.10 Answers to Self learning Exercises

Answers to Self learning Exercise-I

Ans.1: There are four major analytical methods for solving boundary values problems:

1. Image charges
2. Green functions
3. Expansion in orthogonal functions
4. Conformal mappings.

Ans.2 : The method of image charges is a basic problem-solving tool in electrostatics. The name originates from the replacement of certain elements in the original layout with imaginary charges, which replicates the boundary conditions of the problem.

Ans. 3 : $\sigma(\theta) = \frac{3}{4\pi} E_0 cos\theta$

Ans.4 : A fundamental solution of a linear differential equation satisfying homogeneous boundary conditions. (other names include influence function, impulse response, source solution).

Formal solution to a boundary value problem in the form of one or more integrals, each of which contains a Green's function and a nonhomogeneous term (``driving term"). The non-homogeneous terms may be boundary conditions, initial conditions, or volume energy generation.

Answers to Self learning Exercise-II

Ans.1 : $f(x)$ and $g(x)$ are orthogonal in the interval $x = a$ to $x = b$ if

$$\int_a^b f(x)g(x)dx = 0$$

Ans.2 : The functions are also said to be normal if

$$\int_a^b |f(x)|^2 = 1$$

Ans.3 : $\int_a^b dx U_n^*(x) U_m(x) = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases} = \delta_{nm}$

The functions $U_n(x)$ are said to be orthonormal; δ_{nm} is called a Kronecker delta function.

Ans.4 : Begin with a general solution

$$\varphi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(cos\theta)$$

For $a \leq r \leq b$

Apply boundary conditions at both the surfaces:

$$A_l = \frac{(2l+1)V(b^{l+1} + a^{l+1})}{2\,(b^{2l+1} - a^{2l+1})} \int_0^1 P_l(x)dx$$

$$B_l = \frac{(2l+1)Va^{l+1}b^{l+1}(b^l + a^l)}{2\,(b^{2l+1} - a^{2l+1})} \int_0^1 P_l(x)dx$$

4.11 Exercise

Long Answer Type Questions:

Q.1 Explain conducting sphere in an uniform electric field by method of image.

Q.2 **(a)** A charge Q is distributed uniformly along a line from z = −a to z = a at x = y = 0. Show that the electric potential for r>a is

$$\varphi(r,\theta) = \frac{\frac{Q}{r}\sum_n \left(\frac{a}{r}\right)^{2n} P_{2n}(\cos\theta)}{2n+1}$$

(b) A flat circular disk of radius a has charge Q distributed uniformly over its area. Show that the potential for r>a is

$$\varphi(r,\theta) = \frac{Q}{r}[\,1 - \frac{1}{4}\left(\frac{a}{r}\right)^2 P_2(cos\,\theta) + \frac{1}{8}\left(\frac{a}{r}\right)^4 P_4 - \frac{5}{64}\left(\frac{a}{r}\right)^6 P_6 + \ldots.$$

For both examples, also calculate the potential for r < a.

Q.3 A semi-infinite cylinder of radius a about the z axis (z > 0) has grounded conducting walls. The disk at z = 0 is held at potential V. The "top" of the cylinder is open.

Show that the electric potential inside the cylinder is

$$\varphi(r,z) = \frac{2V}{a} \frac{\sum_l \frac{e^{-k_l z}}{k_l} J_0(k_l r)}{J_l(k_l a)}$$

Refer to the notes on Bessel functions for the needed relations.

References and Suggested Readings

1. Classical Electrodynamics by J.D. Jackson, 1962.
2. Boundary-value Problems in Electrostatics I by Karl Friedrich Gauss, 2000.
3. Introduction to Electrodynamics by D.J Griffiths, 1999 .
4. Classical Theory of Field by L.D. Landau and E. M. Lifshitz, 1971.
5. Electrodynamics of Continuous Media by L.D. Landau and E. M. Lifshitz, 1960

UNIT- 5
Multipole Expansion

Structure of the Unit

5.0 Objectives

This chapter deals with the potential at large distance due to localized charge distributions and its expansion in multipoles to understand the mechanism of potentials at large distances. The energy of a charge distribution or multipoles in an external field is also discussed.

5.1 Introduction

If we are far away from a localized charge distribution, it looks like as a point charge and the potential can directly be written as

$$V(\vec{r}) = \frac{1}{4\pi \in_0} \frac{Q}{r}$$

where Q is the total charge of the charge distribution, measured at a distance r. But what happened, if Q is zero, we would certainly conclude that potential is approximately zero, which is quite correct up to a certain extent. As potential at large distances is very small even if Q is not zero.

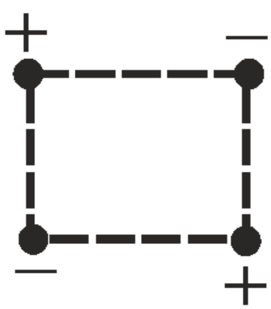
In our earlier classes we have studied the potential due to a dipole (two equal and opposite charges $\pm q$ separated by a small distance) is given by :

$$\boxed{V(\vec{r}) \cong \frac{1}{4\pi \in_0} \frac{p\cos\theta}{r^2} \;\alpha\; \frac{1}{r^2}}$$

which falls off more rapidly than the potential for point charge and for dipole total charge is zero also. ***If we put another pair of equal and opposite charges to make a quadrupole, the potential falls off with*** $\frac{1}{r^3}$ ***and also for a octopole, it falls off with*** $\frac{1}{r^4}$ ***and so on.***

Figure 5.1(a) Monopole $\left[V\alpha\frac{1}{r}\right]$

Figure 5.1(b) Dipole $\left[V\alpha\frac{1}{r^2}\right]$



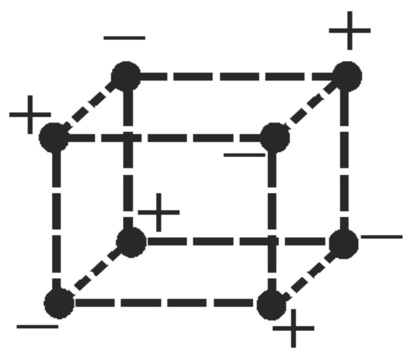


Figure 5.1(c) Quadrupole $\left[V\alpha\frac{1}{r^3}\right]$

Figure 5.1(d) Octopole $\left[V\alpha\frac{1}{r^4}\right]$

5.2 Multipole Expansion

A localized charge distribution is shown in figure 1, which is described by the charge density $\rho(\vec{r}\,')$ non-vanishing only inside the charge distribution. We would like to develop a systematic expansion for the potential of this localized charge distribution in powers of $\frac{1}{r}$.

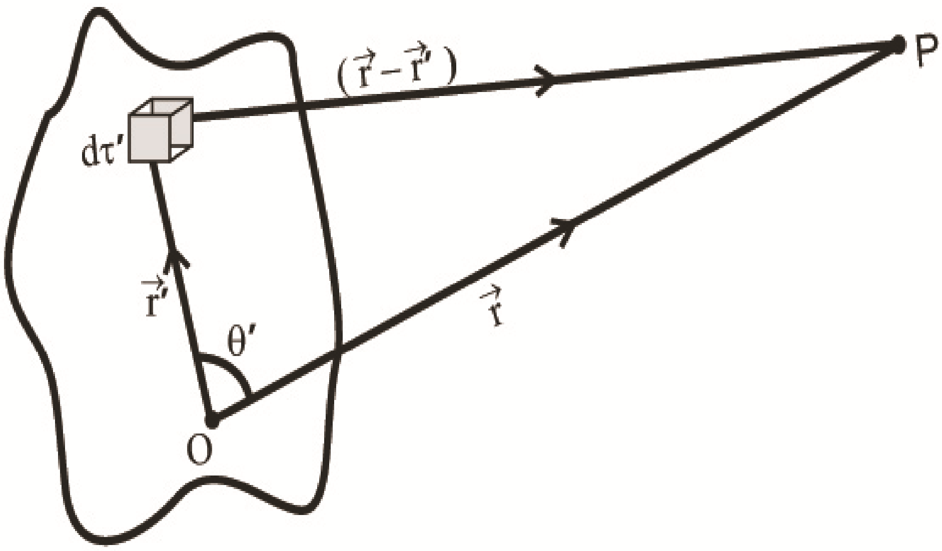


Figure 5.2 Localized Charge distribution

For a charge distribution $\rho(\vec{r}\,')$ consider the potential at r distance will be:

$$\boxed{V(\vec{r}) = \frac{1}{4\pi \in_0} \int \frac{1}{\left|\vec{r}-\vec{r}\,'\right|} \rho(\vec{r}\,') d\tau'} \tag{5.1}$$

where using the law of cosines,

$$\left|\vec{r}-\vec{r}\,'\right| = r^2 + r'^2 - 2rr'\cos\theta'$$

$$= r^2 \left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'\right] \tag{5.2}$$

$$= r\sqrt{1+\in}$$

Where $\in = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right)$ (5.3)

For $r >> r', \in << 1$ for the points well outside the charge

So $\left|\vec{r} - \vec{r}'\right|^{-1} = \frac{1}{r}(1+\in)^{-1/2}$

$$= \frac{1}{r}\left(1 - \frac{1}{2}\in + \frac{3}{8}\in^2 - \frac{5}{16}\in^3 + ...\right)$$

So

$$\frac{1}{\left|\vec{r} - \vec{r}'\right|} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\theta'\right)^2\right.$$
$$\left. - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\theta'\right)^3 + ...\right]$$
$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\theta' + \left(\frac{r'}{r}\right)^2\frac{(3\cos^2\theta' - 1)}{2} + \left(\frac{r'}{r}\right)^3\frac{(5\cos^3\theta' - 3\cos\theta')}{2} + ...\right]$$

..(5.4)

$$\frac{1}{\left|\vec{r} - \vec{r}'\right|} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

The above equation is known as Legendre polynomial. This is the method to obtain Legendre polynomial. Hence $\frac{1}{\left|\vec{r} - \vec{r}'\right|}$ is called the generating function for Legendre polynomial.

So using this expansion in potential term, we get :

$$V(\vec{r}) = \frac{1}{4\pi\in_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\theta)\rho(\vec{r}')d\tau' \quad (5.5)$$

The multipole expansion of V in powers of $\frac{1}{\left|\vec{r} - \vec{r}'\right|}$ will be :

$$V(\vec{r}) = \frac{1}{4\pi \in_0}\left[\left[\frac{1}{r}\int \rho(\vec{r}')d\tau' + \frac{1}{r^2}\int r'\cos\theta'\rho(\vec{r}')d\tau' + \right.\right.$$
$$\left.\frac{1}{r^3}\int (r'^2)\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\vec{r}')d\tau' + ...\right] \qquad (5.6)$$

The equation (5.6) has many number of terms, which can be explained with the numbered terms as described below:

The n = 0 term gives monopole contribution.

The n = 1 term gives dipole contribution

[where $\boxed{\vec{p} = \int \vec{r}'\rho(\vec{r}')d\tau'}$]

The n = 2 term gives Quadrupole contribution [where

$\boxed{Q = \int (r')^2(3\cos^2\theta - 1)\rho(\vec{r}')d\tau'}$]

Hence we can expansion of $V(\vec{r})$ will be :

$$\boxed{V(\vec{r}) = \frac{1}{4\pi \in_0}\left[\frac{Q}{r} + \frac{\vec{p}.\hat{r}}{r^2} + \frac{1}{2}\sum Q\frac{\hat{r}.\hat{r}'}{r^3} + ...\right]} \qquad (5.7)$$

Hence we can see that the coefficients in equation 5.7 are linear combinations of the corresponding multipoles. Let us look at these coefficients more closely. The first term in expansion is : $\frac{1}{4\pi \in_0}\frac{Q}{r'}$, which is nothing but the potential due to the total charge of the distribution. The first term will exist if we measure potential at sufficiently large distance.

If we have a neutral molecule, then the first term will be zero. The second term $\frac{1}{4\pi \in_0}\left(\frac{\vec{p}.\hat{r}}{r^2}\right)$ dominates and has a non zero value for the distribution and hence the potential will vary asymptotically as $\frac{1}{r^2}$ and the electric field strengths behave asymptotically like $\frac{1}{r^3}$. For a dipole of strength $\vec{p}$, the potential is given by :

$$V(\vec{r}) = \frac{\vec{p}.\hat{r}}{4\pi \in_0 r^2} = \frac{\mathrm{p}\cos\theta}{4\pi \in_0 r^2}$$

To calculate the field, we take the negative gradient of V, hence:

$$\boxed{E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi \in_0 r^3}} \quad \text{and}$$

$$\boxed{E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi \in_0 r^3}}$$

Thus $\boxed{\vec{E}(\vec{r}) = \frac{p}{4\pi \in_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})}$ (5.8)

In the coordinate free from it can be written as :

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi \in_0}\frac{1}{r^3}[3(\vec{p}.\hat{r})\,\hat{\mathrm{r}} - \vec{p}]} \tag{5.9}$$

Similarly if first and second terms are zero but the third term is not, the ***potential will behave like*** $\frac{1}{r^3}$ ***at large distances and the field strength will fall off with the*** $\frac{1}{r^4}$.

The advantage of describing a charge distribution by this hierarchy of moments is that it singles out just those features of the charge distribution which determines the field at a great distance. For our understanding to the dielectrics, it turns out that only the ***monopole strength (the total charge)*** and the dipole strength of the molecules matter and hence we can ignore all other moments. ***If the molecule structures are neutral, we left with only dipole moments to be considered.***

5.3 Multipole Expansion of the Energy of a Charge Distribution in an External Field

If a localized charge distribution $\rho(\vec{r}')$ is placed in an external potential $V(\vec{r})$ the electrostatic energy of the system is:

$$\boxed{W = \int \rho(\vec{r}')V(\vec{r})d\tau'} \tag{5.10}$$

If the potential is slowly varying, it can be expanded in Taylor series around a suitably chosen origin like this

$$V(\vec{r}) = V(0) + \vec{r}'.\nabla V(0) + \frac{1}{2}\sum_{i,j} r_i' r_j' \frac{\partial^2 V}{\partial r_i \partial r_j}(0) + ... \tag{5.11}$$

Using the electric field $\vec{E} = -\nabla V$, the last two terms can be rewritten as:

$$V(\vec{r}) = V(0) - \vec{r}.\vec{E}(0) - \frac{1}{2}\sum_i \sum_j r_i' r_j' \frac{\partial E_j}{\partial r_i}(0) + ... \tag{5.12}$$

as $\nabla.\vec{E} = 0$, we can subtract $\frac{1}{6} r^2 \nabla.E(0)$ from the equation, we get:

$$V(\vec{r}) = V(0) - \vec{r}.\vec{\mathrm{E}}(0) - \frac{1}{6}\sum_i \sum_j (3\mathrm{r}_i' \mathrm{r}_j' - \mathrm{r}^2\,\delta_{ij}) \frac{\partial E_j}{\partial r_i}(0) + ... \tag{5.13}$$

Using this in electrostatic energy shown in equation 5.10, we can write:

$$W = qV_{(0)} - \vec{p}.\vec{E}(0) - \frac{1}{6}\sum_i \sum_j Q_{ij} \frac{\partial E_j}{\partial r_i}(0) + ... \tag{5.14}$$

This expansion shows that how various multipoles interact with an external field, the charge with the potential, the dipole with the electric field, the quadrupole with the electric field gradient and so on. Hence with multipole expansion we can show that these multipoles interact with electric potential.

5.4 Self Learning Exercise- I

Very short Answer type questions

Q.1 Write down the potential if we are far away from a localized charge distribution?

Q.2 What do you mean by a dipole?

Q.3 For a quadrupole, what will be the dependence on distance for the potential.

Short Answer type Questions

Q.4 Obtain the form of electric field for a dipole.

Q.5 Charge is uniformly distributed throughout a sphere of radius 'a' at unit density. A redistribution of the charge results in the density function:

$$\rho_v(r) = k\left(3 - \frac{r^2}{a^2}\right) \tag{5.15}$$

Evaluate *k*.

5.5 Illustrative Examples

Example 5.1. Consider a sphere of radius R, centered at origin, having charge density

$$\rho(\vec{r},\theta) = k\frac{R}{r^2}(R-2r)\sin\theta \tag{5.16}$$

where k is a constant and r,θ are the spherical coordinates. Find the different terms of potential for points on the z-axis, far from the sphere.

Sol.

For multipole expansion of potential, we know that first term is a monopole term, for which the charge is :

$$Q = \int \rho d\tau = kR\int \frac{1}{r^2}(R-2r)\sin\theta\, r^2\sin\theta d\theta d\phi dr \tag{5.17}$$

We can see that the *r* integral is :

$$\int_O^R (R-2r)dr = \left[(Rr - r^2)\right]_0^R = 0$$

hence $Q = 0$

The total charge in the sphere is zero, hence the monopole contribution in the potential will be zero. For second term, which is a dipole term :

$$\int rcos\theta\rho d\tau = kR\int (r\cos\theta)\left[\frac{1}{r^2}(R-2r)\sin\theta\, r^2\sin\theta drd\theta d\phi\right] \tag{5.18}$$

Here we can see that in θ integral:

$$\int_0^\pi \sin^2\theta\cos\theta d\theta = \left[\frac{\sin^3\theta}{3}\right]_0^\pi = \frac{1}{3}(0-0) = 0$$

Hence the dipole term contribution in the potential will also be zero.

For third term, which is quadrupole term :

$$\int r^2\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)\rho d\tau$$

$$=\frac{1}{2}kR\iint(3\cos^2\theta-1)\left[\frac{1}{r^2}(R-2r)\sin\theta\right]r^2\sin\theta drd\theta d\phi \tag{5.19}$$

Where r integral is given as :

$$\int_0^R r^2(R-2r)dr=\left[\frac{r^3}{3}R-\frac{r^4}{2}\right]_0^R$$

$$=\frac{R^4}{3}-\frac{R^3}{2}=-\frac{R^4}{6}$$

θ Integral can be calculated as

$$\int_0^\pi(3\cos^2\theta-1)\sin^2\theta d\theta=\int_0^\pi(2-3\sin^2\theta)\sin^2\theta d\theta$$

$$=2\int_0^\pi\sin^2\theta d\theta-3\int_0^\pi\sin^4\theta d\theta=2\left(\frac{\pi}{2}\right)-3\left(\frac{3\pi}{8}\right)$$

$$=\pi\left(1-\frac{9}{8}\right)=-\frac{\pi}{8} \tag{5.20}$$

and the ϕ integral will give :

$$\int_0^{2\pi}d\phi=2\pi$$

The value of the whole integral is given by :

$$\frac{1}{2}kR\left(-\frac{R^4}{6}\right)\left(-\frac{\pi}{8}\right)(2\pi)=\frac{k\pi^2R^5}{48} \tag{5.21}$$

For the point on the z-axis, we can take $r\to z$ and the approximate potential will be :

$$V(z) \cong \frac{1}{4\pi \in_0} \frac{k\pi^2 R^5}{48z^3} \tag{5.22}$$

which will be a Contribution of quadrupole term. Hence we can see that even if the total charge and the dipole are not available in the problem the potential have its physical value.

Example 5.2 Three point charges are placed as shown in figure below, all these charges are separated by a distance ‘a’ from the origin.

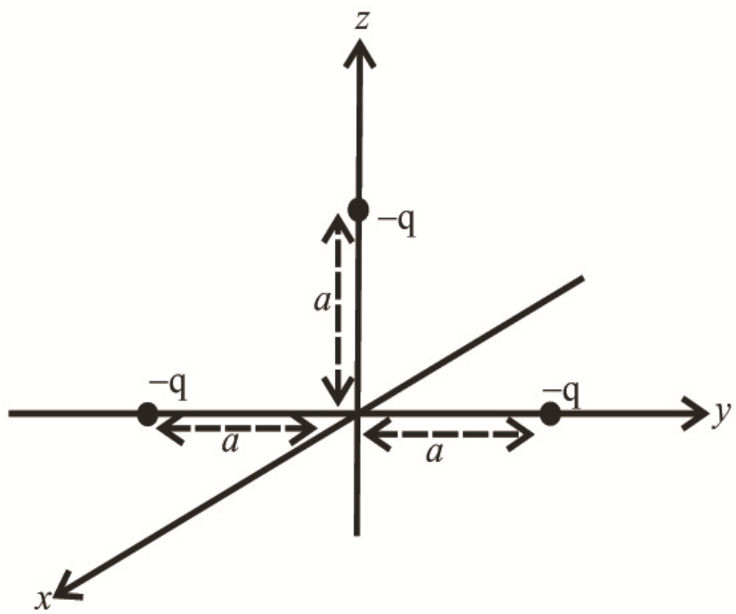


Figure 5.3

Find the approximates potential and electric field at points far from the origin by including first two terms in the multipole expansion of potential.

Sol.

We can see from the configuration given in figure that the total charge will be $-q$ hence $Q = -q$ So

$$V_{(mono)} = \frac{1}{4\pi \in_0} \frac{-q}{r} \tag{5.23}$$

The dipole moment of the configuration will be :

$\vec{p} = qa\hat{z}$ and hence the dipole contribution of the configuration will be :

$$V_{dip} = \frac{1}{4\pi \in_0} \frac{qa\cos\theta}{r^2} \tag{5.24}$$

and hence the total potential is given by :

$$V(\vec{r},\theta) \cong \frac{q}{4\pi \in_0}\left(-\frac{1}{r} + \frac{a\cos\theta}{r^2}\right),$$

which is the approximate potential including monopole and dipole terms.

Now to calculate the electric field of the configuration we use:

$$\boxed{\vec{E} = -\nabla V}$$

In terms of spherical coordinates:

$$\boxed{\vec{E} = -\left(\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}\right)} \tag{5.25}$$

Calculating the $\hat{r}$ and $\hat{\theta}$ terms for the electric field ,the final electric field is given by :

$$\vec{E}(r,\theta) \cong \frac{q}{4\pi \in_0}\left[-\frac{1}{r^2}\hat{r} + \frac{a}{r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right] \tag{5.26}$$

which is the required approximate electric field.

5.6 Self Learning Exercise-II

Q.1 Discuss monopole, dipole and quadrupole for a system of charges.

Q.2 Write down the variation of the potential on the measuring distance for various charge combinations.

5.7 Summary

In this unit the potential at large distance due to localized charge distributions and its expansion in multipoles is developed so that reader can understand the mechanism of potentials at large distances. Initially the mutipole expansion is developed to give an idea of monopoles, dipoles, quadrupoles etc. in the charge distribution available in atoms and molecules, then the energy of a charge distribution or multipoles in an external field is developed to give an idea that how various multipoles interact with an external field.

5.8 Glossary

Induce : to cause something to happen

Multipole expansion: It is a mathematical series representing a function that generally depends on angles — usually the two angles on a sphere. The function being expanded may be complex in general. Multipole expansions are very frequently used in the study of electromagnetic and gravitational fields, where the fields at distant points are given in terms of sources in a small region.

Electric quadrupole: The simplest example of an electric quadrupole consists of alternating positive and negative charges, arranged on the corners of a square. The monopole moment (just the total charge) of this arrangement is zero. Similarly, the dipole moment is zero, regardless of the coordinate origin that has been chosen.

Asymptote: straight line that continually approaches a given curve but does not meet it at any finite distance.

5.9 Answers to Self Learning Exercise

Answers to Self Learning Exercise-I

Ans.1 : $V(\vec{r}) = \dfrac{1}{4\pi \in_0} \dfrac{Q}{r}$

where Q is the total charge of the system.

Ans.2 : A group of two opposite charges having similar magnitude, separated by small distance between them. For such dipole , dipole moment is given by

$\vec{p} = q\vec{d}$

Ans.3 : For large distance, potential is inversely proportional to cube of the distance i.e.

$$\left[V \alpha \frac{1}{r^3} \right]$$

Ans.4 : See equation 5.8

Ans.5 : k=5/12

Answers to Self Learning Exercise-II

Ans.2:

$$V(\vec{r}) = \frac{1}{4\pi \in_0}\left[\left[\frac{1}{r}\int \rho(\vec{r}')d\tau' + \frac{1}{r^2}\int r'\cos\theta'\rho(\vec{r}')d\tau' + \right.\right.$$

$$\left.\frac{1}{r^3}\int (r'^2)\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\vec{r}')d\tau' + ...\right]$$

5.10 Exercise

Section – A (Very short Answer type Questions)

Q.1 What do you mean by multipole expansion of the potential?

Q.2 Write down the electrostatic energy for a charge distribution placed in an external electric field.

Section- B (Short Answer type Questions)

Q.3 A circular disk of radius 'R' has a surface charge density that increases linearly away from the center, the constant of proportionality being k. Determine the total charge on the disk.

Q.4 Calculate the energy required to uniformly charge a sphere of radius 'R' by a total charge Q.

Q.5 Show that the electric field of a dipole can be written in the coordinate free form:

$$E_{dip}(\vec{r}) = \frac{1}{4\pi \in_0}\frac{1}{r^3}\left[3(\vec{p}.\hat{r})\hat{r} - \vec{p}\right]$$

Section – C (Long Answer type questions)

Q.6 Two point charges 3q and –q are separated by a distance 'a' for the given arrangements in the figures 5.3(a),(b),(c) below find :

(i)The monopole and dipole moment

(ii)The approximate potential at large r including both the monopole and dipole contribution.

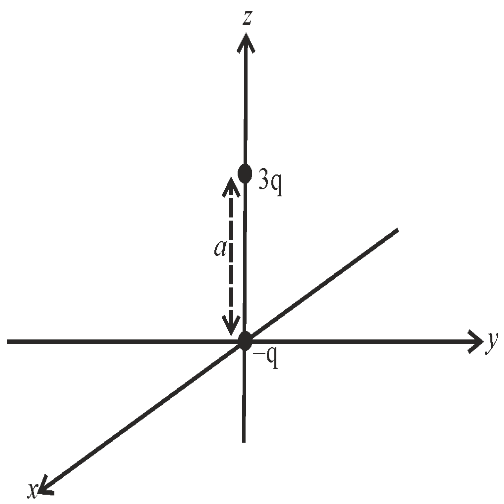


Figure 5.3(a)

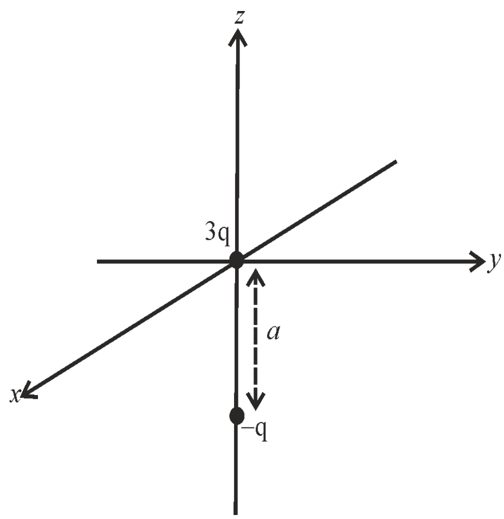


Figure 5.3(b)

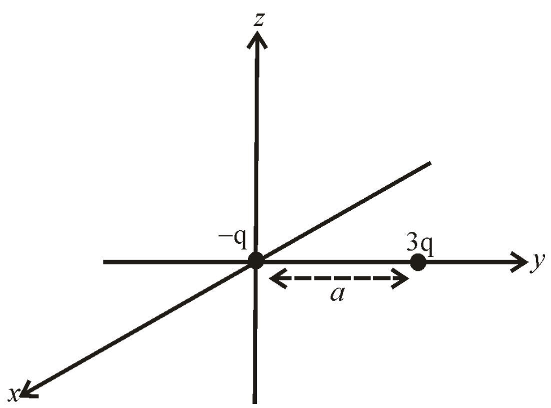


Figure 5.3(c)

Q.7 For a spherical shell of radius R, which carries a surface charge $\sigma = k\cos\theta$

(i) Calculate the dipole moment of this configuration and

(ii) Find the approximate potential at points far from the sphere.

Q.8 Prove that due to a system of point charges, the potential at an external point is given by the sum of the individual potentials due to monopoles, dipoles, quadrupoles etc.

Q.9 Obtain the multipole expansion of energy for a charge distribution in an electric field that how various multipoles interact with an external field.

Q.10 Determine that the interaction energy between two dipoles $\vec{p}_1$ and $\vec{p}_2$ and mutual potential energy will be :

$$W_{12} = \frac{\vec{p}_1 . \vec{p}_2 - 3(\hat{r}.\overrightarrow{p_1})(\hat{r}.\overrightarrow{p_2})}{\left|\vec{r}_1 - \vec{r}_2\right|^3}$$

where $\vec{r}_1 \neq \vec{r}_2$

5.11 Answers to Exercise

Ans.2 : $W = \int \rho(\vec{r}')V(\vec{r})d\tau'$

Ans.3:. $Q = \frac{2\pi k R^3}{3}$

Ans.4 : $\frac{3kQ^2}{5R}$

Ans.6 :

[a] Q = 2q, p=3qa$\hat{z}$

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\cos\theta}{r^2}\right]$$

[b] Q = 2q, p=qa$\hat{z}$

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{qa\cos\theta}{r^2}\right]$$

[c] Q = 2q, p=3qa$\hat{y}$

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{2q}{r} + \frac{3qa\sin\theta\sin\emptyset}{r^2}\right]$$

Ans.7 :

[i] $p = \frac{4\pi k R^3}{3}\hat{z}$

[ii] $V = \frac{kR^3}{3\epsilon_0}\left[\frac{\cos\theta}{r^2}\right]$

References and Suggested Readings

1. David J. Griffiths, Introduction to Electrodynamics, 2nd Edition, Prentice-Hall,2000

2. J.D. Jackson, Classical Electrodynamics, Wiley Eastern Limited,2002.

3. W.K.H. Panofsky and M. Phillips, Classical Electricity and Magnetism, 2^{nd} edition, Addison Wesley, 1962.

4. Mathew N.O. Sadiku, Elements of Electromagnetics, Oxford University Press, 2001.

5. Landau & Lifshitz, Classical theory of Electrodynamics (Pergaman press, New York).

UNIT-6
Elementary Treatment of Electrostatics with Permeable Media

Structure of the Unit

6.0 Objectives

This chapter deals with the boundary value problems in electrostatics with dielectric their treatment to understand the electrostatics quantities molecular polarizability and electric susceptibility, electrostatic energy in dielectric media.

6.1 Introduction

In this unit we will discuss the electrostatic boundary condition at the interface of the dielectric. We will study the displacement vector D and we find the usefulness of the D in case of symmetrical situations with dielectrics. understand the electrostatics quantities molecular polarizability and electric susceptibility are explained in this unit.

6.2 Elementary Treatment of Electrostatics with permeable Media

If a piece of dielectric medium is placed in an electric field. Even if the substance consists of neutral atoms, the field will induce in each a tiny dipole moment, pointing in the same direction of the field.

If the material is made of polar molecules, dipole experiences the torque, to line up them with field. Thus produced in the medium an electric polarization $\vec{p}$ (dipole moment per unit volume) given by

$$\vec{p}(\vec{r}) = \sum_i N_i < \vec{p}_i >$$

[Where $\vec{p}_i$ is the dipole moment if i^{th} molecule, N_i is the average number of i^{th} type molecules per unit volume].

For a single dipole the potential is

$$\phi(\vec{r}) = \frac{1}{4\pi \in_0} \frac{\vec{p}.|\hat{r} - \hat{r}'|}{|\hat{r} - \hat{r}'|^2}$$

For a polarization $\vec{P}$

dipole moment $\vec{p} = \int_V \vec{P} d\tau'$

So the potential is:

$$\phi(\vec{r}) = \frac{1}{4\pi \in_0} \int_V \frac{\hat{r}.P(\vec{r}')}{|\vec{r} - \vec{r}'|^2} d\tau' \text{ [we know that } \nabla'\left(\frac{1}{|\vec{r} - \vec{r}'|}\right) = \frac{|\hat{r} - \hat{r}'|}{|\vec{r} - \vec{r}'|^2}]$$

So $\phi(\vec{r}) = \frac{1}{4\pi \in_0} \int_V \vec{P}.\nabla'\left(\frac{1}{|\vec{r} - \vec{r}|'}\right) d\tau'$

Integrating by parts we get:

$$\phi(\vec{r}) = \frac{1}{4\pi \in_0}\left[\int_V \nabla'.\left(\frac{\vec{P}}{|\vec{r}-\vec{r}'|}\right)d\tau' - \int_V \frac{1}{|\vec{r}-\vec{r}'|}\left(\nabla'.\vec{P}\right)d\tau'\right]$$

Using the divergence theorem:

$$\phi(\vec{r}) = \frac{1}{4\pi \in_0}\oint_S \frac{\vec{P}}{|\vec{r}-\vec{r}'|}.d\vec{S}' - \frac{1}{4\pi \in_0}\int_V \frac{1}{|\vec{r}-\vec{r}'|}\left(\nabla'.\vec{P}\right)d\tau'$$

The first term looks like the potential of a ***surface charge density*** $\boxed{\sigma_b = \vec{P}.\hat{n}}$ [along the surface vector] while the second term looks like the potential of ***volume charge density*** $\boxed{\rho_b = -\nabla.\vec{P}}$ [volume contribution is large] and hence the potential $\phi(\vec{r}) = \int d\tau' \frac{1}{|\vec{r}-\vec{r}'|}\left[\rho(\vec{r}') - \nabla'.\vec{P}(\vec{r}')\right]$, with $\vec{E} = -\nabla\phi$ so first Maxwell's eq. will be : $\nabla.\vec{E} = \frac{\left[\rho - \nabla.\vec{P}\right]}{\in_0}$

The presence of divergence of $\vec{P}$ in effective charge to the density is due to presence of bound charges to the material.

So $\left(\nabla.\vec{E}\right)\in_0 = \rho - \nabla.\vec{P}$

or $\nabla.\left(\in_0 \vec{E} + \vec{P}\right) = \rho$, which can be written as $\boxed{\vec{D} = \in_0 \vec{E} + \vec{P}}$ [electric displacement] and the Gauss's law will be $\boxed{\nabla.\vec{D} = \rho_{free}}$ or in integral from $\oint \vec{D}.d\vec{a} = Q_{free\,enclosed}$.

For simplicity consider medium is ***linear isotropic***, then the induced polarization $\vec{P}$ is parallel to $\vec{E}$, that is

$\boxed{\vec{P} = \in_0 \chi_e \vec{E}}$ [χ_e = Electrical susceptibility]

So in isotropic (linear) medium

$$\boxed{\vec{D} = \in_0 \vec{E} + \vec{P} = \in_0 \vec{E} + \in_0 \chi_e \vec{E} = \in_0 (1+\chi_e)\vec{E}}$$

So $\vec{D}$ is proportional to $\vec{E} \Rightarrow \boxed{\vec{D} = \in \vec{E}}$

where $\boxed{\in = \in_0 (1+\chi_e)}$ and $\boxed{\in_r = 1 + \chi_e = \frac{\in}{\in_0}}$ is called the ***relative permittivity or***

dielectric constant of the medium (material).

For anisotropic medium

$$P_x = \in_0 \left(\chi_{e_{xx}} E_x + \chi_{e_{xy}} E_y + \chi_{e_{xz}} E_z \right)$$

$$P_y = \in_0 \left(\chi_{e_{yx}} E_x + \chi_{e_{yy}} E_y + \chi_{e_{yz}} E_z \right)$$

$$P_z = \in_0 \left(\chi_{e_{zx}} E_x + \chi_{e_{zy}} E_y + \chi_{e_{zz}} E_z \right)$$

constitute the ***Susceptibility Tensor.***

6.3 Boundary Conditions

The electrostatic boundary condition can be written in terms of *D*. The discontinuity in the normal components of *D* will be.

$$\left(D_2 - D_1 \right) = \sigma_f$$

In parallel components

$$\boxed{D_2^{\parallel} - D_1^{\parallel} = P_2^{\parallel} - P_1^{\parallel}}$$

and on $\vec{E}$

$$\left(E_2 - E_1 \right) = \frac{\sigma}{\in_0} \text{ in parallel components}$$

$$\boxed{E_2^{\parallel} - E_1^{\parallel} = 0} \text{(tangential components)}$$

6.4 Boundary Value Problems with Linear Dielectrics

To illustrate the method of images for dielectrics, we consider a point charge q embedded in a semi infinite dielectric ε_1 at a distance d away from a plane interface which separates the medium from another semi infinite ε_2 .We have to find the solutions for this.

$$\left.\begin{array}{ll} \varepsilon_1 \nabla.\vec{E} = \dfrac{l}{\in_0} & z > 0 \\ \varepsilon_2 \nabla.\vec{E} = 0 & z < 0 \\ \nabla \times \vec{E} = 0 & everywhere \end{array}\right\} \qquad (1)$$

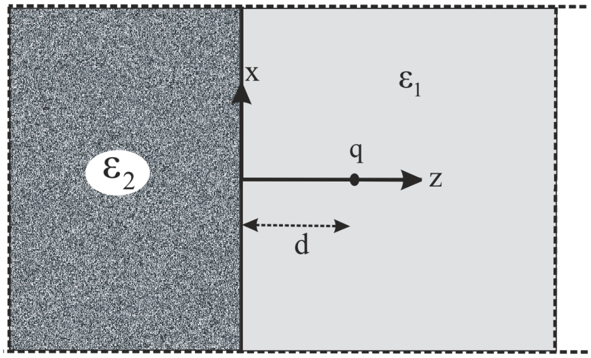


Figure 6.1

Boundary conditions at z = 0

$$\lim_{z\to 0}\begin{Bmatrix} \epsilon_1 E_z \\ E_x \\ E_y \end{Bmatrix} = \lim_{z\to 0}\begin{Bmatrix} \epsilon_2 E_z \\ E_x \\ E_y \end{Bmatrix} \tag{2}$$

Using image method to locate an image charge q' at the symmetrical position A' as shown below.

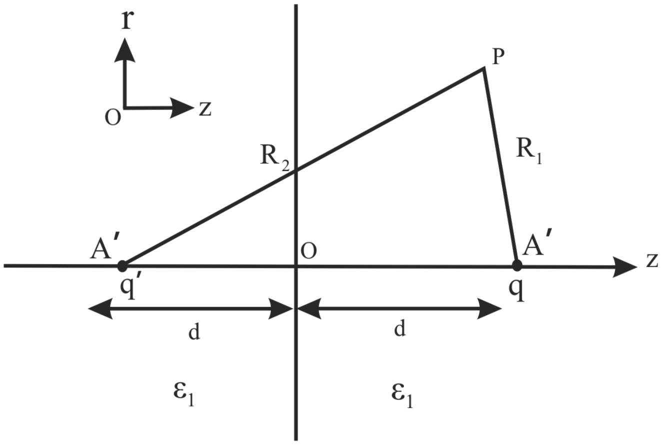


Figure 6.2

Then for $z > 0$ the potential at a point P described by cylindrical coordinates (r,ϕ,z) will be:

$$\phi = \frac{1}{4\pi \epsilon_0 \epsilon_1}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right) \qquad \text{for} \quad z > 0$$

where $R_1 = \sqrt{r^2 + (d-z)^2}$, $R_2 = \sqrt{r^2 + (d+z)^2}$

Since there is no charge in the z < 0 region, it must be a solution of the Laplace equation without singularities in that region. Consider the potential at z<0 is equivalent to the charge q'' at the position of actual charge q.

$$\phi = \frac{1}{4\pi \in_0 \in_2} \frac{q''}{R_2} \qquad z<0 \qquad (4)$$

Since $\frac{\partial}{\partial z}\left(\frac{1}{R_1}\right)\Bigg|_{z=0} = -\frac{\partial}{\partial z}\left(\frac{1}{R_2}\right)\Bigg|_{z=0} = \frac{d}{\left(r^2+d^2\right)^{3/2}}$

While $\frac{\partial}{\partial r}\left(\frac{1}{R_1}\right)\Bigg|_{z=0} = \frac{\partial}{\partial r}\left(\frac{1}{R_2}\right)_{z=0} = -\frac{-r}{\left(r^2+d^2\right)^{3/2}}$

Using the boundary conditions (eqn.(2))

$$q - q' = q''$$

and $\frac{1}{\in_1}\left(q+q'\right) = \frac{1}{\in_2} q''$

after solving the image charges .

$$q' = -\left(\frac{\in_2 - \in_1}{\in_2 + \in_1}\right) q \text{ and } q'' = \left(\frac{2\in_2}{\in_2 + \in_1}\right) q$$

The polarization charge density is $-\nabla.\vec{P}$ For dielectrics $\vec{P} = \in_0 \chi_e \vec{E}$ so that

$$-\nabla.\vec{P} = -\in_0 \chi_e \nabla.\vec{E} = 0, \text{except at the point charge } q.$$

Bound charge density $\left(\rho_b\right)$ proportional to free charge $\left(\rho_f\right)$

$$\rho_b = -\nabla.\vec{P} = -\nabla.\left(\in_0 \frac{\chi_e}{\in} \vec{D}\right) = -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_f$$

The surface found charge

$\sigma_b = \vec{P}.\hat{n} = P_2 = \in_0 \chi_e E_z$, where E_z is z component of total field inside the dielectric, at z=0, that is

$$-\frac{1}{4\pi \in_0} \frac{q}{\left(\rho^2+d^2\right)} \cos\theta = -\frac{1}{4\pi \in_0} \frac{qd}{\left(r^2+d^2\right)^{3/2}}$$

The z component of bound charges is $-\dfrac{\sigma_b}{2\in_0}$

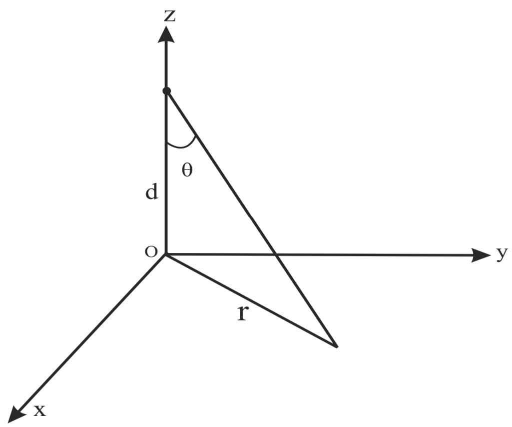


Figure 6.3

So $$\sigma_b = \in_0 \chi_e \left[-\frac{1}{4\pi \in_0} \frac{qd}{\left(r^2+d^2\right)^{3/2}} - \frac{\sigma_b}{2\in_0} \right]$$

Solving for σ_b we get:

$$\sigma_b = -\frac{q}{2\pi}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{d}{\left(r^2+d^2\right)^{3/2}}$$

Using $\sigma_{pol} = -\left(\vec{P}_2 - \vec{P}_1\right).\hat{n}_{21}$ where $\hat{n}_{21}$ is the normal from dielectric 1 to dielectric 2 and for this

$$\sigma_{pol} = -\frac{q}{2\pi \in_1}\left(\frac{\in_2 - \in_1}{\in_2 + \in_1}\right)\frac{d}{\left(r^2+d^2\right)^{3/2}}$$

6.5 Molecular Polarizability & Electrical Susceptibility

The polarization is given by:

$\vec{P} = N\langle \vec{p} \rangle$. This induced dipole moment is proportional to electric field on the molecule .For this we define a molecular prolarizability α as the ratio of the average dipole moment to the applied field at the molecule, this gives:

$\langle p \rangle = \alpha\left(E + E_i\right)$ here E_i is the internal field.

We know polarization is given by $P = N_i \langle p_i \rangle$

Space for a molecule is a sphere of radius R and hence

$$p = \frac{4\pi R^3}{3} P,$$

so average electric field inside the sphere is :

$$E_i = \frac{3}{4\pi R^3}\left[\int_{r<R} \vec{E} d^3 x\right]\frac{1}{4\pi\varepsilon_0} = -\frac{1}{4\pi\varepsilon_0}\frac{p}{R^3}$$

The density of molecules in $\mathrm{N} = \dfrac{1}{(4/\ 3)\pi R^3}$

$$\mathrm{E}_i = -\frac{1}{4\pi \in_0}\frac{\alpha \mathrm{E}}{R^3} \qquad \text{so total field will be :}$$

$$\mathrm{E}_{Total} = \mathrm{E} + \mathrm{E}_i = \left(\mathrm{E} - \frac{1}{4\pi \in_0}\frac{\alpha \mathrm{E}}{R^3}\right) = \left(1 - \frac{\alpha}{4\pi \in_0 R_3}\right)\mathrm{E}$$

$$= \left(1 - \frac{N\alpha}{3 \in_0}\right)\mathrm{E}$$

So $P = \mathrm{N}\alpha \mathrm{E}_{Total} = \dfrac{\mathrm{N}\alpha}{\left(1 - \dfrac{\mathrm{N}\alpha}{3 \in_0}\right)}\vec{\mathrm{E}} = \in_0 \chi_e \vec{\mathrm{E}}$

And hence $\chi_e = \dfrac{N\alpha / \in_0}{\left(1 - \dfrac{\mathrm{N}\alpha}{3 \in_0}\right)}$ solving for α

$$\alpha = \frac{3 \in_0}{\mathrm{N}}\frac{\chi_e}{(3 + \chi_e)} \text{ but } \chi_e = \in_r - 1$$

$$\text{so } \boxed{\alpha = \frac{3 \in_0}{\mathrm{N}}\left(\frac{\in_\gamma - 1}{\in_\gamma + 2}\right)}$$

This is called the ***Clausius –Mossotti*** equation

6.6 Models for the Molecular Polarizability

The polarization of the collection of atoms or molecules can arise in two ways.

1) The applied field distorts the charge distribution and so produces an induced dipole moment in each molecules.

2) The applied field tends to line up the initially randomly oriented permanent dipole moments of the molecules.

To estimate the induced moments, consider harmonically bounded charges(ions and electrons). Each charge ρ is bound under the action of a restoring force

$$F = -mw_0{}^2 x$$

where m is the mass of charge and ω_0 is the frequency of oscillation about equilibrium.

Under the action of an electric field

$\vec{\mathrm{E}}$ the charges displaced from its equilibrium by x ,given by

$$mw_0^2 x = eE \tag{1}$$

The induced dipole moment is

$$p_{mol} = \gamma E = \frac{e^2}{mw_0{}^2}\mathrm{E} \tag{2}$$

This means that the polarizability is $\gamma = \dfrac{e^2}{mw_0{}^2}$

If there are set of charges e_i with mass m_i and oscillation frequencies w_i in each molecule ,then the molecular polariazability is

$$\gamma_{mol} = \sum \frac{e_i^2}{m_i w_i{}^2} \tag{3}$$

We know that the binding frequencies of electrons in atoms must be of the order of light frequencies. Taking a typical λ of light as 3000 A^0, we find $w \approx 6X10^{15}$ $\sec^{-1}$ so electronic contribution is $\gamma_{el} \sim \dfrac{e^2}{mw^2} \sim 6\times10^{-24}\ cm^3$.

The possibility that thermal agitation could modify results (in eqn.(3)). In statistical mechanics the probability distribution of particles in phase space(p,q space) is

$f(H) = e^{-\left(\frac{H}{kT}\right)}$ The Boltzmann's factor.

For the problem of harmonically bound charges with and applied field in the z direction, the Hamiltonian is :

$$\mathrm{H} = \frac{p^2}{2m} + \frac{1}{2} m w^2{}_0 x - e\mathrm{E}z$$

The average value of the dipole moment is the z direction is :

$$\langle p_{mol} \rangle = \frac{\int d^3 p \int d^3 x (ez) f(\mathrm{H})}{\int d^3 p \int d^3 x\, f(\mathrm{H})}$$

For a displaced coordinate $x' = x - eEk / m\omega_0^2$ then,

$$H = \frac{p^2}{2m} + \frac{m\omega_0^2}{2}(x')^2 - \frac{e^2 E^2}{2m\omega_0^2} \text{ and}$$

$$\langle p_{mol} \rangle = \frac{\int d^3 p \int d^3 x' \left(ez' + \frac{e^2 E}{m\omega_0^2} \right) f(H)}{\int d^3 p \int d^3 x' f(H)}$$

Since H is even is z' , the first integral vanishes, hence, we obtain $\langle p_{mol} \rangle = \frac{e^2 \mathrm{E}}{m\omega_0^2}$ same as we have obtained ignoring thermal motion.

The second type of polarizability is that caused by the partial orientation of otherwise permanent dipole moments. As for some polar substances such as HCl and H_2O it is important.

Here all molecules are assumed to passes a permanent dipole moment p_0, which can be oriented in any direction in space. In the absence of field, thermal agitation keeps the molecules randomly oriented so that there is no net dipole moment. With applied field they try to line up along the field to have lowest energy and there will be an average dipole moment. For this case Hamiltonian

$$\mathrm{H} = \mathrm{H}_0 - \vec{p}_0 . \vec{\mathrm{E}}$$

and the average dipole moment will be :

$$\langle p_{mol} \rangle = \frac{\int d\Omega\, p_0 \cos\theta \exp\left(\frac{p_0 \mathrm{E} \cos\theta}{k\mathrm{T}} \right)}{\int d\Omega exp\left(\frac{p_0 \mathrm{E} \cos\theta}{k\mathrm{T}} \right)}$$

As $\frac{p_0\vec{E}}{kT} << 1$ so that

$$\langle p_{mol} \rangle = \frac{p_0^2 E}{3kT}$$

Here we can see that ***orientation polarization depends inversely on the temperature***. In general both polarizations induced (electronic and ionic) and orientation are present and the general form of the molecular polarization is :

$$\gamma_{mol} = \gamma_i + \frac{1}{3}\frac{p_0^2}{kT} \text{ of the form } \left(a + \frac{b}{T}\right)$$

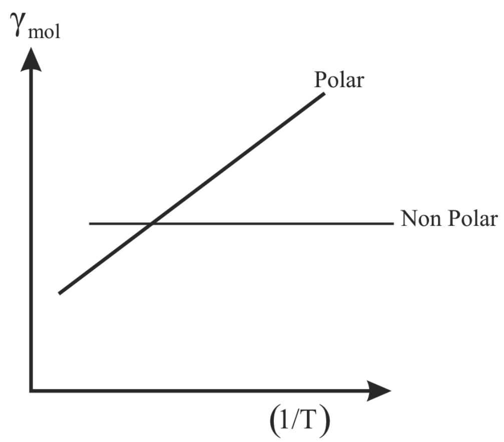


Figure 6.4

6.7 Self Learning Exercise

Very Short Answer type Questions

Q.1 Define dielectric polarisation.

Q.2 What do you mean by polar molecules?

Q.3 Define polarizability.

Q.4 Find the relation between atomic polarizability α and the χ_e

Short Answer type Questions

Q.5 Establish a relation between electric field $\vec{E}$, Polarization $\vec{P}$ and displacement vector $\vec{D}$.

Q.6 A Parallel-Plate Capacitor is filled with insulating material of dielectric constant ε_r . What effect does this have on its capacitance ?

Q.7 Define the Claussius-Mossotti relation.

6.8 Illustrative Examples

Example 1: An interface between two dielectrics is shown as in figure below. Obtain the relationship between the tangential components of E field at either side of the interface.

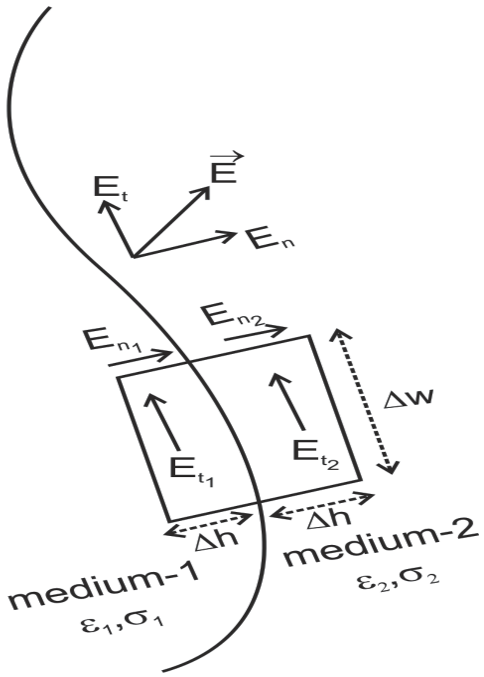


Figure 6.5

Sol. We consider the convention for interface problem as, at any point of the interface the unit normal vector $\hat{n}$ points out of medium 1 and into medium 2.

Since the ***electrostatic field is conservative,*** $\oint \vec{E}.d\vec{l} = 0$ around the rectangular contour of figure 6.5 .Hence:

$$E_{t_2}\Delta w - E_{n_2}\Delta h - E_{n_1}\Delta h - E_{t_1}\Delta w + E_{n_1}\Delta h + E_{n_2}\Delta h = 0$$

In the limit as the rectangular path approaches the surface, i.e. as $\Delta h \to 0$, this becomes

$$\left(E_{t_2} - E_{t_1}\right)\Delta w = 0 \text{ or } E_{t_2} = E_{t_1}$$

Thus the tangential components of $\vec{E}$ are continuous across the interface.

Example 2: If a surface charge density ρ_s exists at the interface of two material media as shown in figure given below, obtain, the relationship between the normal components of the $\vec{D}$-vector at either side of the interface.

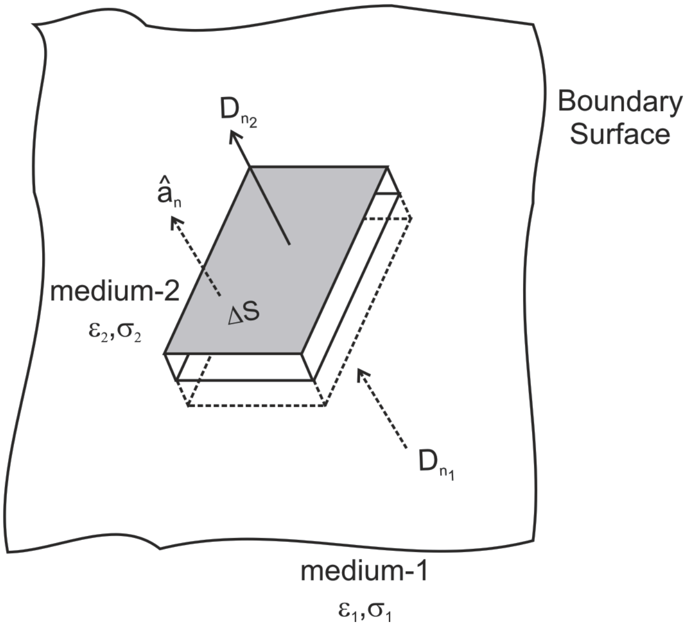


Figure 6.6

Sol. As per the Gauss's law $\oint_S \vec{D}.d\vec{S} = \int_V \rho_V dV$ to the infinitesimal box as shown in figure 6.6 .As the height of this box approaches zero, i.e. $\Delta h \to 0$, only the components of $\vec{D}$ normal to the boundary contributes to the Gauss's law:

$$D_{n_2}\Delta S - D_{n_1}\Delta S = \lim_{\Delta h \to 0} \rho_V \left(\Delta h\, \Delta S\right)$$

hence $D_{n_2}\Delta S - D_{n_1}\Delta S = \Delta h \rho_V\, \Delta S$

which finally written as : $D_{n_2} - D_{n_1} = \rho_S$

Thus at a point of an interface, the jump in the normal components of D equal the local free surface charge density.

As per the examples 1 and 2, the results can also the written as :

$$\frac{D_{t_2}}{\epsilon_2} = \frac{D_{t_1}}{\epsilon_1}$$

and $\epsilon_2 E_{n_2} - \epsilon_1 E_{n_1} = \rho_S$

If both the media are perfect dielectrics, with $\rho_S = 0$, the boundary conditions can be written as :

$$\boxed{E_{t_2} = E_{t_1} \;\; , D_{n_2} = D_{n_1} ,}$$

$$\boxed{\frac{D_{t_2}}{\epsilon_2} = \frac{D_{t_1}}{\epsilon_1} \;\; , \; \epsilon_2 E_{n_2} = \epsilon_1 E_{n_1}}$$

6.9 Electrostatic Energy in Dielectric Media

Electrostatic energy of a system of charges in free space is :

$$W = \frac{1}{2} \epsilon_0 \int e(x)\phi(x) d^3x \tag{1}$$

But for dielectric media this cannot be taken in general. As the work done in dielectric media is not only to bring real charges into position, but also to produce a certain state of polarization in the medium.

Let us consider a small change in energy δW due to some sort of change in δe and this is:

$$\delta W = \epsilon_0 \int \delta e(x)\phi(x) d^3x \tag{2}$$

Where $\phi(x)$ is the potential due to charge density $e(x)$ already present.

Since $\nabla.\vec{D} = \rho$ so $\delta\rho = \nabla.\left(\delta\vec{D}\right)$ and energy change will be:

$$\delta W = \epsilon_0 \int \vec{\mathrm{E}}.\delta\vec{D}\, d^3x \tag{3}$$

Where we have used $\vec{\mathrm{E}} = -\nabla\phi,$ by allowing D variations from 0 to D we see that:

$$W = \epsilon_0 \int d^3x \int_0^D \vec{\mathrm{E}}.\delta\vec{D} \tag{4}$$

If medium is linear $\vec{\mathrm{E}}.\delta\vec{D} = \frac{1}{2}\delta\left(\vec{\mathrm{E}}.\vec{D}\right)$ and the electrostatic energy is:

$$W = \frac{1}{2} \epsilon_0 \int \vec{\mathrm{E}}.\vec{D} d^3x \tag{5}$$

This result in eqn. (5) can be transformed in eq. (1) by using $\vec{\mathrm{E}} = -\nabla\phi$ and $\nabla.\vec{D} = \rho$

So it is dear that eq. (1) is valid macroscopically if the behaviour is linear otherwise energy can be calculated from eq. (4).

6.10 Illustrative Examples

Example 3: A spherical conductor, of radius '*a*', carries Q as charge as shown in the figure below. It is surrounded by linear dielectric material of susceptibility χ_e out of radius *b*, find the energy of this configuration.

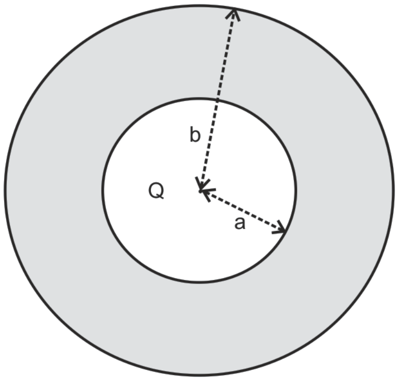


Figure 6.7

Sol. We know that the free charge is Q, and the arrangement is spherically symmetric, so let's start by calculating $\vec{D}$ from equation $\int_S \vec{D}.d\vec{S} = Q$

hence $\vec{D} = \frac{Q}{4\pi r^2}\hat{r}$ for $r > a$

Inside the spherical conductor, of course $\vec{E} = \vec{p} = \vec{D} = 0$ and hence we can write:

$$\vec{D} = \begin{Bmatrix} 0 & (r < a) \\ \frac{Q}{4\pi r^2}\hat{r} & (r > a) \end{Bmatrix} \text{ and}$$

similarly we can obtain $\vec{E}$, which is:

$$\vec{E} = \begin{Bmatrix} 0 & (r < a) \\ \frac{Q}{4\pi \in r^2}\hat{r} & (a < r < b) \\ \frac{Q}{4\pi \in_0 r^2}\hat{r} & (r > b) \end{Bmatrix}$$

Now we will calculate the energy of this configuration.

We know that energy is given as:

$$\boxed{W = \frac{1}{2}\int \vec{D}.\vec{E}\, d\tau}$$

$$= \frac{1}{2}\frac{Q^2}{(4\pi)^2}4\pi\left\{\frac{1}{\in}\int_a^b \frac{1}{r^2}\frac{1}{r^2}r^2 d\tau + \frac{1}{\in_0}\int_b^\infty \frac{1}{r^2}d\tau\right.$$

$$= \frac{Q^2}{8\pi}\left\{\frac{1}{\in}\left(\frac{-1}{r}\right)\Bigg|_a^b + \frac{1}{\in_0}\left(-\frac{1}{r}\right)\Bigg|_b^\infty\right\}$$

$$= \frac{Q^2}{8\pi \in_0}\left\{\frac{1}{(1+\chi_e)}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{b}\right\}$$

$$W = \frac{Q^2}{8\pi \in_0 (1+\chi_e)}\left(\frac{1}{a}+\frac{\chi_e}{b}\right)$$

which is the required energy of the configuration.

Example 4: A metal sphere of radius '*a*' carries a charge Q. The sphere is surrounded by a linear dielectric material of permittivity $\in$ with outer radius 'b'. Find the potential at the centre of the sphere. Compute the polarization, surface and volume bound charges.

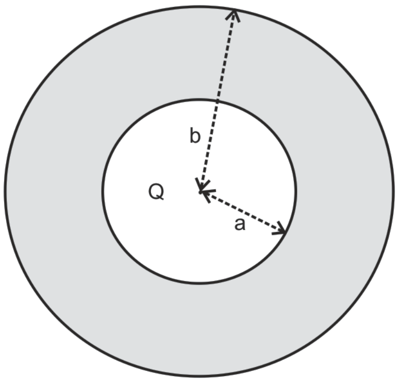


Figure 6.8

Sol. The free charge on the sphere is given as Q and due to spherical symmetric problem the displacement is:

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{r} \text{ for all points } r > a$$

Inside the metal sphere for points $r < a$;

$$\vec{E} = \vec{P} = \vec{D} = 0$$

So we are left with two regions only:

(i) The linear dielectric material of permittivity $\in$ for $a<r<b$; and

(ii) the free space of permittivity $r>b$.

Using the expression for $\vec{D}$, we can write the electric field for the two regions as,

$$\vec{E}=\begin{cases}\dfrac{1}{4\pi\in}\dfrac{Q}{r^2}\hat{r} & a<r<b\\ \dfrac{1}{4\pi\in_0}\dfrac{Q}{r^2}\hat{r} & r>b\end{cases}$$

Therefore the potential at the centre is:

$$V=-\int_{\infty}^{b}\vec{E}.d\vec{r}$$

$$=-\int_{\infty}^{b}\left(\frac{Q}{4\pi\in_0 r^2}\right)dr-\int_{b}^{a}\left(\frac{Q}{4\pi\in r^2}\right)dr-\int_{a}^{0}0.dr$$

$$=-\frac{Q}{4\pi}\left[\frac{1}{\in_0}\left(-\frac{1}{r}\right)_{\infty}^{b}+\frac{1}{\in}\left(-\frac{1}{r}\right)_{b}^{a}\right]$$

$$V=\frac{Q}{4\pi}\left[\frac{1}{\varepsilon_0 b}+\frac{1}{\varepsilon a}-\frac{1}{\varepsilon b}\right]$$

We know that the polarization is the dielectrics is given by :-

$$\vec{P}=\varepsilon_0\chi\vec{E}\ ,\quad \text{thus :}$$

$$\vec{P}=\varepsilon_0\chi.\frac{1}{4\pi\varepsilon}\frac{Q}{r^2}\hat{r}\ ,$$

For the region $a<r<b$, therefore bound changes that appear on the surface will be : - $\sigma_p{}' = \vec{p}.\hat{n} = \begin{cases}\dfrac{\varepsilon_0\chi\,Q}{4\pi\,\varepsilon\,b^2} & \text{at the outer surface}\\ \dfrac{-\varepsilon_0\chi Q}{4\pi\,\varepsilon\,a^2} & \text{at the inner surface}\end{cases}$

while for volume charges, the repression is:-

$$\rho_p' = -\nabla.\vec{P} = 0$$

Hence eqn. represents the desired expression for potential a centre, polarization, surface and volume bound changes respectively.

Example 5: A dielectric sphere is placed in a uniform electrostatic field. Calculate the electric field outside and inside the dielectric sphere.

Sol. A dielectric sphere when introduced in a uniform electric field $\vec{E}_0$ then the sphere will be polarized uniformly so that field inside and outside the dielectric sphere will not be same. The resultant field inside the sphere will be sum of uniform applied field $\vec{E}_0$ and the internal field $\vec{E}_{in}$ generated by polarization say $\vec{P}$,hence.

$$\boxed{\left|\vec{E}\right| = \left|\vec{E}_o + \vec{E}_{in}\right| = E_0 - \frac{\rho}{3\varepsilon_0}}$$

But from the definition of polarization

$$\vec{P} = \varepsilon_0 \left(\varepsilon_r - 1\right) \vec{E}$$

Hence $\vec{E} = \vec{E}_o - \dfrac{\varepsilon_0 \left(\varepsilon_r - 1\right) \vec{E}}{3\varepsilon_0}$

$$\vec{E} + \frac{\left(\varepsilon_r - 1\right) \vec{E}}{3} = \vec{E}_o$$

which gives

$$\vec{E} = \frac{3}{\varepsilon_r + 2} \vec{E}_o$$

We can see that since $\varepsilon_r > 1$, the factor $\left(\dfrac{3}{\varepsilon_r + 2}\right)$ should be less than 1, which implies that $\vec{E} < \vec{E}_0$. The resultant field inside the dielectric sphere is smaller than the applied electric field. The direction of the field inside the sphere is same as $\vec{E}_0$ and it is also uniform within the sphere.

Now we will calculate the resultant field outside the sphere. The field outside the dielectric sphere will the sum of the uniform field and the outside field due to polarization of dielectric. The outside field will be

$$\vec{E}_{out} = \hat{r} \, \frac{2\rho R^3 \cos\theta}{3\varepsilon_0 r^3} + \hat{\theta} \, \frac{\rho R^3 \sin\theta}{3\varepsilon_0 r^3}$$

The R is the radius of the sphere, $\hat{r}$ and $\hat{\theta}$ specify the factor corresponding to distance and direction from the centre .

Where the resultant field outside the sphere will be: -

$$\vec{E} = \vec{E}_o + E_{out}$$

$$\vec{E} = \vec{E}_o + \frac{2\rho R^3 \cos\theta}{3\varepsilon_o r^3}\hat{r} + \frac{\rho R^3 \sin\theta}{3\varepsilon_o r^3}\hat{\theta}$$

Using $\vec{P} = \varepsilon_o(\varepsilon_r - 1)\vec{E}$

and using internal field we may write:

$$\vec{P} = \varepsilon_o \frac{(\varepsilon_r - 1)}{(\varepsilon_r + 2)} 3\vec{E}_o$$

or $$\frac{\vec{P}}{3\varepsilon_o} = \frac{(\varepsilon_r - 1)}{(\varepsilon_r + 2)}\vec{E}_o$$

and hence the outside electric field will be: -

$$\vec{E} = \vec{E}_o + \frac{2(\varepsilon_r - 1)R^3}{(\varepsilon_r + 2)r^3} E_0 \cos\theta\,\hat{r} + \frac{(\varepsilon_r - 1)R^3}{(\varepsilon_r + 2)r^3} E_o \sin\theta\hat{\theta}$$

6.11 Summary

The unit starts with the elementary treatment of electrostatics with permeable media. Boundary value problems with dielectrics, molecular polarizability and electric susceptibility, models for molecular polarizability have been discussed in this unit. Then the energy of a charge distribution with dielectric is developed.

6.12 Glossary

Induce : to cause something to happen

***Insulate*:** Prevent the passage of electricity to or from (something) by covering it in non-conducting material

6.13 Answers to Self Learning Exercise

Ans.4: We know $\vec{P} = \epsilon_0 \chi_e \vec{E}$ and also $\vec{p} = \alpha\vec{E}$,

$$\vec{P} = N\vec{p} = N\alpha\vec{E} \;,$$

So we can say $\chi_e = \dfrac{N\alpha}{\epsilon_0}$

6.14 Exercise

Section A: (Very Short Answer type Questions)

Q.1 How the Polarisation of a collection of atoms or molecules arises?

Q.2 What do you mean by electric susceptibility?

Q.3 Write down the boundary conditions for normal components of $\vec{D}$ and the tangential components of $\vec{E}$.

Q.4 Write down the relation between dielectric constant and electric susceptibility.

Q.5 Write down the Hamiltonian for a harmonically bound electric change with an applied filled in the $Z-$ direction.

Section B: (Short Answer type Questions)

Q.6 Calculate the induced dipole moment per unit volume of gas placed in an electric field of 6×10^5 Volts/ m. the molecular polarizability is 2.33×10^{-41} Farad$-m^2$ and the density of molecules is given 20.60×10^{25} per unit Volume.

Q.7 Determine the total electrostatic energy of a dielectric material .

Q.8 Consider an electric charge $(-e)$, moving in a circular orbit of radius a_o about charge $(+e)$ in a field directed at right angles to the plane of the orbit. Show that the Polarizability α approximately $4\pi\varepsilon_0 a_o^3$.

Q.9 Prove that potential due to a polarized medium is expressible in terms of a sum of volume and surface integral. Explain the physical meaning of the two.

Q.10 A long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation out to a radius 'a' .Find the electric displacement

Q.11 Show that energy stored in a capacitor is given leg $W=\frac{Q^2}{2C}=\frac{1}{2}CV^2$ .If a slab of dielectric $(\varepsilon_r>1)$ is being inserted between the plates of a parallel plate capacitor, with the charge on the capacitor held fixed. Calculate the amount of work to be done for this. $\left(work\ \ done = F_e dx=\frac{1}{2}V^2 dC\right)$

Section C:(Long Answer type Questions)

Q.12 What do you mean by molecular polarzability and electric susceptibility? Establish a relation between these quantities and relate your answer in terms of the result obtained in Claussius-Mossotti equation.

Q.13 Consider a point charge q embedded in a semi-infinite dielectric $\in_1$, at a distance 'd' away from a plane interface which separates the first medium from another semi-infinite dielectric ε_2

Obtain the following:

(i) the potential for $z > 0,$ considering the interface at $z = 0,$

(ii) the potential for $z < 0,$

(iii) the polarization charge density using the method of images.

Q.14 Obtain the electrostatic energy in dielectric media and represent its formulation in terms of the polarization of the dielectric.

Q.15 A nucleus with quadrupole moment Q finds itself in a cylindrically symmetric electric field with a gradient $\left(\frac{\partial E_z}{\partial z}\right)_0$ along the $Z - axis$ at the position of the nucleus. Show that the energy of quadruple interaction is : $W = (-1)\frac{e}{4}Q\left(\frac{\partial E_Z}{\partial z}\right)_o$

Q.16 Two concentric conducting shells of inner and outer radii 'a' and 'b', respectively, carry changes $\pm$ Q. The empty space between the shells is filled by a spherical shell of dielectric of dielectric constant $\left(\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}\right)$ as shown in the figure.

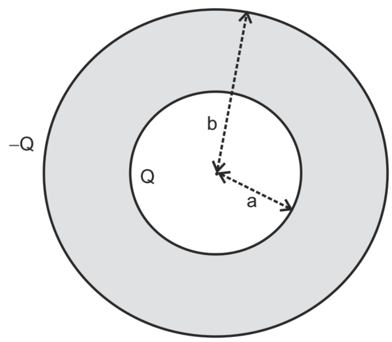


Figure 6.9

(i) Find the electric field between the spheres,

(ii) Find the potential difference between the spheres.

(iii)Calculate the polarization in the dielectric.

6.15 Answers to Exercise

Ans.6: $\left(\vec{P} = 2.88\times10^{-9}\,\frac{Coul.}{m^2}\right)$

Ans.10: $\left(\vec{D} = \frac{\lambda}{2\pi r}\,\hat{r}\right)$

References and Suggested Readings

1. David J. Griffiths, Introduction to Electrodynamics, 2nd Edition, Prentice-Hall,2000
2. J.D. Jackson, Classical Electrodynamics, Wiley Eastern Limited,2002.
3. W.K.H. Panofsky and M. Phillips, Classical Electricity and Magnetism, 2nd edition, Addison Wesley, 1962.
4. Mathew N.O. Sadiku, Elements of Electromagnetics, Oxford University Press, 2001.
5. Landau & Lifshitz, Classical theory of Electrodynamics (Pergaman press, New York).

UNIT-7
Magnetostatics

Structure of the Unit

7.0 Objectives

In previous chapters, we limited our discussions to static electric fields characterized by $\vec{E}$ or $\vec{D}$. We now focus our attention on static magnetic field or magnetostatic fields, which are characterized by H or B. Steady currents produce magnetic fields that are constant in time. These fields are called magnetostatic fields. In this chapter, first we discuss the basic laws of steady magnetic field produced by steady currents in non-magnetic materials, which have permeability $\mu = 4\pi \times 10^{-7}$ H/m. It also covers magnetic vector potential, magnetic induction for a circular current loop and magnetic field of a localized current distribution.

7.1 Introduction

The history of magnetism is fairly old. Naturally occurring substances called lodestones were the first materials in which magnetic forces were observed. In 1819, Hans Christian observed that current carrying wires produced deflections of permanent magnetic dipoles placed in their neighbourhood. Thus, the currents were sources of magnetic induction ($\vec{B}$). Biot-Savart (1820) and Andre-Marie ampere (1820-25) established the basic experimental law relating the magnetic induction $\vec{B}$ to the currents and established the law of force between one current and another.

A magnetostatic field is produced by a constant current flow or direct current. This current flow may be due to magnetization currents as in permanent magnets, electron-beam currents as in vacuum tubes or conduction currents as in current-carrying wires. There are two major laws governing magnetostatic fields: (1) Biot-Savart's law and (2) Ampere's circuit law. Biot-Savart's law is the general law of magnetostatics and Ampere's law is a special case of Biot-Savart's law and is easily applied in problems involving symmetrical current distribution.

7.2 Magnetic field and its direction

All magnetic fields are produced by currents. Even in a permanent magnet, it is the currents at the atomic level which produce the magnetic fields.

A magnetic field can be described by either of two vectors, the magnetic induction (also called magnetic flux density or magnetic field) $\vec{B}$ or magnetic field intensity (also called magnetic field strength) $\vec{H}$. In vacuum, these two variables are related by

$$\boxed{\vec{B}=\mu_0\vec{H}} \quad (1)$$

where the constant μ_0 is the permeability of free space and is given by

$$\mu_0=4\pi \times 10^{-7}\ Hm^{-1}\left(or\ \frac{N}{A^2}\right)$$

The unit of magnetic flux density $\left(\vec{B}\right)$ is Weber/ m^2 or Tesla or $\left(\frac{Newton}{Ampere\ meter}\right)$. The unit of magnetic field intensity $\left(\vec{H}\right)$ is Ampere/meter.

The direction of the magnetic field is easily known using the right hand thumb rule as shown in fig. 7.1

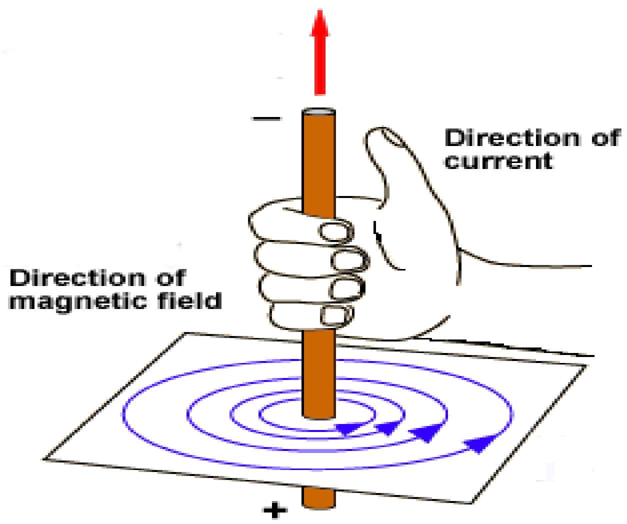


FIG 7.1 Right hand thumb rule

If a current element is held in the right hand with the thumb pointing upwards indicating the direction of current, the direction of the remaining fingers indicate the direction of the magnetic field. If the current in upwards, the direction of magnetic field is anti-clockwise(top view) and if the current is downwards, the direction of magnetic field is clockwise as shown in fig. 7.2

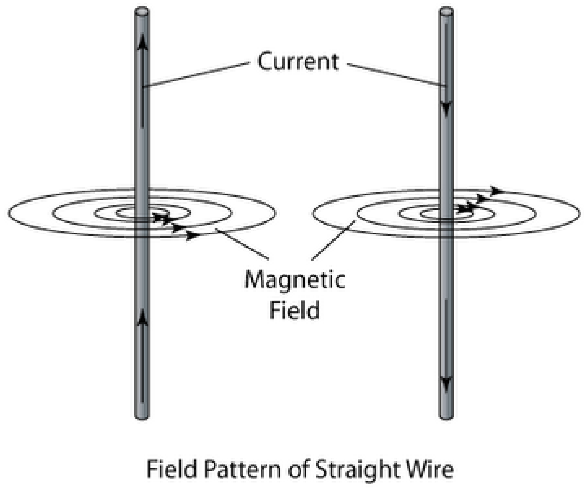


FIG 7.2 Direction of the magnetic field

7.3 Magnetic Force on a Current Carrying Conductor

A current element is a current carrying conductor. It is represented by $I\vec{dl}$. Here I is the current and $\vec{dl}$ is the length of the conductor.

If an electric charge moving with a velocity v is placed in a magnetic field with flux density $\vec{B}$, it will experience a force. This force is given by

$$\boxed{\vec{F} = q\left(\vec{v}\times\vec{B}\right)} \tag{1}$$

The force experienced by the charge dq moving with velocity v is given by

$$d\vec{F} = dq\left(\vec{v}\times\vec{B}\right) \tag{2}$$

Since $I = \dfrac{dq}{dt}$

$$\therefore \quad d\vec{F} = Idt\left(\vec{v}\times\vec{B}\right) \tag{3}$$

Suppose, in time dt, charge dq travels along the length $\vec{dl}$ of the conductor, then

$$\vec{v} = \frac{\vec{dl}}{dt}$$

So that eq. (3) becomes $d\vec{F} = I\,dt\left(\dfrac{\vec{dl}}{dt}\times\vec{B}\right)$

$$\boxed{d\vec{F} = I\left(d\vec{l} \times \vec{B}\right)} \tag{4}$$

Therefore, the force on a current element $\left(Id\vec{l}\right)$ placed in a magnetic field $\vec{B}$ is given by

$$d\vec{F} = I\left(d\vec{l} \times \vec{B}\right) \tag{5}$$

This termed as Ampere's force law.

The force law is illustrated using the Fleming's left hand rule as shown in fig.7.3.

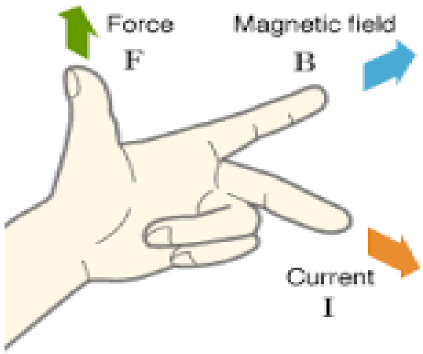


FIG. 7.3 Fleming's Left Hand rule (Showing direction of force)

If the charge q is placed in both electric and magnetic fields, then the total force and the charge will be the vector sum of the electric force (given by coulomb's law) and the magnetic force as in eq.(1)

i.e. $$\boxed{\vec{F} = q\left[\vec{E} + \left(\vec{v} \times \vec{B}\right)\right]} \tag{6}$$

This equation is known as ***Lorentz force*** equation.

7.4 Equation of continuity

It is based on the law of conservation of charge. Conservation of charge demands that the charge density at any point in space must be related to the current density in that neighbourhood by an equation:

$$\boxed{\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0} \tag{1}$$

This equation is known as a ***continuity equation***. This expresses that a decrease in charge inside a small volume with time must correspond to a flow of charge out through the surface of the small volume, since the total amount of charge must be conserved. In the steady state magnetic phenomena $\frac{\partial \rho}{\partial t} = 0$

Therefore, the continuity equation in magnetostatics.

$$\vec{\nabla}.\vec{J} = 0 \tag{2}$$

7.5 Magnetic flux density B and magnetic flux

Magnetic flux density is a measure of the strength of a magnetic field at a given point, expressed by the force per unit length on a conductor carrying unit current at that point. It is also known as magnetic induction. As we know that electric charges moving through a magnetic field are subjected to a force given by

$$\vec{F} = q\left(\vec{v} \times \vec{B}\right) = I\left(d\vec{l} \times \vec{B}\right) \tag{1}$$

$$F_{\max} = I\, dl\, B$$

$$B = \frac{F_{\max}}{I\, dl} = \text{Newton/(Ampere -meter)or Tesla} \tag{2}$$

i.e. magnetic flux density also can be understood as the density of magnetic lines of force or magnetic flux line passing through a unit area or

For uniform magnetic field $$B = \frac{\phi_m}{A} \tag{3}$$

where ϕ_m is magnetic flux and plane of area A is perpendicular to field. The unit of ϕ_m is Weber. Therefore the unit of $\vec{B}$ is also Weber square meter.

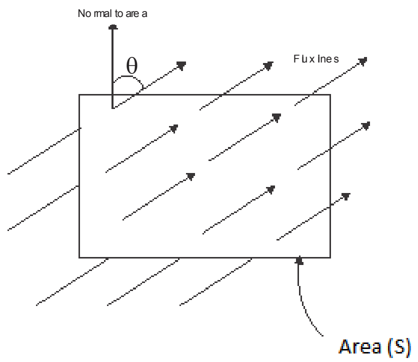


FIG. 7.4 Flux lines through an area

If the magnetic flux density $\vec{B}$ is constant, the magnetic flux passing through a surface of vector area $\vec{S}$ is given by

$$\phi_m = \vec{B}.\vec{S} = B\,S\cos\theta \tag{4}$$

where θ is the angle between the magnetic field lines and area vector $\vec{S}$ .If $\vec{B}$ is not uniform over the area, first we consider the magnetic flux through an infinitesimal area element $d\vec{s}$

i.e. $$d\phi_m = \vec{B}.d\vec{s} \tag{5}$$

Therefore, the total magnetic flux through the surface of vector area $\vec{S}$ is the surface integral

$$\phi_m = \int_S \vec{B}.d\vec{s} \tag{6}$$

In an electrostatic field, the flux passing through a closed surface is the same as the charge enclosed *i.e.*

$$\Psi = \oint \vec{D}.d\vec{s} = Q$$

Thus it is possible to have an isolated electric charge. However, the magnetic flux lines always close upon themselves and they do not start or close on a "magnetic charge"

This is due to the fact that it is not possible to have isolated magnetic charges or poles. Thus the total flux through a closed surface in a magnetic field must be zero,

i.e $$\boxed{\oint_s \vec{B}.d\vec{s} = 0} \tag{7}$$

This equation is referred to as the law of conservation of magnetic flux or Gauss's law for magnetostatic fields. By applying the divergence theorem to eq. (7), we obtain

$$\oint_S \vec{B}.d\vec{s} = 0 = \oint_v \left(\vec{\nabla}.\vec{B}\right) dv = 0$$

$\boxed{\vec{\nabla}.\vec{B} = 0}$ This shows there is no magnetic mono poles. (8)

7.6 Biot-Savart's law

It states that the magnitude of the magnetic induction *dB* produced at a point P by the differential current element *Idl* is proportional to the product *Idl* and the sine of the angle lying between the element and the line joining point P to the element. It is also inversely proportional to the square of the distance *R* from the element to the point P *i.e.*

$$dB\,\alpha\,\frac{Idl\sin\alpha}{R^2} \tag{1}$$

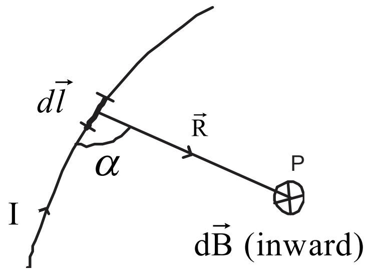


FIG 7.5 Magnetic induction $d\vec{B}$ at P due to current element $Id\vec{l}$

or
$$dB=\frac{KIdl\sin\alpha}{R^2} \tag{2}$$

where K is the constant proportionality. In SI units $K=\frac{\mu_0}{4\pi}$ so eq. (2) becomes

Fig. Magnetic induction $d\vec{B}$ at P due to current element $Id\vec{l}$

$$\boxed{dB=\frac{\mu_0}{4\pi}\frac{Idl\sin\alpha}{R^2}} \tag{3}$$

In vector form

$$d\vec{B}=\frac{\mu_0}{4\pi}\frac{Id\vec{l}\times\vec{a}_R}{R^2}=\frac{\mu_0}{4\pi}\frac{Id\vec{l}\times\vec{R}}{R^3}$$

$$\boxed{d\vec{B}=\frac{Id\vec{l}\times\vec{R}}{R^3}} \tag{4}$$

where $\quad R=\left|\vec{R}\right|$ and $\vec{a}_R = \dfrac{\vec{R}}{R}$

The direction of $d\vec{B}$ can be determined by the right-handed screw rule. The direction of $d\vec{B}$ is normal to the plane containing $Id\vec{l}$ and line drawn from the element to the point P. This normal is in the direction of progress of a right-handed screw turned from $d\vec{l}$ through a small angle to the line from the element to the point P.

It is customary to represent the direction of the magnetic induction $\vec{B}$ (or current I) by a small circle with dot or cross sign depending on whether $\vec{B}$ (or I) is out of, or into, the page as illustrated in fig.7.6.

FIG 7.6 Conventional representation of $\vec{B}$ or I

There are three types of current distributions line current, surface current and volume current as shown in fig. 7.7

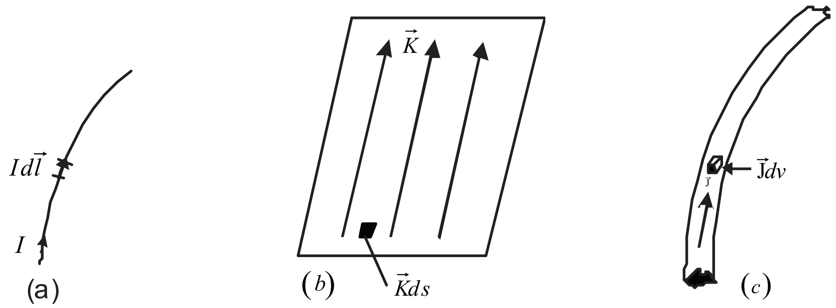


FIG 7.7 Current distributions: (a) line current (b) surface current and (c) volume current

If we define $\vec{K}$ as the surface current density (amperes/meter) and $\vec{J}$ as volume current density (amperes/meter square), the source elements are related as

$$\boxed{Id\vec{l} \equiv \vec{K}ds \equiv \vec{J}dv} \tag{5}$$

Thus in terms of the distributed current sources, the Biot-Savart's law as in eq. (4) becomes

$$\boxed{\vec{B}=\frac{\mu_0}{4\pi}\oint_L \frac{Id\vec{l}\times\vec{R}}{R^3}} \quad \text{(Line current)} \tag{6}$$

$$\boxed{\vec{B}=\frac{\mu_0}{4\pi}\oint_S \frac{\vec{K}ds\times\vec{R}}{R^3}} \quad \text{(Surface current)} \tag{7}$$

$$\boxed{\vec{B}=\frac{\mu_0}{4\pi}\oint_v \frac{\vec{J}dv\times\vec{R}}{R^3}} \quad \text{(Volume current)} \tag{8}$$

7.7 Differential Equation of Magnetostatics and Ampere's law

If we assume that the positions of the source point and the field are at $\vec{r}\,'$ (x',y',z') and $\vec{r}$ (x, y, z), respectively as shown in fig. 7.8, then the basic law for magnetic induction $\vec{B}$ for a volume current density $\vec{J}$ can be written as

$$\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\int_v \vec{J}(\vec{r}\,')\times\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}dv' \tag{1}$$

Here $\vec{r}-\vec{r}\,'=\vec{R}$ is the distance vector from the volume element *dv* at the source point (x',y',z') to the field point (x, y, z) *i.e.*

$$R=|\vec{r}-\vec{r}\,'|=\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2} \tag{2}$$

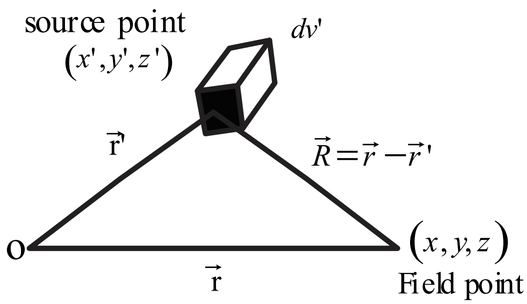


FIG. 7.8 Illustration of the Source point and the field point

Hence $$\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)=-\frac{\vec{R}}{R^3}=-\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \quad (3)$$

where the differentiation is with respect to *x, y* and *z.* Substituting this into eq. (1), we get

$$\vec{B}(\vec{r})=-\frac{\mu_0}{4\pi}\int_v\left[\vec{J}(\vec{r}\,')\times\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\right]dv' \quad (4)$$

Taking the divergence

$$\vec{\nabla}.\vec{B}=-\frac{\mu_0}{4\pi}\int_v\vec{\nabla}.\left[\vec{J}(\vec{r}\,')\times\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\right]dv'$$

Using the identity

$$\vec{\nabla}.(\overrightarrow{A}\times\overrightarrow{C})=\vec{C}.(\vec{\nabla}\times\vec{A})-\vec{A}.(\vec{\nabla}\times\vec{C})$$

We have

$$\vec{\nabla}.\vec{B}=$$

$$-\frac{\mu_0}{4\pi}\int_v\nabla\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right).\left\{\vec{\nabla}\times\vec{J}(\vec{r}\,')\right\}dv'+\frac{\mu_0}{4\pi}\int_v\vec{J}(\vec{r}\,').\left\{\vec{\nabla}\times\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\right\}dv'$$

(5)

Because $\vec{\nabla}$ operates only *r*, the first integral zero. The second term contains a factor curl grad $\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)$ which is identically zero. Therefore

$$\vec{\nabla}.\vec{B}=0 \quad (6)$$

This is the first law in magnetostatics corresponding to the relation $\vec{\nabla}\times\vec{E}=\mathbf{0}$ in electrostatics. The relation shows that the ***magnetic field is solenoidal*** ($\vec{\nabla}.\vec{B}$ =0) in contrast to the ***electrostatic field which is irrotational*** ($\vec{\nabla}\times\vec{E}=0$).

Let us now find the value of $\vec{\nabla}\times\vec{B}$ complete the analogy with electrostatics.

Since $$\vec{B}=\frac{\mu_0}{4\pi}\int_v \frac{\vec{J}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}dv'$$

$$=-\frac{\mu_0}{4\pi}\int_v\left[\vec{J}(\vec{r}\,')\times\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\right]dv'$$

$$\vec{\mathrm{B}}=\frac{\mu_0}{4\pi}\vec{\nabla}\times\int_v\frac{\vec{\mathrm{J}}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dv' \tag{7}$$

Taking the curl

$$\vec{\nabla}\times\vec{B}=\frac{\mu_o}{4\pi}\vec{\nabla}\times\vec{\nabla}\int_v\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dv' \tag{8}$$

Using the identity $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}(\vec{\nabla}.\vec{A})-\nabla^2\vec{A}$, we have

$$\vec{\nabla}\times\vec{\mathrm{B}}=\frac{\mu_o}{4\pi}\vec{\nabla}\left(\vec{\nabla}.\int\frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dv'\right)-\frac{\mu_0}{4\pi}\nabla^2\int\frac{J(r')}{|\vec{r}-\vec{r}\,'|}dv'$$

$$\vec{\nabla}\times\vec{B}=\frac{\mu_0}{4\pi}\vec{\nabla}\int\vec{J}(\vec{r}\,').\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}\,'|}dv'-\frac{\mu_0}{4\pi}\int\vec{J}(\vec{r}\,')\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)dv' \tag{9}$$

We know that $$\boxed{\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)=-4\pi\delta(\vec{r}-\vec{r}\,')} \tag{10}$$

and $$\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)=-\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) \tag{11}$$

where $\vec{\nabla}'$ operates on r' only.

Using eqs. (10) and (11), we can write eq. (9) as

$$\vec{\nabla}\times\vec{B}=-\frac{\mu_0}{4\pi}\vec{\nabla}'\int\vec{J}(\vec{r}\,').\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)dv'+\frac{\mu_0}{4\pi}\int\vec{J}(\vec{r}\,')4\pi\delta(\vec{r}-\vec{r}\,')dv'$$

$$=-\frac{\mu_0}{4\pi}\vec{\nabla}\int\vec{J}\left(\vec{r}\,'\right).\vec{\nabla}'\left(\frac{1}{\left|\vec{r}-\vec{r}\,'\right|}\right)dv'+\frac{\mu_0}{4\pi}\vec{\mathrm{J}}\left(\vec{r}\right)\times 4\pi$$

$$\vec{\nabla}\times\vec{B}=-\frac{\mu_0}{4\pi}\vec{\nabla}\int\frac{\vec{\nabla}'.\vec{J}\left(\vec{r}\,'\right)}{\left|\vec{r}-\vec{r}\,'\right|}dv'+\mu_0\vec{J}\left(\vec{r}\right)$$

But for steady state magnetic phenomena $\vec{\nabla}.\vec{\mathrm{J}}=0$, so that we obtain

$$\boxed{\vec{\nabla}\times\vec{B}=\mu_0\vec{\mathrm{J}}} \tag{12}$$

This is the second law of magnetostatics corresponding to $\vec{\nabla}.\vec{E}=\frac{\rho}{\epsilon_0}$ of electrostatics.

Taking the integral of the normal component of eq.(12) over an open surface S bounded by a closed curve C as shown in fig. 7.9

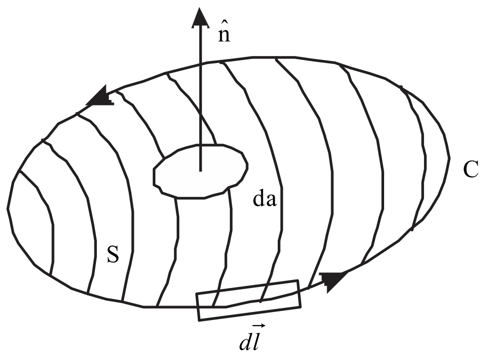


FIG. 7.9 An open surface S bounded by a closed curve

$$\int_s\left(\vec{\nabla}\times\vec{B}\right).\hat{n}da=\mu_0\int_s\vec{J}.\hat{n}da$$

$$\int_s\left(\vec{\nabla}\times\vec{B}\right).d\vec{a}=\mu_0\int_s\vec{J}.d\vec{a}$$

By Stoke's theorem

$$\int_s\left(\vec{\nabla}\times\vec{B}\right).d\vec{a}=\oint_c\vec{B}.d\vec{l}$$

$$\therefore\qquad\oint_c\vec{B}.d\vec{l}=\mu_0\oint_s\vec{J}.d\vec{a} \tag{13}$$

Since the surface integral of the current density is the total current *I* passing through the closed curve C, therefore eq. (13) can be written as

$$\boxed{\oint_c \vec{B}.d\vec{l} = \mu_0 \mathrm{I}} \qquad (14)$$

This is known as Ampere's circuital law. The line integral of magnetic flux density round any closed path is equal to μ_0 time the current flowing through the area enclosed by the path. This law indicates that the magnetic field of a current is non-conservative.

7.8 Applications of Ampere's law

(1) Magnetic field due to a long straight current carrying conductor:- Consider a long straight wire carrying a current *I* as shown in fig 7.10. If wire is vertical then lines of magnetic induction $\vec{B}$ will be concentric circles in horizontal plane. Let P be a point at a distance *r* from the wire, where $\vec{B}$ is to be determined.

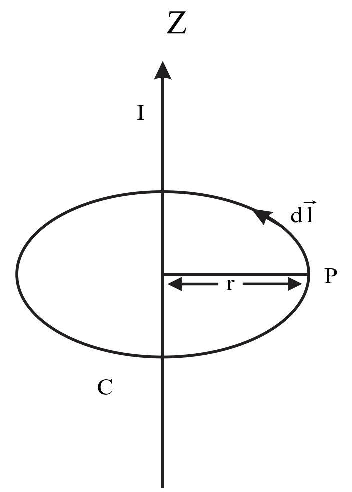


FIG.7.10 Long straight current carrying conductor

Consider a circular path of a radius *r* passing through P. Since $\vec{B}$ and *dl* are always directed along the same direction, therefore line integral of $\vec{B}$ along the boundary C of circular path will be

$$\oint_c \vec{B}.d\vec{l} = \oint_c B\,dl\cos\theta = B\oint_c dl = B2\pi r$$

From Ampere's law

$$\oint_c \vec{B}.d\vec{l} = \mu_0 I$$

$$\therefore \quad B2\pi r = \mu_0 I$$

$$\Rightarrow \quad B = \frac{\mu_0 I}{2\pi r}$$

(2) Magnetic field inside a long solenoid:-

Consider a long solenoid of length l and N be the total number of turns, then the number of turns per unit length is n=$\frac{N}{l}$. Let I be the current flowing in the solenoid as shown in fig 7.11.

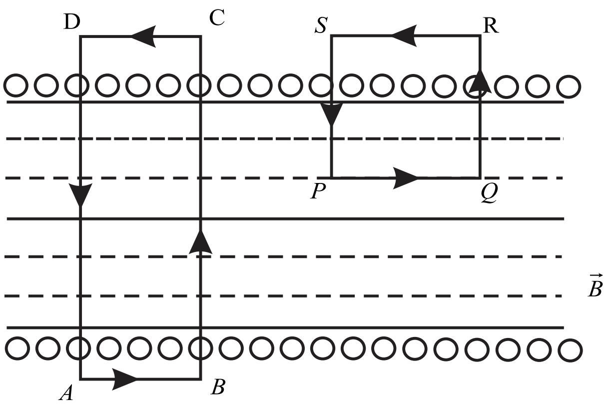


FIG7.11 Solenoid

Field outside the solenoid: - Consider a closed path ABCD. Applying Ampere's law to this path

$$\oint \vec{B}.d\vec{l} = \mu_0 \times 0$$

Since in each turn, equal and opposite current is flowing, therefore net current I enclosed by the total path is zero.

As $\quad d\vec{l} \neq 0 \quad \therefore \quad \vec{B} = 0$

This means that the magnetic induction $\vec{B}$ outside the solenoid is zero.

Field inside the solenoid: - Consider a closed path PQRS, then the line integral of magnetic induction $\vec{B}$ along path PQRS is

$$\oint_{PQRS} \vec{B}.\vec{dl} = \int_{PQ} \vec{B}.\vec{dl} + \int_{QR} \vec{B}.\vec{dl} + \int_{RS} \vec{B}.\vec{dl} + \int_{SP} \vec{B}.\vec{dl} \qquad (1)$$

For path PQ, $\vec{B}$ and $\vec{dl}$ are along the same direction, therefore

$$\oint_{PQ} \vec{B}.\vec{dl} = \int Bdl\cos 0 = B\int dl = Bl \qquad (2)$$

For path QR and SP, $\vec{B}$ and $\vec{dl}$ are mutually perpendicular to each other

$$\oint_{QR} \vec{B}.\vec{dl} = \int_{SP} \vec{B}.\vec{dl} = \int Bdl\cos 90^{0} = 0 \qquad (3)$$

For path RS, $\vec{B}$ =O due to outside a solenoid

$$\therefore \qquad \int_{RS} \vec{B}.\vec{dl} = 0 \qquad (4)$$

Put these values in eq. (1), we get

$$\oint_{PQRS} \vec{B}.\vec{dl} = \int_{PQ} \vec{B}.\vec{dl} = Bl \qquad (5)$$

From Ampere's law $\oint \vec{B}.\vec{dl} = \mu_o \times$ total current enclosed by path

From eq. (5), we get $\vec{B} = \mu_0 NI$

since N in the number of turns in the solenoid, then the total current enclosed = NI

$$B = \frac{\mu_0 NI}{l} \qquad (6)$$

$$\boxed{B = \mu_0 nI} \qquad (7)$$

where n= $\frac{N}{l}$ = number of turns per unit length

7.9 Magnetic Vector Potential

In electrostatics, we know that electric potential depends on the charges which establish the field. The potential is a scalar function and the electric field is vector field. In electrostatics, we know that electric potential depends on the charges which establish the field. The potential is a scalar function and the electric field is expressed as a gradient of the potential *i.e.* $\vec{E} = -\vec{\nabla} V$ (1)

Similarly, a potential may be associated with magnetostatics named as magnetic potential, whose gradient may give the magnetic field.

$$i.e. \qquad \vec{H} = -\vec{\nabla} V_m \tag{2}$$

where V_m is scalar magnetic potential.

But this relation holds good only for $\vec{J}$=0. Since in magnetostatics (from Ampere's Law)

$$\vec{\nabla} \times \vec{H} = \vec{J} \tag{3}$$

$$\vec{\nabla} \times \left(\vec{H}\right) = 0, \qquad \text{If } \vec{J} = 0$$

$$\Rightarrow \quad \vec{H} = -\vec{\nabla} V_m \qquad \text{Since } \vec{\nabla} \times \left(\vec{\nabla} V\right) = 0 \tag{4}$$

Thus the scalar magnetic potential can exist in a region where no current is there.

In magnetostatics, the source for producing magnetic field is a "current element" as is charge in case of electrostatics. Therefore there should be a potential which depends on current element $Id\vec{l}$ (vector quantity).

As we know form Gauss's law of magnetostatics

$$\vec{\nabla} . \vec{B} = 0 \tag{5}$$

If the divergence of a vector is zero, then that vector can be expressed as the curl of another vector $\left(\vec{A}\right)$ *i.e.*

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{6}$$

where $\vec{A}$ is called magnetic vector potential.

$$\text{Since} \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\therefore \quad \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \mu_0 \vec{J}$$

$$\Rightarrow \quad \vec{\nabla} . \left(\vec{\nabla} . \vec{A}\right) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

For steady current $\vec{\nabla} . \vec{A} = 0$

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \tag{7}$$

This is Poisson's equation in magnetostatics. Since for electrostatics the Poisson's eq. $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$, where

$$V = \frac{1}{4\pi\varepsilon_0} \int_v \frac{\rho_v dv}{R} \tag{8}$$

Similarly for magnetostatics

$$\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$

where $$\boxed{\vec{A} = \frac{\mu_0}{4\pi} \int_v \frac{\vec{J} dv}{R}} \tag{9}$$

7.10 Magnetic vector potential from Biot-Savart's law

If we assume that the positions of the source point and the field point are at $\vec{r}\,'(x', y', z')$ and $\vec{r}\,(x, y, z)$, respectively as shown in fig. 7.13, then Biot-Savart's law can be written as

$$\vec{B} = \frac{\mu_0}{4\pi} \int_L \frac{I d\vec{l}\,' \times \vec{R}}{R^3} \tag{1}$$

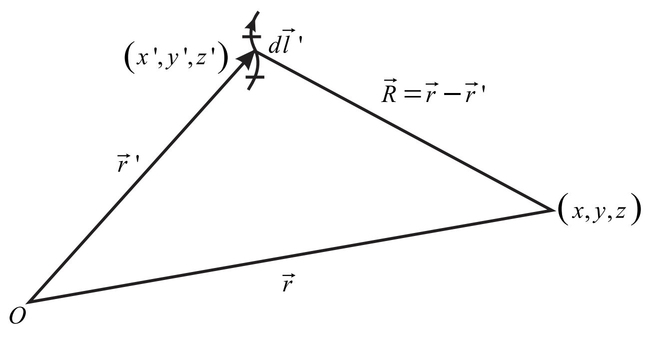


FIG. 7.13 Illustration of the source point (x', y', z') and field point (x, y, z)

where $\vec{R}$ is the distance vector from the line element $d\vec{l}\,'$ at the source point (x', y', z') to the field point *(x, y, z)* and $R = |\vec{R}|$ *i.e.*

$$R=\left|\vec{R}\right|=\left|\vec{r}-\vec{r}\,'\right|=\left[\left(x-x'\right)^2+\left(y-y'\right)^2+\left(z-z'\right)^2\right]^{1/2} \tag{2}$$

Hence

$$\vec{\nabla}\left(\frac{1}{R}\right)=-\frac{\left(x-x'\right)\vec{a}_x+\left(y-y'\right)\vec{a}_y+\left(z-z'\right)\vec{a}_z}{\left[\left(x-x'\right)^2+\left(y-y'\right)^2+\left(z-z'\right)^2\right]^{3/2}}=-\frac{\vec{R}}{R^3} \tag{3}$$

i.e. $$\frac{\vec{R}}{R^3}=-\vec{\nabla}\left(\frac{1}{R}\right) \tag{4}$$

From eqs. (1) & (4), we get

$$\vec{B}=-\frac{\mu_0}{4\pi}\int_L I d\vec{l}\,'\times\vec{\nabla}\left(\frac{1}{R}\right) \tag{5}$$

Using vector identity

$$\vec{\nabla}\times\left(f\vec{F}\right)=f\vec{\nabla}\times\vec{F}+\left(\vec{\nabla}f\right)\times\vec{F} \tag{6}$$

Where f = scalar field $=\frac{1}{R}$ and $\vec{F}$ = vector field = $d\vec{l}\,'$, we have

$$\vec{\nabla}\times\left(\frac{d\vec{l}\,'}{R}\right)=\frac{1}{R}\,\vec{\nabla}\times d\vec{l}\,'+\vec{\nabla}\left(\frac{1}{R}\right)\times d\vec{l}\,'$$

$$\Rightarrow \quad d\vec{l}\,'\times\vec{\nabla}\left(\frac{1}{R}\right)=\frac{1}{R}\vec{\nabla}\times d\vec{l}\,'-\vec{\nabla}\times\left(\frac{d\vec{l}\,'}{R}\right)$$

Since $\vec{\nabla}$ operates with respect to $\left(x,y,z\right)$ while $d\vec{l}\,'$ is function of (x',y',z') therefore $\vec{\nabla}\times d\vec{l}\,'=0$

Hence $$d\vec{l}\,'\times\vec{\nabla}\left(\frac{1}{R}\right)=-\vec{\nabla}\times\left(\frac{d\vec{l}\,'}{R}\right) \tag{7}$$

From eqs. (5) & (7), we get

$$\vec{B}=\vec{\nabla}\times\int_L\frac{\mu_0 I\,d\vec{l}\,'}{4\pi R} \tag{8}$$

But we know that $\vec{B} = \nabla \times \vec{A}$

Therefore magnetic vector potential

$$\vec{A} = \int_L \frac{\mu_0 I\, d\vec{l}\,'}{4\pi R} \qquad \text{Or}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_L \frac{I\, d\vec{l}\,'}{R} \tag{9}$$

Therefore general expression for magnetic vector potential is

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{I\, d\vec{l}}{R} \qquad \text{(for line current)} \tag{10}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_S \frac{\vec{K} ds}{R} \qquad \text{(for surface current)} \tag{11}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_v \frac{\vec{J} dv}{R} \qquad \text{(for volume current)} \tag{12}$$

Since magnetic flux thorough a surface S is given by

$$\psi_m = \int_S \vec{B}.d\vec{s} \tag{13}$$

But magnetic field in terms of vector potential is given by $\vec{B} = \vec{\nabla} \times \vec{A}$ (14)

From eq. (13) and (19) , we get

$$\psi_m = \int_S \left(\nabla \times \vec{A}\right).d\vec{s}$$

Applying stroke's theorem, we get

$$\psi_m = \oint \vec{A}.d\vec{l} \tag{15}$$

Thus the line integral of magnetic vector potential $\vec{A}$ around a closed curve gives the magnetic flux linked with the surface enclosed by the curve.

7.11 Magnetic Induction for a Circular Current loop

Let us determine the magnetic induction $\vec{B}$ at an observation point $\left(r, \theta, \phi\right)$

due to a circular loop carrying current I' as shown in fig 7.14 .

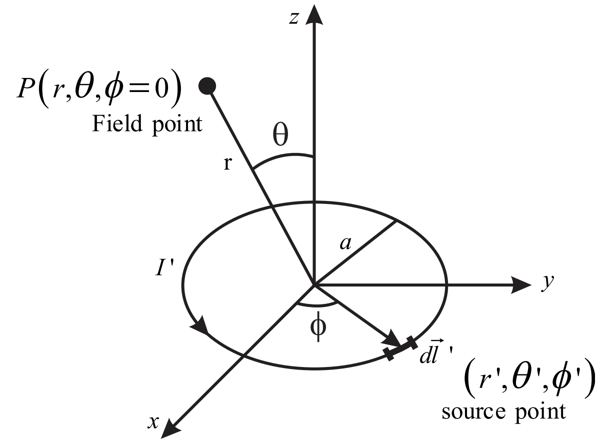


FIG. 7.14 Magnetic induction as P due to a current loop

The current I' flows in the wire and the loop is placed on the X-Y plane. Due to the cylindrical geometry, we may choose the observation point P in the X-Z plane $(\phi=0)$ without loss of generality. The radius of the loop is *a,* then

$$d\vec{l}\,'=ad\phi'\,\vec{a}'_{\phi} \tag{1}$$

The magnetic vector potential at P is

$$\vec{A}(\vec{r})=\frac{\mu_0 I'}{4\pi}\int\frac{d\vec{l}\,'}{|\vec{r}-\vec{r}\,'|} \tag{2}$$

In the spherical coordinate system

$$|\vec{r}-\vec{r}\,'|=\sqrt{r^2+r'^2-2rr'\cos(\phi-\phi')\sin\theta\sin\theta'-2rr'\cos\theta\cos\theta'} \tag{3}$$

Since the loop lies on the X-Y plane , $\theta'=\dfrac{\pi}{2}$ *and* $r'=a$, then eq. (3) becomes

$$|\vec{r}-\vec{r}\,'|=\sqrt{r^2+a^2-2ra\cos(\phi-\phi')\sin\theta} \tag{4}$$

Putting the value of $d\vec{l}\,'$ and $|\vec{r}-\vec{r}\,'|$ from eqs. (1) and (4), respectively into eq. (2), we get

$$\vec{A}(\vec{r})=\frac{\mu_0 I'}{4\pi}\int\frac{ad\phi'\,\vec{a}'_{\phi}}{\sqrt{r^2+a^2-2ar\cos(\phi-\phi')\sin\theta}} \tag{5}$$

We evaluate this integral on the X-Z plane where $\phi = 0$, then eq. (5) becomes

$$\vec{A}(\vec{r}) = \frac{\mu_0 I'}{4\pi} \int \frac{ad\phi' \vec{a}'_\phi}{\sqrt{r^2 + a^2 - 2ar\cos\phi' \sin\theta}} \tag{6}$$

We know from the transformation relation between Cartesian coordinates *(x, y, z)* and spherical coordinates (r, θ, ϕ)

$$\vec{a}_\phi = -\sin\phi' \vec{a}_x + \cos\phi' \vec{a}_y \tag{7}$$

From eqs. (6) and (7), we get

$$\vec{A}(\vec{r}) = \frac{\mu_0 I'}{4\pi r} \int \frac{a\, d\phi' \left(-\sin\phi' \vec{a}_x + \cos\phi' \vec{a}_y\right)}{\sqrt{1 + \left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)\cos\phi' \sin\theta}} \tag{8}$$

where we have used $\vec{a}_x$ and $\vec{a}_y$ since these are constant vectors, otherwise integration with vector which are always changing direction is impossible. Since a << r $\Rightarrow \left(\frac{a}{r}\right)$ << 1, therefore term $\left(\frac{a}{r}\right)^2$ can be neglected in comparison with 1, then eq. (8) becomes

$$\vec{A}(\vec{r}) = \frac{\mu_0 I'}{4\pi r} \int_{\phi'} \frac{a(-\sin\phi' \vec{a}_x + \cos\phi' \vec{a}_y) d\phi'}{\sqrt{1 - 2\left(\frac{a}{r}\right)\cos\phi' \sin\theta}}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I'}{4\pi r} \int_{\phi'} a(-\sin\phi' \vec{a}_x + \cos\phi' \vec{a}_y) \left[1 + \left(\frac{a}{r}\right)\cos\phi' \sin\theta\right] d\phi'$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I'}{4\pi r} \left[\int_0^{2\pi} -a\sin\phi' \left\{1 + \frac{a}{r}\cos\phi' \sin\theta\right\} d\phi' \vec{a}_x + \int_0^{2\pi} a\cos\phi \left\{1 + \frac{a}{r}\cos\phi' \sin\theta\right\} d\phi' \vec{a}_y \right]$$

$$=\frac{\mu_0 I'}{4\pi r}\left[a\left\{\sin\phi'\right\}_0^{2\pi}\vec{a}_y+\frac{a^2\sin\theta}{r}\int_0^{2\pi}\cos^2\phi' d\phi'\vec{a}_y\right]$$

since first integral gives 0 value.

$$=\frac{\mu_0 I'}{4\pi r}\left[0+\frac{a^2\sin\theta}{r}\left\{\frac{\phi'}{2}+\frac{\sin^2\phi'}{4}\right\}_0^{2\pi}\vec{a}_y\right]$$

$$=\frac{\mu_0 I' a^2}{4\pi r^2}\sin\theta\left[\frac{2\pi}{2}\right]\vec{a}_y$$

$$\vec{A}(\vec{r})=\frac{\mu_0 I'\pi a^2}{4\pi r^2}\sin\theta\,\vec{a}_y$$

i.e. $\vec{A}$ has only ϕ component according to eq. (8) and it is given by

$$\vec{A}=\frac{\mu_0 I'\pi a^2}{4\pi r^2}\sin\theta\,\vec{a}_\phi \tag{9}$$

$$\vec{A}=\mu_0\frac{\vec{m}\times\vec{a}_r}{4\pi r^2} \tag{10}$$

where $\vec{m}=I'\pi a^2\vec{a}_z$ is defined as the magnetic dipole moment and $\vec{a}_z\times\vec{a}_r=\sin\theta\vec{a}_\phi$. It is a vector whose magnitude is the product of the current in and the area of the loop and whose direction is the direction of the thumb as the fingers of the right hand follow the direction of the current . The magnetic field produced by a small current loop is similar to the electric field from a small electric field from a small electric dipole. For this reason, a small current loop is called magnetic dipole.

The magnetic flux density is $\vec{B}=\nabla\times\vec{A}$

$$\vec{B}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\vec{a}_r & r\vec{a}_\theta & r\sin\theta\vec{a}_\phi\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ A_r & rA_\theta & r\sin\theta A_\phi\end{vmatrix}$$

$$=\frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{a}_r & r\vec{a}_\theta & r\sin\theta\,\vec{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & \frac{\mu_0 m}{4\pi r}\sin^2\theta \end{vmatrix}$$

$$=\frac{1}{r^2\sin\theta}\left\{\vec{a}_r\left[\frac{\mu_0 m}{4\pi r}2\sin\theta\cos\theta\right]-r\vec{a}_\theta\left[-\frac{\mu_0 m}{4\pi r^2}\sin^2\theta\right]\right\}$$

$$\boxed{\vec{B}=\frac{\mu_0 m}{4\pi r^3}\left[2\cos\theta\,\vec{a}_r+\sin\theta\,\vec{a}_\theta\right]} \qquad (11)$$

The magnetic flux lines of a magnetic dipole are continuous as illustrated in fig 7.15

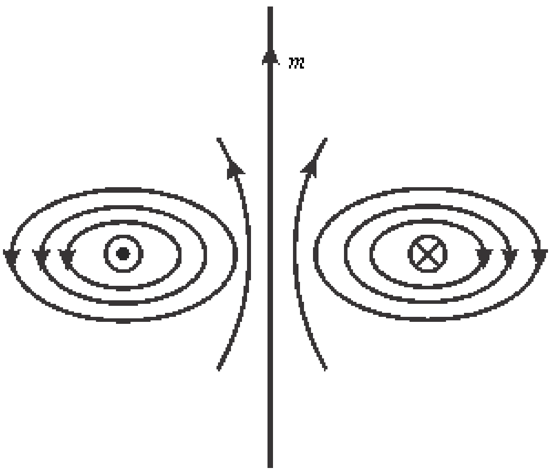


FIG. 7.15 Magnetic induction lines due to small current loop (magnetic dipole)

7.12 Magnetic fields of localized current distribution and magnetic moment

Let us consider a current element I $\vec{dl}'$ on a current loop c' having position vector $\vec{r}'$ relative to origin, then the vector potential $\vec{A}$ at field point P having position vector *r* is given by

$$\vec{A} = \frac{\mu_0}{4\pi} \oint_{c'} \frac{I d\vec{l}\,'}{\left|\vec{r} - \vec{r}\,'\right|} \tag{1}$$

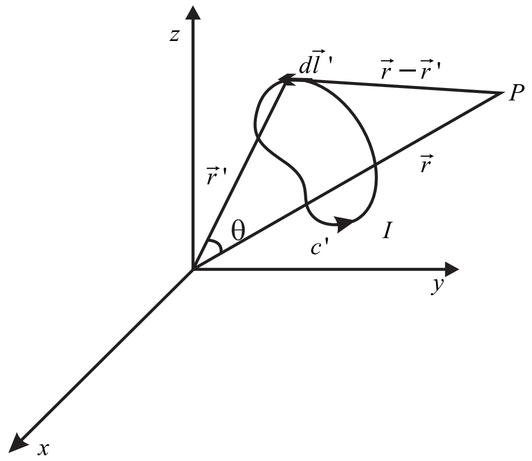


FIG. 7.16 Current Loop

where $\left|\vec{r} - \vec{r}\,'\right| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta}$

$$= \left(r^2 + r'^2 - 2\vec{r}.\vec{r}\,'\right)^{1/2}$$

$$\left|\vec{r} - \vec{r}\,'\right| = \text{r}\left(1 + \frac{r'^2}{r^2} - \frac{2\vec{r}.\vec{r}}{r^2}\right)^{1/2} \tag{2}$$

From eqs. (1) & (2), we get

$$\vec{A} = \frac{\mu_0 I}{4\pi r} \oint_{c'} \frac{d\vec{l}\,'}{\left(1 + \frac{r'^2}{r^2} - \frac{2\vec{r}.\vec{r}\,'}{r^2}\right)^{1/2}}$$

$$\vec{A} = \frac{\mu_0 I}{4\pi r} \oint_{c'} d\vec{l}\,'\left(1 + \frac{r'^2}{r^2} - \frac{2\vec{r}.\vec{r}\,'}{r^2}\right)^{-1/2} \tag{3}$$

Using binomial expansion and considering only first term in $\frac{r'}{r}$ since r'<< r, eq. (3) may write

$$\vec{A}=\frac{\mu_0 I}{4\pi r}\oint_{c'} d\vec{l}\,'\left[1+\frac{\vec{r}.\vec{r}\,'}{r^2}\right]$$

$$\vec{A}=\frac{\mu_0 I}{4\pi r}\left[\oint_{c'} d\vec{l}\,'+\oint_{c'}\frac{\vec{r}.\vec{r}\,'}{r^2}\,d\vec{l}\,'\right] \tag{4}$$

Since $\oint_{c'} d\vec{l}\,' = 0$ for any arbitrary closed loop c' as initial and final positions are same. Therefore eq. (4) becomes

$$\vec{A}=\frac{\mu_0 I}{4\pi r^3}\oint_{c'}\left(\vec{r}.\vec{r}\,'\right) d\vec{l}\,' \tag{5}$$

From vector triple product, we know

$$\vec{r}\times\left(d\vec{l}\,'\times\vec{r}\,'\right)=\left(\vec{r}.\vec{r}\,'\right)d\vec{l}\,'-\left(\vec{r}\,'\times d\vec{l}\,'\right)\vec{r}\,' \tag{6}$$

Let us write the differential of $\vec{r}\,'\left(\vec{r}.\vec{r}\,'\right)$ for a small change in $\vec{r}\,'$ *i.e.*

$$d\left[\vec{r}\,'\left(\vec{r}.\vec{r}\,'\right)\right]=d\vec{r}\,'\left(\vec{r}.\vec{r}\,'\right)+\vec{r}\,'\left(\vec{r}.d\vec{r}\,'\right)$$

But $d\vec{r}\,'=d\vec{l}\,'$

$$d\left[\vec{r}\,'\left(\vec{r}.\vec{r}\,'\right)\right]=d\vec{l}\,'\left(\vec{r}.\vec{r}\,'\right)+\vec{r}\,'\left(\vec{r}.d\vec{l}\,'\right) \tag{7}$$

Add eqs. (6) and (7) , we get

$$\vec{r}\times\left(d\vec{l}\,'\times.\vec{r}\,'\right)+d\left[\vec{r}\,'\left(\vec{r}.\vec{r}\,'\right)\right]=2\left(\vec{r}.\vec{r}\,'\right)d\vec{l}\,'$$

$$\Rightarrow\quad \left(\vec{r}.\vec{r}\,'\right)d\vec{l}\,'=\frac{1}{2}\left[\vec{r}\times\left(d\vec{l}\,'\times\vec{r}\,'\right)\right]+\frac{1}{2}d\left[\vec{r}\,'\left(\vec{r}.\vec{r}\,'\right)\right] \tag{8}$$

From eqs. (5) and (8) , we get

$$\vec{A}=\frac{\mu_0 I}{4\pi r^3}\left\{\oint_{c'}\frac{1}{2}\,\vec{r}\times\left(d\vec{l}\,'\times\vec{r}\,'\right)+\oint_{c'}\frac{1}{2}\,d\left[\vec{r}\,'\left(\vec{r}.\vec{r}\,'\right)\right]\right\} \tag{9}$$

Since the second integral involves perfect differential around a closed counter, hence its integral around a closed loop is always zero. Hence (9) becomes

$$\vec{A} = \frac{\mu_0 I}{4\pi r^3} \oint_{c'} \frac{1}{2} \vec{r} \times \left(d\vec{l}\,' \times \vec{r}\,' \right)$$

$$= \frac{\mu_0 I}{4\pi r^3} \oint_{c'} \frac{1}{2} \left(\vec{r}\,' \times d\vec{l}\,' \right) \times \vec{r}$$

$$\vec{A} = \frac{\mu_0}{4\pi r^3} \oint_{c'} \frac{1}{2} \left(\vec{r}\,' \times I\, d\vec{l}\,' \right) \times \vec{r} \qquad (10)$$

Defining , the magnetic dipole moment of current loop by

$$\vec{m} = \frac{1}{2} \oint_{c'} \left(\vec{r}\,' \times I\, d\vec{l}\,' \right) \qquad (11)$$

Therefore eq. (10) becomes

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}} \qquad (12)$$

This is the lowest non vanishing term in the expansion of $\vec{A}$ for a localized steady state current distribution.

Since $\quad \vec{\nabla}\left(\frac{1}{r} \right) = -\frac{\vec{r}}{r^3}$

$$\vec{A} = -\frac{\mu_0}{4\pi} \left[m \times \vec{\nabla}\left(\frac{1}{r} \right) \right] \qquad (13)$$

The magnetic induction $\vec{B} = \vec{\nabla} \times \vec{A}$

7.13 Illustrative Examples

Example 7.1 A long cable is constructed of a solid conductor of radius r in a co-axial hollow cylindrical conduction of inner and outer radii R_1 and R_2, respectively, The I Amperes current flows in the inner and outer conductors in opposite direction. Determine the magnetic field induction $\vec{B}$ at any point a distance x from the axis of the cable.

Sol. The cross-sectional view of the cable normal to its length is shown in figure

7.20. We have drawn a circle of radius x around the axis of the cable. By symmetry, the magnetic induction $\vec{B}$ produced by the flow of current is tangential at any point on the circumference of this path. From Ampere's law

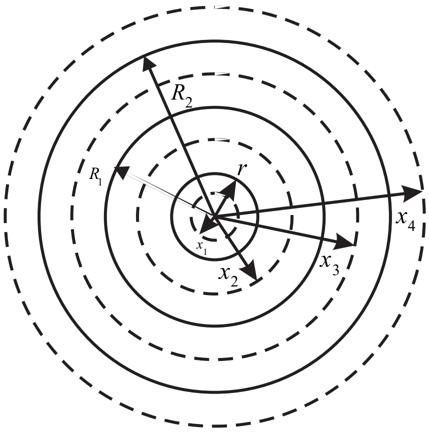


FIG. 7.20 Cross-Sectional view of cable normal to its length

$$\oint \vec{B}.d\vec{l} = \mu_0 I \tag{1}$$

where I is the current enclosed by the path.

Case-I if x <r, *i.e.* the current is enclosed by the circular path of radius x_1. That is

Current = current density $\times$ area of circular path of radius x_1

$$= \frac{I}{\pi r^2} \times \pi x_1^2 = \frac{I x_1^2}{r^2} \tag{2}$$

From eqs. (1) & (2), we get

$$\oint \vec{B}.d\vec{l} = B\oint dl = \mu_0 \frac{I x_1^2}{r^2}$$

$$\Rightarrow \qquad B(2\pi x_1) = \frac{\mu_0 I x_1^2}{r^2}$$

$$B = \frac{\mu_0 I x_1}{2\pi r^2} \tag{3}$$

Case-II r < x < R, then the current enclosed is the entire current enclosed by the circular path of radius x_2, *i.e.* current = I

From eq. (1), we get

$$\oint \vec{B}.d\vec{l} = \mu_0 I$$

$$\Rightarrow \quad B\oint dl = \mu_0 I$$

$$\Rightarrow \quad B\left(2\pi\, x_2\right) = \mu_o I$$

$$B = \frac{\mu_0 I}{2\pi\, x_2} \tag{4}$$

Case-III If $R_1 < x < R_2$, then the current enclosed is the enclosed by the circular path of radius x_3. That is

Current = current density $\times$ area of circular path of radius x_3

$$= \frac{I}{\pi\left(R_2^2 - R_1^2\right)}\pi\left(x_3^2 - R_1^2\right) = \frac{\pi\left[\left(R_2^2 - R_1^2\right) - \left(x_3^2 - R_1^2\right)\right]}{\pi\left(R_2^2 - R_1^2\right)}$$

$$= I\left(\frac{R_2^2 - x_3^2}{R_2^2 - R_1^2}\right) \tag{5}$$

From eqs. (1) & (5), we get

$$B\oint dl = \mu_0 I\left(\frac{R_2^2 - x_3^2}{R_2^2 - R_1^2}\right)$$

$$\Rightarrow \quad B\left(2\pi\, x_3\right) = \mu_0 I\left(\frac{R_2^2 - x_3^2}{R_2^2 - R_1^2}\right)$$

$$B = \frac{\mu_0 I}{2\pi x_3}\left(\frac{R_2^2 - x_3^2}{R_2^2 - R_1^2}\right) \tag{6}$$

Case IV If $x > R_2$ *i.e.* current enclosed by the circular path of radius x_4. That is I = 0. Because of the current in the outer and inner conductors are equal and opposite.

From eq. (1), we get

$$\oint \vec{B}.d\vec{l} = 0 \quad \Rightarrow \quad B\oint dl = 0$$

$$\Rightarrow \quad B\left(2\pi x_4\right)=0 \qquad \Rightarrow \quad B = 0$$

Example 7.2 Find the magnetic flux density at a point on the axis of a circular loop of radius a that carries a direct current *I*.

Sol. From Biot-Savart's Law

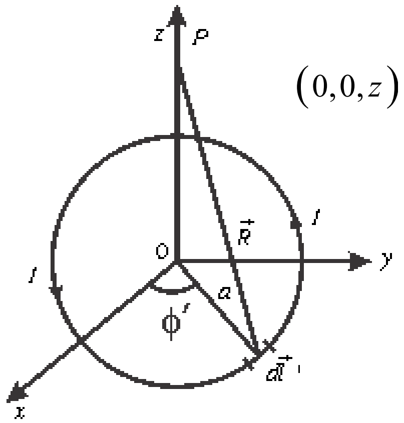


FIG 7.21 A circular loop carrying current I

$$\vec{B}=\frac{\mu_0 I}{4\pi}\int\frac{d\vec{l}'\times\vec{R}}{R^3} \tag{1}$$

where $\vec{R}$ in the vector from the source element $d\vec{l}^{\,1}$ to the field point P.

Here $d\vec{l}\,'=ad\phi'\,\vec{a}_\phi$

$\vec{R}=z\,\vec{a}_z-a\vec{a}_\rho$

and $$R=\left|\vec{R}\right|=\sqrt{z^2+a^2} \tag{2}$$

$\therefore \quad d\vec{l}\,'\times\vec{R}=ad\phi'\vec{a}_\phi\times\left(z\vec{a}_z-a\,\vec{a}_\rho\right)$

$$=az\,d\phi'\,\vec{a}_\rho+a^2d\phi'\vec{a}_z \tag{3}$$

Because of cylindrical symmetry, it is easy to see that $\vec{a}_\rho$ component is cancelled by the contribution of the element located diametrically opposite to $d\vec{l}\,'$, so we need only consider the $\vec{a}_z$ component of this cross product.

From eqs. (1), (2) and (3), we get

$$B = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{a^2 d\phi'}{\left(z^2 + a^2\right)^{3/2}} \vec{a}_z$$

$$B = \frac{\mu I a^2}{2\left(z^2 + a^2\right)^{3/2}} \vec{a}_z$$

Example7.3(a) Find magnetic vector potential and magnetic induction for a long current carrying wire.

(b) Show that the magnetic vector potential for two long straight parallel wires carrying same current *I* in opposite directions is given by

$$\vec{A} = \frac{\mu_0 I}{2\pi} \log\left(\frac{r_2}{r_1}\right) \vec{n}$$

where r_1 and r_2 are the distances from the fixed point P to the wires and $\vec{n}$ is the unit vector parallel to the wires.

Sol. Let P be the field point at a distance x from the wire carrying current *I* and having length L along y-axis as shown in fig.7.20.

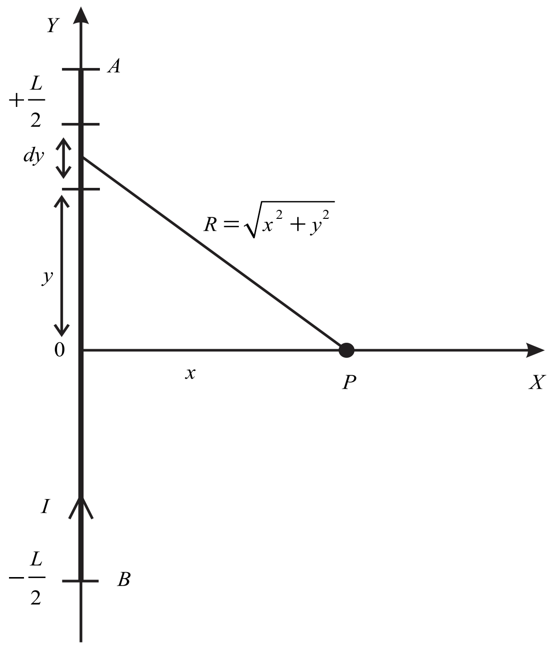


FIG. 7.20 A current carrying wire of length L

The magnetic vector potential at P due to a current element *dy* at a distance *y* from

O is given by $$d\vec{A} = \frac{\mu_0}{4\pi}\frac{Id\vec{y}}{R}$$

$$d\vec{A} = \frac{\mu_0}{4\pi}\frac{I\ dy}{\sqrt{x^2+y^2}}\vec{j}$$

where $\vec{j}$ is unit vector along y-axis.

The magnetic potential due to whole length of wire is

$$\vec{A} = \vec{j}\,\frac{\mu_0 I}{4\pi}\int_{\frac{-L}{2}}^{\frac{L}{2}}\frac{dy}{\sqrt{x^2+y^2}}$$

$$= \vec{j}\,\frac{\mu_0 I}{4\pi}\left[\log\left\{y+\sqrt{x^2+y^2}\right\}\right]_{-L/2}^{L/2}$$

Since $\displaystyle\int\frac{dx}{\sqrt{x^2+a^2}} = \log\left\{x+\sqrt{x^2+a^2}\right\}$

$$\vec{A} = \vec{j}\,\frac{\mu_0 I}{4\pi}\left[\log\left\{\frac{L}{2}+\sqrt{\left(\frac{L^2}{4}+x^2\right)}\right\} - \log\left\{-\frac{L}{2}+\sqrt{\left(\frac{L^2}{4}+x^2\right)}\right\}\right]$$

$$= \vec{j}\,\frac{\mu_o I}{4\pi}\log\left[\frac{\frac{L}{2}+\sqrt{\left(\frac{L^2}{4}+x^2\right)}}{-\frac{L}{2}+\sqrt{\left(\frac{L^2}{4}+x^2\right)}}\right]$$

$$= \vec{j}\,\frac{\mu_0 I}{4\pi}\log\left[\frac{1+\left(1+\frac{4x^2}{L^2}\right)^{1/2}}{-1+\left(1+\frac{4x^2}{L^2}\right)^{1/2}}\right]$$

As wire is infinitely long, then $\frac{x^2}{L^2} << 1$

Hence on using binomial theorem and neglecting higher order terms, we have

$$\vec{A} = \vec{j}\,\frac{\mu_0 I}{4\pi}\log\left[\frac{1+1+\frac{4x^2}{2L^2}}{-1+1+\frac{4x^2}{2L^2}}\right]$$

$$= \vec{j}\,\frac{\mu_0 I}{4\pi}\log\left[\frac{2+\frac{2x^2}{L^2}}{\frac{2x^2}{L^2}}\right]$$

$$= \vec{j}\,\frac{\mu_0 I}{4\pi}\log\left[\frac{L^2}{x^2}+1\right]$$

$$= \vec{j}\,\frac{\mu_0 I}{4\pi}\log\left(\frac{L}{x}\right)^2 \qquad \text{Since } \frac{L^2}{x^2} >> 1$$

$$\vec{A} = \vec{j}\,\frac{\mu_0 I}{2\pi}\log\left(\frac{L}{x}\right) \tag{1}$$

This is the expression for magnetic potential due to a long straight current carrying wire.

The magnetic induction $\vec{B}$ is given by

$$\vec{B} = \nabla \times \vec{A}$$

$$= \nabla \times \left[\vec{j}\,\frac{\mu_0 I}{2\pi}\log\left(\frac{L}{x}\right)\right]$$

$$\vec{B} = \frac{\mu_0 I}{2\pi} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \log\left(\frac{L}{x}\right) & 0 \end{vmatrix}$$

$$= \frac{\mu_0 I}{2\pi} \vec{k} \left[\frac{\partial}{\partial x} \log\left(\frac{L}{x}\right) \right] \qquad (2)$$

(b) In this problem, there are two wires in which same current is flowing in opposite directions and $\vec{n}$ is the unit vector parallel to the wires as shown in fig.7.21. The distance of field point P from one wire is r_1 and from second wire r_2. From eq. (1), the magnetic vector potential at point P due to one wire is

$$\vec{A}_1 = \frac{\mu_0 I}{2\pi} \log\left(\frac{L}{r_1}\right) \vec{n}$$

and due to second wire is

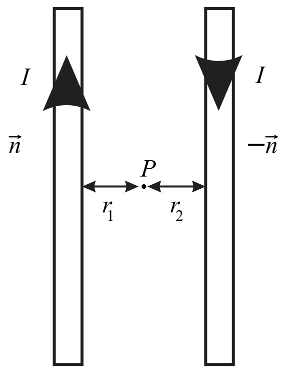


FIG. 7.21 Two long parallel current carrying wires

$$\vec{A}_2 = -\frac{\mu_0 I}{2\pi} \log\left(\frac{L}{r_2}\right) \vec{n}$$

where L is the length of two wires. Thus total magnetic vector potential at point P will be

$$\vec{A}=\vec{A}_1+\vec{A}_2$$

$$\vec{A}=\frac{\mu_0 I}{2\pi}\log\left(\frac{L}{r_1}\right)\vec{n}-\frac{\mu_0 I}{2\pi}\log\left(\frac{L}{r_2}\right)\vec{n}$$

$$=\frac{\mu_0 I}{2\pi}\log\left(\frac{L/r_1}{L/r_2}\right)$$

$$\vec{A}=\frac{\mu_0 I}{2\pi}\log\left(\frac{r_2}{r_1}\right)\vec{n}$$

Example 7.4 Find the magnetic vector potential of an infinite solenoid with *n* turns per unit length, radius *R* and current *I*.

Solution-: We know that the magnetic induction for solenoid is $B=\mu_0 nI$. Since the magnetic flux through a surface S is given by

$$\psi=\int_s \vec{B}.d\vec{s} \qquad \text{since } \vec{B}=\vec{\nabla}\times\vec{A}$$

$$=\int_s\left(\vec{\nabla}\times\vec{A}\right).d\vec{s}$$

$$\psi=\oint_L \vec{A}.d\vec{l} \qquad \text{Using Stoke's theorem}$$

$$\therefore \qquad \oint_L \vec{A}.d\vec{l}=\int_S \vec{B}.d\vec{s}$$

The vector potential is circumferential, it mimics the magnetic field of the wire. Using a circular Amperian loop at radius *r* inside the solenoid, we hence

$$\int_L \vec{A}.d\vec{l}=A\left(2\pi r\right)=\int_s \vec{B}.d\vec{s}=\mu_0 nI\left(\pi r^2\right)$$

$$A=\frac{\mu_0 nI}{2}r \qquad \text{for } r<R$$

$$\text{Or} \qquad \vec{A}=\frac{\mu_0 nI}{2}r\hat{\phi} \qquad \text{for } r<R$$

For an Amperian loop outside the solenoid, the flux is

$$\int_s \vec{B}.d\vec{s} = \mu_0 nI\left(\pi R^2\right)$$

Since the field only extends out to R. Thus

$$\vec{A} = \frac{\mu_0 nI}{2}\frac{R^2}{r}\hat{\phi} \qquad \text{for } r > R$$

7.14 Self Learning Exercise

Q.1 What is the unit of magnetic flux density?

Q.2 What is the magnetic induction due to solenoid?

Q.3 Define relation between magnetic vector potential $\left(\vec{A}\right)$ and magnetic induction$\left(\vec{B}\right)$.

Q.4 The unit of magnetic dipole moment is….

Q.5 What is the utility of magnetic vector potential in magnetostatics.

Q.6 Write the continuity equation for magnetostatics.

Q.7 What is the magnetic dipole moment of a circular coil with n turns *A* cross sectional area and *I* current.

Q.8 Define the magnetic flux through a given area in terms of magnetic vector potential.

Q.9 The direction of the magnetic field in easily known using………

Q.10 Write the Lorentz force equation for a charge *q* placed in both electric and magnetic field.

Q.11 A current of 20 Amperes flow through each of the two parallel long conducting wires. The distance between two parallel wires is 4 *cm*. Determine the force exerted per unit length of each wire.

Q.12 Define magnetic flux and magnetic flux density.

Q.13 Explain that the south and north poles of a magnet cannot be isolated.

Q.14 The magnitude of magnetic field strength $\vec{H}$ at a radius of 1 meter from a long conductor is 2 Amp/m. Determine the current in the conducting wire.

Q.15 Show that the divergence of the magnetic induction is always zero or magnetic field lines are always continuous.

Q.16 A current distribution gives rise to the vector magnetic potential $\vec{A} = x^2 y\,\hat{i} + y^2 x\,\hat{j} - 4xyz\,\hat{k}$ Wb/m. Calculate magnetic induction $\vec{B}$ at (-1, 2, 5).

7.15 Summary

This unit starts with the introduction of magnetostatic fields. By giving the concept of two main governing laws of magnetostatics fields, we have derived differential equations of magnetostatics. Here, we study the different application of Ampere's law. The vector potential, magnetic induction for a circular current loop, magnetic fields of a localized current distribution and magnetic moment have also been studied in this unit. In the end some examples on above concept are given.

7.16 Glossary

Localized: happening in or limited to a particular area

Magnetic Flux : Number of magnetic field lines passing through given area.

7.17 Answer to Self-Learning Exercise

Ans. 1 : Weber/meter^2

Ans. 2 : $B = \mu_0 nI$

Ans. 3 : $\vec{B} = \nabla \times \vec{A}$

Ans. 4 : Amperes $\times \mathrm{m}^2$

Ans. 5 : Define potential related to current element, which is a vector quantity.

Ans. 6 : $\vec{\nabla}.\vec{J} = 0$

Ans. 7 : $\mathrm{m} = \mathrm{n\,I\,A}$

Ans. 8 : $\psi = \oint_L \vec{A}.d\vec{l}$

Ans. 9 : The right hand thumb rule.

Ans. 10 : $\vec{F} = q\left[\vec{E} + \left(\vec{v} \times \vec{B}\right)\right]$

Ans. 11 : $F = \dfrac{\mu_o}{2\pi}\dfrac{I_1 I_2}{d} l \;\&\; \dfrac{F}{l} = 2\times 10^{-3}\, N/m$

Ans. 12 : $B = \frac{F_{\max}}{Idl}$ and $\phi_m = \oint_s \vec{B}.\vec{ds}$

Ans. 13 : The magnetic flux lines always close upon themselves. This is due to the fact that it is not possible to have isolated magnetic poles or magnetic charges.

Ans. 14 : $\oint \vec{H}.\vec{dl} = I \Rightarrow I = 12.56$ Ampere

Ans. 15 : Prove $\vec{\nabla}.\vec{B} = 0$

Ans. 16 : $\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \vec{B} = 20\hat{i} + 40\hat{j} + 3\hat{k}$

7.18 Exercise

Section–A: Very Short Answer type Question

Q.1 Is it possible to have isolated magnetic change?

Q.2 How can produce magnetostatic field?

Q.3 Which are main laws governing magnetostatics.

Q.4 Magnetic field is conservative or not.

Q.5 What is the unit of vector magnetic potential?

Section –B : Short Answer type Questions

Q.6 State and explain Ampere's law both in integral and differential form.

$$\oint_c \vec{B}.\vec{dl} = \mu_0 I \textit{ and } \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Q.7 What is the magnitude of the current flowing in two parallel wires, which are 10 *cm* apart [center to center], if the force between them is $F = 10^{-3} N$. The currents in both wires have the same magnitude.

Hint $F = \frac{\mu_o}{2\pi} \frac{I_1 I_2}{d}, I = 22.36\ Amp.$

Q.8 Prove that, the magnetic force on a volume current is given by

$$\vec{F} = \oint_v \left(\vec{J} \times \vec{B}\right) dv$$

Q.9 Explain the concept of magnetic vector potential.

Q.10 The vector magnetic potential $\vec{A}$ due to a direct current in a conductor in free space is given by $\vec{A} = \left(x^2 + y^2\right)\vec{a}_z \ \mu wb/m$. Determine the magnetic field produced by the current element at (1, 2, 3).

$\vec{H} = \left(3.97\vec{a}_x - 4.7\vec{a}_y\right) Amp/m$

Section- C : Long Answer type Questions

Q.11 Derive the expression for different equation of magnetostatics and Ampere's law.

$$\vec{\nabla}.\vec{B} = 0 \ , \vec{\nabla}\times\vec{B} = \mu_0\vec{J} \quad and \oint_c \vec{B}.d\vec{l} = \mu_0 I$$

Q.12 Show that the magnetic induction can be written in terms of magnetic vector potential. Derive magnetic vector potential from Biot-Savart's law.

$$\vec{B} = \vec{\nabla}\times\vec{A} \ , \ \vec{A} = \frac{\mu_0}{4\pi}\int_L \frac{I\,d\vec{l}}{R}$$

Q.13 A coaxial cable has core of radius *a* and sheath of radius *b*. A current *I* flows along the core, uniformly distributed across it, and returns along the sheath, uniformly distributed around it. Find the magnetic flux density **(i)** within the core (r < a)

(ii) within the core-sheath space $\left(a \le r \le b\right)$ and

(iii) outside the sheath (r > b).

Q.14 Prove that the magnetic induction for a circular current loop is given by

$$\vec{B} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\vec{a}_r + \sin\theta\vec{a}_\theta\right)$$

`

Calculate the expression for vector potential and magnetic induction of a localized current distribution.

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3}, \vec{B} = \frac{\mu_0}{4\pi r^3}\left[\frac{3\left(\vec{m}.\vec{r}\right)\vec{r}}{r^2} - \vec{m}\right]$$

7.19 Answers to Exercise

Ans.1: No, magnetic flux lines always close upon themselves.

Ans.2: If the charges are moving with constant velocity, a magnetostatic field is produced.

Ans.3: (1) Biot-Savart's law and (2) Ampere's circuit law.

Ans.4: No, since $\vec{\nabla}\times\vec{B}=\mu_o\vec{J}$.

Ans.5: Weber/meter.

Ans.13: (i) $B=\dfrac{\mu_0 Ir}{2\pi a^2}$ (ii) $B=\dfrac{\mu_0 I}{2\pi r}$ and (iii) B=0

References and Suggested Readings

1. J.D. Kraus and K.R. Carver, Electromagnetics, 2nd edition, McGraw Hill, 1973.
2. J.D. Jackson, Classical Electrodynamics, Wiley Eastern Limited, 1978.
3. Mathew N.O. Sadiku, Elements of Electromagnetics, Oxford University Press, 2001.
4. David J. Griffiths, Introduction to Electrodynamics, 2nd Edition, Prentice-Hall, 1989.
5. W.K.H. Panofsky and M. Phillips, Classical Electricity and Magnetism, 2nd edition, Addison Wesley, 1962.
6. P.C. Clemmow, An Introduction to Electromagnetic Theory, Cambridge University Press, 1973.
7. E.G. Cullwick, The Fundamentals of Electromagnetism, 3rd edition, Cambridge University Press, 1966.

UNIT - 8
Macroscopic Equations in Magnetostatics

Structure of the unit

8.0 Objectives

We have discussed so far magnetostatic fields produced by steady currents in non-magnetic material *i.e.* in vacuum. How is the magnetic field affected when material media are present. We shall discuss this question in the present section.

First we will discuss about the magnetization bound current densities and Macroscopic equations for magnetostatics. This chapter also covers Magnetic Susceptibility, Relative permeability, classification of magnetic materials, boundary conditions on B& H, method of solving boundary value problems in magnetostatics, uniformly magnetized sphere in an external field, permanent magnetic shielding and spherical shell of permeable material in an uniform field.

8.1 Introduction

We have dealt so far with the basic laws of magnetostatic fields as microscopic equations given in previous chapter, where we have assumed that the current density $\vec{J}$ was a completely known function of position. As we know that a given material is composed of atoms. The atoms in matter have electrons that give rise to effective atomic currents due to their orbital motion and spin, the current density of which is a rapidly fluctuating quantity. Its average over a macroscopic volume in only known. The orbital motion and spin of electrons in atoms provide tiny currents which give rise to the magnetic dipole moments. All these moments can give rise to fields that vary appreciably on the atomic scale of dimensions. This is called a macroscopic effect *i.e.* when we are talking about magnetic field inside matter, we mean the macroscopic field. It is the average over regions large enough to contain many atoms of matter. Ordinarily, the magnetic dipoles of atoms cancel each other out because of the random orientation. But when a magnetic field is applied, a net alignment of these magnetic dipoles occurs and the medium becomes magnetized. It is also called magnetization.

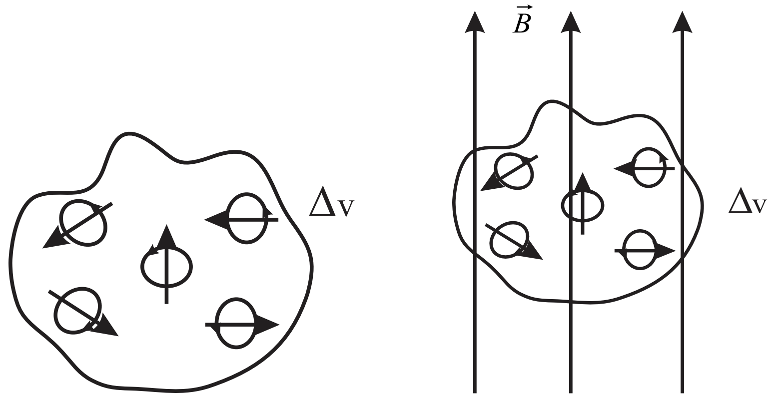
There are some materials acquire a magnetization parallel $\vec{B}$ called paramagnets and some opposite to $\vec{B}$ called diamagnets. A few substances called ferromagnets, retain a substantial magnetization indefinitely after the external field has been removed. The magnetization of these materials is not determined by the present field but by the whole magnetic history of the object.

8.2 Magnetization and Bound Current Densities

According to the elementary atomic model of matter, all materials are composed of atoms. Each atom may be regarded as consisting of electron orbiting about a central positive nucleus. In addition, the electrons of atom also rotate (spin) on their own axes. Both of these electronic motion produce equivalent atomic currents flowing in circular loops. The equivalent current loop has a magnetic dipole moment of $\vec{m}=I_b S \ \vec{a}_n$,where S is the area of the loop and I_b is the bound current (bound to the atom). Where $\vec{a}_n$ is the unit vector perpendicular to plane of the area.

Since the nucleus of an atom also rotate (spin) on their own axis. The magnetic dipole moment of a spinning nucleus is usually negligible in comparison to that of an orbiting or spinning electron because of the much large mass and lower angular velocity of the nucleus.

In the absence of an external magnetic field, the sum of magnetic moments of atoms of material is zero due to random orientations as shown in fig 8.1 (a). When external magnetic field is applied, the magnetic moments of electrons align themselves with applied field as shown in fig 8.1(b).The material then is said to be magnetized or magnetic polarization. The state of magnetic polarization is described by a quantity called magnetization $\vec{M}$, which is defined as the magnetic dipole moment per unit volume.



$\vec{B}=0=\vec{M}=0$

(a) (b)

Fig . 8.1 Magnetic dipole moment in a volume Δv (a) before $\vec{B}$ is applied, (b) after $\vec{B}$ is applied.

Let $\vec{m}_k$ be the magnetic dipole moment of kth atom. If there are N atoms in a given volume Δv, then the magnetization vector $\vec{M}$ is defined as

$$\vec{M} = \lim_{\Delta v \to 0} \frac{\sum_{K=1}^{N} \vec{m}_k}{\Delta v} \left(\frac{Ampere}{meter} \right) \tag{1}$$

A medium for which $\vec{M}$ is not zero everywhere is said to be magnetized. For a differential volume dv', the magnetic moment is $d\vec{m} = \vec{M}\, dv'$. As we know that the magnetic vector potential due to magnetic moment $d\vec{m}$ is given by

$$d\vec{A} = \frac{\mu_0}{4\pi} \frac{d\vec{m} \times \vec{a}_R}{R^2}$$

$$= \frac{\mu_0}{4\pi} \frac{\vec{M} \times \vec{a}_R}{R^2} dv'$$

$$d\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{M} \times \vec{R}}{R^3} dv' \tag{2}$$

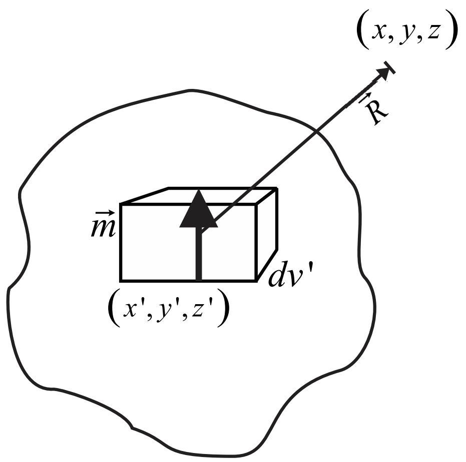


FIG. 8.2

where $\vec{R}$ is the distance vector form the source point (x', y', z') to the field point (x, y, z) and $R = |\vec{R}|$.

Since $\frac{\vec{R}}{R^3} = -\vec{\nabla}\left(\frac{1}{R}\right) = \vec{\nabla}'\left(\frac{1}{R}\right)$ (3)

From eqs. (2) and (3), we get

$$d\vec{A} = \frac{\mu}{4\pi} M \times \vec{\nabla}'\left(\frac{1}{R}\right) dv'$$

Thus, $$\vec{A}=\int_{v'} d\vec{A} = \frac{\mu_0}{4\pi}\int_{v'} M\times\vec{\nabla}'\left(\frac{1}{R}\right)dv' \tag{4}$$

where v' is the volume of the magnetized material.

Apply the vector identity

$$\vec{\nabla}\times\left(f\vec{F}\right)=f\vec{\nabla}\times\vec{F}+\vec{\nabla}f\times\vec{F}$$

where f is a scalar field and $\vec{F}$ is a vector field. Taking $f=\frac{1}{R}$ *and* $\vec{F}=\vec{M}$,

we have $$\vec{\nabla}'\times\left(\frac{\vec{M}}{R}\right)=\frac{1}{R}\vec{\nabla}'\times\vec{M}+\vec{\nabla}'\left(\frac{1}{R}\right)\times\vec{M}$$

$$\Rightarrow \quad \vec{M}\times\vec{\nabla}'\left(\frac{1}{R}\right)=\frac{1}{R}\vec{\nabla}'\times\vec{M}-\vec{\nabla}'\times\left(\frac{\vec{M}}{R}\right) \tag{5}$$

Substituting eq. (5) into eq. (4), we get

$$\vec{A}=\frac{\mu_0}{4\pi}\int_{v'}\frac{\vec{\nabla}'\times\vec{M}}{R}dv'-\frac{\mu_0}{4\pi}\int_{v'}\vec{\nabla}'\times\left(\frac{\vec{M}}{R}\right)dv'$$

Applying the vector identity $\int_{v'}\vec{\nabla}\times\vec{F}\,dv'=-\oint_{s'}\vec{F}\times d\vec{s}\,'$ to the second integral, we obtain $$\vec{A}=\frac{\mu_0}{4\pi}\int_{v'}\frac{\vec{\nabla}'\times\vec{M}}{R}dv'+\frac{\mu_0}{4\pi}\oint_{s'}\frac{\vec{M}\times\vec{a}_n\,'}{R}ds' \tag{6}$$

where $\vec{a}_n\,'$ is the unit outward normal vector from ds' and $\vec{s}\,'$ is the surface bounding the volume v'.

As we know that the vector magnetic potential in terms of surface and volume current is given by

$$\vec{A}=\frac{\mu_0}{4\pi}\oint_{s}\frac{\vec{K}\,ds}{R} \quad and \quad \vec{A}=\frac{\mu_0}{4\pi}\int_{v}\frac{\vec{J}\,dv}{R} \tag{7}$$

Therefore eq. (6) becomes

$$\boxed{\vec{A}=\frac{\mu_0}{4\pi}\int_{v'}\frac{\vec{J}_b\,dv'}{R}+\frac{\mu_0}{4\pi}\oint_{s'}\frac{\vec{K}_b ds'}{R}} \tag{8}$$

Where $\boxed{\vec{J}_b = \vec{\nabla} \times \vec{M}}$ (Ampere/M^2) (9)

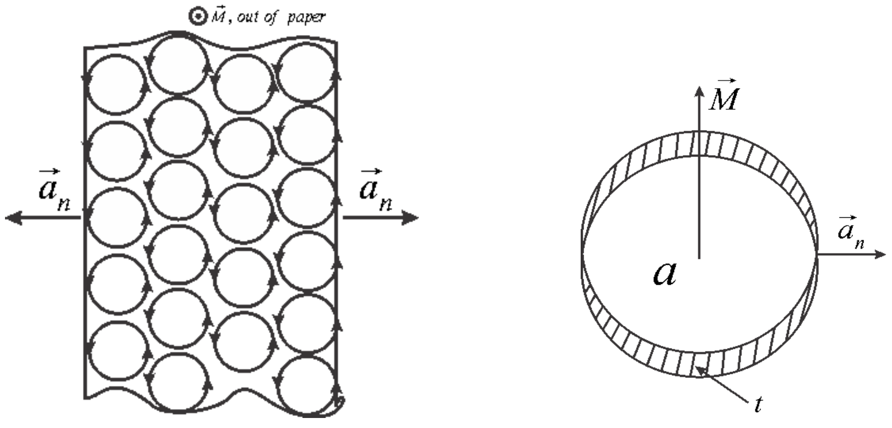
and $\boxed{\vec{K}_b = \vec{M} \times \vec{a}_n}$ (A/m) (10)

(Omitted the prime on $\vec{\nabla}$ and $\vec{a}_n$ for simplicity)

where $\vec{J}_b$ is the bound volume current density or magnetization volume current density and $\vec{K}_b$ is the bound surface current density. Equation (8) shows that the potential of a magnetic body is due to a volume current density $\vec{J}_b$ throughout the body and a surface current $\vec{K}_b$ on the boundary of the magnetic body. Therefore the problem of finding the magnetic flux density $\vec{B}$ caused by a given volume density of magnetic dipole moment $\vec{M}$ is then reduced to finding the equivalent magnetization current densities $\vec{J}_b$ *and* $\vec{K}_b$, these gives magnetic vector potential $\vec{A}$ and then obtaining $\vec{B}$ from the curl of $\vec{A}$.

8.3 Physical Interpretation of Bound Currents

In the last section, we have seen that the field of a magnetized object is identical to the field produced by a certain distribution of bound currents $\vec{J}_b$ and $\vec{K}_b$. These bound currents arise physically. Fig 8.3 (a) shows a cross section of uniformly magnetized material with the dipoles represented by tiny current loops.



(a) (b)

Fig. 8.3 – A cross section of a magnetized material

When external magnetic field is applied, the atomic circulating currents of material align with it and matter becomes magnetized. The Strength of this magnetizing effect is measured by magnetization vector $\vec{M}$. It is clear from fig 8.3 that the all the internal currents cancel, since every time there is one going to the right, a contiguous one is going to the left. However, at the edge there is no adjacent loop to do the canceling. The whole thing is equivalent to a single current *I* flowing around the boundary. Each of the tiny loop has area *a* and thickness *t* as shown in fig 8.3 (b). In terms of the magnetization M, its dipole moment is given by

$$m = M\,a\,t = I\,a$$

$$\Rightarrow \quad I = M\,t$$

$$\Rightarrow \quad \frac{I}{t} = K_b = \text{Surface current density} = \text{M} \tag{1}$$

Therefore, on the surface of the material there will be a surface current density $\vec{K}_b$, whose direction is correctly given by the cross product $\vec{M}\times\vec{a}_n$ *i.e.*

$$\boxed{\vec{K}_b = \vec{M}\times\vec{a}_n} \tag{2}$$

It $\vec{M}$ is uniform inside the material, the space derivatives of a constant $\vec{M}$ vanish *i.e.* $\vec{J}_b = \vec{\nabla}\times\vec{M} = 0$. Therefore the net effect is a macroscopic current flowing over the surface of the magnetized object.

When the magnetization $\vec{M}$ is non-uniform, the internal atomic currents do not completely cancel, resulting in a net volume current density $\vec{J}_b$ i.e. $\vec{J}_b = \vec{\nabla}\times\vec{M}$.

We know that the magnetic vector potential in terms of magnetization vector $\vec{M}$ is given by

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{v'}\frac{\vec{\nabla}\times\vec{M}}{R}dv' + \frac{\mu_0}{4\pi}\oint_{s'}\frac{\vec{M}\times\vec{a}_n}{R}ds' \tag{3}$$

Since $\vec{M}$ is localized, the surface integral taken over a surface outside the region in which the current flows, vanish. Hence

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{v'}\frac{\vec{\nabla}\times\vec{M}}{R}dv'$$

But we have already shown that

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{v'} \frac{\vec{J}\, dv'}{R}$$

Therefore $\quad \vec{J}_b = \vec{\nabla} \times \vec{M} \qquad (4)$

It is net volume current density, when the magnetization $\vec{M}$ is non-uniform and localized.

8.4 Macroscopic Equations

When we speak about the magnetic field in matter, it means, we are talking about the macroscopic fields *i.e.* average over regions large enough to contain many atoms. Therefore we have to write the basic laws of magnetic fields as macroscopic equations.

The first step is to observe that the averaging of the equation $\vec{\nabla} . \vec{B}_{micro} = 0$ leads to the same equation for the macroscopic magnetic induction $\left(\vec{B}\right)$

$$\boxed{\vec{\nabla} . \vec{B} = 0} \qquad (1)$$

The macroscopic equivalent of the microscopic equation, $\vec{\nabla} . \vec{B}_{micro} = \mu_0 \vec{J}_{micro}$ can be read off from the equation of magnetic vector potential in terms of magnetization $\vec{M}$. The macroscopic effect of magnetization can be studied by incorporating the equivalent bound volume current density $\vec{J}_b$ into the free volume current density $\vec{J}_f$ of Ampere's law *i.e.*

$$\frac{I}{\mu_0}\left(\vec{\nabla} \times \vec{B}\right) = \vec{J}_f + \vec{J}_b = \vec{J} \qquad (2)$$

$$= \vec{J}_f + \vec{\nabla} \times \vec{M}$$

$$\Rightarrow \quad \vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J}_f \qquad (3)$$

The $\vec{\nabla} \times \vec{M}$ term can be combined with $\vec{B}$ to define a new macroscopic field $\vec{H}$, called magnetic field intensity $\vec{H}$, such that

$$\boxed{\vec{H}=\frac{\vec{B}}{\mu_0}-\vec{M}} \quad \left(\frac{Ampere}{meter}\right) \tag{4}$$

The use of the vector $\vec{H}$ enables us to write a curl equation relating the magnetic field and the distribution of free currents in any medium.

From eqs. (3) and (4), we get

$$\boxed{\vec{\nabla}\times\vec{H}=\vec{J}_f} \quad \left(\frac{Ampere}{meter^2}\right) \tag{5}$$

or integral from $\oint \vec{H}.\boxed{d\vec{l}=I_{f(enclosed)}}$ (6)

where $I_{f(enclosded)}$ is the total free current passing through the Amperian loop.

It is Ampere's law in terms of $\vec{H}$ or it is Ampere's law in magnetized materials.

Therefore the macroscopic equations are given by eqs. (1) and (5) *i.e.*

$$\left.\begin{aligned}\vec{\nabla}.\vec{B}&=0\\ \vec{\nabla}\times\vec{H}&=\vec{J}_f\end{aligned}\right\} \tag{7}$$

These two are the fundamental governing differential equations for magnetostatics in the macroscopic equivalent.

8.5 Magnetic Susceptibility and Relative Permeability

In free space $\left(\vec{M}=0\right)$, the Ampere's Law is

$$\vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0}\right)=\vec{J}_f \; or \; \vec{\nabla}\times\vec{H}=\vec{J}_f \tag{1}$$

In a material medium $\vec{M}\neq 0$ and as a result $\vec{B}$ changes so that

$$\vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0}\right)=\vec{J}_f+\vec{J}_b \tag{2}$$

Since $\quad \vec{J}_b=\nabla\times\vec{M}$ (3)

Substituting eqs. (1) and (3) into eq. (2), we get

$$\vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0}\right)=\vec{\nabla}\times\vec{H}+\vec{\nabla}\times\vec{M}$$

$$=\vec{\nabla}\times\left(\vec{H}+\vec{M}\right)$$

$$\Rightarrow \quad \boxed{\vec{B}=\mu_0\left(\vec{H}+\vec{M}\right)} \qquad (4)$$

This relationship holds for all materials whether they are linear or not. When the magnetic properties of the medium are linear and isotropic, the magnetization is directly proportional to the magnetic field intensity *i.e.*

$$\vec{M}\propto\vec{H}$$

$$\boxed{\vec{M}=\chi_m\vec{H}} \qquad (5)$$

where χ_m is a dimensionless quantity called magnetic susceptibility of the medium. It is a measure of how susceptible or sensitive the material is to a magnetic field. Substituting eg. (5) into eq. (4) yields

$$\vec{B}=\mu_0\left(1+\chi_m\right)\vec{H} \qquad (6)$$

$$\vec{B}=\mu_0\mu_r\,\vec{H}=\mu\vec{H} \qquad (7)$$

$$\vec{H}=\frac{1}{\mu}\vec{B} \qquad (8)$$

Where $$\boxed{\mu_r=\frac{\mu}{\mu_0}=1+\chi_m} \qquad (9)$$

The quantity μ_r is another dimensionless quantity known as the relative permeability of the medium. The quantity $\boxed{\mu=\mu_0\,\mu_r}$ is called the permeability of the medium and is measured in Henry/meter. The relation $\vec{B}=\mu\,\vec{H}$ holds good only for linear and isotropic material like diamagnetic and paramagnetic substances. If the materials are anisotropic, then the fields $\vec{B},\vec{H}$ and $\vec{M}$ are no longer parallel like ferromagnetic substances *i.e.* $\vec{B}=F\left(\vec{H}\right)$. (10)

The phenomenon of hysteresis implies that $\vec{B}$ is not a single valued function of $\vec{H}$. In fact, the function F ($\vec{H}$) depends on the history of preparation of the material.

8.6 Classification of Magnetic Materials

The behaviour or classification of magnetic materials is described in terms of magnetic susceptibility and relative permeability of the materials. A material is said to be magnetic if $\chi_m \neq 0$ or $\mu_r \neq 1$ and non-magnetic if $\chi_m = 0$ or $\mu_r = 1$. Free space, air and materials with $\chi_m = 0$ are referred as non-magnetic.

Magnetic materials can be roughly classified into three main groups in accordance with their μ_r values. A material is said to be

Diamagnetic, if $\mu_r \leq 1$ (χ_m is a very small negative number).

Paramagnetic, if $\mu_r \geq 1$ (χ_m in a very small positive numbers). Ferromagnetic, if $\mu_r \gg 1$ (χ_m in a very large positive material).

Diamagnetism arises mainly from the orbital motion of the electron within an atom and is present in all materials. Diamagnetism materials exhibit no permanent magnetism and the materials are weekly affected by a magnetic field. For most diamagnetic materials (bismuth, copper, Lead, Diamond, germanium), χ_m ***is of the order of*** 10^{-5}**.**

Paramagnetism arises mainly from the magnetic dipole moments of the spinning electrons. In these materials, the magnetic moments due to electron motion do not cancel completely and the atoms or molecules have a net average magnetic moment. Unlike diamagnetism, paramagnetism is temperature dependent. For most paramagnetic materials (Air, Platinum, tungsten, potassium), χ_m ***is of order of*** 10^{-5} ***to*** 10^{-3} ***and is temperature dependent.***

Ferromagnetism occurs in materials whose atoms have relatively large permanent magnetic moment. ***Iron, Nickel, cobalt*** etc. are mostly used ferromagnetic materials. They retain a considerable amount of their magnetization when removed from the field. Ferromagnetism can be explained in terms of magnetized domain. A ferromagnetic materials is composed of many small domains, each containing about 10^{15} or 10^{16} atoms. These domains are fully magnetized in the absence of an applied magnetic field. Quantum theory states that strong coupling forces exit between the magnetic dipole moments of the atoms in domain, holding the dipole moments in parallel.

They have very large and positive susceptibility and below the Curie temperature relationship between $\vec{B}$ and $\vec{H}$ given by $\vec{B} = F\left(\vec{H}\right)$. The shape of hysteresis loops varies from one material to another. ***The area of a hysteresis loop gives the energy loss per unit volume during one cycle of the periodic magnetization of the ferromagnetic material. This energy loss is in the form of heat.***

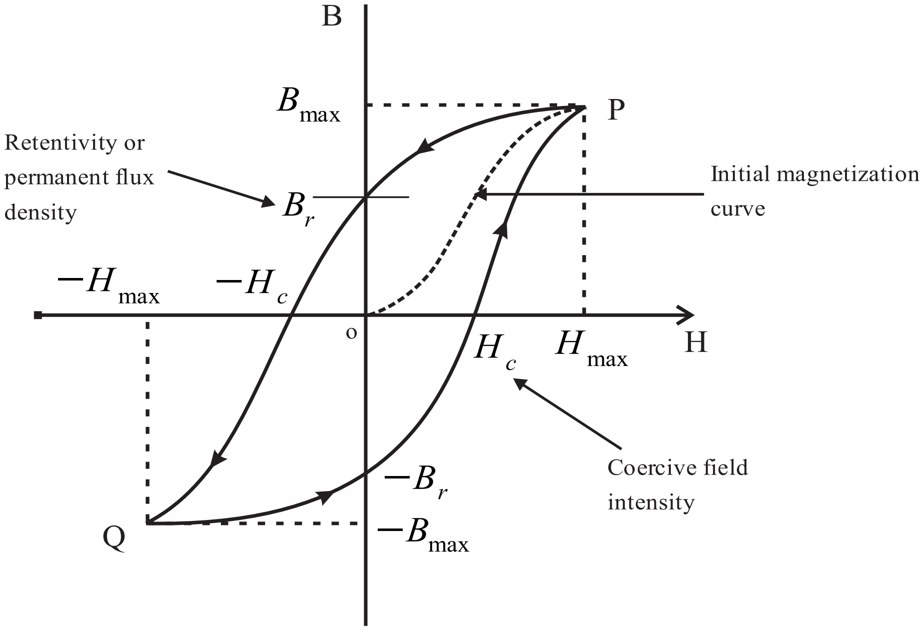


Fig. 8.4 Hysteresis Loop (Typical magnetization B-H curve)

8.7 Boundary Condition on B & H

When the magnetic field $\vec{H}$ or $\vec{B}$ exists in a region of space, which has two different media, the conditions that the magnetic field must satisfy at the boundary or interface of these media are called magnetic boundary conditions. These conditions are derived by applying the integral form of Gauss's Law for magnetostatics and Ampere's Circuit Law over a small (infinitesimal) region at the boundary of two media.

Consider the boundary between two magnetic media 1 and 2 characterized by μ_1 and μ_2 ,respectively as shown in fig 8.5.

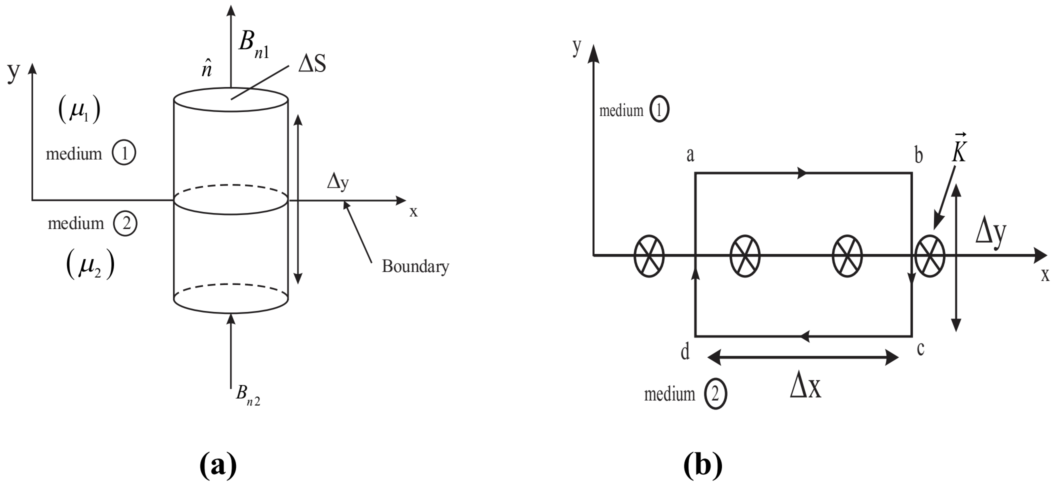


Fig. 8.5 Boundary conditions between two magnetic media (a) for $\vec{B}$ and (b) for $\vec{H}$

Applying Gauss's law for magnetic fields to the Gaussian surface of fig 8.5(a), we have

$$\underset{Top}{\oint}\vec{B}.d\vec{s}+\underset{Bottom}{\oint}\vec{B}.d\vec{s}+\underset{Sides}{\oint}\vec{B}.d\vec{s}=0 \tag{1}$$

In the limit $\Delta y \rightarrow 0$, the contribution due to the sides vanishes, then eq. (1) becomes

$$\int_s B_{n1}\hat{n}.\Delta s\hat{n}+\int_s B_{n2}\hat{n}.\Delta s(-\hat{n})=0$$

$$\Rightarrow \quad B_{n1}\,\Delta S-B_{n2}\,\Delta S=0$$

$$\Rightarrow \quad B_{n1}=B_{n2}\ or\ \left(\vec{B}_1-\vec{B}_2\right).\hat{n}=0$$

$$\Rightarrow \quad \boxed{B_{n1}=B_{n2}} \tag{2}$$

$$\text{or} \quad \mu_1\vec{H}_{n1}=\mu_2\,\vec{H}_{n2} \tag{3}$$

Since $\vec{B}=\mu\vec{H}$ and $\hat{n}$ is a unit vector normal to the boundary directed from medium (2) to medium (1).

Equation (2) shows that the normal component of $\vec{B}$ is continuous at the boundary while that of $\vec{H}$ is discontinuous at the boundary. Similarly we apply Ampere's circuit Law to the closed path abcda of fig. 8.5 (b), where surface current density *K* on the boundary or interface is assumed normal to path. We obtain

$$\oint \vec{H}.d\vec{l} = \int_{ab} + \int_{bc} + \int_{cd} + \int_{da} = I$$

$$\Rightarrow \quad H_{t1}\Delta x + H_{n1}\frac{\Delta y}{2} + H_{n2}\frac{\Delta y}{2} - H_{t2}\Delta x - H_{n2}\frac{\Delta y}{2} - H_{n1}\frac{\Delta y}{2} = I$$

As $\Delta y \rightarrow 0$, we get

$$H_{t1}\Delta x - H_{t2}\Delta x = I$$

$$H_{t1} - H_{t2} = \frac{I}{\Delta x} = K \tag{4}$$

Here H_{t1} and H_{t2} are tangential components in medium 1 and 2, respectively.

Equation (4) shows that the tangential component at boundary of $\vec{H}$ is also discontinuous at boundary by an amount equal to surface current density. This equation may be written in terms of *B* as

$$\frac{B_{t1}}{\mu_1} - \frac{B_{t2}}{\mu_2} = K \tag{5}$$

The vector form of eg. (4) is given by

$$\hat{n} \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{K} \tag{6}$$

$$\text{Or } \left(\vec{H}_1 - \vec{H}_2\right) = \vec{K} \times \hat{n} \tag{7}$$

where $\hat{n}$ is unit vector at the boundary of two media directed from region 2 to 1. If the boundary is free of current or the media are not conductors, K=0 and equation (4) becomes

$$\boxed{\vec{H}_{t1} = \vec{H}_{t2} \; or \; \frac{\vec{B}_{t1}}{\mu_1} = \frac{\vec{B}_{t2}}{\mu_2}} \tag{8}$$

Thus the tangential component of $\vec{H}$ is continuous while that of $\vec{B}$ is discontinuous at the boundary.

If the fields make an angle θ with the normal to the interface, then the normal component of $\vec{B}$ across the boundary can be written as

$$B_1 \cos\theta_1 = B_{n1} = B_{n2} = B_2 \cos\theta_2 \tag{9}$$

while the tangential component of $\vec{H}$ across the boundary with no surface current can be written as

$$\frac{B_1}{\mu_1}\sin\theta_1 = H_{t1} = H_{t2} = \frac{B_2}{\mu_2}\sin\theta_2 \tag{10}$$

Dividing eq. (10) by eq. (9) gives

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2} \tag{11}$$

This is the law of refraction for magnetic flux lines at a boundary with no surface current.

8.8 Methods of Solving Boundary– Value Problems in Magnetostatics

The basic equations of magnetostatics are

$$\vec{\nabla}.\vec{B} = 0 \tag{1}$$

and $$\vec{\nabla}\times\vec{H} = \vec{J} \tag{2}$$

with some constitutive relation between $\vec{B}$ and $\vec{H}$ i.e. $\vec{B} = \mu\vec{H}$ for linear media or $\vec{B} = F\left(\vec{H}\right)$ for non linear media.

The different techniques for solving boundary–value problems in magnetostatics are as follows:-

A. Generally applicable method of the vector potential–Since the divergence of $\vec{B}$ is always equal to zero, therefore we can introduce a vector potential $\vec{A}$ such that

$$\vec{B} = \nabla\times\vec{A} \tag{3}$$

If the relationship between $\vec{B}$ and $\vec{H}$ is non-linear, the second equation becomes very complicated even if the current distribution is simple. For linear media with $\vec{B} = \mu\vec{H}$, the eq. (2) becomes

$$\vec{\nabla}\times\left(\frac{1}{\mu}\vec{\nabla}\times\vec{A}\right)=\vec{J} \tag{4}$$

If μ is constant over a finite region of space, the eq. (4) can be written as

$$\vec{\nabla}\left(\vec{\nabla}.\vec{A}\right)-\nabla^2\vec{A}=\mu\vec{J} \tag{5}$$

With the choice of the coulomb gauge $\vec{\nabla}.\vec{A}=0$, the eq. (5) becomes a Poisson equation *i.e.*

$$\boxed{\nabla^2\vec{A}=-\mu\vec{J}} \tag{6}$$

The solution of eq. (6) in different linear media must be matched across the boundary surface using the boundary conditions.

B. $\vec{J}=0$, magnetic scalar potential

If the current density vanishes in some finite region of space, the eq. (2) becomes

$$\vec{\nabla}\times\vec{H}=0 \tag{7}$$

This implies that we can introduce a magnetic scalar potential ϕ_M such that

$$\vec{H}=-\vec{\nabla}\phi_M \tag{8}$$

If medium is non linear *i.e.* $\vec{B}=F\left[\vec{H}\right]$, again this equation becomes very complicated differential equation. Assuming that the medium is linear and uniform *i.e.* the magnetic permeability is constant in space. The eq. (1) together eq. (8) becomes the Laplace's equation for the magnetic scalar potential.

$$\vec{\nabla}.\left(\mu\vec{\nabla}\phi_M\right)=0$$

$$\Rightarrow \quad \nabla^2\phi_M=0 \tag{9}$$

Again the solutions in different media are connected through the boundary conditions. It is clear from eq. (9) that one can use methods of solving differential equations to find the magnetic scalar potential and therefore the magnetic fields $\vec{B}$ and $\vec{H}$.

C. Hard Ferromagnets $\left(\vec{M}\ given\ and\ \vec{J}=0\right)$

In this case, the magnetization $\vec{M}$ is independent on the magnetic field and therefore we can assume that $\vec{M}$ is a given function of coordinates *i.e.* $\vec{M}\left(\vec{r}\right)$.

(i) Scalar potential

In this case $\vec{J}=0$, we can again use a scalar potential. The first equation can be written as

$$\vec{\nabla}.\vec{B}=\mu_0\vec{\nabla}.\left(\vec{H}+\vec{M}\right)=0$$

Since $\vec{B}=\mu_0\left(\vec{H}+\vec{M}\right)$

Hence $\nabla.\vec{H}=-\vec{\nabla}.\vec{M}$ (10)

Now using the magnetic scalar potential form eq. (8), we obtain a magnetostatic Poisson equation

$$-\nabla^2\phi_M=-\vec{\nabla}.\vec{M}$$

$$\Rightarrow\quad \nabla^2\phi_M=\vec{\nabla}.\vec{M}$$

$$\Rightarrow\quad \nabla^2\phi_M=-\rho_M \tag{11}$$

where the effective magnetic charge density is given by

$$\rho_M=-\vec{\nabla}.\vec{M} \tag{12}$$

The solution for the potential ϕ_M, if there are no boundary surface is

$$\phi_M\left(\vec{r}\right)=\frac{1}{4\pi}\int\frac{\rho_M\left(\vec{r}\,'\right)}{\left|\vec{r}-\vec{r}\,'\right|}dv'=-\frac{1}{4\pi}\int\frac{\vec{\nabla}'.\vec{M}\left(\vec{r}\,'\right)}{\left|\vec{r}-\vec{r}\,'\right|}dv' \tag{13}$$

If $\vec{M}$ is well behaved and localized in space, integration by parts may be performed to yield

$$\phi_M\left(\vec{r}\right)=-\frac{1}{4\pi}\left\{\int\vec{\nabla}'.\left[\frac{\vec{M}\left(\vec{r}\,'\right)}{\left|\vec{r}-\vec{r}\,'\right|}\right]dv'-\int\vec{M}\left(\vec{r}\,'\right).\vec{\nabla}'\frac{1}{\left|\vec{r}-\vec{r}\,'\right|}dv'\right\}$$

$$=\frac{1}{4\pi}\int\vec{M}\left(\vec{r}\,'\right).\vec{\nabla}'\frac{1}{\left|\vec{r}-\vec{r}\,'\right|}dv' \tag{14}$$

Here the first integral vanishes by the divergence theorem is reduces to the integral over the surface where the magnetization is zero. As we know that

$$\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)=-\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) \tag{15}$$

Using eq. (15), eq. (14) can be rewritten as follows –

$$\phi_M\left(\vec{r}\right)=-\frac{1}{4\pi}\vec{\nabla}.\int\frac{\vec{M}\left(\vec{r}\,'\right)}{|\vec{r}-\vec{r}\,'|}dv' \tag{16}$$

Far from the region of nonvanishing magnetization, the potential may be approximated by

$$\phi_M\left(\vec{r}\right)\simeq-\frac{1}{4\pi}\vec{\nabla}\left(\frac{1}{r}\right).\int\vec{M}\left(\vec{r}\,'\right)dv'$$

$$\phi_M\left(\vec{r}\right)=-\frac{1}{4\pi}\left(-\frac{\vec{r}}{r^3}\right).\vec{m}$$

where $\vec{\nabla}\left(\frac{1}{r}\right)=-\frac{\vec{r}}{r^3}$ and

$\vec{m}=\int\vec{M}\left(\vec{r}\,'\right)dv'$ is the total magnetic moment.

$$\Rightarrow\quad \phi_M\left(\vec{r}\right)=\frac{\vec{m}.\vec{r}}{4\pi r^3} \tag{17}$$

In solving magnetostatics problems with a given magnetization distribution which changes abruptly at the boundaries of the specimen, it is convenient to introduce the magnetic surface charge density. If a hard ferromagnet has volume V and surface S, we specify $\vec{M}\left(\vec{r}\right)$ inside V and assume that it falls suddenly to zero at the surface S. Application of the divergence theorem to ρ_M (eq. 12) in a Gaussian pillbox straddling the surface shows that there is an effective magnetic surface charge density, which is given by

$$\sigma_M=\hat{n}.\vec{M} \tag{18}$$

where $\hat{n}$ is the outwardly directed normal. Then instead of eq. (13), the potential is represented as follows

$$\phi_M(\vec{r}) = -\frac{1}{4\pi}\int_v \frac{\vec{\nabla}'.\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}dv' + \frac{1}{4\pi}\oint_S \frac{\hat{n}.\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}ds' \tag{19}$$

As important special case, is that of uniform magnetization throughout the volume V. Then the first term vanishes; only the surface integral over σ_M contributes.

Note: It is important to note that eq. (16) is generally applicable, even for the limit of discontinuous distributions of $\vec{M}$. Because we can introduce a limiting procedure after transforming eq. (13) into eq. (16) in order to discuss discontinuities in $\vec{M}$. Never combine the surface integral of σ_M with eq. (16).

(ii) **Vector potential**

Since from eq. (1), we have

$$\vec{\nabla}.\vec{B} = 0$$

$$\Rightarrow \quad \vec{B} = \vec{\nabla}\times\vec{A} \tag{20}$$

where $\vec{A}$ is magnetic vector potential

From eq. (2), we have

$$\vec{\nabla}\times\vec{H} = 0 \qquad \text{Since } \vec{J} = 0 \text{ in case of hard ferromagnets.}$$

$$\Rightarrow \quad \vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = 0 \quad \text{Since } \vec{B} = \mu_0\left(\vec{H}+\vec{M}\right)$$

$$\Rightarrow \quad \vec{\nabla}\times\frac{\vec{B}}{\mu_0} = \vec{\nabla}\times\vec{M}$$

$$\Rightarrow \quad \vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J}_M\right) \tag{21}$$

where $\boxed{\vec{J}_M = \vec{\nabla}\times\vec{M}}$ is effective current density due to magnetization.

From eqs. (20) and (21), we get

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right) = \mu_0\vec{J}_M$$

$$\Rightarrow \quad \vec{\nabla}\left(\vec{\nabla}.\vec{A}\right) - \nabla^2\vec{A} = \mu_0\vec{J}_M$$

For the coulomb gauge $\vec{\nabla}.\vec{A} = 0$**,** we get

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}_M \tag{22}$$

The solution for the vector potential in the absence of boundary surface is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_v \frac{\vec{\nabla}' \times \vec{M}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} dv' \tag{23}$$

If the distribution of magnetization is discontinuous, it is necessary to add a surface integral to eq. (23), i.e.

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_v \frac{\vec{\nabla}' \times \vec{M}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} dv' + \frac{\mu_0}{4\pi} \oint_S \frac{\vec{M}(\vec{r}\,') \times \vec{n}}{|\vec{r} - \vec{r}\,'|} ds' \tag{24}$$

If $\vec{M}$ is constant throughout the volume, only the surface integral survives.

8.9 Uniformly Magnetised Sphere in an External Field or Permanent Magnet

Let us consider a sphere of radius R with a uniform magnetization $\vec{M}$ along z-axis in the absence of an external magnetic field (because of the existence of permanent magnet) as shown in fig. 8.6.

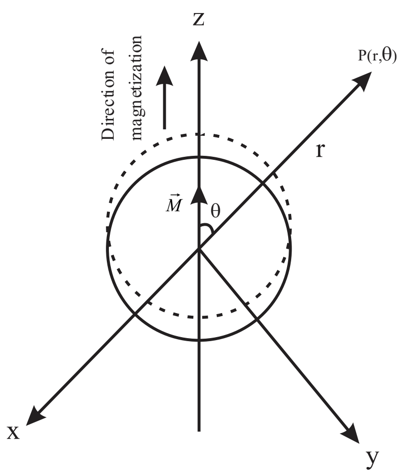


Fig. 8.6 Uniformly Magnetized Sphere

(A) In the absence of external magnetic field

(i) Potential and field at an external point $P(r,\theta)$ - The magnetization sphere may be thought to be equivalent to two sphere – one having a north polarity and another a south polarity, and the two spheres slightly displaced. Such a system is equivalent to a short magnetic dipole of moment

$$\vec{m} = \left(\frac{4\pi}{3} R^3 \right) \vec{M} \tag{1}$$

The magnetic scalar potential at an external distant point P (r, θ) is given by

$$\phi_{out} = \frac{\mu_0}{4\pi} \frac{\vec{m}.\vec{r}}{r^3} = \frac{\mu_0}{4\pi} \frac{m \cos\theta}{r^2} \tag{2}$$

$$= \frac{\mu_0}{4\pi} \frac{\left(\frac{4\pi}{3} R^3 M \right) \cos\theta}{r^2}$$

$$\phi_{out} = \frac{\mu_0 M R^3 \cos\theta}{3r^2} \tag{3}$$

and magnetic field intensity at the outside point P(r, θ) is obtained as [using eq.(3)]

$$\vec{H}_{out} = \frac{\vec{B}_{out}}{\mu_0} = -\frac{\nabla \phi}{\mu_0} = -grad\left(\frac{MR^3 \cos\theta}{3r^2} \right) \tag{4}$$

Thus the radial and transverse components of magnetic field intensity are

$$H_r = -\frac{\partial}{\partial r}\left(\frac{MR^3 \cos\theta}{3r^2} \right) = \frac{2}{3} \frac{MR^3 \cos\theta}{r^3} \tag{5}$$

and $$H_\theta = -\frac{1}{r} \frac{\partial}{\partial \theta}\left(\frac{MR^3 \cos\theta}{3r^2} \right)$$

$$H_\theta = \left(\frac{1}{3} \right) \frac{MR^3 \sin\theta}{r^3} \tag{6}$$

so that $\vec{H}_{out} = H_r \vec{a}_r + H_\theta \vec{a}_\theta$

$$\vec{H}_{out} = \frac{MR^3}{3r^3}\left(2\cos\theta\, \vec{a}_r + \sin\theta\, \vec{a}_\theta \right) \tag{7}$$

$$\Rightarrow \qquad H_{out} = \left|\vec{H}_{out}\right| = \frac{MR^3}{3r^3}\sqrt{4\cos s^2\theta + \sin^2\theta}$$

$$H_{out} = \frac{MR^3}{3r^3}\sqrt{1+3\cos^2\theta} \tag{8}$$

(ii) **Potential and field at an internal point** – The scalar magnetic potential is given by eq. (3) by putting r = R.

$$\phi_{in} = \frac{\mu_0 MR\cos\theta}{3} \tag{9}$$

$$\phi_{in} = \frac{\mu_0 Mz}{3} \qquad \left(\text{since z = R}\cos\theta\right) \tag{10}$$

Since no free poles exist inside the magnetized sphere, the potential ϕ_{in} must satisfy Laplace's equation $\nabla^2\phi_{in} = 0$. According to uniqueness theorem there can be only one solution of Laplace's equation and that it must be as given by eq. (10). Therefore the magnetic field intensity may be obtained as

$$\vec{H}_{in} = \frac{\vec{B}_{in}}{\mu_0} = \frac{-\nabla\phi_{in}}{M_0} = -\nabla\left(\frac{Mz}{z}\right) \tag{11}$$

In terms of components

$$\left(H_x\right)_{in} = -\frac{\partial}{\partial x}\left(\frac{Mz}{3}\right) = 0$$

$$\left(H_y\right)_{in} = -\frac{\partial}{\partial y}\left(\frac{Mz}{3}\right) = 0$$

and $$\left(H_z\right)_{in} = -\frac{\partial}{\partial z}\left(\frac{Mz}{3}\right) = -\frac{M}{3}$$

Therefore $\boxed{\vec{H}_{in} = -\frac{M}{3}\vec{a}_z = -\frac{\vec{M}}{3}}$ or $\boxed{\vec{B}_{in} = \mu_0\left(\vec{H}+\vec{M}\right) = \frac{2}{3}\mu_0\vec{M}}$ (12)

Since the direction of $\vec{M}$ is along z-axis. The magnetic field intensity is constant throughout the magnetized sphere and points opposite the direction of $\vec{M}$. In other words, the $\vec{H}$ field acts to demagnetize the sphere.

The lines $\vec{B}$ and $\vec{H}$ are shown in fig. 8.7. The lines of $\vec{B}$ are continuous closed paths, but those of $\vec{H}$ terminate on the surface because there is an effective surface charge density σ_M.

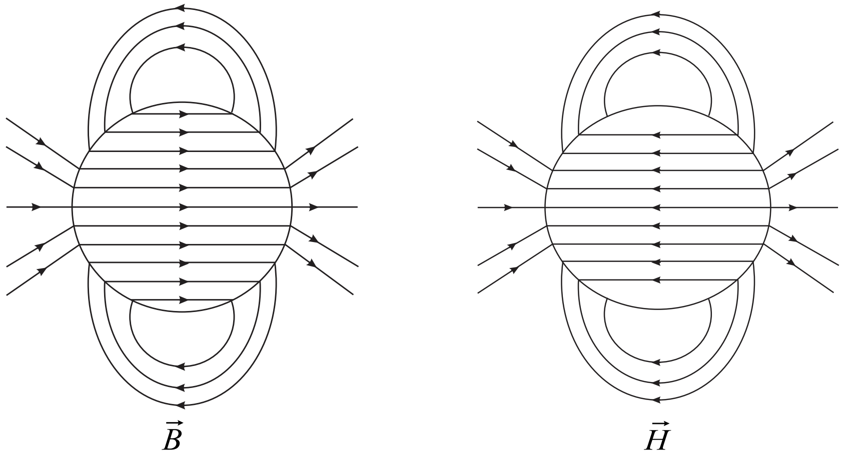


FIG. 8.7 Lines of $\vec{B}$ and lines of $\vec{H}$ for a uniformly magnetized sphere.

(B) In presence of external magnetic field

Let us assume the existence of uniform external magnetic field $\vec{H}_0$ along z-axis, then the resultant magnetic field intensity at external and internal points can be written as

$$\vec{H}_{out} = \left(H_0 \cos\theta + \frac{2MR^3 \cos\theta}{3r^3} \right)\vec{a}_r + \left(-H_0 \sin\theta + \frac{MR^3 \sin\theta}{3r^3} \right)\vec{a}_\theta \qquad (13)$$

and $$\vec{H}_{in} = \vec{H}_0 - \frac{\vec{M}}{3} \qquad (14)$$

Equation (14) shows that the magnetization produces a reverse field inside the sphere, known as diamagnetic field. It is proportional to $\vec{M}$. The factor $\left(\frac{1}{3}\right)$ is known as ***the demagnetizing factor***.

For a diamagnetic or paramagnetic substance, we know that

$$\vec{M} = \chi \vec{H}_{in}$$

$$\vec{M} = \left(\frac{\mu}{\mu_0} - 1 \right)\vec{H}_{in} \qquad (15)$$

Since $\mu = \mu_0(1+\chi)$

Substituting the value of $\vec{M}$ from eq. (15) into eq. (14), we get

$$\vec{H}_{in} = \vec{H}_0 - \frac{1}{3}\left(\frac{\mu}{\mu_0} - 1\right)\vec{H}_{in}$$

$$\vec{H}_{in} = \vec{H}_0 - \frac{1}{3}(\mu_r - 1)\vec{H}_{in} \quad \text{where } \frac{\mu}{\mu_0} = \mu_r \text{ is relative permeability}$$

$$\Rightarrow \quad \vec{H}_0 = \vec{H}_{in} - \frac{\vec{H}_{in}}{3} + \mu_r \frac{\vec{H}_{in}}{3}$$

$$= \frac{2}{3}\vec{H}_{in} + \frac{\mu_r}{3}\vec{H}_{in}$$

$$\vec{H}_0 = \left(\frac{2+\mu_r}{3}\right)\vec{H}_{in}$$

$$\Rightarrow \quad \vec{H}_{in} = \left(\frac{3}{\mu_r + 2}\right)\vec{H}_0 \tag{16}$$

and $$\vec{M} = (\mu_r - 1)\left(\frac{3}{\mu_r + 2}\right)\vec{H}_0 \tag{17}$$

Thus magnetic induction at an internal point is

$$\vec{B}_{in} = \mu \vec{H}_{in}$$

$$\vec{B}_{in} = \mu\left(\frac{3}{\mu_r + 2}\right)\vec{H}_0 \tag{18}$$

Putting $\mu_r = \dfrac{\mu}{\mu_0}$, we get

$$\vec{B}_{in} = \frac{3\mu}{\left(\dfrac{\mu}{\mu_0} + 2\right)}\vec{H}_0$$

$$\vec{B}_{in} = \left(\frac{3\mu}{\mu + 2\mu_0} \right) \mu_0 \vec{H}_0 \tag{19}$$

For a paramagnetic substance, μ_r is slightly greater than unity, so the $\vec{H}_{in}$ is slightly less than $\vec{H}_0$ and induction $\vec{B}_{in}$ is greater than the free space value $\mu_0 \vec{H}_0$. It means lines of magnetic induction are crowded together in a paramagnetic sphere. Reverse is the case of a diamagnetic substance for which $\mu_r < 1$. For a ferromagnetic substance, we consider the case when there is no external field (*i.e.* $\vec{H}_0 = 0$), but a finite magnetization $\vec{M}$ corresponds to a spherical permanent magnet. The external field is then purely that of a point dipole given by equation (7) and the internal field is just the demagnetizing field $\left(-\frac{\vec{M}}{3} \right)$ .For a ferromagnetic substance, eq. (17) implies that the magnetization vanishes when the external field vanishes. The existence of permanent magnets contradicts this result. The non-linear relation $\vec{B} = F\left(\vec{H}\right)$ and the phenomenon of hysteresis allow the creation of permanent magnets.

From eq. (19)

$$\vec{B}_{in}\left(\mu + 2\mu_0\right) = 3\mu\ \mu_0 \vec{H}_0$$

$$\mu \vec{B}_{1n}\left(1 + 2\frac{\mu_0}{\mu}\right) = 3\mu \vec{B}_0$$

$$\Rightarrow \quad \vec{B}_{in} + 2\frac{\vec{B}_{in}}{\mu}\mu_0 = 3\vec{B}_0$$

$$\Rightarrow \quad \vec{B}_{in} + 2\mu_0 \vec{H}_{in} = 3\vec{B}_0 \tag{20}$$

$$\Rightarrow \quad \vec{B}_{in} = -2\mu_0 \vec{H}_{in} + 3\vec{B}_0$$

The hysteresis curve provides the other relation between $\vec{B}_{in}$ and $\vec{H}_{in}$, so that specific values can be found for any external field. Eq. (21) corresponds to a line with slope -2 on the hysteresis diagram with intercept $3B_0$ on the y-axis as shown in fig. 8.8.

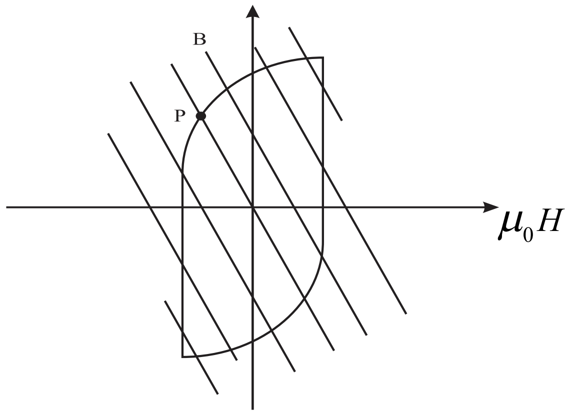


FIG. 8.8 Hysteresis curve

Suppose the external field is increased until the ferromagnetic sphere becomes saturated and then decreased to zero. The internal B and H will then be given by the point marked P in fig. 8.8. The magnetization can be found from eq. (14) with $\vec{B}_0 = 0$.

8.10 Magnetic Shielding: Spherical Shell of Permeable Material in a Uniform Field

Initially a certain magnetic induction $\vec{B}_0 = \mu_0 \vec{H}_0$ exists in a region of empty space. The lines of magnetic induction are modified if a permeable body is placed in the region. The magnetic field is greatly reduced inside a volume covered with a ferromagnetic shell (permeable media) because the magnetic field lines are strongly shielded by the permeable material. *i.e.* a reduction in field is said to be due to the magnetic shielding provided by the permeable material.

Let us consider a spherical shell of inner radius *a* and outer radius *b* made of material of permeability μ and placed in a formerly uniform constant magnetic induction $\vec{B}_0$ as shown in fig. 8.9.

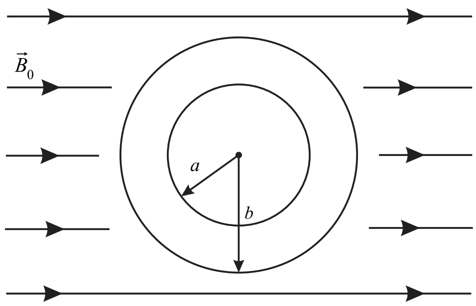


FIG. 8.9 Spherical shell of permeable material in a uniform field

We went to find the fields $\vec{B}$ and $\vec{H}$ everywhere in space, but most particularly in the cavity $(r<a)$ as function of μ. Since there are no conduction currents present.

$$\vec{\nabla}\times\vec{H}=0$$

$$\Rightarrow \quad \vec{H}=-\nabla\phi_M \tag{1}$$

Since $\vec{B}=\mu\vec{H}, \vec{\nabla}.\vec{B}=0$ becomes $\vec{\nabla}.\vec{H}=0$ in the various regions. Hence

$$\nabla.\left(-\nabla\phi_M\right)=0$$

$$\Rightarrow \quad \nabla^2\phi_M=0 \tag{2}$$

Thus the potential ϕ_M satisfies the Laplace equation everywhere. The problem reduces to finding the proper solutions in the different regions to satisfy the boundary conditions at $r=a$ and $r=b$.

For $r>b$, the potential must be the form

$$\phi_{M1}=-B_0 r\frac{\cos\theta}{\mu_0}+\sum_{l=0}^{\infty}\frac{a_l}{r^{l+1}}P_l\left(\cos\theta\right) \tag{3}$$

In order that $\vec{H}\to\vec{B}_0/\mu_0$ as $r\to\infty$

$$\left(a<r<b\right)\ \phi_{M2}=\sum_{l=0}^{\infty}\left(B_l r^l+\gamma l\frac{1}{r^{l+1}}\right)P_l\left(\cos\theta\right) \tag{4}$$

$$\left(r<a\right)\ \phi_{M3}=\sum_{l=0}^{\infty}\delta_l r^l P_l\left(\cos\theta\right) \tag{5}$$

Since ϕ_M must be finite at r = 0. The boundary conditions at $r = a$ and $r = b$ are that H_θ and B_r be continuous. So

$$\left.\frac{\partial\phi_{M1}}{\partial\theta}\right|_{r=b}=\left.\frac{\partial\phi_{M2}}{\partial\theta}\right|_{r=b},\left.\frac{\partial\phi_{M2}}{\partial\theta}\right|_{r=a}=\left.\frac{\partial\phi_{M3}}{\partial\theta}\right|_{r=a}$$

$$\mu_0\left.\frac{\partial\phi_{M1}}{\partial r}\right|_{r=b}=\mu\left.\frac{\partial\phi_{M2}}{\partial r}\right|_{r=b},\mu\left.\frac{\partial\phi_{M2}}{\partial r}\right|_{r=a}=\mu_0\left.\frac{\partial\phi_{M3}}{\partial r}\right|_{r=a}$$

These four conditions which hold for all angles θ are sufficient to determine the unknown constants. All coefficients with $l = 1$ vanish. The $l = 1$ coefficients satisfy the four simultaneous equations.

$$\alpha_1 - b^3 \beta_1 - \gamma_1 = b^3 \frac{B_0}{\mu_0}$$

$$2\alpha_1 \frac{\mu}{\mu_0} b^3 \beta_1 - 2\frac{\mu}{\mu_0}\gamma_1 = -b^3 \frac{B_0}{\mu_0}$$

$$a^3 \beta_1 + \gamma_1 - a^3 \delta_1 = 0$$

$$\frac{\mu}{\mu_0} a^3 \beta_1 - 2\frac{\mu}{\mu_0}\gamma_1 - a^3 \delta_1 = 0$$

The solutions for α_1 and δ_1 are,

$$\alpha_1 = \left[\frac{(2\mu + \mu_0)(\mu - \mu_0)}{(2\mu + \mu_0)(\mu + 2\mu_0) - 2a^3 / b^3 (\mu - \mu_0)^2} \right] (b^3 - a^3)(B_0 / \mu_0) \quad \ldots(6)$$

and $$\delta_1 = -\left[\frac{9\mu\mu_0}{(2\mu + \mu_0)(\mu + 2\mu_0) - 2a^3 / b^3 (\mu - \mu_0)^2} \right] (B_0 / \mu_0) \quad (7)$$

The potential outside the spherical shell corresponds to a uniform field $\vec{B}_0$ plus a dipole field with dipole moment α_1 oriented parallel to $\vec{B}_0$. Inside the cavity, there is a uniform magnetic force field parallel to $\vec{B}_0$ and equal in magnitude to $-\delta_1$. For $\mu \gg \mu_0$ the dipole moment $\propto_1$ and the inner field - δ become

$$\alpha_1 \rightarrow b^3 (B_0 / \mu_0) \quad (8)$$

$$-\delta_1 \rightarrow \frac{9\mu_0}{2\mu\left(1 - \frac{a^3}{b^3}\right)} (B_0 / \mu_0) \quad (9)$$

Thus, the inner field is proportional to $\left(\mu/\mu_0\right)^{-1}$. Consequently, a shield made of high permeability material with $\mu/\mu_0 \sim 10^3$ to 10^6 causes a great reduction in the field inside, even for a relatively thin shell. Thus, the magnetic induction in the cavity is given by

$$B_3 = \mu_0 H_3 = -\mu_0 - \delta_1 \qquad (10)$$

The magnetic shielding factor

$$h_m \equiv \frac{B_0}{B_3} = -\frac{B_0}{\mu_0 \delta_1} \approx \frac{2}{9}\mu_r \left(1 - \frac{a^3}{b^3}\right) \qquad (11)$$

if $\mu_r \gg 1$

The behavior of the lines $\vec{B}$ through permeable media is shown in fig 8.10. The lines tend to pass through the permeable medium if possible.

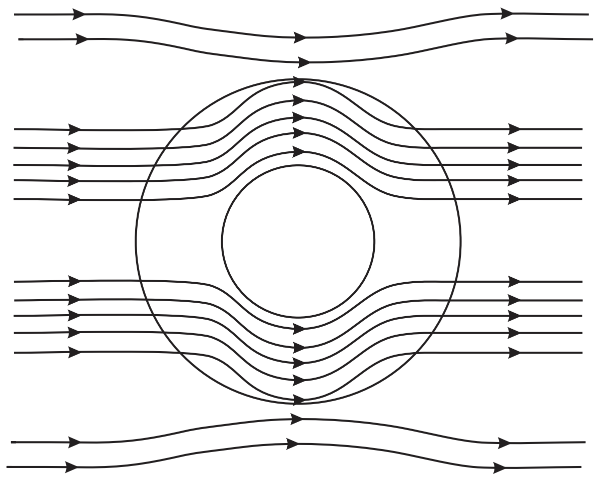


FIG. 8.10 Shielding effect of a shell of highly permeable material

8.11 Illustrative Examples

Example 1 Region $0 \le z \le 1m$ is occupied by an infinite slab of permeable material $\left(\mu_r = 3\right)$. If $\vec{B} = 12y\vec{a}_x - 6x\vec{a}_y \; mWb/m^2$ within the slab, determine

(a) Volume current density $\left(\vec{J}\right)$.

(b) Volume bound current density $\left(\vec{J}_b\right)$

(c) Magnetization $\left(\vec{M}\right)$

(d) Surface bound current density $\left(\vec{K}_b\right)$ on $z=0$

Solution:

Given $\vec{B}=\left(12y\,\vec{a}_x-6x\vec{a}_y\right)\times10^{-3}\,Wb/m^2$

$$\mu_3=3\;\therefore\;\mu=\mu_0\mu_r=4\pi\times10^{-7}\times3=12\pi\times10^{-7}\,H/m$$

(a) $$\vec{J}=\vec{\nabla}\times\vec{H}=\vec{\nabla}\times\frac{\vec{B}}{\mu}=\frac{1}{12\pi\times10^{-7}}\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)\vec{a}_z$$

$$=\frac{10^7}{12\pi}\left(-6-10\right)\times10^{-3}\vec{a}_z$$

$$=\frac{-10^4\times16}{12\pi}\vec{a}_z$$

$$\vec{J}=-4.24\vec{a}_z\;\;KA/m^2$$

(b) $$\vec{J}_b=\vec{\nabla}\times\vec{M}=\vec{\nabla}\times\chi_m\vec{H}$$

$$=\chi_m\left(\vec{\nabla}\times\vec{H}\right)=\chi_m\vec{J}=\left(\mu_r-1\right)\vec{J}=\left(3-1\right)\left(-4.24\;\vec{a}_z\right)\times10^3$$

$$\vec{J}_b=-8.48\;\vec{a}_z\;KA/m^2$$

(c) Magnetization (M)

$$\vec{M}=\chi_m\vec{H}$$

$$=\left(\mu_r-1\right)\frac{\vec{B}}{\mu}$$

$$=\left(3-1\right)\frac{\left(12y\vec{a}_x-6x\vec{a}_y\right)\times10^{-3}}{12\pi\times10^{-7}}$$

$$=\frac{20}{12\pi}\left(12y\vec{a}_x-6x\vec{a}_y\right)\times10^3$$

$$=0.53\left(12y\vec{a}_x-6x\,\vec{a}_y\right)\times10^3$$

$$\vec{M} = \left(6.36y\vec{a}_x - 3.18x\vec{a}_y\right) KA/m$$

(d) $K_b = \vec{M} \times \vec{a}_n$, since z = 0 is the lower side of the slab occupying $0 \le z \le 1, \vec{a}_n = -\vec{a}_z$, Hence

$$\vec{K}_b = \left(6.36y\vec{a}_x - 3.18x\vec{a}_y\right) \times 10^3 \times \left(-\vec{a}_z\right)$$

$$= \left(3.18x\vec{a}_x + 6.36y\vec{a}_y\right) \times 10^3$$

$$= \left(3.18x\vec{a}_x + 6.36y\vec{a}_y\right) KA/m$$

Example 2 Magnetic flux line is received at an iron-air boundary at an angle of incidence 60^o. Determine the angle of refraction at the boundary in air. The relative permeability of iron is 350.

Solution: If θ_1 and θ_2 are the angle of incidence and the angle of refraction, respectively, then at the boundary

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2} = \frac{\mu_0\mu_{r1}}{\mu_0\mu_{r2}}$$

$$\Rightarrow \quad \frac{\tan\theta_2}{\tan\theta_1} = \frac{\mu_{r2}}{\mu_{r1}}$$

Given $\mu_{r2} = 1, \mu_{r1} = 350$ and $\theta_1 = 60^o$

$$\therefore \quad \tan\theta_2 = \frac{1}{350}\tan 60^o$$

$$= \frac{\sqrt{3}}{350}$$

$$\theta_2 = \tan -1\left(\frac{\sqrt{3}}{350}\right)$$

$$\theta_2 = 0.28^o$$

Example 3 The magnetic field intensity is $H = 1200 Amp/m$ in a material when B=2 Wb/m^2. When H is reduced to 400 Amp/m, $B = 1.4\ Wb/m^2$. Calculate the change in the magnetization M.

Solution: For case –I

$$\mu = \frac{B}{H} = \frac{2}{1200} = \frac{1}{600}$$

$$\therefore \qquad \mu_2 = \frac{\mu}{\mu_0} = \frac{1}{600} \times \frac{1}{4\pi \times 10^{-7}}$$

$$\mu_r = 1326.96$$

We know that $\mu_r = 1 + X_m \Rightarrow \chi_m = \mu_r - 1$

$$\therefore \qquad \chi_m = 1325.96$$

$$\Rightarrow \qquad M = \chi_m H = 1325.96 \times 1200$$

$$M = 1591152 = 1591.152 \ K\ A/m \qquad (1)$$

For case –II

$$\mu = \frac{B}{H} = \frac{1.4}{400}$$

$$\therefore \qquad \mu_r = \frac{M}{\mu_0} = \frac{1.4}{400} \times \frac{1}{4\pi \times 10^{-7}}$$

$$= \frac{1.4 \times 10^5}{50.24}$$

$$\mu_r = 2786.62$$

$$\Rightarrow \qquad \chi_m = \mu_r - 1 = 2786.62 \ - 1$$

$$\chi_m = 2785.62$$

$$\therefore \qquad M = \chi_m H = 2785.62 \times 400$$

$$M = 1114248 = 1114.248\, K\ A/m \qquad (2)$$

Therefore change in magnetization

ΔM = eq.(1) – eq.(2)

$$= 1591.152 - 1114.248$$
$$= 476.9\, K\ A/m$$

Example 4 If $\mu_1 = 2\mu_0$ for region $1(0 < \phi < \pi)$ and $\mu_2 = 5\mu_0$ for region $2(\pi < \phi < 2\pi)$ with $\vec{B}_2 = 10\vec{a}_\rho + 15\vec{a}_\phi - 20\vec{a}_z\ Wb/m^2$.Calculate $\vec{B}_1$ for region 1

Solution: This problem is given in cylindrical coordinate system (ρ, ϕ, z). Both the regions are separated by ϕ component, therefore the normal and transverse components of $\vec{B}_2$ are given as

$$\vec{B}_{2n} = 15\,\vec{a}_\phi \tag{1}$$

and $$\vec{B}_{2t} = 10\vec{a}_\rho - 20\vec{a}_z \tag{2}$$

As per boundary condition for normal component of $\vec{B}$, me know that

$$\vec{B}_{1n} = \vec{B}_{2n} = 15\vec{a}_\phi$$

Since the interface between this media carries no current, therefore the transverse component of $\vec{H}$ is also continuous *i.e.*

$$\vec{H}_{1t} = \vec{H}_{2t} \Rightarrow \frac{\vec{B}_{1t}}{\mu_1} = \frac{\vec{B}_{2t}}{\mu_2}$$

$$\Rightarrow \quad \vec{B}_{1t} = \frac{\mu_1}{\mu_2}\vec{B}_{2t}$$

$$= \frac{2\mu_0}{5\mu_0}\left(10\vec{a}_\rho - 20\vec{a}_z\right)$$

$$\vec{B}_{1t} = 4\vec{a}_\rho - 8\vec{a}_z \tag{4}$$

Hence magnetic flux density in region 1 is given by

$\vec{B}_1 = 4\vec{a}_\rho + 15\vec{a}_\phi - 8\vec{a}_z\ Wb/m^2$.

Example 5 The slab of magnetic material is infinite in the plane which has a uniform magnetization $\vec{M}$ oriented either parallel or perpendicular to the surfaces of the slab. Calculate the magnetic flux density $\vec{B}$ and magnetic field intensity $\vec{H}$ everywhere in space.

Solution: Fig. 8.11Slab of magnetic material (a) $\vec{M}$ parallel to the surface and (b) $\vec{M}$ perpendicular to the surface

(a) (b)

FIG.8.11 Slab of magnetic material (a) $\vec{M}$ parallel to the surface and (b) $\vec{M}$ perpendicular to the surface

It is a case of hard ferromagnetic *i.e.* $\vec{M}$ given and $\vec{J}=0$.

$$\Rightarrow \quad \vec{\nabla}\times\vec{H}=0 \text{ and } \vec{\nabla}.\vec{B}=\mu_0\vec{\nabla}\left(\vec{H}+\vec{M}\right)=0$$

$$\Rightarrow \quad \vec{\nabla}.\vec{H}=-\vec{\nabla}.\vec{M}$$

$$\Rightarrow \quad \vec{\nabla}.\vec{H}=\rho_M$$

i.e. $\rho_M=-\vec{\nabla}.\vec{M}$ plays a role of magnetic charge density and $\vec{H}$ can be found.

In case (a) since $\vec{M}$ = constant

$$\Rightarrow \quad \rho_M=0$$

Therefore $\vec{H}=0$ everywhere in space.

$$\Rightarrow \quad \vec{B}=0 \text{ outside the slab and } \vec{B}=\mu_0\vec{H} \text{ inside the slab.}$$

In case (b) magnetization creates positive surface charge $\sigma_M=+M$ on the top surface and negative surface charge $\sigma_M=-M$ on the bottom surface. These charges generate magnetic field intensity $\vec{H}$ and it is given by $\vec{H}=-\vec{M}$ i.e. generated field is opposite to the magnetization within the slab and no field outside, $\vec{H}=0$. This makes magnetic flux density $\vec{B}=0$ everywhere in space.

Example 6 The interface $2x+y=8$ between two media carries no current. Medium $1\left(2x+y\geq 8\right)$ is nonmagnetic with $\vec{H}_1=-4\vec{a}_x+3\vec{a}_y-\vec{a}_z$ A/m. Find $\vec{M}_2$ and $\vec{B}_2$ in medium $2\left(2x+y\leq 8\right)$ with $\mu=10\mu_0$.

Solution: Let the surface of the plane be described by $f\left(x,y\right)=2x+y-8$, a unit vector normal to plane is given by

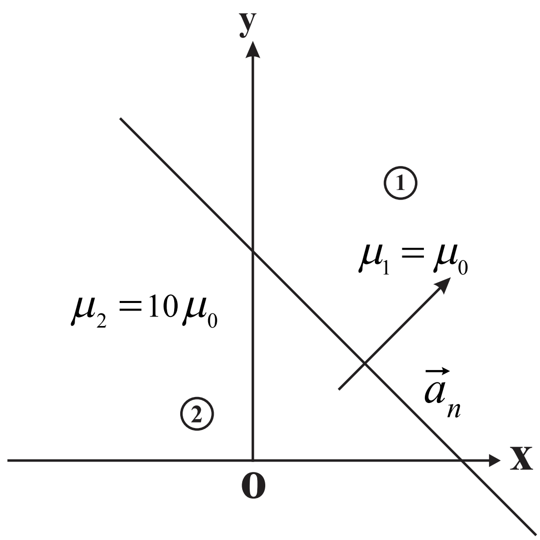


FIG. 8.12

$$\vec{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{2\vec{a}_x + \vec{a}_y}{\sqrt{5}}$$

$$\vec{H}_{in} = \left(\vec{H}_1 . \vec{a}_n\right)\vec{a}_n = \left[\left(-4,3,-1\right).\frac{2,1,0}{\sqrt{5}}\right]\left[\frac{2,1,0}{\sqrt{5}}\right]$$

$$= \left[\frac{-8+3}{\sqrt{5}}\right]\left[\frac{\vec{2}a_x + \vec{a}_y}{\sqrt{5}}\right]$$

$$\vec{H}_{in} = -2\vec{a}_x - \vec{a}_y$$

$$\vec{H}_{1t} = \vec{H}_1 - \vec{H}_{1n}$$

$$= -4\vec{a}_x + 3\vec{a}_y - \vec{a}_z - \left[-2\vec{a}_x - \vec{a}_y\right]$$

$$\vec{H}_{1t} = -2\vec{a}_x + 4\vec{a}_y - \vec{a}_z$$

The interface $2x + y = 8$ between two media carries no current, therefore $\vec{H}_{1t} = \vec{H}_{2t}$

$$\therefore \quad \vec{H}_{2t} = -2\vec{a}_x + 4\vec{a}_y - \vec{a}_z \tag{1}$$

As per boundary condition

$$\vec{B}_{1n} = \vec{B}_{2n} \Rightarrow \mu_1 \vec{H}_{1n} = \mu_2 \vec{H}_{2n}$$

$$\Rightarrow \qquad \vec{H}_{2n} = \frac{\mu_1}{\mu_2}\vec{H}_{1n} = \frac{1}{10}\left(-2\vec{a}_x - \vec{a}_y\right)$$

$$\vec{H}_{2n} = -0.2\vec{a}_x - 0.1\vec{a}_y \qquad (2)$$

From eqs. (1) and (2), we get

$$\vec{H}_2 = \vec{H}_{2t} + \vec{H}_{2n}$$

$$\vec{H}_2 = -2.2\vec{a}_x + 3.9\vec{a}_y - \vec{a}_z$$

$$\vec{M}_2 = \chi_{m2}\vec{H}_2 = \left(\mu_{r2} - 1\right)\vec{H}_2 = \left(10-1\right)\vec{H}_2$$

$$\vec{M}_2 = 9\left(-2.2\vec{a}_x + 3.9\vec{a}_y - \vec{a}_z\right)$$

$$\vec{M}_2 = -19.8\vec{a}_x + 35.1\vec{a}_y - 9\vec{a}_z \ A/m \qquad \text{Ans.}$$

$$\vec{B}_2 = \mu_2\vec{H}_2 = 10\mu_0\vec{H}_2 = 40\pi\times10^{-7}\vec{H}_2$$

$$= 4\pi\times10^{-6}\left(-2.2\vec{a}_x + 3.9\vec{a}_y - \vec{a}_z\right)$$

$$= 12.56\times10^{-6}\left(-2.2\vec{a}_x + 3.9\vec{a}_y - \vec{a}_z\right)$$

$$\vec{B}_2 = -27.63\vec{a}_x + 48.98\vec{a}_y - 12.56\vec{a}_z \ \mu\, Wb/m^2 \qquad \text{Ans.}$$

8.12 Self-Learning Exercise

Q.1 Define macroscopic effect.

Q.2 What is SI unit of magnetization.

Q.3 Write the relation between magnetic field intensity $\vec{H}$ and magnetization $\vec{M}$

Q.4 Define volume current density.

Q.5 How can arise magnetic dipole moments in the atoms of material.

Q.6 Why the magnetic dipole moment of a spinning nucleus is usually negligible.

Q.7 What is the surface current density.

Q.8 In an isotropic medium, magnetic induction $\vec{B}$ and magnetic field intensity $\vec{H}$ at the same point in space ………………………..

Q.9 What is relation between $\vec{B}$ and $\vec{H}$ for ferromagnetic materials.

Q.10 What is the magnetic flux density for magnetic materials.

Q.11 What are the examples of ferromagnetic materials.

Q.12 If the magnetic field intensity $\vec{H}$ is $4\vec{a}_x$ Amp/m, then calculate magnetic flux density in free space.

Q.13 If the normal component of $\vec{B}$ in medium 1 is $2.5\vec{a}_x \, Weber/m^2$, then calculate the normal component of $\vec{B}$ in medium 2.

Q.14 Define demagnetizing field and demagnetizing factor.

Q.15 Explain the magnetic shielding.

8.13 Summary

This unit starts with the introduction of magnetization and bound current densities. By giving the physical interpretation of bound currents, we have derived the macroscopic equations of magnetostatics. Here, we study about magnetic susceptibility, relative permeability and classification of magnetic materials. The boundary conditions on $\vec{B}$ and $\vec{H}$, methods of solving boundary value problems in magnetostatics, uniformly magnetized sphere in an external field, permanent magnetic shielding and spherical shell of permeable material in an uniform field have also been studied in this unit. In the end, some examples on above concept are given.

8.14 Glossary

Ferromagnetic : (Of a body or substance) having a high susceptibility to magnetization, the strength of which depends on that of the applied magnetizing field, and which may persist after removal of the applied field.

Shield: Prevent or reduce the effect of some physical quantity from (something):

8.15 Answer of Self-Learning Exercise

Ans.1 : It is the average over regions large enough to contain many atoms of matter.

Ans.2 : Ampere/meter.

Ans.3 : $\vec{M} = \chi \vec{H}$

Ans.4 : $\vec{J} = \vec{\nabla} \times \vec{M}$, a current throughout the material, when the magnetization is non-uniform.

Ans.5 : There are three sources (i) orbital motion of electrons (ii) electrons spin on their own axes and (iii) Nucleus of an atom spin on their own axis.

Ans.6 : Because of the much larger mass and lower angular velocity of the nucleus.

Ans.7 : $\vec{K} = \vec{M} \times \hat{n}$, a surface current on the boundary, when the magnetization is uniform.

Ans.8 : are parallel but differ in magnitude.

Ans.9 : $\vec{B} = F\left(\vec{H}\right)$ *i.e.* nonlinear functional relationship.

Ans.10 : $\vec{B} = \mu_0\left(\vec{H} + \vec{M}\right)$

Ans.11 : Iron, nickel, cobalt and their alloys.

Ans.12 : $1.6\pi \times 10^{-6}\, \vec{a}_x \ Weber/m^2$

Ans.13 : $2.5\vec{a}_x \ Weber/m^2$

Ans.14 : When uniformly magnetized sphere is placed in an external magnetic field, then the resultant magnetic field intensity at internal point is given by $\vec{H}_{in} = \vec{H}_0 - \frac{\vec{M}}{3}$. This equation shows the magnetization produces a reverse field inside the sphere, known as demagnetizing field and the factor $\left(\frac{1}{3}\right)$ is known as demagnetizing factor.

Ans.15 : Initially, there is a certain magnetic induction $\vec{B}_0 = \mu\vec{H}_0$ exists in a region of empty space. If a permeable body is now placed in the region, the lines of magnetic induction are modified. If the body is hollow, the field in the cavity will be smaller than the external field, vanishing in the limit $\mu \to \infty$. Such a reduction in field is said to be due to the magnetic shielding provided by the permeable material.

8.16 Exercise

Section A :Very Short Answer Type Questions

Q.1 Define magnetization vector.

Q.2 What are magnetization charge densities?

Q.3 Write the relationship between magnetic susceptibility and relative permeability.

Q.4 Write the macroscopic equations for magnetostatics.

Q.5 Define magnetic field intensity in terms of magnetization vector with SI unit.

Section –B: Short Answer type Questions

Q.6 What do you mean by magnetic field inside matter. (Hint: Macroscopic field)

Q.7 Define bound current densities with their physical interpretations.

Q.8 Write the classification of magnetic materials based on their magnetic behavior.

Q.9 What are the boundary conditions for magnetostatic fields at an interface between two different magnetic media. $\left[B_{n1} = B_{n2} \ and \ H_{t1} - H_{t2} = K \right]$

Q.10 The region $y < 0$ (region 1) in a air and y>0 (region 2) has $\mu_r = 10$. If there is a uniform magnetic field $\vec{H} = 5\vec{a}_x + 6\vec{a}_y + 7\vec{a}_z \ A/m$ in region 1. Find $\vec{H}$ and $\vec{B}$ in region 2.

$\left[H^2 = 5\vec{a}_x + 0.6\vec{a}_y + 7\vec{a}_z \ Amp/m \ and \ \vec{B}_2 = \mu_0 \left(50\vec{a}_x + 6\vec{a}_y + 70\vec{a}_z \right) Tesla \right]$

Section –C: Long Answer type Questions

Q.11 Derive the expression of magnetic vector potential $\left(\vec{A}\right)$ in terms of magnetization vector and explain the magnetization current densities.

Q.12 Explain magnetic field inside matter and derive the macroscopic equations for the same.

Q.13 What do you understand by the intensity of magnetization $\vec{M}$. Establish the relation $\vec{J} = \vec{\nabla} \times \vec{M}$ where $\vec{J}$ is current density in a non-uniformly magnetized material at a point where intensity of magnetization vector is $\vec{M}$.

Q.14 Derive the expression for boundary conditions for magnetic field at the interface of two different magnetic media with permeabilities μ_1 and μ_2, respectively, and also show that

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}$$

Where θ_1 and θ_2 are the angles the fields make with the normal to the interface.

Q.15 A magnetized sphere of radius R is placed in uniform external field $\vec{H}_0$. Find out the potential and field inside and outside the sphere.

Q.16 With the help of proper expression, explain the magnetic shielding provided by the permeable media.

8.17 Answers to Exercise

Ans.1 : It is magnetic dipole moment per unit volume.

Ans.2 : $\vec{J}_b = \vec{\nabla} \times \vec{M}$ *and* $\vec{k}_b = \vec{M} \times \vec{a}_n$ are the magnetization volume current density and surface current density, respectively.

Ans.3 : $\mu_r = 1 + \chi_m$

Ans.4 : $\vec{\nabla}.\vec{B} = 0$ *and* $\vec{\nabla} \times \vec{H} = \vec{J}_f$

Ans.5 : $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \quad \frac{Ampere}{meter}$

References and Suggested Readings

1. J.D. Jackson, Classical Electrodynamics, 2nd ed. Wiley Eastern Limited, 1978.
2. E.G. Cullwick, The Fundamentals of Electromagnetism, 3rd ed., Cambridge University Press, 1966.
3. W.K.H. Panofsky and M. Phillips, Classical Electricity and Magnetism, 2nd ed., Addison-Wesley, Reading Mass., 1962.
4. David Griffiths, Introduction to Electrodynamics, 2nd ed., Prentice-Hall, 1989.
5. Mathew N.O. Sadiku, Elements of Electromagnetics, 3rd ed., Oxford University Press, 2001.
6. J. Reitz, F. Milford and R. Christy, Foundations of Electromagnetic Theory, 4th ed., Addison-Wesley 1993.
7. P. Hammond, Applied Electromagnetism, Pergamon press, 1971.

UNIT - 9
Energy in the Magnetic Field , Gauge Transformation

Structure of the Unit

9.0 Objectives

In the previous chapter, we discussed about the magnetic field inside the material. In this chapter, first we will discuss in detail about the energy and energy density in magnetic field. This chapter also covers vector and scalar potential, gauge transformation, Lorentz gauge and Coulomb gauge.

9.1 Introduction

In the previous chapters, we discussed the problems related to steady state magnetic field and did not consider the question of field energy and energy density. The reason was that the creation of a steady state configuration of currents and associated magnetic fields involves an initial momentary period during which the currents and fields are brought from zero to the final values. For such time varying field, there is induced electromotive force (emf) in circuit that opposes the current change. An amount of work must be done to overcome this induced emf. Since the energy in the field is the total work done to establish it. Therefore it can be regarded as energy stored in magnetic field. In the previous chapters, electric and magnetic phenomena were also treated as independent. The independent nature of electric and magnetic phenomena disappears when we consider time dependent problems. Faraday's Law of induction and modified Ampere's Law destroyed the independence. Time-varying magnetic fields give rise to electric fields and vice versa. Therefore we must speak of electromagnetic fields rather than electric and magnetic fields. The set of four equations known as the Maxwell equations describe the behaviour of electromagnetic fields. These equations are as follows:-

$$\boxed{\vec{\nabla}.\vec{D}=\rho} \quad , \quad \boxed{\vec{\nabla}.\vec{B}=0}$$

$$\boxed{\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}} \quad , \quad \boxed{\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}}$$

9.2 Energy in the Magnetic Field

When a current flowing through the circuit, an emf (electromotive force) induced in it opposing the change in the circuit and the applied voltage must overcome this induced emf, if the change in current is to be maintained. Now let the current increases from 0 to I (final value) Amperein t second, then the work done in establishing this current in the circuit is

$$W_m=-\int emf\left(i\right)dt=L\int_0^I i\,di$$

$$\boxed{W_m=\frac{1}{2}L\,I^2=\frac{1}{2}\phi_m I} \tag{1}$$

where $emf = -\frac{d\phi_m}{dt} = -L\frac{di}{dt}$ and the minus sign records the fact this is the work done against the emf. This work is stored as energy of the magnetic field. This energy is released when the current is brought down to zero again. It depends only the geometry of the loop (in the form of *L*) and the final current *I*.

Equation (1)can be generalized to determine the magnetic energy of a continuous distribution of current within a volume. Now consider a current-carrying loop of closed path *C* and bounded surface *S*. The flux ϕ_m is linked with the circuit due to current *I* in itself and is given by

$$\phi_m = \int_s \vec{B}.d\vec{S} = \int_s \left(\vec{\nabla} \times \vec{A}\right).d\vec{S} \tag{2}$$

Since $\boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$

$$\phi = \oint_c \vec{A}.d\vec{l} \text{ [using Stoke's theorem]} \tag{3}$$

From eqs. (1) and (3), we get

$$W_m = \frac{1}{2} I \oint_c \vec{A}.d\vec{l} \tag{4}$$

The vector sign might as well go on the I, then eq. (4) can be written as

$$W_m = \frac{1}{2} \oint_c \left(\vec{A}.\vec{I}\right) dl \tag{5}$$

It is work done in case of linear circuit. The generalization of eq. (5) to volume current is given by

$$\boxed{W_m = \frac{1}{2} \int_v \left(\vec{A}.\vec{J}\right) dv} \tag{6}$$

It is often desirable to express the magnetic energy in term of field quantities $\vec{B}$ and $\vec{H}$ instead of current density $\vec{J}$ and vector potential $\vec{A}$. Therefore from Ampere's Law

$$\vec{\nabla} \times \vec{H} = \vec{J} \text{ or } \vec{\nabla} \times \vec{B} = \mu \vec{J}$$

$$\Rightarrow \qquad \vec{J} = \frac{1}{\mu}\left(\vec{\nabla}\times\vec{B}\right) \tag{7}$$

Put the value of $\vec{J}$ from eq. (7) into eq. (6), we get

$$W_m = \frac{1}{2\mu}\int_v \vec{A}.\left(\vec{\nabla}\times\vec{B}\right)dv \tag{8}$$

Making use of the vector identity

$$\vec{\nabla}.\left(\vec{A}\times\vec{B}\right) = \vec{B}.\left(\vec{\nabla}\times\vec{A}\right) - \vec{A}.\left(\vec{\nabla}\times\vec{B}\right)$$

$$\Rightarrow \qquad \vec{A}.\left(\vec{\nabla}\times\vec{B}\right) = \vec{B}.\left(\vec{\nabla}\times\vec{A}\right) - \vec{\nabla}.\left(\vec{A}\times\vec{B}\right) \tag{9}$$

Substituting eq. (9) into eq. (8), we obtain

$$W_m = \frac{1}{2\mu}\int_v \vec{B}.\left(\vec{\nabla}\times\vec{A}\right)dv - \frac{1}{2\mu}\int_v \vec{\nabla}.\left(\vec{A}\times\vec{B}\right)dv$$

In the first term put $\vec{B} = \vec{\nabla}\times\vec{A}$ and apply divergence theorem in second term, we get

$$W_m = \frac{1}{2\mu}\int_v \vec{B}.\vec{B}\,dv - \frac{1}{2\mu}\oint_s \left(\vec{A}\times\vec{B}\right).d\vec{S}$$

$$W_m = \frac{1}{2}\int_v \vec{H}.\vec{B}\,dv - \frac{1}{2}\oint \left(\vec{A}\times\vec{H}\right).d\vec{S} \tag{10}$$

If v is taken to be sufficiently large, the points on its surface S will be very far from the currents. At those far-away points, the contribution of the surface integral in eq. (10) tends to zero because $\left|\vec{A}\right|$ falls off as $\frac{1}{R}$ and $\left|\vec{H}\right|$ falls off as $\frac{1}{R^2}$. Thus, the magnitude of $\left(\vec{A}\times\vec{H}\right)$ decreases as $\frac{1}{R^3}$, whereas at the same time, the surface S increases only as R^2. When R approaches infinity, the surface integral in eq. (10) vanishes. We have then

$$\boxed{W_m = \frac{1}{2}\int_v \vec{H}.\vec{B}\ dv} \qquad (Joule) \tag{11}$$

Since $\vec{B} = \mu \vec{H}$, we can write eq. (11) in two alternative forms

$$\boxed{W_m = \frac{1}{2}\int_v \frac{B^2}{\mu} dv} \qquad (Joule) \qquad (12)$$

and $$W_m = \frac{1}{2}\int_v \mu H^2 dv \qquad (Joule) \qquad (13)$$

The expressions in eqs. (11), (12) and (13) are for the magnetic energy in a linear medium. If we define a magnetic energy density ω_m, such that its volume integral equals to total magnetic energy $W_m = \int_v \omega_m dv$ (14)

We can write magnetic energy density (ω_m) in three forms:

$$\boxed{\omega_m = \frac{1}{2}\vec{H}.\vec{B}} \qquad (J/m^3) \qquad (15)$$

Or $$\boxed{\omega_m = \frac{B^2}{2\mu}} \qquad (J/m^3) \qquad (16)$$

Or $$\omega_m = \frac{1}{2}\mu H^2 \qquad (J/m^3) \qquad (17)$$

9.3 Scalar and Vector Potentials or Potential Functions

The Maxwell equations consist of a set of coupled first order partial differential equations relating the various components of electric and magnetic fields. These are as follows:-

(i) $\vec{\nabla}.\vec{E} = \frac{\rho}{\in}$ (Gauss's law)

(ii) $\vec{\nabla}.\vec{B} = 0$ (Gauss's law for magnetostatics or non-existence of magnetic monopole)

(iii) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's law)

(iv) $\vec{\nabla}\times\vec{B}=\mu\vec{J}+\mu\in\frac{\partial\vec{E}}{\partial t}$ (Ampere's modified law)

We are already familiar with the concept of the scalar potential ϕ and the vector potential $\vec{A}$ in electrostatics and magnetostatics, respectively. Since in electrostatics $\vec{\nabla}\times\vec{E}=0$ allowed us to write $\vec{E}$ as gradient of a scalar potential *i.e.* $\vec{E}=-\nabla\phi$. In electrodynamics this is no longer possible, because the curl of $\vec{E}$ is non zero as per eq. (iii). But $\vec{\nabla}.\vec{B}=0$ still hold, we can define $\vec{B}$ in terms of a vector potential *i.e.*

$$\boxed{\vec{B}=\nabla\times\vec{A}} \qquad \text{(Tesla)} \qquad (1)$$

Putting this into Faraday's law yields

$$\vec{\nabla}\times\vec{E}=-\frac{\partial}{\partial t}\left(\nabla\times\vec{A}\right)$$

$$\Rightarrow \quad \vec{\nabla}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=0 \qquad (2)$$

This means that the quantity with vanishing curl can be written as the gradient of a scalar potential ϕ *i.e.*

$$\vec{E}+\frac{\partial\vec{A}}{\partial t}=-\vec{\nabla}\phi$$

$$\Rightarrow \quad \boxed{\vec{E}=-\vec{\nabla}\phi-\frac{\partial\vec{A}}{\partial t}} \qquad \left(V/m\right) \qquad (3)$$

The potential representations given by eqs. (1) and (3) automatically fulfil the two homogeneous Maxwell equations (ii) and (iii). The dynamic behavior of $\vec{A}$ and ϕ will be determined by the two inhomogeneous Maxwell equations (i) and (iv).

The electric field in eq. (3) can be viewed as composed of two parts: the first part $\left(-\vec{\nabla}\phi\right)$ is due to charge distribution ρ and the second part $\left(-\frac{\partial\vec{A}}{\partial t}\right)$ is due to time varying current $\vec{J}$. The scalar and vector potentials are given as

$$\phi = \frac{1}{4\pi\varepsilon} \int_{v'} \frac{\rho}{R} dv' \tag{4}$$

and $$\boxed{\vec{A} = \frac{\mu}{4\pi} \int_{v'} \frac{\vec{J}}{R} dv'} \tag{5}$$

These are the solutions of Poisson's equation for electrostatics and magnetostatics, respectively. These solutions may themselves be time dependent because ρ and $\vec{J}$ may be functions of time, but they neglect the time retardation effects associated with the finite velocity of propagation of time varying electromagnetic fields. When ρ and $\vec{J}$ vary slowly with time at very low frequency and the range of interest R is small in comparison with the wavelength, it is allowable to use eqs. (4) and (5) in eqs. (1) and (3) to find quasi-static fields.

Quasi-static fields are approximations. Their consideration leads from field theory to circuit theory. However, when the source frequency is high and the range of interest is no longer small in comparison to the wavelength, quasi-static solutions will not suffice. Time-retardation effects must then be included, as in the case of electromagnetic radiation from antennas.

9.4 Gauge Transformation

The transformation relations under which the physical quantities $\vec{E}$ and $\vec{B}$ are always unchanged, are known as gauge transformations.

Putting eqs. (3) into Gauss's law (i), we find that

$$\nabla^2 \phi + \frac{\partial}{\partial t}\left(\vec{\nabla}.\vec{A}\right) = -\frac{\rho}{\varepsilon} \tag{6}$$

Putting eqs. (1) and (3) into Ampere's modified law (iv), we find

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \mu\vec{J} - \mu\varepsilon\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right) - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2}$$

Using the vector identity $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}\left(\vec{\nabla}.\vec{A}\right) - \nabla^2\vec{A}$, we get

$$\nabla^2\vec{A} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J} + \vec{\nabla}\left(\vec{\nabla}.\vec{A} + \mu\varepsilon\frac{\partial\phi}{\partial t}\right) \tag{7}$$

We have now reduced the set of four Maxwell equations to two equations. But they are still coupled equations. The uncoupling can be accomplished by exploiting the arbitrariness involved in the definition of the potentials. Since eqs. (1) and (3) do not uniquely define the potentials. Therefore we are free to impose extra conditions on ϕ and $\vec{A}$, which are not changing $\vec{E}$ and $\vec{B}$. Suppose we have two sets of potentials $(\phi, \vec{A})$ and $(\phi', \vec{A}')$, which correspond to the same electric and magnetic fields. Therefore the potentials may be written as

$$\vec{A}' = \vec{A} + \vec{\alpha} \tag{8}$$

and $$\phi' = \phi + \beta \tag{9}$$

Since the two vector potentials give the same $\vec{B}$, their curls must be equal and hence

$$\vec{\nabla} \times \vec{\alpha} = 0$$

$$\Rightarrow \quad \vec{\alpha} = \vec{\nabla} \wedge \tag{10}$$

The two scalar potentials also give the same $\vec{E}$, so

$$\vec{\nabla} \beta + \frac{\partial \vec{\alpha}}{\partial t} = 0$$

Using eq. (10), we get

$$\vec{\nabla}\left(\beta + \frac{\partial \wedge}{\partial t} \right) = 0$$

$$\Rightarrow \quad \beta = -\frac{\partial \wedge}{\partial t}$$

Therefore the vector and scalar potentials can be written as

$$\boxed{\vec{A} = A + \vec{\nabla} \wedge} \tag{12}$$

and $$\boxed{\phi' = \phi - \frac{\partial \wedge}{\partial t}} \tag{13}$$

The transformation relations given by eqs. (12) and (13) are called ***gauge transformation*** and the arbitrary scalar function $\wedge$ is called the gauge function.

The physical quantities $\vec{E}$ and $\vec{B}$ are unchanged under the gauge transformation. This invariance of fields is called gauge invariance.

9.5 Lorentz Gauge

Maxwell equation in terms of electromagnetic potentials are given by

$$\left(\nabla^2\vec{A}-\mu\varepsilon\frac{\partial^2\vec{A}}{\partial t}\right)-\vec{\nabla}\left(\vec{\nabla}.\vec{A}+\mu\varepsilon\frac{\partial\phi}{\partial t}\right)=-\mu\vec{J} \qquad (1)$$

and $$\nabla^2\phi+\frac{\partial}{\partial t}\left(\vec{\nabla}.\vec{A}\right)=-\rho/\varepsilon \qquad (2)$$

Since the curl of $\vec{A}$ is designed $\vec{B}\left(\vec{B}=\nabla\times\vec{A}\right)$, we are still at liberty to choose the divergence of $\vec{A}$. There are various ways to choose the $\vec{\nabla}.\vec{A}$. These ways are known as various gauge transformations.

Let $$\boxed{\vec{\nabla}.\vec{A}+\mu\varepsilon\frac{\partial\phi}{\partial t}=0} \qquad (3)$$

This relation between $\vec{A}$ and ϕ is called the Lorentz condition and the gauge is known as Lorentz gauge.

Using Lorentz condition, the eqs. (1) and (2) reduce to

$$\nabla^2\vec{A}-\mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2}=-\mu\vec{J} \qquad (4)$$

and $$\nabla^2\phi-\mu\varepsilon\frac{\partial^2\phi}{\partial t^2}=-\rho/\varepsilon \qquad (5)$$

Hence the Lorentz condition uncouples the wave equations for $\vec{A}$ and for ϕ. Eqs. (4) and (5) are the non-homogeneous or inhomogeneous wave equations for vector potential $\left(\vec{A}\right)$ and scalar potential $\left(\phi\right)$, respectively. These are called wave equations because their solutions represent waves travelling with a velocity equal to $\frac{1}{\sqrt{\mu\varepsilon}}$. Eqs. (3), (4) and (5) form a set of equations equivalent in all respects to Maxwell equations.

Since $\mu\varepsilon = \frac{1}{v^2}$ where v is the phase velocity of wave. In free space $\mu = \mu_0$ and $\varepsilon = \varepsilon_0$, therefore $\mu_0\varepsilon_0 = \frac{1}{c^2}$ where c is the speed of light.

Introducing D' Alembertian or D' Alembert's operator

$$\boxed{\Box^2 = \nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}} \tag{6}$$

The eqs. (4) and (5) take the form

$$\boxed{\Box^2 \vec{A} = -\mu\vec{J}} \tag{7}$$

$$\boxed{\Box^2 \phi = -\rho/\varepsilon} \tag{8}$$

The Lorentz gauge is commonly used because

1) It leads to uncoupled wave equations for potential $\vec{A}$ and ϕ, which treat $\vec{A}$ and ϕ an equivalent footings.
2) It is a concept which is independent of the coordinate system chosen and so fits naturally into the considerations of special relativity.

9.6 Coulomb Gauge (Transverse or Radiation Gauge)

The Maxwell equations in terms of electromagnetic potential are

$$\left(\nabla^2\vec{A} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}.\vec{A} + \mu\varepsilon\frac{\partial\phi}{\partial t}\right) = -\mu\vec{J} \tag{1}$$

and $$\nabla^2\phi + \frac{\partial}{\partial t}\left(\vec{\nabla}.\vec{A}\right) = -\rho/\varepsilon \tag{2}$$

The coulomb gauge restricts the divergence of $\vec{A}$ as $\boxed{\vec{\nabla}.\vec{A} = 0}$ (3)

Form eq. (2), we see that the scalar potential satisfies the Poisson's equation

$$\boxed{\nabla^2\phi = -\rho/\varepsilon} \tag{4}$$

The solution of eq. (4) is given by

$$\phi(\vec{r},t)=\frac{1}{4\pi\varepsilon}\int_{v'}\frac{\rho(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}dv' \tag{5}$$

Thus the scalar potential is just the instantaneous coulomb potential due to the charge density $\rho(\vec{r},t)$. This is the origin of the name "***Coulomb gauge***".

Using Coulomb gauge, equation of vector potential becomes

$$\left(\nabla^2\vec{A}-\mu\varepsilon\frac{\partial^2\vec{A}}{\partial t}\right)-\vec{\nabla}\left(\mu\varepsilon\frac{\partial\phi}{\partial t}\right)=-\mu\vec{J} \tag{6}$$

Since $\mu\varepsilon=\frac{1}{v^2}$, where v is the phase velocity of wave, eq. (6) may be written as

$$\nabla^2\vec{A}-\frac{1}{v^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu\vec{J}+\mu\varepsilon\vec{\nabla}\left(\frac{\partial\phi}{\partial t}\right) \tag{7}$$

This equation may be put in a convenient form by using Poisson's equation (4) with the help of eq. (5) may be expressed as

$$\nabla^2\left[\frac{1}{4\pi\varepsilon}\int\frac{\rho(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}dv'\right]=-\frac{\rho(\vec{r}\,',t)}{\varepsilon} \tag{8}$$

Since Poisson's equation holds for scalars and vectors both, therefore replacing scalar potential source $\rho(\vec{r}\,',t)$ by vector potential source $\vec{J}(\vec{r}\,',t)$, we get

$$\nabla^2\left[\frac{1}{4\pi\varepsilon}\int\frac{\vec{J}(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}dv'\right]=\frac{-\vec{J}(\vec{r}\,',t)}{\varepsilon} \tag{9}$$

Let $$\left[\int\frac{\vec{J}(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}dv'\right]=\vec{G}(\vec{r}\,',t) \tag{10}$$

then we obtain

$$\nabla^2\vec{G}(\vec{r}\,',t)=-4\pi\vec{J}(\vec{r}\,',t) \tag{11}$$

Using the vector identity

$$\vec{\nabla}\times\vec{\nabla}\times\vec{G}=\vec{\nabla}(\vec{\nabla}.\vec{G})-\nabla^2\vec{G}$$

$$\Rightarrow \quad \nabla^2\vec{G} = \vec{\nabla}\left(\vec{\nabla}.\vec{G}\right) - \vec{\nabla}\times\vec{\nabla}\times\vec{G} \tag{12}$$

From eqs. (11) and (12), we get

$$\vec{\nabla}\left(\vec{\nabla}.\vec{G}\right) - \vec{\nabla}\times\vec{\nabla}\times\vec{G} = -4\pi\vec{J} \tag{13}$$

Put the value of $\vec{G}$ from eq. (10) into equation (13), we get

$$\vec{\nabla}\left[\vec{\nabla}.\int\frac{\vec{J}\left(\vec{r}\,',t\right)}{\left|\vec{r}-\vec{r}\,'\right|}dv'\right] - \vec{\nabla}\times\vec{\nabla}\times\int\frac{\vec{J}\left(\vec{r}\,',t\right)}{\left|\vec{r}-\vec{r}\,'\right|}dv' = -4\pi\vec{J}\left(\vec{r}\,',t\right)$$

$$\Rightarrow \vec{J}\left(\vec{r}\,',t\right) = -\frac{1}{4\pi}\vec{\nabla}\left[\vec{\nabla}.\int\frac{\vec{J}\left(\vec{r}\,',t\right)}{\left|\vec{r}-\vec{r}\,'\right|}dv'\right] + \frac{1}{4\pi}\vec{\nabla}\times\vec{\nabla}\times\int\frac{\vec{J}\left(\vec{r}\,',t\right)}{\left|\vec{r}-\vec{r}\,'\right|}dv' \tag{14}$$

The term $\vec{\nabla}.\int\frac{\vec{J}\left(\vec{r}\,',t\right)}{\left|\vec{r}-\vec{r}\,'\right|}dv'$ may be written as

$$\vec{\nabla}.\int\frac{\vec{J}\left(\vec{r}\,',t\right)}{\left|\vec{r}-\vec{r}\,'\right|}dv' = \int\vec{\nabla}.\frac{\vec{J}\left(\vec{r}\,',t\right)}{\left|\vec{r}-\vec{r}\,'\right|}dv' = -\int\vec{J}\left(\vec{r}\,',t\right).\left(\vec{\nabla}'\frac{1}{\left|\vec{r}-\vec{r}\,'\right|}\right)dv' \tag{15}$$

Since $\vec{\nabla}\frac{1}{\left|\vec{r}-\vec{r}\,'\right|} = -\vec{\nabla}'\frac{1}{\left|\vec{r}-\vec{r}\,'\right|}$

Now using vector identity $\vec{\nabla}.\left(f\vec{F}\right) = f\vec{\nabla}.\vec{F} + \vec{F}.\left(\nabla f\right)$

where $f = \frac{1}{\left|\vec{r}-\vec{r}\,'\right|}$ and $\vec{F} = \vec{J}\left(\vec{r}\,',t\right)$

Hence equation (15) may be written as

$$\vec{\nabla}.\int\frac{\vec{J}\left(\vec{r}\,',t\right)}{\left|\vec{r}-\vec{r}\,'\right|}dv' = -\int\left[\vec{\nabla}'.\left(\frac{\vec{J}\left(\vec{r}\,',t\right)}{\left|\vec{r}-\vec{r}\,'\right|}\right) - \frac{\vec{\nabla}'.\vec{J}\left(\vec{r}\,',t\right)}{\left|\vec{r}-\vec{r}\,'\right|}\right]dv'$$

$$= -\int\vec{\nabla}'.\frac{\vec{J}\left(\vec{r}\,',t\right)}{\left|\vec{r}-\vec{r}\,'\right|}dv' + \int\frac{\vec{\nabla}'.\vec{J}\left(\vec{r}\,',t\right)}{\left|\vec{r}-\vec{r}\,'\right|}dv'$$

Using Gauss's divergence theorem in first term, we get

$$\vec{\nabla}.\int \frac{\vec{J}(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}dv' = -\oint_s \frac{\vec{J}(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}.d\vec{S} + \int \frac{\vec{\nabla}.\vec{J}(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}dv' \tag{16}$$

The first term gives zero value as $\vec{J}$ vanishes on the surface, therefore eq. (16) becomes

$$\vec{\nabla}.\int \frac{\vec{J}(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}dv' = \int \frac{\vec{\nabla}'.\vec{J}(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}dv' \tag{17}$$

Putting this value in eq. (14), we get

$$\vec{J}(\vec{r}\,',t) = -\frac{1}{4\pi}\vec{\nabla}\int \frac{\vec{\nabla}'.\vec{J}(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}dv' + \frac{1}{4\pi}\vec{\nabla}\times\vec{\nabla}\times\int \frac{\vec{J}(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}dv'$$

$$\Rightarrow \quad \vec{J}(\vec{r}\,',t) = \vec{J}_l + \vec{J}_t \tag{18}$$

where $$\vec{J}_l = -\frac{1}{4\pi}\vec{\nabla}\int \frac{\vec{\nabla}'.\vec{J}(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}dv' \tag{19}$$

and $$\vec{J}_t = \frac{1}{4\pi}\vec{\nabla}\times\vec{\nabla}\times\int \frac{\vec{J}(\vec{r}\,',t)}{|\vec{r}-r'|}dv' \tag{20}$$

We note that $\vec{\nabla}\times\vec{J}_l = -\frac{1}{4\pi}\vec{\nabla}\times\vec{\nabla}\int \frac{\vec{\nabla}'.\vec{J}(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}dv'$

$\Rightarrow \vec{\nabla}\times\vec{J}_l = 0$ (since curl of gradient of any scalar function always vanishes) (21)

and $$\vec{\nabla}.\vec{J}_t = \frac{1}{4\pi}\vec{\nabla}.\left[\vec{\nabla}\times\vec{\nabla}\times\int \vec{J}\frac{(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}dv'\right]$$

$$\vec{\nabla}.\vec{J}_t = \frac{1}{4\pi}\ \text{div curl}\left(\vec{\nabla}\times\int \frac{\vec{J}(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}dv'\right) = 0 \tag{22}$$

Since divergence of curl of any vector always vanishes.

From eqs. (21) and (22) it is clear that $\vec{\nabla}\times\vec{J}_l = 0$ and $\vec{\nabla}.\vec{J}_t = 0$. Hence $\vec{J}_l$ and

$\vec{J}_t$ are also known as longitudinal (irrotational) and transverse (solenoid) current density, respectively.

Putting the values of ϕ and $\vec{J}$ from eqs. (5) and (18) into eq. (7), we get

$$\nabla^2\vec{A}-\frac{1}{v^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu\left(\vec{J}_l+\vec{J}_t\right)+\mu\varepsilon\vec{\nabla}\left[\frac{\partial}{\partial t}\left(\frac{1}{4\pi\varepsilon}\int\frac{\rho\left(\vec{r}\,',t\right)}{\left|\vec{r}-\vec{r}\,'\right|}\right)dv'\right] \qquad (23)$$

Using continuity equation

$$\vec{\nabla}'.\vec{J}\left(\vec{r}\,',t\right)+\frac{\partial\rho\left(\vec{r}\,',t\right)}{\partial t}=0$$

$$\Rightarrow \quad \frac{\partial\rho\left(\vec{r}\,',t\right)}{\partial t}=-\vec{\nabla}'.\vec{J}\left(\vec{r}\,',t\right) \qquad (24)$$

From eqs. (23) and (24), we get

$$\nabla^2\vec{A}-\frac{1}{v^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu\left(\vec{J}_l+\vec{J}_t\right)-\frac{\mu}{4\pi}\vec{\nabla}\int\frac{\vec{\nabla}'.\vec{J}\left(\vec{r}\,',t\right)}{\left|\vec{r}-\vec{r}\,'\right|}dv'$$

Use eq. (19), we get

$$\nabla^2\vec{A}-\frac{1}{v^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu\left(\vec{J}_l+\vec{J}_t\right)-\mu\vec{J}_l$$

$$\Rightarrow \quad \nabla^2\vec{A}-\frac{1}{v^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu\vec{J}_t \qquad (25)$$

Or $\quad \Box^2\vec{A}=-\mu\vec{J}_t$

where $\Box^2=\nabla^2-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}$

This equation indicates that the source of wave equation for vector potential $\vec{A}$ can be expressed in terms of the transverse current density $\left(\vec{J}_t\right)$. This is the origin of the name "transverse gauge." The name "radiation gauge" stems from the fact that transverse radiation field are given by the vector potential alone. Since the instantaneous coulomb potential ϕ contributes only to near fields. Thus coulomb gauge allows separation of "near" and "radiation" field. The coulomb gauge is often used when there is no source. Then $\phi = 0$, and $\vec{A}$ satisfies the homogeneous equation $\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = 0$ with the fields given by

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} \text{ and } \vec{B} = \nabla \times \vec{A}. \tag{26}$$

9.7 Illustrative Examples

Example 1: A very long solenoid 2×2 cm cross section has an iron core $(\mu_r = 1000)$ and 4000 turns/meter. If it carries a current of 500 mA, find the energy per meter stored in its field.

Solution: Given $S = 2 \times 2 \times 10^{-4}\, m^2$, $\mu_r = 1000$, $n = 4000$ turns/meter and $I = 500 \times 10^{-3}\ Amp$. The magnetic energy is given by $W = \frac{1}{2} L I^2$

where L is the inductance. The inductance per unit length is $L' = \frac{L}{l} = \mu n^2 S$, therefore energy stored per meter in field is given by

$$W = \frac{1}{2} L' I^2$$

$$= \frac{1}{2} \mu_0 \mu_r n^2 S I^2$$

$$= \frac{1}{2} 4\pi \times 10^{-7} \times 1000 \times (4000)^2 \times 2 \times 2 \times 10^{-4} \times \left(500 \times 10^{-3}\right)^2$$

$$W = 2\pi \times 10^{-4} \times 16 \times 4 \times 25$$

$$W = 1.0048\ Joule / meter$$

Example 2 Calculate the energy in joules stored in a magnetic field of a solenoid 30 *cm* long and 3 *cm* in diameter, wound with 100 turns of wire and carrying a current of 10 Amp.

Solution: Given $l = 30\,cm$, $r = 1.5\,cm$, $N = 100$ turns and $I = 10$ Amperes.

The energy stored in the magnetic field is given by

$$W = \frac{1}{2} L I^2$$

where $L = \frac{\mu_0 N^2 S}{l}$ is the inductance

$$\therefore \qquad W = \frac{1}{2} \frac{\mu_0 N^2 \pi r^2}{l} I^2$$

$$= \frac{1}{2} \frac{4\pi \times 10^{-7} \times 10^4 \times 3.14 \times \left(1.5 \times 10^{-2}\right)^2}{30 \times 10^{-2}} \times 100$$

$$W = 1.47 \times 10^{-3} \text{ Joules}$$

Example 3: In a certain material for which $\mu = 6.5\mu_0$ and $\vec{H} = 10\vec{a}_x + 25\vec{a}_y - 40\vec{a}_z A/m,$ calculate the magnetic energy density.

Solution: Given $\mu = 6.5\mu_0$ and $\vec{H} = 10\vec{a}_x + 25\vec{a}_y - 40\vec{a}_z A/m$. The magnetic density

$$\omega = \frac{1}{2} \vec{B}.\vec{H} \qquad (1)$$

$$\vec{B} = \mu \vec{H} = 6.5 \times 4\pi \times 10^{-7} \left(10\vec{a}_x + 25\vec{a}_y - 40\vec{a}_z\right)$$

$$= 81.64 \left(10\vec{a}_x + 25\vec{a}_y - 40\vec{a}_z\right) \times 10^{-7}$$

$$\vec{B} = \left(816.4\vec{a}_x + 2041\vec{a}_y - 3265.6\vec{a}_z\right) \times 10^{-7} \qquad (2)$$

Putting the value of $\vec{B}$ and $\vec{H}$ into eq. (1) , we get

$$\omega = \frac{1}{2}\left[\left(816.4\vec{a}_x + 2041\vec{a}_y - 3265.6\vec{a}_z\right) \times 10^{-7}.\left(10\vec{a}_x + 25\vec{a}_y - 40\vec{a}_z\right)\right]$$

$$= \frac{1}{2}\left[8164 + 51025 + 130624\right] \times 10^{-7}$$

$$w = 94906.5 \times 10^{-7}$$

$$= 9.49 \times 10^{-3} \; Joule/m^3$$

Example 4 Show that the electromagnetic potentials at the position defined by the vector $\vec{r}$ in uniform electric and magnetic fields may be written as

$$\phi = -\vec{E}.\vec{r} \text{ and } A = \frac{1}{2}\left(\vec{B} \times \vec{r}\right)$$

Solution: Let $\vec{E}$ be the electric field, then $\vec{E} = \hat{i}E + \hat{j}E_y + \hat{k}E_z$

$$\Rightarrow \quad \vec{E} = \left(\vec{E}.\vec{\nabla}\right)\vec{r}$$

As $\vec{E}$ is uniform, we can write

$$\vec{E} = \vec{\nabla}\left(\vec{E}.\vec{r}\right) \tag{1}$$

In case of electrostatic field

$$E = -\vec{\nabla}\phi \tag{2}$$

Compare eq. (1) and (2), we get

$$\Rightarrow \quad \phi = -\vec{E}.\vec{r} \qquad \text{Ans.} \tag{3}$$

For second result, use vector identity (vector triple product)

$$\text{Curl } \left(\vec{C} \times \vec{D}\right) = \vec{C} \text{ div } \vec{D} - \left(\vec{C}.grad\right)\vec{D}$$

Put $\vec{C} = \vec{B}$ and $\vec{D} = \vec{r}$, we find

$$\text{curl } \left(\vec{B} \times \vec{r}\right) = \vec{B} \text{ div } \vec{r} - \left(\vec{B}.grad\right)\vec{r}$$

$$= \vec{B}\vec{\nabla}.\vec{r} - \left(\vec{B}.\vec{\nabla}\right)\vec{r}$$

$$\text{Curl } \left(\vec{B} \times \vec{r}\right) = \vec{B}(3) - \vec{B} = 2\vec{B}$$

$$\Rightarrow \quad \vec{B} = \frac{1}{2} \, curl \, \left(\vec{B} \times \vec{r}\right) \tag{4}$$

Form the definition of $\vec{A}$

$$\vec{B} = \text{ curl } \vec{A} \tag{5}$$

Compare eq. (4) and (5), we get

$$\Rightarrow \quad \vec{A} = \frac{1}{2}\left(\vec{B} \times \vec{r}\right) \qquad \text{Ans.}$$

Example 5 Find the electric and magnetic fields corresponding to $\phi = 0$ and $\vec{A} = A_0 \sin\left(kx - wt\right)\vec{a}_y$. Are they satisfied Maxwell's equation in vacuum? What will be the relation between *w* and *k*. Also prove that these potentials are in the coulomb and Lorentz gauge.

Solution : Given $\phi = 0$ and $\vec{A} = A_0 \sin(kx - wt)\vec{a}_y$

Since $\vec{E} = -\nabla\phi - \dfrac{\partial \vec{A}}{\partial t}$

$$= -A_0 \cos(kx - wt)\vec{a}_y(-w)$$

$$\vec{E} = A_0 w \cos(kx - wt)\vec{a}_y$$

and $\vec{B} = \nabla \times \vec{A}$

$$= \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & A_0 \sin(kx - wt) & 0 \end{vmatrix}$$

$$= \vec{a}_z \frac{\partial}{\partial x}\left[A_0 \sin(kx - wt)\right]$$

$$\Rightarrow \quad \vec{B} = A_0 k \cos(kx - wt)\vec{a}_z$$

Hence $\vec{\nabla}.\vec{E} = 0$ and $\vec{\nabla}.\vec{B} = 0$ (1)

$$\vec{\nabla} \times \vec{E} = \vec{a}_z \frac{\partial}{\partial x}\left[A_0 w \cos(kx - wt)\right] = -A_0 wk \sin(kx - wt)\vec{a}_z$$

$$-\frac{\partial \vec{B}}{\partial t} = -A_0 wk \sin(kx - wt)\vec{a}_z$$

Therefore $\vec{\nabla} \times \vec{E} = \dfrac{-\partial \vec{B}}{\partial t}$ (2)

$$\vec{\nabla} \times \vec{B} = -\vec{a}_y \frac{\partial}{\partial x}\left[A_0 k \cos(kx - wt)\right]$$

$$= A_0 k^2 \sin(kx - wt)\vec{a}_y$$

$$\frac{\partial \vec{E}}{\partial t} = A_0 w^2 \sin(kx - wt)\vec{a}_y$$

So $\quad \vec{\nabla}\times\vec{B}=\mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t}$ (3)

$$\Rightarrow \quad A_0k^2\sin\left(kx-wt\right)\vec{a}_y=\mu_0\varepsilon_0A_0w^2\sin\left(kx-wt\right)\vec{a}_y$$

$$\Rightarrow \quad k^2=\frac{w^2}{c^2} \qquad \text{Since } c^2=\frac{1}{\mu_0\varepsilon_0}$$

$$\Rightarrow \quad w=ck$$ Ans.

Therefore $\vec{\nabla}\times\vec{B}=\mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t}$ provided $k^2=\mu_0\varepsilon_0w^2$ i.e. Maxwell's equations are all satisfied with ρ and $\vec{J}$ (both) zero as per equations (1), (2) and (3).

Taking the div. of $\vec{A}$

$$\vec{\nabla}.\vec{A}=0 \text{ (Coulomb gauge)}$$ Ans.

Also calculate $\dfrac{\partial\phi}{\partial t}=0$

Hence $\nabla.\vec{A}+\mu\varepsilon\dfrac{\partial\phi}{\partial t}=0$ (Lorentz gauge)

Example 6 Calculate the gauge transformation using the gauge function $\wedge=-\dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{qt}{r}\right)$ for the potentials $\phi=0$ and $\vec{A}=-\dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{qt}{r^2}\right)\vec{a}_r$.

Solution: Given $\wedge=-\dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{qt}{r}\right),\phi=0$

and $\vec{A}=-\dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{qt}{r^2}\right)\vec{a}_r$

The gauge transformations are $\vec{A}'=\vec{A}+\vec{\nabla}\wedge$

and $\quad \phi'=\phi-\dfrac{\partial\wedge}{\partial t}$

$$\therefore \qquad \vec{A}' = -\frac{1}{4\pi\varepsilon_0}\left(\frac{qt}{r^2}\right)\vec{a}_r + \left(\frac{-qt}{4\pi\varepsilon_0}\right)\left(-\frac{\vec{a}_r}{r^2}\right)$$

$$\vec{A}' = 0 \qquad \text{Ans.}$$

and $$\phi' = 0 - \left(-\frac{1}{4\pi\varepsilon_0}\frac{q}{r}\right) \qquad \Rightarrow \phi' = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

9.8 Self Learning Exercise

Q.1 What is energy stored in an inductor.

Q.2 Write the different forms of magnetic energy density.

Q.3 Maxwell's equations give the relation between……………. .

Q.4 Write the expression for Coulomb gauge.

Q.5 What is the origin of the name "Coulomb gauge."

Q.6 Write the non-homogeneous or inhomogeneous wave equations for scalar and vector potentials.

Q.7 Why are potential functions used in electromagnetics.

Q.8 What do you mean by quasi-static fields. Are they exact solution of Maxwell's equations?

Q.9 Write the Maxwell's equations in terms of electro-magnetic potentials.

Q.10 What is the Lorentz condition for potentials? What is its physical significance.

Q.11 Which gauge allows separation of "near" and "radiation" fields.

9.9 Summary

This unit starts with the introduction of energy in the magnetic field. By giving the concept of quasi-static fields, we have derived the expression for energy and energy density in the magnetic field. Here, we have also introduced the concept of scalar and vector potential and derived the electromagnetic fields in terms of these potentials. The gauge transformation, Lorentz gauge and Coulomb gauge have also been studied in this unit. In the end, some examples on above concept are given.

9.10 Glossary

Invariant : not changing

Homogeneous : Containing terms all of the same degree

9.11 Answer to Self Learning Exercise

Ans.1: $W = \frac{1}{2} L I^2$

Ans.2: $\omega_m = \frac{1}{2} \vec{H}.\vec{B} = \frac{B^2}{2\mu} = \frac{1}{2} \mu H^2$

Ans.3: Electric and magnetic fields.

Ans.4: $\vec{\nabla}.\vec{A} = 0$

Ans.5: The scalar potential is just the instantaneous Coulomb potential due to the charge density $\rho\left(\vec{r}, t\right)$. This is the origin of the name "Coulomb Gauge".

Ans.6: $\nabla^2 \phi - \mu\varepsilon \frac{\partial^2 \phi}{\partial t^2} = -\rho/\varepsilon$ and $\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}$

Ans.7: By the use of electromagnetic potentials $\left(\phi \; and \; \vec{A}\right)$, the four Maxwell's equations are reduced to two equations *i.e.* introducing the potentials has dramatically reduced the complexity of the equations and corresponding increased the ease of finding solution.

Ans.8: When sources $\left(\rho \; and \; \vec{J}\right)$ vary slowly with time at a very low frequency and the range of interest is small in comparison with the wave length, then the fields are called quasi-static fields. These are approximations.

Ans.9: $\nabla^2 \vec{A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla\left(\vec{\nabla}.\vec{A} + \mu\varepsilon \frac{\partial \phi}{\partial t}\right) = -\mu \vec{J}$ and $\nabla^2 \phi + \frac{\partial}{\partial t}\left(\vec{\nabla}.\vec{A}\right) = -\rho/\varepsilon$

Ans.10 : $\vec{\nabla}.\vec{A} + \mu\varepsilon \frac{\partial \phi}{\partial t} = 0$, it is used to uncouple the wave equations for vector and scalar potentials.

Ans.11: Coulomb gauge

9.12 Exercise

Section A : Very Short Answer type Questions

Q.1 Give the expression for the energy stored in the magnetic field in terms of $\vec{B}$ and/or $\vec{H}$.

Q.2 Define the gauge transformations.

Q.3 What is gauge invariance?

Q.4 What is the Lorentz condition for potentials?

Q.5 Write $\vec{E}$ and $\vec{B}$ in terms of potential functions.

Section B : Short Answer type Questions

Q.6 Prove that the electric field vector $\left(\vec{E}\right)$ for time varying field is given by $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$

Q.7 Write the generalized form of Maxwell's equations.

Q.8 Why is the Lorentz gauge commonly used?

Q.9 Are all four Maxwell's equations independent? Explain.

Q.10 Why is Coulomb gauge also known as transverse gauge?

Q.11 A current of 5 Amp. produces a flux of 10 Webers through a coil of 200 turns. Calculate the energy stored in the magnetic field.

Section C : Long Answer type Questions

Q.12 Derive an expression for energy stored per unit volume in a magnetic field.

$$w_m = \frac{1}{2}\vec{H}.\vec{B} = \frac{1}{2}\mu H^2 = \frac{B^2}{2\mu}$$

Q.13 What are electromagnetic potentials? Obtain Maxwell equations in terms of electromagnetic potentials.

Q.14 What do you understand by Lorentz gauge and Coulomb gauge? Show that coupled non-homogeneous or inhomogeneous Maxwell's equations are uncoupled by a gauge transformation.

Q.15 Discuss the non-uniqueness of electromagnetic potentials and hence explain the significance of gauge transformations.

Q.16 Obtain Maxwell's equations in terms of scalar and vector potentials using Coulomb's gauge for potentials. Discuss the usefulness of this gauge also.

9.13 Answers to Exercise

Ans.1: $W_m = \frac{1}{2}\int\left(\vec{H}.\vec{B}\right)dv$

Ans.2: $\vec{A}' = \vec{A} + \vec{\nabla}\wedge$ and $\phi' = \phi - \frac{\partial \wedge}{\partial t}$

Ans.3: The invariance of electric and magnetic fields under gauge transformations is known as gauge invariance.

Ans.4: $\vec{\nabla}.\vec{A} + \mu\varepsilon\frac{\partial\phi}{\partial t} = 0$

Ans.5: $\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla}\times\vec{A}$

Ans.11: $\left[5\times10^{3}\ Joule\right]$

References and Suggested Readings

1. David k. Cheng, Field and wave Electromagnetics, Person, 1989.
2. J. D. Jackson, Classical Electrodynamics, 2nd edition, Wiley eastern limited, 1978.
3. David Griffiths, Introduction to Electrodynamics, 2nd edition, Prentice-Hall, 1989.
4. W.K.H. Panofsky and M. Phillips, Classical Electricity and Magnetism, 2nd ed. Addison-Wesley, 1962.
5. Mathew N.O. Sadiku, Elements of Electromagnetics, 3rded., Oxford University Press, 2001.

UNIT-10
Macroscopic Electromagnetism, Conservation Laws

Structure of the Unit

10.0 Objectives

In the previous chapters, we studied the problem releated to steady state electric and magnetic fields, where electric and magnetic phenomena were treated as independent. This independent nature of electric and magnetic phenomena disappears when we consider time dependent problems. The Faraday's induction law and Ampere's modified law destroyed the independence *i.e.* time varying magnetic fields give rise to electric fields and vice versa. Such type of fields are called electromagnetic fields. In this chaper, first we will derive the macroscopic Maxwell equations of electromagnetism from the microscopic Maxwell equations. This chapter also covers Poynting's theorem (conservation of energy) and conservation of momentum for a system of charged particles and electromagnetic fields and conservation laws for macroscopic media.

10.1 Introduction

The basic laws of electrostatics and magnetostatics are summarized in four differential equations as Gauss's law, Faraday's law, Gauss's law for magnetostatics and Ampere's law, respectively. These static equations will not hold unchanged for time dependent fields, because the Ampere's law was derived for steady state current phenomena with $\vec{\nabla}.\vec{J}=0$. According to the continuity equation for charge and current $\vec{\nabla}.\vec{J}\neq 0$, therefore the Ampere's law should be modified. In 1865, J.C. Maxwell added a term in Ampere's law and this added term was known as displacement current. The modified Ampere's law is the converse of Faraday's law. Without it there would be no electromagnetic radiation, hence it was Maxwell's prediction that light was an electromagnetic wave phenomenon. Thus the set of four equations known as the Maxwell's equations describe the behaviour of electromagnetic fields.

Microscopically, the matter made up of electrons and nuclei. The nuclei can be treated as point systems for dimensions large compared to $10^{-14}\,m$. The equations governing electromagnetic phenomena for these point charges are the microscopic Maxwell equations. The microscopic electromagnetic fields produced by these charges vary rapidly in space and in time. Therefore all the microscopic fluctuations are averaged out to give relative smooth and slowly varying macroscopic quantities, and thereby obtain a set of macroscopic Maxwell equations. The Poynting's theorem will explain about the conservation of energy for a system of charged particles and electromagnetic fields.

10.2 Displacement Current and Maxwell's Equations

From Ampere's law

$$\vec{\nabla}\times\vec{H}=\vec{J} \tag{1}$$

Taking divergence of eq. (1), we get

$$\nabla.\left(\vec{\nabla}\times\vec{H}\right)=\vec{\nabla}.\vec{J}=0 \tag{2}$$

Since divergence of curl of vector always vanishes.

From the equation of continuity for charge and current, we know that

$$\boxed{\vec{\nabla}.\vec{J}+\frac{\partial\rho}{\partial t}=0} \tag{3}$$

i.e. from eqs. (2) and (3), we get

$$\frac{\partial\rho}{\partial t}=0\Rightarrow\rho=\text{constant in time.}$$

Thus eq. (1) provides condition for steady state in which charge density remains constant with respect to time. Hence Ampere's law is not in accordance with the equation of continuity for time varying fields.

Therefore equation (1) should be modified for time varying fields. Maxwell suggested that the definition of total current density is incomplete and advised to add something to $\vec{J}$. Let this something is $\vec{J}_d$. The eq. (1) becomes

$$\vec{\nabla}\times\vec{H}=\vec{J}+\vec{J}_d \tag{4}$$

Taking divergence on both sides

$$\vec{\nabla}.\left(\vec{\nabla}\times\vec{H}\right)=\vec{\nabla}.\left(\vec{J}+\vec{J}_d\right)$$

$$\Rightarrow\quad 0=\vec{\nabla}.\vec{J}+\vec{\nabla}.\vec{J}_d$$

$$\Rightarrow\quad \vec{\nabla}.\vec{J}_d=-\vec{\nabla}.\vec{J}$$

Using eq. (3), we get

$$\vec{\nabla}.\vec{J}_d=\frac{\partial\rho}{\partial t} \tag{5}$$

From Gauss's law $\vec{\nabla}.\vec{D}=\rho$, therefore eq.(5) can be written as

$$\vec{\nabla}.\vec{J}_d=\frac{\partial}{\partial t}\left(\vec{\nabla}.\vec{D}\right)$$

$$\vec{\nabla}.\vec{J}_d = \vec{\nabla}.\frac{\partial \vec{D}}{\partial t}$$

or $$\vec{\nabla}.\left(\vec{J}_d - \frac{\partial \vec{D}}{\partial t}\right) = 0$$

This equation is valid for any volume, hence

$$\vec{J}_d - \frac{\partial \vec{D}}{\partial t} = 0$$

$$\Rightarrow \vec{J}_d = \frac{\partial \vec{D}}{\partial t} \tag{6}$$

Put the value of $\vec{J}_d$ from eq. (6) to eq. (4), we get

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{7}$$

or $$\vec{\nabla} \times \vec{B} = \mu \vec{J} + \mu\varepsilon \frac{\partial \vec{E}}{\partial t} \tag{8}$$

This is the corrected form of Ampere's law for time varying fields. The additional term $\frac{\partial \vec{D}}{\partial t}$ has the dimensions of current density. Since it results from a time varying electric flux density or displacement density, therefore it was termed as a displacement current density by Maxwell. A typical example of such current is the current through a capacitor when an alternating voltage source is applied to its plates.

According to modified Ampere's law a changing electric field produces a magnetic field and according to Faraday's law a changing magnetic field produces an electric field. Thus $\vec{J}_d$ (displacement current density) results into unification of electric and magnetic phenomena. Such type of unification of electric and magnetic phenomena are known as electromagnetic fields. The set of four equations known as the Maxwell equations, describe the behaviour of electromagnetic fields. These equation are as follows:-

Differential form	Integral Form	Remarks
$\vec{\nabla}.\vec{D}=\rho$	$\oint_s \vec{D}.d\vec{s}=\int_v \rho dv=Q$	Gauss's Law
$\vec{\nabla}.\vec{B}=0$	$\oint_s \vec{B}.d\vec{s}=0$	Non existence of isolatedMagnetic charge
$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$	$\oint_L \vec{E}.d\vec{l}=-\int_s \frac{\partial\vec{B}}{\partial t}.d\vec{s}$	Faraday's law
$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$	$\oint_L \vec{H}.d\vec{l}=\int_s\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right).d\vec{s}$	Ampere's modified law

The connections of $\vec{E}$ and $\vec{B}$ with $\vec{D}$ and $\vec{H}$, respectively given by following constitutive relations for linear media

$$\boxed{\vec{D}=\varepsilon_0\vec{E}+\vec{P}}=\varepsilon_0\vec{E}+\chi_e\varepsilon_0\vec{E}$$

$$\boxed{\vec{D}=\varepsilon_0\left(1+\chi_e\right)\vec{E}=\varepsilon\vec{E}}$$

and $$\boxed{\vec{B}=\mu_0\left(\vec{H}+\vec{M}\right)}=\mu_0\left(\vec{H}+\chi_m\vec{H}\right)$$

$$\boxed{\vec{B}=\mu_0\left(1+\chi_m\right)\vec{H}=\mu\vec{H}}$$

Maxwell equations combined with the Lorentz force equation and Newton's second law of motion provide a complete description of the classical dynamics of interacting charged particles and electromagnetic fields.

10.3 Derivation of the Equations of Macroscopic Electromagnetism

Let us consider a microscopic world made up of electrons and nuclei. The nuclei can be treated as point system for the dimensions large then 10^{-14} m. The equations governing electromagnetic phenomena for these point charges are called the microscopic Maxwell equations. These equations are as follows:-

$$\vec{\nabla}.\vec{e}=\frac{\eta}{\varepsilon_0} \qquad \vec{\nabla}\times\vec{e}=-\frac{\partial\vec{b}}{\partial t}$$

$$\vec{\nabla}.\vec{b}=0 \qquad\qquad \vec{\nabla}\times\vec{b}=\mu_0\vec{j}+\mu_0\varepsilon_0\frac{\partial\vec{e}}{\partial t} \qquad (1)$$

where $\vec{e}$ and $\vec{b}$ are the microscopic electric and magnetic fields, respectively and η and $\vec{j}$ are the microscopic charge and current densities. There are no corresponding fields $\vec{d}$ and $\vec{h}$ because all the charges are included in η and $\vec{j}$. The microscopic electromagnetic field functions produced by point charges vary rapidly in space over the atomic distances (of the order of 10^{-10}m or less). These functions can be regarded as sums of delta functions. However, macroscopic functions only measure the averaged quantity. Hence there is a need to develop an averaging method to reduce microscopically fluctuating functions to macroscopically smooth functions, and thereby obtain a set of macroscopic Maxwell equations.

If we replace each delta function $\delta\left(\vec{X}-\vec{X}_0\right)$ in the microscopic distribution function with a smooth function $f\left(\vec{X}-\vec{X}_0\right)$ subject to the condition

$$\int f\left(\vec{X}-\vec{X}_0\right)d^3x=1 \qquad (2)$$

and if the width L of $f\left(\vec{X}-\vec{X}_0\right)$ is much greater than the atomic distance $\left(\sim 10^{-8}m\right)$, then the sum of many such functions will become a smooth function representing the spatially average of microscopic Maxwell equations.

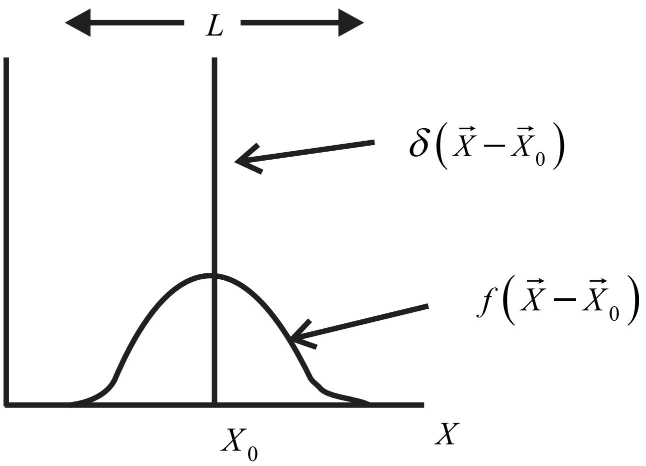


FIG. 10.1

The spatial average of a function $F\left(\vec{X},t\right)$ with respect to a test function $f\left(\vec{X}\right)$ is defined as

$$\left\langle F\left(\vec{X},t\right)\right\rangle = \int f\left(\vec{X}'\right) F\left(\vec{X}-\vec{X}',t\right) d^3x' \tag{3}$$

where $f\left(\vec{X}\right)$ is real, non zero smooth function centered at $\vec{X}=0$.

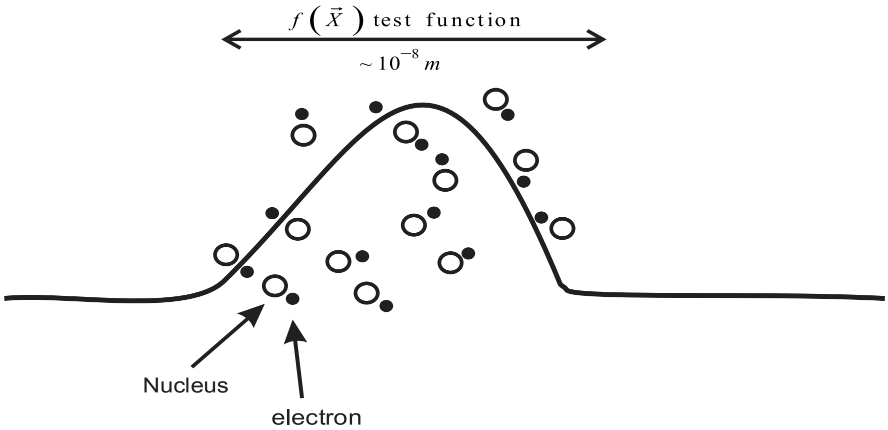


FIG. 10.2 Test function $f\left(\vec{X}\right)$ used in the spatial averaging procedure

Therefore the macroscopic electric and magnetic field quantities $\vec{E}$ and $\vec{B}$ are defined as the average of the microscopic fields $\vec{e}$ and $\vec{b}$ *i.e.*

$$\vec{E}\left(\vec{X},t\right) = \left\langle \vec{e}\left(\vec{X},t\right)\right\rangle$$

$$\vec{B}\left(\vec{X}.t\right) = \left\langle \vec{b}\left(\vec{X},t\right)\right\rangle \tag{4}$$

Then the averages of the two homogenous equations in eq. (1) become the corresponding macroscopic equations

$$\left\langle \vec{\nabla}.\vec{b}\right\rangle = 0 \rightarrow \vec{\nabla}.\vec{B} = 0 \tag{5}$$

$$\left\langle \vec{\nabla}\times\vec{e} + \frac{\partial\vec{b}}{\partial t}\right\rangle = 0 \rightarrow \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \tag{6}$$

The averaged inhomogeneous equations from eq. (1) become

$$\varepsilon_0 \vec{\nabla}.\vec{E} = \left\langle \eta\left(\vec{X},t\right)\right\rangle \tag{7}$$

$$\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \varepsilon_0\frac{\partial\vec{E}}{\partial t} = \left\langle \vec{j}\left(\vec{X},t\right)\right\rangle \tag{8}$$

The derived fields $\vec{D}$ and $\vec{H}$ are introduced by the extraction from $\langle \eta \rangle$ and $\langle \vec{j} \rangle$ of certain contributions that can be identified with the bulk properties of the medium.

We consider a medium made up of molecules composed of nuclei and electrons as well as free charges that are not localized around any particular molecule. Therefore to distinguish the bound charges form the free charges, we can decompose the microscopic charge density η as

$$\eta = \eta_{free} + \eta_{bound} \tag{9}$$

The averaged microscope charge density reduces to

$$\begin{aligned} \langle \eta(\vec{X},t) \rangle &= \rho_f(\vec{X},t) + \rho_b(\vec{X},t) \\ &= \rho_f(\vec{X},t) - \vec{\nabla}.\vec{P}(\vec{X},t) \end{aligned} \tag{10}$$

where ρ_f is the macroscopic charge density and it is given by

$$\rho_f(\vec{X},t) = \left\langle \sum_{j(free)} q_j \delta\left(\vec{X} - \vec{X}_j\right) + \sum_{n(molecule)} q_n \delta\left(\vec{X} - \vec{X}_n\right) \right\rangle \tag{11}$$

and $\vec{P}$ is the macroscopic polarization, which is given as

$$\vec{P}(\vec{X},t) = \left\langle \sum_{n(molecule)} \vec{p}_n \delta\left(\vec{X} - \vec{X}_n\right) \right\rangle \tag{12}$$

Similarly, the averaged microscopic current density can be expressed as

$$\begin{aligned} \langle \vec{j}(\vec{X},t) \rangle &= \vec{J}_f + \vec{J}_b + \vec{J}_P \\ &= \vec{j}_f(\vec{X},t) + \vec{\nabla} \times \vec{M}(\vec{X},t) + \frac{\partial \vec{P}(\vec{X},t)}{\partial t} \end{aligned} \tag{13}$$

where $\vec{J}_f$ is the macroscopic current density, which is given as

$$\vec{J}_f(\vec{X},t) = \left\langle \sum_{j(free)} q_j v_i \delta\left(\vec{X} - \vec{X}_j\right) + \sum_{n(molecule)} q_n v_n \delta\left(\vec{X} - \vec{X}_n\right) \right\rangle \tag{14}$$

and $\vec{M}$ is the macroscopic magnetization, which is given as

$$\vec{M}\left(\vec{X},t\right)=\left\langle \sum_{n(molecule)} \vec{m}_n \delta\left(\vec{X}-\vec{X}_n\right)\right\rangle \tag{15}$$

The polarization current density $\vec{J}_P=\dfrac{\partial\vec{P}\left(\vec{X},t\right)}{\partial t}$ (16)

involves a flow of bound charge as shown in fig. 10.3

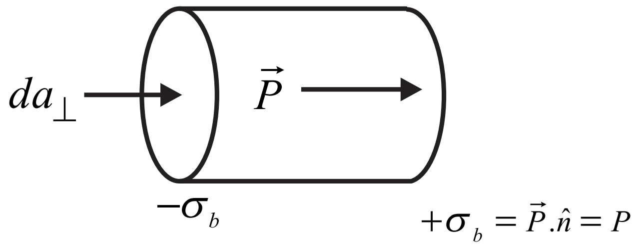


FIG. 10.3 Polarized Material

Inserting eqs. (10) and (13) into eqs. (7) and (8), respectively, we obtain the macroscopic inhomogeneous equations:

$$\varepsilon_0\vec{\nabla}.\vec{E}=\rho_f-\vec{\nabla}.\vec{P}\Rightarrow\vec{\nabla}.\left(\varepsilon_0\vec{E}+\vec{P}\right)=\rho_f$$

$$\Rightarrow \quad \boxed{\vec{\nabla}.\vec{D}=\rho_f} \tag{17}$$

where $\vec{D}=\varepsilon_0\vec{E}+\vec{P}$

$$\frac{1}{\mu_0}\vec{\nabla}\times\vec{B}-\varepsilon_0\frac{\partial\vec{E}}{\partial t}=\vec{J}_f+\vec{\nabla}\times\vec{M}+\frac{\partial\vec{P}}{\partial t}$$

$$\Rightarrow\vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0}-\vec{M}\right)=\vec{J}_f+\frac{\partial}{\partial t}\left(\varepsilon_0\vec{E}+\vec{P}\right)$$

$$\Rightarrow\boxed{\vec{\nabla}\times\vec{H}=\vec{J}_f+\frac{\partial\vec{D}}{\partial t}} \tag{18}$$

where $\boxed{\vec{H}=\dfrac{\vec{B}}{\mu_0}-\vec{M}}$ and $\boxed{\vec{D}=\varepsilon_0\vec{E}+\vec{P}}$

Thus, the macroscopic Maxwell equations or Maxwell equations in matter are as follow:-

$$\boxed{\begin{aligned}&\vec{\nabla}.\vec{D}=\rho_f\\&\vec{\nabla}.\vec{B}=0\\&\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\\&\vec{\nabla}\times\vec{H}=\vec{J}_f+\frac{\partial\vec{D}}{\partial t}\end{aligned}} \qquad \ldots(19)$$

The relations between $\vec{E}$ and $\vec{B}$ with $\vec{D}$ and $\vec{H}$ respectively are given by constitutive relations, *i.e.*

$$\boxed{\vec{D}=\varepsilon\vec{E}} \text{ and } \boxed{\vec{B}=\mu\vec{H}} \qquad (20)$$

10.4 Poynting's theorem(Conservation of Energy for a System of charged particles and Electromagnetic Fields)

Electromagnetic waves carry with them electromagnetic power. Energy is transported through space to distant receiving points by electromagnetic waves. The rate of such energy transportation can be obtained from Maxwell's equations.

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \qquad (1)$$

and
$$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t} \qquad (2)$$

Taking dot product both sides of eq. (2) with $\vec{E}$ gives

$$\vec{E}.\left(\vec{\nabla}\times\vec{H}\right)=\vec{E}.\vec{J}+\vec{E}.\frac{\partial\vec{D}}{\partial t}$$

But $\vec{D}=\varepsilon\vec{E}$ and $\vec{J}=\sigma E$

$$\therefore \quad \vec{E}.\left(\vec{\nabla}\times\vec{H}\right)=\sigma E^2+\vec{E}.\varepsilon\frac{\partial\vec{E}}{\partial t} \qquad (3)$$

As we know that for any two vector fields $\vec{A}$ and $\vec{B}$

$$\vec{\nabla}.\left(\vec{A}\times\vec{B}\right)=\vec{B}.\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}.\left(\vec{\nabla}\times\vec{B}\right)$$

Putting $\vec{A}=\vec{H}$ and $\vec{B}=\vec{E}$ in this vector identity, we get

$$\vec{\nabla}.\left(\vec{H}\times\vec{E}\right)=\vec{E}.\left(\vec{\nabla}\times\vec{H}\right)-\vec{H}.\left(\vec{\nabla}\times\vec{E}\right)$$

$$\Rightarrow \vec{E}.\left(\vec{\nabla}\times\vec{H}\right)=\vec{H}.\left(\vec{\nabla}\times\vec{E}\right)+\vec{\nabla}.\left(\vec{H}\times\vec{E}\right) \tag{4}$$

From eqs. (3) and (4), we obtain

$$\vec{H}.\left(\vec{\nabla}\times\vec{E}\right)+\vec{\nabla}.\left(\vec{H}\times\vec{E}\right)=\sigma E^2+\frac{\varepsilon}{2}\frac{\partial}{\partial t}\left(\vec{E}.\vec{E}\right) \tag{5}$$

From eq. (1) $\vec{H}.\left(\vec{\nabla}\times\vec{E}\right)=\vec{H}\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)=-\frac{\mu}{2}\frac{\partial}{\partial t}\left(\vec{H}.\vec{H}\right)$

Putting the value of $\vec{H}.\left(\vec{\nabla}\times\vec{E}\right)$ into eq. (5), we get

$$-\frac{\mu}{2}\frac{\partial H^2}{\partial t}+\vec{\nabla}.\left(\vec{H}\times\vec{E}\right)=\sigma E^2+\frac{\varepsilon}{2}\frac{\partial E^2}{\partial t}$$

$$\Rightarrow \vec{\nabla}.\left(\vec{E}\times\vec{H}\right)=-\frac{1}{2}\varepsilon\frac{\partial E^2}{\partial t}-\frac{1}{2}\mu\frac{\partial H^2}{\partial t}-\sigma E^2 \tag{6}$$

Taking the volume integral of both sides,

$$\int_v \vec{\nabla}.\left(\vec{E}\times\vec{H}\right)dv=-\frac{\partial}{\partial t}\int_v\left(\frac{1}{2}\varepsilon E^2+\frac{1}{2}\mu H^2\right)dv-\int_v \sigma E^2 dv$$

Applying the divergence theorem to the left-hand side of above mentioned eq. gives-

$$\oint_s\left(\vec{E}\times\vec{H}\right).d\vec{s}=-\frac{\partial}{\partial t}\int_v\left(\frac{1}{2}\varepsilon E^2+\frac{1}{2}\mu H^2\right)dv-\int_v \sigma E^2 dv$$

↓	↓	↓	
Total power leaving the volume through its surface	Rate of decrease in energy stroed in electric and magnetic fields, respectively	Ohmic power dissipated	(7)

Equation (7) is referred ***as Poynting's theorem***. ***It is also known as the conservation of energy for electromagnetic fields.*** The first term on the right hand side of eq. (7) is interpreted as the rate of decrease in energy stored in the electric and magnetic fields, respectively. The second term is the power dissipated due to the fact that the medium is conducting $(\sigma \neq 0)$. Therefore, to be consistent with the law of conservation of energy, this must be equal to the power (rate of energy) leaving the volume through its surface. Thus the quantity $\left(\vec{E} \times \vec{H}\right)$ on the left hand side of eq. (7) is a vector representing the power flow per unit area or energy per unit area per unit time, which is defined as

$$\boxed{\vec{\mathrm{P}} = \vec{E} \times \vec{H}} \qquad \left(Watt / m^2\right) \tag{8}$$

Quantity $\vec{\mathrm{P}}$ is known as the ***Poynting vector***, ***which represent the instantaneous power density vector associated with the electromagnetic field at a given point***. The integration of the Poynting vector over any closed surface gives the net power flowing out of that surface.

Therefore Poynting's theorem states that the net power flowing out of a given volume v is equal to the time rate of decrease in the energy stored within v minus the conduction losses.

Since matter is ultimately composed of charged particles *i.e.* electrons and atomic nuclei. According to the Lorentz force law, the work done on a charge q is

$$\vec{F}.\vec{dl} = q\left(\vec{E} + \vec{v} \times \vec{B}\right).\vec{dl} \tag{9}$$

$$dW = q\vec{E}.\vec{v}dt$$

The magnetic field does no work, since the magnetic force is perpendicular to the velocity.

$$\Rightarrow \frac{dW}{dt} = q\vec{E}.\vec{v}$$

$$= \rho dv\vec{E}.\vec{v}$$

$$= \vec{E}.\rho\vec{v}dv$$

$$\frac{dW}{dt} = \left(\vec{E}.\vec{J}\right)dv = \sigma E^2 dv \tag{10}$$

If there exists a continuous distribution of charge and current, the total rate of doing work by all the charges or fields in a volume v is

$$\frac{dW}{dt}=\int_v \left(\vec{E}.\vec{J}\right)dv=\int_v \sigma E^2 dv \tag{11}$$

The work done per unit time per unit volume by the fields $\left(\vec{E}.\vec{J}\right)$ is conversion of electromagnetic energy into mechanical or heat energy, so that

$$\frac{dW}{dt}=\frac{dE_{mech}}{dt}=\int_v \left(\vec{E}.\vec{J}\right)dv \tag{12}$$

The total energy stored in electromagnetic fields is

$$E_{field}=\frac{1}{2}\int_v \left(\varepsilon E^2+\mu H^2\right)dv \tag{13}$$

Then using eqs. (11), (12) and (13) in Poynting's theorem eq. (7) expresses the conservation of energy for the combined system of charged particles and fields as

$$\frac{dE}{dt}=\frac{d}{dt}\left(E_{mech}+E_{field}\right)=-\oint_s \vec{\mathrm{P}}.d\vec{s} \tag{14}$$

$$\Rightarrow \boxed{\frac{d}{dt}\int_v \left(e_{mech}+e_{field}\right)dv=-\int_v \left(\vec{\nabla}.\vec{\mathrm{P}}\right)dv}$$

Where e_{mech} and e_{field} are the mechanical energy density and energy density of the fields, respectively. Therefore

$$\boxed{\frac{\partial}{\partial t}\left(e_{mech}+e_{field}\right)=-\vec{\nabla}.\vec{\mathrm{P}}} \tag{15}$$

This is the differential version of Poynting's theorem. Compare it with the continuity equation (conservation of charge)

$$\frac{\partial \rho}{\partial t}=-\vec{\nabla}.\vec{J} \tag{16}$$

We can say that the charge density is replaced by the energy density (mechanical plus electromagnetic) and the current density is placed by the Poynting vector.

10.5 Conservation of Momentum for a System of Charged Particles and Electromagnetic Fields

The total electromagnetic force on a charged particle is

$$\boxed{\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)} \tag{1}$$

Denoting $\vec{\mathrm{P}}_{mech}$ as the total momentum of all particles in the volume v, we can write form Newton's second law

$$\vec{F}_{total} = \frac{d\vec{P}_{mech}}{dt} = \int_v \left(\rho\vec{E} + \vec{J} \times \vec{B}\right) dv = \int_v \vec{f} dv \tag{2}$$

We use the Maxwell equations to eliminate ρ and $\vec{J}$ from eq. (2)

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}, \qquad \vec{\nabla} \times \vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \quad \rho = \varepsilon_0\left(\vec{\nabla}.\vec{E}\right), \qquad \vec{J} = \frac{1}{\mu_0}\left(\vec{\nabla} \times \vec{B}\right) - \varepsilon_0\frac{\partial\vec{E}}{\partial t} \tag{3}$$

Substituting eq. (3) into eq.(2) the integrand or force per unit volume or force density becomes

$$\vec{f} = \rho\vec{E} + \vec{J} \times \vec{B} = \varepsilon_0\left(\vec{\nabla}.\vec{E}\right)\vec{E} + \frac{1}{\mu_0}\left(\vec{\nabla} \times \vec{B}\right) \times \vec{B} - \varepsilon_0\frac{\partial\vec{E}}{\partial t} \times \vec{B}$$

$$\vec{f} = \varepsilon_0\left[\left(\vec{\nabla}.\vec{E}\right)\vec{E} + \vec{B} \times \frac{\partial\vec{E}}{\partial t} - c^2\vec{B} \times \left(\vec{\nabla} \times \vec{B}\right)\right] \tag{4}$$

where $c^2 = \dfrac{1}{\mu_0\varepsilon_0}$

The term $\vec{B} \times \dfrac{\partial\vec{E}}{\partial t}$ can be written as

$$\vec{B} \times \frac{\partial\vec{E}}{\partial t} = -\frac{\partial}{\partial t}\left(\vec{E} \times \vec{B}\right) + \vec{E} \times \frac{\partial\vec{B}}{\partial t}$$

$$= -\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right) + \vec{E}\left[-\left(\vec{\nabla}\times E\right)\right] \text{ using Faraday's law}$$

$$\vec{B}\times\frac{\partial \vec{E}}{\partial t} = -\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right) - \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right) \tag{5}$$

Substituting eq. (5) into eq. (4) and adding $c^2\left(\vec{\nabla}.\vec{B}\right)\vec{B} = 0$ to the square bracket for the symmetry of equation because $\vec{\nabla}.\vec{B} = 0$, we get

$$\vec{f} = \varepsilon_0\left[\left(\vec{\nabla}.\vec{E}\right)\vec{E} + c^2\left(\vec{\nabla}.\vec{B}\right)\vec{B} - \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right) - c^2\vec{B}\times\left(\vec{\nabla}\times B\right)\right] - \varepsilon_0\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right) \tag{6}$$

$$\vec{f} = \varepsilon_0\left[\left(\vec{\nabla}.\vec{E}\right)\vec{E} - \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right) + \frac{1}{\mu_0\varepsilon_0}\left\{\left(\vec{\nabla}.\vec{B}\right)\vec{B} - \vec{B}\times\left(\vec{\nabla}\times\vec{B}\right)\right\}\right] - \varepsilon_0\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right) \tag{7}$$

The total electromagnetic momentum $\vec{P}_{field}$ in the volume v defined as

$$\vec{P}_{field} = \varepsilon_0\int_v\left(\vec{E}\times\vec{B}\right)dv$$

$$= \mu_0\varepsilon_0\int_v\left(\vec{E}\times\vec{H}\right)dv = \mu_0\varepsilon_0\int_v\vec{\mathrm{P}}dv \tag{8}$$

The integrand part can be interpreted as a ***density of electromagnetic momentum*** $\left(\vec{g}_{em}\right)$. It is defined by a vector

$$\boxed{\vec{g}_{em} = \mu_0\varepsilon_0\left(\vec{E}\times\vec{H}\right) = \frac{1}{c^2}\vec{\mathrm{P}}} \tag{9}$$

Now identity from vector calculus says

$$\vec{\nabla}\left(\vec{A}.\vec{B}\right) = \vec{A}\times\left(\vec{\nabla}\times\vec{B}\right) + \vec{B}\times\left(\vec{\nabla}\times\vec{A}\right) + \left(\vec{A}.\vec{\nabla}\right)\vec{B} + \left(\vec{B}.\vec{\nabla}\right)\vec{A}$$

If $\vec{A} = \vec{B} = \vec{E}$, we get

$$\nabla\left(E^2\right) = 2\vec{E}\times\left(\vec{\nabla}\times\vec{E}\right) + 2\left(\vec{E}.\vec{\nabla}\right)\vec{E}$$

$$\Rightarrow \vec{E}\times\left(\vec{\nabla}\times\vec{E}\right) = \frac{1}{2}\nabla\left(E^2\right) - \left(\vec{E}.\vec{\nabla}\right)\vec{E} \tag{10}$$

Similarly $\vec{B}\times\left(\vec{\nabla}\times\vec{B}\right)=\frac{1}{2}\nabla\left(B^2\right)-\left(\vec{B}.\vec{\nabla}\right)\vec{B}$ (11)

Putting these values into equation (7), we get

$$\vec{f}=\varepsilon_0\left(\vec{\nabla}.\vec{E}\right)\vec{E}+\frac{1}{\mu_0}\left(\vec{\nabla}.\vec{B}\right)\vec{B}-\frac{1}{2}\nabla\left(\varepsilon_0E^2+\frac{1}{\mu_0}B^2\right)+\varepsilon_0\left(\vec{E}.\vec{\nabla}\right)\vec{E}+\frac{1}{\mu_0}\left(\vec{B}.\vec{\nabla}\right)\vec{B}-\varepsilon_0\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right)$$

$$\vec{f}=\varepsilon_0\left[\left(\vec{\nabla}.\vec{E}\right)\vec{E}+\left(\vec{E}.\vec{\nabla}\right)\vec{E}\right]+\frac{1}{\mu_0}\left[\left(\vec{\nabla}.\vec{B}\right)\vec{B}+\left(\vec{B}.\vec{\nabla}\right)\vec{B}\right]-\frac{1}{2}\nabla\left(\varepsilon_0E^2+\frac{1}{\mu_0}B^2\right)-\varepsilon_0\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right) \tag{12}$$

Now we can introduce the ***Maxwell stress tensor*** $\overleftrightarrow{T}$ which is a 3×3 matrix with components defined by

$$T_{ij}=\varepsilon_0\left(E_iE_j-\frac{1}{2}\delta_{ij}E^2\right)+\frac{1}{\mu_0}\left(B_iB_j-\frac{1}{2}\delta_{ij}B^2\right) \tag{13}$$

where δ_{ij} is Kronecker delta and it is 1 if the indices are the same and zero otherwise. The indices i and j refer to the coordinates x, y and z.

If we define the scalar product of the tensor with an ordinary vector to be another vector.

$$\left[\vec{a}.\overleftrightarrow{T}\right]_j=\sum_i a_iT_{ij} \tag{14}$$

where the subscript j indicates the j^{th} component of the resulting vector, then j^{th} component of divergence of $\overleftrightarrow{T}$ is

$$\left[\vec{\nabla}.\overleftrightarrow{T}\right]_j=\sum_i \partial_iT_{ij} \tag{15}$$

$$\left[\vec{\nabla}.\overleftrightarrow{T}\right]_j=\varepsilon_0\sum_i\left[\left(\partial_iE_i\right)E_j+E_i\left(\partial_iE_j\right)-\frac{1}{2}\delta_{ij}\,\partial_iE^2\right]$$

$$+\frac{1}{\mu_0}\sum_i\left[\left(\partial_iB_i\right)B_j+B_i\left(\partial_iB_j\right)-\frac{1}{2}\delta_{ij}\partial_iB^2\right] \quad 16)$$

$$\left[\vec{\nabla}.\overleftrightarrow{T}\right]_j=\varepsilon_0\left[\left(\vec{\nabla}.\vec{E}\right)E_j+\left(\vec{E}.\vec{\nabla}\right)E_j-\frac{1}{2}\partial_jE^2\right]$$

$$+\frac{1}{\mu_0}\left[\left(\vec{\nabla}.\vec{B}\right)B_j+\left(\vec{B}.\vec{\nabla}\right)B_j-\frac{1}{2}\partial_j B^2\right] \tag{17}$$

Comparing eq.(17) with eq.(12), we can write ***force per unit volume*** $\left(\vec{f}\right)$ in term of Maxwell stress tensor $\left(\overleftrightarrow{T}\right)$ and Poynting vector $\vec{\mathrm{P}}$ as

$$\boxed{\vec{f}=\vec{\nabla}.\overleftrightarrow{T}-\varepsilon_0\mu_0\frac{\partial\vec{\mathrm{P}}}{\partial t}} \tag{18}$$

Then the total force on the volume v is given as

$$\vec{F}_{total}=\int_v \vec{f}dv=\int_v\left[\left(\vec{\nabla}.\overleftrightarrow{T}\right)-\varepsilon_0\mu_0\frac{\partial\vec{\mathrm{P}}}{\partial t}\right]dv \tag{19}$$

Apply the divergence theorem to the first term in the integrand, we get

$$\vec{F}_{total}=\oint_s \overleftrightarrow{T}.\vec{n}ds-\varepsilon_0\mu_0\frac{\partial}{\partial t}\int_v \vec{\mathrm{P}}dv \tag{20}$$

where $\vec{n}$ is the outward normal to the closed surface S.

$$\Rightarrow \vec{F}_{total}+\varepsilon_0\mu_0\frac{\partial}{\partial t}\int_v \vec{\mathrm{P}}dv=\oint_s \overleftrightarrow{T}.\vec{n}ds=\oint_s \overleftrightarrow{T}.d\vec{s} \tag{21}$$

Using eqs. (2) and (8), we can write eq.(21) as follows

$$\frac{d\vec{P}_{mech}}{dt}+\frac{d\vec{P}_{field}}{dt}=\oint_s \overleftrightarrow{T}.d\vec{s}$$

$$\frac{d}{dt}\left(\vec{p}_{mech}+\vec{p}_{field}\right)=\oint_s \overleftrightarrow{T}.d\vec{s} \tag{22}$$

This equation represents a statement of conservation of momentum. $\overleftrightarrow{T}$ is the flow per unit area of momentum across the surface S into the volume or it is the force per unit area transmitted across the surface S and acting on the combined system of particles and fields inside v. More precisely T_{ij} is the force per unit area in the i^{th} direction acting on an element of surface oriented in the j^{th} direction.

Therefore diagonal elements $\left(T_{xx}, T_{yy}, T_{zz}\right)$ represent pressures and off-diagonal elements $\left(T_{xy}, T_{xz}, \ldots\ etc\right)$ are shears.

Using eq. (9), we can write eq. (19) as follows

$$\int_v \left(\vec{f} + \frac{d\vec{g}_{em}}{dt}\right) dv = \int_v \left(\vec{\nabla}.\overleftrightarrow{T}\right) v$$

$$\Rightarrow \int_v \left(\frac{d\vec{p}_{mech}}{dt} + \frac{d\vec{g}_{em}}{dt}\right) dv = \int_v \left(\vec{\nabla}.\overleftrightarrow{T}\right) dv$$

where $\vec{p}_{mech}$ and $\vec{g}$ are the density of mechanical momentum and density of electromagnetic momentum, respectively. Therefore

$$\boxed{\frac{\partial}{\partial t}\left(\vec{p}_{mech} + \vec{g}_{em}\right) = \vec{\nabla}.\overleftrightarrow{T}} \tag{23}$$

This is differential form of conservation law of momentum. Comparing it from the continuity equation, we can say that $-\overleftrightarrow{T}$ is the momentum current density. Thus, the Poynting vector $\left(\vec{\mathrm{P}}\right)$ and Maxwell stress tensor $\left(\overleftrightarrow{T}\right)$ play dual role. $\vec{\mathrm{P}}$ itself is the energy per unit area, per unit time transported by the electromagnetic fields, while $\mu_0 \varepsilon_0 \vec{\mathrm{P}} = \vec{g}_{em}$ is momentum per unit volume stored in those fields. Similarly $\overleftrightarrow{T}$ is itself is the electromagnetic force per unit area acting on a surface and $-\overleftrightarrow{T}$ describes the flow of momentum current density transported by the fields.

10.6 Conservation Laws for Macroscopic Media

The conservation law of energy or Poynting's theorem was derived using the macroscopic Maxwell equations, but the conservation of momentum and the Maxwell stress tensor were discussed only for the microscopic equations. The electromagnetic energy density e_{field}, energy flow $\vec{\mathrm{P}}$, momentum flow $\vec{g}_{em}$ and stress tensor T_{ij} must be defined carefully for bulk matter because considered electromagnetic and mechanical concept are to some extent arbitrary.

There are the Minkowski (1908) results based on the macroscopic Maxwell equations to the conservation of momentum as well as energy. There are the previously obtained expressions for electromagnetic energy density e_{field} and the Poynting's vector $\vec{\mathrm{P}}$, but with the momentum density and stress tensor given by

$$\vec{g}_{em} = \vec{D} \times \vec{B} \tag{1}$$

and $$T_{ij} = \left[E_i D_j + H_i B_j - \frac{1}{2} \left(\vec{E}.\vec{D} + \vec{B}.\vec{H} \right) \delta_{ij} \right] \tag{2}$$

The medium is assumed to be linear, but not necessarily isotropic in its response. Since the stress tensor is not symmetric for anisotropic media. The Hertz and Abraham were first few physicists, who replaced eq. (2) with a symmetrised form. The Minkowski expression given by eq. (1) for $\vec{g}_{em}$ is generally viewed as unacceptable as the electromagnetic momentum density. All workers agree on the definition.

$$\boxed{\vec{g}_{em} = \mu_0 \varepsilon_0 \left(\vec{E} \times \vec{H} \right) = \frac{1}{c^2} \vec{\mathrm{P}}} \tag{3}$$

This result emerges from a statistical mechanical treatment of the system of matter plus fields in which the electromagnetic quantities are defined as the difference between the quantities for the combined system and those for the matter system at the same equilibrium temperature T and density ρ, but with zero fields. With this definition, the energy and momentum flow densities are given by

$$\vec{\mathrm{P}} = \vec{E} \times \vec{H} \tag{4}$$

and $$\vec{g}_{em} = \mu_0 \varepsilon_0 \left(\vec{E} \times \vec{H} \right) \tag{5}$$

For Linear and isotropic medium with $\vec{D} = \varepsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$, the electromagnetic energy density (e) and the electromagnetic stress tensor $\left(T_{ij}\right)$ are given by

$$e = \frac{1}{2} \left\{ E^2 \left[\varepsilon + T \left(\frac{\partial \varepsilon}{\partial T} \right)_\rho \right] + H^2 \left[\mu + T \left(\frac{\partial \mu}{\partial T} \right)_\rho \right] \right\} \tag{6}$$

and $$T_{ij} = \varepsilon E_i E_j + \mu H_i H_j - \frac{1}{2} \delta_{ij} \left[e^2 \left(\varepsilon - \rho \left(\frac{\partial \varepsilon}{\partial \rho} \right)_T \right) + \mu^2 \left(\mu - \rho \left(\frac{\partial \mu}{\partial \rho} \right)_T \right) \right] \tag{7}$$

These reduce to the Minkowski expression for electromagnetic energy density e and Maxwell stress tensor T_{ij} only for the unphysical situation in which ε and μ are independent of temperature and density.

10.7 Illustrative Examples

Example 10.1 Show that equation of continuity is contained in Maxwell's equations.

Solution From Ampere's modified law (IV eq.), we know

$$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t} \tag{1}$$

Taking divergence of either side of eq. (1), we obtain

$$\vec{\nabla}.\left(\vec{\nabla}\times\vec{H}\right)=\vec{\nabla}.\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right) \tag{2}$$

Since $\vec{\nabla}.\left(\vec{\nabla}\times\vec{H}\right)=0$ because divergence of curl of any vector always vanishes, therefore equation (2) becomes

$$\vec{\nabla}.\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)=0$$

$$\Rightarrow \vec{\nabla}.\vec{J}+\vec{\nabla}.\frac{\partial\vec{D}}{\partial t}=0$$

$$\Rightarrow \vec{\nabla}.\vec{J}+\frac{\partial}{\partial t}\left(\vec{\nabla}.\vec{D}\right)=0 \tag{3}$$

From Gauss's law (I^{st} eq.), we know $\vec{\nabla}.\vec{D}=\rho$, therefore eq.(3) becomes

$$\vec{\nabla}.\vec{J}+\frac{\partial\rho}{\partial t}=0$$

This is the equation of continuity.

Example 10.2 Calculate the amplitudes of electric and magnetic field of radiation, when the earth receives 3.8 cal $\min^{-1} cm^{-2}$ solar energy.

Solution Given energy flux per unit area per second at earth is

$$= 3.8\, cal\ \min^{-1}\ cm^{-2} = \frac{3.8 \times 4.2 \times 10^{4}}{60}\, Watt/m^{2}$$

From pointing theorem, the energy per unit area per second is

$$\left|\vec{\mathrm{P}}\right| = \left|\vec{E} \times \vec{H}\right|$$

$$\mathrm{P} = EH$$

$$\therefore\ EH = \frac{3.8 \times 4.2 \times 10^{4}}{60}$$

$$EH = 2660\ Watt/m^{2} \tag{1}$$

But $\quad \dfrac{E}{H} = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = 120\pi = 376.8 \quad (2)$

From eqs. (1) and (2), we get

$$E^{2} = 2660 \times 376.8$$

$$E = \sqrt{2660 \times 376.8}$$

$\therefore \quad E = 1001.14\ Volt/meter$

Substituting this value of E in eq. (1), we get

$$H = \frac{2660}{1001.14}$$

$$H = 2.65\ Amp/meter$$

Therefore the amplitudes of electric and magnetic fields of radiation are

$$E_0 = E\sqrt{2} = 1001.14 \times \sqrt{2} = 1415.82\ Volt/meter$$

and $\quad H_0 = H\sqrt{2} = 2.65 \times \sqrt{2} = 3.74\ Amp/meter$ **Ans.**

Example 10.3 A long coaxial cable carries current I (the current flows down the surface of the inner cylinder, radius a and back along the outer cylinder, radius b) as shown in Fig. 10.4 Calculate the power (energy per unit time) transported down the cables, assuming the two conductors are held at potential difference V and a uniform charge per unit length λ. Also calculate the electromagnetic momentum stored in the fields.

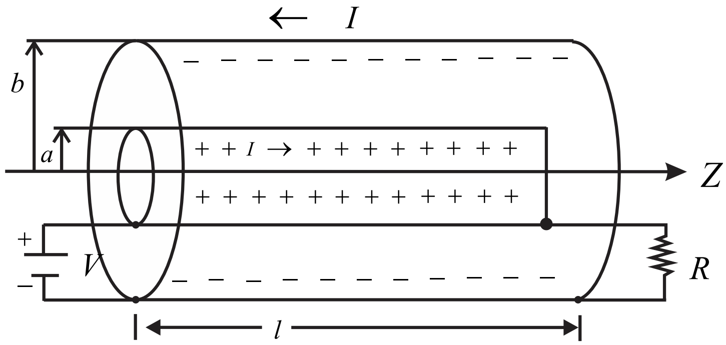


FIG. 10.4

Solution The electric field a distance r from a line charge density λ is

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r} \tag{1}$$

and according to Ampere's law, the magnetic field between the cylinders is

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi} \tag{2}$$

Therefore Poynting vector (energy per unit area per unit time) is

$$\vec{\mathrm{P}} = \vec{E}\times\vec{H} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right) \tag{3}$$

From eqs. (1), (2) and (3), we get

$$\vec{\mathrm{P}} = \frac{1}{\mu_0}\frac{\mu_0 \lambda I}{4\pi^2\varepsilon_0 r^2}\hat{z}$$

$$\vec{\mathrm{P}} = \frac{\lambda I}{4\pi^2\varepsilon_0 r^2}\hat{z}$$

Therefore power (energy per unit time) is given by

$$p = \int \vec{\mathrm{P}}.d\vec{s} = \int_a^b \frac{\lambda I}{4\pi^2\varepsilon_0 r^2}2\pi r dr$$

$$= \frac{\lambda I}{2\pi\varepsilon_0}\int_a^b \frac{dr}{r}$$

$$p = \frac{\lambda I}{2\pi\varepsilon_0} \ln\left(\frac{b}{a}\right) \tag{4}$$

But $$V = \int_a^b \vec{E}.d\vec{r} = \frac{\lambda}{2\pi\varepsilon_0}\int_a^b \frac{dr}{r} = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right) \Rightarrow p = VI \tag{5}$$

The momentum in the field is given by

$$\vec{P}_{field} = \mu_0\varepsilon_0 \int \vec{\mathrm{P}} dv$$

$$= \mu_0\varepsilon_0 \frac{\lambda I}{4\pi^2\varepsilon_0}\int_a^b \frac{l 2\pi r dr}{r^2}\hat{z}$$

$$= \frac{\mu_0 \lambda I l}{2\pi}\int_a^b \frac{dr}{r}$$

$$\vec{P}_{field} = \frac{\mu_0 \lambda I l}{2\pi}\ln\left(\frac{b}{a}\right)\hat{z} \tag{6}$$

The cable is not moving and the fields are static. In fact, if the centre of mass of localized system is at rest, its total momentum must be zero. In this case, it turn out that there is hidden mechanical momentum associated with the flow of current and this exactly cancels the momentum in the fields.

Suppose that we turn up the resistance, so the current decrease. The changing magnetic field will induce an electric field as follows-

$$\vec{E} = \left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln r + k\right]\hat{z} \tag{7}$$

where k is constant and it is a function of time. This field exerts a force on $\pm\lambda$

$$\vec{F} = \lambda l\left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln a + k\right]\hat{z} - \lambda l\left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln b + k\right]\hat{z}$$

$$\vec{F} = -\frac{\mu_0 \lambda l}{2\pi}\frac{dI}{dt}\ln\left(\frac{b}{a}\right)\hat{z} \tag{8}$$

Therefore the total momentum imparted to the cable as the current drops from I to 0, is

$$\vec{P}_{mech} = \int \vec{F} dt$$

$$\vec{P}_{mech} = -\frac{\mu_0 \lambda I l}{2\pi} \ln\left(\frac{b}{a}\right)\hat{z} \tag{9}$$

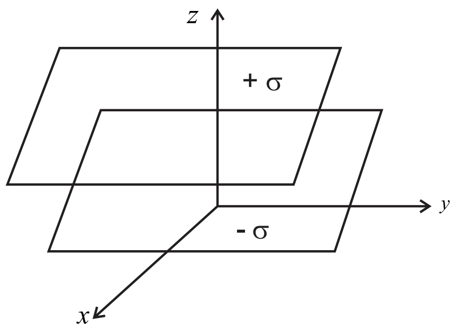
which is precisely the momentum originally stored in the field. The cable will not recoil because an equal and opposite impulse is delivered simultaneously.

Example 10.4 Consider an infinite parallel plate capacitor with the lower plate at $z = -\frac{d}{2}$ carrying the charge density $-\sigma$ and the upper plate at $z = +\frac{d}{2}$ carrying the charge density $+\sigma$ as shown in fig. 10.5

(a) Determine the Maxwell stress tensor in the region between the plates and display in matrix form.

(b) Determine the force per unit area on the top plate and momentum per unit area per unit time crossing the xy plane.

Solution (a) $E_x = E_y = 0, E_z = -\frac{\sigma}{\varepsilon_0}$ and $\vec{B} = 0$



$Fig.$ 10.5

Therefore $\quad T_{xy} = T_{xz} = T_{yz} = ... = 0 \quad (1)$

$$T_{xx} = T_{yy} = -\frac{\varepsilon_0}{2} E^2 = -\frac{\sigma^2}{2\varepsilon_0} \tag{2}$$

$$T_{zz} = \varepsilon_0 \left[E_z^{\ 2} - \frac{1}{2} E^2 \right] = \varepsilon_0 \left[E_z^{\ 2} - \frac{1}{2} E_z^{\ 2} \right] = \frac{\varepsilon_0}{2} E_z^2 = \frac{\sigma^2}{2\varepsilon_0} \tag{3}$$

Therefore the Maxwell stress tensor in matrix form is written as using eqs. (1), (2) and (3)

$$\overleftrightarrow{T} = \frac{\sigma^2}{2\varepsilon_0}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix} \quad (4) \ \textbf{Ans.}$$

(b) $\vec{F} = \oint_s \overleftrightarrow{T}.d\vec{s}$ (Since $\vec{\mathrm{P}} = 0$ because $\vec{B} = 0$)

Integrate one the *xy* plane *i.e.* $d\vec{s} = -dxdy\hat{z}$ (negative sign because of outward with respect to a surface enclosing the upper plate). Therefore

$$F_z = \int T_{zz} ds_z = -\frac{\sigma^2}{2\varepsilon_0} A$$

$$\Rightarrow \vec{f} = \frac{\vec{F}}{A} = -\frac{\sigma^2}{2\varepsilon_0}\vec{z} \quad (5)$$

$$\Rightarrow \quad -T_{zz} = \frac{\sigma^2}{2\varepsilon_0} \quad (6)$$

which is the momentum in the z-direction crossing a surface perpendicular to z.

10.8 Self Learning Exercise

Q.1 Write the integral form of Maxwell's equations and identify each equation with the proper experimental law.

Q.2 Explain the significance of displacement current.

Q.3 Are all four Maxwell's equations independent? Explain.

Q.4 What is the velocity of propagation of electromagnetic waves?

Q.5 Define Poynting vector.

Q.6 What do you mean by microscopic Maxwell's equations?

Q.7 Write the expression for electromagnetic energy density.

Q.8 Write the relation between density of electromagnetic momentum and Poynting vector.

Q.9 What is Kronecker delta?

Q.10 Define the Maxwell's stress tensor.

Q.11 Write the expression for microscopic charge density.

Q.12 Whether the corresponding fields $\vec{d}$ and $\vec{h}$ in the microscopic Maxwell's equations mentioned. Yes or no. Explain it.

10.9 Summary

This unit starts with the introduction of electromagnetic fields. By giving the concept of displacement current, we have introduced Maxwell's equations for electrodynamics and derive the macroscopic Maxwell's equations for electromagnetism. We have also discussed about Poynting theorem (conservation of energy) and momentum for a system of charged particles and electromagnetic fields. The conservation laws for macroscopic media have also been discussed in the last section of unit. In the end, some examples on above concepts are given.

10.10 Glossary

Invariant: not changing

Homogeneous: Containing terms all of the same degree

Localized: happening in or limited to a particular area

10.11 Answers to Self Learning Exercise

Ans.1: $$\oint_s \vec{D}.d\vec{s} = Q$$ Gauss's law

$$\oint_s \vec{B}.d\vec{s} = 0$$ no isolated magnetic charge

$$\oint_L \vec{E}.d\vec{l} = -\int_s \frac{\partial \vec{B}}{\partial t}.d\vec{s} = -\frac{d\phi}{dt}$$ Faraday's law

$$\oint_L \vec{H}.d\vec{l} = I + \int_s \frac{\partial \vec{D}}{\partial t}.d\vec{s}$$ Ampere's modified law

Ans.2: The presence of displacement current means that a changing electric field causes a magnetic field, even in the absence of a current flow. It provides the converse of Faraday's law. Thus the displacement current results into unification of electric and magnetic phenomena.

Ans.3: No, as a matter of face, the two divergence equations can be derived from the two curl equations by making use of the equation of continuity.

Ans.4: $v = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}} = c$

Ans.5: The energy per unit time, per unit area, transported the fields is called Poynting vector.

Ans.6: The equations governing electromagnetic phenomena for charges and currents at the atomic level are called Microscopic Maxwell equations.

Ans.7: $e_{em} = \dfrac{1}{2}\left(\varepsilon_0 E^2 + \dfrac{B^2}{\mu_0}\right)$

Ans.8: $\vec{g}_{em} = \mu_0 \varepsilon_0 \left(\vec{E} \times \vec{H}\right) = \dfrac{1}{c^2}\vec{\mathrm{P}}$

Ans.9: δ_{ij} is the Kronecker delta. It is 1 if the indices are the same and zero otherwise.

Ans.10: Maxwell stress tensor $\left(\vec{T}\right)$ is the force per unit area acting on the surface.

Ans.11: $\eta\left(\vec{X}, t\right) = \underset{j}{\varepsilon}\, q_j \delta\left[\vec{X} - \vec{X}_j\left(t\right)\right]$

Ans.12: There are no corresponding fields $\vec{d}$ and $\vec{h}$ because all the charges are included in η (microscopic charge density) and $\vec{j}$ (microscopic current density).

10.12 Exercise

Section A: Very Short Answer Type Questions

Q.1 How can be obtained the macroscopic Maxwell equations from microscopic Maxwell equations.

Q.2 Write the expression for Poynting vector and its SI unit.

Q.3 Define the Maxwell stress tensor more precisely.

Q.4 How Poynting vector plays a dual role.

Q.5 Write the differential form of conservation law of momentum.

Q.6 How Maxwell Tensor $\left(\overleftrightarrow{T}\right)$ plays a dual role.

Section B: Short Answer Type Questions

Q.7 Write down the microscopic Maxwell equations.

Q.8 State the Poynting's theorem.

Q.9 Write the expression for Maxwell stress tensor.

Q.10 Write the express for conservation of energy for the combined system of particles and fields.

Q.11 Define the statement of conservation of momentum for the combined system of particles and fields.

Q.12 Write down the Macroscopic Maxwell equations.

Section C: Long Answer Type Questions

Q.13 Define the equations of macroscopic electromagnetism from microscopic Maxwell equations.

Q.14 Obtain the Poynting theorem for the conservation of energy in an electromagnetic field and discuss the physical meaning of each term in the resulting equation.

Q.15 State and establish Poynting theorem for conservation of energy in an electromagnetic field. What is the physical significance of the Poynting vector.

Q.16 Derive the expression for conservation of momentum for a system of charged particles and electromagnetic fields.

Q.17 Establish Maxwell's equations for the electromagnetic fields and obtain an expression for Poynting vector.

10.13 Answers to Exercise

Ans.1: The macroscopic Maxwell equations are obtained by taking spatial average of the microscopic Maxwell equations.

Ans.2: $\vec{P} = \vec{E} \times \vec{H} \left(Watt / m^2 \right)$

Ans.3: $\overleftrightarrow{T}$ or T_{ij} (Maxwell stress tensor) is the force per unit area in the i^{th} direction acting on an element of surface oriented in the j^{th} direction.

Ans.4: Poynting vector $\left(\vec{P}\right)$ itself is the energy per unit area, per unit time transported by the electromagnetic fields, while $\mu_0 \varepsilon_0 \vec{P}$ is the momentum per unit volume stored in these fields.

Ans.5: $\dfrac{\partial}{\partial t}\left(\vec{P}_{mech} + \vec{g}_{em}\right) = \vec{\nabla}.\overleftrightarrow{T}$

Ans.6: $\overleftrightarrow{T}$ itself is the electromagnetic force per unit area acting on a surface and $-\overleftrightarrow{T}$ describe the flow of momentum current density transported by the fields.

References and Suggested Readings

1. J.D. Jackson, Classical Electrodynamics, 2nd edition, Wiley eastern Limited, 1978.
2. David Griffiths, Introduction to electrodynamics, 2nd edition, Prentice Hall – 1989.
3. David k. Cheng, Field and Wave Electromagnetics, Pearson, 1989.
4. W.K.H. Panofsky and M. Phillips, Classical Electricity and Magnetism, 2nd edition, Addison- Wesley, 1962.
5. D.S. Jones, The Theory of Electromagnetism, Pergamon, Oxford, 1964.
6. J.A. Stratton, Electromagnetic Theory, McGraw-Hill New York, 1941.
7. E.G. Cullwick, Electromagnetism and Relativity, Longmans, London, 1964.

UNIT-11
Plane Wave in a Non-Conducting Medium

Structure of the Unit

11.0 Objectives

In this chapter our objectives are

(i)To derive wave equation, with the help of Maxwell's equations, that propagates through a non-conducting medium.

(ii)Solutions of the wave equation for good conductors; skin depth.

(iii)Intrinsic impedance of the medium

(iv)Solutions of the wave equation for perfect dielectrics.

11.1 Introduction

This chapter is concerned with plane waves in unbounded or semi-infinite media. The basic properties of plane electromagnetic waves in non conducting media-their transverse nature, are treated on the basis of Maxwell's equations. In this chapter we shall start with Maxwell's equations which are the fundamental equations of electromagnetic field theory. When fields are time – variable, the magnetic field $\vec{H}$ cannot exist without are $\vec{E}$ field nor can $\vec{E}$ exist without a corresponding $\vec{H}$ field. All these facts are best illustrated in the form of a complete set of equations, called Maxwell's equations.

11.2 Basic Review of Maxwell's Equations

The equations grouped below were separately developed in the form of Ampere's law, Faraday's law, Gauss's law, and non existence of monopole. However these laws were extended so as to include time-varying fields by Maxwell. Maxwell's equations are:

Differential form	Integral form
$\vec{\nabla}\times\vec{H}=\vec{J}_C+\dfrac{\partial\vec{D}}{\partial t}$	$\oint\vec{H}.d\vec{l}=\int_S\left(\vec{J}_C+\dfrac{\partial\vec{D}}{\partial t}\right).d\vec{S}$ (Ampere's l)
$\vec{\nabla}\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}$	$\oint\vec{E}.d\vec{l}=\int_S\left(-\dfrac{\partial\vec{B}}{\partial t}\right).d\vec{S}$
$\vec{\nabla}.\vec{D}=\rho_f$	$\oint\vec{D}.d\vec{S}=\int_V\rho\,dV$ (Gauss's law)
$\vec{\nabla}.\vec{B}=0$	$\oint\vec{B}.d\vec{S}=0$ (non existence of monopole)

For free space, where there are no charge $(\rho=0)$ and no conduction currents $(\vec{J}_C=0)$, Maxwell's equations take the form shown below:

Maxwell's equations, Free-space

Differential form	Integral form
$\vec{\nabla}\times\vec{H}=\dfrac{\partial\vec{D}}{\partial t}$	$\oint\vec{H}.d\vec{l}=\int_S\left(\dfrac{\partial\vec{D}}{\partial t}\right).d\vec{S}$

$\vec{\nabla}\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}$	$\oint\vec{E}.d\vec{l}=\int_S\left(-\dfrac{\partial\vec{B}}{\partial t}\right).d\vec{S}$
$\vec{\nabla}.\vec{D}=0$	$\oint\vec{D}.d\vec{S}=0$
$\vec{\nabla}.\vec{B}=0$	$\oint_S\vec{B}.d\vec{S}=0$

The first and second equations (differential form) in the free-space set can be used to show that time variable $\vec{E}$ and $\vec{H}$ fields cannot exist independently. For example, if $\vec{E}$ is a function of time, then $\vec{D}=\epsilon_0\vec{E}$ will also be a function of time, so that $\dfrac{\partial\vec{D}}{\partial t}$ will be non zero. Consequently, $\vec{\nabla}\times\vec{H}$ is non zero, and so a none zero $\vec{H}$ must exist. In a similar way, the second equation can be used to show that if $\vec{H}$ is a function of time, then there must be an $\vec{E}$ field present.

The differential form of Maxwell's equations is used most frequently in the problems. However, the integral form is important in that it better displays the underlying physical laws.

11.3 Derivation of the Wave Equation

In deriving the wave equations, it will be assumed that charge density $\rho=0$. Moreover, linear isotropic material will be assumed, with $\vec{D}=\epsilon\vec{E}$, $\vec{B}=\mu\vec{H}$ and $\vec{J}=\sigma\vec{E}$ with the above assumptions and with the time dependence $e^{j\omega t}$ for both $\vec{E}$ and $\vec{H}$, Maxwell's equations become

$$\vec{\nabla}\times\vec{H}=(\sigma+j\omega\epsilon)\vec{E} \tag{1}$$

$$\vec{\nabla}\times\vec{E}=-j\omega\mu\vec{H} \tag{2}$$

$$\vec{\nabla}.\vec{E}=0 \tag{3}$$

$$\vec{\nabla}.\vec{H}=0 \tag{4}$$

Taking the curl of (1) and (2)

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{H}\right)=(\sigma+j\omega\epsilon)\left(\vec{\nabla}\times\vec{E}\right)$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=-j\omega\mu\left(\vec{\nabla}\times\vec{H}\right)$$

Now using the identity

$$\nabla\times\left(\nabla\times\vec{E}\right)=\nabla\left(\nabla.\vec{E}\right)-\nabla^2\vec{E}$$

and $$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{H}\right)=\vec{\nabla}\left(\vec{\nabla}.\vec{H}\right)-\nabla^2\vec{H},$$

we find the vector wave equations:

$$\nabla^2\vec{H}=j\,\omega\,\mu\left(\sigma+j\,\omega\in\right)\vec{H}\equiv\gamma^2\vec{H}$$

$$\nabla^2\vec{E}=j\,\omega\,\mu\left(\sigma+j\,\omega\in\right)\vec{E}\equiv\gamma^2\vec{E}$$

γ is called the propagation constant and is the square root of γ^2 whose real and imaginary parts are positive:

$$\gamma=\alpha+j\,\beta \qquad \left(j=\sqrt{-1}\right)$$

or $$\alpha+j\,\beta=\sqrt{j\,\omega\,\mu\,\sigma-\omega^2\mu\in}$$

or $$\left(\alpha+j\beta\right)^2=\gamma^2=j\omega\mu\sigma-\omega^2\mu\in$$

or $$\alpha^2-\beta^2+j2\alpha\beta=j\omega\mu\sigma-\omega^2\mu\in$$

Now, equating the real and imaginary parts

$$\alpha^2-\beta^2=-\omega^2\mu\in$$

and $$2\alpha\beta=\omega\mu\sigma$$

Solving for α and β we get

$$\alpha=\omega\sqrt{\frac{\mu\in}{2}\left(\sqrt{1+\left(\frac{\sigma}{\in\omega}\right)^2}-1\right)} \tag{5}$$

$$\beta=\omega\sqrt{\frac{\mu\in}{2}\left(\sqrt{1+\left(\frac{\sigma}{\in\omega}\right)^2}+1\right)} \tag{6}$$

Solutions in Cartesian coordinates

The familiar scalar wave equation in one-dimension,

$$\frac{\partial^2F}{dz^2}=\frac{1}{u^2}\frac{\partial^2F}{\partial t^2}$$

Above equation has solutions of the form $F=f\left(z-ut\right)$ and $F=g\left(z+ut\right)$ where f and g are arbitrary functions.

These represent waves travelling with speed u in the +z and –z directions respectively.

For particular choices

$$f(x) = Ce^{-j\omega\frac{x}{u}} \text{ and } g(x) = De^{+j\omega\frac{x}{u}}$$

Harmonic waves of angular frequency ω are obtained

$$F = Ce^{+j(\omega t - \beta z)} \text{ and } F = De^{j(\omega t + \beta z)}$$

In which $\beta = \dfrac{\omega}{u}$

Of course, the real and imaginary parts are also solutions to the wave equation.

At any fixed *t*, wave form repeats itself when x changes by $\dfrac{2\pi}{\beta}$;the distance

$\lambda = \dfrac{2\pi}{\beta}$ is called the wavelength

The wavelength and the frequency $f = \dfrac{\omega}{2\pi}$ has the relation

$$\lambda f = u \text{ or } \lambda = Tu$$

where $T = \dfrac{1}{f} = \dfrac{2\pi}{\omega}$ is the period of the harmonic wave.

11.4 Solution to the Wave Equation for partially Conducting Media

For a region in which there is some conducting but not much (e.g. moist earth, sea water),the solution to the wave equation in $\vec{E}$ is to taken to be $\vec{E} = E_0 E^{-\gamma z}\hat{a}_x$

Then, $\vec{H}$ is obtained from the Maxwell's equation

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\vec{H} = \sqrt{\frac{\sigma + j\omega\in}{j\omega\mu}}\, E_0 e^{-\gamma z}\hat{a}_y$$

The ratio $\dfrac{E}{H}$ is characteristic of the medium (it is also frequency – dependent). More specifically for waves $\vec{E} = E_x\hat{a}_x$, $\vec{H} = H_y\hat{a}_y$ ***which propagate in the +z direction, the intrinsic impedance,*** η***, of the medium is defined by***

$$\boxed{\eta = \frac{E_x}{H_y}}$$

Thus $\eta = \sqrt{\dfrac{j\omega\mu}{\sigma + j\omega \in}}$

where the square root may be written in the polar form, $|\eta| < \theta$, with

$$|\eta| = \frac{\sqrt{\mu / \in}}{4\sqrt{1+\left(\dfrac{\sigma}{\omega \in}\right)^2}}, \tan 2\theta = \frac{\sigma}{\omega \in}$$

If the wave propagates in the $-z$ direction, $\dfrac{E_x}{H_y} = -\eta$

Inserting the time factor $e^{j\omega t}$ and writing $\gamma = \alpha + j\beta$ results in the following equations for the fields in a partially conducting region:

$$\vec{E}(z,t) = E_0 e^{-\alpha z} e^{j(\omega t - \beta z)} \hat{a}_x$$

$$\vec{H}(z,t) = \frac{E_0}{\eta} e^{-\alpha z} e^{j(\omega t - \beta z - \theta)} \hat{a}_y$$

The ***factor*** $e^{-\alpha z}$ ***attenuates the magnitudes of both*** $\vec{E}$ ***and*** $\vec{H}$ ***as they propagate*** in the $+z$ direction.

The expression for α in eq. (5) shows that ***there will be some attenuation unless the conductivity*** σ ***is zero, which would be the case only for perfectly dielectric or free space.***

Likewise, the phase difference θ between $\vec{E}(z,t)$ and $\vec{H}(z,t)$ vanishes only when σ is zero.

The velocity of propagation and the wavelength are given by

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\dfrac{\mu \in}{2}\left(\sqrt{1+\left(\dfrac{\sigma}{\omega \in}\right)^2}+1\right)}}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\dfrac{\mu \in}{2}\left(\sqrt{1+\left(\dfrac{\sigma}{\omega \in}\right)^2}+1\right)}}$$

If the propagation velocity is known, $\lambda f = u$ may be used to determine the wavelength λ. The term $\left(\frac{\sigma}{\omega \in}\right)^2$ has the effect of reducing both the velocity and wavelength from what they would be in either free space or perfect dielectrics, where $\sigma = 0$. Note that the ***medium is dispersive: waves with different frequencies*** ω ***have different velocities*** u.

11.5 Solution to the Wave Equation for Perfect Dielectrics

For perfect dielectric $\sigma = 0$, and so $\alpha = 0$

$$\beta = \omega\sqrt{\mu \in},\ \eta = \sqrt{\frac{\mu}{\in}} < 0$$

Since $\alpha = 0$***, there is no attenuation of*** $\vec{E}$ ***and*** $\vec{H}$ ***waves. The zero angle on*** η ***results in*** $\vec{H}$ ***being in time phase with*** $\vec{E}$ ***at each fixed location.***

Assuming the $\vec{E}$ of the wave in x-direction and propagation of the wave in $z-$ direction, then the field equations may be obtained as

$$\vec{E}(z,t) = E_0 e^{j(\omega t - \beta z)}\hat{a}_x$$

$$\vec{H}(z,t) = \frac{E_0}{\eta} e^{j(\omega t - \beta z)}\hat{a}_y$$

The velocity and the wavelength are

$$\boxed{u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}},\ \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu \in}}$$

11.6 Solution to the wave equation in Free Space

Free space in nothing more than the perfect dielectric for which

$$\mu = \mu_0 = 4\pi \times 10^{-7}\,\frac{H}{m}$$

$$\in = \in_0 = 8.854 \times 10^{-12}\,\frac{F}{m} = \frac{10^{-9}}{36\pi}\frac{F}{m}$$

For free space,

$$\eta = \eta_0 \approx 120\pi\,\Omega \text{ and } u = c = 3\times 10^8\,\frac{m}{s}$$

11.7 Solution to the Wave equation for Good Conductors: Skin Depth

Materials are ordinarily classified as good conductors if $\sigma \gg \omega \in$ ***in the range of practical frequencies.***

Therefore, the propagation constant and the intrinsic impedance are

$$\gamma = \alpha + j\beta\,,\ \gamma = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu\sigma}$$

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}} < 45^0$$

It is seen that for all conductors the $\vec{E}$ and $\vec{H}$ waves are attenuated. This is a very rapid attenuation. α will always be equal to β. At each fixed location $\vec{H}$ is out of time phase with $\vec{E}$ by $\frac{\pi}{4}$ rad. Once again assuming $\vec{E}$ in $\hat{a}_x$ and propagation in $\hat{a}_z$, the field equations are

$$\vec{E}(z,t) = E_0 e^{-\alpha z} e^{j(\omega t - \beta z)} \hat{a}_x$$

$$\vec{H}(z,t) = \frac{E_0}{|\eta|} e^{-\alpha z} e^{j\left(\omega t - \beta z - \frac{\pi}{4}\right)} \hat{a}_y$$

Moreover $u = \dfrac{\omega}{\beta} = \sqrt{\dfrac{2\omega}{\mu\sigma}} = \omega\delta,$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{\pi f \mu\sigma}} = 2\pi\delta$$

The velocity and wavelength in a conducting medium are written here in terms of the ***skin depth or depth of the penetration***

$$\boxed{\delta = \frac{1}{\sqrt{\pi f \mu\sigma}}}$$

11.8 Illustrative Examples

Example1: Write the Maxwell's equations for electromagnetic fields in a homogeneous medium with constant $\in$ and μ. Deduce the wave equation for $\vec{H}$.

Sol. In a homogeneous medium with constant $\in$and μ, the Maxwell's equation.

$div\vec{B}=0$ becomes $div\vec{H}=0$ ($\because \vec{B}=\mu\vec{H}$ and μ is constant).

Similarly the equation $div\vec{D}=0$ becomes $div\vec{E}=0$ (because $\in$ is constant and $\vec{D}=\in\vec{E}$)

and $Curl\vec{H}=\frac{1}{c}\frac{\partial\vec{D}}{\partial t}$ becomes $Curl\vec{H}=\frac{\in}{c}\frac{\partial\vec{E}}{\partial t}$.

The equation $Curl\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$, becomes $Curl\vec{E}=-\frac{\mu}{c}\frac{\partial\vec{H}}{\partial t}$.

So the Maxwell's equations for the homogeneous medium with constant $\in$ and μ are:

$$div\vec{E}=0,$$

$$div\vec{H}=0$$

$$Curl\vec{E}=-\frac{\mu}{c}\frac{\partial\vec{H}}{\partial t},$$

$$Curl\vec{H}=\frac{\in}{c}\frac{\partial\vec{E}}{\partial t}$$

Eliminating $\vec{E}$ in the usual manner, we obtain

$$CurlCurl\vec{H}=\frac{\in}{c}\frac{\partial}{\partial t}Curl\vec{E}=-\frac{\in\mu}{c^2}\frac{\partial^2\vec{H}}{\partial t^2}$$

and since $CurlCurl\vec{H}=grad\,Div\vec{H}-\nabla^2\vec{H}$,

we reach the wave equation

$$\nabla^2\vec{H}-\frac{\mu\in}{c^2}\frac{\partial^2\vec{H}}{\partial t^2}=0.$$

Example 2: A plane electromagnetic wave is propagating in an infinite homogeneous material medium having permittivity $\in$ and permeability μ. Find the general relationship between the three vectors $\vec{k},\vec{E}$ and $\vec{H}$.

Sol. In a plane wave in a vacuum, the spatial dependence of the field is given by a factor $e^{i\vec{k}.\vec{r}}$, with a real wave vector $\vec{k}$. In considering wave propagation in a matter, however, it is in ***general necessary to take*** $\vec{k}$ ***complex:***

$$\vec{k} = \vec{k'} + i\vec{k''},$$

where the vectors $\vec{k'}$ and $\vec{k''}$ are real.

Taking $\vec{E}$ and $\vec{H}$ as proportional to $e^{i\vec{k}.\vec{r}}$, and carrying out the differentiation with respect to coordinates in Maxwell's equations for monochromatic field, viz.

$$i\omega\mu\vec{H} = c\,Curl\,\vec{E} \qquad i\omega\in\vec{E} = -c\,Curl\,\vec{H}$$

We obtain

$$i\omega\mu\vec{H} = c(i\vec{k}\times\vec{E}) \quad \text{and} \quad i\omega\in\vec{E} = -c(i\vec{k}\times\vec{H})$$

(Here Curl operator is replaced by $i\vec{k}$)

$$\text{or } \omega\mu\vec{H} = c\vec{k}\times\vec{E} \qquad \omega\in\vec{E} = -c\vec{k}\times\vec{H}$$

This is how $\vec{k}, \vec{H}$ and $\vec{E}$ are related.

In particular , taking the scalar product of these formulae with $\vec{k}$, we obtain

$$\vec{k}.\vec{E} = 0 \qquad , \vec{k}.\vec{H} = 0$$

Using $\vec{k} = \vec{k'} + i\vec{k''}$ we obtain for the square of the wave vector

$$k^2 = k'^2 - k''^2 + 2i\vec{k'}.i\vec{k''} = \in\frac{\mu\omega^2}{c^2}$$

We see that $\vec{k}$ can be real only if $\in$ and μ are real and positive.

11.9 Self Learning Exercise

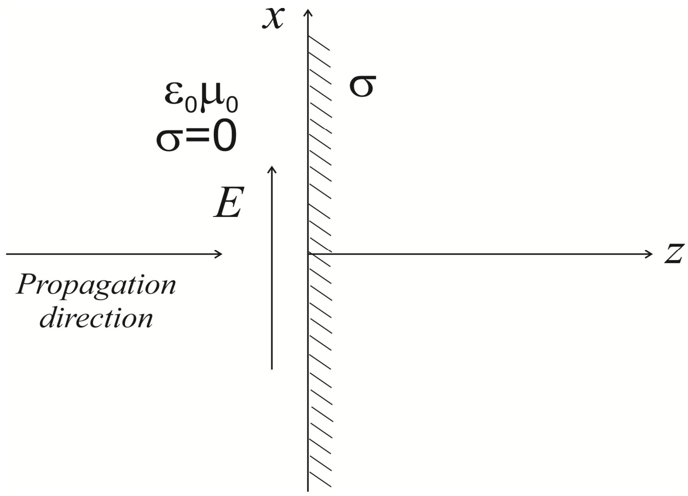
Q.1 Deduce the wave equation for $\vec{E}$ in a homogeneous medium with constant $\in$ and μ .

Q.2 Write Maxwell's equations for a monochromatic field (i.e. a plane monochromatic wave) and deduce wave equation.

Q.3 Write Maxwell's equation in free space in the integral form.

11.10 Illustrative Examples

Example 3: Assume a field $\vec{E} = 1.0e^{-\alpha z}e^{j(\omega t - \beta z)}\hat{a}_x\left(\frac{V}{m}\right)$; with $f = \frac{\omega}{2\pi} = 100MHz$, at the surface of a conductor, $\sigma = 58$ Mega Siemens (MS)/m, located at z > 0, as shown in figure given below. Calculate the skin depth as the wave propagates into the conductor.



Sol. At depth z the magnitude of the field is

$$\left|\vec{E}\right| = 1.0e^{-\alpha z} = 1.0e^{-z/\delta}$$

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \equiv \frac{6.6\times 10^{-2}}{\sqrt{f}}$$

For copper $\mu_r = 1$, so that $\mu = \mu_0 = 1.26\times 10^{-6}\frac{H}{m}$

$$\therefore \delta = \frac{1}{\left[3.14\times 100\times 10^6\times 1.26\times 10^{-6}\times 58\times 10^6\right]^{\frac{1}{2}}}m \qquad = 6.61\mu m$$

Example 4: In fee space, $\vec{E}(z,b) = 10^3\sin(\omega t - \beta z)\hat{a}_y\left(\frac{V}{m}\right)$. Obtain $\vec{H}(z,b)$.

Sol. Examination of the phase, $\omega t - \beta z$, shows that the direction of propagation is +z. since $\vec{E}\times\vec{H}$ must also be in the +z-direction, $\vec{H}$ must have the direction $-\hat{a}_x$. Consequently

$$\frac{E_y}{-H_x} - \eta_0 = 120\pi\Omega$$

or $\quad H_x = -\dfrac{10^3}{120\pi}\sin(\omega t - \beta z)\dfrac{A}{m}$ $\quad$ or $H_x = -\dfrac{10^3}{120\pi}\sin(\omega t - \beta z)\hat{a}_x\dfrac{A}{m}$

Example 5: For the wave given in problem (6), determine the propagation constant γ, given that the frequency if $f = 95.5$ MHz.

Sol. In general $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$ $\qquad (j = \sqrt{-1})$

In free space, $\sigma = 0$, so that

$$\gamma = j\omega\sqrt{\mu_0 \in_0} = j\left(\frac{2\pi f}{c}\right)$$

$$= j\frac{2\pi(95.5\times10^6)}{3\times10^8} = j(2.0)m^{-1}$$

Note that this result shows that the attenuation factor is $\alpha = 0$ and the phase shift constant in $\beta = 2.0\dfrac{rad}{m}$.

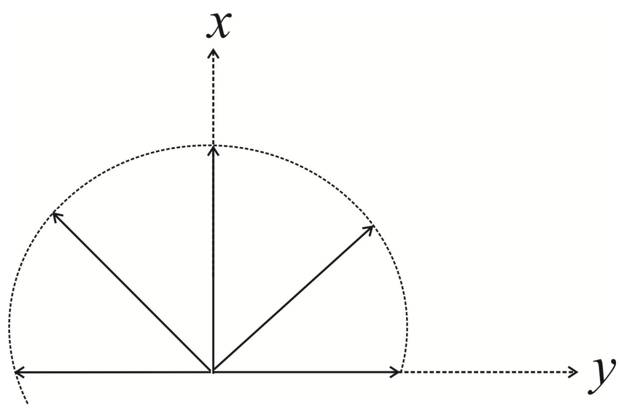
Example 6: Examine the field

$$\vec{E}(z,t) = 10\sin(\omega t + \beta z)\hat{a}_x + 10\cos(\omega t + \beta z)\hat{a}_y$$

In the plane $z = 0$ for $\omega t = 0, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{3\pi}{4}, \pi$,what is its polarization?

Sol. At different instants, E_x, E_y and $\vec{E}$ are shown below

ωt	$E_x = 10\sin\omega t$	$E_y = 10\cos\omega t$	$\vec{E} = E_x\hat{a}_x + E_y\hat{a}_y$
0	0	10	$10\hat{a}_y$
$\dfrac{\pi}{4}$	$\dfrac{10}{\sqrt{2}}$	$\dfrac{10}{\sqrt{2}}$	$10\left(\dfrac{\hat{a}_x + \hat{a}_y}{\sqrt{2}}\right)$
$\dfrac{\pi}{2}$	10	0	$10\hat{a}_x$
$\dfrac{3\pi}{4}$	$\dfrac{10}{\sqrt{2}}$	$-\dfrac{10}{\sqrt{2}}$	$10\left(\dfrac{\hat{a}_x + \hat{a}_y}{\sqrt{2}}\right)$
π	0	-10	$10(-\hat{a}_y)$



As shown in the figure $\vec{E}(x,t)$ is circularly polarized. In addition, the wave travels in the $-\hat{a}_z$ direction.

Example 7: Calculate the ocean depths at which a $1\mu Vm^{-1}$ field will be obtained with E at the surface equal to $1Vm^{-1}$ at frequencies of 1, 10, 100 and 1000 KHz. What is the most suitable frequency for communication with submerged submarines?

Given that $\sigma = 4$ Siemen per meter (or Sm^{-1}) and $\in_r = 80$ for sea water.

Sol. At the highest frequency (1000 K Hz), the value of

$$\omega\varepsilon \approx 2\pi \times 10^6 \times 8.85 \times 10^{-12} \times 80$$
$$= 4.4 \times 10^{-3}$$

Therefore at 1000 K Hz, $\sigma >> \omega\varepsilon$, so that

$$\alpha = \sqrt{\frac{\omega\mu\sigma}{2}}$$ can be used at all.

Four frequencies

At 1 KHz $\alpha = \sqrt{\frac{2\pi \times 10^3 \times 4\pi \times 10^{-7} \times 4}{2}} = 0.13 Npm^{-1}$

Since $\frac{E}{E_0} = 10^{-6} = e^{-\alpha x}$, $\therefore x = \frac{6}{\alpha}\log e = \frac{13.8}{\alpha}$

and at 1 KHz, $x = \frac{13.8}{0.13} = 106m$

At 10 KHz, x = 35 m ;

At 100 KHz, x = 11 m ; and

At 1000 KHz, x = 3.5 m, where x = depth

Although 1 KHz would appear to be the best of the above four frequencies, an even lower frequency might be desirable depending on other factors including the efficiencies of the antennas for transmitting and receiving.

11.11 Summary

Starting with Maxwell's equations we derived the wave equations for propagation in dielectric/conducting media. We deduced the value of the complex propagation constant. We solved the wave equation for fields in conducting and dielectric media. We obtained solutions for perfect dielectrics, free-space and for good conductors. Finally we obtained expression for skin depth of penetration into a conductor. The unit ended with illustrative problems with solution.

11.12 Glossary

Induce : to cause something to happen

Homogeneous: Containing terms all of the same degree

Monochromatic: having single frequency

Dispersive: waves with different frequencies have different velocities .

11.13 Answers to Self Learning Exercise

Ans.1: The Maxwell's equation become

$$div\vec{E} = 0,\ div\vec{H} = 0$$

$$Curl\vec{E} = -\frac{\mu}{c}\frac{\partial \vec{H}}{\partial t},\ Curl\vec{H} = \frac{\in}{c}\frac{\partial \vec{E}}{\partial t}$$

Eliminating $\vec{H}$,We obtain

$$CurlCurl\vec{E} = -\frac{\mu}{c}\frac{\partial}{\partial t}Curl\vec{H}$$

or $$grad\, Div\vec{E} - \nabla^2\vec{E} = -\frac{\mu}{c}\frac{\partial}{\partial t}\left(\frac{\in}{c}\frac{\partial \vec{E}}{\partial t}\right)$$

or $-\nabla^2 \vec{E} = -\frac{\mu \in}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$

or $\nabla^2 \vec{E} - \frac{\mu \in}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$

We see that the velocity of propagation of electromagnetic waves in a homogeneous dielectric is

$$\frac{c}{\sqrt{\mu \in}}$$

Ans.2: Maxwell's equations for a monochromatic field are obtained by replacing the operator $\frac{\partial}{\partial t}$ by $(-i\omega)$.

Curl $\vec{E} = -\frac{\mu}{c} \frac{\partial \vec{H}}{\partial t}$ becomes Curl $\vec{E} = -\frac{\mu}{c} (-i\omega) \vec{H}$

or $\quad i\omega\mu(\omega)\vec{H} = c\, Curl\, \vec{E}$ (1)

and $\quad$ Curl $\vec{H} = \frac{\in}{c} \frac{\partial \vec{E}}{\partial t}$ becomes Curl $\vec{H} = \frac{\in}{c} (-i\omega) \vec{E}$

or $\quad i\omega \in (\omega) \vec{E} = -c\, Curl\, \vec{H}$ (2)

So, the equations for a monochromatic field are :

$$i\omega\mu(\omega)\vec{H} = c\, Curl\, \vec{E}$$

and $\quad i\omega \in (\omega)\vec{E} = -c\, Curl\, \vec{H}$

Eliminating $\vec{H}$ from these equations in the usual way we find

$$Curl Curl\, \vec{E} = \frac{i\omega\mu(\omega)}{c} Curl\, \vec{H}$$

or $\quad grad\; Div \vec{E} - \nabla^2 \vec{E} = \frac{i\omega\mu(\omega)}{c} - \left(\frac{i\omega \in (\omega)}{c} \right) \vec{E}$

or $\quad -\nabla^2 \vec{E} = \frac{\omega^2 \mu \in \vec{E}}{c^2}$

or $\quad \nabla^2 \vec{E} + \frac{\omega^2 \mu \in}{c^2} \vec{E} = 0$ Wave equation

Ans.3: The equation are (For free space $\rho = 0, \vec{j} = 0$)

$$\oint \vec{H}.\vec{dl} = \int_s \frac{\partial \vec{D}}{\partial t}.\vec{ds} \tag{1}$$

$$\oint \vec{E}.\vec{dl} = \int_s \frac{\partial \vec{B}}{\partial t}.\vec{ds} \tag{2}$$

$$\oint \vec{D}.d\vec{s} = 0 \tag{3}$$

$$\oint \vec{B}.d\vec{s} = 0 \tag{4}$$

11.14 Exercise

Q.1 Determine the propagation constant γ for a material having $\mu_r = 1, \in_r = 8$, and $\sigma = 0.25 \ {pS}/{m}$, if the wave frequency is $1.6 MHz$.

Q.2 Write Maxwell's equations in the differential form in free-space.

Q.3 Write Maxwell's equations for harmonically varying fields.

Q.4 In free space $\vec{E}(z,t) = 50\cos(\omega t - \beta z)\hat{a}_x \left(\frac{V}{m}\right)$.

Find the average power crossing a circular area of radius 2.5 m in the plane z = constant.

11.15 Answers to Exercise

Ans.1: In this case

$$\frac{\sigma}{\omega \in} = \frac{0.25 \times 10^{-12}}{2\pi(1.6 \times 10^6)(8)\left(10^{-9}/36\pi\right)} \approx 10^{-9} \approx 0$$

So that $\alpha \approx 0, \beta \approx \omega\sqrt{\mu \in} = 2\pi f \frac{\sqrt{\mu_r \in_r}}{c} = 9.48 \times 10^{-2} \frac{rad}{s}$

and $\gamma = \alpha + j\beta \approx j9.48 \times 10^{-2} m^{-1}$.

The material behaves like a perfect dielectric at the given frequency. Conductivity of the order of $1\frac{pS}{m}$ indicates that the material is more like an insulator than a conductor.

Ans.2: The equations are for free space ($\rho = 0, \vec{j} = 0$)

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \qquad (1)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (2)$$

$$\vec{\nabla}.\vec{D} = 0 \qquad (3)$$

$$\vec{\nabla}.\vec{B} = 0 \qquad (4)$$

Ans.3: For harmonic variation, the phasor form of Maxwell's integral and differential equations are :

Integral form		Differential form	
$\oint \vec{H}.\vec{dl} = (\sigma + j\omega \in)\int_s \vec{E}.d\vec{s}$	(1)	$\vec{\nabla} \times \vec{H} = (\sigma + \text{jw} \in)\vec{E}$	(1)
$\oint \vec{E}.\vec{dl} = -j\omega\mu\int_s \vec{H}.d\vec{s}$	(2)	$\vec{\nabla} \times \vec{H} = -j\omega\mu\vec{H}$	(2)
$\oint \vec{D}.\vec{ds} = \int_v \rho_f dv$	(3)	$\vec{\nabla}.\vec{D} = \rho_f$	(3)
$\oint \vec{B}.\vec{ds} = 0$	(4)	$\vec{\nabla}.\vec{B} = 0$	(4)

Ans.4: In complex form

$$\vec{E} = 50e^{j(\omega t - \beta z)}\hat{a}_x \left(\frac{V}{m}\right)$$

and since $\eta = 120\pi\Omega$ and propagation is in +z-direction.

$$\vec{H} = \frac{5}{12\pi}e^{j(\omega t - \beta z)}\hat{a}_y \left(\frac{A}{m}\right)$$

Then $P_{average} = \frac{1}{2}\text{Re}(\text{E} \times \text{H}^*)$

$$= \frac{1}{2}(50)\left(\frac{5}{12\pi}\right)\hat{a}_z \frac{W}{m^2}$$

The flow is normal to the area, and so

$$P_{avg} = \frac{1}{2}(50)\left(\frac{5}{12\pi}\right)\left(\pi\left(2.5\right)^{2}\right)W = 65.1W$$

References and Suggested Readings

1. John David Jackson ,Classical Electrodynamics ,Third Edition ,John-Wiley Sons,2010
2. L.D.Landau ,E.M.Lifshitz and L.P.Pitaevskii, Electrodynamics of Continuous Media (Second Edition)Butterworth-Heinemann,2010

UNIT-12
Waves in a Conducting or Dissipative Medium

Structure of the Unit

12.0 Objectives

After interacting with the material presented here students will be able to understand

5. Waves in a dissipative medium
6. Superposition of Waves, and
7. Causality connection between D and E, Kramers-Kroning relation

12.1 Waves in a Conducting or Dissipative Medium

Let us consider some linear medium with $D = \epsilon E, B = \mu H, J = \sigma E;\ \epsilon, \mu, \sigma$ are taken as real. Then the Maxwell equations become

$$\nabla . B = 0,$$
$$\nabla . E = 0$$
$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$$
and
$$\nabla \times B = \frac{4\pi\mu}{c}\,\sigma E + \frac{\epsilon\mu}{c}\frac{\partial E}{\partial t}$$
We have set ρ equal to zero in these equations. It may be that there is initially some macroscopic charge density within a conductor. If this is the case, that density will decay to zero with a characteristic time on the order of γ^{-1} where γ is the damping constant.

Let us look for plane wave solutions to the field equations.
$$E(x,t) \;=\; E_0 e^{i(k.x-\omega t)}$$
and
$$B(x,t) \;=\; B_0 e^{i(k.x-\omega t)}$$
The divergence equations then tell us that $\vec{E}.\vec{k} = 0 \; and \, \vec{B}.\vec{k} = 0$ as in a nondissipative medium. From Faraday's law we find the familiar result
$$\boxed{\vec{B}_0 = \frac{1}{c}\left(\hat{k}\times\vec{E}_0\right)}$$
and from the Ampere's law we find
$$i\left(\vec{k}\times\vec{B}_0\right) = \frac{4\pi\mu\sigma}{c}\vec{E}_0 - \frac{i\omega\mu\epsilon}{c}\vec{E}_0$$
From above two equations and identity ($k \times k \times E_0 = -k^2 E_0$)
$$-i\frac{4\pi\mu\sigma}{c}E_0 - \frac{\omega\mu\epsilon}{c}E_0 = -\frac{ck^2}{\omega}E_0$$
or
$$k^2 \;= i\frac{4\pi\mu\sigma\omega}{c^2} + \frac{\omega^2\mu\epsilon}{c^2}$$
Taking the point of view that ω is some given real frequency, we can solve this relation for the corresponding wave number k, which is complex.

If we write $k = k_0 + i\alpha$, then the real and imaginary parts of above equation give us two equations which may be solved for k_0 and α

$$k_0^2 - \alpha^2 = \frac{\omega^2 \epsilon\mu}{c^2}\ , 2k_0\alpha = \frac{\omega^2 \mu\epsilon}{c^2}\left(\frac{4\pi\sigma}{\epsilon\omega}\right)$$

The solution is

$$\begin{Bmatrix} k_0 \\ \alpha \end{Bmatrix} = \sqrt{\mu\epsilon}\left(\frac{\omega}{c}\right)\left\{\frac{\sqrt{1+\left(\frac{4\pi\sigma}{\epsilon\omega}\right)^2} \pm 1}{2}\right\}^{\frac{1}{2}}$$

where the + sign refers to k_0 and the - sign to α.

This expression appears somewhat impenetrable although it doesn't say anything unexpected or remarkable. It takes on simple forms in the limits of high and low conductivity. The relevant dimensionless parameter is $\frac{4\pi\sigma}{\epsilon\omega}$. It if is much larger than unity, corresponding to a good conductor, then

$$k_0 \approx \alpha \approx \frac{\sqrt{2\pi\omega\mu\sigma}}{c} = \frac{1}{\delta}\ \ \frac{4\pi\sigma}{\epsilon\omega} \gg 1$$

Where we have introduced the penetration depth δ. This is the distance that an electromagnetic wave will penetrate into a good conductor before being attenuated to a fraction 1/e of its initial amplitude. Since the wavelength of the wave is $\lambda = 2\pi/k_0$, δ is also a measure of the wavelength in the conductor.

For a poor conductor, by which we mean $\frac{4\pi\sigma}{\epsilon\omega} \ll 1$, one has

$$k_0 + i\alpha \approx \frac{\sqrt{\mu\epsilon}\omega}{c} + \frac{i2\pi}{c}\sqrt{\frac{\mu}{\epsilon}}\sigma$$

Note:- In the latter case, the real part of the wave number is the same as in a nonconducting medium and the imaginary part is independent of frequency so that waves of all frequencies are attenuated by equal amounts over a given distance. Also, $\alpha \ll k_0$ which tells us that the wave travels many wavelengths before being attenuated significantly.

For a given σ, α is an increasing function of ω and saturates at high frequencies. Therefore, if one wants a wave to travel as far as possible, one wants to use as low frequency a wave as possible. Then one should be in the good-conductor limit where the attenuation varies as $\sqrt{\omega}$ and vanishes as $\omega \to 0$.

Given that we have found the complex wave number, and letting k point in the z-direction, we have

$$\boxed{E(x,t) = E_0 e^{i(k_0 . z - \omega t)} e^{-z}}$$

$$B(x,t) = \frac{c}{\omega}(k_0 + i\alpha)(\hat{z} \times E_0) e^{i(k_0 z - \omega t)} e^{-\alpha z}$$

Define the complex index of refraction

$$\boxed{n \equiv \frac{c}{\omega} k = \frac{c}{\omega}(k_0 + i\alpha)}$$

so that

$$B = n(\hat{z} \times E_0)$$

Note that because n is complex, B is not in phase with E; to make the phase difference explicit, let us write n in polar form:

$$n = |n| e^{i\varphi}$$

Where

$$\varphi = \arctan\left(\frac{\alpha}{k_0}\right)$$

We can find |n| and φ in terms of other parameters; let $\gamma \equiv \left(\frac{4\pi\sigma}{\epsilon\omega}\right)^2$. Then

$$\varphi = \arctan\left[\frac{\sqrt{1+\gamma} - 1}{\sqrt{1+\gamma} + 1}\right]^{\frac{1}{2}}$$

We know

$$\tan 2\varphi = \frac{2\, tan\, \varphi}{1 - tan^2\, \varphi} = 2\left[\frac{\frac{\sqrt{1+\gamma}-1}{\sqrt{1+\gamma}+1}}{1 - \frac{(\sqrt{1+\gamma}-1)}{\sqrt{1+\gamma}} + 1}\right]^{\frac{1}{2}} = \sqrt{\gamma}$$

Thus,

$$\varphi = \frac{1}{2}\arctan \gamma^{\frac{1}{2}} = \frac{1}{2}\arctan\left(\frac{4\pi\sigma}{\epsilon\omega}\right)$$

And

$$|n| = \frac{c}{\omega}\sqrt{k_0^2 + \alpha^2} = \sqrt{\mu\epsilon}\left[1 + \left(\frac{4\pi\sigma}{\epsilon\omega}\right)^2\right]^{\frac{1}{4}}$$

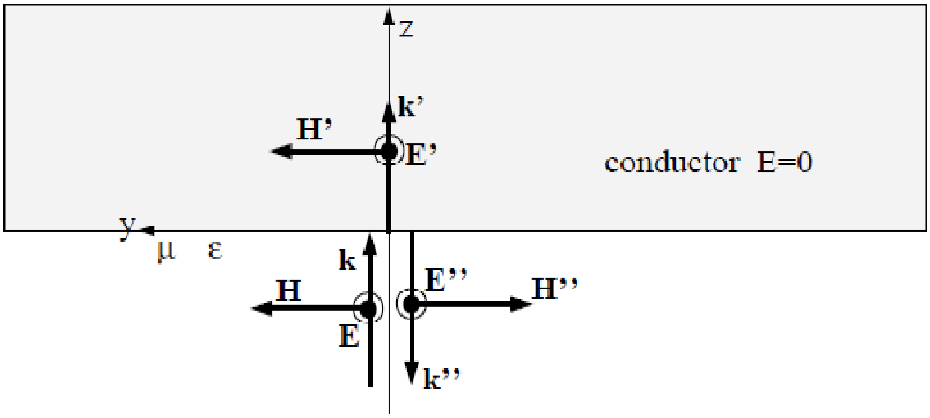
Then we have

$$B(x,t) = \sqrt{\mu\epsilon}\left[1 + \left(\frac{4\pi\sigma}{\epsilon\omega}\right)^2\right]^{\frac{1}{4}} e^{\frac{i}{2}\arctan\left(\frac{4\pi\sigma}{\epsilon\omega}\right)}(\hat{z} \times E_0)$$

The amount by which B(x,t) is phase-shifted from E(x,t) is easily seen from this expression to lie between 0 and $\pi/4$, it is zero in the small σ/ω limit and π/4 in the large σ/ω limit.

Another significant feature of the expression for B(x,t) is that in the small σ/ω limit, the amplitude of B relative to that of E is just $\sqrt{\mu\epsilon}$ as for insulators. But in the opposite limit, one finds that the relative amplitude is $\sqrt{\frac{4\pi\sigma\mu}{\omega}}$ which is much larger than unity. Here the wave has, relatively speaking, a much larger magnetic induction than electric field.

12.2Reflection of a Wave Normally Incident on a Conductor



Let us consider a wave normally incident on a conductor from vacuum.

We know

$$k = \frac{\omega}{c}\hat{z}, k` = \frac{\omega}{c}n\hat{z}$$

$$n = \sqrt{\mu\epsilon}(1+\gamma)^{\frac{1}{4}}e^{i\varphi}$$

The relevant boundary conditions are H_t and E_t continuous.

Let $E_0=E_0\hat{x}$, $E`_0=E`_0\hat{x}, E``_0 = E``_0\hat{x}$. The corresponding magnetic field amplitudes are $H_0=E_0\hat{y}$, $H``_0= - E``_0\hat{y}$, and, for the transmitted wave in the conductor,

$$H_0' = \sqrt{\frac{\epsilon}{\mu}}(1+\gamma)^{\frac{1}{4}}e^{i\varphi}E_0'\hat{y}$$

Our boundary conditions give immediately

$$E_0 + E_0`` = E_0`$$

$$E_0 - E_0`` = \sqrt{\frac{\epsilon}{\mu}}(1+\gamma)^{\frac{1}{4}}e^{i\varphi}E_0'$$

Then we get

$$E_0' = \frac{2}{1+\sqrt{\mu\epsilon}(1+\gamma)^{\frac{1}{4}}e^{i\varphi}}E_0$$

$$E_0`` = \frac{1-\sqrt{\frac{\epsilon}{\mu}}(1+\gamma)^{\frac{1}{4}}e^{i\varphi}}{1+\sqrt{\frac{\epsilon}{\mu}}(1+\gamma)^{\frac{1}{4}}e^{i\varphi}}E_0$$

Let us calculate the Poynting vector in the conductor. Its time average is

$$< S` \geq \frac{c}{8\pi}\mathcal{R}(E` \times H`) = \frac{c}{8\pi}\mathcal{R}\left\{\frac{4|E_0|^2\sqrt{\frac{\epsilon}{\mu}}(1+\gamma)^{\frac{1}{4}}e^{-i\varphi}}{\left|1+\sqrt{\frac{\epsilon}{\mu}}(1+\gamma)^{\frac{1}{4}}e^{i\varphi}\right|^2}\right\}e^{-2\alpha z}\hat{z}$$

Using the interpretation of this vector as the energy current density, we may find the power per unit area transmitted into the conductor by evaluating < S` > .$\hat{z}$ at z = 0,

$$P` = \frac{c}{2\pi}|E_0|^2\sqrt{\frac{\epsilon}{\mu}}\left\{\frac{(1+\gamma)^{\frac{1}{4}}cos\varphi}{1+2\sqrt{\frac{\epsilon}{\mu}}cos\varphi(1+\gamma)^{\frac{1}{4}}+\left(\frac{\epsilon}{\mu}(1+\gamma)^{\frac{1}{2}}\right)}\right\}$$

The incident power per unit area is $P = \frac{c}{8\pi}|E_0|^2$, so the fraction of the incident power which enters the conductor, where it is dissipated as Joule heat, is

$$T = \frac{P`}{P} = 4\sqrt{\frac{\epsilon}{\mu}}\left\{\frac{(1+\gamma)^{\frac{1}{4}}cos\varphi}{1+2\sqrt{\frac{\epsilon}{\mu}}cos\varphi(1+\gamma)^{\frac{1}{4}}+\left(\frac{\epsilon}{\mu}(1+\gamma)^{\frac{1}{2}}\right)}\right\}$$

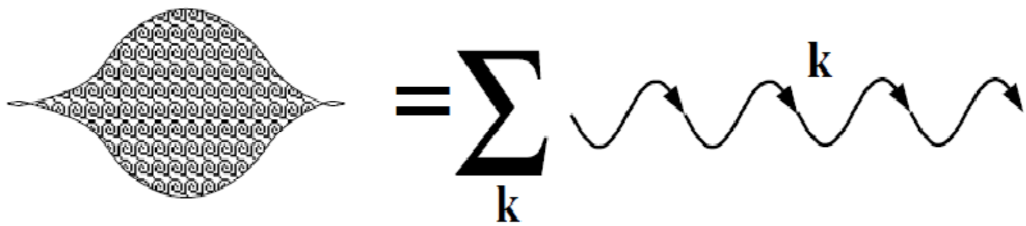
For a good conductor

$\varphi=\pi/4$, $\cos\varphi = 1/\sqrt{2}$, and $\gamma >> 1$. Then

$$T \approx 4\sqrt{\frac{\epsilon}{\mu}}\left\{\frac{\gamma^{\frac{1}{4}}(1/\sqrt{2})}{\left(\frac{\epsilon}{\mu}\gamma^{\frac{1}{2}}\right)}\right\} = 2\sqrt{2}\sqrt{\frac{\mu}{\epsilon}}\sqrt{\frac{\omega\epsilon}{4\pi\sigma}} = \frac{2\mu\omega}{c}\frac{c}{\sqrt{2\pi\sigma\omega\mu}} = \frac{2\mu\omega}{c}\delta$$

12.3 Superposition of Waves

No wave is truly monochromatic, although some waves, such as those produced by lasers, are exceedingly close to being so. Fortunately, in the case of linear media, the equations of motion for electromagnetic waves are completely linear and so any sum of harmonic solutions is also a solution.



Superposition procedure amounts to making a Fourier transform of the pulse. For simplicity we shall work in one spatial dimension which simply means that we will superpose waves whose wave vectors are all in the same direction (the z-direction). One such wave has the form $e^{i(k_0.z-\omega t)}$, where we shall not initially restrict ω(k) to any particular form. Given a set of such waves, we can build a general solution of this kind (wave vector parallel to the z-axis) by integrating over some distribution A(k) of them:

$$u(z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, A(k)e^{i(kz-\omega t)}$$

At time t = 0, this function is simply

$$u(z,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\, A(k)e^{i(kz)}$$

and the inverse transform gives A in terms of the zero-time wave

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dz\, u(z,0)e^{-i(kz)}$$

All of the standard rules of Fourier transforms are applicable to the functions A(k) and u(z,0). For example, if A(k) is a sharply peaked function with width Δk, then the width of u(z,0) must be of order 1/Δk or larger, and conversely. One may make this statement more precise by defining.

$$(\Delta z)^2 = < z^2 > - < z >^2$$

and

$$(\Delta \mathrm{k})^2 = < \mathrm{k}^2 > - < k >^2$$

Where

$$< f(k) > \equiv \frac{\int_{-\infty}^{\infty} dk\, f(k)|A(k)|^2}{\int_{-\infty}^{\infty} dk\, |A(k)|^2}$$

$$< f(z) > \equiv \frac{\int_{-\infty}^{\infty} dz\, f(z)|u(z,0)|^2}{\int_{-\infty}^{\infty} dz\, |u(z,0)|^2}$$

The relation between these widths which must be obeyed is

$$\Delta z\, \Delta k \geq .5$$

Now, given a "reasonable" initial wave form u(z,0) with some Δz and a Fourier transform A(k) with some Δk, the nature of calculated by the Fourier transform. One can always do these integrals numerically if all else fails. Here we shall do some approximate calculations designed to demonstrate a few general points.

Suppose that we have found A(k) and that it is some peaked function centered at k_0 with a width Δk. If $\omega(k)$ is reasonably well approximated by a truncated Taylor's series expansion for k within Δk of k_0, then we may write

$$\omega(k) \approx \omega_0 + \frac{d\omega}{dk}_{k_0}\,(k - k_0) \equiv \omega_0 + v_g(k - k_0)$$

Where $\omega_0 \equiv \omega(k_0)$ and group velocity of the packet $v_g = \frac{d\omega}{dk}_{k_0}$

In this approximation, one finds

$$u(z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, A(k) e^{i(k(z - v_g t))} e^{-i\omega_0 t} e^{i v_g k_0 t} = u(z - v_g t, 0) e^{i(v_g k_0 - \omega_0)t}$$

This result tells us that the wave packet retains its initial form and translates in space at a speed v_g. It does not spread (disperse) or distort in any way. In particular, the energy carried by the wave will move with a speed v_g. The group velocity is evidently an important quantity. We may write it in terms of the index of refraction by using the defining relation $k = \frac{\omega n(\omega)}{c}$. Take the derivative of this with respect to k:

$$1 = \left(\frac{n}{c} + \frac{\omega}{c}\frac{dn}{d\omega}\right)\frac{d\omega}{dk}$$

Or

$$v_g = \frac{c}{n + \omega \frac{dn}{d\omega}}$$

As an example consider the collisionless plasma relation $\boxed{n = \sqrt{1 - \omega_p^2 / \omega^2}}$. One easily finds that

$$\boxed{v_g = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}$$

For $\omega < \omega_p$, the group velocity is imaginary which corresponds to a damped wave; for $\omega > \omega_p$, it is positive and increases from zero to c as ω increases.

For Example

Let's treat a simple example in which A(k) is a Gaussian function of k - k_0,

$$A(k) = \left(\frac{A_0}{\delta}\right) e^{-\frac{(k-k_0)^2}{2\delta^2}}$$

Further, let ω(k) be approximated by

$$\omega(k) \;=\; \omega_0 \;+\; v_g(k - \;k_0) \;+\alpha(k - \;k_0)^2$$

The corresponding u(z,t) is

$$u(z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\; \left(\frac{A_0}{\delta}\right) e^{-\frac{(k-k_0)^2}{2\delta^2}}\; e^{i\left(k\left(z-v_g t\right)\right)} e^{-i\omega_0 t} e^{i v_g k_0 t}$$

$$= \left(\frac{A_0}{\delta}\right)\frac{1}{\sqrt{2\pi}} e^{i(k_0 z-\omega_0 t)} \int_{-\infty}^{\infty} dk e^{i(k-k_0)(z-v_g t)}\; e^{-\left(\frac{1}{2\delta^2 i\alpha t}\right)(k-k_0)^2}$$

$$= \frac{A_0}{\sqrt{1+2i\alpha\delta^2 t}} e^{i(k_0 z-\omega_0 t)} e^{-\frac{\left(z-v_g t\right)^2\delta^2}{[2(1+2i\alpha\delta^2 t)]}}$$

- If $\alpha = 0$, this is a Gaussian-shaped packet which travels at speed v_g with a constant width equal to δ^{-1}.
- If $\alpha \neq 0$, it is still a Gaussian-shaped packet traveling at speed v_g; however, it does not have a constant width any longer.

To make the development of the width completely clear, consider $|u(z, t)|^2$ which more nearly represents the energy density in the wave:

$$|u(z,t)|^2 = \frac{A_0^2}{\sqrt{1+4\alpha^2\delta^4t^2}} e^{-\frac{(z-v_g t)^2\delta^2}{[(1+4\alpha^2\delta^4t^2)]}}$$

The width of this travelling Gaussian is easily seen to be

$$\omega(t) = \frac{\sqrt{(1+4\alpha^2\delta^4t^2)}}{\delta}$$

At short times the width increases as the square of the time, while at long times it becomes linear with t.

12.4 Self Learning Exercise -I

Q.1 What is penetration depth δ of a conductor?

Q.2 Define good and poor conductors in terms of σ and ω.

Q.3 State Poynting vector.

Q.4 Relate group velocity and phase velocity.

12.5 A Pulse in the Ionosphere

Let us consider a wave packet propagating in the ionosphere, treating the ionosphere as a collisionless plasma and with k parallel to B_0, that $\epsilon(\omega) = 1 + \frac{\omega_p^2}{\omega(\omega_B-\omega)}$ for one particular polarization of the wave. If ω is small enough compared to other frequencies, we may approximate in such a way that $n(\omega) = \frac{\omega_p}{\sqrt{\omega\omega_B}}$, which gives rise to anomalous dispersion indeed. Defining $\omega_0 = \frac{\omega_p^2}{\omega_B}$, one finds that the group velocity of a signal is $v_g = 2c\sqrt{\omega}/\omega_0$

We have $u(z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk \left(\frac{A_0}{\delta}\right) e^{-\frac{(k-k_0)^2}{2\delta^2}+ikz-ic^{2k^{2t}}/\omega_0}$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk \left(\frac{A_0}{\delta}\right) e^{-\frac{(k-k_0)^2}{2\delta^2}+i(k-k_0)z+ik_0z-\frac{ic^2t(k-k_0)^2}{\omega_0}-\frac{i2c^2k_0t(k-k_0)}{\omega_0}-ic^2k_0^2t/\omega_0}$$

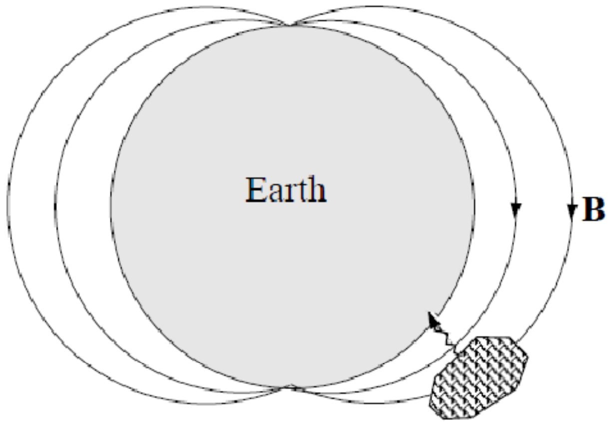
$$= \frac{A_0}{\sqrt{1+2i\delta^2c^2t/\omega_0}} e^{i\left(k_0z-\frac{c^2k_0^2t}{\omega_0}\right)} e^{-\frac{\left(z-\frac{2c^2k_0^2t}{\omega_0}\right)^2\delta^2}{[2(1+2ic^2\delta^2t/\omega_0)]}}$$

This is a travelling, dispersing Gaussian. Its speed is the group velocity $v_g(k_0)$. The width of the Gaussian is

$$\omega(t) = \sqrt{1 + \frac{4c^2\delta^4 t^2}{\omega_0^2}}/\delta \rightarrow 2\delta c^2 t/\omega_0$$

at long times. The packet spreads at a rate given by $v_w = 2\delta c^2/\omega_0$. The ratio of this spreading rate to the group velocity is δ/k_0 and so we retain a well-defined pulse provided the spread in wavenumber is small compared to the central wavenumber.

Pulses of this general type are generated in the ionosphere by thunderstorms. They have a very broad range of frequencies ranging from very low ones up into at least the AM radio range. The electromagnetic waves tend to be guided along lines of the earth's magnetic induction, and so, if for example the storm is in the southern hemisphere, the waves travel north in the ionosphere along lines of B and then come back to earth in the northern hemisphere.



By this time they are much dispersed, with the higher frequency components arriving well before the lower frequency ones since $v_g = 2c\sqrt{\omega/\omega_0}$ for $\omega << \omega_0$. Frequencies in the audible range, $\omega \sim 10^2$ or 10^3 sec^{-1} take one or more seconds (a long time for electromagnetic waves) to arrive. If one receives the signal and converts it directly to an audio signal at the same frequency, it sounds like a whistle, starting at high frequencies and continuing down to low ones over a time period of several seconds. This characteristic feature has caused such waves to be known as whistlers.

12.6 Causality and the Dielectric Function

A linear dispersive medium is characterized by a dielectric function $\epsilon(\omega)$ having physical origins. One consequence of having such a relation between D(x,ω) and E(x,ω), that is,

$$D(x,\omega) = \epsilon(\omega)E(x,\omega)$$

is that the relation between D(x,t) and E(x,t) is nonlocal in time. To see this we have only to look at the Fourier transforms of D and E. One has

$$D(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, D(x,\omega)e^{-i\omega t}$$

And its inverse

$$D(x,\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, D(x,t)e^{-i\omega t}$$

similar relations hold for E(x,t) and E(x,ω). Using the relation $D(x,\omega) = \epsilon(\omega)E(x,\omega)$,we have

$$D(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, \epsilon(\omega)E(x,\omega)e^{-i\omega t}$$

We can write E(x,ω) here as a Fourier integral and so have

$$D(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \epsilon(\omega)e^{-i\omega t}\int_{-\infty}^{\infty} dt`\, e^{i\omega t`}E(x,t`)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, d\omega\, [\epsilon(\omega) - 1 + 1]E(x,t`)e^{-i\omega(t-t`)}$$

$$= E(x,t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, d\omega\, [\epsilon(\omega) - 1]E(x,t`)e^{-i\omega(t-t`)}$$

$$= E(x,t) + 4\pi P(x,t)$$

The final term, 4πP(x,t), can be written in terms the Fourier transform of $\epsilon(\omega) - 1$; introduce the function

$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, [\epsilon(\omega) - 1]e^{-i\omega t}$$

Then we have

$$D(x,t) = E(x,t) + \int_{-\infty}^{\infty} dt`\, G(t-t`)E(x,t`)$$

$$D(x,t) = E(x,t) + \int_{-\infty}^{\infty} dT\, G(T) E(x, t-T)$$

This equation makes it clear that when the medium has a frequency-dependent dielectric function, as all materials do, then the electric displacement at time t depends on the electric field not only at time t but also at times other than t. This is somewhat disturbing because one can see that, depending on the character of G, we could get a polarization P(x,t) that depends on values of E(x,t`) for t` > t, which means we get an effect arising from a cause that occurs at a time later than the effect. This behaviour can be avoided if the function G(T) vanishes when T< 0, and that is what in fact happens.

Example, with

$$\epsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

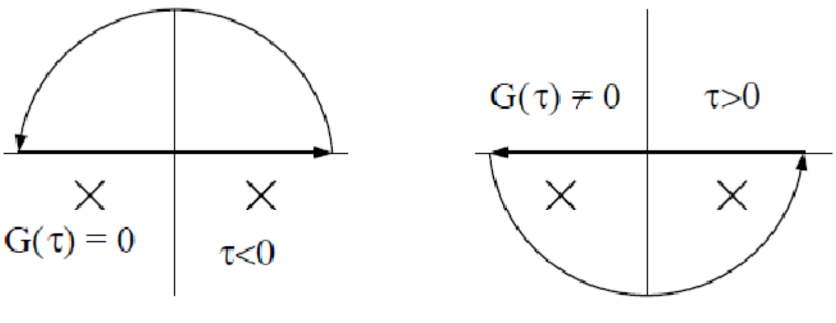
Then

$$G(T) = \frac{\omega_p^2}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega T}}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

This integral was made for contour integration techniques. The poles of the integrand are in the lower half-plane in complex frequency space at

$$\omega_{\pm} = \frac{1}{2}\left[\pm\sqrt{4\omega_0^2 - \gamma^2} - i\gamma\right]$$

without producing a contribution to the integral, we can close the contour in the upper (lower) half-plane when T is smaller (larger) than zero. Because there are poles only in the lower half-plane, we can see immediately that G(T) will be zero for T < 0. That is pleasing since we don't want the displacement (that is, the polarization) to respond at time t to the electric field at times later than t.

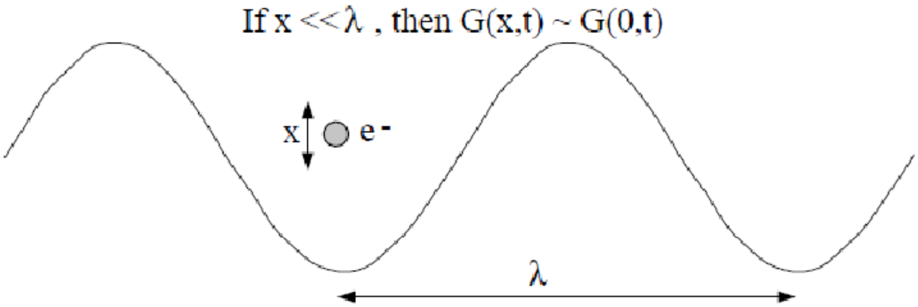


Applying Cauchy's theorem to the case of T > 0, one finds that, for all T,

$$G(T) = \omega_p^2 e^{-\frac{\gamma z}{2}} \frac{sin(v_0 T)}{v_0} \theta(T)$$

where $\theta(x)$ is the step function, equal to unity for x > 0 and to zero otherwise, and $v_0 = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$. The characteristic range in time of this function is γ^{-1} and hence the nonlocal (in time) character of the response is not important for frequencies smaller than about γ; it becomes important for larger ones.

One may naturally wonder whether there should also be nonlocal character of the response in space as well as in time. In fact there should and will be under some conditions. If we look back at our derivation of the model dielectric function, we see that the equation of motion of the particle was solved using E(0,t) instead of E(x,t); the latter is of course the more correct choice. The difference is not important so long as the excursions of the charge from the point on which it is bound are much smaller than the wavelength of the radiation, which is the case for any kind of wave with frequencies up to those of soft X-rays. Hence the response can be expected to be local in space in insulating materials. However, if an electron is free, it can move quite far during a cycle of the field and if it does so, the response will be nonlocal in space as well as time.



Returning to the question of causality, we have seen that the simple model dielectric function produces a function G(t) which is zero for t < 0, as is necessary if “causality" is to be preserved, by which we mean there is no response in advance of the “cause" of that response. It is easy to see what are the features of the dielectric function that give rise to the result G(t) = 0 for t < 0. One is that there are no simple poles of the dielectric function in the upper half of the complex frequency plane. Another is that the dielectric function goes to zero for large ω fast enough that we can do the contour integral as we did it.

More generally, if one wants to have a function G(t) which is consistent with the requirements of causality, this implies certain conditions on any $\epsilon(\omega)$. Additional conditions can be extracted from such simple things as the fact that G(t) must be real so that D is real if E is. Without going into the details of the matter let us make some general statements. The reality of G requires that

$$\epsilon(-\omega) = \epsilon^*(\omega^*)$$

That G is zero for negative times requires that $\epsilon(\omega)$ be analytic in the upper half of the frequency plane. Assuming that G(t)$\rightarrow$ 0 as t $\rightarrow\infty$, one finds that $\epsilon(\omega)$ is analytic on the real axis. This last statement is actually not true for conductors which give a contribution to $\epsilon\sim i\sigma/\omega$ so that there is a pole at the origin. Finally, from the small-time behavior of G(t), one can infer that at large frequencies the real part of $\epsilon(\omega) - 1$ varies as ω^{-2} while the imaginary part varies as ω^{-3}. This is accomplished by repeatedly integrating by parts

$$\epsilon(\omega) - 1 = \int_0^\infty dTG(T)e^{i\omega T} \approx \frac{iG(0^+)}{\omega} - \frac{G`(0^+)}{\omega^2} + \frac{iG`(0^+)}{\omega^3} + \cdots$$

This series is convergent for large ω. The first term vanishes if G(T) is continuous across T = 0. Thus

$$R(\epsilon(\omega) - 1)\sim\frac{1}{\omega^2}$$

$$T(\epsilon(\omega) - 1)\sim 1/\omega^3$$

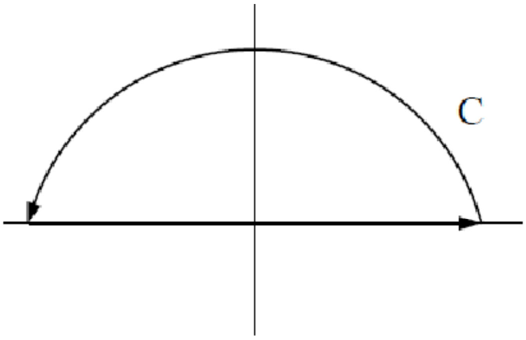
From inspection, one may see that the various dielectric functions we have contrived satisfy these conditions.

Given that the dielectric function has the analyticity properties described above, it turns out that by rather standard manipulations making use of Cauchy's integral theorem, one can write the imaginary part of $\epsilon(\omega)$ in terms of an integral of the real part over real frequencies and conversely. That one can do so is important because it means, for example, that if one succeeds in measuring just the real (imaginary) part, the imaginary (real) part is then known. The downside of this apparent miracle is that one has to know the real or imaginary part for all real frequencies in order to obtain the other part.

To see how this works, notice that as a consequence of the analytic properties of the dielectric function, it obeys the relation

$$\epsilon(z) = 1 + \frac{1}{2\pi i} \int d\omega` \, (\epsilon(\omega`) - 1))/(\omega` - z)$$

where the contour does not enter the lower half-plane (where ϵ may have poles) anywhere and where z is inside of the contour. Let C consist of the real axis and a large semicircle which closes the path in the upper half-plane.



Then, given that ϵ falls off fast enough, as described above, at large ω, the semicircular part of the path does not contribute to the integral. Hence we find that

$$\epsilon(z) = 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega` \, \frac{(\epsilon(\omega`) - 1))}{\omega` - z}$$

At this juncture, z can be any point in the upper half-plane. Let's use $z = \omega + i\eta$ and take the limit of $\eta \rightarrow 0$, finding

$$\epsilon(\omega + i\eta) = 1 + \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega` \, \frac{(\epsilon(\omega`) - 1))}{\omega` - \omega - i\eta}$$

The presence of the η in the denominator means that at the integration point $\omega` = \omega$, we must be careful to keep the singularity inside of, or above, the contour. Here we pick up $2\pi i$ times the residue, and the residue is just $\epsilon(\omega) - 1$. This relation shows identity but is not useful otherwise. However, one can also pull the following trick: If we integrate right across the singularity, taking the principal part (denoted P) of the integral plus an infinitesimal semicircle right below the singularity that amounts to taking $i\pi$ times the residue. Hence we can make the replacement

$$\frac{1}{\omega` - \omega - i\eta} \rightarrow P\left(\frac{1}{\omega` - \omega}\right) + i\pi\delta(\omega` - \omega)$$

where P stands for the principal part; this substitution leads to

$$\epsilon(\omega) = 1 + \frac{1}{\pi i} P \int_{-\infty}^{\infty} d\omega` \, \frac{(\epsilon(\omega`) - 1))}{\omega` - \omega}$$

Let us write separately the real and imaginary parts of this expression:

$$R\big(\epsilon(\omega)\big) \sim 1 + \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega` \, \frac{T\big(\epsilon(\omega`)\big)}{\omega` - \omega}$$

$$T\big(\epsilon(\omega)\big) \sim \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega` \, \frac{R(\epsilon(\omega`) - 1)}{\omega` - \omega}$$

These equations are known as the Kramers-Kronig relations for the dielectric function. They may be written as integrals over only positive frequencies because of the fact that the real part of $\epsilon(\omega)$ is an even function of ω while the imaginary part is odd. It should also be pointed out that we have assumed there is no pole in $\epsilon(\omega)$ at ω = 0; if there is one (conductors have dielectric functions with this property) some modification of these expressions will be necessary.

12.7 Self Learning Exercise- II

Very Short Answer Type Questions

Q.1 Define Causality

Q.2 Explain the phenomena of a Pulse in the Ionosphere.

Q.3 Write down Kramers-Kronig relations for the dielectric function.

Q.4 Write down the the Maxwell equations.

Long Answer Type Questions

Q.5 State the applications of Kramers–Kronig Relations.

12.8 Summary

In this chapter we firstly introduce waves in a conducting or dissipative medium followed by superposition of waves in one dimensional group velocity, causality connection between D and E, Kramers-Kroning relation.

12.9 Glossary

Dielectric: A substance in which an electric field may be maintained with zero or near-zero power dissipation, i.e. , the electrical conductivity is zero or near zero. ***Note 1***: A dielectric material is an electrical insulator. ***Note 2***: In a dielectric, electrons are bound to atoms and molecules, hence there are few free electrons.

***Electromagnet:* -** A magnet consisting of a solenoid with an iron core, which has a magnetic field only during the time of current flow through the solenoid.

Resistivity: Resistance between the terminal of unit are and unit length conductor is known as resistivity of that material, its unit is OHM-meter.

Conductivity: Conductivity=1/resistivity (MHO-m-1)

Independent equations: A system of equations is said to be independent if the system has exactly one solution.

Differential equation: An equation that expresses a relationship between functions and their derivatives.

12.10 Answer to Self Learning Exercises

Answer to Self Learning Exercise-I

Ans.1: Penetration depth is the distance that an electromagnetic wave will penetrate into a good conductor before being attenuated to a fraction 1/e of its initial amplitude.

Ans. 2: Good conductor

$$\frac{4\pi\sigma}{\epsilon\omega} \gg 1$$

Poor conductor

$$\frac{4\pi\sigma}{\epsilon\omega} \ll 1$$

Ans.3: Poynting vector is a quantity describing the magnitude and direction of the flow of energy in electromagnetic waves. The Poynting vector S is defined as to be equal to the cross product $\frac{1}{\mu}\left(\vec{E}\times\vec{B}\right)$, where μ is the permeability of the medium through which the radiation passes, E is the amplitude of the electric field, and B is the amplitude of the magnetic field.

Ans.4: $v_g = c\sqrt{1-\frac{\omega_p^2}{\omega^2}}$

Answer to Self Learning Exercise-II

Ans.1: Connection between two events or states such that one produces or brings about the other, where one is the cause and the other its effect, also called causation.

Ans. 2: Section 12.5

Ans. 3: $R\big(\epsilon(\omega)\big) \sim 1 + \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega` \ \frac{T\big(\epsilon(\omega`)\big)}{\omega` - \omega}$

$$T\big(\epsilon(\omega)\big) \sim \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega` \ \frac{R(\epsilon(\omega`) - 1)}{\omega` - \omega}$$

These equations are known as the Kramers-Kronig relations for the dielectric function.

Ans. 4: Let us consider some linear medium with

$D = \epsilon E, B = \mu H, J = \sigma E;\ \epsilon, \mu, \sigma$ are taken as real. Then the Maxwell equations become

$$\nabla . B = 0,$$

$$\nabla . E = 0$$

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$$

and

$$\nabla \times B = \frac{4\pi\mu}{c}\ \sigma E + \frac{\epsilon\mu}{c}\frac{\partial E}{\partial t}$$

We have set ρ equal to zero in these equations.

Ans.5:

- The Kramers–Kronig relations allow one to calculate the refractive index profile and thus also the chromatic dispersion of a medium solely from its frequency-dependent losses, which can be measured over a large spectral range. Note that a similar relation, allowing the calculation of the absorption from the refractive index, is much less useful because it is much more difficult to measure the refractive index in a wide frequency range.
- Modified Kramers–Kronig relations are also very useful in nonlinear optics . The basic idea is that the *change* in the refractive index caused by some

excitation of a medium (e.g. generation of carriers in a semiconductor) is related to the change in the absorption. As the change in the absorption is normally significant only in a limited range of optical frequencies, it is relatively easily measured. Such methods can also be applied to laser gain media, e.g. for calculating phase changes in fiber amplifiers associated with changes of the excitation level. Note that in the case of rare-earth-doped gain media, for example, it is not sufficient to consider only the changes in gain and loss around a certain laser transition, because changes in strong absorption lines in the ultraviolet spectral region are also important.

12.11 Exercise

Q.1 State and Prove Kramers-Kronig relations.

Q.2 Explain Reflection of a wave which is normally incident on a conductor

References and Suggested Readings

1. Classical Electrodynamics by J.D. Jackson, 1962

2. Classical Electricity and Magnetism by W. K. H. Panofsky and M. Philips, 2005

3. Introduction to Electrodynamics by D.J Griffiths, 1999

4. Classical Theory of Field by L.D. Landau and E. M. Lifshitz, 1971

5. Electrodynamics of Continuous Media by L.D. Landau and E. M. Lifshitz, 1960

6.http://www.phys.lsu.edu/~jarrell/COURSES/ELECTRODYNAMICS/Chap7/chap7.pdf

UNIT-13
Wave Guides

Structure of the Unit

13.0 Objective

After interacting with the material presented here students will be able to

1. Describe the development of the various types of waveguides in terms of their advantages and disadvantages.
2. Compare Waveguide and Transmission Line
3. Describe the physical dimensions of the various types of waveguides
4. Identify the modes of operation in waveguides.
5. Describe the basic principles of TE wave in rectangular wave guides.

13.1 Introduction

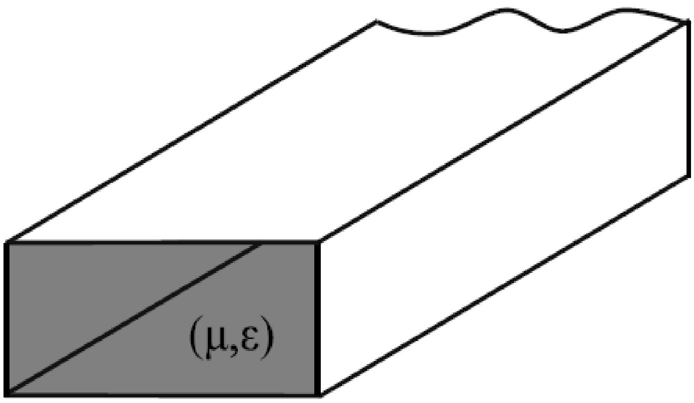
Waveguides, like transmission lines, are structures used to guide electromagnetic waves from one point to another point. However, the fundamental characteristics of waveguide and transmission line are quite different. The differences in these modes result from the basic differences in geometry for a transmission line and a waveguide.

Waveguides can be generally classified as either metal waveguides or dielectric waveguides. Metal waveguides normally take the form of an enclosed conducting metal pipe. The waves propagating inside the metal waveguide may be characterized by reflections from the conducting walls. The dielectric waveguide consists of dielectrics only and employs reflections from dielectric interfaces to propagate the electromagnetic wave along the waveguide.

1. Metal Wave guides

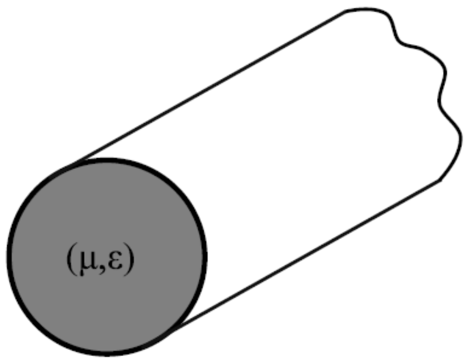
a. Rectangular Wave guide

As shown in the given diagram, the rectangular wave guide is designed from conducting material in rectangular shape which is hollow from the center and fully polished from interior. The outer surface of the wave guide is coded with insulating material or paint in order to avoid dust and rust. These types of wave guides are available in different lengths and sizes in order to fulfill the requirements of the circuit.



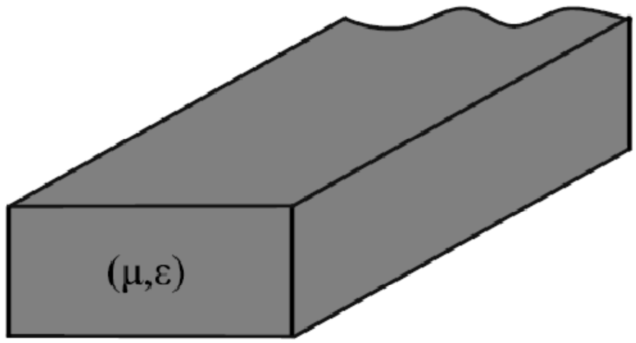
b. Circular Wave guide

As shown in the given diagram the circular waveguide is designed from a conducting pipe which is hollow from the center and polished from interior portion. The outer surface of the wave guide is coded with the insulated paint in order to avoid dust and rust. These types of wave guide are available in different lengths and sizes in order to fulfill the requirement of the circuit.

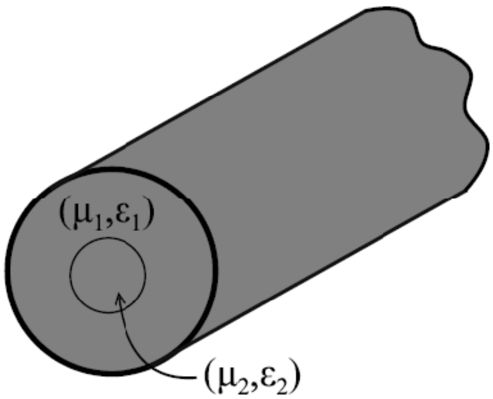


2.Dielectric Wave guides

a. Dielectric Slab Wave guide



b. Optical Fiber



Uses of Wave Guide

There are the following uses of Wave Guide.

1. It is used where the transmission or reception is in the range of microwave frequencies.
2. It is also used for handling the high power of energy.
3. It is mostly used in the airborne radar.

4. In ground radar's we also use the wave guide.
5. The circular wave guide is mostly used in the ground radar to transmit or receive the energy from antenna which revolves in 360° bearing continuously.
6. The wave guide is also used in communication system.
7. In satellite communication the wave guide is mostly used.
8. We also use the wave guide in the devices of navigation aids.
9. In some cases the wave guide is used as attenuator where very high frequencies are involved.
10. The wave guides are also used with the cavity resonators to carry the input and output signals.

Comparison of Waveguide and Transmission Line Characteristics

Transmission line	Waveguide
Two or more conductors separated by some insulating medium (two-wire, coaxial, microstrip, etc.).	Metal waveguides are typically one enclosed conductor filled with an insulating medium (rectangular, circular) while a dielectric waveguide consists of multiple dielectrics.
Normal operating mode is the TEM or quasi-TEM mode.	Operating modes are TE or TM modes
No cutoff frequency for the TEM mode. Transmission lines can transmit signals from DC up to high frequency.	Must operate the waveguide at a frequency above the respective TE or TM mode cutoff frequency for that mode to propagate.
Significant signal attenuation at high frequencies due to conductor and dielectric losses.	Lower signal attenuation at high frequencies than transmission lines.
Small cross-section transmission lines	Metal waveguides can transmit high

(like coaxial cables) can only transmit low power levels due to the relatively high fields concentrated at specific locations within the device	power levels. The fields of the propagating wave are spread more uniformly over a larger cross-sectional area than the small cross-section transmission line.
Large cross-section transmission lines (like power transmission lines) can transmit high power levels.	Large cross-section (low frequency) waveguides are impractical due to large size and high cost.

13.2 General Wave Characteristics

Given any time-harmonic source of electromagnetic radiation, the phasor electric and magnetic fields associated with the electromagnetic waves that propagate away from the source through a medium characterized by (μ,ε) must satisfy the source-free Maxwell's equations given by

$$\nabla \times \tilde{E} = -j\omega\mu\tilde{H}$$
$$\nabla \times \tilde{H} = -j\omega\mu\tilde{E}$$

The source-free Maxwell's equations can be manipulated into wave equations for the electric and magnetic fields. These wave equations are

$$\nabla^2\tilde{E} + k^2\tilde{E} = 0$$
$$\nabla^2\tilde{H} + k^2\tilde{H} = 0$$

Where $k = \omega\sqrt{\mu\varepsilon}$

where the ***wave number k is real valued for lossless media and complex valued for lossy media.***

The electric and magnetic fields of a general wave propagating in the +z-direction (either unguided, as in the case of a plane wave or guided, as in the case of a transmission line or waveguide) through an arbitrary medium with a propagation constant of γ are characterized by a ***z-dependence of*** $e^{-\gamma z}$ **.**

The electric and magnetic fields of the wave may be written in rectangular coordinates as

$$\tilde{E}(x, y, z) = \tilde{e}(x, y)e^{-\gamma z}$$
$$\tilde{H}(x, y, z) = \tilde{h}(x, y)e^{-\gamma z}$$

Where $\gamma = \alpha + \beta$, and α is the wave attenuation constant and β is the wave phase constant. The propagation constant is purely imaginary (α= 0, γ=jβ) when the wave travels without attenuation (no losses) or complex-valued when losses are present.

The transverse vectors $[\tilde{e}(x,y), \tilde{h}(x,y)]$ in the general wave field expressions may contain both transverse field components and longitudinal field components. By expanding the curl operator of the source free Maxwell's equations in rectangular coordinates, we note that the derivatives of the transverse field components with respect to z are

$$\frac{\partial \widetilde{E_x}}{\partial z} = -\gamma \tilde{E}_x, \frac{\partial \widetilde{E_y}}{\partial z} = -\gamma \tilde{E}_y$$
$$\frac{\partial \widetilde{H_x}}{\partial z} = -\gamma \widetilde{H}_x, \frac{\partial \widetilde{H_y}}{\partial z} = -\gamma \widetilde{H}_y$$

If we equate the vector components on each side of the two Maxwell curl equations, we find

$$j\omega\varepsilon \tilde{E}_x = \frac{\partial \widetilde{H_z}}{\partial y} + \gamma \widetilde{H}_y$$
$$j\omega\varepsilon \tilde{E}_y = -\frac{\partial \widetilde{H_z}}{\partial x} - \gamma \widetilde{H}_x$$
$$j\omega\varepsilon \tilde{E}_z = \frac{\partial \widetilde{H_y}}{\partial x} - \frac{\partial \widetilde{H_x}}{\partial y}$$
$$-j\omega\mu \widetilde{H}_x = \frac{\partial \widetilde{E_z}}{\partial y} + \gamma \tilde{E}_y$$
$$-j\omega\mu \widetilde{H}_y = -\frac{\partial \widetilde{E_z}}{\partial x} - \gamma \tilde{E}_x$$
$$-j\omega\mu \widetilde{H}_z = \frac{\partial \widetilde{E_y}}{\partial x} - \frac{\partial \widetilde{E_x}}{\partial y}$$

From above equations we can solve the longitudinal field components in terms of the transverse field components and we get

$$\tilde{E}_x = \frac{1}{h^2}\left(-\gamma \frac{\partial \widetilde{E_z}}{\partial x} - j\omega\mu \frac{\partial \widetilde{H_z}}{\partial y}\right)$$
$$\tilde{E}_y = \frac{1}{h^2}\left(-\gamma \frac{\partial \widetilde{E_z}}{\partial y} + j\omega\mu \frac{\partial \widetilde{H_z}}{\partial x}\right)$$
$$\widetilde{H}_x = \frac{1}{h^2}\left(-\gamma \frac{\partial \widetilde{H_z}}{\partial x} + j\omega\varepsilon \frac{\partial \widetilde{E_z}}{\partial y}\right)$$

$$\tilde{H}_y = \frac{1}{h^2}\left(-\gamma\frac{\partial\widetilde{H_z}}{\partial y} - j\omega\varepsilon\frac{\partial\widetilde{E_z}}{\partial x}\right)$$

Where $h^2 = \gamma^2 + \omega^2\mu\varepsilon = \gamma^2 + k^2$

The equations for the transverse fields in terms of the longitudinal fields describe the different types of possible modes for guided and unguided waves.

Mode	Electric field	Magnetic field	
Transverse electromagnetic (TEM modes)	$\tilde{E}_z = 0$	$\tilde{H}_z = 0$	Plane wave transmission line modes
Transverse electric (TE modes)	$\tilde{E}_z = 0$	$\tilde{H}_z \neq 0$	Waveguide modes
Transverse magnetic (TM modes)	$\tilde{E}_z \neq 0$	$\tilde{H}_z = 0$	Waveguide modes
Hybrid (HE or EH modes)	$\tilde{E}_z \neq 0$	$\tilde{H}_z \neq 0$	Waveguide modes

For simplicity, consider the case of guided or unguided waves propagating through an ideal (lossless) medium where k is real-valued.

For TEM modes, the only way for the transverse fields to be non-zero with $\tilde{E}_z = 0$ $\tilde{H}_z = 0$ is for h = 0, which yields

$\gamma = \sqrt{h^2 - k^2} = \sqrt{-k^2} = jk = \alpha + j\beta$

or $\beta = k$

Thus, for unguided TEM waves (plane waves) moving through a lossless medium or guided TEM waves (waves on a transmission line) propagating on an ideal transmission line, we have $\gamma = j\beta = jk$

For the waveguide modes (TE, TM or hybrid modes), h cannot be zero since this would yield unbounded results for the transverse fields. Thus, $\beta \neq k$ for waveguides and the waveguide propagation constant can be written as

$\gamma = \sqrt{h^2 - k^2} = \sqrt{-k^2(1 - h^2/k^2\,)} = jk\sqrt{(1 - h^2/k^2}$

The propagation constant of a wave in a waveguide (TE or TM waves) has very different characteristics than the propagation constant for a wave on a transmission line (TEM waves). The ratio of h/k in the waveguide mode propagation constant equation can be written in terms of the cutoff frequency f_c for the given waveguide mode as follows.

$$\frac{h}{k} = \frac{h}{\omega\sqrt{\mu\varepsilon}} = \frac{h}{2\pi f\sqrt{\mu\varepsilon}} = \frac{f_c}{f}$$

$$f_c = \frac{h}{2\pi\sqrt{\mu\varepsilon}}$$

The waveguide propagation constant in terms of the waveguide cutoff frequency is

$$\gamma = jk\sqrt{1 - \frac{f_c^2}{f^2}}$$

An examination of the waveguide propagation constant equation reveals the cutoff frequency behaviour of the waveguide modes.

If $f < f_c, \gamma = \alpha$ (purely real) $e^{-\gamma z} = e^{-\alpha z}$ waves are attenuated (evanescent modes)

If $f > f_c, \gamma = j\beta$ (purely imaginary) $e^{-\gamma z} = e^{-j\beta z}$ waves are unattenuated (propagating modes)

Therefore, in order to propagate a wave down a waveguide, the source must operate at a frequency higher than the cutoff frequency for that particular mode. If a waveguide source is operated at a frequency less than the cutoff frequency of the waveguide mode, then the wave is quickly attenuated in the vicinity of the source

13.3 Self Learning Exercise -I

Q.1 Express Maxwell's equations in cylindrical coordinate system.

(1) $\nabla\times\vec{H} = \left(\sigma + j(\omega)(\varepsilon)\right)\vec{E}$

(2) $\nabla\times\vec{E} = \left(\sigma - j(\omega)(\mu)\right)\vec{H}$

Q.2 Using the equations of Q.1, find all cylindrical field components in terms of E_z and H_z.

Q.3 Differentiate between Metallic and Dielectric Wave Guides.

Q.4 Define Transverse Electric Field, Transverse Magnetic Field, Transverse Electomagnetic Fields.

13.4 TE Mode in Rectangular Wave Guides

Waves propagate along the waveguide (+z-direction) within the waveguide through the lossless dielectric. The electric and magnetic fields of the guided waves must satisfy the source-free Maxwell's equations**.**

Assumptions:

(1) the waveguide is infinitely long, oriented along the z-axis, and uniform along its length.

(2) the waveguide is constructed from ideal materials [perfectly conducting pipe (PEC) is filled with a perfect insulator (lossless dielectric)].

(3) fields are time-harmonic.

The cross-sectional size and shape of the waveguide dictates the discrete modes that can propagate along the waveguide. That is, there are only discrete electric and magnetic field distributions that will satisfy the appropriate boundary conditions on the surface of the waveguide conductor.

If the single non-zero longitudinal field component associated with a given waveguide mode can be determined for a TM mode, for a TE mode), the remaining transverse field components can be found using the general wave equations for the transverse fields in terms of the longitudinal fields.

General Waves in an arbitrary Medium

$$\tilde{E}_x = \frac{1}{h^2}\left(-\gamma\frac{\partial\widetilde{E_z}}{\partial x} - j\omega\mu\frac{\partial\widetilde{H_z}}{\partial y}\right)$$

$$\tilde{E}_y = \frac{1}{h^2}\left(-\gamma\frac{\partial\widetilde{E_z}}{\partial y} + j\omega\mu\frac{\partial\widetilde{H_z}}{\partial x}\right)$$

$$\widetilde{H}_x = \frac{1}{h^2}\left(-\gamma\frac{\partial\widetilde{H_z}}{\partial x} + j\omega\varepsilon\frac{\partial\widetilde{E_z}}{\partial y}\right)$$

$$\widetilde{H}_y = \frac{1}{h^2}\left(-\gamma\frac{\partial\widetilde{H_z}}{\partial y} - j\omega\varepsilon\frac{\partial\widetilde{E_z}}{\partial x}\right)$$

TE Modes in an ideal Waveguide

$\tilde{E}_z = 0, \gamma = j\beta$

$\tilde{E}_x = \frac{1}{h^2}\left(-j\omega\mu\frac{\partial\widetilde{H_z}}{\partial y}\right), \tilde{E}_y = \frac{1}{h^2}\left(j\omega\mu\frac{\partial\widetilde{H_z}}{\partial x}\right)$

$\widetilde{H}_x = \frac{1}{h^2}\left(-\gamma\frac{\partial\widetilde{H_z}}{\partial x}\right), \widetilde{H}_y = \frac{1}{h^2}\left(-\gamma\frac{\partial\widetilde{H_z}}{\partial y}\right)$

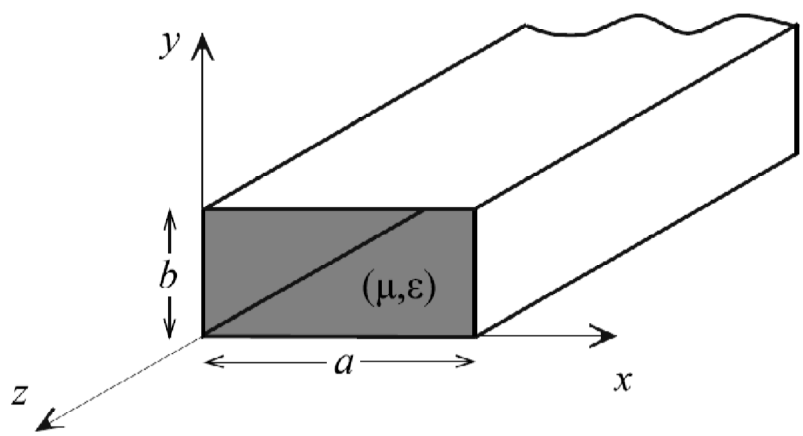
TM Modes in an Ideal Waveguide

$\widetilde{H}_z = 0, \gamma = j\beta$

$\tilde{E}_x = \frac{1}{h^2}\left(-\gamma\frac{\partial\widetilde{E_z}}{\partial x}\right), \tilde{E}_y = \frac{1}{h^2}\left(-\gamma\frac{\partial\widetilde{E_z}}{\partial y}\right)$

$\widetilde{H}_x = \frac{1}{h^2}\left(j\omega\varepsilon\frac{\partial\widetilde{E_z}}{\partial y}\right), \widetilde{H}_y = \frac{1}{h^2}\left(-j\omega\varepsilon\frac{\partial\widetilde{E_z}}{\partial x}\right)$

The longitudinal magnetic field of the TE mode and the longitudinal electric field of the TM mode are determined by solving the appropriate boundary value problem for the given waveguide geometry.



The rectangular waveguide can support either TE or TM modes*. *The rectangular cross-section (a > b) allows for single-mode operation. Single-mode operation means that only one mode propagates in the waveguide over a given frequency range. A square waveguide cross-section does not allow for single-mode operation.

Rectangular Waveguide TE Modes

The longitudinal magnetic field of the TE modes within the rectangular waveguide must satisfy the same wave equation as the longitudinal electric field of the TM modes:

$$\nabla^2\widetilde{H}_z^{TE} + k^2\widetilde{H}_z^{TE} = 0$$

which expanded in rectangular coordinates is

$$\frac{\partial^2 \widetilde{H}_z^{TE}}{\partial x^2} + \frac{\partial^2 \widetilde{H}_z^{TE}}{\partial y^2} + \frac{\partial^2 \widetilde{H}_z^{TE}}{\partial z^2} + k^2 \widetilde{H}_z^{TE} = 0$$

The magnetic field function may be determined using the separation of variables technique by assuming a solution of the form

$$\widetilde{H}_z^{TE} = X(x)Y(y)e^{-j\beta z}$$

$+\hat{z}$ traveling wave.

Inserting the assumed solution into the governing differential equation gives

$$Y(y)\frac{d^2X(x)}{dx^2}e^{-j\beta z} + X(x)\frac{d^2Y(y)}{dy^2}e^{-j\beta z} + (k^2 - \beta^2)X(x)Y(y)e^{-j\beta z} = 0$$

$$Y(y)\frac{d^2X(x)}{dx^2}e^{-j\beta z} + X(x)\frac{d^2Y(y)}{dy^2}e^{-j\beta z} + h^2X(x)Y(y)e^{-j\beta z} = 0$$

Where $h^2 = (k^2 - \beta^2)$

Dividing this equation by the assumed solution gives

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + h^2 = 0$$

Note that the first term in above equation is a function of x only while the second term is a function of y only.

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -k_x^2 \Rightarrow \frac{d^2X(x)}{dx^2} + k_x^2X(x) = 0$$

$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -k_y^2 \Rightarrow \frac{d^2Y(y)}{dy^2} + k_y^2Y(y) = 0$$

Where $h^2 = \left(k_x^2 + k_y^2\right)$

The original second order partial differential equation dependent on two variables has been separated into two second order ordinary differential equations each dependent on only one variable. The general solutions to the two separate differential equations are

$$X(x) = Asink_xx + Bcosk_xx$$

$$Y(y) = Csink_yy + Dcosk_yy$$

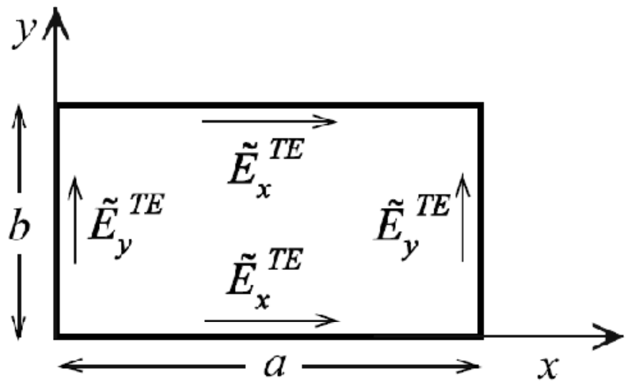
The resulting longitudinal magnetic field for a rectangular waveguide TE mode is

$$\widetilde{H}_z^{TE} = (Asink_xx + Bcosk_xx)(Csink_yy + Dcosk_yy)e^{-j\beta z}$$

To determine the unknown coefficients, we apply the TE boundary conditions. Given no longitudinal electric field for the TE case, the boundary conditions for the transverse electric field components on the walls of the waveguide must be enforced. The TE boundary conditions are:

$$\tilde{E}_y^{TE}(0,y,z) = \tilde{E}_y^{TE}(a,y,z) = 0 \text{ (Vertical walls)}$$

$$\tilde{E}_x^{TE}(x,0,z) = \tilde{E}_x^{TE}(x,b,z) = 0 \text{ (Horizontal walls)}$$



The transverse components of the TE electric field are related to longitudinal magnetic field by the standard TE equations.

$$\tilde{E}_x^{TE} = \frac{1}{h^2}\left(-j\omega\mu\frac{\partial\widetilde{H_z}}{\partial y}\right)$$

$$= \frac{k_y}{h^2}\left(-j\omega\mu(Asink_x x + Bcosk_x x)\,(Ccosk_y y - Dsink_y y)e^{-j\beta z}\right)$$

$$\tilde{E}_y^{TE} = \frac{1}{h^2}\left(j\omega\mu\frac{\partial\widetilde{H_z}}{\partial x}\right)$$

$$= \frac{k_x}{h^2}\left(j\omega\mu(Acosk_x x - Bsink_x x)\left(Csink_y y + Dcosk_y y\right)e^{-j\beta z}\right)$$

The application of the TE boundary conditions yields

$$\tilde{E}_x^{TE}(x,0,z) = 0 \Rightarrow C = 0$$

$$\tilde{E}_x^{TE}(x,b,z) = 0 \Rightarrow k_y b = n\pi\ (n = 0,1,2,\dots) \Rightarrow k_y = \frac{n\pi}{b}$$

$$\tilde{E}_y^{TE}(0,y,z) = 0 \Rightarrow A = 0$$

$$\tilde{E}_y^{TE}(a,y,z) = 0 \Rightarrow k_x a = m\pi\ (m = 0,1,2,\dots) \Rightarrow k_x = \frac{m\pi}{a}$$

Combining the constants B and D into the constant H_o, the resulting longitudinal magnetic field of the TE_{mn} mode is

$$\widetilde{H}_z^{TE_{mn}}(x,y,z) = H_0\left(cos\frac{m\pi}{a}x\right)(cos\frac{n\pi}{b}y)e^{-j\beta z}$$

Note that the indices include $m = 0$ and $n = 0$ in the TE solution since these values still yield a non-zero longitudinal magnetic field. However, the case of $n = m = 0$ is not allowed since this would make all of the transverse field components zero. The resulting transverse fields for the waveguide TE modes are

$$\tilde{E}_x^{TE_{mn}} = \frac{1}{h^2}\left(-j\omega\mu\frac{\partial \widetilde{H}_z^{TE_{mn}}}{\partial y}\right) = \frac{j\omega\mu}{h^2}\left(\frac{n\pi}{b}\right) H_0 \left(cos\frac{m\pi}{a}x\right)(sin\frac{n\pi}{b}y)e^{-j\beta z}$$

$$\tilde{E}_y^{TE_{mn}} = \frac{1}{h^2}\left(j\omega\mu\frac{\partial \widetilde{H}_z^{TE_{mn}}}{\partial x}\right) = \frac{j\omega\mu}{h^2}\left(\frac{m\pi}{a}\right) H_0 \left(sin\frac{m\pi}{a}x\right)(cos\frac{n\pi}{b}y)e^{-j\beta z}$$

$$\widetilde{H}_x^{TE_{mn}} = \frac{1}{k_c^2}\left(-j\beta\frac{\partial \widetilde{H}_z^{TE_{mn}}}{\partial x}\right) = -\left(\frac{j\beta m\pi}{k_c^2 a}\right) H_0 \left(sin\frac{m\pi}{a}x\right)(cos\frac{n\pi}{b}y)e^{-j\beta z}$$

$$\widetilde{H}_y^{TE_{mn}} = \frac{1}{k_c^2}\left(-j\beta\frac{\partial \widetilde{H}_z^{TE_{mn}}}{\partial x}\right) = \left(\frac{j\beta n\pi}{k_c^2 b}\right) H_0 \left(cos\frac{m\pi}{a}x\right)(sin\frac{n\pi}{b}y)e^{-j\beta z}$$

where ($m = 0, 1, 2, ...$) and ($n = 0, 1, 2, ...$) but $m = n \neq 0$ for the TE$_{mn}$ mode.

Rectangular waveguide mn index pairs (TE$_{mn}$)

	$n=0$	$n=1$	$n=2$	
$m=0$	$\times$	01	02	...
$m=1$	10	11	12	...
$m=2$	20	21	22	...
....	...	...	...	...

13.5 TE Mode Parameters

The propagation constant in the rectangular waveguide for both the TE$_{mn}$ and TM$_{mn}$ waveguide modes (γ_{mn}) is defined by

$$\boxed{\gamma_{mn} = \sqrt{h_{mn}^2 - k^2} = \sqrt{\left(k_x^2 + k_y^2\right) - k^2}}$$

$$\boxed{h_{mn} = \sqrt{\left(k_x^2 + k_y^2\right)} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}}$$

$$\boxed{\gamma_{mn} = \sqrt{h_{mn}^2 - k^2} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon}}$$

The equation for the waveguide propagation constant γ_{mn} can be used to determine the cutoff frequency for the respective waveguide mode. The

propagation characteristics of the wave are defined by the relative sizes of the parameters h_{mn} and k. The propagation constant may be written in terms of the attenuation and phase constants as

$$\gamma_{mn} = \alpha_{mn} + j\beta_{mn}$$

so that,

if $\quad h_{mn} = k, \gamma_{mn} = 0 \ (\alpha_{mn} = \beta_{mn} = 0)$ ***cutoff frequency***

if $\quad h_{mn} > k \ \gamma_{mn}$ (real), $[\gamma_{mn} = \alpha_{mn}]$ evanescent modes

if $h_{mn} < k$ $\quad \gamma_{mn}$ (imag.), $[\gamma_{mn} = j\beta_{mn}]$ propagating modes

Therefore, the cutoff frequencies for the TE mode in the rectangular waveguide are found by solving

$$h_{mn} = \sqrt{\left(k_x^2 + k_y^2\right)} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = k_{c_{mn}} = 2\pi f_{c_{mn}}\sqrt{\mu\varepsilon}$$

$$\boxed{f_{c_{mn}} = \left(\frac{1}{2\pi\sqrt{\mu\varepsilon}}\right)\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}}$$

Where $f_{c_{mn}}$ is rectangular waveguide cutoff frequency.

Note that the cutoff frequency for a particular rectangular waveguide mode depends on the dimensions of the waveguide (a,b), the material inside the waveguide (μ,ε), and the indices of the mode (m,n). The rectangular waveguide must be operated at a frequency above the cutoff frequency for the respective mode to propagate.

According to the cutoff frequency equation, the cutoff frequencies of both the TE_{10} and TE_{01} modes are less than that of the lowest order TM mode (TM11). Given $a > b$ for the rectangular waveguide, the TE_{10} has the lowest cutoff frequency of any of the rectangular waveguide modes and is thus the dominant mode (the first to propagate). Note that the TE_{10} and TE_{01} modes are ***degenerate modes (modes with the same cutoff frequency)*** for a square waveguide. The rectangular waveguide allows one to operate at a frequency above the cutoff of the dominant TE10 mode but below that of the next highest mode to achieve single mode operation. A waveguide operating at a frequency where more than one mode propagates is said to be *overmoded*.

Example 1 A rectangular waveguide (a = 2 cm, b = 1 cm) filled with deionised water (μ_r =1, ε_r = 81) operates at 3 GHz. Determine all propagating modes and the corresponding cutoff frequencies.

Solution

$$f_{c_{mn}} = \left(\frac{1}{2\pi\sqrt{\mu\varepsilon}}\right)\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$= \frac{c}{2\sqrt{(1)(81)}}\sqrt{\left(\frac{m}{0.02}\right)^2 + \left(\frac{n}{0.01}\right)^2}$$

$$= \frac{c}{18}\sqrt{\left(\frac{m}{0.02}\right)^2 + \left(\frac{n}{0.01}\right)^2}$$

Cutoff frequencies - TE modes (GHz)

	$n = 0$	$n = 1$	$n = 2$
$m = 0$	$\times$	1.667	3.333...
$m = 1$	0.833	1.863	3.436...
$m = 2$	1.667	2.357	3.727...
....	...	...	... ...

13.6 Self Learning Exercise- II

Q.1 Consider a length of air-filled copper X-band waveguide, with dimensions a=2.286cm, b=1.016cm. Find the cut-off frequencies of the first four propagating modes.

Q.2 The cutoff frequency of an air-filed rectangular waveguide is 2.4 GHz for the TE_{10} mode. What would be the cutoff frequency if the same guide were filled with a lossless nonmagnetic material whose dielectric permittivity is six times that of air?

Q.3 In an air-filled rectangular waveguide, the cutoff frequency of a TE_{10} mode is 5GHz, whereas that of TE_{01} mode is 12 GHz. Calculate

(I) the dimensions of the guide

(II) the cutoff frequencies of the next three higher TE modes

Q.4 In an air-filled square waveguide with *a* =1.2 cm,

E_x= -10 sin (2ω y/*a*) sin(ωt-150z) V/m

(a) the mode of propagation

(b) frequency of operation

(c) the field components Hz and Ez

13.7 Summary

This chapter has presented information on waveguide theory and application. Waveguides are the primary methods of transporting microwave energy. Waveguides have fewer losses and greater power-handling capability than transmission lines. The "a," dimension determines the frequency range of the waveguide, and the "b," dimension determines power-handling capability. Waveguides handle a small range of frequencies both above and below the primary operating frequency. Energy is transported through waveguides by the interaction of electric and magnetic fields, abbreviated E FIELD and H FIELD, respectively. In this chapter we firstly discussed details of wave guide and their characteristic which is followed by TE mode in a rectangular waveguide.

13.8 Glossary

Cutoff frequency: The frequency either above which or below which the output of a circuit, such as a line, amplifier, or filter, is reduced to a specified level.

Transmission line: The material medium or structure that forms all or part of a path from one place to another for directing the transmission of energy, such as electric currents, magnetic fields, acoustic waves, or electromagnetic waves. Examples of transmission lines include wires, optical fibers, coaxial cables, rectangular closed waveguides, and dielectric slabs.

Waveguide: A material medium that confines and guides a propagating electromagnetic wave.

Dielectric: A substance in which an electric field may be maintained with zero or near-zero power dissipation, *i.e.* , the electrical conductivity is zero or near zero. ***Note 1***: A dielectric material is an electrical insulator. ***Note 2:*** In a dielectric, electrons are bound to atoms and molecules, hence there are few free electrons.

13.9 Answers to Self Learning Exercises

Answers to Self learning Exercise-I

Ans.1: We know that

$$\nabla \times A = \left(\frac{1}{r} \frac{\partial \mathrm{A}_z}{\partial \phi} - \frac{\partial \mathrm{A}_\phi}{\partial z} \right) a_r + \left(\frac{\partial \mathrm{A}_r}{\partial z} - \frac{\partial \mathrm{A}_z}{\partial r} \right) a_\phi$$

$$+ \frac{1}{r} \left[\frac{\partial}{\partial r} \left(r * \mathrm{A}_\phi \right) - \frac{\partial \mathrm{A}_r}{\partial \phi} \right] a_z$$

Equation (1) yields (σ = 0)

$$j(\omega)(\varepsilon)\mathrm{E}_r = \frac{1}{r} \frac{\partial \mathrm{H}_z}{\partial \phi} + j(k * \mathrm{H}_\phi)$$

$$j(\omega)(\varepsilon)\mathrm{E}_\phi = - j(k * \mathrm{H}_r) - \frac{\partial \mathrm{H}_z}{\partial r}$$

$$j(\omega)(\varepsilon)\mathrm{E}_z = \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r} \frac{\partial \mathrm{H}r}{\partial \phi}$$

Equation (2) yields:

$$- j(\omega)(\mu)\mathrm{H}_r = \frac{1}{r} \frac{\partial \mathrm{E}_z}{\partial \phi} + j(k * \mathrm{E}_\phi)$$

$$- j(\omega)(\mu)\mathrm{H}_\phi = - j(k * \mathrm{E}_r) - \frac{\partial \mathrm{E}_z}{\partial r}$$

$$- j(\omega)(\mu)\mathrm{H}_z = \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r} \frac{\partial \mathrm{E}r}{\partial \phi}$$

Ans.2:

$$\mathrm{E}_r = - \frac{j(\omega)(\mu)}{k_c^2} \frac{1}{r} \frac{\partial \mathrm{H}_z}{\partial \phi} - \frac{j(k)}{k_c^2} \frac{\partial \mathrm{E}_z}{\partial r}$$

$$H_r = \frac{j(\omega)(\varepsilon)}{k_c^2}\frac{1}{r}\frac{\partial E_z}{\partial \phi} - \frac{j(k)}{k_c^2}\frac{\partial H_z}{\partial r}$$

$$H_\phi = -\frac{j(\omega)(\varepsilon)}{k_c^2}\frac{1}{r}\frac{\partial E_z}{\partial r} - \frac{j(k)}{k_c^2}\frac{1}{r}\frac{\partial H_z}{\partial \phi}$$

$$E_\phi = \frac{j(\omega)(\varepsilon)}{k_c^2}\frac{1}{r}\frac{\partial E_z}{\partial \phi} + \frac{j(\omega)(\mu)}{k_c^2}\frac{1}{r}\frac{\partial H_z}{\partial r}$$

Ans.3: Metallic Wave Guide is used in high frequency, microwaves and millimeter waves transmission. Co-axial cables and hollow rectangular or circular wave guide's fall in this category.

Dielectric Wave Guide is used at sub millimeter wavelengths and optical frequencies. Optical fibers and thin film integrated optical devices fall in this category.

Ans.4: ***Transverse Electric Field, the electric field is perpendicular to the direction of wave propagation***. That is to say that the electric field does not have any component in the direction of wave propagation.

For Transverse Magnetic Field, the magnetic field is transverse to the direction of wave propagation. That is to say that the electric field does not have any component in the direction of wave propagation.

For Transverse Electromagnetic Fields, the electric and magnetic fields both do not have a component in the direction of wave propagation.

Answers to Self learning Exercise-II

Ans. 1 : Air Filled cut off frequency

$$f_c = \frac{c}{2}\sqrt{\left(\frac{m}{0.02286}\right)^2 + \left(\frac{n}{0.01016}\right)^2}$$

Ans. 2 : 980MHz

Ans. 3 : **(I)** a = 3cm b = 1.25cm

(II) TE_{20}(10Ghz), TE_{11}(13GHz), TE_{30}(15GHz)

Ans. 4 :

(a) since m=0 it has to be TE. n=2 so it's TE02.

(b) Since a=b, fc02= (c/2π) sqrt(k_x^2 +k_y^2) = 25GHz , where k_y=0. β=150 rad/m (is given from inside the sin(ωt-150z),

$k^2 = \beta^2 + k_x^2 + k_y^2 = \omega^2\mu\varepsilon$, then f of operation is=26.00665 = 26 GHz

(c) $E_y = 0 = E_z = H_z$

$$H_y = -7.286\sin\left(\pi\frac{y}{a}\right)\sin\left(\omega t - 150z\right)\ mA/m$$

$$H_z = -25.43\cos\left(2\pi\frac{y}{a}\right)\cos\left(\omega t - 150z\right)\ mA/m$$

13.10 Exercise

Section A:Very Short Answer Type Questions

Q.1 An air-filled 1.5 cm×3 cm waveguide is operated at a frequency that lies in the middle of its TE_{10} mode band. Determine cut-off frequencies in GHz.

Section B: Short Answer Type Questions

Q.2 In an air-filled rectangular waveguide with a = 2.286cm and b = 1.016cm, the y-component of the TE mode is given by

$$E_y = \sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{3\pi y}{b}\right)\cos(10\pi \times 10^{10}t - \beta z)\,V/m$$

Find:

a) the operating mode.

b) the propagation constant

Section C: Long answer Type Questions

Q.3 Write down disadvantages of waveguides over conventional transmission lines.

Q.4 In an air-filled rectangular waveguide, a TE mode operating at 6GHz has

$$E_y = \sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)\cos\left(\omega t - 12z\right)\,V/m$$

Find:

a) the mode of operation

b) the cutoff frequency

c) H_x

References and Suggested Readings

1. Electromagnetic Waves and Antennas by Sophocles J. Orfanidis, 2008.

2. Fundamentals of Applied Electromagnetics by Fawwaz T. Ulaby, 2010.

3. Microwave Devices & Circuits Paperback by Samuel Y. Liao, 2000.

UNIT -14
Lienard –Wiechert Potentials, Power Radiated by an Accelerated Charge

Structure of the Unit

14.0 Objectives

(i) To obtain Lienard- Wiechert Potentials for a moving point charge.

(ii) To determine total power radiated by an accelerated point charge.

14.1 Introduction

A moving charge generates electric and magnetic fields hence scalar and vector potentials associated with these fields. These potential and fields must be known to obtain radiated power by the moving point charge. This unit is written for this purpose.

14.2 Lienard – Wiechert Potentials

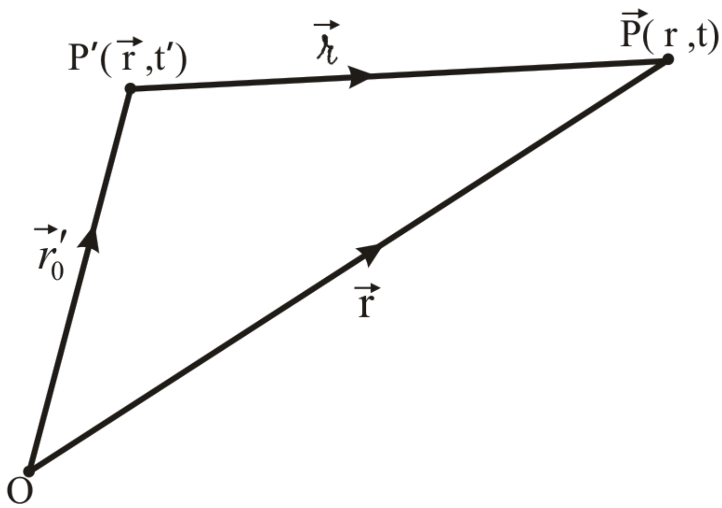


Fig.-1

Due to the motion of point charge(source point $\vec{r}'$) , potential at a position $\vec{r}$ (field point)at present time t are actually associated with the fields that generated at an earlier position $\vec{r}'$ at an earlier time $t' = t - \left|\frac{\vec{r} - \vec{r}'}{c}\right|$ are called retarded potential, given as

$$\Phi(\vec{r},t) = \frac{1}{4\pi \epsilon_0}\int\int \frac{\rho(\vec{r}', t')d^3r'\, dt'}{|\vec{r}-\vec{r}'|}\delta\left[t' - \left(t - \frac{|\vec{r}-\vec{r}'|}{c}\right)\right] \tag{1}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\int \frac{\vec{J}(\vec{r}',t')d^3r'\, dt'}{|\vec{r}-\vec{r}'|}\delta\left[t' - \left(t - \frac{|\vec{r}-\vec{r}'|}{c}\right)\right] \tag{2}$$

The charge and current densities of a moving point charge of magnitude q at a position r_0' at time t' is given as

$$\rho(\vec{r}',t') = q\ \delta(\vec{r}' - \vec{r}_0') \tag{3}$$

$$\vec{J}(\vec{r}',t') = q\ \vec{v}\ \delta(\vec{r}' - \vec{r}_0') \tag{4}$$

Where $\vec{v}(\vec{r}_0',t')$ is the instantaneous velocity of point charge along the path.

Using equations (3) & (4) into (1) & (2) respectively, then

$$\Phi(\vec{r},t) = \frac{1}{4\pi \in_0} \iint \frac{q\ \delta(\vec{r}' - \vec{r}_0')\delta\left\{t' - \left(t - \frac{|\vec{r}-\vec{r}'|}{c}\right)\right\}d^3r'\ dt'}{|\vec{r}-\vec{r}'|}$$

And $$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi} \iint \frac{q\vec{v}\delta\left\{t' - \left(t - \frac{|\vec{r}-\vec{r}'|}{c}\right)\right\}\delta(\vec{r}' - \vec{r}_0')d^3r'\ dt'}{|\vec{r}-\vec{r}_0'|}$$

Using the property of Dirac-delta function solutions of above relation is

$$\Phi(\vec{r},t)) = \frac{q}{4\pi \in_0} \int \frac{dt'}{|\vec{r}-\vec{r}_0'|}\delta\left\{t' - \left(t - \frac{|\vec{r}-\vec{r}_0'|}{c}\right)\right\} \tag{5}$$

and $$\vec{A}(\vec{r},t) = \frac{\mu_0 q}{4\pi} \int \frac{\vec{v}\ dt'}{|\vec{r}-\vec{r}_0'|}\delta\left\{t' - \left(t - \frac{|\vec{r}-\vec{r}_0'|}{c}\right)\right\} \tag{6}$$

For determination of integral over t',

Let

$$\chi = t' - \left(t - \frac{|\vec{r}-\vec{r}\,'_0|}{c}\right) = t' - t + \frac{|\vec{r}-\vec{r}\,'_0|}{c} \tag{7}$$

So, $$\frac{d\chi}{dt'} = 1 - \frac{1}{c}\ \frac{d}{dt'}|\vec{r}-\vec{r}\,'_0|$$

$$= 1 - \frac{1}{c}\ \frac{d}{dt'}\sqrt{r^2 + r_0'^2 - 2\vec{r}.\vec{r}\,'_0}$$

$$= 1 - \frac{1}{c}.\left(\frac{-1}{2}\right).\left[\frac{d}{dt'}\left(r^2 + \vec{r}\,'_0\ .\ \vec{r}\,'_0 - 2\vec{r}.\vec{r}\,'_0\right)\right].\frac{1}{\sqrt{r^2 + \vec{r}_0'^2 - 2\vec{r}.\vec{r}\,'_0}}$$

$$= 1 - \frac{1}{c}.\left(\frac{-1}{2}\right).\frac{\left(2\vec{r}\,'_0\ .\frac{d\vec{r}\,'_0}{dt'} - 2\vec{r}.\frac{d\vec{r}\,'_0}{dt'}\right)}{\sqrt{r^2 + \vec{r}\,'_0 - 2\vec{r}.\vec{r}\,'_0}}$$

$$= 1 - \frac{1}{c}.(-1).\frac{\left(\vec{r}\,'_0\ .\vec{v} - \vec{r}.\vec{v}\right)}{\sqrt{r^2 + r_0'^2 - 2\vec{r}.\vec{r}\,'_0}}$$

$$= 1 - \frac{1}{c}\frac{\left(\vec{r}-\vec{r}'_0\right)}{\left|\vec{r}-\vec{r}'_0\right|}.\vec{v}$$

$$\Rightarrow \frac{d\chi}{dt'} = 1 - \hat{\imath}\,.\,\vec{\beta} \text{ Where } \hat{\imath} = \frac{\vec{r}-\vec{r}'_0}{\left|\vec{r}-\vec{r}'_0\right|},\ \vec{\beta} = \frac{\vec{v}}{c}$$

$$\Rightarrow dt' = \frac{d\chi}{1-\hat{\imath}.\vec{\beta}} \tag{8}$$

Using equations (7) and (8) into (5) and (6),

Gives

$$\Phi\left(\vec{r},t\right) = \frac{q}{4\pi \in_0}\int \frac{\delta(\chi)\,d(\chi)}{\imath\left(1-\hat{\imath}.\vec{\beta}\right)}$$

and $$\vec{A}\left(\vec{r},t\right) = \frac{\mu_0 q}{4\pi}\int \frac{\vec{v}\,\delta(\chi)\,d\chi}{\imath\left(1-\hat{\imath}.\vec{\beta}\right)}$$

Using property of Dirac-delta function

$$\boxed{\Phi\left(\vec{r},t\right) = \frac{q}{4\pi \in_0}\ \frac{1}{\imath\left(1-\hat{\imath}.\vec{\beta}\right)}} \tag{9}$$

$$\boxed{\vec{A}\left(\vec{r},t\right) = \frac{\mu_0}{4\pi}\ \frac{q\vec{v}}{\imath\left(1-\hat{\imath}.\vec{\beta}\right)}} \tag{10}$$

$$= \frac{\mu_0}{4\pi}\frac{\in_0}{\in_0}\frac{q}{\imath\left(1-\hat{\imath}.\vec{\beta}\right)}\vec{v}$$

$$\boxed{\vec{A}\left(\vec{r},t\right) = \frac{\vec{v}}{c^2}\Phi\ \left(\vec{r},t\right)} \tag{11}$$

$$\left\{\because \mu_0\varepsilon_0 = \frac{1}{c^2}\right.$$

Equation (9) & (10) are ***the Lienard Wiechart potential for moving point charge***.

14.3 Self Learning Exercise –I

Q.1 What is charge density?

Q.2 What is current density?

Q.3 Write the properties of Dirac delta function?

Q.4 What do you mean by retarded time?

Q.5 How you get electric and magnetic fields from scalar and vector potentials?

14.4 Total Power Radiated by an Accelerated Point Charge

Since the electric and magnetic field generated by an accelerated point charge of magnitude q are

$$\vec{E}\left(\vec{r},t\right)=\frac{q}{4\pi \in_0}\frac{\left[c^2\left(1-\beta^2\right)\hat{\imath}+\vec{\imath}\times\hat{\imath}\times\vec{a}\right]}{\imath^2c^2\left(1-\hat{\imath}.\vec{\beta}\right)^2} \tag{12}$$

where $\vec{a}=\frac{d\vec{v}}{dt'}$ acceleration of charge,

$\vec{\imath}$ =vector along the direction of emitted fields.

$\boxed{\vec{\beta}=\frac{\vec{v}}{c}}$, ε_0 Permittivity of free space

and $\boxed{\vec{B}\left(\vec{r},t\right)=\frac{1}{c}\hat{\imath}\times\vec{E}}$ (13)

Since power radiated per unit area is given by Poynting vector

$$\boxed{\vec{S}=\frac{\vec{E}\times\vec{B}}{\mu_0}}$$

Using equation (13), gives

$$\vec{S}=\frac{\vec{E}\times\vec{\imath}\times\vec{E}}{\mu_0c}=\frac{1}{\mu_0c}\left\{\hat{\imath}E^2-\vec{E}\left(\hat{\imath}\ .\ \vec{E}\right)\right\}$$

Since radiation fields are perpendicular to direction of propagation,

So, $\hat{\imath}.\vec{E}=0$

and $$\vec{S}_{rad}=\frac{1}{\mu_0c}\ \hat{\imath}\ E_{rad}^2 \tag{14}$$

Since total power radiated is given as $P=\iint\vec{S}_{rad}\ .\ d\vec{A}$ where integration is taken over area surrounding to point charge, so

$$P=\int_0^{\pi}\int_0^{2\pi}\frac{1}{\mu_0c}\left(\hat{\imath}\ .\ \hat{\imath}\right)\mathrm{E}_{rad}^2\imath^2\sin\theta d\theta d\Phi$$

$$P=\frac{1}{\mu_0 c}\int_0^{\pi}\int_0^{2\pi}\left(\mathscr{r}\,E_{rad}\right)^2\sin\theta\, d\theta\, d\Phi \tag{15}$$

where θ is the angle between direction of motion of charge and direction of emitted radiation.

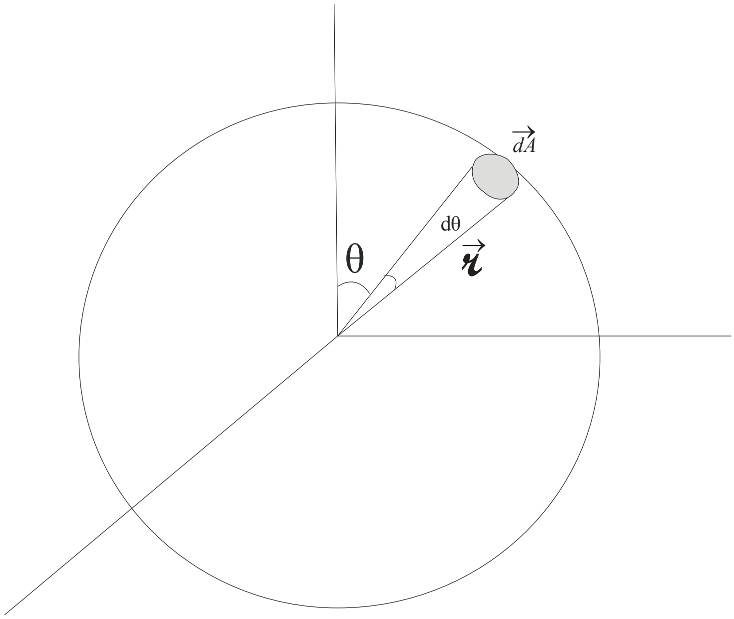


Figure 2

Since total power radiated is given as $P=\iint \vec{S}_{rad}\ .\ d\vec{A}$ where integration is taken over area surrounding to point charge, so

$$P=\int_0^{\pi}\int_0^{2\pi}\frac{1}{\mu_0 c}\left(\hat{\mathscr{r}}\,.\,\hat{\mathscr{r}}\right)E_{rad}^2\,\mathscr{r}^2\sin\theta\, d\theta\, d\Phi$$

$$P=\frac{1}{\mu_0 c}\int_0^{\pi}\int_0^{2\pi}\left(\mathscr{r}\,E_{rad}\right)^2\sin\theta\, d\theta\, d\Phi \tag{15}$$

where θ is the angle between direction of motion of charge and direction of emitted radiation.

Now multiply equation (12) by $\mathscr{r}$, gives

$$\mathscr{r}\vec{E}\left(\vec{r},t\right)=\frac{q}{4\pi\in_\circ}\frac{1}{c^2\left(1-\hat{\mathscr{r}}\ .\ \vec{\beta}\right)^2}\left[\frac{c^2\left(1-\beta^2\right)\hat{\mathscr{r}}}{\mathscr{r}}+\frac{\vec{\mathscr{r}}\times\hat{\mathscr{r}}\times\vec{a}}{\mathscr{r}}\right]$$

$$= \frac{q}{4\pi \in_\circ} \frac{1}{c^2\left(1-\hat{\imath}\ .\ \vec{\beta}\right)^2}\left[\frac{c^2\left(1-\beta^2\right)\hat{\imath}}{\imath} + \hat{\imath}\times\hat{\imath}\times\vec{a}\right]$$

I term II Term

Since I term varies as $\frac{1}{\imath}$, so at large distances this term contributes nothing to power ,but II term is independent of distance, so it gives fields which responsible for power radiating at large distances, hence these fields are called ***radiation fields,***

So,

$$\imath\vec{E}_{rad} = \frac{q}{4\pi \in_\circ c^2\left(1-\hat{\imath}\ .\ \vec{\beta}\right)^2}\left(\hat{\imath}\times\hat{\imath}\times\vec{a}\right)$$

If point change is at rest then $\vec{\beta} = 0$, so

$$\imath\vec{E}_{rad} = \frac{q}{4\pi\varepsilon_0 c^2}\left(\hat{\imath}\times\hat{\imath}\times\vec{a}\right)$$

$$\Rightarrow \imath\vec{E}_{rad} = \frac{q}{4\pi\varepsilon_0 c^2}\left\{\hat{\imath}\left(\hat{\imath}\ .\vec{a}\right)-\left(\hat{\imath}\ .\hat{\imath}\right)\vec{a}\right\}$$

$$\Rightarrow \imath\vec{E}_{red} = \frac{q}{4\pi\varepsilon_0 c^2}\left\{\hat{\imath}\left(\hat{\imath}.\vec{a}\right)-\vec{a}\right\}$$

Using this into equation (15), gives

$$P = \frac{1}{\mu_0 c}\int_0^{\pi}\int_0^{2\pi}\frac{q^2}{16\pi^2\varepsilon_0 c^4}\left\{\hat{\imath}\left(\hat{\imath}.\vec{a}\right)-\vec{a}\right\}^2\sin\theta d\theta d\Phi$$

$$= \frac{q^2}{16\pi^2\varepsilon_0 c^5}\int_0^{\pi}\int_0^{2\pi}\left\{\left(\hat{\imath}.\hat{\imath}\right)\left(\hat{\imath}.\vec{a}\right)+a^2-2\left(\hat{\imath}.\vec{a}\right)^2\right\}\sin\theta d\theta d\Phi$$

$$= \frac{q^2}{16\pi^2\varepsilon_0 c^3}\int_0^{\pi}\int_0^{2\pi}\left\{a^2-\left(\hat{\imath}.\vec{a}\right)^2\right\}\sin\theta d\theta d\Phi$$

$$= \frac{q^2}{16\pi^2\varepsilon_0 c^3}\int_0^{\pi}\int_0^{2\pi}\left\{a^2-a^2\cos^2\theta\right\}\sin\theta d\theta d\Phi$$

$$= \frac{q^2}{16\pi^2\varepsilon_0 c^3}\int_0^{\pi}\int_0^{2\pi}\left(a^2\sin^2\theta\right)\sin\theta d\theta d\Phi$$

$$= \frac{q^2 a^2}{16\pi^2 \varepsilon_0 c^3} \left[\int_0^{\pi} \sin^3 \theta d\theta \right] \left[\int_0^{2\pi} d\Phi \right]$$

$$= \frac{q^2 a^2}{16\pi^2 \varepsilon_0 c^3} \left(\frac{4}{3} \right) (2\pi)$$

$$\boxed{P = \frac{2}{3} \left(\frac{1}{4\pi\varepsilon_0} \right) \frac{q^2 a^2}{c^3}}$$

This is ***Larmor formula***, this gives total power radiated by an accelerated point charge which instantaneously at rest.

14.5 Relativistic Generalization of Power Radiated by a Point Charge-Lienard's Generalization of the Larmor Formula

The Larmor's formula is based on the assumption that the point charge is instantaneously at rest, so this result holds good as long as $v \ll c$***.***

When $v \neq 0$, then rate of energy passes through surface, is not same as the rate of energy at which it is lost from the particle. If $\frac{dW}{dt}$ is the rate of energy passes through the surface at a distance r from point charge, then the rate of energy left from the particle was

$$\frac{dW}{dt'} = \frac{dW}{dt} . \left(\frac{dt'}{dt} \right) \tag{18}$$

Where t' is ***retarded time***, given as

$$t' = t - \frac{r}{c}$$

After differentiation

$$\frac{dt'}{dt} = 1 - \frac{1}{c} \frac{dr}{dt}$$

Or $$\frac{dt'}{dt} = 1 - \frac{1}{c} \left(\frac{dr}{dt'} \right) \frac{dt'}{dt}$$

Or $$\frac{dt'}{dt} + \frac{1}{c} \frac{dr}{dt'} \frac{dt'}{dt} = 1$$

$$\Rightarrow \qquad \frac{dt'}{dt}\left[1+\frac{1}{c}\frac{dr}{dt'}\right]=1$$

$$\Rightarrow \qquad \frac{dt'}{dt}=\frac{1}{1+\frac{1}{c}\frac{dr}{dt'}} \tag{19}$$

Since $r=\left|\vec{r}-\vec{r}_0'\right|$

Hence

$$\frac{dr}{dt'}=\frac{d}{dt'}\sqrt{r^2+r_0'^2-2\vec{r}.\vec{r}_0'}$$

$$=\left[\left(\frac{1}{2}\right)\right]\left[\frac{d}{dt'}\left(r^2+r_0'^2-2\vec{r}.\vec{r}_0'\right)\right].\frac{1}{\sqrt{r^2+r_0'^2-2\vec{r}.\vec{r}_0'}}$$

$$=\left[\left(\frac{1}{2}\right)\right]\left[2\vec{r}_0'.\frac{d\vec{r}_0'}{dt'}-2\vec{r}.\frac{d\vec{r}_0'}{dt'}\right].\frac{1}{\sqrt{r^2+r_0'^2-2\vec{r}.\vec{r}_0'}}$$

$$=\left[\left(\frac{1}{2}\right)\right]2\left[\vec{r}_0'.\vec{\mathrm{v}}-\vec{r}.\vec{\mathrm{v}}\right].\frac{1}{\sqrt{r^2+r_0'^2-2\vec{r}.\vec{r}_0'}}$$

$$=-\frac{\left(\vec{r}-\vec{r}_0'\right).\vec{v}}{\left|\vec{r}-\vec{r}_0'\right|}$$

$$=-\frac{\vec{r}}{r}.\vec{v}$$

$$\frac{dr}{dt'}=-\hat{r}.\vec{v}$$

Hence from eq (19)

$$\frac{\partial t'}{\partial t}=\frac{1}{1-\hat{r}.\frac{\vec{v}}{c}}=\frac{1}{1-\hat{r}.\vec{\beta}}$$

where $\vec{\beta}=\frac{\vec{v}}{c}$

Using this into eq. (18), gives

$$\frac{dW}{dt'}=\left(\frac{1}{1-\hat{r}.\vec{\beta}}\right).\frac{dW}{dt'}$$

Hence power radiated by the point particle into an element of area $r^2 \sin\theta \, d\theta d\Omega$

$$dP = \left(\frac{1}{1-\hat{r}.\vec{\beta}}\right).\frac{1}{\mu_0 c} \quad E_{rad}^2.r^2 \sin\theta d\theta d\phi$$

$$\Rightarrow \qquad \frac{dP}{d\Omega} = \left(\frac{1}{1-\hat{r}.\vec{\beta}}\right).\frac{1}{\mu_0 c}\left(r\,E_{rad}\right)^2$$

Where $d\Omega = \sin\theta d\theta d\phi$

Since from eq. (16)

$$r\,\vec{E}_{rad} = \frac{q}{4\pi \in_0 c^2 \left(1-\hat{r}.\beta\right)^2}\left(\hat{r}\times\hat{r}\times\vec{a}\right)$$

Then

$$\frac{dP}{d\Omega} = \frac{1}{\mu_0 c}.\left(\frac{q}{4\pi \in_0 c^2}\right)^2 \frac{1}{\left(1-\hat{r}.\vec{\beta}\right)^5}\left|\hat{r}\times\hat{r}\times\vec{a}\right|^2$$

or

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2 \in_0 c^3}.\left(\frac{\left|\hat{r}\times\hat{r}\times\vec{a}\right|^2}{\left(1-\hat{r}.\vec{\beta}\right)^5}\right)$$

and the total power

$$P = \int_0^{2\pi}\int_0^{\pi}\frac{dP}{d\Omega}\sin\theta d\theta d\phi$$

$$P = \frac{q^2}{16\pi^2 \in_0 c^3}.\int_0^{2\pi}\int_0^{\pi}\left(\frac{\left|\hat{r}\times\hat{r}\times\vec{a}\right|^2}{\left(1-\hat{r}.\vec{\beta}\right)^5}\right)\sin\theta d\theta d\phi$$

After solving

$$\boxed{P = \frac{q^2}{4\pi \in_0 c^3}.\frac{2}{3}.\gamma^6\left[a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right]}$$

where $\gamma = \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$

This is ***Lienard formula for power radiated by the relativistic point particle***.

14.6 Illustrative Example

Example1: Calculate total power radiated by a proton initially at rest and having acceleration $10^6 \, m/\sec^2$

Sol. Larmor formula

P= $(1/4\pi\varepsilon_0)(2/3)(q^2a^2/c^3)$ where π= 3.14

$\varepsilon_0 = 8.85\text{x}10^{-12}$ (M.K.S.)

q= electronic charge=$1.6\text{x}10^{-19}$ C

a= acceleration of the charge particle= 10^6 m/s^2

C= velocity of light= $3\text{x}10^8$ m/s

Hence

P= $(1/4\text{x}3.14\text{x}8.85\text{x}10^{-12})$ x (2/3) x $(1.6\text{x}10^{-19})^2$ x $(10^6)^2$ x $(3\text{x}10^8)^{-3}$

=$9\text{x}10^9\text{x}2\text{x}1.6\text{x}1.6/ (3\text{x}3\text{x}3\text{x}3)\text{ x}10^{-38+12-24}$

=$5.69\text{x}10^{-42}$ W

=$5.69\text{x}10^{-42}/1.6\text{x}10^{-19}$ eV/s

= $(56900/1600)\text{ x}10^{-24}$

=$35.56\text{x}10^{-24}$ eV/s

14.7 Self learning Exercise -II

Q.1 Is a point charge with a constant velocity, radiate energy? Give reason.

Q.2 Give unit of Poynting vector?

Q.3 Write unit of power?

Q.4 What is meant by unit vector?

Q.5 Define Poynting vector?

14.8 Summary

Lienard Wiechert Potential related with moving charge are derived which shows velocity dependence.

Second part of unit is dedicated to determination of Larmor formula, i.e. total power radiated by an accelerated point change that initially at rest. Larmor formula shows that radiated power depends on square of acceleration of charge, so an accelerating or decelerating charge particle radiates energy.

14.9 Glossary

Charge density: Charge density is defined as charge per unit volume

Current density: Current density is defined as current per unit area

Scalar potential: Potential defined in scalar field

Vector potential: Potential defined in vector field

Radiated Power: Energy radiated per unit time

Poynting vector: Power radiated per unit area taken along perpendicular to the propagation of radiation

14.10 Answer to Self-Learning Exercises

Answer to Self-Lesrning Exercise -I

Ans.1: Charge density is defined as charge per unit volume.

Ans.2: Current density is defined as current per unit area.

Ans.3: $\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$, $\vec{B} = \vec{\nabla} \times \vec{A}$, symbol has their usual meaning.

Answer to Self-Learning Exercise -II

Ans.1: No,

Ans.2: $J/m^2.s$, W/m^2

Ans.3: Watt

14.11 Exercise

Section-A (Very Short Answer Type Questions)

Q.1 What is scalar potential?

Q.2 What is vector potential?

Q.3 How velocity of light in vacuum related with $\mu_0\varepsilon_0$?

Q.4 What is the value of ε_0 in MKS system?

Q.5 What is the value of μ_0 in MKS system?

Section-B (Short Answer Type Questions)

Q.6 Define radiation.

Q.7 Write properties of electromagnetic fields.

Q.8 What is point charge?

Q.9 Explain the term 'retarded potential'.

Section-C (Long Answer Type Questions)

Q.10 Describe Linard Wiechert potential for a moving point charge.

Q.11 Derive electromagnetic fields for a moving point charge.

Q.12 Discuss the Poynting vector for an accelerated point charge.

Q.13 Obtain Larmor formula.

14.12 Answers to Exercise

Ans.1: Potential defined in scalar field.

Ans.2: Potential defined in vector field.

Ans.3: $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

Ans.4: $\varepsilon_0 = 8.85 \times 10^{-12} \; coul^2/N\text{-}m^2$

Ans.5: $\mu_0 = 4\pi \times 10^{7} \; N/A^2$

Ans.10: $35.56\text{x}10^{-24}$ eV/s

References and Suggested Readings

1. D.J. Griffiths, Introduction to Electrodynamics (Second Edition), Prentice – Hall of India Private Limited, 1993
2. J.D. Jackson, Classical Electrodynamics (Second edition), Wiley Eastern Limited, 1989.
3. S.P. Puri, Classical Electrodynamics (Second Edition), Tata McGraw Hill Publishing Company Limited, 1997.

UNIT-15
Radiation Emitted by a Charge in Arbitrary Extremely Relativistic Motion

Structure of the unit

15.0 Objectives

Our objectives in this chapter are :

1. Radiation emitted by a charge in extremely relativistic case
2. Angular distribution of the radiation from a rapidly moving charge
3. Thomson scattering by free charges
4. Illustrative problems.

15.1 Introduction

We now consider the radiation emitted by a charged particle moving with a velocity which is not small compared with the velocity of light. The motion of charged particles in external force fields necessarily involves the emission of radiation whenever the charges are accelerated. The emitted radiation carriers off energy, momentum, and angular momentum and so must influence the subsequent motion of the charged particles. Consequently the motion of the sources of radiation is determined, in part, by the manner of emission of the radiation. We derive an expression for the total four momentum radiated during the time of passage of the particle through a given electromagnetic field. We also derive an expression for the effective cross-section for scattering by a system of free charges.

15.2 Radiation Emitted by a Charge in Arbitrary Extremely Relativistic Motion

We now consider the radiation emitted by a charge particle moving with a velocity which is not small compared with the velocity of light.

The formulas derived under the assumption v<<c, are not applicable to this case.

We can however, consider the particle in that system of reference in which the particle is at rest at a given moment; in this system of reference the formulas referred to are of course valid (we call attention to the fact that can be done only for the case of a single moving particle; for a system of several particles there is generally no system of reference in which all the particles are at rest simultaneously.

Thus in this particular system of reference the particle radiates, in time dt, the energy $d\varepsilon$,where $d\varepsilon = \frac{2}{3}\ \frac{e^2}{c^3} w^2 dt$

where e is the charge, c is the speed of light and w is the acceleration of the particle in this system of reference.

In this system of reference, the total radiated momentum is zero: $dP = 0$

Infact, the radiated moment is given by the integral of the momentum flux density in the radiation field over a closed surface surrounding the particle. But because of the symmetry of the dipole radiation, the momenta carried off in opposite directions are equal in magnitude and opposite in direction; therefore the integral is identically zero.

For the transformation to an arbitrary frame, we rewrite the formulas (1) and (2) in four dimensional form. It is easy to see that "the radiated four momentum" dP_i must be written as

$$dP^i = \frac{2}{3}\ \frac{e^2}{c}\ \frac{du^k}{ds}\ \frac{du_k}{ds}dx^i$$
$$= \frac{-2}{3}\ \frac{e^2}{c}\ \frac{du^k}{ds}\ \frac{du_k}{ds}u^i ds$$

where $\frac{du^k}{ds}$ is four acceleration, and

$ds = cdt\sqrt{1-\frac{v^2}{c^2}}$ is the differential interval

and $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is the relativistic gamma factor γ,

$\frac{dx^i}{ds} = u^i$ is four velocity.

In fact, in the reference frame in which the particle is at rest, the space components of the four velocity u^i are equal to zero: $\left[u^i = \left(\frac{1}{\sqrt{1-v^2/c^2}}, \frac{\vec{v}/c}{\sqrt{1-v^2/c^2}}\right)\right]$

and $\frac{du^k}{ds}\ \frac{du_k}{ds} = -\frac{w^2}{c^4}$.

This is because in the reference frame in which the particle velocity is $\vec{v} = 0$, the components of the four acceleration $w^i = \left(0, \frac{w}{c^2}, 0, 0\right)$(where w is the ordinary three dimensional acceleration, which here is assumed to be directed along the x-axis).

From (3) it follows that the space components of dP^i become zero and time component gives equation (1):

To see this we write

$$dP^0 \equiv \frac{1}{c} d\varepsilon = -\frac{2}{3}\frac{e^2}{c}\left(\frac{-w^2}{c^4}\right) c\ dt$$

$$\boxed{d\varepsilon = \frac{2e^2}{3c^3} w^2\ dt}$$

The total four momentum radiated during the time of passage of the particle through a given electromagnetic field is equal to the integral, i.e.

$$\Delta P^i = -\frac{2}{3}\frac{e^2}{c}\int \frac{du^k}{ds}\ \frac{du_k}{ds}\ dx^i$$

We rewrite this formula in another form expressing the four-acceleration $\frac{du^i}{ds}$ in terms of the electromagnetic field tensor, using the equation of motion.

$$mc\frac{du_k}{ds} = \frac{e}{c} F_{kl} u^l,$$

where $F_{kl} = \frac{\partial A_l}{\partial x^k} - \frac{\partial A_k}{\partial x^l}$ is electromagnetic field tensor .

We then obtains $\Delta P^i = \frac{-2e^4}{3m^2c^5}\int \left(F_{kl}u^l\right)\left(F^{km}u_m\right) dx^i$

The time component gives the total radiated energy $\Delta\varepsilon$. Substituting for all the four dimensional quantities their expressions in terms of three-dimensional quantities, we find.

$$\Delta\varepsilon = \frac{2e^4}{3c^3}\int_{-\infty}^{\infty} \frac{w^2 - \frac{\left(\vec{v}\times\vec{w}\right)}{c^2}}{\left(1-\frac{v^2}{c^2}\right)^3} dt \tag{6}$$

Where $\vec{w} = \dot{\vec{v}}$ is the acceleration of the particle.

In terms of the external electric and magnetic fields:

$$\Delta\varepsilon = \frac{2e^4}{3m^2c^3}\int_{-\infty}^{\infty}\frac{\left\{\vec{E}+\frac{1}{c}\vec{v}\times\vec{H}\right\}^2-\frac{1}{c^2}\left(\vec{E}.\vec{v}\right)^2}{1-\frac{v^2}{c^2}}dt \tag{7}$$

The expressions for the total radiated momentum $(eq.(4))$ differ by having an extra factor $\vec{v}$ in the integrand.

Note that

(i) From formula (7) we note that for velocities close to the velocity of light, the total energy radiated per unit time varies with the velocity essentially like $\left[1-\left(\frac{v^2}{c^2}\right)\right]^{-1}$, that is, proportionally to the square of the energy of the moving particle.

(ii) The only exception is motion in an electric field, along the direction of the field. In this case the factor $\left(1-\frac{v^2}{c^2}\right)$ standing in the denominator is cancelled by an identical factor in the numerator, and, therefore, the radiation does not depend on the energy of the particle.

15.3 Angular Distribution of the Radiation from a Rapidly Moving Charge

In order to solve the question of the angular distribution of the radiation from a rapidly moving charge, it is convenient to use the Linear-Wiechert expression for the field, namely,

$$\vec{E} = \frac{e\left(1-\frac{v^2}{c^2}\right)\left(\vec{R}-\frac{\vec{v}}{c}R\right)}{\left(R-\vec{R}.\frac{\vec{v}}{c}\right)^3}\left(\vec{R}-\frac{\vec{v}}{c}R\right)+\frac{e\vec{R}\times\left\{\left(\vec{R}-\frac{\vec{v}}{c}R\right)\times\vec{v}\right\}}{c^2\left(R-\frac{\vec{R}.\vec{v}}{c}\right)^3} \tag{8}$$

and $$\vec{H} = \frac{1}{R}\vec{R}\times\vec{E} \tag{9}$$

At large distances we must retain only the term of lowest order in $\frac{1}{R}$ (the second terms). Introducing the unit vector $\vec{n}$ in the direction of the radiation. $\vec{R} = \vec{n} R$,

we get $$\vec{E} = \frac{e}{c^2 R} \frac{\vec{n} \times \left\{ \left(\vec{n} - \frac{\vec{v}}{c} \right) \times \vec{w} \right\}}{\left(1 - \frac{\vec{n} . \vec{v}}{c} \right)^3},$$

$$\vec{H} = \vec{n} \times \vec{E} \tag{10}$$

where all the quantities on the right side of the equations refer to the ***retarded time***

$$t' = t - \frac{R}{c}$$

The intensity radiated into the solid angle $d\Omega$ is

$$dI = \left(\frac{c}{4\pi} \right) E^2 R^2 d\Omega .$$

Expanding E^2, we get

$$dI = \frac{e^2}{4\pi c^3} \left\{ \frac{2\left(\vec{n} . \vec{w} \right) \left(\vec{v} . \vec{w} \right)}{c\left(1 - \frac{\vec{v} . \vec{n}}{c} \right)^5} + \frac{w^2}{\left(1 - \frac{\vec{v} . \vec{n}}{c} \right)^4} - \frac{\left(1 - \frac{v^2}{c^2} \right)\left(\vec{n} . \vec{w} \right)^2}{\left(1 - \frac{\vec{v} . \vec{n}}{c} \right)^6} \right\} d\Omega \tag{12}$$

If we want to determine the angular distribution of the total radiation throughout the whole motion of the particle, we must integrate the intensity over the time.

In doing this, it is important to remember that the integrand is a function of time t'; therefore we must write

$$t' + \frac{R(t')}{c} = t ,$$

where $R(t') = c(t - t')$ (13)

Differentiating the relation $R(t') = c(t - t')$ with respect to t, we get

$$\frac{\partial R}{\partial t} = \frac{\partial R}{\partial t'}\ \frac{\partial t'}{\partial t}$$

$$= \frac{-\vec{R}.\vec{v}}{R}\ \frac{\partial t'}{\partial t}$$

$$\Rightarrow \frac{\partial R}{\partial t} = C\left(\frac{\partial t'}{\partial t}\right) \tag{14}$$

(the value of $\dfrac{\partial R}{\partial t'}$ is obtained by differentiating the identity $R^2 = \vec{R}.\vec{R}$ and substituting $\dfrac{\partial \vec{R}(t')}{\partial t'} = -\vec{v}(t')$.

The minus sign is present because $\vec{R}$ is the radius vector from the charge e to the point P, and not the reverse.)

From Eq.(14) we get

$$dt = \frac{\partial t}{\partial t'} dt'$$

$$\Rightarrow dt = \left(1 \frac{\vec{n}.\vec{v}}{c}\right) dt' \tag{15}$$

Making use of (15), the integration over t' is immediately done. Thus we have the following expression for the total radiation into the solid angle $d\Omega$:

$$d\varepsilon_n = \frac{e^2}{4\pi c^3} d\Omega \int \left\{ \frac{2\left(\vec{n}.\vec{w}\right)\left(\vec{v}.\vec{w}\right)}{c\left(1 - \frac{\vec{v}.\vec{n}}{c}\right)^4} + \frac{w^2}{\left(1 - \frac{\vec{v}.\vec{n}}{c}\right)^3} - \frac{-\left(1 - \frac{v^2}{c_2}\right)\left(\vec{n}.\vec{w}\right)^2}{\left(\frac{1 - \vec{v}.\vec{n}}{c}\right)^5} \right\} dt' \tag{16}$$

As we see here, the general case of angular distribution of the radiation is quite complicated.

In the ultra-relativistic Case, $\left(\left(1 - \frac{v}{c}\right) << 1\right)$ it has a characteristic appearance, which is related to the presence of high powers of the difference $\left(1-\frac{\vec{n}.\vec{v}}{c}\right)$ in the denominators of the various terms in this expression.

Thus, the intensity is large within a narrow range of angle in which the difference $\left(1-\frac{\vec{n}.\vec{v}}{c}\right)$ is small.

Denoting by θ the small angle between $\vec{n}$ and $\vec{v}$ we have

$$\left(1-\frac{\vec{n}.\vec{v}}{c}\right)=1-\frac{v}{c}\cos\theta$$

$$\cong 1-\frac{v}{c}+\frac{\theta^2}{2}$$

$$\cong \frac{1}{2}\left(1+\frac{v}{c}\right)\left\{1-\frac{v}{c}+\frac{\theta^2}{2}\right\} \quad (\because v\approx c)$$

$$\cong \frac{1}{2}\left\{1-\frac{v^2}{c^2}+\theta^2\right\}$$

This difference is small for

$$\theta \sim \sqrt{1-\frac{v^2}{c^2}} \tag{17}$$

Thus ***an ultra-relativistic particle radiates mainly along the direction of its own motion,*** within the small range (17)of angles around the direction of its velocity.

It may also be pointed out that, for arbitrary velocity and acceleration of the particle, there are always two directions for which the radiated intensity is zero. These are the directions for which the vector $\vec{n}-\left(\frac{\vec{v}}{c}\right)$ is parallel to the vector $\vec{w}$, so that the field(10) becomes zero.

Finally, we give the simpler formulas to which (12) reduces in two special cases:

If the velocity and acceleration of the particle are parallel,

$$\vec{H} = \frac{e}{c^2 R} \quad \frac{\vec{w} \times \vec{n}}{\left(1 - \frac{\vec{n}.\vec{v}}{c}\right)^3} \text{ and}$$

the intensity is $dI = \frac{c}{4\pi} \quad H^2 R^2 \ d\Omega$

or $\qquad dI = \frac{e^2}{4\pi c^3} \quad \frac{w^2 \sin^2 \theta}{\left(1 - \frac{v}{c}\cos\theta\right)^6} d\Omega$

It is naturally, symmetric around the common direction of $\vec{v}$ and $\vec{w}$ vanishes along $(\theta = 0)$ and opposite to $(\theta = \pi)$ the direction of the velocity.

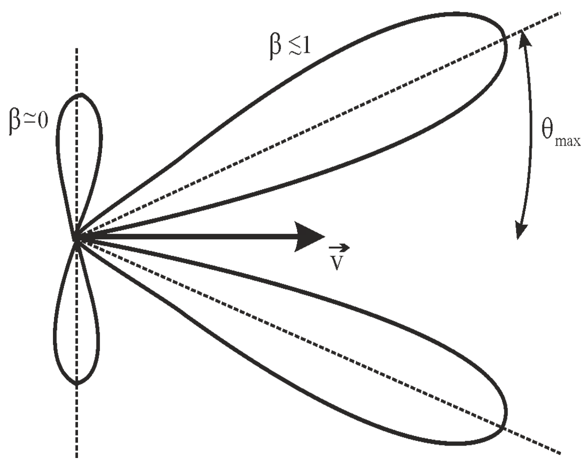


Figure: *Radiation pattern for charge accelerated in its direction of motion. The two patterns are not to scale.*

In the ultra-relativistic case, the intensity as a function of θ has a sharp double maximum in the region $\theta \sim \sqrt{1 - v^2/c^2}$ with a steep drop to zero for $\theta = 0$

If the velocity and acceleration are perpendicular to one another, we have form (12):

$$dI = \frac{e^2 w^2}{4\pi c^3} \left[\frac{1}{\left(1 - \frac{v}{c}\cos\theta\right)^4} - \frac{\left(1 - \frac{v^2}{c^2}\right)\sin^2\theta\cos^2\varphi}{\left(1 - \frac{v}{c}\cos\theta\right)^6} \right] d\Omega \tag{19}$$

where θ is the angle between $\vec{n}$ and $\vec{v}$, and ϕ is the azimuthal angle of the vector $\vec{n}$ relative to the plane passing through $\vec{v}$ and $\vec{w}$.

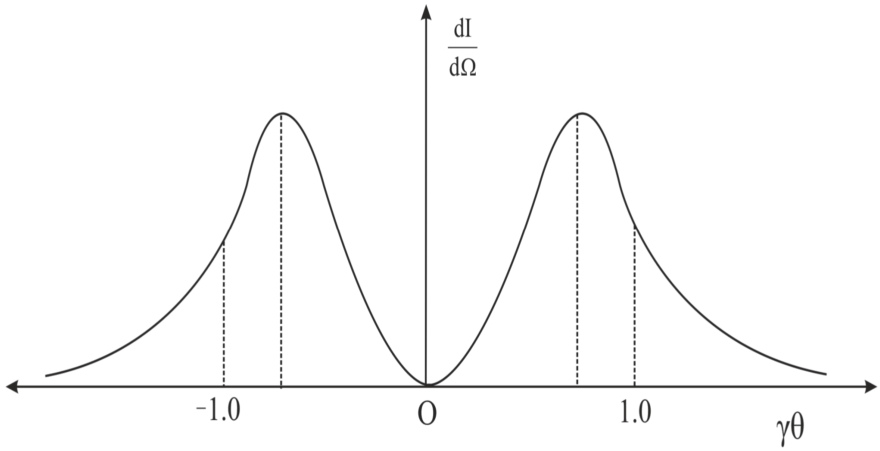


Figure: Angular distribution of radiation

The intensity is symmetric only with respect to the plane of $\vec{v}$ and $\vec{w}$,vanishes along the two directions in this plane which from the angle $\theta = \cos^{-1}\left(\frac{v}{c}\right)$ with the velocity.

15.4 Thomson Scattering by Free Charges

If an electromagnetic wave falls on a system of charges, then under its action the charges are set in motion. This motion in turn produces radiation in all directions. There occurs, as we say, a scattering of the original wave.

The scattering is most conveniently characterized by the ratio of the amount of energy emitted by the scattering system in a given direction per unit time, to the energy flux density of the incident radiation. This ratio clearly has dimensions of area, and is called the effective cross-section (or simply the cross-section).

Let dI be the energy radiated by the system into solid angle $d\Omega$ per second for an incident wave with pointing vector $\vec{S}$. Then the effective gross-section for scattering into the solid angle $d\Omega$ is

$$d\sigma = \frac{d\overline{I}}{S} \qquad (1)$$

where the dash over a symbol means a time average.

We consider the scattering produced by a free charge at rest.

Let a plane monochromatic linearly polarized wave be incident on this charge. Its electric field can be written as

$$\vec{E} = \vec{E}_0 \cos\left(\vec{k}.\vec{v} - wt + \alpha\right)$$

We shall assume that the velocity acquired by the charge under the influence of the incident wave is small compared with the velocity of light $\left(\frac{v}{c} \ll 1\right)$. This is usually the case.

Then the force acting on the charge due to electromagnetic field incident on it can be taken to be e$\vec{E}$; we can neglect the force $\frac{e}{c}\vec{v} \times \vec{H}$ due to magnetic field.

In this case we can also neglect the effect of the displacement of the charge during its vibrations under the influence of the field.

If the charge carries out vibrations around the coordinates origin, then we can assume that the field which acts on the charge at all times is the same as that at the origin, that is, $\vec{E} = \vec{E}_0 \cos\left(wt - \alpha\right)$. $\left(\vec{r} = \vec{0}\right)$

Since the equation of motion of the charge is therefore $m\ddot{\vec{r}} = e\vec{E}$, $\ddot{\vec{r}} = \frac{e\vec{E}}{m}$

Since the dipole moment of the charge is $\vec{d} = e\vec{v}$ ($\vec{d}$ stands for dipole moment of the charge), then

$$\ddot{\vec{d}} = e\ddot{\vec{r}} = \frac{e^2\vec{E}}{m} \tag{2}$$

To calculate the scattered radiation, we use formula for dipole radiation, namely

$$dI = \frac{1}{4\pi c^3}\left(\ddot{\vec{d}} \times \vec{n}\right)^2 d\Omega = \frac{\ddot{\vec{d}}^2 \sin^2\theta}{4\pi c^3} d\Omega \tag{3}$$

where θ is the angle between $\ddot{\vec{d}}$ and $\vec{n}$.

This is the amount of energy radiated of the charge in unit time into the element of solid angle $d\Omega$. The use of the formula for dipole rotation is justified since the velocity acquired by the charge is assumed to be small.

We also note that the frequency of the wave radiated by the charge (i.e scattered by it) is clearly the same as the frequency of the incident wave. Substituting(2) into (3), we find

$$dI = \frac{1}{4\pi c^3}\left(\frac{e^2\vec{E}}{m}\times\vec{n}'\right)^2 d\Omega$$

$$dI = \frac{e^4}{4\pi m^2 c^3}\left(\vec{E}\times\vec{n}'\right)^2 d\Omega$$

where n' is a unit vector in the scattering direction.

On the other hand, the Poynting vector of the incident wave is

$$S = \frac{c}{4\pi}\ E^2 \tag{4}$$

From this we find, for the cross-section for scattering into the solid angle $d\Omega$,

$$d\sigma = \frac{dI}{S} = \frac{e^4}{4\pi m^2 c^3}\frac{E^2\sin^2\theta}{\frac{c}{4\pi}E^2}d\Omega$$

$$d\sigma = \left(\frac{e^2}{mc^2}\right)^2 \sin^2\theta\, d\Omega \tag{5}$$

where θ is the angle between the direction of scattering (the vector $\vec{n}$), and the direction of the electric field $\vec{E}$ of the incident wave. We see that the ***effective scattering cross-section of a free charge is independent of frequency.***

We now determine the total cross-section σ. To do this, we choose the polar axis along $\vec{E}$. Then $d\Omega = \sin\theta d\theta d\varphi$; substituting this and integrating with respect to θ from 0 to π; and over ϕ from 0 to 2π,

$$\sigma = \left(\frac{e^2}{mc^2}\right)^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin^2\theta.\sin\theta\ d\theta$$

$$= 2\pi\left(\frac{e^2}{mc^2}\right)^2 \int_0^{\pi} \sin^3\theta \; d\theta$$

Using the value of the integral

$$\int_0^{\pi} \sin^3\theta \; d\theta = \frac{4}{3}, \text{ we get}$$

$$\boxed{\sigma = \left(\frac{8\pi}{3}\right)\left(\frac{e^2}{mc^2}\right)^2} \tag{6}$$

This is the ***Thomson's formula***

15.5 Thomson's Scattering Cross-Section in the case where the Incident Wave is Unpolarized (Ordinary light)

If the incident wave is ordinary light i.e. unpolarized, then to calculate $d\sigma$ we must average (5) over all directions of the vector $\vec{E}$ in a plane perpendicular to the direction of propagation of the incident wave (direction of the wave vector $\vec{k}$).

Denoting by $\hat{e}$ the unit vector along the direction of $\hat{E}$, we write:

$$\overline{\sin^2\theta} = 1 - \left(\overline{\bar{n}.\hat{e}}\right)^2$$

$$\overline{\sin^2\theta} = 1 - n_\alpha n_\beta \overline{e_\alpha e_\beta} \tag{7}$$

The averaging is done using the formula

$$\overline{e_\alpha e_\beta} = \frac{1}{2}\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right) \tag{8}$$

and gives

$$\sin^2\theta = \frac{1}{2}\left(1 + \frac{\left(\vec{n}.\vec{k}\right)^2}{k^2}\right) = \frac{1}{2}\left(1 + \cos^2\Theta\right)$$

where Θ is the angle between the directions of the incident and scattered waves (the scattering angle)

Thus the elective cross-section for scattering of an unpolarized wave by a free charge.

$$d\sigma = \frac{1}{2}\left(\frac{e^2}{mc^2}\right)^2 \left(1+\cos^2\Theta\right)d\Omega$$

15.6 Illustrative Examples

Example1: Determine the effective cross-section for scattering of a linearly polarized wave by a charge carrying out small vibrations under the influence of on elastic force (oscillator).

Sol. Let the incident field be representation by

$$\vec{E} = \vec{E}_0 \cos(wt+\alpha)$$

Under the influence of this field, the equation of motion of the charge is

$$m\ddot{\vec{r}} = -k\,\vec{r} + eE_0\cos(wt+\alpha)$$

$$\text{or } \ddot{\vec{r}} + w_0^2\,\vec{r} = \frac{e}{m}\vec{E}_0\cos(wt+\alpha)$$

where $w_0^2 = \frac{k}{m}$ is the frequency of its free vibrations.

For the forced vibrations, we then have

$$\vec{r} = \frac{e\vec{E}_0\cos(wt+\alpha)}{m\left(w_0^2-w^2\right)}$$

Calculating the dipole moment $\vec{d}$ from this we get

$$\vec{d} = e\,\vec{r} = \frac{e^2\vec{E}_0\cos(wt+\alpha)}{m\left(w_0^2-w^2\right)}$$

The effective cross-Sections is $d\sigma = \frac{\overline{dI}}{S}$ using the formula $\overline{dI} = \frac{\ddot{\vec{d}}^2}{4\pi c^3}\sin^2\theta\, d\Omega$

where θ is the angle between $\ddot{\vec{d}}$ and $\vec{n}'$

$$\overline{dI} = \frac{1}{4\pi c^3}\left(\frac{e^2E_0.w^2}{m\left(w_0^2-w^2\right)}\right)^2\cos^2(wt+\alpha)\sin^2\theta\, d\Omega$$

$$\overline{dI} = \frac{1}{8\pi c^3}\left(\frac{e^2 E_0.w^2}{m\left(w_0^2 - w^2\right)}\right)^2 \sin^2\theta \; d\Omega$$

On the other hand, the Poynting vector of the incident wave is

$$S = \frac{e}{4\pi}E^2$$

$$\Rightarrow S = \frac{e}{4\pi}E_0^2 \cos^2\left(wt + \alpha\right)$$

$$\therefore \overline{S} = \frac{c}{4\pi}E_0^2 \frac{1}{2} = \frac{c}{8\pi}E_0^2$$

We find

$$d\sigma = \frac{\overline{dI}}{\overline{S}}$$

$$= \frac{1}{8\pi c^3}\left(\frac{e^2 w^2}{m\left(w_0^2 - w^2\right)}\right)^2 \frac{E_0^2 \sin^2\theta d\Omega}{\frac{c}{8\pi}E_0^2}$$

$$= \left(\frac{e^2}{mc^2}\right)^2 \frac{w^4}{\left(w_0^2 - w^2\right)^2} .\sin^2\theta \; d\Omega$$

θ is the angle between $\vec{E}$ and $\vec{n}'$

Example 2: Determine the frequency w' of the light scattered by a moving charge.

Sol. In a frame of reference in which the charge is at rest, the frequency of the light does not change on scattering ($w = w'$). This relation can be written in invariant form as

$$k'_i \; u^i = k_i \; u^i$$

where u^i is the four velocity of the charge. From this we find without difficulty.

$$w'\left[1 - \frac{v}{c}\cos\theta'\right] = w\left[1 - \frac{v}{c}\cos\theta\right]$$

where θ and θ' are the angle made by the incident and scattered waves with the direction of motion (v is the velocity of change).

Note that $\quad k^i \equiv \left(\frac{w}{c}, \vec{k}\right)$, $u^i = \left(\frac{1}{\sqrt{\frac{1-v^2}{c^2}}}, \frac{\vec{v}/c}{\sqrt{\frac{1-v^2}{c^2}}},\right)$

Therefore

$$k_i' u^i = \frac{w'}{c} \cdot \frac{1}{\sqrt{\frac{1-v^2}{c^2}}} - \frac{\vec{k}' \cdot \vec{v}/c}{\sqrt{\frac{1-v^2}{c^2}}}$$

Similarly $k_i u^i = \frac{w}{c} \cdot \frac{1}{\sqrt{\frac{1-v^2}{c^2}}} - \frac{\vec{k}' \cdot \vec{v}/c}{\sqrt{\frac{1-v^2}{c^2}}}$

Equating these two, we get

$$w'\left[1 - \frac{v}{c}\cos'\theta\right] = w\left[1 - \frac{v}{c}\cos\theta\right]$$

15.7 Self Learning Exercise

Q.1 Define scattering cross section for scattering by a free charge at rest. Show that scattering cross-section of a free charge is independent of frequency.

Q.2 Show that an ultra-relativistic particle radiates mainly along the direction of its motion, within the small angle range $\theta \sim \sqrt{1 - v^2/c^2}$ around the direction of its velocity

15.8 Illustrative Examples

Example 3: Determine the effective cross-section for scattering of an elliptically polarized wave by a free charge.

Sol. The electric field of the elliptically polarized wave can be represented as

$$\vec{E} = \vec{A}\cos(wt + \alpha) + \vec{B}\sin(wt + \alpha),$$

Where $\vec{A}$ and $\vec{B}$ are mutually perpendicular vectors.

Since the equations of motion of the charge is

$$m\ddot{\vec{r}} = e\vec{\mathrm{E}},$$

and its dipole moment $\vec{d} = e\vec{v}$ is then

$$\ddot{\vec{d}} = e\ddot{\vec{v}} = \frac{e^2\vec{\mathrm{E}}}{m} = \frac{e^2}{m}\left[\vec{\mathrm{A}}\cos\left(\mathrm{wt}+\alpha\right)+\vec{\mathrm{B}}\sin\left(\mathrm{wt}+\alpha\right)\right]$$

$\therefore$ Scattered radiation

$$dI = \frac{e^4}{4\pi m^2 c^3}\left[\vec{E}\times\vec{n}'\right]^2 d\Omega$$

On the other hand, the Poynting vector of the incident wave is

$$S = \frac{C}{4\pi}E^2$$

Using the formula $d\sigma = \frac{\overline{dI}}{S}$,

we find,

$$d\sigma = \left(\frac{e^2}{mc^2}\right)^2 \frac{\left(\vec{A}\times\vec{n}\right)^2 + \left(\vec{B}\times\vec{n}\right)^2}{\left(A^2+B^2\right)} d\Omega$$

Example 4: Determine the effective cross-section for scattering of a linearly polarized wave by an oscillator, taking into account the radiation damping.

Sol. We write the equation of motion of the charge in the incident field in the form

$$\ddot{\vec{r}} + w_0^2 r = \frac{e}{m}\vec{E}_0 e^{-iwt} + \frac{2e^2}{3mc^3}\dddot{\vec{r}}$$

In the damping force, we can substitute approximately

$$\dddot{\vec{r}} = -w_0^2\dot{\vec{r}};$$

then we find

$$\ddot{\vec{r}} + \gamma\dot{\vec{r}} + w_0^2\vec{r} = \frac{e}{m}\vec{E}_0 e^{-iwt},$$

where $\gamma = \frac{2e^2}{3mc^3}w_0^2$.

From this we obtain

$$\vec{r} = \frac{e}{m}\vec{E}_0 \frac{e^{-iwt}}{w_0^2 - w^2 - iw\gamma}$$

The effective cross-section is

$$\sigma = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \frac{w^4}{\left(w_0^2 - w^2\right)^2 + w^2\gamma^2}$$

15.9 Summary

In this chapter we have derived an expression for the total four-momentum radiated during the passage of the particle through a given electromagnetic field. We have also studied the question of angular distribution of the radiation from a rapidly moving charge. We have found that for arbitrary velocity and acceleration of the particle, there are always two directions for which the radiated intensity is zero. These are the directions for which the vector $\vec{n} - \frac{\vec{v}}{c}$ is parallel to the vector $\vec{W}$, so that the field becomes zero. We have also derived an expression for Thomson scattering cross-section by free charges when incident radiation is unpolarized and also when the incident radiations is polarized.

15.10 Glossary

Relativistic: speed is order of speed of light

Elliptical polarization**:** It is the polarization of electromagnetic wave such that the tip of the electric field vector describes an ellipse in any fixed plane intersecting, and normal to, the direction of propagation.

15.11 Exercise

Q.1 Derive an expression for the angular distribution of the radiation from a rapidly moving charge. Discuss the case when the velocity and acceleration of the particle are parallel.

Q.2 Consider a charge moving along the direction of the election field. Show that in this case the radiation does not depend on the energy of the particle.

References and Suggested Readings

1. L.D.Landau &E.M.Lifshitz Classical Theory Fields(Fourth Revised Edition) Butterworth-Heinemann,2010
2. John David Jackson ,Classical Electrodynamics ,Third Edition ,John-Wiley Sons,2010
3. I.E.Tamm,Fundamental of the theory of Electricity(Translated from the Russian by G.Leib), Mir Publishers,Moscow.

UNIT -16
Special Theory of Relativity

Structure of the Unit

16.0 Objectives

The objectives of this unit are :

(i) To study postulates of special theory of relativity

(ii) To derive Lorentz Transformation

(iii) To study Relativistic equation of motion

(iv) To study application of energy momentum conservation

16.1 Introduction

Galileo and Newton through experiments arrived at certain laws covering motion of bodies, known as classical mechanics. They proposed that length, time and mass are fundamental and absolute quantities. These remain the same for all moving and stationary observers. Also they found that physical laws of mechanics remains invariant in all inertial frames of reference. However this did not hold true for the laws of electrodynamics.

In 1905, Albert Einstein proposed a revolutionary theory of relativity in which he postulated that physical Laws are universal and a single theory should govern mechanics and electrodynamics. He also postulated that the velocity of light in vacuum is universal and is the highest achievable velocity for moving objects. This theory revolutionized the world of science. Infact, all major researches in the 20^{th} century are based on this theory of relativity.

No doubt, this theory reveals much more newer ideas and questions of the absolute character of length, mass and time. But its results are not much different from classical results. ***When velocity of object v is far less than the speed of light c. This is called classical limit***. When v = c ,we call it relativistic limit and Mechanics is known as relativistic mechanics.

16.2 Preliminary

(a) Galilean Transformations

The classical theory mainly involves Galilean transformations. Consider two frames of references S and S' such that their origins O and O' coincides at $t = t' = 0$ i.e. initially. The frame S' moves with constant velocity v relative to S along x-axis as shown in figures (16.1). After time t, S' travels a distance vt along x, From figure (16.1) it is seen that for given point P

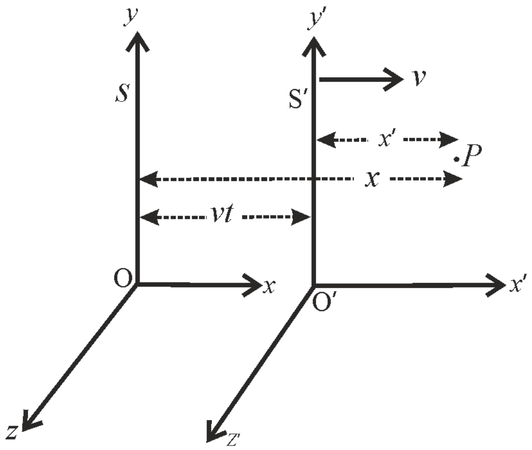


Figure16.1

$$\boxed{\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ \& t' &= t \end{aligned}} \qquad (16.1)$$

Where x', y', z' are the coordinates observed from frame S' at time t' whereas x, y, z are the coordinates observed for the same point P from frame S at time t.

Equations (16.1) establish interrelation between coordinates observed in the two frames. These are known as ***coordinate transformations***. Differentiating eq. (16.1) w.r. to time, we find

$$\because \frac{dx'}{dt'} = v'_x, \frac{dy'}{dt'} = v'_y, \frac{dz'}{dt'} = v'_z$$

$$\boxed{\begin{aligned} v'_x &= v_x - v \\ v'_y &= v_y \\ v'_z &= v_z \end{aligned}} \qquad (16.2)$$

Equations (16.2) are known as ***Galilean transformation for velocity*** or the law of Galilean addition of velocities.

Differentiating eq. (16.2) with respect to time, we find

$$\begin{aligned} a'_x &= a_x \\ a'_y &= a_y \\ a'_z &= a_z \end{aligned} \qquad (16.3)$$

Equations (16.3) are known as Galilean transformation for acceleration. It is clear that acceleration remains same or force remains same in both frames. So all physical laws also remain invariant under Galilean Transformation.

(b) Michelson-Morley Experiment

Classical mechanics postulates absolute length , mass and time and so there was a search for absolute frame of reference. Michelson Morley attempted to identify the hypothetical ether proposed by Fizeau to be an absolute frame of reference. They modified Michelson's interferometer to

(i) Identify ether as an absolute frame of reference and

(ii) determine the velocity of the earth with respect to stationary ether using Galilean transformations.

They oriented the instrument so that one arm of the interferometer becomes parallel to the tangential velocity of the earth while other remains perpendicular to it but in the plane of the earth's velocity. While going in the direction of the earth's motion, the velocity of light will be (c+v) and in opposite direction it will be (c−v) Also the velocity of light is the perpendicular direction will be $\sqrt{c^2+v^2}$. Thus there will be a phase difference $\Delta\phi$ and a definite interference pattern due to earth's revolution. On rotating the instrument by 90^0 in its plane, because arms are interchanged a phase difference of 2 $\nabla\phi$ is introduced, this will shift the fringe pattern

(c) Conclusions from Michelson-Morley Experiment

The instrument was sensitive to measure $\left(\frac{1}{100}\right)^{th}$ of a fringe shift and the expected fringe shift was 0.37. ***All attempts made to identify expected fringe shift failed; there was no measurable fringe shift .The result of Michelson-Morley experiment was negative.*** The outcome of the experiment was formulated as follows :

(i) ***Absolute frame does not exist***. Motion relative to some material object only is meaningful.

(ii) Velocity of light in free space is absolute, it is not subjected to relative motion.

As we know that Michelson and Morley performed experiment in relation with velocity of light. In the similar manner many experiments were performed such as Fizeau's experiment, Aberration of star and Noble and Trouton experiment etc. But nobody was able to find a principle which can explain all these experiments. So in 1905 Einstein had given the following new idea "***The motion through ether is a meaningless concept, only the motion relative to material bodies has physical significance".***

Due To this reason we cannot find the velocity of earth in ether experimentally. On the basis of this concept, The special theory of relativity was brought by Einstein in 1905. Einstein modified the Newton's space-Time concept and put forward the new principle, known as special theory of relativity. In classical mechanics, equations of motion are applicable only on those particles whose velocity is less than velocity of light. While equations of motion in Newton's mechanics modified by Einstein is also applicable on those particles which travel with velocity of light. In classical Physics the space and time are same for all observers, but it varies for moving observers in relativity theory.

16.3 Postulates of Special theory of Relativity

Einstein introduced his special theory of relativity proposing drastic revision in Newtonian concepts of space and time. The special theory of relativity has made wide ranging change in our understanding of nature, but Einstein based it on just two simple postulates.

(a) **Principle of physical Equivalence :** ***According to this all laws of physics are the same in all inertial frame of reference.*** The consequence of this postulate is that all inertial frames are completely equivalent.

(b) **Constancy of speed of Light :** ***According to this the speed of light in vacuum (free space) is same in all inertial frames and is independent of the motion of the observer or its source.***

The universality of Laws of physics and the absolute character of the speed of light in free space immediately questions the absoluteness of length, mass and time. This could be understood using mathematics of Lorentz transformation.

16.4 Lorentz Transformation

We know that Galilean transformation is used to transform the coordinates of an event from one inertial frame to another inertial frame in classical principle.

But these transformations are not applicable in Michelson's Morley experiment with regard to velocity of light. So Galilean transformation was replaced by a new set of transformation to preserve the invariance of Maxwell's equations (Electrodynamics) under relative motion as well as invariance of Mechanical laws should also be established as earlier. Such type of transformation equations are known as Lorentz transformation. These transformation express the fundamental properties of space & time.

Derivation of Lorentz Transformation :

Suppose S and S' are two inertial frames. S' is moving along x-axis with uniform velocity v with respect to S. Initially at $t = t' = 0$ the origins of both frames coincide with each other. The coordinates of a given point P observed from S are (x, y, z, t) where as in S', these are x', y', z', t'

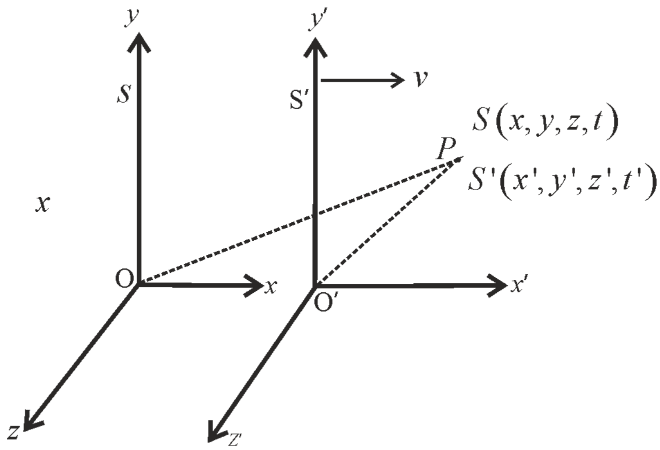


Figure16.2

Now the basic requirements for transformations between S and S' are

(i) Principle of Homogeneity of space and time requires that transformation should be linear.
(ii) Transformation should obey both the postulates of special theory of relativity.
(iii) All the coordinates perpendicular to the direction of motion of frame should remain constant.
(iv) Under non-relativistic limit $v << c$, these transformations must coincide with the Galilean transformation.

Let a flash of light is generated at time $t = t' = 0$ at the origin O which grows in form of spherical wave front in the space. If the time taken by this light flash to reach at point P is t and t' for the observer sitting at the origin O and O' in frames S and S' respectively. Let (x, y, z, t) and (x', y', z', t') are the position and time coordinate of the event (flash) in frame S and S' respectively. When the flash is observed from origin O of the frame S, then we have

$$\text{Velocity of light} = \frac{dis\tan ce}{time}$$

$$\therefore c = \frac{OP}{t} = \frac{\left(x^2 + y^2 + z^2\right)^{1/2}}{t} \Rightarrow x^2 + y^2 + z^2 - c^2 t = 0 \qquad (16.4)$$

When the same flash is observed from origin O' of frame S' then we have

$$c = \frac{OP'}{t'} = \frac{\left(x'^2 + y'^2 + z'^2\right)^{1/2}}{t'} \qquad (16.5)$$

Moreover $y = y', z = z'$ (16.6)

From equations (16.4) and (16.5) using equations (16.6) we have

$$x^2 - y^2 t^2 = x'^2 - c^2 t'^2 \qquad (16.7)$$

The transformation between x and x' can be represented by the simple relationship

$$\boxed{x' = \gamma(x - vt)} \qquad (16.8)$$

Where γ being independent of x and t.

The Law of equivalence tells us that the motion of S' w.r.t. S with velocity v is the same as the motion of S w.r.t. S' with velocity –v, then

$$x = \gamma(x' + vt') \qquad (16.9)$$

Putting the value of x' from equation (16.8) in equation (16.9) we have

$\mathrm{x} = \gamma\left[\gamma(x - vt) + vt'\right]$ solving this for t', we get

$$t' = \gamma\left[t - \frac{x}{v}\left(1 - \frac{1}{\gamma^2}\right)\right] \qquad (16.10)$$

Putting the value of x' from equation (16.8) and t' from equation (16.10) in equation (16.7) we get

$$x^2 - c^2t^2 = \gamma^2(x - vt)^2 - c^2\gamma^2\left[t - \frac{x}{v}\left(1 - \frac{1}{\gamma^2}\right)\right]^2 \qquad (16.11)$$

This is an identity and hence comparing the coefficient of t^2 and $2xt$ on both side of equation (16.11) we shall get

$$-c^2 = \gamma^2v^2 - c^2v^2$$

So $\gamma^2 = \dfrac{1}{1 - \dfrac{v^2}{c^2}}$ or $\boxed{\gamma = \dfrac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}}$ (16.12)

Now substituting the value of γ from equation (16.12) in equation (16.8) and (16.10), we have Lorentz transformation of space i.e.

$$\boxed{x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}} \qquad (16.13)$$

and $\boxed{t' = \dfrac{\left(t - \dfrac{v}{c^2}x\right)}{\sqrt{1 - \dfrac{v^2}{c^2}}}}$ (16.14)

If we assume that the system S is moving with velocity $-v$ relative to S' along the x-direction. Then the Lorentz transformation equations can be expressed as

$$\boxed{x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}, \; y = y', \; z = z' \; and \; t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1 - \frac{v^2}{c^2}}}} \qquad (16.15)$$

So equations (16.15) are known as inverse Lorentz transformation equations. It can be easily seen that if v <<c then $\dfrac{v}{c} \to 0$ then $x' = x - vt, y' = y, z' = z$ and $t' = t$

These are the Galilean transformation. Thus Lorentz transformation reduce to Galilean transformation if v <<c (Non-relativistic).

From equations (16.13) (16.14) and (16.15) it is seen that in the domain of the theory of relativity, space and time cannot be separated. ***In other words, space is***

four dimensional in which three space coordinate and fourth is time coordinate. The fourth coordinate time is imaginary and it is equal to ict, where $i=\sqrt{-1}$. This follows from the equation.

$$x^2+y^2+z^2-c^2t^2=x'^2+y'^2+z'^2-c^2t'^2=0 \tag{16.16}$$

16.5 The Quasi-Euclidean Geometry of the four dimensional World

We have already seen that the fundamental invariant of the homogeneous Lorentz transformation is the quantity

$$\boxed{S^2=x^2+y^2+z^2-c^2t^2=x'^2+y'^2+z'^2-c^2t'^2} \tag{16.17}$$

If we employ the new coordinates

$$\boxed{x_1=x, x_2=y, x_3=z \; and \; x_4=ict} \tag{16.18}$$

in S and the corresponding ones

$$x_1'=x', x_2'=y', x_3'=z' \text{ and } x_4'=ict' \tag{16.19}$$

In S' ,then (16.17) assumes the form

$$\boxed{S^2=x_1^2+x_2^2+x_3^2+x_4^2=x_1'^2+x_2'^2+x_3'^2+x_4'^2} \tag{16.20}$$

Written in this manner, the formal analogy with the three dimensional case becomes rather complete, except for the fact that the fourth square in the sum here is actually negative. Nevertheless, the form (16.20) enables one to apply rather freely the rules of the usual Euclidean geometry to that of the four-dimensional continuum of x, y, z and ict. However, because of the imaginary character of the fourth coordinate employed in this continuum, its geometry is not realty ,but only formally, identical with the Euclidean geometry: that is why it is usually referred to as quasi-Euclidean.

The immediate advantage of the transition from the form (16.17) to the form (16.20) is that now the rotation of the axes of one observer with respect to the other would not lead to an oblique coordinate system. On the other hand ,these rotations would now be concerned with only rectangular coordinate system. The Lorentz transformation would therefore, become linear orthogonal.

Confining ourselves to the case of homogeneous transformation i.e. those not involving any displacement of the origin, we can write

$$\boxed{x_i' = \sum_{i=1}^{4} a_{ik} x_k} \qquad \text{where } i = 1, 2, 3, 4$$

Where the a_{ik} are the coefficient of our transformation, geometrically the coefficient a_{ik} may be understood as the cosine of the angle between the x_i'-axis and the x_k-axis. In view of the very nature of the coordinate (16.18) and (16.19), the coefficients a_{pq} (p,q=1 ,2, 3, 4) and a_{44} must be real, while the coefficients a_{p4} and a_{4q} must be purely imaginary.

Our problem now consists in studying the question of the invariance of the quantity.

$$S^2 = x_i x_i = x_i' x_i' \tag{16.21}$$

Under the transformations $x_i' = a_{ik} x_k$, i = 1, 2 , 3, 4 (16.22)

Substituting (16.22) in (16.21) we get

$$x_i x_i = (a_{ik} x_k)(a_{il} x_l)$$
$$= (a_{ik} a_{il}) x_k x_l$$

When a comparison of coefficients on the two sides gives

$$a_{ik} a_{il} = \delta_{kl} \tag{16.23}$$

Here, δ_{kl} is the well known Kronecker delta symbol

$$\delta_{kl} = 1 \qquad \text{if } k = l$$
$$= 0 \qquad \text{if } k \neq l \tag{16.24}$$

Condition (16.23) are the so called orthogonality conditions which our coefficients of transformation must satisfy in order that S^2 be invariant, they also imply that the transformation under consideration are orthogonal, i.e. the ones among rectangular coordinates. These conditions are, in all ,ten in number, four for k = l and six for $k \neq l$,consequently, they leave for the sixteen coefficients a_{ik} six degree of freedom, as it must be for the case of homogeneous transformation. From (16.22) , (16.23) and (16.24) we obtain

$$a_{ik} x_i' = a_{ik} (a_{il} x_l)$$

$$= \left(a_{ik} a_{il} \right) x_l = \delta_{kl} x_l = x_k$$

$$k=1,2,3,4 \tag{16.25}$$

which are the transformations inverse to the former ones. The invariance of (16.21), when required under the transformation (16.25) leads to the orthogonality conditions.

$$a_{ik} a_{jk} = \delta_{ij} \tag{16.26}$$

These conditions, however, are not materially different from those embodied in (16.23)

From the transformation coefficients a_{ik} we can construct the determinant

$$a = \left| a_{ik} \right| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \tag{16.27}$$

For the evaluation of this determinant, let us consider its square :

$$a^2 = \left| a_{il} a_{ik} \right| \tag{16.28}$$

Of course if we write one of the two factor determinants with its rows and columns interchanged it would not make any quantitative difference to the result. Equation (16.28) would, however, become

$$a^2 = \left| a_{li} a_{ik} \right| or \left| \left(a_{il} a_{kl} \right) \right| \tag{16.28'}$$

Which on making use of the conditions (16.23) or (16.26) gives

$$a^2 = \left| \delta_{ik} \right| = 1 \tag{16.29}$$

Thus the value of the determinant, a is equal either to +1 or -1.

16.6 Illustrative Examples

Example 16.1 If (x, y, z, t) be the coordinates of an event in S-frame and (x', y', z', t') be the coordinates of the same event in S'-frame which moves relative to S-frame with a uniform velocity v along x-direction. ***Show that*** $\mathrm{ds}^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$ ***is invariant under Lorentz Transformation.***

Sol. Using the Lorentz transformation equations in differential form

$$dx = \frac{dx' + v dt'}{\sqrt{1 - v^2/c^2}}, \; dy = dy', dz = dz'$$

$$dt = \frac{dt' + \frac{v}{c^2} dx'}{\sqrt{1 - v^2/c^2}}$$ Where velocity v is constant

$$\therefore ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$$

$$= \left(\frac{dx' + v\,\mathrm{dt}'}{\sqrt{1 - v^2/c^2}} \right)^2 + dy'^2 + dz'^2 - c^2 \left(\frac{dt' + \frac{v}{c^2} dx'}{\sqrt{1 - v^2/c^2}} \right)^2$$

$$= \frac{1}{1 - v^2/c^2} \left\{ dx'^2 + v^2 dt'^2 + 2v dx' dt' - c^2 dt'^2 - \frac{v^2}{c^4} dx'^2 - 2v dx' dt' \right\} + dy'^2 + dz'^2$$

$$= \frac{dx'^2 \left(1 - v^2/c^2\right)}{1 - v^2/c^2} + dy'^2 + dz'^2 - c^2 dt'^2 \frac{\left(1 - v^2/c^2\right)}{1 - v^2/c^2}$$

$$= dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2 = ds'^2$$

Thus ds^2 ***is invariant under Lorentz transformation.***

16.7 Self Learning Exercise

Section A: Very Short Answer Type Questions

Q.1 What is Inertial frame?

Q.2 Write down Relativistic equation of motion?

Section B:Short Answer Type Questions

Q.3 State and explain the fundamental postulates of special theory of relativity.

Q.4 Prove that three dimensional volume element $dxdydz$ is not Lorentz invariant but four dimensional volume element $dxdydzdt$ is Lorentz Invariant.

16.8 Relativistic Equation of Motion Minkowski Force

We know that Newton's equation of motion is invariant with respect to the Galilean transformations but are not invariant under Lorentz transformations. Thus in order to make Newton's second law conform to Einstein's principle of relativity, we have to seek its generalization. However, these generalized equations must reduce to the following Newtonian equation in the limit $v << c$

$$\frac{d}{dt}(mv_i) = F_i \tag{16.30}$$

The four dimensional generalization of equation (16.30) is obviously.

$$\boxed{\vec{F}_\mu = \frac{d\,\vec{p}_\mu}{d\tau}} \qquad \text{Where } \mu = 1,2,3,4 \tag{16.31}$$

τ is the proper time, $\vec{P}_\mu$ is four momentum of particle and F_μ is a force four-vector known as Minkowski force.

$$\boxed{F_\mu = \gamma_u \frac{dP_\mu}{dt}} = \gamma_u \frac{dP_\mu}{ds}.\frac{ds}{dt} = \gamma_u \frac{ds}{dt}.\frac{dP_\mu}{ds} \tag{16.32}$$

The interval between two events in the four-dimensional space

$$\boxed{ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2}$$

$$= dt^2\left[c^2 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2\right]$$

$$= dt^2\left[c^2 - u_x^2 - u_y^2 - u_z^2\right]$$

$$= dt^2[\mathrm{c}^2 - \mathrm{u}^2]$$

$$\therefore \mathrm{ds}^2 = c^2dt^2\left[1 - \frac{u^2}{c^2}\right] \qquad \text{Where } \boxed{\gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}}$$

$$\therefore \left(\frac{ds}{dt}\right)^2 = \frac{c^2}{\gamma_u^2} \qquad \text{So } \frac{ds}{dt} = \frac{c}{\gamma_u}$$

$$\therefore \gamma_u \frac{ds}{dt} = c \tag{16.33}$$

From equations (16.32) and (16.33) we have

$$\vec{F}_\mu = c\frac{d}{ds}\vec{P}_\mu \tag{16.34}$$

We know that

$$\boxed{\vec{P}_\mu = m_0 \frac{dx_\mu}{d\tau} = m_0 \gamma_u \frac{dx_\mu}{dt}}$$

So $$\vec{P}_\mu = m_0 c \left(\frac{dx_\mu}{ds} \right) \quad (16.35)$$

From equations (16.34) and (16.35) we have

$$\vec{F}_\mu = c \frac{d}{ds} \left(m_0 c \frac{dx_\mu}{ds} \right) \quad (16.36)$$

Thus equation (16.36) is the fundamental relativistic equation of motion.

16.9 Applications of Energy-Momentum Conservation

It is proposed to treat the application of the energy momentum conservation in the following two problems:

(1) Kinematics of decay products of an unstable particle.

(2) Centre of momentum system and Threshold energy.

Let us take these applications one by one :

(1)Decay of an Unstable Particle

In particle physics, the study of decay process of unstable particles constitutes an intensive field of research. In order to illustrate the method, we consider the two body decay of an unstable particle at rest. These are exemplified by the decay of charged meson's $\pi^\pm, K^\pm$, hyperons Λ etc. A charged pi-meson decays into a μ meson and a neutrino.

$$\pi^\pm \to \mu^\pm + v \quad (16.37)$$

Experimentally it found that the rest energy of $\pi^\pm$ is 139.6 MeV and that of $\mu^\pm = 105.7 MeV$. ***The rest mass of neutrino is zero***.

Thus the energy balance for the decay is 33.9 MeV. This energy is to be shared between the products. Let us treat the ***kinematics of decay through the use of the energy-momentum conservation.***

Let the rest mass of the unstable particle be m and those of the products m_1 and m_2 . Defining the excess of mass Δm as

$$\Delta m = m - (m_1 + m_2) \quad (16.38)$$

From equation (16.38) it clear that such a spontaneous decay is possible only if the mass excess is positive. Since the decay takes place at rest, the decay particles must have equal and opposite momenta i.e. $\overrightarrow{p_1} = -\overrightarrow{p_2} = \overrightarrow{p}$

According to the law of conservation of energy we get

$$\boxed{\sqrt{c^2 p_1^2 + m_1^2 c^4} + \sqrt{c^2 p_2^2 + m_2^2 c^4} = mc^2} \tag{16.39}$$

We can utilize this result to find the magnitude of momentum $|p|$ and the energy of decay particles. For this we can make use of the invariance of the scalar product of two four-vectors.

The conservation of energy and momentum in two body decay can be expressed through a four-vector equation.

$$\vec{p}_\mu = \vec{p}_{1\mu} + \vec{p}_{2\mu} \tag{16.40}$$

Where $p_\mu, p_{1\mu}$ *and* $p_{2\mu}$ stand for the unstable particle, the decay particle no. 1 and decay particle no. 2 respectively. We get

$$p_{2\mu} = p_\mu - p_{1\mu} \tag{16.41}$$

and forming the Lorentz invariant of the 4-vectors on both sides

$$p_{2\mu}.p_{2\mu} = p_\mu.p_\mu + p_{1\mu}.p_{1\mu} - p_\mu.2p_{1\mu} \tag{16.42}$$

Putting the values of these terms which are invariants as

$$\begin{aligned} p_{2\mu}.p_{2\mu} &= m_2^2 c^2 \\ p_{1\mu}.p_{1\mu} &= m_1^2 c^2 \\ p_\mu.p_\mu &= m^2 c^2 \\ p_\mu.p_{1\mu} &= mE_1 \end{aligned} \tag{16.43}$$

We know that the term $p_\mu.p_{1\mu}$ is also Lorentz invariant and in the rest frame of *m*, its space part vanishes. E_1 is the total energy of particle of mass m_1 .From equation (16.42) we get

$m_2^2 c^2 = m^2 c^2 + m_1^2 c^2 - 2mE_1$ which gives

$$E_1 = \frac{m^2c^2 + m_1^2c^2 - m_2^2c^2}{2m} \quad \text{and similarly} \quad E_2 = \frac{m^2c^2 + m_2^2c^2 - m_1^2c^2}{2m} \tag{16.44}$$

Let us obtain the expressions for the kinetic energies T_1 and T_2 of the decay

Products . Now

$$T_1 = E_1 - m_1c^2$$

$$T_1 = \frac{m^2c^2 + m_1^2c^2 - m_2^2c^2}{2m} - m_1c^2$$

$$= \frac{m^2c^2 + m_1^2c^2 - m_2^2c^2 - 2mm_1c^2}{2m}$$

$$T_1 = \frac{m^2 + m_1^2 - m_2^2 - 2mm_1}{2m}c^2 = \frac{(m - m_1)^2 - m_2^2}{2m}c^2$$

$$= (m - m_1 - m_2)\frac{\left(m - m_1 + m_2\right)}{2m}c^2$$

$$= \Delta mc^2\left[\frac{2m - 2m_1 - m + m_1 + m_2}{2m}\right]$$

$$\therefore T_1 = \Delta mc^2\left[1 - \frac{m_1}{m} - \frac{\Delta m}{2m}\right] \qquad (16.45)$$

Similarly $\therefore T_2 = \Delta mc^2\left[1 - \frac{m_2}{m} - \frac{\Delta m}{2m}\right]$ (16.46)

Here Δm is the excess of mass, $\frac{\Delta m}{2m}$ is the relativistic correction. If $\frac{\Delta m}{2m}$ is not negligible as compared to unity, then the product particles must be treated relativistically

For example , We can take the case of $\pi^{\pm} \to \mu^{\pm} + \nu$ decay ,we have

$$m_{\pi^{\pm}} = 139.6MeV, m_{\mu^{\pm}} = 105.7MeV \text{ and } m_{\nu} = 0$$

Therefore from equation (16.45) the kinetic energy of the $\mu^{\pm}$ meson

$$T_{\mu} = 33.9\left[1 - \frac{105.7}{139.6} - \frac{33.9}{2(139.6)}\right] = 4.1MeV$$

It was the unique value of the μ meson kinetic energy (4.1 MeV) from π meson decay that led Powell and coworkers in 1947 to the discovery of π meson through the nuclear emulsion technique. The mass of the incoming particle m is determined with the help of equations (16.42) by putting the values of different terms from equation (16.43) and evaluating the scalar product $p_{1\mu}.p_{2\mu}$ in the lab Frame.

$$m^2 = m_1^2 + m_2^2 + \frac{2E_1E_2}{c^2} - 2p_1p_2\cos\theta \qquad (16.47)$$

However in a three or more body decay, the resulting decay products do not have unique momentum and are distributed in energy ,but these decays have some upper end points which pertain to the maximum values of energy. These maximum energies can, however be determined in the manner as illustrated for a two-body decay.

(2) Centre of momentum system and threshold energy

A common problem in nuclear or high energy physics is the study of scattering of a projectile from a Target. Incident particle called projectile of mass m_1, momentum p_1 and energy E_1 is made to impinge on particle 2 called target of mass m_2 at rest in the lab frame. The collision may give rise to elastic scattering when the incident particle is scattered at a certain angle and target recoils at some other angle. By applying the laws of conservation of momentum and energy this process can be analyzed to have complete information about the particles involved. However, the collision could also give rise to a reaction resulting in the production of two or more particles at least one of which is different from the incident particles. The study of such problems many times becomes much easier if we transform the energy and momentum of the interacting particles from Lab frame to the centre of mass frame. This system called the zero momentum system has the advantage that the projectile and target have equal and oppositely directed momenta. Alternatively we can employ the concept of the invariance of the scalar product of two four-vectors.

Let us consider the invariant scalar products of the four vectors in the Lab and the C.M. systems, we get

$$(p_{1\mu} + p_{2\mu}).(p_{1\mu} + p_{2\mu}) = (\mathrm{p}'_{1\mu} + \mathrm{p}'_{2\mu}).(\mathrm{p}'_{1\mu} + \mathrm{p}'_{2\mu}) \qquad (16.48)$$

The unprimed quantities refer to the lab system. Where the spatial momentum $\mathrm{p}_2 = 0$ and the primed quantities on the right hand, pertain to the CM system where the total spatial momenta $\mathrm{p}'_1 + p'_2 = 0$.

Putting the values of different terms, we get

$$\frac{1}{c^2}\left(E_1 + m_2c^2\right)^2 - p^2 = \frac{1}{c^2}\left(E'_1 + E'_2\right)^2$$

$$\text{or } \left(E_1 + m_2c^2\right)^2 - c^2p^2 = \left(E_1' + E_2'\right)^2 \tag{16.49}$$

Putting $E_1 = \left(m_1^2c^4 + c^2p^2\right)^{1/2}$, The total energy in the CM system is given by

$$E' = E_1' + E_2'$$

$$E' = \left(m_1^2c^2 + m_2^2c^2 + 2m_2E_1\right)^{1/2} c \tag{16.50}$$

The separate energies E_1' and E_2' can be determined from the scalar products like

$$p_{1\mu}(p_{1\mu} + p_{2\mu}) = \mathrm{p}_{1\mu}'(\mathrm{p}_{1\mu}' + \mathrm{p}_{2\mu}') \tag{16.51}$$

Putting the values for different Lorentz invariants, yields the result.

$$\mathrm{E}_1' = \frac{E'^2 + m_1^2c^4 - m_2^2c^4}{2E'} \tag{16.52}$$

$$\text{Similarly } \mathrm{E}_2' = \frac{E'^2 + m_2^2c^4 - m_1^2c^4}{2E'} \tag{16.53}$$

Lastly we apply the concept of the invariance of the scalar product of two four-vectors to the problem of the calculation of the threshold energy for the production of particles.

In a reaction, the initial particles of mass m_1 and m_2 are transformed into two or more particles with masses m_i, i =3,4… etc.

Defining Δm as the difference between the sum of the masses of the product and reactants.

$$\Delta m = (m_3 + m_4 + ...) - (m_1 + m_2) \tag{16.54}$$

If Δm is positive the reaction will not take place unless the projectile has certain minimum kinetic energy T_{th},called the threshold energy of reaction. At the threshold, the products are produced with zero kinetic energy. This implies that

$$E_{th}' = m_1c^2 + m_2c^2 + \Delta mc^2 \tag{16.55}$$

Substituting this value of E_{th}' in Eq (16.50), we get

$$(\mathrm{m}_1 + \mathrm{m}_2 + \Delta\,\mathrm{m})\,\mathrm{c}^2 = (\mathrm{m}_1^2\mathrm{c}^2 + \mathrm{m}_2^2\mathrm{c}^2 + 2\,\mathrm{m}_2\mathrm{E}_1)^{1/2} c \tag{16.56}$$

The incident kinetic energy of the projectile at threshold is

$$T_{th} = E_1 - m_1c^2 \tag{16.57}$$

Evaluating the value of E_1 from equation (16.56), we get

$$E_1 = \Delta m \left[1 + \frac{m_1}{m_2} + \frac{\Delta m}{2m_2}\right] c^2 + m_1 c^2 \tag{16.58}$$

therefore $$T_{th} = \Delta m \left[1 + \frac{m_1}{m_2} + \frac{\Delta m}{2m_2}\right] \tag{16.59}$$

Let us applying Equation (16.59) to the production of a proton -antiproton pair in proton-proton collisions

$$p + p \rightarrow p + p + p + \overline{p} \tag{16.60}$$

Where p and $\overline{p}$ stand for proton and antiproton respectively.

The mass difference $\Delta mc^2 = 2m_p c^2 = 1.8777\, BeV$

Hence from Eq. (16.59) we have

$$T_{th} = 1.877[1+1+1] = 5.631 \mathrm{Be}V$$

Lastly let us calculate the threshold energy for the production of a π^0 meson according to the reaction when a high energy photon strikes a proton at rest.

$$\gamma + p \rightarrow p + \pi^0 \tag{16.61}$$

The rest mass of the π^0 is 135 MeV. Hence

$$T_{th} = 135.0\left[1 + 0 + \frac{135.0}{2(938.5)}\right] = 144.7 MeV$$

Therefore the minimum energy that the photon must have in the Laboratory for producing π^0 by striking a proton at rest is 144.7 MeV.

16.10 Summary

Newton's equations of motion governing the dynamics of particles are invariant under Galilean transformation, whereas Maxwell's equations governing the electrodynamics are not. This implies that the velocity of light (in vacuum) is not the same in all inertial frames ,while Michelson- Morley experiment established that the velocity of light is not affected by the motion of the reference frame i.e. the velocity of light is the same in all inertial frames and has the value obtained from Maxwell's electromagnetic wave equations

$\boxed{v = \dfrac{1}{\sqrt{\mu_0 \in_0}} = 3\times10^8\, m/\sec = c}$. This led to doubt the correctness of Galilean transformation and hence Newton's law say motion. Einstein Sought to resolve the contradictions by enunciating two postulates of the special theory of relativity:

(i) All physical laws are same in all inertial reference frames.

(ii)The velocity of light in free space has the same value equal to c in all inertial frames.

The above postulates easily explain negative result of the Michelson-Morley experiment. Using his postulates, Einstein in 1905, rederived Lorentz transformation equations:

$$\boxed{x' = \gamma(x - vt)} \quad \text{Where } \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$y' = y$$

$$z' = z$$

$$\boxed{t' = \gamma\left(t - \frac{v^2}{c^2}x\right)}$$

In Lorentz Transformation space and time co-ordinates are intermixed. We note that the L.T. reduces to G.T. for $\frac{v}{c} << 1$. The special theory of relativity limits the maximum attainable velocity of a particle to the velocity of light (c) in empty space.

According to the principle of the special theory of relativity, if momentum is conserved in one inertial frame, then it must be conserved in all inertial frames. Energy will also conserved in all frames if momentum is conserved. Similarly it can be shown that momenta is conserved in all frames, if the energy is conserved. Thus conservation of energy and momentum go together in the relativity Theory. ***The laws of conservation of momentum and energy can now be coalesced to give a composite law of conservation of four momentum.*** The law of conservation of energy-momentum is used in the kinematics of decay products of an unstable particle and centre of momentum system and threshold energy for the production of particles in a reaction. The quasi-Euclidean geometry of the four dimensional world is discussed to explain ***Lorentz transformation as orthogonal transformation*** in 4 dimensions. The chapter includes the relativistic equation of motion using the concept of proper time $d\tau$ and four momentum four vector by modifying the classical equation of motion $F = \frac{dp}{dt}$ as $F_\mu = \frac{dp_\mu}{d\tau}$.

16.11 Glossary

Equivalence : The condition of being equal or equivalent in value ,worth ,function etc.

Invariant: A function, quantity, or property which remains unchanged when a specified transformation is applied.

16.12 Answers to Self Learning Exercise

Ans.1 : The frame in which Newton's Law of Inertia holds good.

Ans.2 : $\vec{F}_\mu = c\frac{d}{ds}\left(m_0 c\frac{dx_\mu}{ds}\right)$

16.13 Exercise

Section – A (Very Short Answer Type Questions)

Q.1 What is Relativistic Mechanics ?

Q.2 What do you mean by proper time?

Q.3 In what condition Lorentz Transformation reduces to Galilean Transformation?

Section – B (Short Answer Type Questions)

Q.4 Derive relativistic equation of motion.

Q.5 If a photon strikes a stationary electron giving rise to an electron position pair as well as a recoil electron, show that the threshold energy for the reaction in $4m_0c^2$ where m_0 is the rest mass of an electron.

Q.6 Calculate the threshold kinetic energy in MeV for the following process

$\gamma + p = p + \pi^o$

Rest mass of p and π^0 are 1836 and 264 electron masses respectively. (Ans. 145 MeV)

Section – C (Long Answer Type Question)

Q.7 What was the dead lock between theoretical conclusions and experimental results in classical electrodynamics and how did Einstein resolve it by revising our fundamental ideas of space and time.

Q.8 State the fundamental postulates of special theory of relativity and deduce the Lorentz transformation.

Q.9 State Lorentz transformation and show that the result of two successive Lorentz transformation is a Lorentz transformation form a group.

Q.10 Discuss the spontaneous decay of unstable particles with particular reference of charged pi-mesons.

16.14 Answers to Exercise

Ans.1: When $v \simeq c$ then this type of Mechanics in known as Relativistic.

Ans.2: It is the time measured by an observer which is at rest with respect to event.

Ans.3: When velocity of frame v is very-very less than c i.e. $\frac{v}{c} << 1$ then L.T. reduces to G.T.

References and Suggested Readings

1. Classical electrodynamics by J.D. Jackson (John Wiley &Sons)
2. Classical electricity and magnetism by Panofsky and Philips (Indian Book, New Delhi)
3. Introduction to Electrodynamics by Griffiths.
4. Element of Electromagnetics by Mathew N.O. and Sadiku (Oxford Univ. Press)
5. Classical theory of Electrodynamics by Landau-Lifshitz (Pergaman press, New York)
6. Electrodynamics of continuous media by Landau&Lifshitz (Pergaman Press, New York)
7. Electrodynamics by S.P. Puri.

UNIT -17
Four Vectors in Electrodynamics

Structure of the Unit

17.0 Objectives

The objectives of this unit are

- To study Four vectors in electrodynamics
- To study 4-current density and 4-potential
- To study covariant continuity equation and wave equation
- To study covariance of Maxwell's equations

17.1 Introduction

Through the Lorentz transformation equations for space and time coordinates we have learnt about the basic concepts of space-time continuum. The physical phenomena do not appear the same to observers in relative motion with respect to each other, although the physical laws must be the same for all observers. ***The equations of electrodynamics must be invariant i.e. retain their form on transformation from one inertial frame to another under Lorentz transformation***. However ,we will first show that equations of electrodynamics can be formulated in the four dimensional form as relations between four vectors and four tensors which posses the invariance properties under L.T(Lorentz transformation). These sets of four components will be introduced in the pseudo-Euclidean space which puts time on a different footing than the space coordinates. To this end let us develop the four vector formalism which is ideally suited for electrodynamics.

17.2 Minkowski Space and Space Time Continuum

The idea of four dimensional space was first of all suggested by Minkowski to which he called as space-time continuum. According to Minkowski, the external world is not formed of ordinary three dimensional space known as Euclidean space, but it is four dimensional space time continuum known as Minkowski space, where the time or more conveniently *ict* may be regarded to be fourth dimension. Thus an event in Minkowski space can be represented by four coordinates (x_1, x_2, x_3, x_4) out of which the first three are space co-ordinate. This four dimension al Minkowski space can more conveniently be represented (3+1) dimensional space time continuum. Let (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) are the coordinates of two events in four dimensional space, then the quantity.

$$\boxed{S_{12} = \sqrt{c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2}} \tag{17.1}$$

is called the ***interval between the two events*** . The interval between two infinitesimally close event is

$$\boxed{dS = \sqrt{c^2 dt^2 - dx^2 - dy^2 - dz^2}} \tag{17.2}$$

The interval between two events is Lorentz invariant in inertial frames. The invariance of an interval is a mathematical expression of the invariance of the velocity of light.

17.3 Four Vectors

Having introduced the idea of four dimensional space it is possible to extend ordinary vector analysis to four dimensions to derive generally valid laws in the form of equations between four dimensional vectors, these four dimensional vectors are called four vectors.

The coordinates of a point in a reference frame S at time t is given by (x, y, z). The coordinates (x, y, z) are the space components of ordinary vector $\vec{r}$ in three dimensional space. If (ict) is supposed to be the fourth coordinate, then the space expressed by (x, y, z, ict) is known as four dimensional space. Where (x, y, z) are position component and (ict) is time component. For these components we can use tensor notation in which we represent $x = x_1$, $y = x_2$, $z = x_3$ and $\boxed{ict = x_4}$. For the length of four vector x_μ

$$\boxed{S^2 = x^2 + y^2 + z^2 - c^2t^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = \sum_{\mu=1}^{4} x_\mu{}^2}$$

Similarly in frame S'

$$S'^2 = x'^2 + y'^2 + z'^2 - c'^2t'^2 = x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2 = \sum_{\mu=1}^{4} x'_\mu{}^2$$

Since $\boxed{S^2 = S'^2}$

Therefore

$$\sum_{\mu=1}^{4} \mathrm{x}_\mu^2 = \sum_{\mu=1}^{4} \mathrm{x}_\mu'^2 \tag{17.3}$$

From Lorentz Transformation the components of the four dimensional radius vector, transform according to

$x' = \gamma(x - vt)$

$x' = \gamma[\mathrm{x} + \mathrm{i}\,\beta(\mathrm{ict})]$

Where $\boxed{\beta = \frac{v}{c}}$ and $\boxed{\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}}}$

$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{17.4}$$

Or $ict' = \gamma(ict - i\beta x)$

Using tensor notation system use Lorentz transformation reduces to

$$\boxed{\begin{aligned} x_1' &= \gamma(x_1 + i\beta x_4) \\ x_2' &= x_2 \\ x_3' &= x_3 \\ x_4' &= \gamma(-i\beta x_1 + x_4) \end{aligned}} \tag{17.5}$$

Writing the following Lorentz equations in Matrix form

$$\boxed{\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} \tag{17.6}$$

Any set of four components of vector $A_\mu(A_1, A_2, A_3, A_4)$ ***which transform under Lorentz transformations like the four components*** $(x_1 x_2 x_3 x_4)$ ***i.e. Equation (17.6) is called a four vector***.

In four dimensional space the four vector A_μ should posses the following properties

(1) $\boxed{\sum_{\mu=1}^{4} A_\mu^2 = \text{Lorentz Invarient}}$

(2) It should follow the following transformation $\boxed{A_u' = a_{\mu\upsilon} A_\upsilon}$

(3) Three components of it are real and one component is imaginary.

(4) The scalar product of this four vector with another four vector is a Lorentz Invariant quantity.

So we can write the transformation equation of four vector $\vec{A}$ in accordance with equation (17.6) as follows :

$$\boxed{\begin{bmatrix} A'_1 \\ A'_2 \\ A'_3 \\ A'_4 \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}} \tag{17.7}$$

$$or \boxed{A'_\mu = \sum_{\mu=1,\upsilon=1}^{4} a_{\mu\upsilon} A_\upsilon} \tag{17.8}$$

Examples of Four Vectors :

(1) Position four vector $x_\mu(x_1, x_2, x_3, x_4)$

(2) Displacement four vector $dx_\mu(dx, dx_2, dx_3, dx_4)$

(3) Velocity four vector : for this first of all we define proper time. It is the time measured by an observers which is at rest with respect to the event and it is denoted by τ. ***In relativistic mechanics time t is not absolute but the proper time τ is invariant.*** The proper time of a particle is a measure of the length of the time track. If v is the velocity of a particle, the proper time interval $\Delta\tau$ is given by

$$\boxed{\Delta\tau = \Delta t\sqrt{1-\frac{v^2}{c^2}}} \tag{17.9}$$

The components of velocity four vector or four velocity are given by

$$\boxed{u_i = \frac{dx_i}{d\tau}} \tag{17.10}$$

where $dx_i \rightarrow$ displacement four vector.

$$\boxed{u_1 = \frac{dx_1}{d\tau} = \frac{dx_1}{dt}.\frac{dt}{d\tau} = \frac{u_x}{\sqrt{1-\frac{v^2}{c^2}}}}$$

$$u_2 = \frac{dx_2}{d\tau} = \frac{dx_2}{dt}.\frac{dt}{d\tau} = \frac{u_y}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$u_3 = \frac{dx_3}{d\tau} = \frac{dx_3}{dt}.\frac{dt}{d\tau} = \frac{u_z}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$u_4 = \frac{dx_4}{d\tau} = \frac{d(ict)}{d\tau} = ic\frac{dt}{d\tau} = \frac{ic}{\sqrt{1-\frac{v^2}{c^2}}}$$

Or in brief $u_i = \left(\frac{u}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{ic}{\sqrt{1-\frac{v^2}{c^2}}} \right)$

(4) Acceleration four vector: the acceleration four vectors is defined as

$$a_i = \frac{dv_i}{d\tau} = \frac{du_i}{dt}\left(1-\frac{v^2}{c^2}\right)^{-1/2} \tag{17.11}$$

(5) Energy – Momentum four vector **:**

The momentum four vector is obtained by multiplying the velocity four vector by the rest mass m_o so that

$$\boxed{p_1 = m_o u_1 = \frac{m_o u_x}{\sqrt{1-\frac{v^2}{c^2}}}}$$

$$p_2 = m_o u_2 = \frac{m_o u_y}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$p_3 = m_o u_3 = \frac{m_o u_z}{\sqrt{1-\frac{v^2}{c^2}}}$$

and $\boxed{p_4 = m_o u_4 = \frac{i m_o c}{\sqrt{1-\frac{v^2}{c^2}}} = imc = \frac{iE}{c}}$

In brief the momentum four vector is written as $\boxed{p_\mu = \left(p, \frac{iE}{c}\right)}$ (17.12)

17.4 Current Density Four Vector

In electrodynamics in the context of special theory of relativity that a charge distribution that is static in one frame, will appear to be a current distribution in another interval frame. It implies that the current and charge densities are not distinct entities and their relationship may be presented though the definition of the four current density four vector J_μ.

$$\boxed{J_\mu = (J_1, J_2, J_3, J_4) = (J, ic\rho)} \tag{17.13}$$

To justify this consider the charge contained in a small volume dV i.e.

$$dq = \rho dV \tag{17.14}$$

Multiplying both sides of the equation (17.14) by dx_μ, we get

$$dq dx_\mu = \rho dx_\mu dV = \rho \frac{dx_\mu}{dt}.dVdt \tag{17.15}$$

Now as dq is a scalar and dx_μ is displacement four vector, so L.H.S. of equation (17.15) is a four vector. So R.H.S. must also be a four vector. But as

$$dVdt = dx_1 dx_2 dx_3 dt = \frac{1}{ic}[dx_1 dx_2 dx_3 d(ict)]$$

$$= \frac{1}{ic} dx_1 dx_2 dx_3 dx_4 \tag{17.16}$$

So $dVdt$ ***is Lorentz invariant***. So $\rho \frac{dx_\mu}{dt}$ must be a four vector

Let $\boxed{j_\mu = \rho \frac{dx_\mu}{dt}}$ is 4-current density four vector

Then $\boxed{J_1 = \rho \frac{dx_1}{dt} = \rho u_1}$

$$J_2 = \rho \frac{dx_2}{dt} = \rho u_2$$

$$J_3 = \rho \frac{dx_3}{dt} = \rho u_3$$

$$\boxed{J_4 = \rho \frac{dx_4}{dt} = \rho \frac{d}{dt}(ict) = ic\rho}$$

i.e. the components of the 4-current density four vector J_μ are given by

$$\boxed{J_\mu = (J, ic\rho)} \tag{17.17}$$

As J_μ has been specified as four vector it must transform from one inertial frame S to the other inertial frame S' moving with velocity v relative to S along x-axis under Lorentz transformations as

$$\boxed{J'_\mu = a_{\mu\upsilon} J_\upsilon}$$

So that

$$J'_1 = a_{1\upsilon} J_\upsilon = a_{11} J_1 + a_{12} J_2 + a_{13} J_3 + a_{14} J_4$$

$$J'_1 = \gamma J_1 + 0J_2 + 0J_3 + i\beta\gamma J_4$$

$$= \gamma \left[J_1 + i\beta J_4 \right] = \frac{J_1 - v\rho}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{17.18 a}$$

$$J'_2 = a_{2\upsilon} J_\upsilon = a_{21} J_1 + a_{22} J_2 + a_{23} J_3 + a_{24} J_4$$

$$J'_2 = 0J_1 + 1J_2 + 0J_3 + 0J_4$$

$$\therefore J'_2 = J_2 \tag{17.18 b}$$

$$J'_3 = a_{3\upsilon} J_\upsilon = a_{31} J_1 + a_{32} J_2 + a_{33} J_3 + a_{34} J_4$$

$$J'_3 = 0J_1 + 0J_2 + 1J_3 + 0J_4$$

$$\therefore J'_3 = J_3 \tag{17.18 c}$$

$$J'_4 = a_{4\upsilon} J_\upsilon = a_{41} J_1 + a_{42} J_2 + a_{43} J_3 + a_{44} J_4$$

$$ic\rho' = -i\frac{v}{c}\gamma J_1 + 0J_2 + 0J_3 + \gamma ic\rho$$

$$\therefore \rho' = \frac{\rho - \frac{v}{c^2} J_1}{\sqrt{1 - v^2/c^2}} \tag{17.18 d}$$

Discussion of Results :

(1) Equation of continuity in covariant form. The continuity equation is

$$\boxed{\nabla . J + \frac{\partial \rho}{\partial t} = 0}$$

can be written as $\nabla . J + \dfrac{\partial (ic\rho)}{\partial (ict)} = 0$

i.e. $\boxed{\dfrac{\partial J_1}{\partial x_1} + \dfrac{\partial J_2}{\partial x_2} + \dfrac{\partial J_3}{\partial x_3} + \dfrac{\partial J_4}{\partial x_4} = 0}$

where $ic\rho = J_4$

and $ict = x_4$

$$\boxed{\frac{\partial J_\mu}{\partial x_\mu} = \Box . J_\mu = 0} \tag{17.19}$$

Where $\boxed{\Box = \dfrac{\partial}{\partial x_\mu}}$ is the ***four dimensional divergence operator.***

Equation (17.19) is covariant form of continuity equation. This is unaltered under Lorentz transformation. This equation also shows that four divergence of the current density four vector J_μ vanishes.

(2) Special case : Let us consider that charge distribution is at rest in frame S. The current density J in frame S is zero. i.e. $J=0$, $J_1 = J_2 = J_3 = 0$ Then transformation equations (17.18) take the form

$$J_1' = -\frac{v\rho}{\sqrt{1 - \frac{v^2}{c^2}}}, J_2' = 0, J_3' = 0 \; and \; \boxed{\rho' = \frac{\rho}{\sqrt{1 - \frac{v^2}{c^2}}}} \tag{17.20}$$

Invariance of charge : If $d\tau' = dx_1' \, dx_2' \, dx_3'$ is the volume element in frame S', then charge contained in the volume element in system S' is

$$dq' = \rho' dx_1' \, dx_2' \, dx_3' = \frac{\rho' dx_1 dx_2 dx_3}{\sqrt{1 - \frac{v^2}{c^2}}} = \rho dx_1 dx_2 dx_3 = dq$$

i.e. charge measured in frame S' is the same as that in frame S i.e. electric charge is invariant under Lorentz transformations, but $\rho' = \dfrac{\rho}{\sqrt{1-\dfrac{v^2}{c^2}}}$; the electric charge density is not relativistically invariant.

17.5 4-Potential Four Vector

As we have study that magnetic vector potential $\vec{A}$ and scalar potential ϕ are known as electromagnetic potentials because their variations with space and time are responsible for electromagnetism. The Lorentz condition relates the space variation of $\vec{A}$ (magnetic vector potential) with time variation of ϕ (scalar potential). This condition for free space is

$$\boxed{\nabla.\vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0} \tag{17.21}$$

We know that source of ϕ is ρ (static charges) and that of A is $\vec{J}$ (moving charges). Thus ρ and J are the two different forms of charge and expressible in terms of four current J_μ. The electric field in any inertial frame appears as magnetic field in another frame moving with constant velocity with respect to first frame. In this way in four dimensional system ϕ and $\vec{A}$ can be expressed as the components of a four vector A_μ. This four vector is known as four vector potential or electromagnetic four potential. We can then define this four vector potential (A_μ) as

$$\boxed{A_\mu = \left(A, \frac{i\phi}{c} \right)} \tag{17.22}$$

Note : The Lorentz condition $\nabla.A + \dfrac{1}{c^2}\dfrac{\partial \phi}{\partial t} = 0$

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} + \frac{\partial \frac{i\phi}{c}}{\partial (ict)} = 0$$

or $$\boxed{\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} + \frac{\partial A_4}{\partial x_4} = 0}$$

So $\boxed{\Box.A_\mu = 0}$ or $A_\mu = \left(A, \dfrac{i\phi}{c} \right)$

Since the Lorentz transformations govern the law of four vector transformation hence electromagnetic four potential must transform as

$$A'_u = a_{\mu\upsilon} A_\upsilon \qquad (17.23)$$

So that

$$A'_1 = a_{1\upsilon} A_\upsilon = a_{11}A_1 + a_{12}A_2 + a_{13}A_3 + a_{14}A_4$$

$$A'_1 = \gamma A_1 + 0A_2 + 0A_3 + iB\gamma A_4$$

$$A'_1 = \gamma\left(A_1 + i\frac{v}{c}(ic\rho)\right) = \gamma(A_1 - v\rho) \qquad (17.23\text{ a})$$

$$A'_2 = A_2 \; and \; A'_3 = A_3 \qquad (17.23\text{ b})$$

But

$$A'_4 = a_{4\upsilon} A_\upsilon = a_{41}A_1 + a_{42}A_2 + a_{43}A_3 + a_{44}A_4$$

$$\frac{i\phi'}{c} = -i\beta\gamma A_1 + 0A_2 + 0A_3 + \gamma\frac{i\phi}{c}$$

$$\therefore \phi' = \gamma(\phi - vA_1) \qquad (17.23\text{ c})$$

Lorentz condition in Covariant form will be $\frac{\partial A_\mu}{\partial x_\mu} = 0$ where $\mu = 1,2,3,4$

$$\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} + \frac{\partial A_4}{\partial x_4} = 0 \qquad \text{or} \qquad \Box . A_u = 0 \qquad (17.24)$$

Similarly

$$\boxed{\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial (ict)^2}\right]\left(\vec{A}, \frac{i\phi}{c}\right) = -\mu_0(\vec{J}, ic\rho)} \qquad (17.25)$$

Or $\boxed{\Box^2 A_u = -\mu_0 J_\mu}$ (17.26)

Where $\boxed{\Box^2 = \sum_{\mu=1}^{4} \frac{\partial^2}{\partial x_\mu^2}}$ = ***D'Alembert operator*** and equation (17.26) is known as D'Alembert equation and this is covariant form of Maxwell's equation. ***D'Alembert operator is Lorentz invariant, where*** $\Box = \frac{\partial}{\partial x_\mu}$ ***is four dimensional divergence operator***. Equation (17.24) is covariant equation and is known as Lorentz condition in covariant form. This equation expresses that the four divergence of the electromagnetic four potential vanishes.

17.6 Covariant Continuity Equation

The law of conservation of charge is mathematically expressed by the continuity equation

$$divJ + \frac{\partial \rho}{\partial t} = 0 \qquad (17.27)$$

Where J is current density and represents flow of charge per unit area per sec ,whereas ρ is volume charge density and represents charge per unit volume. So it is clear that ρ and $\vec{J}$ are merely two forms of charge, hence can be represented as components of current density four vector (J_μ). The equation (17.27) contains space and time derivatives ∇ and $\frac{\partial}{\partial t}$ respectively. Thus it can be conveniently transformed into covariant form.

Equation (17.2) can be written as

$$\nabla . J + \frac{\partial (ic\rho)}{\partial (ict)} = 0$$

Or $\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} + \frac{\partial (ic\rho)}{\partial (ict)} = 0$

If we use the following four dimensional system in which we have

$x_1 = x, x_2 = y, x_3 = z, x_4 = ict$ and $J_1 = \mathrm{J}_x, J_2 = \mathrm{J}_y, J_3 = \mathrm{J}_z, J_4 = ict$

$$\frac{\partial J_1}{\partial x_1} + \frac{\partial J_2}{\partial x_2} + \frac{\partial J_3}{\partial x_3} + \frac{\partial J_4}{\partial x_4} = 0 \qquad \text{or}$$

$$\sum_{\mu=1,2,3,4}^{4} \frac{\partial \vec{J_\mu}}{\partial x_\mu} = 0 \qquad (17.28)$$

$$\boxed{\square . J_\mu = 0}$$

Where $\frac{\partial}{\partial x_\mu}$ ***is four dimensional divergence operator and*** $\vec{J_\mu}$ ***is four current density vector.*** Equation (17.28) is covariant continuity equation i.e. its form in unaltered under Lorentz transformation. This equation expresses that the four divergence of the current density four vector J_μ vanishes.

17.7 Wave Equations

Consider two systems S and S'. Where S' is moving with velocity v relative to S along (+ve) direction of x-axis. If a wave is travelling in a space with velocity v in systems S, then the propagation wave Equation for such a wave is of the form

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi=0 \tag{17.29}$$

Where ψ is known as wave function and differential operator $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)$ is called D'Alembert's operator and it is denoted by $\Box^2$. Here ψ is function of x, y, z and t and thus it may be written as $\psi(x,y,z,t)$.

Now in frame S' which is moving relative to S, the propagation wave equation of same wave is given by

$$\left(\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right)\psi=0 \tag{17.30}$$

where c is not primed, because according to the second postulates of special theory of relativity c is always constant. Thus ψ may be written as

$$\psi(x',y',z',t')$$

If (x',y',z',t') and (x,y,z,t) are coordinates of any event in S and S' respectively then D'Alembertian operator in system S is

$$\Box^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$ and D'Alembertian operator in S' is

$$\Box'^2=\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}$$

According to transformation of differential operator we have

$$\frac{\partial}{\partial x'}=\frac{1}{\sqrt{1-\beta^2}}\left(\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}\right)$$

$$\frac{\partial}{\partial y'}=\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial z'}=\frac{\partial}{\partial z}$$

and

$$\frac{\partial}{\partial t'}=\frac{1}{\sqrt{1-\beta^2}}\left(v\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)$$

Which gives

$$\frac{\partial^2}{\partial x'^2} = \frac{1}{(1-\beta^2)}\left(\frac{\partial^2}{\partial x^2} + \frac{v^2}{c^4}\frac{\partial^2}{\partial t^2} + \frac{2v}{c^2}\frac{\partial^2}{\partial x \partial t}\right)$$

$$\frac{\partial^2}{\partial y'^2} = \frac{\partial^2}{\partial y^2} \quad \text{and} \quad \frac{\partial^2}{\partial z'^2} = \frac{\partial^2}{\partial z^2}$$

And $\frac{\partial^2}{\partial t'^2} = \frac{1}{(1-\beta^2)}\left(v^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2} + 2v\frac{\partial^2}{\partial x \partial t}\right)$

Therefore $\square'^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial z'^2}$

$$= \left(\frac{1}{1-\beta^2}\right)\left[\frac{\partial^2}{\partial x^2} + \frac{v^2}{c^4}\frac{\partial^2}{\partial t^2} + \frac{2v}{c^2}\frac{\partial^2}{\partial x \partial t}\right] + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$-\frac{1}{c^2}\frac{1}{(1-\beta^2)}\left[v^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2} + 2v\frac{\partial^2}{\partial x \partial t}\right]$$

$$\therefore \square'^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \square^2$$

Thus we may say that $\square^2$ ***is invariant under Lorentz transformation. But*** $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ ***i.e. Laplacian operator is not invariant under Lorentz transformation.***

17.8 Self Learning Exercise

Section A : Very Short Answer Type Questions

Q.1 What is Minkowski space?

Q.2 What is four vector?

Section B : Short Answer Type Questions

Q.3 Express Lorentz condition and equation of continuity in covariance form.

Q.4 What is a four vector? Explain with examples.

17.9 Covariance of Maxwell's equations (Four Tensor form)

In order to obtain covariance of Maxwell's equations we have to represent these equations in terms of four vectors. Maxwell's field equations in free space are

$$\left.\begin{array}{ll}\boxed{div\vec{E} = \frac{\rho}{\in_0}} & (a) \\ \boxed{div\vec{B} = 0} & (b) \\ \boxed{Curl\vec{E} = \frac{\partial \vec{B}}{\partial t}} & (c) \\ \boxed{Curl\vec{B} = \mu_0\left(\vec{J} + \in_0 \frac{\partial \vec{E}}{\partial t}\right)} & (d)\end{array}\right\} (17.31)$$

Writing these equations in component form by introducing the coordinates

$x = x_1, \mathrm{y} = x_2, \mathrm{z} = x_3$ and $ict = x_4$.

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\in_0} \quad 17.32(a)$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \quad 17.32(b)$$

$$curlE + ic\frac{\partial B}{x_4} = 0 \quad 17.32(c)$$

$$curlB - \frac{i}{c}\frac{\partial E}{\partial x_4} = u_0 J \quad 17.32(d)$$

$$\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3} = \frac{\rho}{\in_0} \quad 17.32(a')$$

$$\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} = 0 \quad 17.32(b')$$

Now considering the non homogeneous pair of equations i.e equation (17.32 a) and (17.32 d) in terms of components we get

$$0 + \frac{\partial B_z}{\partial x_2} - \frac{\partial B_y}{\partial x_3} - \frac{i}{c}\frac{\partial E_x}{\partial x_4} = \mu_0 J_1 \quad 17.33\ (a)$$

$$-\frac{\partial B_z}{\partial x_1} + 0 + \frac{\partial B_x}{\partial x_3} - \frac{i}{c}\frac{\partial E_y}{\partial x_4} = \mu_0 J_2 \quad 17.33\ (b)$$

$$\frac{\partial B_y}{\partial x_1} - \frac{\partial B_x}{\partial x_2} + 0 - \frac{i}{c}\frac{\partial E_z}{\partial x_4} = \mu_0 J_3 \quad 17.33\ (c)$$

and $\frac{\partial E_x}{\partial x_1}+\frac{\partial E_y}{\partial x_2}+\frac{\partial E_z}{\partial x_3}+0=\frac{\rho}{\in_0}$ This can be written as

$$\frac{i}{c}\frac{\partial E_x}{\partial x_1}+\frac{i}{c}\frac{\partial E_y}{\partial x_2}+\frac{i}{c}\frac{\partial E_z}{\partial x_3}+0=\mu_0 J_4 \tag{17.33 (d)}$$

Considering J_1, J_2, J_3 and J_4 in the R.H.S. of equations (17.33) as the components of current density four vector J_μ. Now we introduce the ***electromagnetic antisymmetric field*** tensor by

$$F_{\mu\upsilon}=\frac{\partial A_\upsilon}{\partial x_\mu}-\frac{\partial A_\mu}{\partial A_\upsilon} \tag{17.34}$$

where $\vec{A}_\mu$ *or* $\vec{A}_\upsilon$ is electromagnetic potential four vector.

$$F_{\mu\upsilon}=\begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix}=\begin{bmatrix} 0 & B_z & -B_y & -\frac{iE_x}{c} \\ -B_z & 0 & B_x & -\frac{iE_y}{c} \\ B_y & -B_x & 0 & -\frac{iE_z}{c} \\ \frac{iE_x}{c} & \frac{iE_y}{c} & \frac{iE_z}{c} & 0 \end{bmatrix} \tag{17.35}$$

Now equations (17.33 a, b, c,d) may be written in a more compact form by single equation :

$$\sum_{\upsilon=1}^{4}\frac{\partial}{\partial x_\upsilon}F_{\mu\upsilon}=\mu_0 J_\mu \tag{17.36}$$

For example if $\mu=2$ equation (17.36) takes the form.

$$\sum_{\upsilon=1}^{4}\frac{\partial}{\partial x_\upsilon}F_{2\upsilon}=\mu_0 J_2$$

Or $$\frac{\partial F_{21}}{\partial x_1}+\frac{\partial F_{22}}{\partial x_2}+\frac{\partial F_{23}}{\partial x_3}+\frac{\partial F_{24}}{\partial x_4}=\mu_0 J_2$$

Now putting the values of F_{21}, F_{22}, F_{23} *and* F_{24} and form (17.35)

We shall get

$$-\frac{\partial B_z}{\partial x_1}+0+\frac{\partial B_x}{\partial x_3}+\frac{\partial}{\partial x_4}\left(-\frac{iE_y}{c}\right)=\mu_0 J_2$$

$$-\left(\frac{\partial B_z}{\partial x_1}\right)+\frac{\partial B_x}{\partial x_3}-\frac{i}{c}\frac{\partial E_y}{\partial x_4}=\mu_0 J_2$$

This is same as equation (17.33b). Similarly for $\mu = 1,3\, and\, 4$ we get equations (17.33a), (17.33c) and (17.33d) respectively. Now writing homogeneous pair of Maxwell's equation i.e. equation (17.32b) and (17.32c) in terms of four dimensional components.

$$\frac{\partial B_x}{\partial x_1}+\frac{\partial B_y}{\partial x_2}+\frac{\partial B_z}{\partial x_3}=0 \qquad 17.37(a)$$

$$-\frac{\partial B_x}{\partial x_4}+\frac{\partial}{\partial x_2}\left(\frac{iE_z}{c}\right)-\frac{\partial}{\partial x_3}\left(\frac{iE_y}{c}\right)=0 \qquad 17.37(b)$$

$$-\frac{\partial B_y}{\partial x_4}-\frac{\partial}{\partial x_1}\left(\frac{iE_z}{c}\right)+\frac{\partial}{\partial x_3}\left(\frac{iE_x}{c}\right)=0 \qquad 17.37(c)$$

$$-\frac{\partial B_z}{\partial x_4}+\frac{\partial}{\partial x_1}\left(\frac{iE_y}{c}\right)-\frac{\partial}{\partial x_2}\left(\frac{iE_x}{c}\right)=0 \qquad 17.37(d)$$

Using electromagnetic field tensor $F_{\mu\upsilon}$ these equations (17.37) can be written as.

$$\frac{\partial F_{23}}{\partial x_1}+\frac{\partial F_{31}}{\partial x_2}+\frac{\partial F_{12}}{\partial x_3}+0=0 \qquad 17.38(a)$$

$$0+\frac{\partial F_{34}}{\partial x_2}+\frac{\partial F_{42}}{\partial x_3}+\frac{\partial F_{23}}{\partial x_4}=0 \qquad 17.38(b)$$

$$\frac{\partial F_{43}}{\partial x_1}+0+\frac{\partial F_{14}}{\partial x_3}+\frac{\partial F_{31}}{\partial x_4}=0 \qquad 17.38(c)$$

$$\frac{\partial F_{24}}{\partial x_1}+\frac{\partial F_{41}}{\partial x_2}+0+\frac{\partial F_{12}}{\partial x_4}=0 \qquad 17.38(d)$$

All the equations (17.38) can be written by a single equation in tensor form as

$$\frac{\partial F_{\lambda\mu}}{\partial x_\upsilon}+\frac{\partial F_{\mu\upsilon}}{\partial x_\lambda}+\frac{\partial F_{\upsilon\lambda}}{\partial x_\mu}=0 \qquad 17.39$$

For example if $\lambda, \mu\, and\, \upsilon$ and take the values as any combination of (1,2,3) we always get

$$\frac{\partial F_{12}}{\partial x_3}+\frac{\partial F_{23}}{\partial x_1}+\frac{\partial F_{31}}{\partial x_2}=0$$

$$\frac{\partial B_z}{\partial x_3}+\frac{\partial B_1}{\partial x_1}+\frac{\partial B_2}{\partial x_2}=0$$

This is same as equation (17.37a). Similarly we may get equations (17.37b) (17.37c) and (17.37d) from equation (17.39).

Hence equation (17.36) and(17.39) represents Maxwell's equations in terms $F_{\mu\upsilon}$ ***(electromagnetic field Tensor). As tensor equations are invariant under Lorentz transformation. So Maxwell's equations (17.36) and (17.39) are invariant under Lorentz Transformation***. So equation (17.36) , (17.39) and consequently equations (17.33) and (17.37) represent Maxwell's field equations in covariant form.

17.10 Illustrative Examples

Example. 1 Prove that the law of conservation of charge i.e. continuity equation is self contained in the inhomogeneous pair of Maxwell's field equations.

Sol. The inhomogeneous pair of Maxwell's field equation in terms of electromagnetic tensor $F_{\mu\upsilon}$ is given by $\frac{\partial F_{\mu\upsilon}}{\partial x_\upsilon}=\mu_0 J_\mu$ (1)

where J_μ is the current density four vector.

Now differentiating equation (1) w.r.t. x_μ we get

$$\frac{\partial^2 F_{\mu\upsilon}}{\partial x_\mu \partial x_\upsilon}=\mu_0\frac{\partial J_\mu}{\partial x_\mu} \tag{2}$$

Since $F_{\mu\upsilon}$ ***is antisymmetric*** i.e. $F_{\mu\upsilon}=-F_{\mu\upsilon}$ then equation (2) reduces to

$$-\frac{\partial^2 F_{\mu\upsilon}}{\partial x_\mu \partial x_\upsilon}=\mu_0\frac{\partial J_\mu}{\partial x_\mu} \tag{3}$$

Interchanging dummy indices $\mu\, and\, \upsilon$ in equation (3) we get

$$-\frac{\partial^2 F_{\mu\upsilon}}{\partial x_\upsilon \partial x_\mu}=\mu_0\frac{\partial J_\mu}{\partial x_\mu} \tag{4}$$

Using the property of perfect differentials i.e. $\left(\frac{\partial^2}{\partial x_\mu \partial x_\upsilon}=\frac{\partial^2}{\partial x_\upsilon \partial x_\mu}\right)$

Equation (4) can be written as

$$-\frac{\partial^2 F_{\mu\upsilon}}{\partial x_\mu \partial x_\upsilon} = \mu_0 \frac{\partial J_\mu}{\partial x_\mu} \tag{5}$$

Now adding equation (2) and (5) we get

$$2\mu_0 \frac{\partial J_\mu}{\partial x_\mu} = 0 \qquad \text{i.e.} \qquad \frac{\partial J_\mu}{\partial x_\mu} = 0$$

i.e. $$\frac{\partial J_1}{\partial x_1} + \frac{\partial J_2}{\partial x_2} + \frac{\partial J_3}{\partial x_3} + \frac{\partial J_u}{\partial x_u} = 0$$

or $$\frac{\partial J_1}{\partial x_1} + \frac{\partial J_2}{\partial x_2} + \frac{\partial J_3}{\partial x_3} + \frac{\partial (ic\rho)}{\partial (ict)} = 0$$

or $divJ + \frac{\partial \rho}{\partial t} = 0$ This is continuity equation

17.11 Summary

In this unit we have learnt about Minkowski space and space time continuum. We develop the four vector formalism which is ideally suited for electrodynamics e.g. four current density four vector and four potential four vector. Then using the law of conservation of charge we have derived covariant continuity equation. Then we derived covariance of Maxwell's equations in four dimensional form. We show that these equations of electrodynamics are invariant i.e. retain their form on transformation from one inertial frame to another under Lorentz transformation.

17.12 Glossary

Continuum **:** A continuous sequence in which adjacent elements are not perceptibly different from each other, but the extremes are quite distinct

Invariant : A function quantity, or property which remains unchanged when a specified transformation is applied

Antisymmetric : Unaltered in magnitude but changed in sign by exchange of two variables or by a particular symmetry operation.

D'Alembert : French physicist and mathematician Jean le Rond d'Alembert.

17.13 Answers to Self Learning Exercise

Ans.1: Four dimensional space time continuum is known as Minkowski space.

Ans.2: Any set of four components A_μ which transform under Lorentz transformation like the four components $\left(x_1, x_2, x_3, x_4\right)$ is called a four vector.

17.14 Exercise

Section – A (Very Short Answer Type Questions)

Q.1 Give two examples of four vectors in electrodynamics.

Q.2 Write covariant continuity equation.

Q.3 Write homogeneous pair of Maxwell's field equations.

Q.4 Write inhomogeneous pair of Maxwell's field equations.

Section – B (Short Answer Type Questions)

Q.5 Discuss Minkowski's four dimensional space-time continuum.

Q.6 Derive expression for electrmagnetic potential four vector and give its Lorentz transformation.

Q.7 Show that D'Alembertian operator $\square^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is invariant in mathematical form for Lorentz transformation.

Section – C (Long Answer Type Questions)

Q.8 What is a four vector? Obtain Lorentz transformation of the components current density and charge density .Hence show that these form a four vector.

Q.9 Explain how the Minkowski's four dimensional space time description of events and intervals is consistent with the postulates of special theory of relativity. State Lorentz transformation in a four vector dimensional space representation.

Q.10 Using continuity equation, define four vector of current density. Write the equation in terms of operator $\square$. Interrelate the components in two inertial frames and establish the invariance of charge.

Q.11 Define electromagnetic field tensor and derive Maxwell's field equations in covariance form.

17.15 Answers to Exercise

Ans.1: (1) J_μ i.e. current density four vector

(2) A_μ i.e. electromagnetic potential four vector

Ans.2 : $\square . J_\mu = 0$ or $\frac{\partial J_1}{\partial x_1} + \frac{\partial J_2}{\partial x_2} + \frac{\partial J_3}{\partial x_3} + \frac{\partial J_4}{\partial x_4} = 0$ where $J_\mu = ic\rho$ & $x_\mu = ict$

Ans.3: $\frac{\partial F_{\lambda\mu}}{\partial x_\upsilon} + \frac{\partial F_{\mu\upsilon}}{\partial x_\lambda} + \frac{\partial F_{\upsilon\lambda}}{\partial x_\mu} = 0$ where $\lambda, \mu \, and \, \upsilon$ and can take the values any combination of (1,2,3)

Ans.4: $\sum_{\upsilon=1}^{4} \frac{\partial}{\partial x_\upsilon} F_{\mu\upsilon} = \mu_0 J_\mu$ where J_μ is the current density four vector.

Reference Books and Suggested Readings

1. Classical electrodynamics by J.D. Jackson (John Wiley &Sons)
2. Classical electricity and magnetism by Panofsky and Philips (Indian Book, New Delhi)
3. Introduction to Electrodynamics by Griffiths.
4. Element of Electromagnetics by Mathew N.O. and Sadiku (Oxford Univ. Press)
5. Classical theory of Electrodynamics by Landau-Lifshitz (Pergaman press, New York)
6. Electrodynamics of continuous media by Landau&Lifshitz (Pergaman Press, New York)
7. Electrodynamics by S.P. Puri.

UNIT- 18
Electromagnetic Field Tensor

Structure of Unit

18.0 Objectives

The objectives of this unit are :

- To study electromagnetic field tensor.
- To study Lorentz transformation of Electric and magnetic fields.
- To study invariants of the electromagnetic fields.
- To represent Maxwell's Equations in Tensor form.

18.1 Introduction

We know that the Lorentz transformation was introduced by consideration of the propagation of an electromagnetic wave. Actually the homogeneous equation governing electromagnetic wave propagation is already in covariant form, since ***D'Alembertian operator*** $\Box^2 = \frac{\partial}{\partial x_\mu}.\frac{\partial}{\partial x_\mu}$ ***is invariant***. In general Maxwell's equation and their consequences lend themselves very simply to covariant description. This follows from the fact that no modifications at all are necessary in the laws of electrodynamics to make them agree with the requirements of relativity. The covariant formulation of space-time coordinates in the equations automatically puts the rest of equations into covariant form. Therefore now we introduced the electromagnetic field tensor which gives the correct description of the electromagnetic field, since it accounts for the intermingling of electric and magnetic fields.

18.2 The Electromagnetic Field Tensor

The electromagnetic field vectors $\vec{E}$ and $\vec{B}$ are written in terms of electromagnetic potentials $\vec{A}$ and ϕ as

$$\boxed{\vec{B} = \nabla \times \vec{A} = curl\vec{A}} \tag{18.1}$$

$$\boxed{\vec{E} = -\frac{\partial \vec{A}}{\partial t} - grad\phi = -\frac{\partial \vec{A}}{\partial t} - \nabla\phi} \tag{18.2}$$

Here $\vec{E}$ ***and*** $\vec{B}$ ***are not four vectors, but the six components*** $E_x, \mathrm{E_y}, \mathrm{E_z}(\mathrm{E_1}, \mathrm{E_2}, \mathrm{E_3})$ ***and*** $B_x, \mathrm{B_y}, \mathrm{B_z}(\mathrm{B_1}, \mathrm{B_2}, \mathrm{B_3})$ ***may be used to develop an antisymmetric tensor of rank two by relating it with four electromagnetic potentials*** A_μ**.** This Tensor is known as electromagnetic field Tensor $\mathrm{F}_{\mu\upsilon}$.

From equation (18.2) *x* component of *E* is written as

$$E_x = -\frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x} \tag{18.3}$$

Now $E_x = -\frac{\partial A_1}{\partial t} - \frac{\partial \phi}{\partial x_1}$

$$\frac{iE_x}{c} = -\frac{i}{c}\frac{\partial A_1}{\partial t} - \frac{i}{c}\frac{\partial \phi}{\partial x_1}$$

$$\frac{iE_x}{c} = \frac{\partial A_1}{\partial (ict)} - \frac{\partial}{\partial x_1}\frac{i\phi}{c} = \frac{\partial A_1}{\partial x_4} - \frac{\partial A_4}{\partial x_1}$$

$$\Rightarrow \frac{iE_x}{c} = \frac{\partial A_1}{\partial x_4} - \frac{\partial A_4}{\partial x_1} \tag{18.4}$$

Similarly $\frac{iE_y}{c} = \frac{\partial A_2}{\partial x_4} - \frac{\partial A_4}{\partial x_2}$ (18.5)

$$\frac{iE_z}{c} = \frac{\partial A_3}{\partial x_4} - \frac{\partial A_4}{\partial x_3} \tag{18.6}$$

Similarly from equation (18.1) the components of $\vec{B}$ can be expressed as in terms of the electromagnetic four potential as

$$\vec{B} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \tag{18.7}$$

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \tag{18.8}$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \tag{18.9}$$

From equations (18.4) to (18.9)it is clear that these equations can be expressed by a single equation as

$$\boxed{F_{\mu\upsilon} = \frac{\partial A_\upsilon}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\upsilon}} \tag{18.10}$$

With

$$\begin{aligned} \frac{iE_x}{c} &= F_{41} \qquad , B_z = F_{12} \\ \frac{iE_y}{c} &= F_{42} \qquad , B_x = F_{23} \\ \frac{iE_z}{c} &= F_{43} \qquad , B_y = F_{31} \end{aligned} \tag{18.11}$$

Also $F_{\mu\upsilon} = -F_{\upsilon\mu}$ and $F_{\mu\mu} = 0$

So $F_{\mu\upsilon}$ ***is anti symmetric Tensor of rank two*** where $\mu = 1,2,3,4$ and $\upsilon = 1,2,3,4$ and

$$F_{11} = F_{22} = F_{33} = F_{44} = 0 \text{ and}$$

$$F_{41} = -F_{14} = \frac{iE_x}{c},$$

$$F_{42} = -F_{24} = \frac{iE_y}{c},$$

$$F_{43} = -F_{34} = \frac{iE_z}{c},$$

$$F_{23} = -F_{32} = B_x,$$

$$F_{31} = -F_{13} = B_y \text{ and}$$

$$F_{12} = -F_{21} = B_z.$$

So

$$F_{\mu\upsilon} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} = \begin{bmatrix} 0 & B_z & -B_y & -\frac{iE_x}{c} \\ -B_z & 0 & B_x & -\frac{iE_y}{c} \\ B_y & -B_x & 0 & -\frac{iE_z}{c} \\ \frac{iE_x}{c} & \frac{iE_y}{c} & \frac{iE_z}{c} & 0 \end{bmatrix} \tag{18.12}$$

This tensor is called the electromagnetic anti symmetric tensor of rank two. This is the covariant tensor form of equations (18.1) and (18.2).

As an example let $\mu = 1$ and $\upsilon = 3$ or $\mu = 3$ $\upsilon = 1$ then equation (18.10) yields $F_{13} = \frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3} = -B_2$ from equation (18.8)

Similarly $F_{31} = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} = B_2$

In this way we can write field equations for any components of $\vec{B}$ and $\vec{E}$.

18.3 Lorentz Transformation of Electric Field and Magnetic Field

Since the fields $\vec{E}$ and $\vec{B}$ are the elements of a second rank tensor $F_{\mu\upsilon}$ and for the ***Maxwell's field equations to be invariant under Lorentz Transformation,*** then necessary condition for this is that the electromagnetic field tensor $F_{\mu\upsilon}$ must have the same form in all inertial frames. The values of $F_{\mu\upsilon}$ in frame S' can be expressed in terms of the values in another inertial frame S according to

$$F'_{\mu\upsilon} = \frac{\partial x'_\mu}{\partial x_\alpha}\frac{\partial x'_\upsilon}{\partial x_\beta}F_{\alpha\beta} = a_{\mu\alpha}a_{\upsilon\beta}F_{\alpha\beta} \qquad (18.13)$$

Equation (18.13) can be derived in this manner. We know that the transformation of x_μ and A_υ are written as

$$x'_\mu = a_{\mu\alpha}x_\alpha \qquad (18.14)$$

And $$A'_\upsilon = a_{\upsilon\beta}x_\beta \qquad (18.15)$$

The inverse transformation of x_μ is written as

$$x_\alpha = a_{\mu\alpha}x'_\mu \qquad (18.16)$$

i.e. $$\frac{\partial x_\alpha}{\partial x'_\mu} = a_{\mu\alpha} \qquad (18.17)$$

Therefore

$$F'_{\mu\upsilon} = \frac{\partial A'_\upsilon}{\partial x'_\mu} - \frac{\partial A'_\mu}{\partial x'_\upsilon}$$

$$a_{\upsilon\beta} = a_{\upsilon\beta}\frac{\partial A_\beta}{\partial x'_\mu} - a_{\mu\alpha}\frac{\partial A_\alpha}{\partial x'_\upsilon}$$

Using(18.15)

$$a_{\upsilon\beta} = a_{\upsilon\beta}\frac{\partial A_\beta}{\partial x_\alpha}\frac{\partial x_\alpha}{\partial x'_\mu} - a_{\mu\alpha}\frac{\partial A_\alpha}{\partial x_\beta}\frac{\partial x_\beta}{\partial x'_\upsilon}$$

Using(18.17)

$$= a_{\upsilon\beta}a_{\mu\alpha}\frac{\partial A_\beta}{\partial x_\alpha} - a_{\mu\alpha}a_{\upsilon\beta}\frac{\partial A_\alpha}{\partial x_\beta}$$

or $$F'_{\mu\upsilon} = a_{\mu\alpha} a_{\upsilon\beta} \left[\frac{\partial A_\beta}{\partial x_\alpha} - \frac{\partial A_\alpha}{\partial x_\beta} \right]$$

i.e. $$F'_{\mu\upsilon} = a_{\mu\alpha} a_{\upsilon\beta} F_{\alpha\beta} \qquad (18.18)$$

Where $a_{\mu\upsilon} = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix}$ and

$$F_{\mu\upsilon} = \begin{bmatrix} 0 & B_z & -B_y & -\frac{iE_x}{c} \\ -B_z & 0 & B_x & -\frac{iE_y}{c} \\ B_y & -B_x & 0 & -\frac{iE_z}{c} \\ \frac{iE_x}{c} & \frac{iE_y}{c} & \frac{iE_z}{c} & 0 \end{bmatrix}$$

So

Transformation equations for magnetic field components :-

(1) X Component of Magnetic Field

$B_x = F_{23}$

So in equation (18.18) taking $\mu = 2$ and $\upsilon = 3$ we get

$$\begin{aligned} F'_{23} &= a_{2\alpha} a_{3\beta} F_{\alpha\beta} \\ &= a_{2\alpha}\left[a_{31}F_{\alpha 1} + a_{32}F_{\alpha 2} + a_{33}F_{\alpha 3} + a_{34}F_{\alpha 4}\right] \\ &= a_{21}\left[a_{31}F_{11} + a_{32}F_{12} + a_{33}F_{13} + a_{34}F_{14}\right] \\ &+ a_{22}\left[a_{31}F_{21} + a_{32}F_{22} + a_{33}F_{23} + a_{34}F_{24}\right] \\ &+ a_{23}\left[a_{31}F_{31} + a_{32}F_{32} + a_{33}F_{33} + a_{34}F_{34}\right] \\ &+ a_{24}\left[a_{31}F_{41} + a_{32}F_{42} + a_{33}F_{43} + a_{34}F_{44}\right] \end{aligned}$$

$F'_{23} = a_{22} a_{33} F_{23}$ Since $a_{22} = a_{33} = 1$ So $F'_{23} = F_{23}$

And all other coefficients are zero.

Or $B'_x = B_x$ (18.19)

(2) y Component of Magnetic Field

$B_y = F_{31}$

So in equation (18.18) taking $\mu = 3$ and $\upsilon = 1$ we shall get

$F'_{31} = a_{3\alpha} a_{1\beta} F_{\alpha\beta}$

The only surviving coefficient are those for which $\alpha = 3$ and $\beta = 1$ and $\beta = 4$

$\therefore F'_{31} = a_{33} a_{11} F_{31} + a_{33} a_{14} F_{34}$

Putting the values of different coefficients

$$F'_{31} = 1\gamma\beta_2 + 1i\gamma\beta\left(\frac{-iE_z}{c}\right)$$

$$\therefore B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right) \qquad (18.20)$$

(3) z Component of Magnetic Field

$B_z = F_{12}$

So in equation (18.18) taking $\mu = 1$ and $\upsilon = 2$ we shall get

$F'_{12} = a_{1\alpha} a_{2\beta} F_{\alpha\beta}$

The only surviving coefficients are those for which $\alpha = 1$, $\alpha = 4$ and $\beta = 2$.

$F'_{12} = a_{11} a_{22} F_{12} + a_{14} a_{22} F_{42}$

Putting the value of different coefficients

$$F'_{12} = \gamma 1 B_3 + (\mathrm{i}\gamma\beta).\frac{iE_z}{c}$$

$$\text{Or } B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right) \qquad (18.21)$$

Transformation equations for Electric field components :

(1) X Component of Electric Field

$\because \frac{iE_x}{c} = F_{41}$ So in equation (18.18) taking $\mu = 4$ and $\upsilon = 1$ we shall get

$F'_{41} = a_{4\alpha} a_{1\beta} F_{\alpha\beta}$ the only surviving coefficients are those for which $\alpha = 4$, $\beta = 1$ and $\alpha = 1$, $\beta = 4$

$$F'_{41} = a_{44} a_{11} F_{41} + a_{41} a_{14} F_{14}$$

$$\because \frac{iE'_x}{c} = \gamma\gamma\frac{iE_x}{c} + (-i\beta\gamma)(i\beta\gamma)\left[-i\frac{E_x}{c}\right]$$

$$E'_x = \gamma^2 (E_x)(1-\beta^2) \qquad \because 1-\beta^2 = \frac{1}{\gamma^2}$$

$$\therefore E'_x = \gamma^2 (E_x)\frac{1}{\gamma^2} \qquad \therefore E'_x = E_x \tag{18.22}$$

(2) Y Component of Electric Field

$\because \frac{iE_y}{c} = F_{42}$ So in equation (18.18) taking $\mu = 4$, $\upsilon = 2$ we shall get

$F'_{42} = a_{4\alpha}a_{2\beta}F_{\alpha\beta}$ the only serving coefficients are those for which

$\alpha = 1, \beta = 2 \quad and\ \alpha = 4, \beta = 2$

$$F'_{42} = a_{41}a_{22}F_{12} + a_{44}a_{22}F_{41}$$

$$\frac{iE'_y}{c} = (-iB\gamma)(1)B_z + \gamma(1)\frac{iE_y}{c}$$

$$\because E'_y = \gamma\left(E_y - vB_z\right) \tag{18.23}$$

(3) Z Component of Electric Field

$\because \frac{iE_z}{c} = F_{43}$, so in equation (18.18) taking $\mu = 4$ and $\upsilon = 3$ we shall get

$F'_{43} = a_{4\alpha}a_{3\beta}F_{\alpha\beta}$ the only surviving coefficients are those for which

$\alpha = 1, \beta = 3\ and\ \alpha = 4, \beta = 3$

$$F'_{43} = a_{41}a_{33}F_{13} + a_{44}a_{33}F_{43}$$

$$\frac{iE'_z}{c} = (-iB\gamma)(1)(-B_y) + \gamma(1)\frac{iE_z}{c}$$

$$\because E'_z = \gamma\left(E_z + vB_y\right) \tag{18.24}$$

Equations (18.19)(18.20), (18.21), (18.22) , (18.23) and (18.24) represents required transformation (Lorentz) equations for magnetic and electric fields $\vec{B}$ and $\vec{E}$. These equations can be inverted to give inverse Lorentz transformation of magnetic and electric fields $\vec{B}$ and $\vec{E}$ i.e.

$$\boxed{\begin{array}{ll} B_x = B'_x & ,E_x = E'_x \\ B_y = \gamma\left(B'_y - \dfrac{v}{c^2}E'_z\right) & ,E_y = \gamma\left(E'_y + vB'_z\right) \\ B'_z = \gamma\left(B_z + \dfrac{v}{c^2}E'_y\right) & ,E_z = \gamma\left(E'_z - vB'_y\right) \end{array}} \tag{18.25}$$

18.4 The Invariants of the Electromagnetic Fields

As we know these are two invariants of the electromagnetic field which are

(i) $\vec{E}.\vec{B}$

(ii) $c^2B^2 - E^2$

(1) ***Invariance of*** $\vec{E}.\vec{B}$ ***:***

According to transformations of magnetic and electric field components.

$$B'_x = B_x$$

$$B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$$

$$B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

$$E'_x = E_x$$

$$E'_y = \gamma\left(E_y - vB_z\right)$$

$$E'_z = \gamma\left(E_z + vB_y\right)$$

Therefore $\vec{E}'.\vec{B}' = (\hat{i}E'_x + \hat{j}E'_y + \hat{k}E'_z).(\hat{i}B'_x + \hat{j}B'_y + \hat{k}B'_z)$

$$= E'_x\,\mathrm{B}'_x + E'_y\,\mathrm{B}'_y + E'_z\,\mathrm{B}'_z$$

$$= E_x.\mathrm{B}_x + \left\{\gamma\left(E_y - v\,B_z\right)\gamma\left(B_y + \frac{v}{c^2}E_z\right)\right\} + \left\{\gamma\left(E_z + v\,B_y\right)\gamma\left(B_z - \frac{v}{c^2}E_y\right)\right\}$$

$$E'.B' = E_xB_x + \left\{\frac{E_y\,B_y + \frac{v}{c^2}E_yE_z - vB_zB_y - \frac{v^2}{c^2}B_zE_z}{1-\beta^2}\right\}$$

$$+\left\{\frac{E_z\,B_z - \frac{v}{c^2}E_yE_z + vB_yB_z - \frac{v^2}{c^2}B_yE_y}{1-\beta^2}\right\}$$

$$= E_xB_x + \frac{E_y\,B_y + E_z\,B_z}{\left(1-\beta^2\right)}\left[1 + \frac{v^2}{c^2}\right]$$

$$= E_xB_x + E_y\,B_y + E_z\,B_z = \vec{E}.\vec{B} \qquad (18.26)$$

i.e. $\vec{E}.\vec{B}$ ***is invariant under Lorentz transformation***. The importance of this result lies in the facts that if $\vec{E}.\vec{B}=0$ (as in the case of a plane electromagnetic wave) in one frame, it will be zero in all inertial frames i.e. if vectors $\vec{E}\,and\,\vec{B}$ and are mutually perpendicular in any frame S then they are mutually perpendicular in another inertial frame S' .

(2) Invariance of ($c^2B^2-E^2$) :

Therefore $c^2B'^2-E'^2=c^2\left(B_x'^2+B_y'^2+B_z'^2\right)-\left(E_x'^2+E_y'^2+E_z'^2\right)$

$$=c^2\left\{B_x^2+\gamma^2\left(B_y+\frac{v}{c^2}E_z\right)^2+\gamma^2\left(B_z-\frac{v}{c^2}E_y\right)^2\right\}$$

$$-\left[E_x^2+\gamma^2\left(E_y-vB_z\right)^2+\gamma^2\left(E_z+vB_y\right)^2\right]$$

$$=c^2B_x^2+\gamma^2c^2\left\{\left(B_y^2+\frac{v^2}{c^4}E_z^2+2\frac{v}{c^2}B_yE_z\right)+B_z^2+\frac{v^2}{c^4}E_y^2-\frac{2v}{c^2}E_yB_z\right\}$$

$$\left\{-E_x^2+\gamma^2\left(E_y^2+v^2B_z^2-2vE_yB_z+E_z^2+v^2B_y^2+2vE_zB_y\right)\right\}$$

$$=c^2B_x^2+\gamma^2\left(c^2B_y^2+c^2B_z^2-E_y^2-E_z^2\right)-\frac{v^2\gamma}{c^2}\left(-E_z^2-E_y^2+c^2B_z^2+c^2B_y^2\right)-E_x^2$$

$$=c^2B_x^2+\gamma^2\left(c^2B_y^2+c^2B_z^2-E_y^2-E_z^2\right)\left(1-\frac{v^2}{c^2}\right)-E_x^2$$

$$=c^2B_x^2+c^2B_y^2+c^2B_z^2-E_y^2-E_z^2-E_x^2$$

$$=c^2(B_x^2+B_y^2+B_z^2)-(E_x^2+E_y^2+E_z^2)$$

$$=c^2B^2-E^2 \qquad (18.27)$$

i.e. ***the quantity*** $\left(c^2B^2-E^2\right)$ ***is invariant under Lorentz transformation***. The significance of this result lies in the fact that if the magnitude of $\vec{E}$ and $\vec{B}$ vectors in any reference system are given by $E=cB$ i.e. $H=\frac{E}{\mu_0 c}=\epsilon_0 cE$ (as in the case of a plane electromagnetic wave in free space) then they are related to each other by the same relation in any other system.

So the orthogonality of $\vec{E}$ and $\vec{B}$ i.e. $\vec{E}.\vec{B}=0$ and the relation $\mathrm{E}=c\mathrm{B}$ i.e. $H=\frac{1}{\mu_0 c}E=\epsilon_0 cE$ are the invariant properties of a plane wave.

Nevertheless the frequency and direction of the waves will vary with frame of reference and this leads to the phenomenon of the Doppler effect and aberration.

18.5 Maxwell's Equations in Tensor Form

Maxwell's equation are :

$$\left.\begin{aligned}
&\nabla.\vec{D} = \rho && (a)\\
&\nabla.\vec{B} = 0 && (b)\\
&\nabla\times\vec{H} - \frac{\partial\vec{D}}{\partial t} = \vec{J} && (c)\\
&\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 && (d)
\end{aligned}\right\} \qquad (18.28)$$

If for simplicity we take $\mu_r = 1$ and $\in_r = 1$ i.e. in free space Maxwell's equations reduces to

$$\left.\begin{aligned}
&\nabla.\vec{E} = \frac{\rho}{\in} && (a)\\
&\nabla.\vec{B} = 0 && (b)\\
&\nabla\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J} && (c)\\
&\nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 && (d)
\end{aligned}\right\} \qquad (18.29)$$

Now non-homogeneous pair of equations i.e. equations (18.29 (a)) and (18.29(c)) may be written more compactly in a single equation in terms of electromagnetic field tensor $F_{\mu\upsilon}$ as follows :

$$\sum_{\upsilon=1}^{4}\frac{\partial F_{\mu\upsilon}}{\partial x_\upsilon} = \mu_0 J_\mu \qquad (18.30)$$

Now homogeneous pair of equations i.e. equation (18.29(b)) and (18.29(d)) may be written more compactly in a single equation in terms of electromagnetic field tensor $F_{\mu\upsilon}$ as follows :

$$\frac{\partial F_{\mu\upsilon}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\upsilon} + \frac{\partial F_{\upsilon\lambda}}{\partial x_\mu} = 0 \qquad (18.30)$$

Where $\lambda, \mu \,\&\, \upsilon$ can take the values of any combination of (1,2,3). These equations we have already derived in unit (17). Thus equations (18.29) and (18.30) represent Maxwell's equations in Tensor form and these equations are the covariant form of Maxwell's field equations.

18.6 Illustrative Examples

Example.1 Show that the self product of electromagnetic field tensor is given by

$$F_{\mu\upsilon}^2 = 2\left(B^2 - \frac{E^2}{c^2}\right)$$

Where B, E and c are magnetic field, electric field and velocity of light.

Sol. The self product of electromagnetic field tensor $F_{\mu\upsilon}$ is given by

$$F_{\mu\upsilon}.F_{\mu\upsilon} = F_{\mu\upsilon}^2 = F_{1\upsilon}^2 + F_{2\upsilon}^2 + F_{3\upsilon}^2 + F_{4\upsilon}^2$$

$$= F_{11}^2 + F_{12}^2 + F_{13}^2 + F_{14}^2 + F_{21}^2 + F_{22}^2 + F_{23}^2 + F_{24}^2 + F_{31}^2 + F_{32}^2 + F_{33}^2 + F_{34}^2$$
$$+ F_{41}^2 + F_{42}^2 + F_{43}^2 + F_{44}^2$$

Since electromagnetic field tensor $F_{\mu\upsilon}$ is an anti symmetric tensor, hence

$$F_{11} = F_{22} = F_{33} = F_{44} = 0$$

$$F_{\mu\upsilon} = -F_{\upsilon\mu}$$

For $\mu \neq \upsilon$ or $F_{\mu\upsilon}^2 = F_{\upsilon\mu}^2$

$$\therefore F_{\mu\upsilon}^2 = 2[F_{12}^2 + F_{13}^2 + F_{14}^2 + F_{23}^2 + F_{24}^2 + F_{34}^2]$$

But $F_{12} = B_3, F_{13} = -B_2, F_{14} = -\frac{iE_1}{c}$,

$$F_{23} = B_1, F_{24} = -\frac{iE_2}{c}, F_{34} = -\frac{iE_3}{c}$$

or $F_{\mu\upsilon}^2 = 2\left[B_3^2 + B_2^2 - \frac{E_1^2}{c^2} + B_1^2 - \frac{E_2^2}{c^2} - \frac{E_3^2}{c^2}\right]$

$$= 2\left[\left(B_1^2 + B_2^2 + B_3^2\right) - \frac{1}{c^2}\left(E_1^2 + E_2^2 + E_3^2\right)\right]$$

$$= 2\left[B^2 - \frac{E^2}{c^2}\right]$$ Hence proved

Example 2 Show that a purely electric field in one frame appears both as an electric and magnetic field to an observer moving with respect to first.

Sol. Suppose in frame S, $E \neq 0$ but $B = 0$. Then in the S' frame, we have from transformation equations

$$E'_{\parallel} = E_{\parallel}$$

and $$E'_{\perp} = \gamma E_{\perp}$$

$$B'_{\parallel} = 0$$

But $\vec{B}'_{\perp} = -\gamma \vec{B} \times \vec{E}_{\perp}$

Thus $\vec{B}' = \vec{B}'_{\perp} + \vec{B}'_{\parallel} = -\gamma\vec{\beta} \times \vec{E}_{\perp} = -\gamma\vec{\beta} \times \vec{E}$

$$\because \ \vec{E} = \vec{E}_{\parallel} + \vec{E}_{\perp}$$

Since $\vec{v} \times \vec{E}_{\parallel} = 0$

So electric field in frame S, appears as electromagnetic field in frame S'

18.7 Self Learning Exercise

Section A : Very Short Answer Type Questions

Q.1 What is the electromagnetic field tensor?

Q.2 Write down the transformation formula for $F_{\mu\upsilon}$.

Section B : Short type Answer Type Questions

Q.3 Show that $\vec{E}.\vec{B}$ is Lorentz invariant ?

Q.4 Give physical significance of $\vec{E}.\vec{B}$ and $c^2B^2 - E^2$.

Example 3 Prove that a field that is purely magnetic in one frame cannot he transformed into one that is purely electric in a different reference frame.

Sol. Suppose in frame S, $E = 0$, but $B \neq 0$, then in frame S',

We have $E'_{\parallel} = 0$

$$\vec{E}'_{\perp} = \gamma\vec{\beta} \times \vec{B}_{\perp} \text{ and } B'_{\parallel} = B_{\parallel}$$

$$B'_{\perp} = \gamma B_{\perp}$$

$$\because B = B_{\parallel} + B_{\perp}$$

So that $\vec{E}' = \vec{E}'_{\perp} = \gamma\vec{\beta} \times \vec{B}_{\perp} = \gamma\vec{\beta} \times \vec{B}$

Thus a purely magnetic field to an observer in one frame appears both as an electric and a magnetic field to a relatively moving observer.

Example 4 Show that the four-tensor $F_{\mu\upsilon}$ for the electromagnetic field must be totally anti symmetric.

Sol. In the instantaneous rest frame of a particle of charge q, the force acting on it must be $q\vec{E}$. Since $\vec{E} = -\nabla\phi$, we can rewrite it as :

$$\left.\begin{aligned} -E_x &= \frac{\partial A_4}{\partial x_1} \\ -E_y &= \frac{\partial A_4}{\partial x_2} \\ -E_z &= \frac{\partial A_4}{\partial x_3} \end{aligned}\right\} \tag{18.31}$$

Equating this force to the time rate of change of the momentum P of the particle in this frame of reference

$$\frac{dP_k}{dt} = qF_{k4} \quad (k = 1,2,3) \tag{18.32}$$

Generalizing this result for the suffixes $(\mu = 1,2,3,4)$, we get

$$\frac{dP_\mu}{d\tau} = qF_{\mu 4} \tag{18.33}$$

Now $P_\mu = (0,0,0,i\,m_0c)$, the above equation (18.33) is rewritten as

$$\frac{dP_\mu}{d\tau} = \frac{q}{m_0c}\sum_{\upsilon=1}^{4} P_\upsilon F_{\mu\upsilon} \tag{18.34}$$

Multiplying both sides of the equation (18.34) by $2P_\mu$, we get

$$\sum_{\mu=1}^{4} 2P_\mu \frac{dP_\mu}{d\tau} = \frac{2q}{m_0c}\sum_{\upsilon=1}^{4}\ \sum_{\mu=1}^{4} P_\upsilon F_{\mu\upsilon} P_\mu \tag{18.35}$$

The L.H.S. of Eq. (18.35) is $\sum_{\mu=1}^{4} 2P_\mu \frac{dP_\mu}{d\tau} = \frac{d}{d\tau}\sum_{\mu=1}^{4} P_\mu^2 = \frac{d}{d\tau}\left(-m_0^2c^2\right) = 0$ giving

$$\sum_{\upsilon=1}^{4}\sum_{\mu=1}^{4} P_\upsilon F_{\mu\upsilon} P_\mu = 0 \tag{18.36}$$

Since the L.H.S. of this equation (18.35) is a scalar. Thus the equation holds in any reference frame which can related to our rest frame through a Lorentz transformation. However, it is possible only if $F_{\mu\upsilon}$ is antisymmetric. Hence Proved.

Example 5 Starting from the four dimensional form of homogeneous Maxwell's equations, viz

$\sum_{\mu=1}^{4}\sum_{\upsilon=1}^{4}\frac{\partial F_{\mu\upsilon}}{\partial x_\upsilon}=0\ \left(\mu=1,2,3,4\right)$ obtain the wave equation for the field in a vacuum in the tour dimensional form. Further show that this equation reduces to the following equations for the potentials in the absence of charges and currents (i.e. $\rho=0$ and $J=0$), i.e. $\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=0$ and $\nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=0$.

Sol. We know that Maxwell's equations in the absence of charges and currents are

$$\sum_{\upsilon=1}^{4}\frac{\partial F_{\mu\upsilon}}{\partial x_\upsilon}=0 \text{ and } \left(\mu=1,2,3,4\right) \qquad (18.37)$$

Now putting the values of $F_{\mu\upsilon}$ in terms of electromagnetic potential Since

$$F_{\mu\upsilon}=\frac{\partial A_\upsilon}{\partial x_\mu}-\frac{\partial A_\mu}{\partial x_\upsilon} \qquad (18.38)$$

We get $$\sum_{\upsilon=1}^{4}\frac{\partial}{\partial x_\upsilon}\left(\frac{\partial A_\upsilon}{\partial x_\mu}-\frac{\partial A_\mu}{\partial x_\upsilon}\right)=0$$

Or $$\sum_{\upsilon=1}^{4}\frac{\partial^2 A_\upsilon}{\partial x_\upsilon\partial x_\mu}-\sum_{\upsilon=1}^{4}\frac{\partial^2 A_\mu}{\partial x_\upsilon\partial x_\upsilon}=0 \qquad (18.39)$$

Now $$\sum_{\upsilon=1}^{4}\frac{\partial^2 A_\upsilon}{\partial x_\upsilon\partial x_\mu}=\frac{\partial}{\partial x_\mu}\sum_{\upsilon=1}^{4}\frac{\partial A_\upsilon}{\partial x_\upsilon} \qquad (18.40)$$

If the four potential is subject to the Lorentz condition then

$$\sum_{\upsilon=1}^{4}\frac{\partial^2 A_\upsilon}{\partial x_\upsilon}=0\ \because\ \frac{\partial A_1}{\partial x_1}+\frac{\partial A_2}{\partial x_2}+\frac{\partial A_3}{\partial x_3}+\frac{\partial A_4}{\partial x_4}=0$$

So equation (18.39) becomes $$\sum_{\upsilon=1}^{4}\frac{\partial^2 A_\mu}{\partial x_\upsilon\partial x_\upsilon}=0\ \left(\mu=1,2,3,4\right) \qquad (18.41)$$

Making use of D' Alembertian operator $\boxed{\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}}$ (or Four dimensional Laplacian operator), the above equation (18.41) can be expressed in the four dimensional form, as

$$\boxed{\Box^2 A_\mu = 0} \qquad (\mu = 1,2,3,4)$$

for $\mu = 4$, it reduces to $\Box^2 \phi = 0$ and by putting $\mu = 1$, we get $\nabla^2 A_1 - \frac{1}{c^2}\frac{\partial^2 A_1}{\partial t^2} = 0$

Similarly for $\mu = 2$ and $\mu = 3$. We get

$$\nabla^2 A_2 - \frac{1}{c^2}\frac{\partial^2 A_2}{\partial t^2} = 0$$

$$\nabla^2 A_3 - \frac{1}{c^2}\frac{\partial^2 A_3}{\partial t^2} = 0$$

Combining these three equations. We shall get $\Box^2 A = 0$

18.8 Summary

In this unit we have defined electromagnetic field Tensor and derived expression for this in terms of electromagnetic potential. Then we derive expression for components of electromagnetic field vectors $\vec{E}$ *and* $\vec{B}$ in terms of electromagnetic field tensor $F_{\mu\upsilon}$. Hence we obtain Lorentz Transformation of electric and magnetic field vectors. Then we discuss the invariants of the electromagnetic fields. In the last we obtain Maxwell's equations in Tensor form.

18.9 Glossary

Invariant : A function quantity, or property which remains unchanged when a specified transformation is applied

Antisymmetric : Unaltered in magnitude but changed in sign by exchange of two variables or by a particular symmetry operation.

Inertial Frame : a frame of reference) in which bodies continue at rest or in uniform straight motion unless acted on by a force

18.10 Answers To Self Learning Exercise

Ans.1: It is an anti symmetric tensor of rank two.

Ans.2: $F'_{\mu\upsilon} = a_{\mu\alpha} a_{\upsilon\beta} F_{\alpha\beta}$

$\mu = 1,2,3,4 \;\&\; \alpha = 1,2,3,4$

$\upsilon = 1,2,3,4 \;\&\; \beta = 1,2,3,4$

Ans.4: The importance of this result lies in the facts that if $\vec{E}.\vec{B} = 0$ (as in the case of a plane electromagnetic wave) in one frame, it will be zero in all inertial frames i.e. if vectors $\vec{E}\, and\, \vec{B}$ and are mutually perpendicular in any frame S then they are mutually perpendicular in another inertial frame S' .

The quantity $\left(c^2B^2 - E^2\right)$ is invariant under Lorentz transformation. The significance of this result lies in the fact that if the magnitude of $\vec{E}$ and $\vec{B}$ vectors in any reference system are given by $E = cB$ i.e. $H = \dfrac{E}{\mu_0 c} = \epsilon_0 cE$ (as in the case of a plane electromagnetic wave in free space) then they are related to each other by the same relation in any other system.

18.11 Exercise

Section A : Very Short Answer Type Questions

Q.1 What is anti symmetric tensor?

Q.2 What are invariants of electromagnetic fields?

Q.3 What is covariant form of Maxwell's equation?

Section B : Short Answer Type Questions

Q.4 Define electromagnetic field tensor ?

Q.5 Write down Maxwell's equation in Tensor form.

Q.6 Derive the transformation formula for $F_{\mu\upsilon}$.

Section C : Long Answer Type Questions

Q.7 Define electromagnetic field tensor and obtain an expression for it and demonstrate its each element.

Q.8 Using the transformation property of electromagnetic field tensor, obtain the Lorentz transformation equations for electric and magnetic fields.

Q.9 What are the invariants of the electromagnetic field ? Prove their invariance and give their physical significance.

Q.10 Define electromagnetic field tensor and derive Maxwell's equations in tensor form.

18.12 Answers to Exercise

Ans.1: $F_{\mu\upsilon} = -F_{\upsilon\mu}$

Ans.2: (i) $\vec{E}.\vec{B}$ (ii) $\left(B^2c^2 - E^2\right)$

Ans.3: Tensor form of Maxwell's equation is covariant form.

References and Suggested Readings

1. Classical Electrodynamics by J.D. Jackson (John Wiley & Sons).

2. Classical Electricity and magnetism by Panofsky & Philips. (Indian Book, New Delhi).

3. Introduction to electrodynamics by Griffiths.

4. Elements of Electromagnetic by Mathew N.O. Sadiku (oxford Univ. Press).

5. Classical theory of Electrodynamics by Landau & Lifshitz. (Pergaman press, New York).

6. Electrodynamics of continuous Media by Landau & Lifshitz. (Pergaman press, New York).

7. Relativistic mechanics by Prakash. (Pragati Prakashan Meerut (India))

8. Electromagnetics by B.B. Laud (Wiley Eastern Limited,New Delhi)

UNIT -19

Energy and Momentum Tensor of the EM Fields and Conservation Laws

Structure of the Unit

19.0 Objectives

The objectives of this unit are :

- To derive Lorentz force in covariant form
- To obtain expression for energy and momentum tensor of the EM fields
- To study conservation Laws

- To obtain expression for Lagrangian and Hamiltonian of a charged particle in EM fields.

19.1 Introduction

In the previous unit (18) we introduced the electromagnetic field tensor which gives the correct description of the electromagnetic field, since it accounts for the intermingling of electric and magnetic fields. Subsequently, we introduce the energy momentum tensor of the electromagnetic field and will deduce the Law of conservation of linear momentum and energy for a combined system consisting of the electromagnetic field and the charge particles. Then we will also find expression for Lagrangian and Hamiltonian of a charged particle in an electromagnetic field.

19.2 Lorentz Force in Covariant Form or (Force density four Vector)

We know that when a charged particle is placed in an electromagnetic field, it experiences a force given by ***Lorentz force equation***

$$\boxed{\vec{F} = q\vec{E} + q\vec{u} \times \vec{B}} \tag{19.1}$$

$$= q\left(\vec{E} + \vec{u} \times \vec{B}\right)$$

Where q is electric charge on the particle and $\vec{u}$ is the velocity of the particle. In order to obtain Lorentz force equation in covariant form, we consider the force acting on a unit volume of charge density ρ ; if *f* is the force per unit volume then equation (1) yields

$$\frac{\vec{F}}{V} = \frac{q}{V}\left(\vec{E} + \vec{u} \times \vec{B}\right)$$

or $\vec{F} = \rho\left(\vec{E} + \vec{u} \times \vec{B}\right)$

$$= \rho\vec{E} + \rho\vec{u} \times \vec{B}$$

$$\boxed{\vec{F} = \rho\vec{E} + \vec{J} \times \vec{B}} \tag{19.2}$$

In terms of components above equation (19.2) may be written as

$$f_1 = \rho E_x + J_2 B_z - J_3 B_y$$
$$f_2 = \rho E_y + J_3 B_x - J_1 B_z \qquad (19.3)$$
$$f_3 = \rho E_z + J_1 B_y - J_2 B_x$$

Using electromagnetic field tensor $F_{\mu\upsilon}$ given by $F_{\mu\upsilon} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix}$

$$F_{\mu\upsilon} = \begin{bmatrix} 0 & B_z & -B_y & -\frac{iE_x}{c} \\ -B_z & 0 & B_x & -\frac{iE_y}{c} \\ B_y & -B_x & 0 & -\frac{iE_z}{c} \\ \frac{iE_x}{c} & \frac{iE_y}{c} & \frac{iE_z}{c} & 0 \end{bmatrix} \qquad (19.4)$$

Equation (19.3) can be written as

$$f_1 = F_{11}J_1 + F_{12}J_2 + F_{13}J_3 + F_{14}J_4$$
$$f_2 = F_{21}J_1 + F_{22}J_2 + F_{23}J_3 + F_{24}J_4 \qquad (19.5)$$
$$f_3 = F_{31}J_1 + F_{32}J_2 + F_{33}J_3 + F_{34}J_4$$

Where current density four vector $J_\mu = (J, ic\rho)$

Equation (19.5) can be written in the form of a single equation as

$$f_k = F_{k\upsilon} J_\upsilon \quad \text{(with k=1,2,3)} \qquad (19.6)$$

It is clear that the R.H.S. of equation (19.6) is evidently the space component of a four vector. So f_k must be a space component of a four vector f_μ such that

$$f_\mu = F_{\mu\upsilon} J_\upsilon \qquad (19.7)$$

Here f_μ is called the force density four vector. This equation can also be written as (using $\frac{\partial F_{\upsilon\lambda}}{\partial x_\lambda} = \mu_0 J_\upsilon$)

$$f_\mu = F_{\mu\upsilon} \frac{1}{\mu_0}\left(\frac{\partial F_{\upsilon\lambda}}{\partial x_\lambda}\right) = \frac{1}{\mu_0} F_{\mu\upsilon} \frac{\partial F_{\upsilon\lambda}}{\partial x_\lambda} \qquad (19.8)$$

Equation (19.7)or (19.8) is a tensor equation, So it is invariant under Lorentz transformations i.e. these equations (19.7) and (193.8) represents the covariance form of the Lorentz force equation.

Physical meaning of the fourth component of the force density four vector :

We can write the fourth component of force density four vector

$$f_4 = F_{4\upsilon}J_\upsilon = F_{41}J_1 + F_{42}J_2 + F_{43}J_3 + F_{44}J_4$$

$$= \frac{iE_x}{c}J_1 + \frac{iE_y}{c}J_2 + \frac{iE_z}{c}J_3 + 0$$

$$= \frac{i}{c}[\mathrm{E}_x\mathrm{J}_1 + \mathrm{E}_y\mathrm{J}_2 + \mathrm{E}_z\mathrm{J}_3]$$

$$= \frac{i}{c}\left(\vec{E}.\vec{J}\right) = \frac{i\rho}{c}\left(\vec{E}.\vec{u}\right) \qquad (19.9)$$

Since the fourth component of force density four vector is imaginary and contains $\frac{i}{c}$ factor, So it represents the amount of work done by the electric field on the charge per unit volume per unit time. Hence the Lorentz force equation in Covariance form, gives the rate of change of mechanical momentum per unit volume as its space part and rate of change of mechanical energy per unit volume as its time part.

19.3 Energy and Momentum Tensor of the EM fields

We have derived Lorentz force equation in covariant form in (19.2) article as follows :

$$\mathrm{f}_\mu = F_{\mu\upsilon}J_\upsilon \qquad (19.7)$$

Using equation for Maxwell covariant form of non-homogeneous pair we get

$$\frac{\partial F_{\mu\upsilon}}{\partial x_\upsilon} = \mu_0 J_\mu$$

$$\text{or } J_\mu = \frac{1}{\mu_0}\frac{\partial F_{\mu\upsilon}}{\partial x_\upsilon} \qquad \text{or } J_\upsilon = \frac{1}{\mu_0}\frac{\partial F_{\upsilon\lambda}}{\partial x_\lambda} \qquad (19.10)$$

Thus equation (19.7) takes the form

$$f_\mu = \frac{1}{\mu_0}F_{\mu\upsilon}\frac{\partial F_{\upsilon\lambda}}{\partial x_\lambda}$$

$$\mu_0 f_\mu = \frac{\partial}{\partial x_\lambda}(F_{\mu\upsilon}F_{\upsilon\lambda}) - F_{\upsilon\lambda}\frac{\partial}{\partial x_\lambda}(F_{\mu\upsilon})$$

$$\text{or } \mu_0 f_\mu - \frac{\partial}{\partial x_\lambda}(F_{\mu\upsilon}F_{\upsilon\lambda}) = -\frac{1}{2}\left[F_{\upsilon\lambda}\frac{\partial F_{\mu\upsilon}}{\partial x_\lambda} + F_{\upsilon\lambda}\frac{\partial F_{\mu\lambda}}{\partial x_\lambda}\right]$$

$$\text{or } \mu_0 f_u - \frac{\partial}{\partial x_\lambda}(F_{\mu\upsilon}F_{\upsilon\lambda}) = -\frac{1}{2}\left[F_{\upsilon\lambda}\frac{\partial F_{\mu\upsilon}}{\partial x_\lambda} + F_{\upsilon\lambda}\frac{\partial F_{\mu\lambda}}{\partial x_\upsilon}\right]$$

(We can inter change($\mu\upsilon$) dummy suffixes)

$$\text{or } \mu_0 f_\mu - \frac{\partial}{\partial x_\lambda}(F_{\mu\upsilon}F_{\upsilon\lambda}) = -\frac{1}{2}F_{\upsilon\lambda}\left[\frac{\partial F_{\mu\upsilon}}{\partial x_\lambda} + \frac{\partial F_{\mu\lambda}}{\partial x_\upsilon}\right] \tag{19.11}$$

But we know that homogeneous Maxwell's equations in Covariant form is given by

$$\frac{\partial F_{\mu\upsilon}}{\partial x_\lambda} + \frac{\partial F_{\upsilon\lambda}}{\partial x_\mu} + \frac{\partial F_{\lambda\mu}}{\partial x_\upsilon} = 0$$

$$\frac{\partial F_{\mu\lambda}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\upsilon} = -\frac{\partial F_{\upsilon\lambda}}{\partial x_\mu}$$

Therefore equation (19.11) becomes

$$\mu_0 f_\mu - \frac{\partial}{\partial x_\lambda}(F_{\mu\upsilon}F_{\upsilon\lambda}) = -\frac{1}{2}F_{\upsilon\lambda}\left[-\frac{\partial F_{\upsilon\lambda}}{\partial x_\mu}\right]$$

$$= \frac{1}{4}\left[2F_{\upsilon\lambda}\frac{\partial F_{\upsilon\lambda}}{\partial x_\mu}\right]$$

$$= \frac{1}{4}\frac{\partial}{\partial x_\mu}\left[F_{\upsilon\lambda}.F_{\upsilon\lambda}\right]$$

$$\mu_0 f_u - \frac{\partial}{\partial x_\lambda}\left(F_{\mu\upsilon}F_{\upsilon\lambda}\right) = \frac{1}{4}\frac{\partial}{\partial x_\mu}\left(F_{\upsilon\lambda}F_{\upsilon\lambda}\right)$$

$$= \frac{1}{4}\frac{\partial}{\partial x_\lambda}\left(\delta_{\mu\lambda}F_{\upsilon\lambda}F_{\upsilon\lambda}\right)$$

$$\left[Where \frac{\partial}{\partial x_\mu} = \delta_{\mu\lambda}\frac{\partial}{\partial x_\lambda}\right]$$

When $\mu = \lambda$ then $\delta_{\mu\lambda} = 1$

If $\mu \neq \lambda$ then $\delta_{\mu\lambda} = 0$

$$\therefore f_\mu = \frac{\partial}{\partial x_\lambda}\left[\frac{1}{\mu_0}\left\{F_{\mu\upsilon}F_{\upsilon\lambda} + \frac{1}{4}\delta_{\mu\lambda}F_{\upsilon\lambda}F_{\upsilon\lambda}\right\}\right]$$

$$\text{or } \therefore f_\mu = \frac{\partial}{\partial x_\lambda}\left[T_{\mu\lambda}\right] \tag{19.12}$$

Where

$$T_{\mu\lambda} = \left\{\frac{1}{\mu_0}\left(F_{\mu\upsilon}F_{\upsilon\lambda} + \frac{1}{4}\delta_{\mu\lambda}F_{\upsilon\lambda}F_{\upsilon\lambda}\right)\right\} \tag{19.13}$$

is called electromagnetic energy momentum tensor. The characteristics of this tensor are

(i) ***It is symmetric tensor i.e.*** $T_{\mu\lambda} = T_{\lambda\mu}$

(ii) It has only nine independent components because sum of diagonal elements is zero.

i.e. $T_{11} + T_{22} + T_{33} + T_{44} = 0 \therefore T_{\mu\mu} = 0$

$$L.H.S. = \frac{1}{\mu_0}\left[F_{\mu\upsilon}F_{\mu\upsilon} - \frac{1}{4}\left(\delta_{11} + \delta_{22} + \delta_{33} + \delta_{44}\right)F_{\upsilon\lambda}F_{\upsilon\lambda}\right]$$

(iii) Evaluation of different elements :-

$$\text{Electromagnetic field Tensor is } F_{\mu\upsilon} = \begin{bmatrix} 0 & B_z & -B_y & -\frac{iE_x}{c} \\ -B_z & 0 & B_x & -\frac{iE_y}{c} \\ B_y & -B_x & 0 & -\frac{iE_z}{c} \\ \frac{iE_x}{c} & \frac{iE_y}{c} & \frac{iE_z}{c} & 0 \end{bmatrix} \tag{19.14}$$

Now we have for $\mu = 4$ and $\lambda = 4$

$$F_{\mu\upsilon}F_{\upsilon\lambda} = F_{41}F_{14} + F_{42}F_{24} + F_{43}F_{34} + F_{44}F_{44}$$

$$= \left(\frac{iE_x}{c}\right)\left(-\frac{iE_x}{c}\right) + \left(\frac{iE_y}{c}\right)\left(-\frac{iE_y}{c}\right) + \left(\frac{iE_z}{c}\right)\left(-\frac{iE_z}{c}\right) + 0$$

$$= \frac{E_x^2 + E_y^2 + E_z^2}{c^2} = \frac{E^2}{c^2}$$ and

$$F_{\upsilon\lambda} F_{\upsilon\lambda} = F_{11}^2 + F_{12}^2 + F_{13}^2 + F_{14}^2 + F_{21}^2 + F_{22}^2 + F_{23}^2 + F_{24}^2$$

$$+F_{31}^2 + F_{32}^2 + F_{33}^2 + F_{34}^2 + F_{41}^2 + F_{42}^2 + F_{43}^2 + F_{44}^2$$

$$\therefore F_{\upsilon\lambda} F_{\upsilon\lambda} = 2\left(B_x^2 + B_y^2 + B_z^2 - \frac{E_x^2}{c^2} - \frac{E_y^2}{c^2} - \frac{E_z^2}{c^2} \right)$$

$$= 2\left(B^2 - \frac{E^2}{c^2} \right)$$

$$\therefore T_{44} = \frac{1}{\mu_0}\left[F_{4\upsilon}\, F_{\upsilon 4} + \frac{1}{4}\delta_{44}\left(F_{\upsilon\lambda} F_{\upsilon\lambda} \right) \right]$$

$$= \frac{1}{\mu_0}\left[\frac{E^2}{c^2} + \frac{1}{4} \times 1 \times 2\left(B^2 - \frac{E^2}{c^2} \right) \right]$$

$$= \frac{1}{2\mu_0}\left(B^2 + \frac{E^2}{c^2} \right) = \frac{1}{2}\frac{\mu_0^2 H^2}{\mu_0} + \frac{\mu_0 \in_0 E^2}{2\mu_0}$$

$$T_{44} = \boxed{\frac{1}{2} \in_0 E^2 + \frac{1}{2}\mu_0 H^2 = U} \tag{19.15}$$

This is called ***energy density of electromagnetic field.***

(v) Now if we put $\mu = 4$ and $\lambda = 1$ then

$$T_{41} = \frac{1}{\mu_0}\left[F_{4\upsilon} F_{\upsilon 1} - \frac{1}{4}\delta_{41} F_{\upsilon\lambda}\, F_{\upsilon\lambda} \right] \qquad \text{Since } \delta_{41} = 0$$

$$T_{41} = \frac{1}{\mu_0}\left[F_{41}F_{11} + F_{42}F_{21} + F_{43}F_{31} + F_{44}F_{41} - 0 \right]$$

$$= \frac{1}{\mu_0}\left[0 + \left(\frac{iE_y}{c} \right)\left(-B_z \right) + \left(\frac{iE_z}{c} \right)\left(B_y \right) + 0 \right]$$

$$= -\frac{i}{\mu_0 c}\left[E_y B_z - E_z B_y \right]$$

$$= -\frac{i}{c}\left[E_y H_z - E_z H_y \right]$$

$$= -\frac{i}{c}\left[\vec{E} \times \vec{H} \right]_x = -\frac{1}{c} N_x$$

or $T_{41} = -\frac{i}{c}$ (x component of Poynting vector $\vec{N}$) (19.16)

Similarly $T_{42} = -\frac{i}{c} N_y$ and $T_{43} = -\frac{i}{c} N_z$

So this given the momentum density. Hence energy momentum tensor is given by

$$T_{\mu\lambda} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & -\frac{iN_x}{c} \\ T_{21} & T_{22} & T_{23} & -\frac{iN_y}{c} \\ T_{31} & T_{32} & T_{33} & -\frac{iN_z}{c} \\ \frac{iN_x}{c} & \frac{iN_y}{c} & \frac{iN_z}{c} & U \end{bmatrix} \tag{19.17}$$

Where N_x, N_y and N_z represents the Poynting vector along x, y and z direction respectively and U is energy density of electromagnetic field.

19.4 Conservation Laws

(a) Law of Conservation of energy

We know that Lorentz force tensor is given by

$$f_\mu = \frac{\partial T_{\mu\lambda}}{\partial x_\lambda} \quad \text{Let } \mu = 4 \text{ then}$$

$$f_4 = \frac{\partial T_{4\lambda}}{\partial x_\lambda} = \frac{\partial T_{41}}{\partial x_1} + \frac{\partial T_{42}}{\partial x_2} + \frac{\partial T_{43}}{\partial x_3} + \frac{\partial T_{44}}{\partial x_4}$$

$$= -\frac{i}{c}\frac{\partial N_x}{\partial x_1} - \frac{i}{c}\frac{\partial N_y}{\partial x_2} - \frac{i}{c}\frac{\partial N_z}{\partial x_3} + \frac{\partial U}{\partial (ict)}$$

$$= -\frac{i}{c}.\nabla.\vec{N} - \frac{i}{c}\frac{\partial U}{\partial t}$$

We know that $f_4 = \frac{i}{c}\left(\vec{E}.\vec{J}\right)$ from covariant form of Lorentz force equation.

$$\text{Or } \frac{i}{c}\left(\vec{E}.\vec{J}\right) = -\frac{i}{c}\nabla.\overrightarrow{N} - \frac{i}{c}\frac{\partial U}{\partial t}$$

$$\text{Or } \overrightarrow{E}.\overrightarrow{J} + \nabla.\overrightarrow{N} = -\frac{\partial U}{\partial t} \tag{19.18}$$

Integrating above equation (19.18) overall space volume we get

$$\int_V \vec{E}.\vec{J}dV + \int_V \nabla.\vec{N}dV = -\frac{\partial}{\partial t}\int_V UdV \qquad (19.19)$$

Using Gauss's divergence theorem to change volume integral of second term of equation (19.19) into Surface integral

$$\int_V \vec{E}.\vec{J}dV + \int_S (\vec{E}\times\vec{H}).\mathrm{d}\vec{s} = -\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\in_0 E^2 + \frac{1}{2}\mu_0 H^2\right)dV \qquad (19.20)$$

This equation (19.20) is known as Poynting theorem i.e. conservation of energy in electromagnetism*. *Here term, $\int_S (\vec{E}\times\vec{H}).\mathrm{d}\vec{s}$ ***represents the energy flowing out from the surface per second and*** $\int_V \vec{E}.\vec{J}dV$ ***represents work done by the electric field on moving charge.***

(b) Law of Conservation of Momentum

For this substituting $\mu = 1$ in Lorentz force equation

$$f_1 = \frac{\partial T_{1\lambda}}{\partial x_\lambda} = \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} + \frac{\partial T_{14}}{\partial x_4}$$

$$= \nabla.\vec{T}_1 - \frac{i}{c}\frac{\partial N_x}{\partial(ict)}$$

$$\text{Or } f_1 = \nabla.\vec{T} - \frac{1}{c^2}\frac{\partial N_x}{\partial t} \qquad (19.21)$$

Integrating equation (19.21) over space volume, we get

$$\int_V f_1 dV + \frac{1}{c^2}\frac{\partial}{\partial t}\int_V N_x dV = \int \nabla.\vec{T}_{iJ}dV \qquad (19.22)$$

Using Gauss divergence theorem in R.H.S. of equation (19.22) we get

$$\int_V f_1 dV + \frac{1}{c^2}\frac{\partial}{\partial t}\int_v N_x dV = \int_S T_{iJ}ds \qquad (19.23)$$

Here first term in equation (19.23) represents mechanical momentum and second term represents electromagnetic field momentum.

Now equation (19.23) becomes.

$$\frac{\partial}{\partial t}\left[P_1 + \frac{1}{c^2}\int_V N_x dV\right] = \int_S T_{iJ} ds \qquad (19.24)$$

This equation (19.24) represents the conservation of momentum. The volume integral of the force density f_1 gives the total force which is expressed as the time derivative of the mechanical momentum P_1 Thus $\int_V f_1 dV = \frac{dP_1}{dt}$. In case the field vanishes outside the volume V, then it also vanishes at the boundary surface S which encloses the volume V. Thus.

$$\frac{d}{dt}\left[P_1 + \frac{1}{c^2}\int_V N dV\right] = 0 \qquad \therefore P_1 + \frac{1}{c^2}\int_V N dV = \text{constant} \quad (19.25)$$

Which expresses the law of conservation of the momentum for the combined system of particles and fields. However, if the field does not vanish on the boundary of V, the $\int_S T_{iJ} .ds$ represents the outward flow of momentum per unit area of the surface S surrounding the volume V. Term $\frac{N}{c^2}$ represents the momentum density of electromagnetic field.

19.5 Self Learning Exercise

Section A: Very Short Answer Type Questions

Q.1 What is Lorentz force?

Q.2 What is electromagnetic energy momentum tensor ?

Section B: Short Answer Type Questions

Q.3 Using expression of electromagnetic energy momentum tensor, explain Law of conservation of energy.

19.6 Lagrangian and Hamiltonian of a Charged Particle in EM Fields

(a) Non Relativistic Case:

We know that the total force on a charged particle moving with velocity $\vec{v}$ in an electromagnetic field is given by

$$\vec{F} = q\left[\vec{E} + \vec{v}\times\vec{B}\right] \tag{19.26}$$

Where q is the charge of the moving particle, $\vec{E}$ is the electric field and $\vec{B}$ is the magnetic field induction.

The field vectors $\vec{E}$ and $\vec{B}$ in terms of electromagnetic potential $\vec{A}$ and ϕ are given by $\vec{B} = curl\vec{A}$ and $\vec{E} = -grad\phi - \frac{\partial\vec{A}}{\partial t}$ (19.27)

Now putting the value of $\vec{B}$ and $\vec{E}$ from (19.27) in (19.26) we get

$$\vec{F} = q\left[\left(-\frac{\partial\vec{A}}{\partial t} - grad\phi\right) + \left(\vec{v}\times curl\vec{A}\right)\right]$$

$$= q\left[-\frac{\partial\vec{A}}{\partial t} - grad\phi + \vec{v}\times\left(\nabla\times\vec{A}\right)\right]$$

$$= q\left[-\frac{\partial\vec{A}}{\partial t} - grad\phi + \nabla\left(\vec{v}.\vec{A}\right) - \left(\vec{v}.\nabla\right)\vec{A}\right]$$

$$i.e.\vec{F} = q\left[-grad\phi - \left\{\frac{\partial\vec{A}}{\partial t} + \left(\vec{v}.\nabla\right)\vec{A}\right\} + \nabla\left(\vec{v}.\vec{A}\right)\right] \tag{19.28}$$

Since vector potential $\vec{A}$ is the function of both space and time

i.e. $\vec{A} = \vec{A}(x, y, z, t)$

We have $\frac{dA}{dt} = \frac{\partial A}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial A}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial A}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial A}{\partial t}$

$$= \frac{\partial A}{\partial x}v_x + \frac{\partial A}{\partial y}v_y \frac{\partial A}{\partial z}v_z + \frac{\partial A}{\partial t}$$

$$= \left(\hat{i}v_x + \hat{j}v_y + \hat{k}v_z\right).\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)A + \frac{\partial A}{\partial t}$$

$$= \left(\vec{v}.\nabla\right)A + \frac{\partial A}{\partial t} \tag{19.29}$$

Putting values from (19.29) in (19.28) we get

$$\vec{F} = q\left[-\nabla\phi - \frac{d\vec{A}}{dt} + \nabla\left(\vec{v}.\vec{A}\right)\right]$$

$$\text{i.e. } \vec{F} = q\left[-\nabla\left(\phi - \vec{v}.\vec{A}\right) - \frac{d\vec{A}}{dt}\right] \tag{19.30}$$

Equation (19.30) expresses the Lorentz force in terms of electromagnetic potential $\vec{A}$ and ϕ . Since Force $\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(m\vec{v}\right)$

$$\text{So. } = \frac{d}{dt}\left(m\vec{v}\right) = q\left[-\nabla\left(\phi - \vec{v}.\vec{A}\right) - \frac{d\vec{A}}{dt}\right]$$

$$\text{i.e. } \frac{d}{dt}\left(m\vec{v} + q\vec{A}\right) + \nabla\left\{q\left(\phi - \vec{v}.\vec{A}\right)\right\} = 0 \tag{19.31}$$

This equation (19.31) has the general form of a set of Lagrangian equation given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\alpha}\right) - \left(\frac{\partial L}{\partial x_\alpha}\right) = 0 \tag{19.32}$$

Now comparing equations (19.31) and (19.32) we get

$$\left.\begin{aligned} &\boxed{\frac{\partial L}{\partial \dot{x}_\alpha} = mv_\alpha + qA_\alpha} && (a) \\ &\frac{\partial L}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}\left\{-q\left(\phi_\alpha - v_\alpha.A\right)\right\} && (b) \end{aligned}\right\} \tag{19.32}$$

Where $\dot{x}_\alpha = \frac{\partial x_\alpha}{\partial t}$

Now integrating equations(19.32a) and (19.32b) we get

$$L = \frac{1}{2}mv^2 + q\left(\vec{v}.\vec{A}\right) + c_1 \tag{19.33}$$

$$L = -q\phi + \left(\vec{v}.\vec{A}\right) + c_2 \tag{19.34}$$

Where c_1 and c_2 are constants of integration such that constant c_1 is independent of position and constant c_2 is independent of velocity . A glance at equations (19.33) and (19.34) reveals that the proper Lagrangian for the charged particle is

$$L = \frac{1}{2}mv^2 + q\left(\vec{v}.\vec{A}\right) - q\phi$$

i.e. $$\boxed{L = \frac{1}{2}mv^2 - q\left(\phi - \vec{v}.\vec{A}\right)} \tag{19.35}$$

This is the desired value of Lagrangian

As we know Hamiltonian function is defined as

$$\boxed{H = \sum_{\alpha} p_\alpha \dot{x}_\alpha - L = \vec{p}.\vec{v} - L}$$

$$= \left(m\vec{v} + q\vec{A}\right).\vec{v} - \left\{\frac{1}{2}mv^2 + q\left(\vec{v}.\vec{A}\right) - q\phi\right\}$$

$$= \frac{1}{2}mv^2 + q\phi$$

$$= \frac{(mv)^2}{2m} + q\phi$$

$$= \frac{\left(\vec{p} - q\vec{A}\right)^2}{2m} + q\phi$$

So Hamiltonian $$\boxed{H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\phi} \tag{19.36}$$

Equation (19.35) and (19.36) represent expression for Non relativistic Lagrangian and Hamiltonian of a charged particle in E.M. field.

(b) Relativistic Lagrangian and Hamiltonian of a charged particle in an electromagnetic field :

The x-component of force is given by equation (19.30)

$$F_x = q\left[\left\{-\nabla\left(\phi - \vec{v}.\vec{A}\right)\right\}_x - \frac{dA_x}{dt}\right]$$

Or $$F_x = q\left[-\frac{\partial}{\partial x}\left(\phi - \vec{v}.\vec{A}\right) - \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{x}}\left(\vec{v}.\vec{A}\right)\right\}\right] \tag{19.37}$$

$$\left[Since \frac{\partial}{\partial \dot{x}}\left(\vec{v}.\vec{A}\right) = \frac{\partial}{\partial v_x}\left(\vec{v}.\vec{A}\right) = A_x\right]$$

As the scalar potential ϕ is independent of velocity $\dot{x}$, i.e. $\frac{\partial \phi}{\partial \dot{x}} = 0$ Therefore equation (19.37) is equivalent to $\boxed{F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right)}$ (19.38)

Where $\boxed{U = q\left(\phi - \vec{v}.\vec{A}\right)}$

Clearly U is the function of x and $\dot{x}$ This is known as velocity dependent potential.

$$U = q\left(\phi - \vec{v}.\vec{A}\right) \tag{19.39}$$

The Lagrangian L given by $\boxed{L = E^* - U}$

Where E^* is kinetic energy given by $\boxed{E^* = \mathrm{mc}^2 - m_0c^2 = (\mathrm{m} - \mathrm{m}_0)\mathrm{c}^2}$

Or $E^* = \left[\frac{m_0}{\sqrt{1 - v^2/c^2}} - m_0\right]c^2$

$$\boxed{E^* = \left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right]m_0c^2} \tag{19.40}$$

The relativistic Lagrangian of the charge particle in an electromagnetic field is

$$L = \left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right]m_0c^2 - q\phi + q\left(\vec{v}.\vec{A}\right) \tag{19.41}$$

Or $L = \left[1 - \sqrt{1 - v^2/c^2}\right]mc^2 - q\phi + q\left(\vec{v}.\vec{A}\right)$ (19.42)

Differentiating equation (19.41) with respect to v the relativistic momentum of the charged particle in a electromagnetic field is

$$p = \frac{\partial L}{\partial v} = \frac{mv}{\left(1 - v^2/c^2\right)^{\frac{3}{2}}} + qA \tag{19.43}$$

or $\vec{p} = \frac{m\vec{v}}{\left(1 - v^2/c^2\right)^{\frac{3}{2}}} + q\vec{A}$

The Hamiltonian is defined as $H = \sum_{\alpha} p_{\alpha} \dot{x}_{\alpha} - L$ (19.44)

Substituting value of p and L in equation (19.44) we get an expression for the relativistic Hamiltonian of a charged particle in electromagnetic field as

$$H = \left[\frac{m_0 v}{\left(1 - v^2/c^2\right)^{\frac{3}{2}}} + qA \right] v - \left[\left(\frac{1}{\left(1 - v^2/c^2\right)^{\frac{3}{2}}} - 1 \right) m_0 c^2 - q\phi + q(\vec{v}.\vec{A}) \right]$$

$$H = \frac{m_0 v^2}{\left(1 - v^2/c^2\right)^{\frac{3}{2}}} - \left[\left(\frac{1}{1 - v^2/c^2} - 1 \right) m_0 c^2 \right] + q\phi \tag{19.45}$$

So, Equations (19.42) and (19.45) represents expression for relativistic Lagrangian and Hamiltonian of a charged particle in E.M. field.

19.7 Illustrative Examples

Example 19.1 Express the Lorentz force formula in terms of electromagnetic potentials.

Sol. The force on a charged particle in electromagnetic field is given by

$$\vec{F} = q[\vec{E} + \vec{v} \times \vec{B}] \tag{19.46}$$

The field vector $\vec{E}$ and $\vec{B}$ in terms of electromagnetic potential $\vec{A}$ and ϕ are given by

$$\vec{E} = -grad\phi - \frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \text{curl}\,\vec{A} \tag{19.47}$$

So equation (19.46) reduces by using (19.47)

$$\vec{F} = q\left[\left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t} \right) + \vec{v} \times \left(\nabla \times \vec{A} \right) \right]$$

$$\vec{F} = q\left[-\nabla\phi - \frac{\partial \vec{A}}{\partial t} + \nabla\left(\vec{v}.\vec{A} \right) - \left(\vec{v}.\nabla \right)\vec{A} \right]$$

$$\vec{F} = q\left[-\nabla\phi - \left\{\frac{\partial\vec{A}}{\partial t} + \left(\vec{v}.\nabla\right)\vec{A}\right\} + \nabla\left(\vec{v}.\vec{A}\right)\right] \tag{19.48}$$

Since vector potential $\vec{A}$ is a functions of time and space both .So

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial A}{\partial y}.\frac{\partial y}{\partial t} + \frac{\partial A}{\partial z}.\frac{\partial z}{\partial t}$$

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + v_x.\frac{\partial A}{\partial x} + v_y\frac{\partial A}{\partial y} + v_z\frac{\partial A}{\partial z}$$

i.e. $$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \left(\vec{v}.\nabla\right)A \tag{19.49}$$

Putting value from Equation (19.49) in Equation (19.48) we get

$$\vec{F} = q\left[-\nabla\phi - \frac{d\vec{A}}{dt} + \nabla\left(\vec{v}.\vec{A}\right)\right]$$

or $$\vec{F} = q\left[-\nabla\left(\phi - \vec{v}.\vec{A}\right) - \frac{d\vec{A}}{dt}\right] \tag{19.50}$$

This is the required result.

Example 19.2 Using the Lorentz force equation in covariant form derive the transformation law for the force.

Sol. Lorentz force equation is covariant form is

$$f_\mu = F_{\mu\upsilon}J_\upsilon \tag{19.51}$$

Here $F_{\mu\upsilon}$ is electromagnetic field tensor, f_μ is force density (force per unit volume) four vector and J_υ is current density four vector.

As we know that in a frame in which charges are at rest with respect to the frame, no work is done on moving charges i.e. fourth component of the force density four vector f_4 is zero i.e.

$$f_4 = 0$$

Hence according to law of transformation of force density four vector,

$$f'_\mu = a_{\mu\upsilon}f_\upsilon \tag{19.52}$$

Where $a_{\mu\upsilon}$ are element of transformation matrix.

$$f'_1 = a_{11}f_1 + a_{12}f_2 + a_{13}f_3 + a_{14}f_4 = \gamma f_1 + i\beta\gamma f_4$$

Or $f_1' = \gamma f_1$ since $f_4 = 0$

$f_2' = a_{22} f_2 = f_2$ and $f_3' = a_{33} f_3 = f_3$

Now the components of the total force exerted on a given volume of the charge distribution is given

$$F_k' = \int_{V'} f_k' dV'$$

So that

$$F_1' = \int_{V'} f_1' dV' = \int_{V'} \gamma f_1 dV \sqrt{1 - \frac{v^2}{c^2}} \qquad \because dV' = dV \sqrt{1 - \frac{v^2}{c^2}}$$

[Here V is volume and *v* is velocity of frame S in the respect to S']

$$\text{Or } F_1' = \int_V f_1 dV = F_1 \qquad \text{or} \qquad F_x' = F_x$$

$$F_2' = \int_{V'} f_2' dV' = \int_V f_2 dV \sqrt{1 - \frac{v^2}{c^2}} = \sqrt{1 - \frac{v^2}{c^2}} F_2$$

$$or\ F_y' = \sqrt{1 - \frac{v^2}{c^2}} F_y$$

$$F_3' = \int_{V'} f_3' dV' = \int_V f_3 dV \sqrt{1 - \frac{v^2}{c^2}} = \sqrt{1 - \frac{v^2}{c^2}} F_3$$

$$or\ F_z' = \sqrt{1 - \frac{v^2}{c^2}} F_z$$

We can write $F_{\|}' = F_{\|}$ and $F_\perp' = \sqrt{1 - \frac{v^2}{c^2}} F_\perp$ (19.53)

Where F_{11} and $F_\perp$ are the components of force parallel and perpendicular to the direction of motion of the frame S'

Example 19.3 Show that

(i) The momentum of charged particle is an electromagnetic field is given by

$$\vec{P} = m\vec{v} + q\vec{A}$$

(ii) The Lagrangian function of the charged particle in an electromagnetic field is given by $L = \frac{1}{2}mv^2 - q(\phi - \vec{v}.\vec{A})$

(iii) The Hamiltonian function of the charged particle in a electromagnetic field given by $\mathrm{H} = \frac{1}{2m}\left(\vec{P} - q\vec{A}\right)^2 + q\phi$

Sol. **(i)** The Lorentz force on a charged particle in electromagnetic field in terms of electromagnetic potential is given by

$$\vec{F} = q\left[-\nabla\left[\phi - \vec{v}.\vec{A}\right] - \frac{d\vec{A}}{dt}\right] \tag{19.50}$$

According to Newton's second law $\vec{F} = \frac{d}{dt}\left(m\vec{v}\right)$,so equation (19.50) reduces to

$$\frac{d}{dt}\left(m\vec{v}\right) = q\left[-\nabla\left[\phi - \vec{v}.\vec{A}\right] - \frac{d\vec{A}}{dt}\right]$$

i.e. $$\frac{d}{dt}\left(m\vec{v} + q\vec{A}\right) = -q\nabla\left[\phi - \vec{v}.\vec{A}\right] \tag{19.54}$$

This equation (19.54) has the general form of a set of Lagrangian equation given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_\alpha}\right) - \frac{\partial L}{\partial x_\alpha} = 0 \tag{19.55}$$

So comparing equations (19.54) and (19.55) we shall get

$$\left.\begin{aligned} \frac{\partial L}{\partial \dot{x}_\alpha} &= mv_\alpha + qA & (a) \\ \frac{\partial L}{\partial x_\alpha} &= \frac{\partial}{\partial x_\alpha}\left[-q\left(\phi_\alpha - v_\alpha.A\right)\right] & (b) \end{aligned}\right\} \tag{19.56}$$

From classical mechanics by the definition of Lagrangian we know that $\frac{\partial L}{\partial \dot{x}_\alpha}$ is the α^{th} component of momentum So $p_\alpha = \frac{\partial L}{\partial \dot{x}_\alpha}$

Hence $p_\alpha = \frac{\partial L}{\partial \dot{x}_\alpha} = mv_\alpha + qA$

or $$\vec{p} = m\vec{v} + q\vec{A} \qquad (19.57)$$

(ii) Integrating equations (19.56) (a) and (b) we get

$$L = \frac{1}{2}mv^2 + q(\vec{v}.\vec{A}) + c_1 \qquad (19.58)$$

$$L = -q\phi + q(\vec{v}.\vec{A}) + c_2 \qquad (19.59)$$

Where c_1 and c_2 are constants of integration such that c_1 is independent of position ,while c_2 is independent of velocity. A glance at equations (19.58) and (19.59) reveals that the proper Lagrangian for the charged particle in electromagnetic field is

$$L = \frac{1}{2}mv^2 + q(\vec{v}.\vec{A}) - q\phi = \frac{1}{2}mv^2 - q(\phi - \vec{v}.\vec{A}) \qquad (19.60)$$

(iii) As Hamiltonian function is defined as

$$\mathrm{H} = \sum p_\alpha.\dot{\mathrm{x}}_\alpha - L = \vec{p}.\vec{v} - L \qquad (19.61)$$

So substituting the values of p and L from equations (19.57) and (19.60) in equation (19.61)

we shall get

$$H = \left(m\vec{v} + q\vec{A}\right).\vec{V} - \left[\frac{1}{2}mv^2 - q(\phi - \vec{v}.\vec{A})\right]$$

$$= \frac{1}{2}mv^2 + q\phi$$

$$= \frac{(\mathrm{mv})^2}{2m} + q\phi \qquad \because mv = p - qA$$

$$= \frac{\left(\vec{p} - q\vec{A}\right)^2}{2m} + q\phi$$

So $$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\phi \qquad (19.62)$$

This is the required result.

19.8 Summary

In this unit we have derived Lorentz force in covariant form. Then using this covariant form we derive expression for energy and momentum tensor of the E.M. fields. Then we discuss law of conservation of energy and law of conservation of momentum, using energy and momentum tensor of the EM fields. In the last we derive Lagrangian and Hamiltonian of a charged particle in EM fields, in both non relativistic and relativistic form as follows.

(a) Non relativistic expressions for Lagrangian, Momentum and Hamiltonian for a charged particle in an electromagnetic field are

Lagrangian, $\boxed{L = \frac{1}{2} m_0 v^2 - q\phi + q\left(\vec{v}.\vec{A}\right)}$

Momentum, $\boxed{\vec{p} = m_0 \vec{v} + q\vec{A}}$

Hamiltonian, $\boxed{H = \frac{1}{2} m_0 v^2 + q\phi}$

(b) Relativistic expression for Lagrangian, momentum and Hamiltonian for a charged particle in an electromagnetic field are

Lagrangian, $L = \left\{1 - \sqrt{\left(1 - v^2/c^2\right)}\right\} mc^2 - q\phi + q\left(\vec{v}.\vec{A}\right)$

Momentum, $\vec{p} = \dfrac{mv}{\left(1 - v^2/c^2\right)^{\frac{3}{2}}} + q\vec{A}$

Hamiltonian, $H = \dfrac{m_0 v^2}{\left(1 - v^2/c^2\right)^{\frac{3}{2}}} - \left[\left(\dfrac{1}{\sqrt{1 - v^2/c^2}} - 1\right) m_0 c^2\right] + q\phi$

19.9 Glossary

Homogeneous: Consisting of parts all of the same kind, in mathematics containing terms all of the same degree.

Antisymmetric : Unaltered in magnitude but changed in sign by exchange of two variables or by a particular symmetry operation.

19.10 Answer to Self Learning Exercise

Ans.1: It is the force experienced by a charge particle moving in electromagnetic field.

Ans.2: $T_{\mu\lambda} = \left\{ \frac{1}{\mu_0} \left(F_{\mu\upsilon} F_{\upsilon\lambda} + \frac{1}{4} \delta_{\mu\lambda} \, F_{\upsilon\lambda} F_{\upsilon\lambda} \right) \right\}$

19.11 Exercise

Section –A (Very Short Answer Type Questions)

Q.1 What is covariant form of Lorentz force?

Q.2 Give relativistic expression for Lagrangian.

Q.3 Give relativistic expression for Hamiltonian.

Section –B (Short Answer Type Questions)

Q.4 Derive Lorentz force formula for a charged particle moving in electromagnetic field.

Q.5 Define electromagnetic energy momentum tensor and gives its various properties.

Q.6 Using expression for electromagnetic energy momentum tensor, explain Law of conservation of momentum.

Q.7 Derive $L = \frac{1}{2} m_0 v^2 - q\phi + q(\vec{v}.\vec{A})$

Section –C (Long Answer Type Questions)

Q.8 Derive the Lorentz force equation in covariant form and explain the meaning of the fourth component of the force density four vector.

Q.9 Derive an expression for electromagnetic energy momentum tensor of the E.M. Field using covariant form of Lorentz force and discuss it.

Q.10(a) Starting from the Lorentz force equation $f_\upsilon = J_\mu F_{\upsilon\mu}$ and using covariant form of Maxwell's equations. Show that the Lorentz force equation can be written as $f_v = \frac{\partial T_{\upsilon\mu}}{\partial x_\mu}$ Where J_μ is the current density four vector, $F_{\upsilon\mu}$ is the electromagnetic field tensor and $T_{\upsilon\mu}$ is the energy momentum tensor. (Einstein's summation convention used)

(b) Obtain the various components of the energy momentum tensor.

Q.11 Choose a suitable Lorentz invariant Lagrangian for the relativistic description of motion of a classical particle of mass m and charge q in an electromagnetic field given by four vector potential $A_\mu(x)$.

19.12 Answers to Exercise

Ans.1: $f_\mu = F_{\mu\upsilon}J_\upsilon = F_{\mu\upsilon}\frac{1}{\mu_0}\left(\frac{\partial F_{\upsilon\lambda}}{\partial x_\lambda}\right) = \frac{1}{\mu_0}F_{\mu\upsilon}\frac{\partial F_{\mu\lambda}}{\partial x_\lambda}$ is the covariant form of Lorentz force.

Ans.2: $L = \left\{1-\sqrt{1-\frac{v^2}{c^2}}\right\}mc^2 - q\phi + q\left[\vec{v}.\vec{A}\right]$ is required expression.

Ans.3: $H = \frac{m_0 v^2}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}} - \left[\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}-1\right)m_0c^2\right] + q\phi$ is required expression.

References and Suggested Readings

1. Classical electrodynamics by J.D. Jackson (John Wiley &Sons)
2. Classical electricity and magnetism by Panofsky and Philips (Indian Book, New Delhi)
3. Introduction to Electrodynamics by Griffiths.
4. Element of Electromagnetics by Mathew N.O. and Sadiku (Oxford Univ. Press)
5. Classical theory of Electrodynamics by Landau-Lifshirz (Pergaman press, New York)
6. Electrodynamics of continuous media by Landau&Lifshitz (Pergaman Press, New York)
7. Electrodynamics by S.P. Puri.