UNIT-1
Vector Analysis

Structure of the Unit

1.0 Objectives

Vector analysis is a mathematical shorthand. The vector form helps to provide a clearer understanding of the physical laws. This makes the calculus of the vector functions the natural instrument for the physicist and engineers in solid mechanics, electromagnetism, and so on. To meet objectives ,we emphasize the physical interpretation of vector functions.

1.1 Introduction

Vector algebra is introduced early in the text. The unit deals with vector functions and extends the differential calculus to these vector functions. We finally discuss physical meaning of three important concepts namely ,the gradient, divergence and curl.

1.2 Scalar Product

Scalar Product (dot product) of two vectors $\vec{A}$ and $\vec{B}$ is defined as

$$\boxed{\vec{A}.\vec{B} = AB\cos\theta} \qquad \text{where } 0 \le \theta \le \pi$$

Here θ is the angle between $\vec{A}$ and $\vec{B}$. Note that $\vec{A}.\vec{B}$ is a scalar quantity.

General Properties of Scalar Product:-

(i) $\vec{A}.\vec{B} = \vec{B}.\vec{A}$ (Commutative Law)

(ii) $\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B}+\vec{A}.\vec{C}$

(iii) $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$

$\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

(iv) If $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$, then

$$\vec{A}.\vec{B} = A_1B_1 + A_2B_2 + A_3B_3$$

$$\vec{A}.\vec{A} = A^2 = A_1^2 + A_2^2 + A_3^2$$

(v) $\left|\vec{A}.\vec{B}\right| \le \left|\vec{A}\right|\left|\vec{B}\right|$

(vi) $\vec{A}.\vec{B}$ is independent of co-ordinate system.

Typical Applications of Scalar Product:-

(i) If $\vec{A} \ne 0$, $\vec{B} \ne 0$ and $\vec{A}.\vec{B} = 0$, then $\vec{A}$ and $\vec{B}$ will be perpendicular.

(ii) Component of vector $\vec{A}$ along $\hat{n}$ direction is $\boxed{\vec{A}.\hat{n}}$, where $\hat{n}$ is unit vector.

(iii) Angle between $\vec{A}$ and $\vec{B}$ can be found out by $\boxed{\cos\theta = \frac{\vec{A}.\vec{B}}{AB} = \hat{A}.\hat{B}}$

Example1.1 Find the angle between side AC and side AB of a triangle ABC. Coordinates of the vertices A,B,C are $(1+2\sqrt{3},1,2)$, $(1,1,2)$, $(1,3,2)$ respectively.

Sol. $\vec{AB}$ =Position vector of $\vec{B}$ – Position Vector of $\vec{A}$

$$\overrightarrow{AB} = \left(\hat{i}+\hat{j}+2\hat{k}\right) - \left[\left(1+2\sqrt{3}\right)\hat{i}+\hat{j}+2\hat{k}\right] = -2\sqrt{3}\,\hat{i}$$

$$\Rightarrow \quad \left|\vec{AB}\right| = 2\sqrt{3}$$

$$\vec{AC} = \text{P.V. of } \vec{C} \text{ - P.V. of } \vec{A}$$

$$\overrightarrow{AC} = \left(\hat{i}+3\hat{j}+2\hat{k}\right) - \left[\left(1+2\sqrt{3}\right)\hat{i}+\hat{j}+2\hat{k}\right]$$

$$= -2\sqrt{3}\hat{i} + 2\hat{j}$$

$$\Rightarrow \quad \left|\vec{AC}\right| = \sqrt{\left(2\sqrt{3}\right)^2 + 2^2} = 4$$

By dot product

$$\cos\theta = \frac{\vec{AB}.\vec{AC}}{\left|\vec{AB}\right|\left|\vec{AC}\right|} = \frac{\left(-2\sqrt{3}\,\hat{i}\right).\left(-2\sqrt{3}\hat{i}+2\hat{j}\right)}{\left(2\sqrt{3}\right).4} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta = 30^0$$

1.3 Vector Product (Cross Product)

Vector product of two vectors $\vec{A}$ and $\vec{B}$ is defined as $\boxed{\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}}$ where $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector in direction of $\left(\vec{A}\times\vec{B}\right)$. Direction of unit victor $\hat{n}$ is perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$ and it is given by right handed system.

General Properties of Vector Product:-

(i) If $\vec{A}$ and $\vec{B}$ are parallel or antiparallel i.e. collinear , then $\vec{A}\times\vec{B}=0$

$$\vec{A}\times\vec{A}=0$$

$$\hat{i}\times\hat{i}=0,\ \hat{j}\times\hat{j}=0,\ \hat{k}\times\hat{k}=0$$

(ii) $\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$ (Anti commutative law)

(iii) $\vec{A}\times\left(\vec{B}+\vec{C}\right)=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$

(iv) $\hat{i}\times\hat{j}=\hat{k},\ \ \hat{j}\times\hat{k}=\hat{i},\ \ \hat{k}\times\hat{i}=\hat{j},$

$$\hat{j}\times\hat{i}=-\hat{k},\ \ \hat{k}\times\hat{j}=-\hat{i},\ \hat{i}\times\hat{k}=-\hat{j},$$

(v) If $\vec{A}=A_1\hat{i}+A_2\hat{j}+A_3\hat{k}$ and $\vec{B}=B_1\hat{i}+B_2\hat{j}+B_3\hat{k}$,Vector product of two vectors $\vec{A}$ and $\vec{B}$ as a determinant is represented as given below

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\end{vmatrix}$$

$$\vec{A}\times\vec{B}=\hat{i}\left(A_2B_3-A_3B_2\right)-\hat{j}\left(A_1B_3-A_3B_1\right)+\hat{k}\left(A_1B_2-A_2B_1\right)$$

Typical Applications of Vector Product:-

(i) Area of the parallelogram with sides $\vec{A}$ and $\vec{B}$ is $\left|\vec{A}\times\vec{B}\right|$

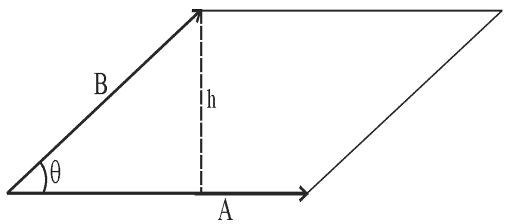

Figure 1.1

Area=Ah=AB $\sin\theta$

$$=\left|\vec{A}\times\vec{B}\right|$$

(ii) Area of the Triangle with sides $\vec{A}$ and $\vec{B}$ is $\frac{1}{2}\left|\vec{A}\times\vec{B}\right|$

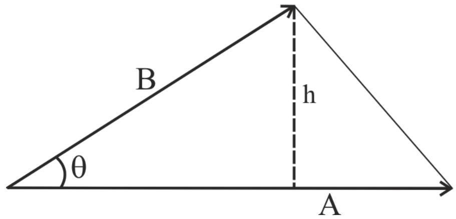

Figure 1.2

$$\text{Area} = \frac{1}{2}Ah = \frac{1}{2}AB\sin\theta = \frac{1}{2}\left|\vec{A}\times\vec{B}\right|$$

(iii) Unit vector that is perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$ is given by

$$\boxed{\hat{n} = \pm\frac{\vec{A}\times\vec{B}}{\left|\vec{A}\times\vec{B}\right|}}$$

Example1.2 A solid sphere is rotating with an angular velocity 30 r.p.m. about a fixed axis MN. Position vectors of the points M and N of the sphere are $\left(\hat{i}+2\hat{j}+3\hat{k}\right)m$ and $\left(4\hat{i}+5\hat{j}+6\hat{k}\right)m$ respectively. There is an insect at a point $\left(2\hat{i}-2\hat{j}+5\hat{k}\right)m$ on the surface of the sphere. Calculate the speed of the insect.

Sol. Angular Velocity ω =30 rev. per minute $=30\times\frac{2\pi}{60}\frac{rad}{\sec}=\pi\frac{rad}{\sec}$

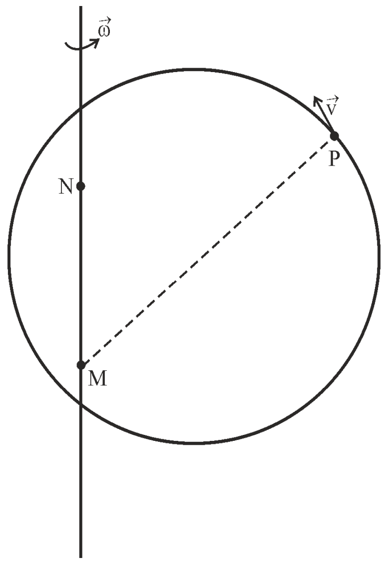

Figure 1.3

Angular velocity $\vec{\omega}$ is an axial vector. Let it be along $\vec{MN}$.

$$\vec{MN} = \vec{N} - \vec{M} = \left(4\hat{i} + 5\hat{j} + 6\hat{k}\right) - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$$

$$= 3\hat{i} + 3\hat{j} + 3\hat{k}$$

$$\frac{\vec{MN}}{\left|\vec{MN}\right|} = \frac{3\hat{i} + 3\hat{j} + 3\hat{k}}{\sqrt{3^2 + 3^2 + 3^2}} = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$$

$$\vec{\omega} = \omega.\frac{\vec{MN}}{\left|\vec{MN}\right|} = \frac{\pi}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right)$$

Position vector of the point P with respect to point M is

$$\vec{r} = \vec{P} - \vec{M}$$

$$\vec{r} = \left(2\hat{i} - 2\hat{j} + 5\hat{k}\right) - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) = \hat{i} - 4\hat{j} + 2\hat{k}$$

Linear velocity of P is

$$\vec{v} = \vec{\omega} \times \vec{r} = \frac{\pi}{\sqrt{3}}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -4 & 2 \end{vmatrix}$$

$$= \frac{\pi}{\sqrt{3}}\left[\hat{i}\{2-(-4)\} - \hat{j}(2-1) + \hat{k}(-4-1)\right]$$

$$= \frac{\pi}{\sqrt{3}}\left[6\hat{i} - \hat{j} - 5\hat{k}\right]$$

$$v = \frac{\pi}{\sqrt{3}}\sqrt{36+1+25} = \pi\sqrt{\frac{62}{3}}\,\frac{m}{\sec}$$

1.4 Scalar Triple Product $\vec{A}.\left(\vec{B} \times \vec{C}\right)$

$$\vec{A}.\left(\vec{B} \times \vec{C}\right) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

where $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$\vec{B} = B_1\hat{i} + B_2\hat{j} + B\,\hat{k}$

$\vec{C} = C_1\hat{i} + C_2\hat{j} + C_3\hat{k}$

Note that $\vec{A}.\left(\vec{B}\times\vec{C}\right)$ is scalar quantity. We can write scalar triple product $\vec{A}.\left(\vec{B}\times\vec{C}\right)$ as $\left[ABC\right]$ we read $\left[ABC\right]$ as box product.

General Properties of Scalar Triple Product:-

(i) $\vec{A}.\left(\vec{B}\times\vec{C}\right)=\vec{B}.\left(\vec{C}\times\vec{A}\right)=\vec{C}.\left(\vec{A}\times\vec{B}\right)$ i.e. $\left[ABC\right]=\left[BCA\right]=\left[CAB\right]$

(ii) $\left[\hat{i}\,\hat{j}\,\hat{k}\right]=\left[\hat{j}\,\hat{k}\,\hat{i}\right]=\left[\hat{k}\,\hat{i}\,\hat{j}\right]=1$ & $\left[\hat{i}\,\hat{k}\,\hat{j}\right]=-1$

(iii) Magnitude of the scalar triple product $\vec{A}.\left(\vec{B}\times\vec{C}\right)$ of three vectors is equal to the volume of a parallelepiped having sides $\vec{A},\vec{B}$ and $\vec{C}$

(iv) $\vec{A}.\left(\vec{A}\times\vec{C}\right)=0$ *i.e.* $\left[A\,A\,C\right]=0$

$\vec{C}.\left(\vec{A}\times\vec{A}\right)=0$ *i.e.* $\left[C\,A\,A\right]=0$

(v) If scalar triple product vanishes i.e. $\vec{A}.\left(\vec{B}\times\vec{C}\right)=0$ then $\vec{A},\vec{B}$ and $\vec{C}$ are coplanar. In that case volume of parallelepiped formed by them is zero

Note: $\left(\vec{A}.\vec{B}\right)\times\vec{C}$ is meaningless. Similarly $\left(\vec{A}.\vec{B}\right).\vec{C}$ is also meaningless.

Geometrical Interpretation of Scalar Triple Product: -

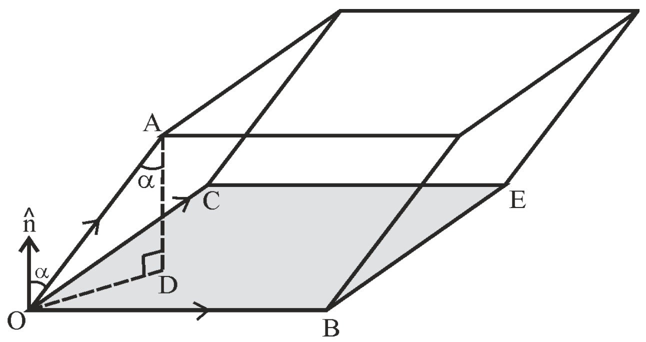

Figure 1.4

$\vec{OA}=\vec{A}$, $\vec{OB}=\vec{B}$, $\vec{OC}=\vec{C}$

$\vec{B}\times\vec{C}=B\;C\sin\theta\;\hat{n}$

$= (\text{Area of Parallelogram OBEC})\ \hat{n}$

$= S\hat{n}$

Here $\vec{A}.\hat{n} = A\,.\,1.\cos\alpha = AD = height = h$

Thus $\vec{A}.\left(\vec{B}\times\vec{C}\right) = \vec{A}.\left(S\hat{n}\right)$

$= S\,\vec{A}.\,\hat{n} = Sh$

= Volume of parallelepiped

1.5 Vector Triple Product $\vec{A}\times\left(\vec{B}\times\vec{C}\right)$

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \vec{B}\left(\vec{A}.\vec{C}\right) - \vec{C}\left(\vec{A}.\vec{B}\right)$$

Note that $\vec{A}\times\left(\vec{B}\times\vec{C}\right)$ is a vector quantity

General Properties:-

(i) $$\vec{A}\times\left(\vec{B}\times\vec{C}\right) + \vec{B}\times\left(\vec{C}\times\vec{A}\right) + \vec{C}\times\left(\vec{A}\times\vec{B}\right) = 0$$

(ii) We have in general $\vec{A}\times\left(\vec{B}\times\vec{C}\right) \neq \left(\vec{A}\times\vec{B}\right)\times\vec{C}$

1.6 Gradient $\vec{\nabla}\phi$

Vector differentiable operator $\vec{\nabla}$ is defined as $$\vec{\nabla} \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$\vec{\nabla}$ is called 'del'

Gradient: if $\phi(x, y, z)$ is a differentiable scalar field then gradient of ϕ is defined by

$$grad\phi = \vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Note that $\vec{\nabla}\phi$ is a vector field .

General Properties:-

(i) $\vec{\nabla}(\phi_1 + \phi_2) = \vec{\nabla}\phi_1 + \vec{\nabla}\phi_2$

(ii) $\vec{\nabla}(C\phi) = C\vec{\nabla}\phi$ where C is constant

(iii) If ϕ =constant then $\vec{\nabla}\phi = 0$

Geometrical Interpretation of gradient:-

"The magnitude of this $\vec{\nabla}\phi$ is equal to maximum value of rate of change of ϕ with distance."

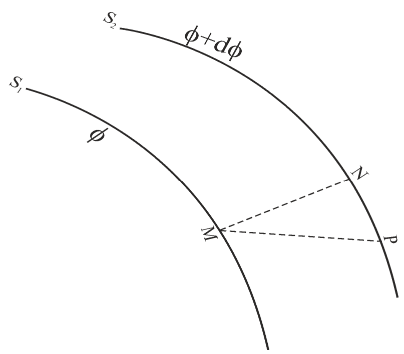

Figure 1.5

Consider a surface S_1 that has constant potential ϕ. At distance MN, there is another surface S_2 which has constant potential $\phi + d\phi$. Here MN is the shortest distance between the two surface s_1 and s_2. If we move from s_1 to s_2, then change in ϕ *is* $d\phi$.Rate of change of ϕ with distance is highest along the normal MN. So grad ϕ at point M is directed along MN.

"$\vec{\nabla}\phi$ points in the direction of maximum rate of increase of the function ϕ with space"

"For any point on the constant surface, direction of vector $\vec{\nabla}\phi$ at that point will be normal to the constant ϕ surface."

Directional Derivate:

The component of grad ϕ in the direction of vector $\vec{b}$ is equal to $\vec{\nabla}\phi.\hat{b}$, it is called directional derivate of ϕ in direction of vector $\vec{b}$

Example1.3 Scalar Potential ϕ is given by $\phi = x + y + z$ and an ellipsoid is given by $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3$

Find

(i) $\vec{\nabla}\phi$ at point (1,2,3)

(ii) Unit vector, that is normal to ellipsoid surface at the point (1,2,3)

(iii) Directional derivative of ϕ in the direction of the outward normal of the given ellipsoid at the point (1,2,3)

Sol. (i) $\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x + y + z)$

$$= \hat{i}\frac{\partial}{\partial x}(x + y + z) + \hat{j}\frac{\partial}{\partial y}(x + y + z) + \hat{k}\frac{\partial}{\partial z}(x + y + z)$$

$$= \hat{i} + \hat{j} + \hat{k}$$

which is constant vector and independent of the position of point

(ii) For ellipsoid surface $\psi(x, y, z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9} =$ constant, then $\vec{\nabla}\psi$ will be perpendicular to the surface $\psi(x, y, z)$=constant

$$\vec{\nabla}\psi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2 + \frac{y^2}{4} + \frac{z^2}{9}\right)$$

$$= \hat{i}\frac{\partial}{\partial x}\left(x^2 + \frac{y^2}{4} + \frac{z^2}{9}\right) + \hat{j}\frac{\partial}{\partial y}\left(x^2 + \frac{y^2}{4} + \frac{z^2}{9}\right) + \hat{k}\frac{\partial}{\partial z}\left(x^2 + \frac{y^2}{4} + \frac{z^2}{9}\right)$$

$$= 2x\hat{i} + \frac{y}{2}\hat{j} + \frac{2z}{9}\hat{k}$$

At the point $(1, 2, 3)$

$$\vec{\nabla}\psi = 2\hat{i} + \hat{j} + \frac{2}{3}\hat{k}$$

$$\left|\vec{\nabla}\psi\right| = \frac{7}{3}$$

Unit vector $\hat{n} = \frac{\vec{\nabla}\psi}{\left|\vec{\nabla}\psi\right|} = \frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{2}{7}\hat{k}$

Another unit vector normal to the surface is $-\left(\frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{2}{7}\hat{k}\right)$

Its direction is opposite to that above.

(iii) Required directional derivate $\nabla\phi \cdot \hat{n}$

$$= \left(\hat{i} + \hat{j} + \hat{k}\right) \cdot \left(\frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{2}{7}\hat{k}\right)$$

$$= \frac{6 + 3 + 2}{7} = \frac{11}{7}$$

Example1.4 Electric potential due to positive point charge Q is given by $V = \frac{kQ}{r}$ .Find grad V

Sol. $\vec{\nabla}V = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\frac{kQ}{r}$

$$\vec{\nabla}V = kQ\left[\hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right]$$

$$\vec{\nabla}V = kQ\left[-\frac{\hat{i}}{r^2}\left(\frac{\partial r}{\partial x}\right) - \frac{\hat{j}}{r^2}\left(\frac{\partial r}{\partial y}\right) - \frac{\hat{k}}{r^2}\left(\frac{\partial r}{\partial z}\right)\right]$$

$$\vec{\nabla}V = -\frac{kQ}{r^2}\left[\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right]$$

Here $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r^2 = x^2 + y^2 \; z^2$ Partial differentiation with respect to x gives

$$2r\frac{\partial r}{\partial x} = 2x + 0 + 0 \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

Thus $\vec{\nabla} V = -\frac{kQ}{r^2}\left[\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right]$

$$= -\frac{kQ}{r^3}\vec{r}$$

$$\vec{\nabla} V = -\frac{kQ}{r^3}\vec{r} \quad \text{where } \hat{r} = \frac{\vec{r}}{r}$$

Physical Interpretation:

Potential V decreases as r increases. Potential $V = \frac{kQ}{r}$ remains same for all points having distance r from the charge Q. Thus equipotential surfaces are spherical type and they are shown with their values of potential.

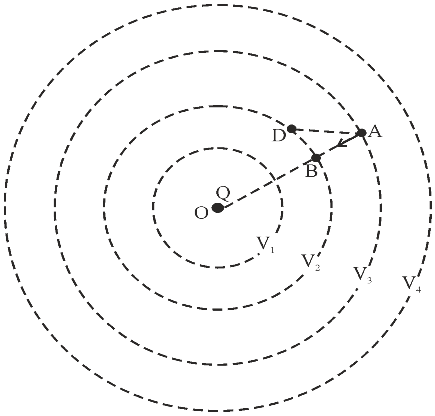

Figure 1.6

Gradient will be in the direction of increasing values of V

Here $V_2 > V_3 > V_4$ so grad$V$ at point A is directed towards V_2 surface. Rate of change of ϕ with distance is highest along the normal AB. Thus at point A, grad$V$ will be in direction of $\vec{AB}$, that is $-\hat{r}$ direction.

Let the charge Q be located at the origin O then

$$\frac{V_2 - V_3}{AB} = \frac{\frac{kQ}{OB} - \frac{kQ}{OB}}{\Delta r} = \frac{\frac{k}{r} - \frac{kQ}{(r+\Delta r)}}{\Delta r} = \frac{kQ}{r(r+\Delta r)}$$

At point A

$\left|\vec{\nabla} V\right|$ =maximum rate of change in V per unit distance

$$= \lim_{\Delta r \to 0} \left(\frac{V_2 - V_3}{AB} \right) = \frac{kQ}{r^2}$$

1.7 Self Learning Exercise-I

Very Short Answer Type Questions

Q.1 Find the angle made by vector $\vec{A} = 4\hat{i} + 4\hat{j} + 3\hat{k}$ with x axis.

Q.2 "If $\vec{A}, \vec{B}, \vec{C}$ are not null vectors and $\vec{A}.\vec{B} = \vec{A}.\vec{C}$ then $\vec{B}$ need not be equal to $\vec{C}$" give an example in favour of above statement.

Short Answer Type Questions

Q.3 If $\vec{a}_1 = \frac{a}{2}\left(-\hat{i} + \hat{j} + \hat{k}\right)$, $\vec{a}_2 = \frac{a}{2}\left(\hat{i} - \hat{j} + \hat{k}\right)$ and $\vec{a}_3 = \frac{a}{2}\left(\hat{i} + \hat{j} - \hat{k}\right)$ represent the primitive translation vectors (sides of primitive cell) of the BCC lattice then find the volume of the primitive cell. Here a is the side of conventional cell.

Q.4 Find $\vec{\nabla} r^2$

1.8 Divergence

If $\vec{F}$ is differentiable vector field then divergence of $\vec{F}$ is $\vec{\nabla}.\vec{F}$ which is defined as

$$\vec{\nabla}.\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) . \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k} \right)$$

$$\boxed{\vec{\nabla}.\vec{F} = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right)}$$

where $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

General Properties of Divergence:-

(i) $\vec{\nabla}.\left(C\vec{A} \right) = C\,\vec{\nabla}.\vec{A}$ where C is constant

(ii) $\vec{\nabla}.\left(\vec{A} + \vec{B} \right) = \vec{\nabla}.\vec{A} + \vec{\nabla}.\vec{B}$

(iii) $\vec{\nabla}.\left(\phi \vec{A}\right) = \vec{\nabla}\phi.\vec{A} + \vec{\nabla}\phi.\vec{A}$

(iv) $\vec{\nabla}.\vec{A} \neq \vec{A}.\vec{\nabla}$

Note that $\vec{A}.\vec{\nabla} = \left(A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z}\right)$ is an operator

(v) If $\vec{A}$=constant, then $\vec{\nabla}.\vec{A} = 0$

(vi) $\vec{\nabla}.(\nabla\phi) = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

where $\boxed{\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$ is Laplacian operator

Physical Interpretation of Divergence:

$$\boxed{div\,\vec{F} = \underset{\Delta V \to 0}{Lim}\frac{\oint_S \vec{F}.d\vec{S}}{\Delta V}}$$

Here volume element ΔV is bounded by the infinitesimal surface S in the neighbourhood of a point P. $\oint_S \vec{F}.d\vec{S}$ is the net out flow flux of $\vec{F}$ through infinitesimal surface S.

Thus "divergence of vector field $\vec{F}$ at the point P is equal to net outward flux per unit volume as the volume shrinks to zero in the neighbourhood of the point P."

If div $\vec{F}$ is positive, it means net flux is coming out through infinitesimal volume element at the point P and that point acts as a source.

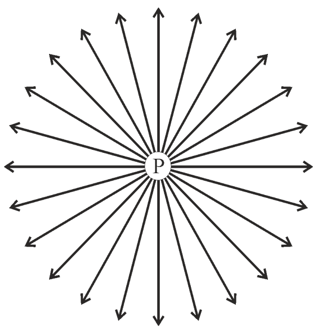

Figure 1.7

In the given figure 1.7, the point P acts as source.

Negative value of div $\vec{F}$ means net flux is going into infinitesimal volume element at the point P and that point acts as a sink.

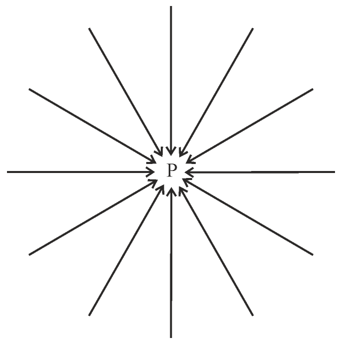

Figure 1.8

If $\vec{\nabla}.\vec{F} = 0$ *then* $\vec{F}$ *is called **Solenoidal vector**.*

Magnetic field $\vec{B}$ is a solenoid vector $\vec{\nabla}.\vec{B} = 0$ means there is neither source nor sink for field $\vec{B}$. Magnetic field lines always make closed loop. Due to that fact net out flow flux any infinitesimal volume is zero.

Example1.5 Electric field inside a uniformly charged solid sphere is $\vec{E} = \dfrac{\rho \vec{r}}{3 \in_0}$.

Find div $\vec{E}$.

Sol. $\vec{\nabla}.\vec{E} = \vec{\nabla}\left(\dfrac{\rho \vec{r}}{3 \in_0}\right) = \dfrac{\rho}{3 \in_0}\vec{\nabla}.\vec{r}$

$$= \frac{\rho}{3 \in_0}\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{\rho}{3 \in_0}\left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right]$$

$$= \frac{\rho}{3 \in_0}[3]$$

$\vec{\nabla}.\vec{E} = \dfrac{\rho}{\in_0}$ which is Gauss's in electrostatics

1.9 Curl

If $\vec{F}$ (x,y,z) is a differentiable vector field, then curl of $\vec{F}$ is defined as

$$\text{Curl } \vec{F} = \vec{\nabla}\times\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$\boxed{Curl\,\vec{F} = \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \hat{j}\left(\frac{\partial F_1}{\partial y} - \frac{\partial F_3}{\partial z}\right) + \hat{k}\left(\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial z}\right)}$$

$$\boxed{Curl\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}}$$

General Properties:-

(i) Curl $\left(\vec{A}+\vec{B}\right)$=curl $\vec{A}$+curl $\vec{B}$

(ii) $\boxed{\text{Curl Grad}\,\phi = \vec{0}}$ i.e. $\vec{\nabla}\times\vec{\nabla}\phi = \vec{0}$

(iii) If $\vec{c}$ is constant vector, then curl $\vec{c} = \vec{0}$

(iv) $\vec{\nabla}\times(\phi A) = \vec{\nabla}\phi\times\vec{A} + \phi\left(\vec{\nabla}\times\vec{A}\right)$

(v) $\boxed{\text{Div Curl}\,\vec{A} = \vec{0}}$ i.e. $\vec{\nabla}.\left(\nabla\times\vec{A}\right) = \vec{0}$

(vi) $\vec{\nabla}\times\left(\nabla\times\vec{A}\right) = \vec{\nabla}\left(\vec{\nabla}.\vec{A}\right) - \nabla^2\vec{A}$

(vii) If curl $\vec{F} = 0$ then field $\vec{F}$ is called ***irrotional field*** and line integral $\int_A^B \vec{F}.d\vec{r}$ is independent of the path joining any two points A and B. In above case, circulation $\oint \vec{F}.d\vec{r}$ zero for any closed path in that region.

(viii) If curl $\vec{F} = 0$, then three components of $\vec{F}$ are interrelated as

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

(ix) If curl $\vec{F} = 0$ it follows $\vec{F} = \vec{\nabla}\phi$ i.e. vector field $\vec{F}$ can be expressed as gradient of scalar field ϕ.

(x) If div $\vec{F} = 0$ it follows $\vec{F} = \vec{\nabla} \times \vec{A}$ $\left[\because divcurl\,\vec{A} = 0\right]$

Example1.6 A field $\vec{F}$ is given by $\vec{F} = x^2\hat{j}$ Calculate curl $\vec{F}$.

Sol.

$$Curl\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x^2 & 0 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(x^2\right)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(0)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}\left(x^2\right) - \frac{\partial}{\partial y}(0)\right\}$$

$$= 0\hat{i} + 0\hat{j} + 2x\hat{k}$$

Curl $\vec{F} = 2x\hat{k}$

Physical Interpretation:

Value of F increases with x and $\vec{F}$ is directed along positive y direction.It is obvious from figure1.9, higher value of F is represented by larger arrow. Now we calculate the line integral $\oint \vec{F}.d\vec{r}$ along closed loop ABCDA in anticlockwise direction.

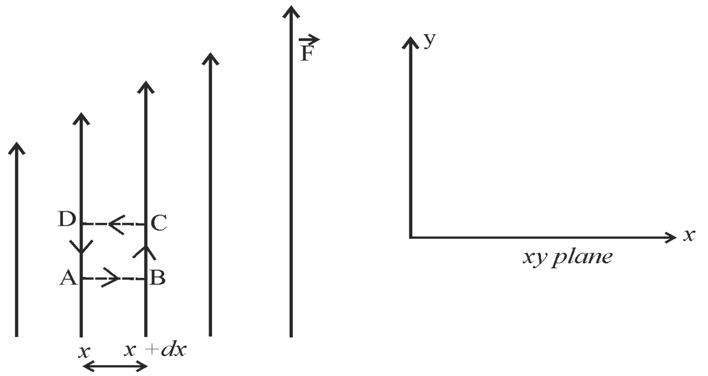

Figure 1.9

$$\oint \vec{F}.d\vec{r} = \int_{AB} \vec{F}.d\vec{r} + \int_{BC} \vec{F}.d\vec{r} + \int_{CD} \vec{F}.d\vec{r} + \int_{DA} \vec{F}.d\vec{r}$$

$$= \int_{AB} F\,dr\cos 90^0 + \int_{BC} F\,dr\cos 0^0 + \int_{CD} F\,dr\cos 90^0 + \int_{DA} F\,dr\cos 180^0$$

$$= 0 + F_{BC}(BC) + 0 + F_{DA}(-DA)$$

$$\because BC = DA = \Delta y$$

$$AB = \Delta x$$

$$\oint \vec{F}\,.\,d\vec{r} = (F_{BC} - F_{DA})\Delta y$$

$$= \left[(x+\Delta x)^2 - x^2\right]\Delta y \qquad \because F = x^2$$

$$= \left[x^2 + (\Delta x)^2 + 2x\,\Delta x - x^2\right]\Delta y$$

$$= [2x + \Delta x]\,\Delta x\Delta y$$

$$\frac{\oint \vec{F}\,.\,d\vec{r}}{\Delta x\Delta y} = 2x + \Delta x$$

$$\frac{\oint \vec{F}\,.\,d\vec{r}}{Area\ ABCD} = 2x + \Delta x$$

When $\Delta x \to 0,\ \Delta y \to 0$, Area ABCD become infinitesimally small and component of $curl\vec{F}$ along z direction is given by $\frac{\oint \vec{F}\,.\,d\vec{r}}{Area\ ABCD} \cong 2x$

We know that area vector is perpendicular to plane of area. Here, we have calculated in $\oint \vec{F}\,.\,d\vec{r}$ in xy plane and for anti-clockwise rotation, outward unit vector is $\vec{k}$ which is perpendicular to xy plane.

Thus $curl\vec{F}$ has component $2x$ along $\hat{k}$ direction. Similarly we can show that $curl\vec{F}$ do not have components along $\hat{i}$ and $\hat{j}$ directions.

Example1.7 If electrostatic field $\vec{E} = \frac{a\vec{r}}{r^3}$,then find curl $\vec{E}$ where *a* is constant

Sol. $\vec{E} = a\left(\dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{r^3}\right) = \dfrac{ax}{r^3}\hat{i} + \dfrac{ay}{r^3}\hat{j} + \dfrac{az}{r^3}\hat{k}$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{ax}{r^3} & \dfrac{ay}{r^3} & \dfrac{az}{r^3} \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}\left(\frac{az}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{ay}{r^3}\right)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}\left(\frac{az}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{ax}{r^3}\right)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}\left(\frac{ay}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{ax}{r^3}\right)\right\}$$

$$\vec{\nabla} \times \vec{E} = \hat{i}\left\{az\left(-3r^{-4}\right)\frac{\partial r}{\partial y} - ay\left(-3r^{-4}\right)\frac{\partial r}{\partial z}\right\} - \hat{j}\left\{az\left(-3r^{-4}\right)\frac{\partial r}{\partial x} - ax\left(-3r^{-4}\right)\frac{\partial r}{\partial z}\right\}$$

$$+\hat{k}\left\{ay\left(-3r^{-4}\right)\frac{\partial r}{\partial x} - ax\left(-3r^{-4}\right)\frac{\partial r}{\partial y}\right\}$$

We know that $r^2 = x^2 + y^2 + z^2$ partial differential $w.r.to\ x$ gives

$$2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{r}{x}$$

Similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$, $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$ putting these values, $\vec{\nabla} \times \vec{E}$ becomes

$$\vec{\nabla} \times \vec{E} = \hat{i}\left\{az\left(-3r^{-4}\right)\frac{y}{r} - ay\left(-3r^{-4}\right)\frac{z}{r}\right\} - \hat{j}\left\{az\left(-3r^{-4}\right)\frac{x}{r} - ax\left(-3r^{-4}\right)\frac{z}{r}\right\}$$

$$+\hat{k}\left\{ay\left(-3r^{-4}\right)\frac{x}{r} - ax\left(-3r^{-4}\right)\frac{y}{r}\right\}$$

$$= \hat{i}\left(0\right) + \hat{j}\left(0\right) + \hat{k}\left(0\right) = \vec{0}$$

Note: Here $\vec{E} = \dfrac{a\vec{r}}{r^3} = \dfrac{a}{r^2}\hat{r} = f\left(r\right)\hat{r}$ where $f\left(r\right) = \dfrac{a}{r^2}$ is function of radial distance. $\vec{\nabla} \times \vec{E} = 0 = \vec{\nabla} \times f\left(r\right)\hat{r}$

Similarly we can prove that any vector field which can be written as $f\left(r\right)\hat{r}$,curl of that field will be zero. That type of field is known as central field. So curl of central field is always zero and it is conservative in nature. Electrostatic field and gravitational field are central fields.

Example1.8 A vector field is given by $\vec{F} = f(x)\,\hat{i} + f(y)\,\hat{j} + f(z)\hat{k}$, then prove that curl $\vec{F} = 0$, where $f(x)$ is function of x only, $f(y)$ is function of y only, $f(z)$ function of z only.

Sol.

$$Curl\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(x) & f(y) & f(z) \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y} f(z) - \frac{\partial}{\partial z} f(y)\right\} - \hat{j}\left\{\frac{\partial}{\partial x} f(z) - \frac{\partial}{\partial z} f(x)\right\} + \hat{k}\left\{\frac{\partial}{\partial x} f(y) - \frac{\partial}{\partial y} f(x)\right\}$$

$$= \hat{i}\{0-0\} - \hat{j}\{0-0\} + \hat{k}\{0-0\} = \vec{0}$$

1.10 Self Learning Exercise-II

Very Short Answer Type Questions

Q.1 If $\vec{F} = (x^2+1)\hat{i} + 3y^3\hat{j} + z^{1/2}\hat{k}$ then find curl $\vec{F}$

Q.2 For particular path $\oint \vec{F}.d\vec{r} = 0$ .Does it imply $\vec{\nabla} \times \vec{F} = 0$

Short Answer Type Questions

Q.3 A vector field is $\vec{F} = \hat{i} \times \vec{r}$. Is this field solenoidal?

Q.4 Continuity equation is given by $\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0$,where $\vec{J} = ax\hat{i} + by\hat{j}$ then find ρ where *a* and *b* are constants.

1.11 Summary

(i) Scalar Triple Product $\vec{A}.(\vec{B} \times \vec{C})$

$$\vec{A}.\left(\vec{B}\times\vec{C}\right)=\begin{vmatrix}A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix}$$

(ii) Vector Triple Product $\vec{A}\times\left(\vec{B}\times\vec{C}\right)$

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{A}.\vec{C}\right)-\vec{C}\left(\vec{A}.\vec{B}\right)$$

(iii) $grad\phi=\vec{\nabla}\phi=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\phi$

(iv) $div\vec{F}=\vec{\nabla}\,.\,\vec{F}=\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)$ Where $\vec{F}=F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$

(v) Curl $\vec{F}=\vec{\nabla}\times\vec{F}=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\left(F_1\hat{i}+F_2\hat{j}+F_3\hat{k}\right)$

$$Curl\,\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}$$

1.12 Glossary

Collinear : Lying in the same straight line

Equipotential surfaces: All points on an equipotential surface have the same potential.

Ellipsoid : a geometric surface described by standard equation: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ where ±*a*, ±*b*, ±*c* are the intercepts on the *x, y,* and *z* axes.

1.13 Answer to Self Learning Exercises

Answer to Self Learning Exercise-I

Ans.1: $\cos^{-1}\frac{4}{\sqrt{41}}$

Ans.2: Let $\vec{A} = \hat{i} + \hat{j} + \hat{k}$, $\vec{B} = \hat{i} + \hat{j}$, $\vec{C} = \hat{j} + \hat{k}$

Ans.3: $\left|\vec{a}_1 \cdot \left(\vec{a}_2 \times \vec{a}_3\right)\right| = \frac{a^3}{2}$

Ans.4: $2\vec{r}$

Answer to Self Learning Exercise-II

Ans.1: $Crul\ \vec{F} = \vec{0}$ since $Crul\ \vec{F} = f(x)\hat{i} + f(y)\hat{j} + f(z)\hat{k}$

Ans.2: No, if $\oint \vec{F}.d\vec{r} = 0$ for all paths then $\nabla \times \vec{F} = 0$

Ans.3: Yes, because $\nabla.\vec{F} = 0$

Ans.4: $\rho_0 - (a+b)t$ where ρ_0 is constant

1.14 Exercise

Section A : Very Short Answer Type Questions

Q.1 Diamond unit cell consists a basis in which one atom at origin 'O' and another atom at point P $\left(\frac{a}{4}, \frac{a}{4}, \frac{a}{4}\right)$. Find the angle made by $\vec{OP}$ with x, y, z axes.

Q.2 $\vec{F} = 2r\hat{r} + \frac{3a\vec{r}}{r^2}$,Is this field $\vec{F}$ irrotational?

Q.3 If $\vec{A} = x^2\hat{i} + y^2\hat{j}$,then find $Div\,\vec{A}$

SectionB : Short Answer Type Questions

Q.4 A particle is displaced from position $\vec{r}_1 = \left(2\hat{i} + 3\hat{j} + 4\hat{k}\right)$ to $\vec{r}_2 = \left(4\hat{i} + 2\hat{j} + \hat{k}\right)$ under a constant force $\vec{F} = \left(\hat{i} + \hat{j} + \hat{k}\right)$ .All units are in S.I. Calculate the work done by the force on the particle for given displacement.

Q.5 Reciprocal lattice vector of a unit cell are $\frac{2\pi}{a}\hat{i}$, $\frac{2\pi}{b}\hat{j}$, $\frac{2\pi}{c}\hat{k}$. Find the volume of the cell formed by these reciprocal lattice vectors.

Q.6 A solenoidal vector field is given by $\vec{F} = x\hat{i} - by\hat{j} + cz\hat{k}$. Find the relation between c and b.

Q.7 Position vector of a moving particle is given by $\vec{r} = \left(t\hat{i} + t^2\hat{j}\right)$. Find the areal velocity of the particle about origin at time t=2 sec. All units are in SI and areal velocity is $\frac{d\vec{A}}{dt} = \frac{1}{2}\left(\vec{r}\times\vec{v}\right)$

Section C : Long Answer Type Questions

Q.8 A particle of mass m is moving with velocity $\vec{v} = \vec{w}\times\vec{r}$, where $\vec{w}$ is constant vector. Angular momentum of the particle is $\vec{L} = m\,\vec{r}\times\left(\vec{w}\times\vec{r}\right)$ then find curl $\vec{L}$.

Q.9 Find curl of the following vector fields-

(i) $\vec{F_1} = y\,\hat{i}$ (ii) $\vec{F_2} = x\,\hat{j}$ (iii) $\vec{F_3} = \left(\vec{F_1} + \vec{F_2}\right) = y\hat{i} + x\hat{j}$

Q.10 A wire of radius 'R' carries current along positive z direction. Magnetic field inside the wire is $\vec{B} = \frac{\mu_0}{2}\left(\vec{J}\times\vec{r}\right)$ where $\vec{J} = J\hat{k}$ is uniform current density. Calculate the curl $\vec{B}$ inside the wire.

Q.11 If $\vec{v} = \vec{\omega}\times\vec{r}$ then find $\vec{\nabla}.\vec{v}$ and $\nabla\left|\vec{v}\right|$ where $\vec{\omega} = \omega\hat{k}$

1.15 Answers to Exercise

Ans.1: $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ with each axis

Ans.2: Yes ,because $\vec{F} = f(r)\hat{r}$

Ans.3: $2x + 2y$

Ans.4: $\vec{W} = \hat{F}.\left(\vec{r_2} - \vec{r_1}\right) = -2\ Joule$

Ans.5: $\dfrac{(2\pi)^3}{abc}$

Ans.6: $\vec{\nabla}.\vec{F} = 0$ gives $b - c = 1$

Ans.7: Hint $\vec{v} = \dfrac{d\vec{r}}{dt}$, $\dfrac{d\vec{A}}{dt} = 2\hat{k}$

Ans.8: $-3m\vec{v}$

Ans.9: (i) $curl\ \vec{F_1} = -\hat{k}$ (ii) $curl\ \vec{F_2} = \hat{k}$ (iii) $curl\ \left(\vec{F_1} + \vec{F_2}\right) = 0$

Note that all these three fields $\vec{F_1}, \vec{F_2}$ and $\vec{F_3}$ are not in the form of $\left[f(x)\hat{i} + f(y)\hat{j} + f(z)\hat{k}\right]$, $curl\,\vec{F_1} \neq 0$, $curl\,\vec{F_2} \neq 0$, $curl\,\vec{F_3} = 0$. Thus field which is not in the form of $\left[f(x)\hat{i} + f(y)\hat{j} + f(z)\hat{k}\right]$ its curl may be zero or may not be zero.

Ans.10: $Curl\,\vec{B} = u_0\,\vec{J}$

Ans.11: $\vec{\nabla}.\vec{v} = 0$, $\vec{\nabla}\left|\vec{v}\right| = \dfrac{x\hat{i} + y\hat{k}}{\sqrt{x^2 + y^2}}$

References and Suggested Readings

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UNIT- 2
Coordinate Systems

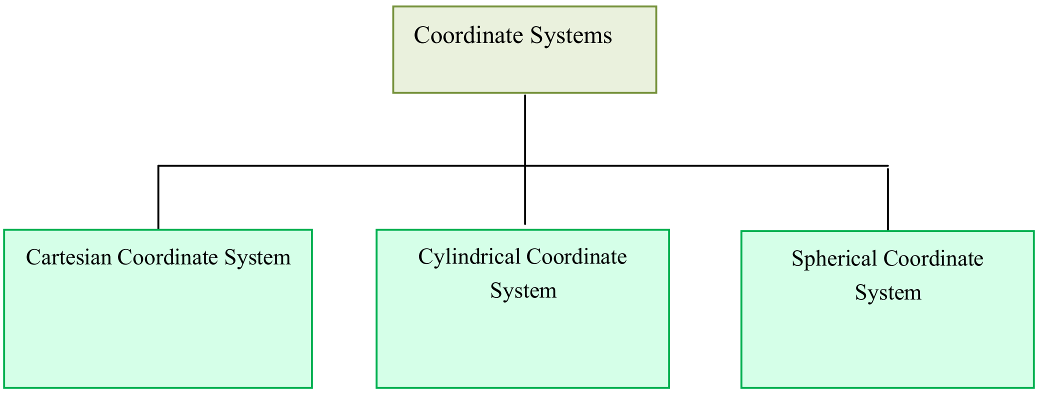
Structure of the Unit

2.0 Objectives

The chapter provides a simple formalism for expressing certain basic ideas about the coordinates system. The aim of this chapter is to enable the reader to understand the relationship and formalisms between different coordinates system. Topics have covered the algebra and differential calculus of various operations. Transformation between coordinates system is also well explained. Added feature on curvilinear coordinates system are extremely useful in the study of this chapter.

2.1 Introduction

Coordinate system is a basic idea which is used to define the position of an object in given space. The position of an object is identified by coordinates of its concerned coordinate system. These coordinates are based on measurements of position (displacements, directions, projections, angle etc.) from a given location. There are mainly three types of coordinate systems.



We are quite familiar with Cartesian coordinate system. For systems exhibiting cylindrical or spherical symmetry, it is easy to use the cylindrical and spherical coordinate systems respectively.

2.2 Cartesian or Rectangular Coordinate System

The idea of Cartesian coordinate system was invented by (and is named after) French philosopher, physicist, physiologist, and mathematician René Descartes in the 17th century. A Cartesian or Rectangular co- ordinate system usually consists of three mutually intersecting perpendicular co – ordinate axes are set – up which are labeled as X, Y and Z axes. The point where three axis (X,Y,Z) cross or intersect is called the origin (0,0,0).

In rectangular coordinates system a point P is identified by coordinates x_i, y_i, and z_i (three dimensions) where these values are all measured from the origin (see figure.1).

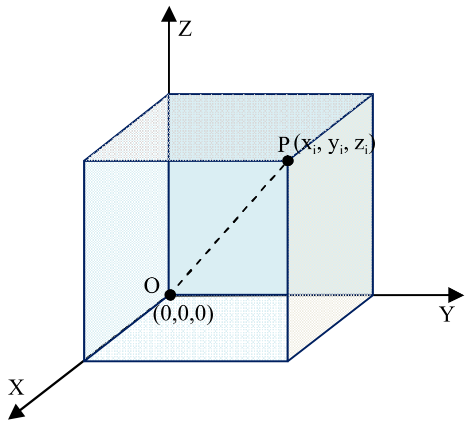


Fig.1: Location of point P in Cartesian

In a three dimensional space a point P can be identified as the intersection of three surfaces as shown in fig. When the surfaces intersect perpendicularly we have an orthogonal coordinate system. The point P (x_i, y_i, z_i) is located at the intersection of three constant surfaces i.e.,

	Ranges of Variables
x = const. (Planer Surface)	$(-\infty \leq z < \infty)$
y = const. (Planer Surface)	$(-\infty \leq z < \infty)$
z = const. (Planer Surface)	$(-\infty \leq z < \infty)$

Unit vectors $\hat{a}_x$, $\hat{a}_y$ and $\hat{a}_z$ are perpendicular to the planer surfaces.

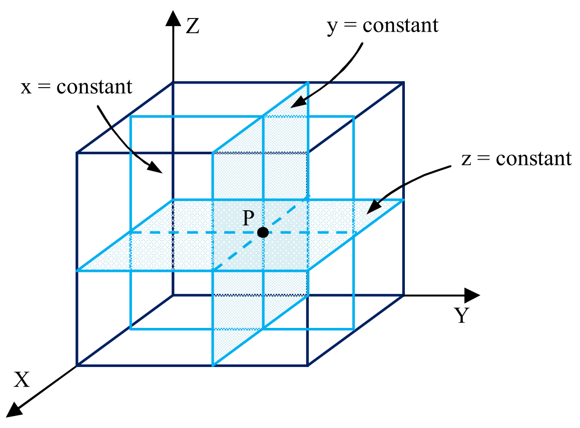


Fig.2: Location of point P in Cartesian coordinate constant surfaces

2.2.1 Differential Elements in Cartesian Coordinates

In Mathematical physics, there are three differential elements corresponding to length (l), area (A) and volume (V). These three differential elements provide an integrating function to different coordinating system.

The definition of the proper differential elements of length (*dl* for line integrals) and area (*ds* for surface integrals) can be characterized directly from the definition of the differential volume or parallelepiped (*dv* for volume integrals) in a particular coordinate system respectively.

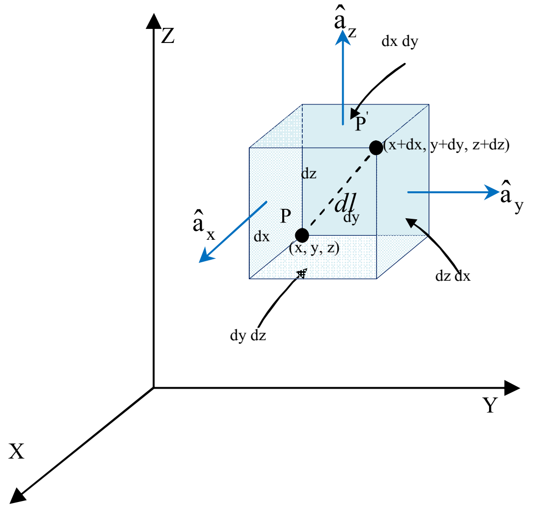


Fig.3: Constants surfaces and unit vectors in Cartesian coordinates system

In order to determine the differential element (parallelepiped) corresponding to length, area and volume in Cartesian coordinate system, let us consider a point P at the location (x,y,z). If now each coordinate's value is increased by a differential amount by (x+dx, y+dy, z+dz) as shown in fig.3, then

(1)Differential Length Element

$$\boxed{d\vec{l} = \hat{a}_x dx + \hat{a}_y dy + \hat{a}_z dz}$$

The distance between two points P and P' (diagonal length).

$$\boxed{dl = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}}$$

(2) Differential surface area element

$$d\vec{s}_x = dy\,dz\,\hat{a}_x$$
$$d\vec{s}_y = dx\,dz\,\hat{a}_y$$
$$d\vec{s}_z = dx\,dy\,\hat{a}_z$$

or

$$ds_x = dy\,dz$$
$$ds_y = dx\,dz$$
$$ds_z = dx\,dy$$

(3) Differential volume element $dv = dx\,dy\,dz$

2.3 Cylindrical Coordinate System

In cylindrical coordinates system a point P is specified by three coordinates (ρ, φ , z), where ρ represents a radial distance, φ an angular displacement (Azimuth Angle) and z an axial displacement (see figure.4).

In circular cylindrical coordinate system a point P can also be identified as intersection of three mutually perpendicular surfaces as follows:

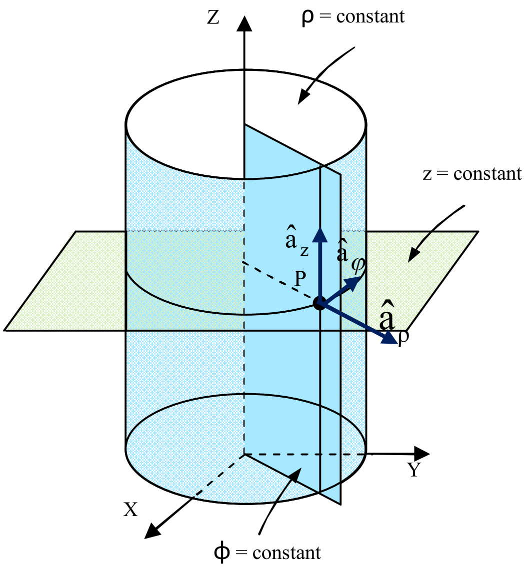


Fig.4: Constants surfaces and unit vectors in cylindrical coordinates system

	Ranges of Variables
ρ = const. (A circular cylinder);	$(0 \leq \rho < \infty)$
φ = const. (A Plane);	$(0 \leq \varphi \leq 2\pi)$
z = const. (Another plane).	$(-\infty \leq z < \infty)$

Unit vectors in cylindrical coordinates system are characterized as follow:

$\hat{a}_\rho \perp \rho = \text{const.}$ (Perpendicular to the circular cylindrical Surface)

$\hat{a}_\phi \perp \phi = \text{const.}$(Perpendicular to the planer Surface)

$\hat{a}_z \perp z = \text{const.}$(Perpendicular to the planer Surface)

2.3.1 Differential Elements in Cylindrical Coordinate System

The differential element corresponding to length, area and volume in cylindrical coordinate system can be found as shown in fig below;

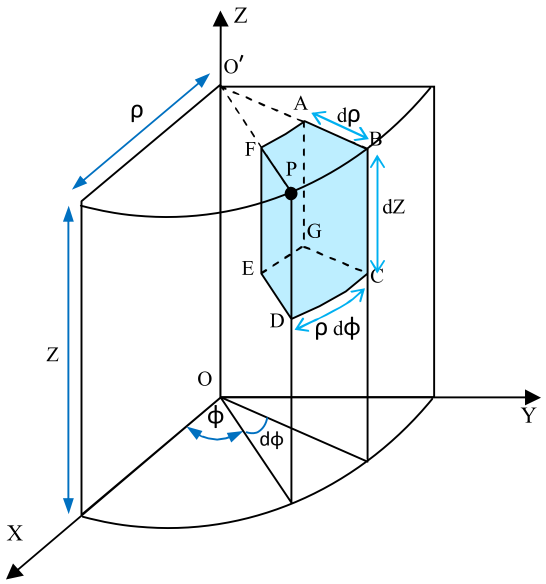


Fig.5: Differential element in cylindrical coordinates system

(1)Differential Length Element

$$d\vec{l} = \hat{a}_\rho d\rho + \hat{a}_\varphi r\, d\varphi + \hat{a}_z dz$$

(2) Differential surface area element

$$d\vec{s}_r = \rho\, d\varphi\, dz\, \hat{a}_\rho \qquad\qquad ds_r = \rho\, d\varphi\, dz$$

$$d\vec{s}_\varphi = d\rho\, dz\, \hat{a}_\varphi \qquad \text{or} \qquad ds_\varphi = d\rho\, dz$$

$$d\vec{s}_z = \rho\, d\rho\, d\varphi\, \hat{a}_z \qquad\qquad ds_z = \rho\, d\rho\, d\varphi$$

(3) Differential volume element

$$dv = \rho\, d\rho\, d\varphi\, dz$$

2.4 Spherical Coordinate System

In spherical coordinates system a point P is specified by three coordinates (r,θ,ϕ), where r represents a radius vector (distance between origin to any point), the second coordinate θ is angle between y – axis and the radius vector r. Therefore θ is also known as "Co–latitude" or "***Polar Angle***". The third coordinate φ an angular displacement between z – axis and y=0 plane. Therefore angle φ is also known as longitude or "***Azimuthal Angle***" (see figure.6).

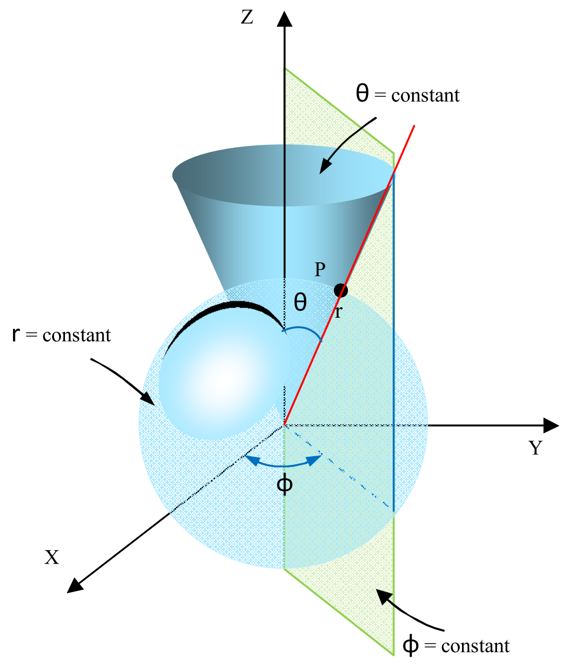


Fig.6: constant surfaces in spherical coordinate system

In spherical coordinate system any point can be identified by intersection of three mutually perpendicular surfaces i.e., a cone, a sphere and a plane.

	Ranges of Variables
r = const. (Spherical Surface)	$(0 \leq r < \infty)$
θ = const. (Conical Surface)	$(0 \leq \theta \leq \pi)$
φ = const. (Plane)	$(0 \leq \varphi \leq 2\pi)$

Unit vectors in spherical coordinates system are characterized as follow:

$\hat{a}_r \perp r = \text{const.}$ (Perpendicular to the spherical cylindrical Surface)

$\hat{a}_\theta \perp \theta = \text{const.}$(Perpendicular to the conical Surface)

$\hat{a}_\varphi \perp \varphi = \text{const.}$(Perpendicular to the planer Surface)

2.4.1 Differential Elements in Spherical Coordinate System

The differential elements corresponding to length, area and volume in cylindrical coordinate system can be found as shown in fig.5;

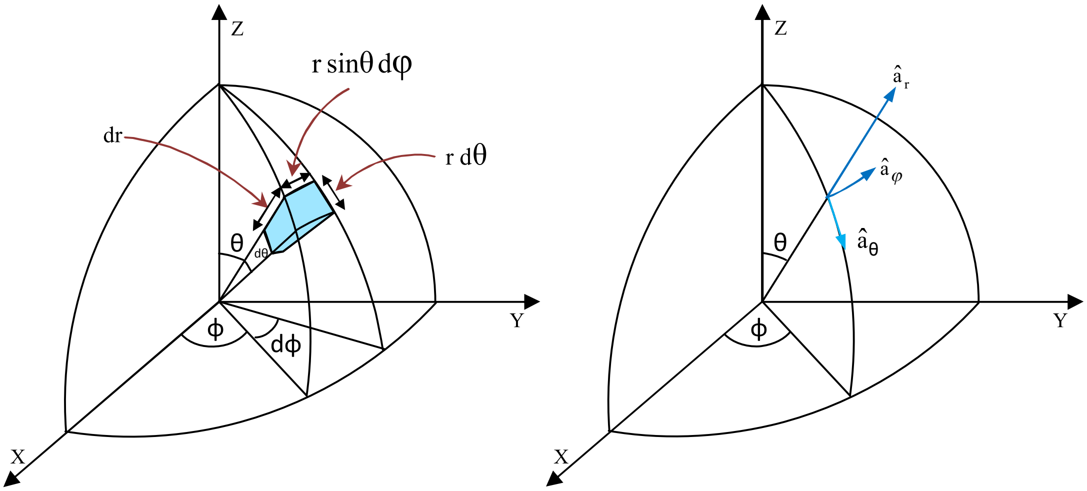


Fig.7: Differential Elements in spherical coordinate system

(1)Differential Length Element

$$\boxed{d\vec{l} = \hat{a}_r dr + \hat{a}_\theta r\, d\theta + \hat{a}_\varphi\, r \sin\theta\, d\varphi}$$

(2) Differential surface area element

$d\vec{s}_r = r^2 \sin\theta\, d\theta\, d\varphi\, \hat{a}_r$ $\qquad ds_r = r^2 \sin\theta\, d\theta\, d\varphi$

$d\vec{s}_\theta = r \sin\theta\, dr\, d\varphi\, \hat{a}_\theta$ $\qquad$ or $\qquad ds_\theta = r \sin\theta\, dr\, d\varphi$

$d\vec{s}_\varphi = r\, dr\, d\theta\, \hat{a}_\varphi$ $\qquad ds_\varphi = r\, dr\, d\theta$

(3) Differential volume element

$$\boxed{dv = r^2 \sin\theta\, dr\, d\theta\, d\varphi}$$

Example 1 Given two points $P_1$and P_2 are located with the position vectors r_1 and r_2 respectively. What is the distance between them in the following coordinate systems?

(a) Cartesian coordinate system

(b) Cylindrical coordinate system

(c) Spherical coordinate system

Sol. (a) In Cartesian coordinate system

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

(b) In Cylindrical coordinate system

$$d = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos(\varphi_2 - \varphi_1) + (z_2 - z_1)^2}$$

(c) In Spherical coordinate system

$$d = \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos\theta_1\cos\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2\cos(\varphi_2 - \varphi_2)}$$

2.5 Illustrative Examples

Example 2 Determine the volume of a sphere of radius '2a' from the differential volume element.

Sol. By eqn. $dv = r^2 \sin\theta \, dr \, d\theta \, d\varphi$

$$\text{Volume} = \int dv = \int_{r=0}^{r=2a} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\varphi$$

$$V = \int_{r=0}^{r=2a} r^2 dr \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\varphi=0}^{2\pi} d\varphi = \left.\frac{r^3}{3}\right|_0^{2a} \left.(\cos\theta)\right|_0^{\pi} \left.(\varphi)\right|_0^{2\pi} = \frac{32\pi a^3}{3}$$

Example 3 Determine the surface area of a sphere of radius 'a' from the differential volume element.

Sol. Consider an infinitesimal area element on the surface of a sphere of radius a (see fig)

The area of this element has magnitude

$$dA = (r\,d\theta)(r\sin\theta \; d\varphi) = r^2 \sin\theta \, d\theta \, d\varphi$$

$$\text{Surface Area} = \int dA = a^2 \int_{-\pi}^{\pi} \sin\theta \, d\theta \int_{\varphi=0}^{2\pi} d\varphi$$

$$A = a^2 \left.(\cos\theta)\right|_{-\pi}^{\pi} \left.(\varphi)\right|_0^{2\pi} = 4\pi a^2$$

2.6 Self Learning Exercises-I

Very Short Answer Type Questions

Q.1 What are coordinates of origin in Cartesian coordinate system?

Q.2 What is the range of azimuthal angle in cylindrical coordinate system?

Q.3 Who invented the idea of Cartesian coordinate system?

Short Answer Type Questions

Q.4 Write the differential volume element for the Cartesian coordinate system?

Q.5 Write the differential volume element for the cylindrical coordinate system?

Q.6 Write the differential volume element for the spherical coordinate system?

2.7 Transformations Between Coordinate systems

2.7.1 Transformation between Cartesian and Cylindrical coordinates system

If you are given Cylindrical coordinates (ρ, φ , z) of a point in the plane, the Cartesian coordinates (x,y,z) can be determined from the coordinate transformations and vice versa.

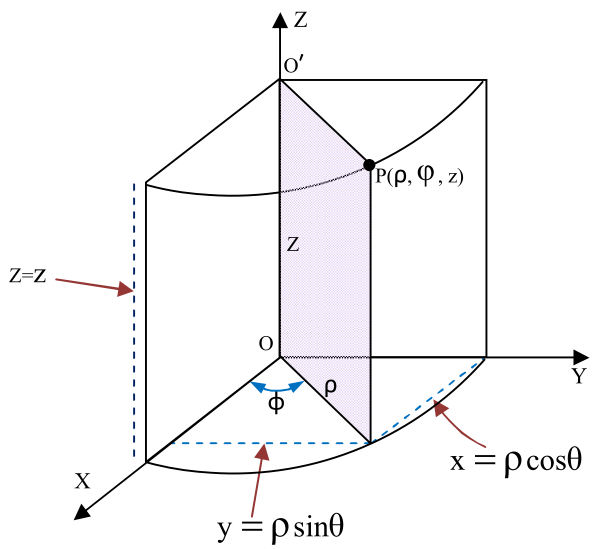


Fig.8. Transformation between cylindrical to Cartesian coordinates

Cartesian to Cylindrical	Cylindrical to Cartesian
$x = \rho\cos\varphi$	$\rho = \sqrt{x^2 + y^2}$
$y = \rho\sin\varphi$	$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$
$z = z$	$z = z$

The unit vectors also are related by the coordinate transformations

(a) Rectangular to Cylindrical unit vectors Transformation

$$(A_x, A_y, A_z) \rightarrow (A_\rho, A_\varphi, A_z)$$

The transformation of unit vectors from rectangular to cylindrical coordinates requires the components of the rectangular coordinate vector 'A' in the direction of the cylindrical coordinate unit vectors (using the dot product). The required dot products are

$$A_\rho = \vec{A}_{Cartesian}.\hat{a}_\rho = (A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z).\hat{a}_\rho = A_x\hat{a}_x.\hat{a}_\rho + A_y\hat{a}_y.\hat{a}_\rho + A_z\hat{a}_z.\hat{a}_\rho$$
$$= A_x\hat{a}_x.\hat{a}_\rho + A_y\hat{a}_y.\hat{a}_\rho \qquad (\because \hat{a}_z.\hat{a}_\rho = 0)$$
$$A_\varphi = \vec{A}_{Cartesian}.\hat{a}_\varphi = (A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z).\hat{a}_\varphi = A_x\hat{a}_x.\hat{a}_\varphi + A_y\hat{a}_y.\hat{a}_\varphi + A_z\hat{a}_z.\hat{a}_\varphi$$
$$= A_x\hat{a}_x.\hat{a}_\varphi + A_y\hat{a}_y.\hat{a}_\varphi \qquad (\because \hat{a}_z.\hat{a}_\varphi = 0)$$
$$A_z = \vec{A}_{Cartesian}.\hat{a}_z = (A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z).\hat{a}_z = A_x\hat{a}_x.\hat{a}_z + A_y\hat{a}_y.\hat{a}_z + A_z\hat{a}_z.\hat{a}_z$$
$$= A_z \qquad (\because \hat{a}_x.\hat{a}_z = \hat{a}_y.\hat{a}_z = 0; \hat{a}_z.\hat{a}_z = 1)$$

where the $\hat{a}_z$ unit vector is identical in both orthogonal coordinate systems. The four remaining unit vector dot products can be determined with the help of geometry relationships between two coordinate systems.

$$\hat{a}_\rho = \cos\varphi\,\hat{a}_x + \sin\varphi\,\hat{a}_y$$
$$\hat{a}_\varphi = -\sin\varphi\,\hat{a}_x + \cos\varphi\,\hat{a}_y$$

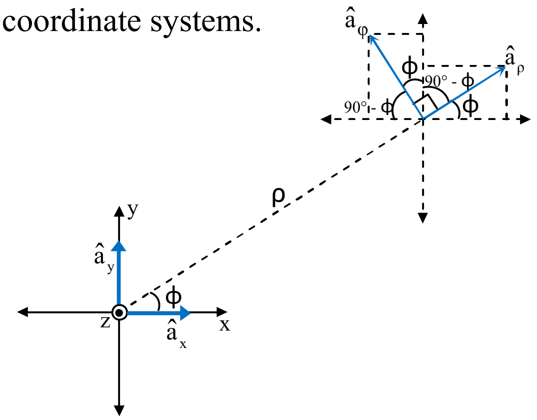


Fig.9: Transformations of cylindrical unit vectors in terms of Cartesian unit vectors

$$\hat{a}_x.\hat{a}_\rho = \hat{a}_x.(\cos\varphi\,\hat{a}_x + \sin\varphi\,\hat{a}_y) = \cos\varphi$$

$$\hat{a}_y.\hat{a}_\rho = \hat{a}_y.(\cos\varphi\,\hat{a}_x + \sin\varphi\,\hat{a}_y) = \sin\varphi$$

$$\hat{a}_x.\hat{a}_\varphi = \hat{a}_x.(-\sin\varphi\,\hat{a}_x + \cos\varphi\,\hat{a}_y) = -\sin\varphi$$

$$\hat{a}_y.\hat{a}_\varphi = \hat{a}_y.(-\sin\varphi\,\hat{a}_x + \cos\varphi\,\hat{a}_y) = \cos\varphi$$

Substituting these values in above eqn.,we have

$$A_\rho = A_x\hat{a}_x.\hat{a}_\rho + A_y\hat{a}_y.\hat{a}_\rho = A_x\cos\varphi + A_y\sin\varphi$$

$$A_\varphi = A_x\hat{a}_x.\hat{a}_\varphi + A_y\hat{a}_y.\hat{a}_\varphi = A_y\cos\varphi - A_x\sin\varphi$$

$$A_z = \hat{a}_\varphi$$

The resulting cylindrical coordinate vector is

$$\begin{aligned}\vec{A}_{Cylindrical} &= A_r\hat{a}_\rho + A_\varphi\hat{a}_\varphi + A_z\hat{a}_z \\ &= (A_x\cos\varphi + A_y\sin\varphi)\hat{a}_\rho + (A_y\cos\varphi - A_x\sin\varphi)\hat{a}_\varphi + A_z\hat{a}_z\end{aligned}$$

Also, in matrix form

$$\begin{bmatrix} A_\rho \\ A_\varphi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Similarly, the transformation from cylindrical to rectangular coordinates can be found as the inverse of the rectangular to cylindrical transformation.

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_\rho \\ A_\varphi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\varphi \\ A_z \end{bmatrix}$$

The cylindrical coordinate variables in the transformation matrix must be expressed in terms of rectangular coordinates.

$$\cos\varphi = \frac{x}{\rho} = \frac{x}{\sqrt{x^2 + y^2}} \qquad \text{and} \qquad \sin\varphi = \frac{y}{\rho} = \frac{y}{\sqrt{x^2 + y^2}}$$

The resulting transformation is

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{\sqrt{x^2+y^2}} & 0 \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\varphi \\ A_z \end{bmatrix}$$

The cylindrical to rectangular transformation can be written as

$$\vec{A}_{Cartesian} = \left(A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z\right).\hat{a}_\rho = A_x\hat{a}_x.\hat{a}_\rho + A_y\hat{a}_y.\hat{a}_\rho + A_z\hat{a}_z.\hat{a}_\rho$$

$$= (A_\rho\cos\varphi - A_\varphi\sin\varphi)\,\hat{a}_x + (A_\rho\sin\varphi + A_\varphi\cos\varphi)\,\hat{a}_y + A_z\hat{a}_z$$

$$= \left(A_\rho\frac{x}{\sqrt{x^2+y^2}} - A_\varphi\frac{y}{\sqrt{x^2+y^2}}\right)\hat{a}_x + \left(A_\rho\frac{y}{\sqrt{x^2+y^2}} + A_\varphi\frac{x}{\sqrt{x^2+y^2}}\right)\hat{a}_y + A_z\hat{a}_z$$

2.7.2 Transformation between Cartesian and Spherical coordinates system

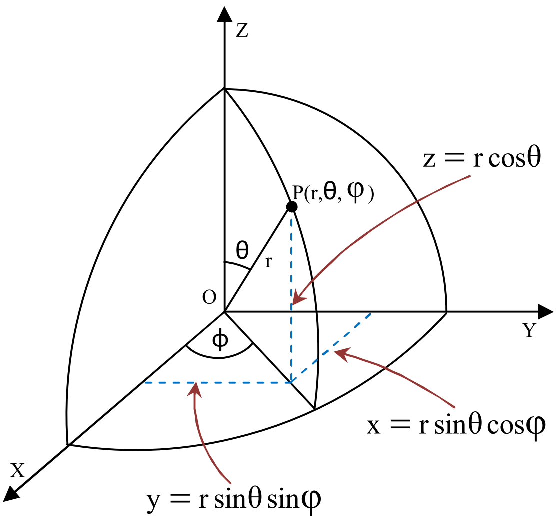


Fig.10: Transformation between Spherical to Cartesian coordinates

Cartesian to Spherical	Spherical to Cartesian
$x = r\sin\theta\cos\varphi$	$r = \sqrt{x^2+y^2+z^2}$
$y = r\sin\theta\sin\varphi$	$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$
$z = r\cos\theta$	$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$

The unit vectors transformations from Cartesian to spherical coordinates can be determined in the same way as done earlier.

(a) Cartesian to Spherical unit vectors Transformation

$$\left(A_x, A_y, A_z\right) \rightarrow \left(A_r, A_\theta, A_\varphi\right)$$

The required dot products of unit vectors to determine the transformation from rectangular coordinates to spherical coordinates are

$$A_r = \vec{A}_{Cartesian}.\hat{a}_r = \left(A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z\right).\hat{a}_r = A_x\hat{a}_x.\hat{a}_r + A_y\hat{a}_y.\hat{a}_r + A_z\hat{a}_z.\hat{a}_r$$
$$A_\theta = \vec{A}_{Cartesian}.\hat{a}_\theta = \left(A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z\right).\hat{a}_\theta = A_x\hat{a}_x.\hat{a}_\theta + A_y\hat{a}_y.\hat{a}_\theta + A_z\hat{a}_z.\hat{a}_\theta$$
$$A_\varphi = \vec{A}_{Cartesian}.\hat{a}_\varphi = \left(A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z\right).\hat{a}_\varphi = A_x\hat{a}_x.\hat{a}_\varphi + A_y\hat{a}_y.\hat{a}_\varphi + A_z\hat{a}_z.\hat{a}_\varphi$$

The unit vector dot products can be determined with the help of geometry relationships between Cartesian and spherical coordinate systems as follows;

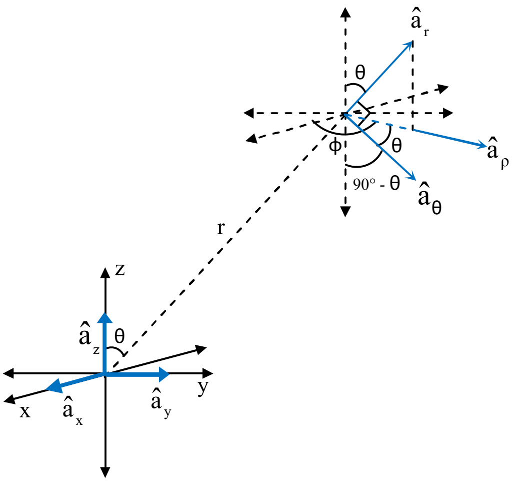


Fig.11: Transformations of Spherical unit vectors in terms of Cartesian unit vectors

Here it is considered the projection of ($\hat{a}_r, \hat{a}_\theta$) into the unit vectors ($\hat{a}_\rho, \hat{a}_z$), where $\hat{a}_\rho$ is the unit vectors taken from cylindrical coordinates (See fig),

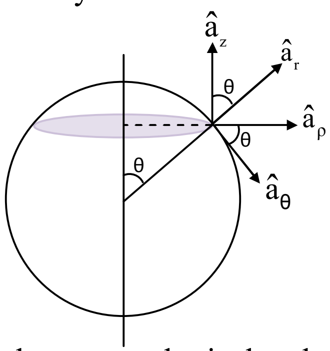


Fig.12: Relationship between spherical and cylindrical unit vectors

The vector decomposition of $\hat{a}_\rho$ in to the Cartesian unit vectors ($\hat{a}_x, \hat{a}_y$);

$$\hat{a}_\rho = \cos\varphi\,\hat{a}_x + \sin\varphi\,\hat{a}_y$$

Therefore, $\hat{a}_r. = \sin\theta\,\hat{a}_\rho + \cos\theta\,\hat{a}_z = \sin\theta\left(\cos\varphi\,\hat{a}_x + \sin\varphi\,\hat{a}_y\right) + \cos\theta\,\hat{a}_z$

$$= \sin\theta\cos\varphi\,\hat{a}_x + \sin\theta\sin\varphi\,\hat{a}_y + \cos\theta\,\hat{a}_z$$

$$\hat{a}_\theta. = \cos\theta\,\hat{a}_\rho + \cos(90^\circ - \theta)\,\hat{a}_z = \cos\theta\left(\cos\varphi\,\hat{a}_x + \sin\varphi\,\hat{a}_y\right) - \sin\theta\,\hat{a}_z$$

$$= \cos\theta\cos\varphi\,\hat{a}_x + \cos\theta\sin\varphi\,\hat{a}_y - \sin\theta\,\hat{a}_z$$

$$\hat{a}_\varphi. = -\sin\varphi\,\hat{a}_x + \cos\varphi\,\hat{a}_y$$

The dot products relationships are then

$$\hat{a}_x.\hat{a}_r = \sin\theta\cos\varphi \qquad \hat{a}_y.\hat{a}_r = \sin\theta\sin\varphi \qquad \hat{a}_z.\hat{a}_r = \cos\theta$$

$$\hat{a}_x.\hat{a}_\theta = \cos\theta\cos\varphi \qquad \hat{a}_y.\hat{a}_\theta = \cos\theta\sin\varphi \qquad \hat{a}_z.\hat{a}_\theta = -\sin\theta$$

$$\hat{a}_x.\hat{a}_\varphi = \sin\varphi \qquad \hat{a}_y.\hat{a}_\varphi = \cos\varphi\sin\varphi \qquad \hat{a}_z.\hat{a}_\varphi = 0$$

and the rectangular to spherical unit vector transformation may be written as

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

2.8 Illustrative Examples

Example 4 Deduce the Spherical to Cartesian unit vectors transformation.

Sol. The unit vector transformation from Spherical to Cartesian coordinates system can be found as the inverse of the rectangular to cylindrical transformation.

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & \sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix}$$

We can write the spherical coordinate variables in terms of the Cartesian coordinate variables.

$$\sin\theta = \frac{\rho}{r} = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \qquad \text{and} \qquad \cos\theta = \frac{z}{\rho} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\sin\varphi = \frac{y}{\rho} = \frac{y}{\sqrt{x^2+y^2}} \qquad \text{and} \qquad \cos\varphi = \frac{x}{\rho} = \frac{x}{\sqrt{x^2+y^2}}$$

The resulting transformation is

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & -\frac{y}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \frac{x}{\sqrt{x^2+y^2}} \\ \frac{z}{\sqrt{x^2+y^2+z^2}} & -\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix}$$

Example 5 Find the location of the point (1,1,1) in cylindrical and spherical coordinate systems.

Sol. (a) In Cylindrical Coordinate System.

$$r = \sqrt{x^2+y^2} = \sqrt{(1)^2+(1)^2} = \sqrt{2}\text{units}$$

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \varphi = \tan^{-1}\left(\frac{1}{1}\right) = 45^\circ \text{ and}$$

$$z = 1$$

(b) In Spherical Coordinate System.

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{(1)^2+(1)^2+(1)^2} = \sqrt{3}\text{ units}$$

$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = 54^\circ 74'$$

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \varphi = \tan^{-1}\left(\frac{1}{1}\right) = 45^\circ$$

2.9 Orthogonal Curvilinear Coordinates

Curvilinear coordinate system is simply a general way to represent all coordinate systems (Cartesian, Cylindrical and spherical etc) which may be orthogonal and nonorthogonal. Cartesian, cylindrical and Spherical coordinate systems are only special cases of generalized curvilinear coordinates system.

Let us proceed to develop a general formula in generalized curvilinear coordinate system from which the all specific coordinate system can be easily obtained by simply putting suitable parameters.

Let us consider the eqn. of surface as

$$u\,(x,y,z) = c\ (\text{ constant}) \qquad \ldots (1)$$

This eqn (1) represents the surface in space. It is well known that intersection of two surfaces is line i.e., the system of two surfaces $u_1 = c_1$ and $u_2 = c_2$ represent a line where the two surfaces intersect. Intersection of three surfaces is a point in space i.e., the system of three surfaces $u_1 = c_1$, $u_2 = c_2$ and $u_3 = c_3$ represent a point where the three surfaces intersect.

Therefore, in generalized coordinate system, three family of surfaces, described by $u_1=c_1$ (constant), $u_2=c_2$ (constant), $u_3=c_3$ (constant) which intersect at point P. Consider these three such surfaces

$$u_1(x,y,z) = c_1 \quad ; \quad u_2(x,y,z) = c_2 \quad \text{and} \quad u_3(x,y,z) = c_3 \qquad \ldots (2)$$

The value of u_1,u_2,u_3 for the three surfaces intersecting at P are called ***curvilinear co-ordinates*** or curvilinear surfaces. For example the u_1 coordinate curves are defined as the intersection of the coordinate surfaces u_2 =constant and u_3 =constant.

If these three coordinate surfaces intersect mutually perpendicular at every point P, then the curvilinear coordinates (u_1,u_2,u_3) are said to be *orthogonal curvilinear coordinates*.

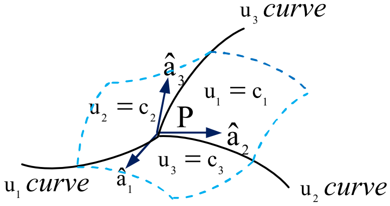


Fig.13: A General Curvilinear Coordinates system

A point P can be described by curvilinear coordinate (u_1,u_2,u_3) same as Cartesian coordinate system.

$$\left.\begin{aligned} u_1 &= u_1(x,y,z) \\ u_2 &= u_2(x,y,z) \\ u_3 &= u_3(x,y,z) \end{aligned}\right\} \qquad \ldots(3)$$

Also, we can associate a unit vector $\hat{a}_i$ normal to the surface u_i (x,y,z) = c (constant) and in the direction on increasing u_i. In generalized curvilinear coordinates system, the variables u_1, u_2, and u_3 are not measures of length directly and hence each variable should be multiplied by a general function of u_1, u_2, and u_3, in order to determine sides of the parallelepiped as shown in fig. Therefore, we define three new quantity h_1, h_2, and h_3 (function of u_1, u_2, and u3) are known as scale factors. The scale factor h_i gives the magnitude of elemental length ds when we make infinitesimal change in coordinate u_i from u_i to ui+du_i i.e., scale factor relating elemental length of the sides of parallelepiped s to coordinate increments.

Hence the elemental length of the sides of differential volume (parallelepiped) can be found as.

$$ds_i = h_i(u_1, u_2, u_3)du_i \qquad ...(4)$$

$$\boxed{ds_i = h_i du_i}$$

The infinitesimal volume element is therefore

$$dV = ds_1\, ds_2\, ds_3 = h_1 du_1\ h_2 du_2\ h_3 du_3 = h_1\, h_2\, h_3\ du_1 du_2 du_3 \qquad ...(5)$$

The scale factors and variables for three coordinate systems (Cartesian, Cylindrical and Spherical) are tabulated as

Table.1: variables, scale factors and unit vectors for three coordinate systems

S.No.	Curvilinear		Cartesian	Cylindrical	Spherical
1	u_1	Variables	x	r	r
	u_2		y	φ	θ
	u_3		z	z	φ
2	h_1	Scale Factors	1	1	1
	h_2		1	r	r
	h_3		1	1	$r\sin\theta$
3	$\hat{e}_1$	Unit Vector	$\hat{a}_x$	$\hat{a}_r$	$\hat{a}_r$
	$\hat{e}_2$		$\hat{a}_y$	$\hat{a}_\varphi$	$\hat{a}_\theta$
	$\hat{e}_3$		$\hat{a}_z$	$\hat{a}_z$	$\hat{a}_\varphi$

From above results, differential volume element for three coordinate systems (Cartesian, Cylindrical and Spherical) can be tabulated as follow:

Table.2. Differential volume element for different coordinate systems

Coordinate System	**Volume Element**
Curvilinear	$h_1 h_2 h_3 \; du_1 du_2 du_3$
Cartesian	$dx dy dz$
Cylindrical	$r\, dr\, d\varphi\, dz$
Spherical	$r^2 \sin\theta\, dr\, d\theta\, d\varphi$

2.10 Differential Vector Operators

(1) Gradient. In curvilinear co-ordinates system grad f is

$$\text{grad f} = \nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\hat{a}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\hat{a}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\hat{a}_3$$

The ‘grad*f*’ in Cartesian coordinates, we have h_1=1,h_2=1,h_3=1,u_1=x,u_2=y,u_3=z; so we have

$$\text{grad f (cartesian)} = \frac{\partial f}{\partial x}\hat{a}_x + \frac{\partial f}{\partial y}\hat{a}_y + \frac{\partial f}{\partial z}\hat{a}_z$$

Similarly, in cylindrical coordinates and spherical coordinates the gradf can be written as follows;

$$\text{grad f (cylinderical)} = \frac{\partial f}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial f}{\partial \varphi}\hat{a}_\varphi + \frac{\partial f}{\partial z}\hat{a}_z$$

$$\text{grad f (spherical)} = \frac{\partial f}{\partial u_1}\hat{a}_r + \frac{1}{r}\frac{\partial f}{\partial u_2}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial d\varphi}\hat{a}_\varphi$$

(2) Divergence. In orthogonal curvilinear co-ordinates system divA is

$$\text{div}\vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(A_1 h_2 h_3\right) + \frac{\partial}{\partial u_2}\left(A_2 h_1 h_3\right) + \frac{\partial}{\partial u_3}\left(A_3 h_1 h_2\right)\right]$$

$$\text{div}\vec{A}\text{(cartesian)} = \left[\frac{\partial}{\partial x}\left(A_x\right) + \frac{\partial}{\partial y}\left(A_y\right) + \frac{\partial}{\partial z}\left(A_z\right)\right]$$

$$\text{div}\vec{A}(\text{cylindrical}) = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(A_r r\right) + \frac{\partial}{\partial \varphi}\left(A_\varphi\right) + \frac{\partial}{\partial z}\left(A_z r\right)\right]$$

$$\text{div}\vec{A}(\text{cylindrical}) = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(A_r r^2 \sin\theta\right) + \frac{\partial}{\partial \theta}\left(A_\theta r \sin\theta\right) + \frac{\partial}{\partial \varphi}\left(A_\varphi r\right)\right]$$

(3) Curl. In orthogonal curvilinear co-ordinates system curlA is

$$\text{curl}\vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{a}_1 & h_2\hat{a}_2 & h_3\hat{a}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$$

(4) Laplacian. In orthogonal curvilinear co-ordinates system $\nabla^2 f$ is

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right]$$

Curl and Laplacian in Cartesian, cylindrical and spherical coordinate system can be obtained by substitution of suitable parameters.

2.11 Self Learning Exercises-II

Very Short Type Questions

Q.1 What is the shape of constant surface corresponding polar angle in spherical coordinate system?

Q.2 What is the shape of constant surface corresponding X-axis in Cartesian coordinate system?

Q.3 What is the shape of constant surface corresponding to coordinate ρ in cylindrical coordinate system?

Short Answer Type Questions

Q.4 Calculate the distance between two points $P_1(1,1,2)$ and $P_2(1,2,4)$ in the In Cartesian coordinate system.

Q.5 Calculate the distance between two points $P_1(1,\pi/2,2)$ and $P_2(2, 3\pi/2,4)$ in the In cylindrical coordinate system.

Q.6 Calculate the distance between two points $P_1(2, \pi/2, \pi/4)$ and $P_2(4, 3\pi/2, \pi/2)$ in the In spherical coordinate system.

2.12 Summary

- Coordinate system is used to define the position of an object in given space.
- There are mainly three types of coordinate systems.
- In Cartesian, cylindrical and spherical coordinates system a point P is identified by intersection of three mutually perpendicular surfaces as follows:

Coordinate Systems

Cartesian	Cylindrical	Spherical
x = const. (Plane)	ρ = const. (circle)	r = const. (Sphere)
y = const. (Plane)	φ = const. (Plane)	θ = const. (Cone)
z = const. (Plane)	z = const. (Plane)	φ = const. (Plane)

- The distance between two points in a coordinate system can be expressed as

Coordinate systems

Cartesian	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
Cylindrical	$d = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos(\varphi_2 - \varphi_1) + (z_2 - z_1)^2}$
Spherical	$d = \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos\theta_1\cos\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2\cos(\varphi_2 - \varphi_2)}$

- The transformation relationship between Cartesian and Cylindrical coordinates system can be expressed as

Cartesian to Cylindrical	Cylindrical to Cartesian
$x = \rho\cos\varphi$	$\rho = \sqrt{x^2 + y^2}$
$y = \rho\sin\varphi$	$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$
$z = z$	$z = z$

● The transformation relationship between Cartesian and Spherical coordinates system can be expressed as

Cartesian to Spherical	Spherical to Cartesian
$x = r\sin\theta\cos\varphi$	$r = \sqrt{x^2 + y^2 + z^2}$
$y = r\sin\theta\sin\varphi$	$\theta = \cos^{-1}\left(\dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$
$z = r\cos\theta$	$\varphi = \tan^{-1}\left(\dfrac{y}{x}\right)$

● Generalized curvilinear coordinate system is simply a general way to represents all coordinate systems (Cartesian, Cylindrical and spherical etc) which may be orthogonal and nonorthogonal.

● The scale factors, variables and unit vectors for three coordinate systems (Cartesian, Cylindrical and Spherical) can be tabulated as table.1.

● Differential volume element for different coordinate systems can be expressed as

Coordinate System	**Volume Element**
Curvilinear	$h_1 h_2 h_3 \; du_1 du_2 du_3$
Cartesian	$dx dy dz$
Cylindrical	$r\, dr\, d\varphi\, dz$
Spherical	$r^2 \sin\theta\, dr\, d\theta\, d\varphi$

● Gradient for different coordinate systems can be expressed as

Coordinate System	**Gradient**
Orthogonal Curvilinear	$\text{grad } f = \nabla f = \dfrac{1}{h_1}\dfrac{\partial f}{\partial u_1}\hat{a}_1 + \dfrac{1}{h_2}\dfrac{\partial f}{\partial u_2}\hat{a}_2 + \dfrac{1}{h_3}\dfrac{\partial f}{\partial u_3}\hat{a}_3$
Cartesian	$\text{grad } f = \dfrac{\partial f}{\partial x}\hat{a}_x + \dfrac{\partial f}{\partial y}\hat{a}_y + \dfrac{\partial f}{\partial z}\hat{a}_z$
Cylindrical	$\text{grad } f = \dfrac{\partial f}{\partial r}\hat{a}_r + \dfrac{1}{r}\dfrac{\partial f}{\partial \varphi}\hat{a}_\varphi + \dfrac{\partial f}{\partial z}\hat{a}_z$
Spherical	$\text{grad } f = \dfrac{\partial f}{\partial u_1}\hat{a}_r + \dfrac{1}{r}\dfrac{\partial f}{\partial u_2}\hat{a}_\theta + \dfrac{1}{r\sin\theta}\dfrac{\partial f}{\partial d\varphi}\hat{a}_\varphi$

● Divergence for different coordinate systems can be expressed as

Coordinate System	Divergence
Orthogonal Curvilinear	$\text{divA} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3)+\frac{\partial}{\partial u_2}(A_2h_1h_3)+\frac{\partial}{\partial u_3}(A_3h_1h_2)\right]$
Cartesian	$\text{divA} = \left[\frac{\partial}{\partial x}(A_x)+\frac{\partial}{\partial y}(A_y)+\frac{\partial}{\partial z}(A_z)\right]$
Cylindrical	$\text{divA} = \frac{1}{r}\left[\frac{\partial}{\partial r}(A_r r)+\frac{\partial}{\partial \varphi}(A_\varphi)+\frac{\partial}{\partial z}(A_z r)\right]$
Spherical	$\text{divA} = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(A_r r^2\sin\theta)+\frac{\partial}{\partial \theta}(A_\theta r\sin\theta)+\frac{\partial}{\partial \varphi}(A_\varphi r)\right]$

●In orthogonal curvilinear co-ordinates system curlA can be written as

$$\text{curlA} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{a}_1 & h_2\hat{a}_2 & h_3\hat{a}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1h_1 & A_2h_2 & A_3h_3 \end{vmatrix}$$

●In orthogonal curvilinear co-ordinates system Laplacian vector $\nabla^2 f$ can be written as

$$\nabla^2 f = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial u_1}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right]$$

2.13 Glossary

Orthogonal coordinate system: When the surfaces intersect perpendicularly we have an orthogonal coordinate system.

Curvilinear co-ordinates: The value of u_1,u_2,u_3 for the three surfaces intersecting at P are called ***curvilinear co-ordinates*** or curvilinear surfaces

2.14 Answers to Self Learning Exercises

Answers to Self Learning Exercise -I

Ans.1 : (0,0,0) **Ans.2 :** $(0 \le \varphi \le 2\pi)$

Ans.3 : René Descartes **Ans.4 :** $dv = dx\,dy\,dz$

Ans.5 : $dv = \rho\,d\rho\,d\varphi\,dz$ **Ans.6 :** $dv = r^2 \sin\theta\,dr\,d\theta\,d\varphi$

Answers to Self Learning Exercise -II

Ans.1 : Conical **Ans.2 :** Plane

Ans.3 : A circular cylinder **Ans.4 :** $\sqrt{5}$ units

Ans.5 : $\sqrt{13}$ units **Ans.6 :** $2\sqrt{5+2\sqrt{2}}$ units

2.15 Exercises

Section-A (Very Short Answer Type Questions)

Q.1 In orthogonal curvilinear coordinate system three axes are ……… to each other.

Q.2 Write the formula to determine the base vector for coordinates system?

Q.3 In which coordinate system uses two angles and one distance?

Q.4 In which coordinate system uses two distances and one angle?

Q.5 In which coordinate system uses only distance?

Section-B (Short Answer Type Questions)

Q.6 Compute the vector directed from (1,1,1) to (2,2,2) in Cartesian coordinates system.

Q.7 Determine the value of $\nabla\vec{A}$ at point (1,-1,1). If $\vec{A} = xy\,\hat{i} - x^2z\,\hat{j} + z\hat{k}$

Q.8 Find the location of the point (1,2,3) in cylindrical coordinates system.

Q.9 Write the formula of the base vectors for a coordinate system.

Q.10 Find the location of the point (1,2,3) in spherical coordinates system.

Section C (Long Answer Type Questions)

Q.11 Determine the base vectors for the cylindrical coordinate system.

Q.12 Determine the base vectors for the spherical coordinate system.

Q.13 Derive the expressions for the distance between two points in the cylindrical and spherical coordinate systems.

Q.14 Evaluate the transformation relationship between cylindrical to spherical coordinate system.

Q.15 Transform the vector $\vec{A} = xy\,\hat{i} - x^2z\,\hat{j} + z\hat{k}$ from Cartesian coordinates to cylindrical and spherical coordinate systems.

2.16 Answers to Exercise

Ans.1 : Mutually Perpendicular

Ans.2 : Plane

Ans.3: Spherical

Ans.4 : Cylindrical

Ans.5 : Cartesian

Ans.6 : $\vec{A} = \hat{a}_x + \hat{a}_y + \hat{a}_z$

Ans.7 : -1

Ans.8 : $r = \sqrt{5}$ units, $\varphi = 63^\circ 43'$, z =3

Ans.9 : $\vec{b}_i = \dfrac{\partial \vec{R}}{\partial u_i} \qquad i = 1,2,3$

Ans.10 : $r = \sqrt{14}$ units, $\theta = 0.99^\circ, \varphi = 63^\circ 43'$

References and Suggested Readings

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UNIT-3
Gauss's theorem, Stokes's theorem

Structure of the Unit

3.0 Objectives

After gone through this unit learner will able to solve any physical problem in which vector integration is used. Learner can apply Gauss divergence theorem & convert surface integral into volume integral.

3.1 Introduction

In this unit integral calculus part of vector calculus is discussed. Vector line integral, vector surface integral & volume integral are explained. Using of Gauss divergence theorem & Stoke theorem with various examples are explained.

3.2 Line Integrals

Suppose a continuous vector function $F(x, y, z)$ defined at each point of the curve C

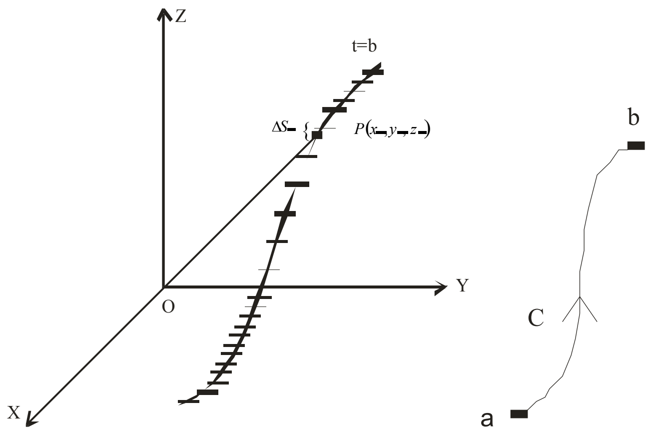
$$r(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}, \; a \le t \le b.$$

We partition the curve into a finite number of sub arcs. The typical sub arc has length Δs_k in each sub arc we choose we point (x_k, y_k, z_k) and from the sum

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\, \Delta s_k \qquad (1)$$

The sum in (1) approaches a limit as n increases, and the length Δs_k approach zero. We call this limit the **integral of F over the Curve C from a to b.**

$$\therefore \qquad \int_C f(x, y, z) = \lim_{n\to\infty} \sum_{k=1}^{n} F(x_k, y_k, z_k)\, \Delta s_k \qquad (2)$$



Let $F(r)$ be a continuous vector function, then component of $F(r)$ along the tangent at P is

$$F(r).\frac{dr}{ds} \qquad (\because \frac{dr}{ds} \text{ is unit tangent vector at P})$$

and $\int_C \left(F(r).\frac{dr}{ds} \right)$ or $\int_C f(r)\,.\,dr$

is called the tangent line integral of $F(r)$ along the curve C.

Let $F(r) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$r = x\hat{i} + y\hat{j} + z\hat{k}$

$$\therefore \quad \int_C \vec{F}.d\vec{r} = \int_C \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$= \int_C \left(F_1dx + F_2dy + F_3dz\right) = \int_C \left[F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt} \right] dt$$

$$= \int_C F(r).\frac{dr}{dt}\,dt$$

3.3 Properties of the Line Integral

Let F and G be two continuous vector point function and k is any constant, then

1. $\int_C k\vec{F}.d\vec{r} = k\int_C \vec{F}.d\vec{r}$
2. $\int_C \left(\vec{F} + \vec{G}\right).d\vec{r} = \int_C \vec{F}.d\vec{r} + \int_C \vec{G}.d\vec{r}$
3. $\int_C \vec{F}.d\vec{r} = -\int_{C_1} \vec{F}.d\vec{r}$

   Where direction of c_1 is opposite to curve C.
4. $\int_C \vec{F}.d\vec{r} = \int_{C_1} \vec{F}.d\vec{r} + \int_{C_2} \vec{F}.d\vec{r}$
5. ***If the line integral depends only on the end points of the curve, not on the path joining them then vector field is called conservative vector field***.

Let $F = \nabla\phi$; F is conservative field and ϕ is its scalar potential.

$$\int_C \vec{F}.d\vec{r} = \int_a^b \vec{F}.d\vec{r} = (\phi)_a^b$$

$$= \int_a^b \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} \right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$= \int_a^b \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right)$$

$$= \int_a^b d\phi = [\phi]_a^b = \phi(b) - \phi(a)$$

Thus, if the curve is closed, then the ***line integral of conservative vector field*** $F(r)$ ***is zero*** i.e. $\oint_C F.dr = 0$.

3.4 Application of the Line Integral

(a) Circulation: If F represent the velocity of a fluid and C is a closed curve, then the integral $\oint_C F.dr$ is called circulation of F around the curve C i.e.

$$\text{Circulation} = \oint_C F.dr$$

(b) Work done by a Force : If F represents the force acting on a particle moving along an arc AB, then the work done by the force F during the displacement from A to B is

$$\text{Work done} = \int_A^B F.dr$$

If F is a conservative vector field and ϕ is scalar potential of F, then

$$\text{Work done} = \int_A^B F.dr$$

$$= \int_A^B \nabla \phi .dr$$

$$= \phi(B) - \phi(A)$$

If curve is closed, then work done $\oint F.dr = 0$.

Example 1 Find the total work done in moving a particle in a force field given by $F = 3xy\hat{i} - 5y\hat{j} + 10x\hat{k}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$

Sol. Total work done $= \int_C F.dr = \int_C \left(3xy\hat{i} - 5z\hat{j} + 10x\hat{k}\right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$

Since $x = t^2 + 1$, $y = 2t^2$ and $z = t^3$

$$\therefore \quad dr = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$= 2tdt\hat{i} + 4tdt\hat{j} + 3t^2 dt\hat{k}$$

Now work done $= \int_1^2 \left[3\left(t^2+1\right)2t^2\hat{i} - 5t^3\hat{j} + 10\left(t^2+1\right)\hat{k}\right] . \left[2tdt\hat{i} + 4tdt\hat{j} + 3t^2dt\hat{k}\right]$

$$= \int_1^2 \left[12t^3\left(t^2+1\right) - 20t^4 + 30t^2\left(t^2+1\right)\right] dt$$

$$= \int_1^2 \left(12t^3 + 10t^4 + 12t^3 + 30t^2\right) dt$$

$$= \int_1^2 \left[12.\frac{t^6}{6} + 10.\frac{t^5}{5} + 12.\frac{t^4}{4} + 30.\frac{t^3}{3}\right]_1^2 = 303 \text{ units.}$$

Example 2 If $F = \left(2x + y^2\right)\hat{i} + \left(2y - 4x\right)\hat{j}$. Evaluate $\oint_C F.dr$ around a triangle ABC in the xy-plane with A(0, 0), B(2, 0), C(2, 1) in counter clockwise direction. What is its value in clockwise direction.

Sol. The curve C is union of three curves C_1, C_2 and C_3

$$\oint_C F.dr = \int_{C_1} F.dr + \int_{C_2} F.dr + \int_{C_3} F.dr$$

$$= I_1 + I_2 + I_3 \text{ (say)}$$

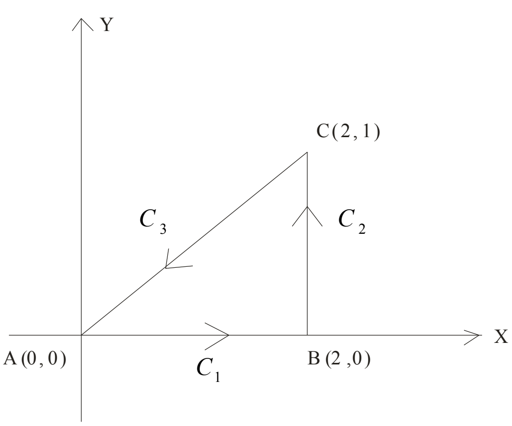
Along C_1 : Straight line AB, y=0, z=0 and x varies from 0 to 2.

$$\therefore r = xi \Rightarrow dr = dxi$$

$$I_1 = \int_C F.dr = \int_{C_1} \left[\left(2x + y\right)i + \left(3y - 4x\right)j\right] . \left(dxi\right)$$

$$= \int_{C_1} \left(2x + y^2\right)dx = \int_0^2 2xdx$$

$$= \left(x^2\right)_0^2 = 4$$



Along C_2 : The straight line BC, x=2, z=0 and y varies from 0 to 1.

$$\therefore r = 2i + yj \Rightarrow dr = dy\, j$$

Thus, $I_2 = \int_{C_2} F.dr = \int_{C_2} (3y - 4x)\, j.dy\, j$

$$= \int_0^1 (3y - 8)dy$$

$$= \left(\frac{3}{2}y^2 - 8y\right)_0^1 = \frac{3}{2} - 8 = \frac{-13}{2}$$

Along C_3 : The straight line CA, z = 0, 2y = x and x varies from 2 to 0

$$\therefore r = xi + \frac{x}{2}j + 0k \Rightarrow dr = dxi + \frac{dx}{2}j = \left(i + \frac{j}{2}\right)dx$$

Thus, $I_3 = \int_C F.dr = \int_C \left[\left(2x + y^2\right)i + (3y - 4x)\, j\right].\left(i + \frac{j}{2}\right)dx$

$$= \int_{C_3} \left\{\left(2x + y^2\right) + \frac{1}{2}(3y - 4x)\right\} dx$$

$$= \int_2^0 \left(2x + \frac{x^2}{4} + \frac{3}{4}x - \frac{4x}{2}\right)dx$$

$$= \left(\frac{x^3}{12} + \frac{3x^2}{8}\right)_2^0 = -\frac{8}{12} - \frac{12}{8} = \frac{-13}{6}$$

The required integral in counter clockwise direction is

$$\oint_C F.dr = I_1 + I_2 + I_3 = 4 - \frac{13}{2} - \frac{13}{6} = \frac{-14}{3}$$

The value of the integral in clockwise direction.

$$\int_{C_1} F.dr = -\int_C F.dr = \frac{14}{3}$$

3.5 Surface Integral

Let R is the shadow region of a surface S defined by the equation $f(x, y, z) = C$ and g is a continuous function defined at the points of S, then the integral of g over S is the integral

$$\iint_S g(x,y,z)dS = \iint_R g(x,y,z)\frac{|\nabla f|}{|\nabla f.|}dA \qquad \ldots (3)$$

Where P is a unit vector normal to R and $\nabla f.P \neq 0$. The integral itself is called a surface integral.

3.6 Surface Integral for Flux

Suppose that F is a continuous vector field defined over a two-sided surface S and $\hat{n}$ is the unit normal field on the surface. The integral of $F.\hat{n}$ over S is called the flux across S in the positive direction

$$\therefore \qquad \text{Flux} = \int\int_S F.\hat{n}dS$$

$$= \iint_R \left(F.\frac{\nabla g}{|\nabla g|}\right)\frac{|\nabla g|}{|\nabla g.P|}dA$$

$$= \iint_R F.\frac{\nabla g}{|\nabla g.P|}dA \qquad \ldots (4)$$

Example 3 : Find the flux of the vector field

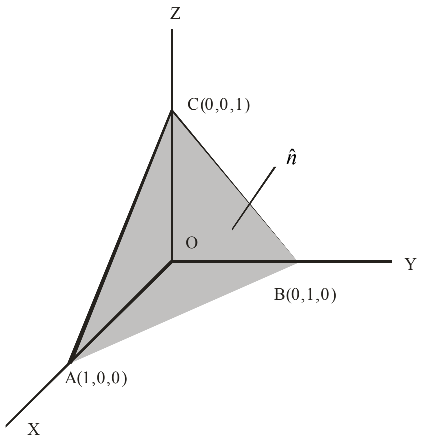
$A = (x-2z)\hat{i} + (x+3y+z)\hat{j} + (5x+y)\hat{k}$ through the upper side of the triangle ABC with vertices at the points A(1, 0, 0), B(0, 1, 0) and C(0, 0, 1).

Sol. Equation of the plane containing the give triangle ABC is

$$f(x,y,z) \equiv x+y+z = 1$$

Unit normal $\hat{n}$ to ABC is

$$\boxed{\hat{n} = \frac{\nabla f}{|\nabla f|}} = \frac{i+j+k}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}(i+j+k)$$



Flux of $A = \iint_S A.\hat{n} ds = \iint_S A.\hat{n} \frac{dA}{|\hat{n}.k|}$

$$= \iint_{AOB} [(x-2)i + (x+3y+z)j + (5x+y)k].\frac{(i+j+k)}{\sqrt{3}}.\frac{dxdy}{\frac{1}{\sqrt{3}}}$$

$$= \int_0^1 \int_0^{1-x} (x - 2z + x + 3y + z + 5x + y) dxdy$$

$$= \int_0^1 \int_0^{1-x} [7x + 4y - (1 - x - y)] dxdy$$

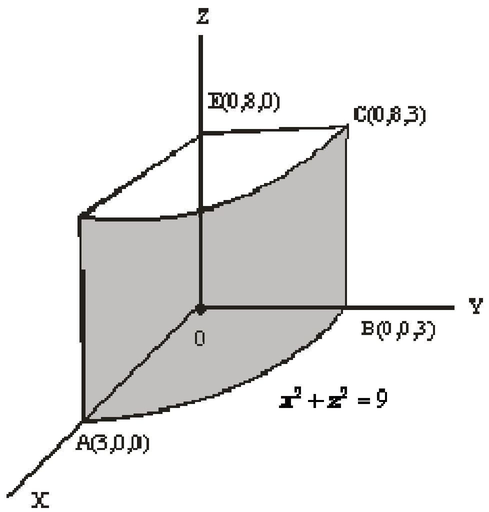
$$= \int_0^1 \left[(8x-1)y + \frac{5y^2}{2} \right]_0^{1-x} dx$$

$$= \int_0^1 \left[(8x-1)(1-x) + \frac{5}{2}(1-x^2) \right] dx$$

$$= \left[\frac{-11}{2}\frac{x^3}{3} + 2x^2 + \frac{3}{2}x \right]_0^1 = \frac{-11}{6} + 2 + \frac{3}{2} = \frac{5}{3}$$

Example 4 : Evaluate $\iint_S A.\hat{n} ds$ over the entire surface S of the region bounded by the cylinder $x^2 + z^2 = 9$, x=0, y=0, z=0 and y=8 where $A = 6zi + (2x + y)i - xk$.

Sol. Here the entire surface S consist of 5 surfaces, namely. S_1 : lateral surface of the cylinder ABCD, S_2 : AOED, S_3 : OBCE, S_4 : OAB, S_5 : CDE.



Thus, $\iint\limits_S A\,.\,\hat{n}dS = \int\int_{S_1+S_2+S_3+S_4+S_5} A\,.\,\hat{n}dS$

$$= \int\limits_{S_1}\int A\,.\,\hat{n}dS + \int\limits_{S_2}\int A\,.\,\hat{n}dS + \ldots + \int\limits_{S_5}\int A\,.\,\hat{n}dS +$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 \text{ (say)}$$

S_1 : **ABCD :** The curved surface S_1 is $f = x^2 + z^2 = 9$. The unit outward normal to S_1 is

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2xi + 2zk}{\sqrt{4x^2 + 4z^2}} = \frac{xi + zk}{3}$$

$$\therefore \quad A\,.\,\hat{n} = [6zi + (2x + y)j - zk]k\,.\frac{(xi + zj)}{3}$$

$$= \frac{1}{3}(6xz - xz) = \frac{5}{3}xz$$

$$\text{and} \quad n\,.\,\hat{k} = \frac{z}{3}$$

$$\therefore \iint\limits_{S_1} A\,.\,\hat{n}dS = \int\limits_{y=0}^{8}\int\limits_{x=0}^{3} \frac{5}{3}xz\frac{dxdy}{z/3}$$

$$= 5\int\limits_0^8\int\limits_0^3 xdxdy = \frac{5\times 9\times 8}{2} = 180$$

S_2 : **AOED :** The surface S_2 is xy-plane i.e. z=0. Unit outward normal to the surface is $\hat{n} = -k$

$$\therefore \iint\limits_{S_2} A\,.\,\hat{n}dS = \iint [6zi + (2x + y)j - xk].(-k)\frac{dxdy}{|-k\,.\,k|}$$

$$= \int\limits_{y=0}^{8}\int\limits_{x=o}^{3} x\,dxdy$$

$$= \left(\frac{x^2}{2}\right)_0^3 .(y)_0^8 = \frac{9}{2}\times 8 = 36$$

S_3 : **OBCE :** Surface S_3 is yz-plane i.e. x=0. Unit outward normal to S_3 is $\hat{n} = -i$.

$$\therefore \iint\limits_{S_3} A.\hat{n}dS = \int\limits_{z=0}^{3}\int\limits_{y=0}^{8} [6zi + (2x + y)j - xk].(-i)\frac{dydz}{|-i\,.\,i|}$$

$$= \int_0^3 \int_0^8 -6z\,dydz = -6\left(\frac{z^2}{2}\right)_0^3 (y)_0^8 = -216$$

S_4 : **OAB :** The section OAB is in xz-plane i.e. y=0. The unit outward normal to S_4 is $\hat{n} = -j$.

$$\therefore \iint_{S_4} A.\hat{n}dS = \int_0^3 \int_0^{\sqrt{9-x^2}} \left[6zi + (2x+y)j - xk\right].(-j)\frac{dxdz}{|-j.j|}$$

$$= \int_0^3 \int_0^{\sqrt{9-x^2}} -2x\,dxdz$$

$$= \int_0^3 -2x(z)_0^{\sqrt{9-x^2}}\,dx$$

$$= \int_0^3 -2x\sqrt{9-x^2}\,dx$$

$$= \left[\frac{\left(9-x^2\right)^{3/2}}{\frac{3}{2}}\right] = \frac{2}{3}(-27) = -18$$

S_5 : **CDE :** The section S_5 is parallel to xz-plane, y=8. The outward normal to S_5 is $\hat{n} = j$.

$$\iint_{S_5} A.\hat{n}dS = \int_0^3 \int_0^{\sqrt{9-x^2}} \left[6zi + (2x+y)j - xk\right].(j)\frac{dxdz}{|j.j|}$$

$$= \int_0^3 \int_0^{\sqrt{9-x^2}} (2x+8)dxdz$$

$$= \int_0^3 (2x+8)(z)_0^{\sqrt{9-x^2}}\,dx$$

$$= 2\int_0^3 (x+4)\sqrt{9-x^2}dx$$

$$= 2\left[\frac{\left(9-x^2\right)^{3/2}}{3/2.(-2)} + 4.\frac{x}{2}\sqrt{9-x^2} + 4.\frac{9}{2}sin^{-1}\frac{x}{3}\right]_0^1$$

$$= 18(1+\pi)$$

Thus, the required surface integral is

$$\iint_S A.\hat{n}dS = 180 + 36 - 216 - 18 + 18 + 18\pi = 18\pi$$

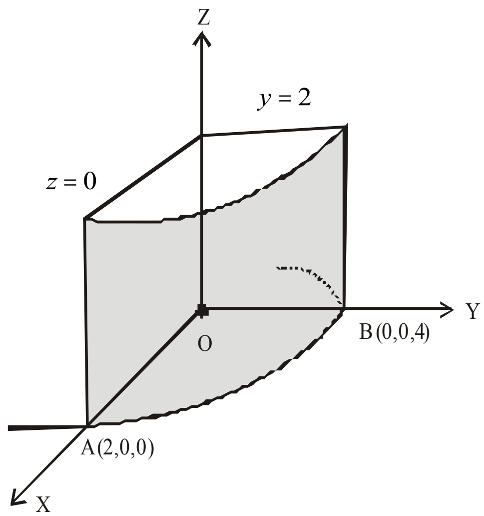
3.7 Volume Integral

Let V be a region in space enclosed by a closed surface S. Let ϕ be a scalar point function and F be a vector point function, then the triple integral

$$\iiint_V \phi\, dV \text{ and } \iiint_V F dV$$

and called volume integrals.

Example 5 : Evaluate $\iiint_V f dV$ where $f = 2x + y$, V is the closed region bounded by the cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 0$, $y = 2$ and $z = 0$.



Sol. $$\iiint_V f dV = \int_{y=0}^{2}\int_{x=0}^{2}\int_{z=0}^{4-x^2}(2x - y)dzdxdy$$

$$= \int_{y=0}^{2}\int_{x=0}^{2}(2xz + yz)_0^{4-x^2} dxdy$$

$$= \int_{y=0}^{2} \int_{x=0}^{2} \left\{2\left(4-x^2\right)x + y\left(4-x^2\right)\right\} dxdy$$

$$= \int_{y=0}^{2} \left(16-8+8y-\frac{8}{3}y\right) dy$$

$$= \left(8y+\frac{16}{3}\frac{y^2}{2}\right)_0^2 = 16+\frac{32}{3} = \frac{80}{3}$$

Example 6 : Evaluate the value of $\iiint_V div\ \vec{F}\, dV$, where $\vec{F} = 4x\hat{i} - 2\hat{j} + z^2\hat{k}$ and V is the region bounded by $x^2+y^2=4$; $z=0$ and $z=3$.

Sol. First we take

$$div\vec{F} = \nabla.\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) . \left(4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}\right)$$

Now $\iiint_V div\vec{F} dV = \iint_R \int_{z=0}^{3} (4-4y+2z)dx\, dy\, dz$

$$= \iint_R \left[4z - 4yz + z^2\right]_0^3 dx\, dy$$

$$= \iint_R \left[12(1-y)+9\right] dx\, dy$$

(Taking parametric equation of the curve x^2+y^2+4 i.e., $x = r\, cos\theta$, $y = r\, sin\theta$ $\Rightarrow$ $dx\, dy = r\, d\theta dr$)

$$= \iint_R (21-12r\, sin\theta) rdr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (21-12r\, sin\theta) rdr\, d\theta$$

$$= \int_0^{2\pi} \left[21\frac{r^2 2}{} - 4r^3\, sin\theta\right]_0^2 d\theta$$

$$= \int_0^{2\pi} (42-32\, sin\theta) d\theta$$

$$= (42\theta + 32\, cos\theta)_0^{2\pi}$$

$= 84\pi + 32 - 32 - 84\pi$

$= 0$

3.8 Self Learning Exercise

Q.1 Work done by a particle in a force field $\vec{F}$ on moving particle from point A to point B is given by..

Q.2 Write the unit normal vector to surface $x^2 + y^2 + z^2 = a^2$ at point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$.

Section – B (Short Answer type Question)

Q.3 Find the circulation of $\vec{F} = \frac{-y}{x^2 + y^2}\hat{i} + \frac{x}{x^2 + y^2}\hat{j}$ round the circle $x^2 + y^2 = 1$ in xy plane.

Q.4 Evaluate $\iint_S \vec{F}.\hat{n}dS$ where $\vec{F} = \left(x^2 + y^2\right)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ & S is surface of plane $2x + y + 2z = 6$ in first octant.

Section – C (Long Answer type Question)

Q.5 Evaluate $\iiint_V diV\vec{F}dV$ where $\vec{F} = \left(y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k}\right)$ & V is volume bounded by sphere $x^2 + y^2 + z^2 = 1$ above xy-plane.

Q.6 Evaluate $\iint_S Curl\,\vec{F}ds$ where $\vec{F} = xy\hat{i} - 2yz\hat{j} - zx\hat{k}$ & S is open surface of rectangle parallel piped formed by planes $x = 0$, $x = 1$, $y = 0$, $y = 2$ & $z = 3$ above xy-plane.

3.9 Gauss Divergence Theorem

The Gauss divergence theorem transforms double (surface) integral into volume Integral with the help of divergence of a vector point function. Gauss's divergence theorem is also known as **Ostogradsky's theorem**.

Statement : If $\vec{F}$ be a continuously differential vector point function and S is a closed, smooth and orientable surface enclosing a region V, then

$$\int_S \vec{F}.\hat{n}dS = \int_S \text{div}\,\vec{F}.dV = \int_V \left(\vec{\nabla}.\vec{F}\right).dV$$

or $$\iint_S \vec{F}.\hat{n}\,dxdy = \iiint_V \text{div}\,\vec{F}\,dxdydz,$$

where $\hat{n}$ is the unit outward drawn normal vector on the surface S.

3.10 Applications of Gauss's divergence theorem

The divergence theorem finds applications in evaluating the integrals of dot and cross products of vector fields and scalar fields.

(A) Product of a scalar function $g(x,y,z)$ and a vector field $\vec{F}(x,y,z)$

The surface integral, with respect to a surface S, of the scalar product $g\vec{F}$ is evaluated by using the following result:

$$\oiint_S g\,\vec{F}.\hat{n}ds = \iiint_V \left[\vec{F}.\left(\vec{\nabla}g\right) + g\left(\vec{\nabla}.\vec{F}\right)\right]dV$$

(B) Cross product of two vector fields $\vec{F}\times\vec{G}$:

The surface integral, with respect to a surface S, of the cross product $\vec{F}\times\vec{G}$ is evaluated by using the following result :

$$\oiint_S \left(\vec{F}\times\vec{G}\right).dS = \iiint_V \left[\vec{G}.\left(\vec{\nabla}\times\vec{F}\right) - \vec{F}.\left(\vec{\nabla}\times\vec{G}\right)\right]dV$$

(C) Product of scalar function $f(x,y,z)$ and a non zero constant vector

Following result exists for the evaluation of surface integral of product of a scalar function, f, and a non zero constant vector.

$$\oiint_S f dS = \iiint_V \vec{\nabla}f\,dV$$

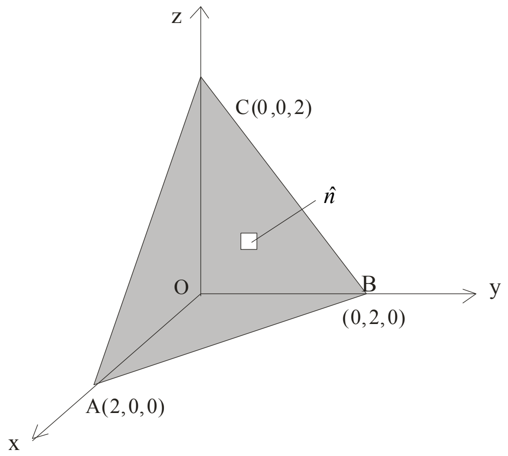
(D) Cross product of a vector field $\vec{F}$ and a non-zero constant vector

Application of divergence theorem to the cross product of a vector field $\vec{F}$ and a non-zero constant vector, gives following result:

$$\oiint_S d\vec{S}\times\vec{F} = \iiint_V \left(\vec{\nabla}\times\vec{F}\right)dV$$

Example 7 : Evaluate the surface integral $\iint_S \vec{F}.\hat{n}ds$, where $\vec{F} = (x^2 + y^2 + z^2)(\hat{i} + \hat{j} + \hat{k})$, S is the tetrahedron $x = 0$, $y = 0$, $z = 0$, $x + y + z = 2$ and $\hat{n}$ is the unit outward drawn normal to the closed surface S .

Sol. It is convenient to use Gauss's theorem for the evaluation of the integral.



By Gauss's theorem

$$\iint_S \vec{F}.\hat{n}ds = \iiint_V div\,\vec{F}.dV$$

Here $\vec{F} = (x^2 + y^2 + z^2)(\hat{i} + \hat{j} + \hat{k})$

$\therefore$ $div\,\vec{F} = \sum \frac{\partial}{\partial x}(x^2 + y^2 + z^2) = 2x + 2y + 2z$

$$\iint_S \vec{F}.\hat{n}ds = \iiint_V 2(x + y + z)dV$$

$$= \int_0^2 \int_0^{2-x} \int_0^{2-x-y} 2(x + y + z)\,dx\,dy\,dz$$

$$= 2\int_0^2 \int_0^{2-x} \left[(x + y)z + \frac{z^2}{2}\right]_0^{2-x-y} dx\,dy$$

$$= 2\int_0^2 \int_0^{2-x} \left[(x + y)(2 - x - y) + \frac{1}{2}(2 - x - y)^2\right] dx\,dy$$

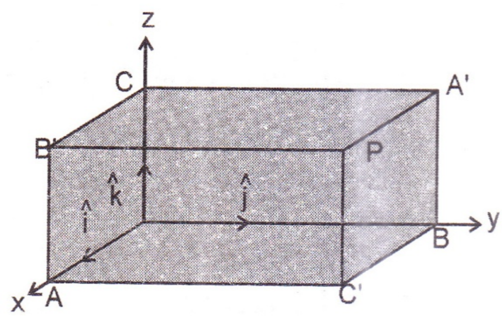
$$\iint_S \vec{F}.\hat{n}ds = 2\int_0^2 \int_0^{2-x} \left[2 - \frac{(x + y)^2}{2}\right] dx\,dy$$

$$= 2\int_0^2 \left[2y - \frac{(x+y)^3}{6}\right]_0^{2-x} dx$$

$$= 2\int_0^2 \left[2(2-x) - \frac{8}{6} + \frac{x^3}{6}\right] dx \quad = 2\left(\frac{8}{3}x - x^2 + \frac{x^4}{24}\right)_0^2 = 4$$

Example 8 : Verify divergence theorem for $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ taken over the rectangular parallelepiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

Sol.



For verification of divergence theorem, we shall evaluate the volume and surface integrals separately and show that they are equal

Now, div $\vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy) = 2(x + y + z)$

$$\therefore \qquad \iiint_V div\,\vec{F}dV = \int_0^c \int_0^b \int_0^a 2(x+y+z)dx\,dy\,dz$$

$$= \int_0^c \int_0^b 2\left[\left(\frac{x^2}{2} + yx + zx\right)\right]_0^a dydz$$

$$= \int_0^c \int_0^b 2\left(\frac{a^2}{2} + ya + za\right)dydz = \int_0^c 2\left[\left(\frac{a^2}{2}y + \frac{ay^2}{2} + azy\right)\right]_0^b dz$$

$$= 2\int_0^c \left(\frac{a^2b}{2} + \frac{ab^2}{2} + abz\right)dz = 2\left[\frac{a^2b}{2}z + \frac{ab^2}{2}z + ab\frac{z^2}{2}\right]_0^c$$

$$= a^2bc + ab^2c + abc^2 = abc(a+b+c) \qquad \ldots (1)$$

To evaluate the surface integrals, divide the closed surface S of the rectangular parallelepiped into 6 parts.

S_1 : the face $OAC'B$, S_2 : the face $OBA'C$, S_3 : the face $AC'PB'$, S_4 : the face $AC'PB'$, S_5 : the face $OCB'A$, S_6 : the face $BA'PC'$

Also

$$\iint_S \vec{F}.\hat{n}dS = \iint_{S_1} \vec{F}.\hat{n}dS + \iint_{S_2} \vec{F}.\hat{n}dS + \iint_{S_3} \vec{F}.\hat{n}dS$$
$$+ \iint_{S_4} \vec{F}.\hat{n}dS + \iint_{S_5} \vec{F}.\hat{n}dS + \iint_{S_6} \vec{F}.\hat{n}dS$$

On $S_1(z=0)$, we have $\hat{n} = -\hat{k}$, $\vec{F} = x^2\hat{i} + y^2\hat{j} - xy\hat{k}$

So that $\vec{F}.\hat{n} = \left(x^2\hat{i} + y^2\hat{j} - xy\hat{k}\right).\left(-\hat{k}\right) = xy$

$\therefore$ $\iint_S \vec{F}.\hat{n}dS = \int_0^b\int_0^a xydxdy = \int_0^b \left[y\frac{x^2}{2} \right] dy = \frac{a^2}{2}\int_0^b ydy = \frac{a^2b^2}{4}$

On $S_2(z=c)$, we have $\hat{n} = \hat{k}$, $\vec{F} = \left(x^2 - cy\right)\hat{i} + \left(y^2 - cy\right)\hat{j} + \left(c^2 - xy\right)\hat{k}$

So that $\vec{F}.\hat{n} = \left[\left(x^2 - cy\right)\hat{i} + \left(y^2 - cy\right)\hat{j} + \left(c^2 - xy\right)\hat{k}\right].\hat{k} = c^2 - xy$

$$\iint_{S_2} \vec{F}.\hat{n}dS = \int_0^b\int_0^a \left(c^2 - xy\right)dx\,dy = \int_0^b \left[c^2a - \frac{a^2}{2}y \right] dy = abc^2 - \frac{a^2b^2}{4}$$

On $S_3(x=0)$, we have $\hat{n} = -\hat{i}$, $\vec{F} = -yz\hat{i} + y^2\hat{j} + z^2\hat{k}$

So that $\vec{F}.\hat{n} = \left(-yz\hat{i} + y^2\hat{j} + z^2\hat{k}\right).\left(-\hat{i}\right) = yz$

$\therefore$ $\iint_{S_3} \vec{F}.\hat{n}dS = \int_0^c\int_0^b yz\,dy\,dz = \int_0^c \frac{b^2}{2}zdz = \frac{b^2c^2}{4}$

On $S_4(x=a)$, we have $\hat{n} = \hat{i}$, $\vec{F} = \left(a^2 - yz\right)\hat{i} + \left(y^2 - az\right)\hat{j} + \left(z^2 - ay\right)\hat{k}$

So that $\vec{F}.\hat{n} = \left[\left(a^2 - yz\right)\hat{i} + \left(y^2 - az\right)\hat{j} + \left(z^2 - ay\right)\hat{k}\right].\,\hat{i} = a^2 - yz$

$\therefore$ $\iint_{S_4} \vec{F}.\hat{n}dS = \int_0^c\int_0^b \left(a^2 - yz\right)dydz = \int_0^c \left(a^2b - \frac{b^2}{2}z \right) dz = a^2bc - \frac{b^2c^2}{4}$

On $S_5(y=0)$, we have $\hat{n} = -\hat{j}$, $\vec{F} = x^2\hat{i} - zx\hat{j} + z^2\hat{k}$

So that $\vec{F}.\hat{n} = \left(x^2\hat{i} - zx\hat{j} + z^2\hat{k}\right).\left(-\hat{j}\right) = zx$

$\therefore$ $\iint_{S_5} \vec{F}.\hat{n}dS = \int_0^a\int_0^c zx\,dz\,dx = \int_0^a \frac{c^2}{2}x\,dx = \frac{a^2c^2}{4}$

On $S_6(y=b)$, we have $\hat{n} = \hat{j}$, $\vec{F} = \left(x^2 - bz\right)\hat{i} + \left(b^2 - zx\right)\hat{j} + \left(z^2 - bx\right)\hat{k}$

So that $\quad \vec{F}.\hat{n} = \left[\left(x^2 - bz\right)\hat{i} + \left(b^2 - zx\right)\hat{j} + \left(z^2 - bx\right)\hat{k}\right].\hat{j} = b^2 - zx$

$$\therefore \quad \iint_{S_6} \vec{F}.\hat{n}dS = \int_0^a \int_0^c \left(b^2 - zx\right) dz\, dx$$

$$= \int_0^c \left(b^2 c - \frac{c^2}{2} x \right) dx = ab^2 c - \frac{a^2 c^2}{4}$$

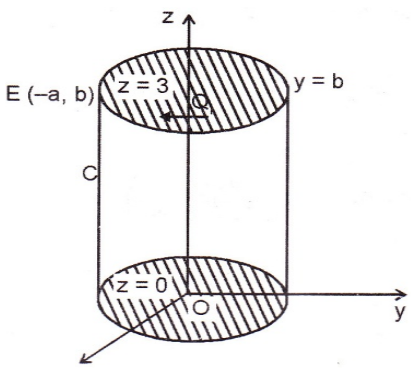
$$\therefore \quad \iint_S \vec{F}.\hat{n}dS = \frac{a^2 b^2}{4} + abc^2 + \frac{a^2 b^2}{4} + \frac{b^2 c^2}{4} + a^2 bc - \frac{b^2 c^2}{4} + \frac{a^2 c^2}{4} + ab^2 c - \frac{a^2 c^2}{4}$$

… (2)

The equality of (1) and (2) verifies divergence theorem.

Example 9 : Verify divergence theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by the cylinder $x^2 + y^2 = 4$, $z = 0$, $z = 3$.

Sol. Since $\quad$ div $\vec{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}\left(-2y^2\right) + \frac{\partial}{\partial z}\left(z^2\right) = 4 - 4y + 2z$



$$\therefore \quad \iiint_V div\,\vec{F}dV = \iiint_V (4 - 4y + 2z)\,dx\,dy\,dz$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^3 (4 - 4y + 2z)\,dz\,dy\,dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4y + z^2\right]_0^3 dy\,dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9)\,dy\,dx = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21\,dy\,dx$$

[since $12y$ is an old function $\therefore \int_{-a}^{a} 12y\,dy = 0$]

$$= \int_{-2}^{2} 42\sqrt{4-x^2}\, dx = 84\int_0^2 \sqrt{4-x^2}\, dx$$

$$= 84\left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 = 84\left[2\sin^{-1} 1\right]$$

$$= 84\left[2\times\frac{\pi}{2}\right] = 84\pi \qquad \ldots (1)$$

To evaluate the surface integral divide the closed surface S of the cylinder into 3 parts.

S_1 : the circular base in the plane $z = 0$

S_2 : the circular top in the plane $z = 3$

S_3 : the curved surface of the cylinder, given by the equation $x^2 + y^2 = 4$

Also $\iint_S \vec{F}.\hat{n}dS = \iint_{S_1} \vec{F}.\hat{n}dS + \iint_{S_2} \vec{F}.\hat{n}dS + \iint_{S_3} \vec{F}.\hat{n}dS$

On $S_1(z = 0)$, we have $\hat{n} = -\hat{k}$, $\vec{F} = 4x\hat{i} - 2y^2\hat{j}$

So that $\vec{F}.\hat{n} = \left(4x\hat{i} - 2y^2\hat{j}\right).\left(-\hat{k}\right) = 0$

$\therefore \quad \iint_{S_1} \vec{F}.\hat{n}dS = 0$

On $S_2(z = 3)$, we have $\hat{n} = \hat{k}$, $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + 9\hat{k}$

So that $\vec{F}.\hat{n} = \left(4x\hat{i} - 2y^2\hat{j} + 9\hat{k}\right).\hat{k} = 9$

$\therefore \quad \iint_{S_2} \vec{F}.\hat{n}dS = \iint_{S_2} 9\,dx\,dy = 9\iint_{S_2} dx\,dy$

$= 9\times$ area of surface $S_2 = 9\left(\pi.2^2\right) = 36\pi$

On S_3, $x^2 + y^2 = 4$

A vector normal to the surface S_3 is given by $\nabla\left(x^2 + y^2 - 4\right) = 2x\hat{i} + 2y\hat{j}$

$\therefore \quad \hat{n}$ = a unit vector normal to surface S_3

$$= \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2 + 4y^2}} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4\times 4}}, \qquad [\text{since } x^2 + y^2 = 4]$$

$$= \frac{x\hat{i} + y\hat{j}}{2}$$

$$\vec{F}.\hat{n} \quad = \left(4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}\right).\left(\frac{x\hat{i} + y\hat{j}}{2}\right) = 2x^2 - y^3$$

Also, on S_3, i.e., $x^2 + y^2 = 4$, $x = 2\cos\theta$, $y = \sin\theta$ and $dS = 2d\theta\, dz$.

To cover the whole surface S_3, z varies from 0 to 3 and θ varies from 0 to 2π.

$$\therefore \qquad \iint_{S_3} \vec{F}.\hat{n}dS = \int_0^{2\pi}\int_0^3 \left[2(2\cos\theta)^2 - (2\sin\theta)^3\right]2dz\,d\theta$$

$$= \int_0^{2\pi} 16\left(\cos^2\theta - \sin^3\theta\right)\times 3d\theta$$

$$= 48\int_0^{2\pi}\left(\cos^2\theta - \sin^3\theta\right) = d\theta = 48\pi$$

$$\left(\text{since } \int_0^{2\pi} \cos^2\theta\, d\theta = \int_0^{\pi/2} \cos^2 d\theta = 4\times\frac{1}{2}\times\frac{\pi}{2} = \pi, \int_0^{2\pi} \sin^3\theta d\theta = 0\right)$$

$$\therefore \qquad \iint_S \vec{F}.\hat{n}dS = 0 + 36\pi + 48\pi = 84\pi$$

The equality of (1) and (2) verifies divergence theorem.

3.11 Stoke's Theorem

The Stoke's theorem transforms line integral into surface integral with the help of curl of a vector point function. Stoke's theorem is the vector form of Green's theorem or generalized Green's theorem.

Statement : If S is the open surface bounded by a closed curve C and $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ is any continuously differentiable vector function, then

$$\boxed{\int_C \vec{F}.d\vec{r} = \int_S Curl\,\vec{F}.\hat{n}\,ds = \int\left(\nabla\times\vec{F}\right).\hat{n}ds},$$

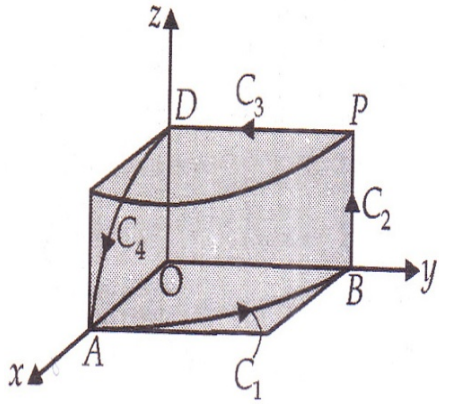
Where $\hat{n}$ is the unit outward normal drawn to the surfaces.

Example 10 : Evaluate $\iint_S \left(\nabla\times\vec{A}\right).\hat{n}\,ds$ over the surface of intersection of cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$, which is included in the first octant given that $\vec{A} = 2yz\hat{i} - (x - 3y - 2)\hat{j} + \left(x^2 + z\right)\hat{k}$.

Sol. By Stoke's theorem

$$\iint_S \left(\nabla\times\vec{A}\right).\hat{n}\,ds = \int_C \vec{A}.d\vec{r}$$

Here C is the curve consisting of four arcs namely C_1: AB, C_2 : BP, C_3 : PD, C_4 : DA. Thus, we evaluate RHS of (1), along these four arcs one by one.



Along C_1 : $z=0$, $x^2+y^2=a^2$ y varies from 0 to a.

$$\int_{C_1} \vec{A}.d\,\vec{r} = \int_{C_1}\left[2yzdx-(x+3y-2)dy+\left(x^2+z\right)dz\right]$$

$$=-\int_{C_1}(x+3y-2)dy$$

$$=-\int_0^a\left[\sqrt{a^2-y^2}+3y-2\right]dy$$

$$=-\left[\frac{y}{2}\sqrt{a^2-y^2}+\frac{a^2}{2}sin^{-1}\frac{y}{2}+\frac{3}{2}y^2-2y\right]_0^a$$

$$\int_{C_1}\vec{A}.d\,\vec{r}=-\frac{\pi a^2}{2}-\frac{3a^2}{2}+2a \qquad (2)$$

Along C_2 : $x=0$, $y=a$; $dx=0$; $dy=0$ and z varies from 0 to a

$$\therefore \quad \int_{C_2}\vec{A}.d\,\vec{r}=\int_{C_2}zdz=\int_0^a zdz=\left(\frac{z^2}{2}\right)_0^a=\frac{a^2}{2} \qquad \dots (3)$$

Along C_3 : $x=0$, $z=a$; $dx=0$; $dz=0$ and y varies from a to 0

$$\therefore \quad \int_{C_3}\vec{A}.d\,\vec{r}=\int_{C_3}(3y-2)dy=-\int_0^a(3y-2)dy$$

$$=\left[-\frac{3y^2}{2}+2y\right]_a^0=\frac{3a^2}{2}-2a \qquad \dots (4)$$

Along C_4 : $y = 0$, $x^2 + z^2 = a^2$, z varies from a to 0

$$\therefore \quad \int_{C_4} \vec{A} . d\vec{r} = \int_a^0 \left(x^2 + z\right) dz = \int_a^0 \left(a^2 - z^2 + z\right) dz$$

$$= \left(a^2 z - \frac{1}{3} z^3 + \frac{z^2}{2}\right)_a^0 = -\frac{2a^3}{3} - \frac{a^2}{2} \qquad \ldots (5)$$

Thus, the desired integral is sum of (2), (3), (4), (5)

$$\text{i.e.} \quad \iint_S \left(\nabla \times \vec{A}\right) . \hat{n}\, ds = -\frac{\pi a^2}{4} - \frac{3a^2}{2} + 2a + \frac{a^2}{2} + \frac{3a^2}{2} - 2a - \frac{2a^3}{3} - \frac{a^2}{2}$$

$$= -\frac{\pi a^2}{4} - \frac{2a^3}{3} = -\frac{a^2}{12}(3\pi + 8a) \qquad \text{Ans.}$$

Example 11 : Verify Stoke's theorem for $F = \left(x^2 + y - 5\right) i + 3xyj + \left(2xz + z^3\right) k$ over the surface of the hemisphere $x^2 + y^2 + z^2 = 16$ above the xy-plane.

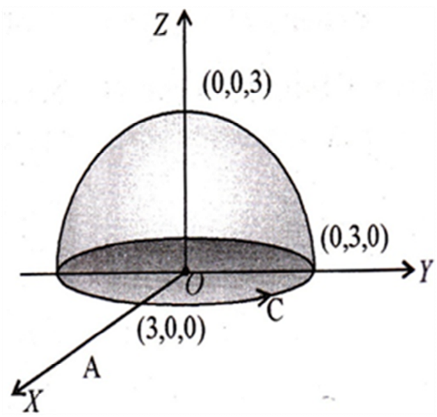
Sol. Here S is the surface $x^2 + y^2 + z^2 = 16$ and C is the boundary of the hemispherical surface and is given by $C : x^2 + y^2 = 16$.

$\therefore \; x = 4\,cos\,t$, $y = 4\,sin\,t$, $z = 0$, $0 \le t \le 2\pi$

$\Rightarrow \; r = 4\,cos\,ti + 4\,sin\,tj + 0k$

$dr = \left(-4\,sin\,ti + 4\,cos\,tj\right) dt$

and $\quad F = \left(16\,cos^2 t + 4\,sin\,t - 4\right) i + 48\,sin\,t . cos\,tj$



Stoke's theorem is

$$\oint_C F.dr = \iint_R (\nabla \times F).\hat{n}\frac{dA}{|n.k|}$$

Where R is the region in xy-plane bounded by curve C

L.H.S. of Stoke's theorem

$$= \oint_C F.dr$$

$$= \int_0^{2\pi}\left[\left(-64\,sin\,t\,cos^2 t + 16\,sin^2 t - 16\,sin\,t\right) + 192\,sin\,t\,cos^2 t\right]dt$$

$$= \int_0^{2\pi}\left\{128\,cos^2 t\,sin\,t - \frac{16(1-cos\,t)}{2} + 16\,sin\,t\right\}dt$$

$$= \left[\frac{-128\left(cos^3 t\right)}{3} - 8t + 4\,sin\,2t - 16\,cos\,t\right]_0^{2\pi} = -16\,\pi \qquad \ldots (1)$$

$$\text{Now} \quad \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y - 4 & 3xy & 2xz + z^2 \end{vmatrix}$$

$$= i(0-0) - j(2z-0) + k(3y-1)$$

$$= -2zj + (3y-1)k$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2xi + 2yj + 2zk}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{(xi + yj + zk)}{4}$$

$$= \frac{1}{4}\left[-2yz + (3y-1)z\right]$$

$$= \frac{yz - z}{4} = \frac{3}{4}(y-1)$$

$$\text{R.H.S. of Stokes theorem} = \iint_R (\nabla \times F).\hat{n}\frac{dxdy}{|n.k|}$$

$$= \iint_R \frac{z}{4}(y-1).\frac{dxdy}{z/4}$$

$$= \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (y-1)\,dydx$$

$$= \int_{-4}^{4} \left(y^2 - y\right)_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} dx$$

$$= \int_{-4}^{4} -2\sqrt{16-x^2}\, dx$$

$$= -2\left(\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2} sin^{-1}\frac{x}{4}\right)_{-4}^{4}$$

$$= 16\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$= -16\pi =$ L.H.S. Hence verified.

Example 12 If $F = \left(y^2 + z^2 + x^2\right)i + \left(z^2 + x^2 - y^2\right)j + \left(x^2 + y^2 - z^2\right)k$ Evaluate $\iint_S (\nabla \times F).\hat{n}ds$ taken over the surface $S = x^2 + y^2 - 2ax + az = 0$, $z = 0$

Sol. The given surface $S = x^2 + y^2 - 2ax + az = 0$ is bounded by the curve C : $x^2 + y^2 - 2ax = 0$, $z = 0$.

or $\quad (x-a)^2 + y^2 = a^2$, $z = 0$

or $\quad x = a + a\,cos\,t$, $y = a\,sin\,t$, $z = 0$

$$\therefore r = a(1 + cos\,t)i + a\,sin\,tj$$

$$\Rightarrow dr = (-a\,sin\,tj + a\,cos\,tj)dt$$

and

$$F = \left[a^2 sin^2 t + a^2(1+cos\,t)^3\right]i + \left(a^2(1+cos\,t)^2 - a^2 sin^2 t\right)j + 2a^2(1+cos\,t)k$$

By Stokes theorem

$$\iint_S (\nabla \times F).nds = \oint F.dr$$

$$= \int_0^{2\pi}\left[a^2\left\{sin^2 t + (1+cos\,t)^2\right\}\left(-a\,sin\,t + a^2\left\{(1+cos\,t)^2 - sin^2 t\right\}(a\,cos\,t)\right]dt$$

$$= a^3\int_0^{2\pi}\left[-2\,sin\,t(1+cos\,t)^2 + 2\left(1 - sin^2 t + cos\,t\right)cos\,t\right]dt$$

$$= 2a^3\left[\int_0^{2\pi} -\left(sin\,t + \frac{sin\,2t}{2}\right)dt + \int_0^{2\pi}\left(cos\,t + \frac{(1+cos\,t)}{2} - 2\,sin^2 t\,cos\,t\right)dt\right]$$

$$= 2a^3\left[\int_0^{2\pi} -\left(\sin t + \frac{\sin 2t}{2}\right)dt + \int_0^{2\pi}\left(\cos t + \frac{(1+\cos t)}{2} - 2\sin^2 t\cos t\right)dt\right]$$

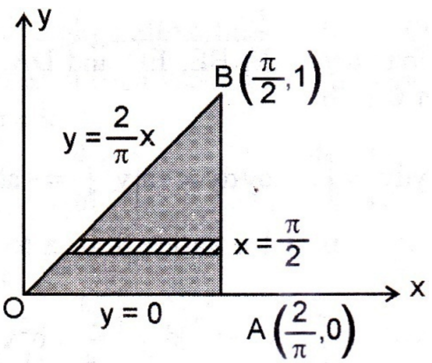
$$= 2a^2\left(\cos t + \frac{\cos 2t}{4}\right)_0^{2\pi} + \left[2a^3 \sin t + a^3\left(t + \frac{\sin 2t}{2}\right) - 2a^3\frac{\sin^3 t}{3}\right]_0^{2\pi}$$

$$= 2\pi a^3$$

Example 13 : Use Stoke's theorem to evaluate $\int_C \vec{F}.d\,\vec{r}$ where $\vec{F} = (\sin x - y)\hat{i} - \cos x\,\hat{j}$ and C is the boundary of the triangle whose vertices are $(0,0)$, $(\pi/2,\,0)$ and $(\pi/2,\,1)$.

Sol. Evaluating $\oint_C \vec{F}.d\,\vec{r}$ by using Stoke's theorem means expressing the line integral in terms of its equivalent surface integral and then evaluating the surface integral.

By Stoke's theorem $\oint_C \vec{F}.d\,\vec{r} = \iint_S crul\,\vec{F}.d\,\vec{S}$, where S is any open two-sided surface bounded by C.



To simplify the work, we shall choose S as the plane surface R in the XY-plane bounded by C.

$\therefore \oint_C \vec{F}.\vec{r} = \iint_S curel\,\vec{F}.\hat{k}\,dx\,dy$ [$\because$ for the XOY-plane , $\hat{n} = \hat{k}$ and $dS = dx\,dy$]

For this problem,

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\sin x - y0) & -\cos x & 0 \end{vmatrix} = (\sin x + 1)\hat{k}$$

$\therefore$ The given line integral

$$= \iint_R (1 + \sin x)\, dx\, dy$$

$$= \int_0^1 \int_{\pi y/2}^{\pi/2} (1 + \sin x)\, dx\, dy = \int_0^1 [x - \cos x]_{\pi y/2}^{\pi/2}$$

$$= \int_0^1 \left(\frac{\pi}{2} - \frac{\pi y}{2} + \cos \frac{\pi y}{2} \right) dy$$

$$= \left[\frac{\pi}{2} y - \frac{\pi y^2}{4} + \frac{2}{\pi} \sin \frac{\pi y}{2} \right]_0^1 = \frac{\pi}{4} + \frac{2}{\pi}$$

3.12 Summary

In this unit line integral in vector calculus is discussed. After that volume & surface integral are discussed. Conversion of line integral into surface integral & surface integral into volume integral is explained. Use of Gauss divergence theorem & Stokes theorem is discussed by using various examples.

3.13 Glossary

Vector : A physical quantity having both magnitude & directions.

Line Integral : An integral calculated along a curve.

Surface Integral : An integral calculated on the surface of any curve.

3.14 Answer to Self Learning Exercise

Ans.1 : $\int_A^B \vec{F}.d\vec{r}$

Ans.2 : $\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

Ans.3 : 2π

Ans.4 : $-\ 37$

Ans.5 : $\frac{\pi}{12}$

Ans.6 : -1

3.15 Exercise

Section – A (Very Short Answer Type Question)

Q.1 State Gauss divergence theorem.

Q.2 State Stoke's theorem.

Q.3 Write value of $\iint_S (\nabla \times \vec{F}).d\vec{S}$ in line integral.

Q.4 Write value of line integral of $\vec{F} = 2x\hat{i} + x^2\hat{j}$ along x-axis from $x = 1$ to $x = 2$.

Q.5 If C is closed curve then write value of $\oint_C \vec{F}.d\vec{r}$ where $\vec{F}$ is conservative field.

Section – B (Short Answer type Question)

Q.6 Find the work done in the force field $\vec{F} = e^{y+2z}(\hat{i} + x\hat{j} + 2x\hat{k})$ in moving the particle from $(0, 0, 0)$ to $(2, 2, 1)$.

Q.7 Evaluate $\int_C (xydx + xy^2dy)$ where C is boundary of square in xy-plane with vertices $(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, -1)$ using Stoke's theorem.

Q.8 Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F} = -y\hat{i} + x\hat{j}$ & C is boundary of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $z = 0$.

Q.9 Evaluate $\int_S \vec{F}.\hat{n}dS$ where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ & S is region bounded by $x^2 + y^2 = 4$ & plane $z = 0$ to $z = 3$.

Q.10 Evaluate $\iint_S \vec{F}.\hat{n}dS$ where $\vec{F} = \frac{\vec{r}}{r^3}$ & S is surface $x^2 + y^2 + z^2 = a^2$.

Section – C (Long Answer type Question)

Q.11 Verify Gauss divergence theorem for $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ over the surface of cube bounded by $n = 0$, $y = 0$, $z = 0$, $x = 1$, $y = 1$, $z = 1$.

Q.12 Verify Gauss divergence theorem for $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$ over the region bounded by co-ordinate planes & $2x + 2y + z = 4$.

Q.13 Verify Gauss divergence theorem for $\vec{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ over the surface of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Q.14 Verify Stoke's theorem for vector point function $\vec{F} = x^2\hat{i} + yx\hat{j}$ round the square in plane $z = 0$ whose sides are along straight lines $x = y = 0$ & $x = y = a$.

Q.15 Verify Stoke's theorem for $\vec{F} = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$ over the surface of sphere $x^2 + y^2 + z^2 = a^2$ above xy-plane.

3.16 Answers to Exercise

Ans.3 : $\int_C \vec{F}.d\vec{r}$

Ans.4 : 3

Ans.5 : 0

Ans.6 : $2e^4$

Ans.7 : $\frac{4}{3}$

Ans.8 : 0

Ans.9 : 84π

Ans.10 : 4π

References and Suggested Readings

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UNIT- 4
Tensor Analysis

Structure of the unit

4.0 Objectives

In this unit we are going to discuss about tensors and its properties.

After going through this unit you will be able to learn

- N-dimensional space
- Contravariant and covariant vectors

- Algebraic operations with tensors
- Contraction ,Direct product, Quotient rule
- Symmetric and anti-symmetric tensor, Pseudo-tensor

4.1 Introduction

The fundamental postulate of Physics is that the laws of nature are covariant. ***The meaning of covariant is that they have the same form in all reference frames***. Tensor formulation is a mathematical tool in which all the physical laws can be formulated in a covariant way.

The tensor formulation was originally given by G. Ricci and it became popular when Albert Einstein used it as a natural tool for the description of his general theory of relativity. It has became an important mathematical tool in almost every branch of theoretical physics such as Mechanics, Electrodynamics, Elasticity, Fluid mechanics etc. Tensor analysis is the generalization of vector calculus.

4.2 N-dimensional space

In three dimensional space (Cartesian system), the coordinates of a point are given by (x, y, z) where x, y, z are numbers. For the generalization of concept of space, from three dimensions to N-dimensions this representation is not suitable. An ordered set of N real variables $x^1, x^2, \ldots\ldots x^N$ can be associated with a point in space and will be called the coordinates of the point. All the points corresponding to all of the coordinates are said to form an N-dimensional space, denoted byV_N.

Transformation of Coordinates:-

The process of obtaining one set of numbers from the other is known as coordinate transformation. Consider two different N-dimensional spaces. Let us consider two sets of variables $(x^1, x^2, \ldots . x^N)$ and $(x'^1, x'^2, \ldots . x'^N)$. A transformation from $(x^1, x^2, \ldots . x^N)$ to the new set of variables $(x'^1, x'^2, \ldots . x'^N)$ through the equations

$$x'^i = f^i(x^1, x^2, \ldots . x^N) \qquad (1)$$

gives a transformation of coordinates. Here f^i is assumed to be single valued real function of the coordinates and possess continuous partial derivatives. This ensures the existence of inverse transformation and is given as

$$x^i = g^i(x'^1, x'^2, \dots . x'^N) \qquad (2)$$

The suffixes or indices i, j in A^i_j are called superscript and subscript respectively.

Concepts of Scalar, Vector and Tensor: -

Scalar: A physical quantity that can be completely described by a real number. Example:- Temperature, mass, density, potential etc. The expression of its component is independent of the choice of the coordinate system.

Vector: A physical quantity that has both direction and length. Example:- Displacement, velocity, force, heat flow etc. The expression of its component is dependent of the choice of the coordinate system.

Tensor: A tensor defines an operation that transforms a vector to another vector. A tensor contains the information about the directions and the magnitudes in those directions.

4.3 Contravariant and Covariant Vectors

Contravariant vectors:-

A set of N functions $A^i (i = 1 \dots N)$ of the N coordinates $x^i (i = 1 \dots N)$ are said to be components of a contravariant vector if they transform according to the equation

$$\boxed{A'^i = \frac{\partial x'^i}{\partial x^j} A^j} \qquad (3)$$

on change of the coordinates x^i to x'^i.

Covariant vectors:-

A set of N functions $A_i (i = 1 \dots N)$ of the N coordinates $x^i (i = 1 \dots N)$ are said to be components of a covariant vector if they transform according to the equation

$$\boxed{A'_i = \frac{\partial x^j}{\partial x'^i} A_j} \qquad (4)$$

on change of the coordinates x^i to x'^i.

Only in Cartesian coordinate

$$\frac{\partial x'^i}{\partial x^j} = \frac{\partial x^j}{\partial x'^i} \qquad \text{... (5)}$$

So that ***there is no difference between covariant and contravariant transformations in Cartesian coordinates***. In other coordinate systems, eqn. (5) in general doesn't apply and there is difference between covariant and contravariant transformations in other coordinate systems. This is important in the curved Riemannian space of general relativity.

Definitions of tensors of rank 2:-

The rank goes as the number of partial derivatives in the definition:

0 for scalar, 1 for vector, 2 for a second rank tensor and so on.

A covariant tensor A_{ij} of rank two is transformed as

$$\boxed{A'_{ij} = \sum_{kl} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} A_{kl}} \qquad (6)$$

A contravariant tensor A^{ij} of rank two is transformed as

$$\boxed{A'^{ij} = \sum_{kl} \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} A^{kl}} \qquad (7)$$

A mixed tensor is contravariant in some indices and covariant in the others. A mixed tensor A^i_j of rank two is transformed as

$$\boxed{A'^i_j = \sum_{kl} \frac{\partial x'^i}{\partial x^k} \frac{\partial x^l}{\partial x'^j} A^k_l} \qquad (8)$$

The components of a vector transform according to eqns. (6), (7), (8) yield entities that are independent of the choice of reference frame. That's why tensor analysis is important in physics.

Example 4.1 A covariant tensor has components xy , $2y - z^2, xz$ in rectangular co-ordinates. Find its covariant components in spherical co-ordinates.

Sol. Let A_j denote the covariant component in rectangular co-ordinates

$$x^1 = x, x^2 = y, x^3 = z.$$

Then
$$A_1 = xy = x^1x^2$$
$$A_2 = 2y - z^2 = 2x^2 - (x^3)^2$$
$$A_3 = xz = x^1x^3$$

Let A'_k denote the covariant component in spherical co-ordinates
$$x'^1 = r, x'^2 = \theta, x'^3 = \varphi.$$

Then
$$A'_k = \frac{\partial x^j}{\partial x'^k} A_j \qquad (1)$$

In spherical coordinates
$$x = r \sin\theta \cos\varphi$$
$$or \quad x^1 = x'^1 \sin x'^2 \cos x'^3$$
$$y = r \sin\theta \sin\varphi$$
$$or \quad x^2 = x'^1 \sin x'^2 \sin x'^3$$
$$z = r\cos\theta$$
$$or \quad x^3 = x'^1 \cos x'^2$$

Therefore equation (1) yields the covariant component.

$$A'_1 = \frac{\partial x^1}{\partial x'^1} A_1 + \frac{\partial x^2}{\partial x'^1} A_2 + \frac{\partial x^3}{\partial x'^1} A_3$$
$$= (\sin x'^2 \cos x'^3)(x^1x^2) + (\sin x'^2 \sin x'^3)(2x^2 - (x^3)^2) + \cos x'^2 (x^1x^3)$$
$$= (\sin\theta\cos\varphi)(r^2 \sin^2\theta \sin\varphi\cos\varphi) + (\sin\theta\sin\varphi)(2r\sin\theta\sin\varphi - r^2\cos^2\theta) + (\cos\theta)(r^2\sin\theta\cos\theta\cos\varphi)$$

$$A'_2 = \frac{\partial x^1}{\partial x'^2} A_1 + \frac{\partial x^2}{\partial x'^2} A_2 + \frac{\partial x^3}{\partial x'^2} A_3$$
$$= (r\cos\theta\cos\varphi)(r^2\sin^2\theta\sin\varphi\cos\varphi) + (r\cos\theta\sin\varphi)(2r\sin\theta\sin\varphi - r^2\cos^2\theta) + (-r\sin\theta)(r^2\sin\theta\cos\theta\cos\varphi)$$

$$A'_3 = \frac{\partial x^1}{\partial x'^3} A_1 + \frac{\partial x^2}{\partial x'^3} A_2 + \frac{\partial x^3}{\partial x'^3} A_3$$
$$= (-r\sin\theta\sin\varphi)(r^2\sin^2\theta\sin\varphi\cos\varphi) + (r\sin\theta\cos\varphi)(2r\sin\theta\sin\varphi - r^2\cos^2\theta) + 0.$$

4.4 Self Learning Exercise-I

Q.1 What is the rank of tensor A_{ij}^{k} ?

Q.2 Write the transformation of covariant tensor B_{ij}?

Q.3 Write one example of a mixed tensor of rank 3.

4.5 Algebraic Operations with Tensors

Kronecker Delta: -

The Kronecker delta is defined as

$$\delta_j^i = 1 \quad \text{If } i = j$$
$$\delta_j^i = 0 \quad \text{If } i \neq j$$

So by definition of Kronecker delta we can write

$$\delta_1^1 = \delta_2^2 = \delta_3^3 = \cdots = \delta_N^N = 1$$
$$\delta_2^1 = \delta_3^2 = \cdots = 0$$

The coordinates $x^1, x^2, \dots, x^N$ are independent so Kronecker delta can also be written as

$$\frac{\partial x^i}{\partial x^j} = \delta_j^i$$

Similarly we can write

$$\frac{\partial x'^i}{\partial x'^j} = \delta_j^i$$

Addition and Subtraction of Tensors:-

The sum of two or more tensors of the same rank and type (i.e. same number of contra-variant and same numbers of covariant indices) is also a tensor of the same rank and type. Thus the sum of two tensors

$$C_j^i = A_j^i + B_j^i$$

is also a tensor.

We can also subtract two tensors provided they are of the same rank and type. The difference of two tensors of same rank and type is another tensor of the same rank and type.

$$C^i_j = A^i_j - B^i_j$$

Theorem: - The sum or difference of two tensors of the same rank and type is again a tensor of the same rank and type.

Proof: - Let A^i_j and B^i_j are two tensors in coordinate system x^i and having the following transformation relations in the coordinate system x'^i

$$A'^p_q = A^i_j \frac{\partial x'^p}{\partial x^i} \frac{\partial x^j}{\partial x'^q} \qquad (1)$$

$$B'^p_q = B^i_j \frac{\partial x'^p}{\partial x^i} \frac{\partial x^j}{\partial x'^q} \qquad (2)$$

From (1) and (2) we get

$$(A'^p_q \pm B'^p_q) = (A^i_j \pm B^i_j) \frac{\partial x'^p}{\partial x^i} \frac{\partial x^j}{\partial x'^q}$$

Which shows that $(A \pm B)$ follows the same law of transformation as in A and B. Hence $(A \pm B)$ is also a tensor of the same rank and type.

4.6 Contraction

If in a tensor we put one contra-variant and one covariant indices equal (i.e. same) then the summation over equal indices is to be taken according to the summation convention. The process is called contraction of a tensor.

Consider a tensor A^{ij}_{pqr} of rank five. If we put$j = i$, we get A^{ij}_{pqr} and is a tensor of rank 3 obtained by contracting A^{ij}_{pqr}.

Theorem :- If in a mixed tensor, contra variant of rank p and covariant of rank q, we equate a covariant and contra variant index and sum with regard to that index then the resulting set of N^{p+q-2} sums is mixed tensor, contra variant of rank (p-1) and covariant of rank (q-1).

Proof: Consider a mixed tensor A^{ij}_{lmn} of rank five, contra variant of rank two and covariant of rank three.

$$A'^{ij}_{lmn} = A^{st}_{uvw}\frac{\partial x'^i}{\partial x^s}\frac{\partial x'^j}{\partial x^t}\frac{\partial x^u}{\partial x'^l}\frac{\partial x^v}{\partial x'^m}\frac{\partial x^w}{\partial x'^n}$$

Let $j = n$, we get

$$A'^{in}_{lmn} = A^{st}_{uvw}\frac{\partial x'^i}{\partial x^s}\frac{\partial x'^n}{\partial x^t}\frac{\partial x^u}{\partial x'^l}\frac{\partial x^v}{\partial x'^m}\frac{\partial x^w}{\partial x'^n}$$

$$= A^{st}_{uvw}\frac{\partial x'^i}{\partial x^s}\frac{\partial x^u}{\partial x'^l}\frac{\partial x^v}{\partial x'^m}\left(\frac{\partial x^w}{\partial x'^n}\frac{\partial x'^n}{\partial x^t}\right)$$

$$= A^{st}_{uvw}\frac{\partial x'^i}{\partial x^s}\frac{\partial x^u}{\partial x'^l}\frac{\partial x^v}{\partial x'^m}\frac{\partial x^w}{\partial x^t}$$

$$= A^{st}_{uvw}\frac{\partial x'^i}{\partial x^s}\frac{\partial x^u}{\partial x'^l}\frac{\partial x^v}{\partial x'^m}\delta^w_t$$

$$= A^{kw}_{uvw}\frac{\partial x'^i}{\partial x^k}\frac{\partial x^u}{\partial x'^l}\frac{\partial x^v}{\partial x'^m}$$

$$= A^{k}_{uv}\frac{\partial x'^i}{\partial x^k}\frac{\partial x^u}{\partial x'^l}\frac{\partial x^v}{\partial x'^m}$$

In this last expression we have put $A^{kw}_{uvw} = A^k_{uv}$. This is the law of transformation of a tensor of rank three.

Thus $A'^{in}_{lmn} = A'^{i}_{lm}$ is a tensor of rank 3 contra variant of rank 2-1 i.e. 1 and covariant of rank 3-1 i.e. 2.

4.7 Direct Product

Let A_i is a covariant vector (first rank tensor) and B^j is a contra-variant vector (first rank tensor) then component of A_i and B^j may be multiplied component by component to give the general term $A_i B^j$.

$$A_i B^j = \frac{\partial x^k}{\partial x'^i} A_k \frac{\partial x'^j}{\partial x^l} B^l$$

$$= \frac{\partial x^k}{\partial x'^i}\frac{\partial x'^j}{\partial x^l}\left(A_k B^l\right)$$

Contracting, we get

$$A'_i B'^i = A_k B^k$$

The operation of adjoining two vectors A_i and B^j is known as the direct product of tensors. The product of two vectors is a tensor of rank two. In general, the direct product of two tensors is a tensor whose rank is the sum of the ranks of the given tensors.

$$A^i_j B^{kl} = C^{ikl}_j$$

Where C^{ikl}_j is a tensor of fourth rank. The direct product is a technique for creating new higher-rank tensors.

The word "tensor product" refers to a way of constructing a big vector space out of two (or more) smaller vector spaces. If a vector V is n-dimensional and a vector is m-dimensional then the product of these two vector spaces is nm-dimensional.

In Quantum mechanics, for each dynamical degree of freedom we associate a Hilbert space. For example, a free particle in has three dynamical degrees of freedom p_x, p_y, p_z in three dimensional system. Note that we can specify only p_x or x, but not both and hence each dimension gives only one degree of freedom but in classical mechanics you have two

Example 4.2 Prove that $A_i B^i$ is invariant if A_i is covariant tensor and B^i is contravariant tensor.

Sol. By the law of transformation of tensors we have

$$A'_i = \frac{\partial x^k}{\partial x'^i} A_k$$

$$B'^i = \frac{\partial x'^i}{\partial x^l} B^l$$

Then we can get

$$A'_i B'^i = A_k B^l \frac{\partial x^k}{\partial x'^i} \frac{\partial x'^i}{\partial x^l}$$

$$= A_k B^l \delta^k_l$$

$$= A_k (\delta^k_l B^l)$$

$$A'_i B'^i = A_k B^k = A_i B^i$$

Hence $A_i B^i$ is an invariant.

Theorem: - The product of two tensors is a tensor of whose rank (or order) is the sum of the ranks of the two tensors.

Proof: Consider two tensors A_k^{ij} and B_q^p. Let the product of these two tensors be a tensor C_{kq}^{ijp}, that is

$$C_{kq}^{ijp} = A_k^{ij} B_q^p \qquad (1)$$

Now we have to show that C_{kq}^{ijp} is a tensor of rank 5. We know that

$$A_w'^{uv} = A_k^{ij} \frac{\partial x'^u}{\partial x^i} \frac{\partial x'^v}{\partial x^j} \frac{\partial x^k}{\partial x'^w} \qquad (2)$$

$$B_s'^r = B_q^p \frac{\partial x'^r}{\partial x^p} \frac{\partial x^q}{\partial x'^s} \qquad (3\ )$$

Multiplying equation (2) and (3), we get

$$A_w'^{uv} B_s'^r = A_k^{ij} B_q^p \frac{\partial x'^u}{\partial x^i} \frac{\partial x'^v}{\partial x^j} \frac{\partial x^k}{\partial x'^w} \frac{\partial x'^r}{\partial x^p} \frac{\partial x^q}{\partial x'^s}$$

Using (1)

$$= C_{kq}^{ijp} \frac{\partial x'^u}{\partial x^i} \frac{\partial x'^v}{\partial x^j} \frac{\partial x^k}{\partial x'^w} \frac{\partial x'^r}{\partial x^p} \frac{\partial x^q}{\partial x'^s}$$

We may write above equation as

$$C_{ws}'^{uvr} = C_{kq}^{ijp} \frac{\partial x'^u}{\partial x^i} \frac{\partial x'^v}{\partial x^j} \frac{\partial x^k}{\partial x'^w} \frac{\partial x'^r}{\partial x^p} \frac{\partial x^q}{\partial x'^s} \qquad (4)$$

Relation (4) is the law of transformation of a mixed tensor of rank five. Hence C_{kq}^{ijp} is a mixed tensor of rank 5.

Hence the theorem is proved.

Example 4.3 If A^μ and B_υ are compononents of a contravariant and covariant tensor of rank one, the prove that

$C_\upsilon^\mu = A^\mu B_\upsilon$ are the components of a mixed tensor of rank two.

Sol. Using tensor transformation

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha} A^\alpha$$

$$B'_{\upsilon} = \frac{\partial x^{\beta}}{\partial x'^{\upsilon}} B_{\beta}$$

Thus $C'^{\mu}_{\upsilon} = A'^{\mu} B'_{\upsilon} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} A^{\alpha} \frac{\partial x^{\beta}}{\partial x'^{\upsilon}} B_{\beta}$

$$= \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\upsilon}} A^{\alpha} B_{\beta}$$

$$\Rightarrow C'^{\mu}_{\upsilon} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\upsilon}} C^{\alpha}_{\beta}$$

Hence above equation is transformation equation for a mixed tensor of rank two.

4.8 Quotient rule

By this law we can test whether a given quantity is a tensor or not. Suppose we are given a quantity X and we don't know whether X is a tensor or not. To test X, we take product of X with an arbitrary tensor, if this product is tensor then X is also a tensor. This is called Quotient law. The Quotient law is a simple indirect test which can be used to ascertain whether a set of quantities form the components of a tensor.

Theorem: - If the product of a set of quantities A^{ijk} with an arbitrary tensor B^{p}_{ij} yield a non-zero tensor C^{pk} then the quantities A^{ijk} are the components of a tensor.

Proof: We consider an arbitrary co-ordinate transformation $x^{i} \rightarrow x'^{i}$, from this transformation $A^{ijk} \rightarrow A'^{ijk}, B^{p}_{ij} \rightarrow B'^{p}_{ij}, C^{pk} \rightarrow C'^{pk}$.

Consider the product

$$C^{pk} = A^{ijk} B^{p}_{ij} \qquad (1)$$

In transformed coordinates x'^{i}, equation (1) becomes

$$C'^{pk} = A'^{ijk} B'^{p}_{ij} \qquad (2)$$

But we have

$$C'^{pk} = C^{qr} \frac{\partial x'^{p}}{\partial x^{q}} \frac{\partial x'^{k}}{\partial x^{r}} \qquad (3)$$

$$B'^{p}_{ij} = B^{l}_{mn} \frac{\partial x'^{p}}{\partial x^{l}} \frac{\partial x^{m}}{\partial x'^{i}} \frac{\partial x^{n}}{\partial x'^{j}} \qquad (4)$$

Substituting equation (3) and (4) into (2), we get

$$C^{qr}\frac{\partial x'^p}{\partial x^q}\frac{\partial x'^k}{\partial x^r} = A'^{ijk}\,B^l_{mn}\frac{\partial x'^p}{\partial x^l}\frac{\partial x^m}{\partial x'^i}\frac{\partial x^n}{\partial x'^j}$$

$$A^{ijk}B^q_{ij}\frac{\partial x'^p}{\partial x^q}\frac{\partial x'^k}{\partial x^r} = A'^{ijk}\,B^l_{mn}\frac{\partial x'^p}{\partial x^l}\frac{\partial x^m}{\partial x'^i}\frac{\partial x^n}{\partial x'^j}$$

By changing dummy indices $q \to l; i \to m$ and $j \to n$, we get

$$A^{mnr}B^l_{mn}\frac{\partial x'^p}{\partial x^l}\frac{\partial x'^k}{\partial x^r} = A'^{ijk}\,B^l_{mn}\frac{\partial x'^p}{\partial x^l}\frac{\partial x^m}{\partial x'^i}\frac{\partial x^n}{\partial x'^j}$$

$$\frac{\partial x'^p}{\partial x^l}\left[A'^{ijk}\frac{\partial x^m}{\partial x'^i}\frac{\partial x^n}{\partial x'^j} - A^{mnr}\frac{\partial x'^k}{\partial x^r}\right]B^l_{mn} = 0 \qquad (5)$$

We know that

$$\frac{\partial x^s}{\partial x'^p}\left[\frac{\partial x'^p}{\partial x^l}B^l_{mn}\right] = \frac{\partial x^s}{\partial x^l}B^l_{mn} = B^s_{mn} \qquad (6)$$

Multiplying equation (5) by $\frac{\partial x^s}{\partial x'^p}$ and substituting equation (6), one gets

$$\left[A'^{ijk}\frac{\partial x^m}{\partial x'^i}\frac{\partial x^n}{\partial x'^j} - A^{mnr}\frac{\partial x'^k}{\partial x^r}\right]B^s_{mn} = 0 \qquad (7)$$

Since equation is valid for arbitrary B^s_{mn}, therefore the quantity under bracket is zero i.e.

$$A'^{ijk}\frac{\partial x^m}{\partial x'^i}\frac{\partial x^n}{\partial x'^j} = A^{mnr}\frac{\partial x'^k}{\partial x^r}$$

Multiplying both sides by $\frac{\partial x'^s}{\partial x^m}\frac{\partial x'^t}{\partial x^n}$, one gets

$$A'^{ijk}\frac{\partial x'^s}{\partial x'^i}\frac{\partial x'^t}{\partial x'^j} = A^{mnr}\frac{\partial x'^s}{\partial x^m}\frac{\partial x'^t}{\partial x^n}\frac{\partial x'^k}{\partial x^r}$$

L.H.S. is non zero for $i = s$ and $j = t$ and hence

$$A'^{ijk} = A^{mnr}\frac{\partial x'^i}{\partial x^m}\frac{\partial x'^j}{\partial x^n}\frac{\partial x'^k}{\partial x^r} \qquad (8)$$

From equation (8), we see that A^{mnr} is a tensor of third rank and contra-variant in all indices. This completes the proof of Quotient law.

4.9 Symmetric and Anti-Symmetric Tensor

Based on permutation of the indices a tensor can be of two types:-

1. Symmetric tensor
2. Anti-symmetric tensor

A tensor is said to be symmetric in two indices of the same type i.e. both covariant or both contravariant, if the value of any component is not changed by permuting them. If for a tensor A_{ij}

$$\boxed{A_{ij} = A_{ji}}$$

Then it is called ***symmetric tensor***.

A tensor is said to be anti-symmetric in two indices of the same type i.e. both covariant or both contravariant, if the value of any component changes its sign by permuting them. If for a tensor A_{ij}

$$\boxed{A_{ij} = -A_{ji}}$$

Then it is called ***anti-symmetric*** tensor. A general tensor can be split up into a symmetric and an anti-symmetric part:

$$\boxed{A_{ij} = \frac{1}{2}\left(A_{ij} + A_{ji}\right) + \frac{1}{2}\left(A_{ij} - A_{ji}\right)}$$

Where the first part in the right hand side is symmetric part and second part is anti-symmetric part.

Theorem:- A symmetric tensor of rank two has at most $\frac{1}{2}N(N+1)$ different components in N - dimensional vector space.

Proof: Let A_{ij} be a tensor of rank two. The number of its all components in N-dimensional vector space is N^2. All the components of A_{ij} are

$$\begin{matrix} A_{11} & A_{12} & A_{13} & A_{14} \dots & A_{1N} \\ A_{21} & A_{22} & A_{23} & A_{24} \dots & A_{2N} \\ A_{31} & A_{32} & A_{33} & A_{34} \dots & A_{3N} \\ \dots & \dots & \dots & \dots\; \dots & \dots \\ A_{N1} & A_{N2} & A_{N3} & A_{N4} \dots & A_{NN} \end{matrix}$$

Number of independent components is N (A_{11} A_{22} A_{33} A_{44} ... A_{NN}). Hence the number of components corresponding to distinct subscripts are $N^2 - N$. But the components are symmetric i.e. $A_{12} = A_{21}$etc.

Number of different components of this form is $\frac{1}{2}(N^2 - N)$. Thus total number of different (i.e. independent) components

$= \frac{1}{2}(N^2 - N) + N = \frac{1}{2}N(N + 1)$.

Theorem: A skew-symmetric (anti-symmetric) tensor of rank two has $\frac{1}{2}N(N - 1)$ different components in N - dimensional vector space.

Proof: Let A_{ij} be a tensor of rank two. The number of its all components in N-dimensional vector space isN^2. All the components of A_{ij} are

$$\begin{matrix} A_{11} & A_{12} & A_{13} & A_{14} \ldots A_{1N} \\ A_{21} & A_{22} & A_{23} & A_{24} \ldots A_{2N} \\ A_{31} & A_{32} & A_{33} & A_{34} \ldots A_{3N} \\ \ldots & \ldots & \ldots & \ldots\ldots \;\ldots\ldots \;\ldots\ldots \\ A_{N1} & A_{N2} & A_{N3} & A_{N4} \ldots A_{NN} \end{matrix}$$

Number of independent components N (A_{11} A_{22} A_{33} A_{44} ... A_{NN}) will be zero. Hence the number of components corresponding to distinct subscripts are $N^2 - N$. But the components are symmetric i.e. $A_{12} = -A_{21}$etc.

Number of different components of this form is $\frac{1}{2}(N^2 - N)$. Thus total number of different (i.e. independent) components

$= \frac{1}{2}(N^2 - N) + 0 = \frac{1}{2}N(N - 1)$.

Example 4.4 Show that any tensor of rank two can be expressed as a sum of a symmetric and an antisymmetric tensor, both of rank two.

Sol. A tensor of $A^{\alpha\beta}$ of rank two can be expressed as

$$A^{\alpha\beta} = \frac{1}{2}\left(A^{\alpha\beta} + A^{\beta\alpha}\right) + \frac{1}{2}\left(A^{\alpha\beta} - A^{\beta\alpha}\right)$$

$$= B^{\alpha\beta} + C^{\alpha\beta}$$

Where $B^{\alpha\beta} = \frac{1}{2}\left(A^{\alpha\beta} + A^{\beta\alpha}\right)$ and $C^{\alpha\beta} = \frac{1}{2}\left(A^{\alpha\beta} - A^{\beta\alpha}\right)$

By addition and subtraction laws of tensors, it is evident that $B^{\alpha\beta}\ and\ C^{\beta\alpha}$ are tensors of rank two.

By interchanging the indices in $B^{\alpha\beta}\ and\ C^{\beta\alpha}$,we have

$$B^{\beta\alpha} = \frac{1}{2}\left(A^{\beta\alpha} + A^{\alpha\beta}\right)$$

$$= \frac{1}{2}\left(A^{\alpha\beta} + A^{\beta\alpha}\right) = B^{\alpha\beta}$$

Hence $B^{\alpha\beta}$ is symmetric tensor of rank two.

Similarly

$$C^{\beta\alpha} = \frac{1}{2}\left(A^{\beta\alpha} - A^{\alpha\beta}\right)$$

$$= -\frac{1}{2}\left(A^{\alpha\beta} - A^{\beta\alpha}\right) = -C^{\alpha\beta}$$

Hence $C^{\alpha\beta}$ is Anti-symmetric tensor of rank two.

Thus any tensor of rank two can be expressed as a sum of a symmetric and an antisymmetric tensor, both of rank two.

Example 4.5 If A^{μ} is arbitrary contravariant vector and $C_{\mu\nu}A^{\mu}A^{\upsilon}$ is an invariant then prove that $\left(C_{\mu\upsilon} + C_{\upsilon\mu}\right)$ is a covariant tensor of second order.

Sol. Here A^{μ} is an arbitrary contravariant vector and $C_{\mu\nu}A^{\mu}A^{\upsilon}$ is invariant, so

$$C'_{\mu\nu}A'^{\mu}A'^{\upsilon} = C_{\mu\nu}A^{\mu}A^{\upsilon} \qquad (1)$$

By tensor transformation

$$C'_{\mu\nu}A'^{\mu}A'^{\upsilon} = C'_{\mu\nu}\frac{\partial x'^{\mu}}{\partial x^{\alpha}}A^{\alpha}\frac{\partial x'^{\upsilon}}{\partial x^{\beta}}A^{\beta}$$

$$\Rightarrow C'_{\mu\nu}A'^{\mu}A'^{\upsilon} = C'_{\mu\nu}\frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\upsilon}}{\partial x^{\beta}}A^{\alpha}A^{\beta}$$

By interchanging the dummy indices μ and υ ,we have

$$\Rightarrow C'_{\upsilon\mu} A'^{\upsilon} A'^{\mu} = C'_{\upsilon\mu} \frac{\partial x'^{\upsilon}}{\partial x^{\alpha}} \frac{\partial x'^{\mu}}{\partial x^{\beta}} A^{\alpha} A^{\beta} \qquad (2)$$

By interchanging the dummy indices α and β ,we have

$$\Rightarrow C'_{\mu\upsilon} A'^{\mu} A'^{\upsilon} = C'_{\mu\upsilon} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\upsilon}}{\partial x^{\beta}} A^{\alpha} A^{\beta} \qquad (3)$$

By adding (2)&(3)

$$\Rightarrow \left(C'_{\mu\upsilon} + C'_{\upsilon\mu} \right) A'^{\mu} A'^{\upsilon} = \left(C'_{\mu\upsilon} + C'_{\upsilon\mu} \right) \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\upsilon}}{\partial x^{\beta}} A^{\alpha} A^{\beta} \qquad (4)$$

By eq.(1)

$$C'_{\mu\nu} A'^{\mu} A'^{\upsilon} = C_{\mu\nu} A^{\mu} A^{\upsilon} = C_{\alpha\beta} A^{\alpha} A^{\beta}$$

And

$$C'_{\upsilon\mu} A'^{\upsilon} A'^{\mu} = C_{\upsilon\mu} A^{\upsilon} A^{\mu} = C_{\beta\alpha} A^{\beta} A^{\alpha}$$

By adding

$$\left(C'_{\mu\upsilon} + C'_{\upsilon\mu} \right) A'^{\mu} A'^{\upsilon} = \left(C_{\alpha\beta} + C_{\beta\alpha} \right) A^{\alpha} A^{\beta} \qquad (5)$$

By using eq.(5),we can write eq.(4) as

$$\left(C_{\alpha\beta} + C_{\beta\alpha} \right) A^{\alpha} A^{\beta} = \left(C'_{\mu\upsilon} + C'_{\upsilon\mu} \right) \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\upsilon}}{\partial x^{\beta}} A^{\alpha} A^{\beta}$$

$$\Rightarrow \left[\left(C'_{\mu\upsilon} + C'_{\upsilon\mu} \right) \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\upsilon}}{\partial x^{\beta}} - \left(C_{\alpha\beta} + C_{\beta\alpha} \right) \right] A^{\alpha} A^{\beta} = 0$$

Here A^{α} *and* A^{β} are arbitrary, so

$$\left[\left(C'_{\mu\upsilon} + C'_{\upsilon\mu} \right) \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\upsilon}}{\partial x^{\beta}} - \left(C_{\alpha\beta} + C_{\beta\alpha} \right) \right] = 0$$

On multiplication of $\frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\upsilon}}$ wit above equation ,we have

$$\left(C'_{\mu\upsilon} + C'_{\upsilon\mu} \right) \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\upsilon}}{\partial x^{\beta}} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\upsilon}} - \left(C_{\alpha\beta} + C_{\beta\alpha} \right) \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\upsilon}} = 0$$

$$\left(C'_{\mu\upsilon} + C'_{\upsilon\mu} \right) \delta^{\mu}_{\mu} \delta^{\upsilon}_{\upsilon} = \left(C_{\alpha\beta} + C_{\beta\alpha} \right) \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\upsilon}}$$

$$\left(C'_{\mu\upsilon} + C'_{\upsilon\mu}\right) = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\upsilon}} \left(C_{\alpha\beta} + C_{\beta\alpha}\right)$$

That is transformation law for covariant tensor of second rank.

Hence $\left(C_{\mu\upsilon} + C_{\upsilon\mu}\right)$ is a covariant tensor of second order.

4.10 Pseudo Tensors

Levi-civita tensor: -

The Levi-civita tensor ϵ_{ijk} is defined as follows:-

$\epsilon_{ijk} = 0$	If any two of the indices are equal.
$\epsilon_{ijk} = 1$	If is an even permutation of 1, 2, 3.
$\epsilon_{ijk} = -1$	If is an odd permutation of 1, 2, 3.

For example

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$$
$$\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$$
$$\epsilon_{112} = \epsilon_{122} = \epsilon_{233} = \cdots = 0$$

A change in orientation i.e. from a left-handed to a right handed system as in reflection produces a change in sign. Such tensors which changes sign under a change of orientation are called Pseudo tensor.

From every anti-symmetric tensor $A^{\alpha\beta}$ of the second rank a pseudo tensor of the same rank can be obtained by multiplying the former with a pseudo-tensor of rank 4,

$i.e.$ $\quad A^{*}_{\mu\sigma} = \frac{1}{2}\sum_{\alpha,\beta=0}^{3} \epsilon_{\mu\sigma\alpha\beta} A^{\alpha\beta}$

Properties of Pseudo tensors:-

1. The sum or difference of two Pseudo tensors f the same rank is a Pseudo tensor.
2. The product of a tensor with a Pseudo tensor is a Pseudo tensor.
3. The product of two Pseudo tensors is a tensor.
4. The partial derivative of a Pseudo tensor w. r. t. x^{i} is a Pseudo tensor.
5. A contracted Pseudo tensor is a Pseudo tensor.

The relation between pseudo tensor ϵ_{ijk} and Kronecker delta tensor δ_{ij} is given as

$$\boxed{\epsilon_{ijm}\epsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}}$$

Example 4.6 Prove that

$$\epsilon_{ilm}\epsilon_{jlm} = 2\delta_{ij}.$$

Sol. We know

$$\epsilon_{ilm}\epsilon_{jkm} = \delta_{ij}\delta_{lk} - \delta_{ik}\delta_{lj}$$

Taking $k = l$ we get

$$\epsilon_{ilm}\epsilon_{jlm} = \delta_{ij}\delta_{ll} - \delta_{il}\delta_{lj}$$

Now $\delta_{ll} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$

$$\delta_{il}\delta_{lj} = \delta_{ij}$$

So we get

$$\epsilon_{ilm}\epsilon_{jlm} = 3\delta_{ij} - \delta_{ij}$$

$$= 2\delta_{ij}$$

4.11 Self Learning Exercise-II

Q.1 What is the value of Levi-civita tensor ϵ_{121}?

Q.2 What is the sum of two tensors of the same rank and type?

Q.3 What is the value of Kronecker delta δ_i^i ?

4.12 Summary

This unit is started with the introduction about the tensor analysis. In this we have defined N-dimensional space and the transformation of coordinates. In tensor algebra sum, contraction, direct product etc. are defined with the examples and theorems. We also studied the quotient rule and definition and properties of pseudo tensor.

4.13 Glossary

Contravariant: For a vector (such as a direction vector or velocity vector) to be basis-independent, the components of the vector must *contra-vary* with a change of basis to compensate. The components of vectors (as opposed to those of dual vectors) are said to be contravariant

N-dimensional space: (mathematics) A vector space having n vectors as its basis

Permutation: the notion of permutation relates to the act of rearranging, or permuting, all the members of a set into some sequence or order (unlike combinations, which are selections of some members of the set where order is disregarded)

Covariant: The meaning of covariant is that they have the same form in all reference frames.

4.14 Answers to Self Learning Exercises

Answers of self learning exercise-I

Ans.1: 3

Ans.2: $B'_{ij} = \sum_{kl} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} B_{kl}$

Ans.3 : C^i_{jk} is a mixed tensor of rank 3.

Answers of self learning exercise-II

Ans.1: Zero.

Ans.2: Tensor of same rank and type.

Ans.3: 1

4.15 Exercise

Section A (Very short answer type questions)

Q.1 Write the number of components of a rank two tensor in 3-dimensional space.

Q.2 Write the total number of independent components of a rank two anti-symmetric tensor in 3-dimensional space.

Q.3 What is rank of the tensor B^{ip}_{ij}?

Section B (Short answer type questions)

Q.4 What is the difference between covariant and contravariant tensor?

Q.5 What do you mean by a mixed tensor and give one example of mixed tensor?

Q.6 Define the product of two tensors with example.

Section C (Long answer type questions)

Q.7 Prove that the contracted tensor B_i^i is a scalar.

Q.8 Show that the sum and difference of two tensors B_r^{pq} and B_r^{pq} are also tensors.

Q.9 $A^{ij}$is a contravariant tensor and B_i is a covariant tensor. Show that $A^{ij}B_k$ is a tensor of rank three, but $A^{ij}B_j$ is a tensor of rank one.

4.16 Answers to Exercise

Ans.1: The number of components of a rank two tensor in 3-dimensinal space is 9.

Ans.2: 3

Ans.3: 2

References and Suggested Readings

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UNIT-5
Matrices

Structure of the Unit

5.0 Objectives

In this unit we will learn the basic concepts related to Orthogonal ,Hermitian ,Unitary ,Eigenvectors etc. This unit is concerned with the most important matrices of physics and engineering. These matrices are frequently used in classical Mechanics and Quantum Mechanics.

5.1 Introduction

The theory and applications of the matrices have connection with the solution of linear system of equations in engineering problems. There is a great importance of the study of the properties of matrices. A matrix ia a rectangular array of numbers. In this chapter emphasis is given on Hermitian and Unitary Matrices. We finally discuss the important concepts related to eigenvectors**.**

5.2 Matrix and its Transpose

Matrix:

A matrix of order $m \times n$ is a rectangular array of numbers having m rows and n columns. We can write it in the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Each number a_{ij} in this matrix is called an element of the matrix A. Here subscripts i and j represent respectively the row and column of the matrix in which the element exists.

Transpose of Matrix :

By interchanging of rows and corresponding columns of matrix A, we obtain A^T (i.e. transpose of matrix)

If matrix $A = \left(a_{ij}\right)_{m\times n}$ then $A^T = \left(a_{ji}\right)_{n\times m}$

Transpose of matrix is denoted by $A', \tilde{A}, A^T$

For example

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 6+i & 7 & 9 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 6+i \\ 3 & 7 \\ 5 & 9 \end{bmatrix}$$

Important properties of Transpose of a Matrix

(i) $(A^T)^T = A$

(ii) $(kA)^T = kA^T$ where k is any scalar

(iii) $(A+B)^T = A^T + B^T$

(iv) $(AB)^T = B^T A^T$

5.3 Orthogonal Matrix

We consider a real square matrix A. *The matrix A is called orthogonal if*

$$AA^T = I = A^T A \tag{5.1}$$

Where *I* is unit matrix (Identity Matrix)

We can write the condition $A^T = A^{-1}$ *for A to be an orthogonal matrix.*

We can prove this condition in following way

$$AA^T = I$$

or $A^{-1}\left(AA^T\right) = A^{-1}(I)$

or $\left(A^{-1}A\right)A^T = A^{-1}$

or $IA^T = A^{-1}$

or $A^T = A^{-1}$ (5.2)

Any one of these relations (5.1) & (5.2) is both the necessary and the sufficient condition for matrix A to be orthogonal.

Here $|A| \neq 0$,so A is nonsingular matrix and A^{-1} exists. ***Important property of orthogonal matrix is determinant of an orthogonal matrix can only have values +1 or -1.***

Proof : $AA^T = I$

$\Rightarrow \quad \left|AA^T\right| = |I|$

$\Rightarrow \quad |A|\left|A^T\right| = 1$

$\Rightarrow \quad |A||A| = 1 \qquad \because |A| = A^T$

$\Rightarrow \quad |A|^2 = 1 \quad \Rightarrow \quad |A| = \pm 1$

5.4 Illustrative Examples

Example 5.1 If A and B are orthogonal matrices then prove that both AB & BA are also orthogonal matrices.

Sol. Since A and B are orthogonal matrices, so

$$AA^T = A^T A = I$$

and $BB^T = B^T B = I$

Let $Z = AB$

$$ZZ^T = (AB)(AB)^T$$

$$= AB(B^T A^T) = \mathrm{A(BB}^T)A^T$$

$$= A(I)A^T = (AI)A^T$$

$$= AA^T$$

$$\Rightarrow \quad ZZ^T = I$$

Hence Z i.e. AB is orthogonal matrix

Similarly BA is also orthogonal matrix.

Example 5.2 Show that matrix $A = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$ is an orthogonal matrix

Sol. $$AA^T = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \left(\cos^2\phi + \sin^2\phi\right) & (-\cos\phi\sin\phi + \sin\phi\cos\phi) \\ (-\sin\phi\cos\phi + \cos\phi\sin\phi) & (\sin^2\phi + \cos^2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad \text{Hence proved}$$

5.5 Symmetric & Antisymmetric Matrix

Symmetric matrix :

We consider real matrix A

A square matrix A is called symmetric if $\boxed{A = A^T}$

i.e. matrix elements $a_{ij} = a_{ji}$ for all values of *i* and *j*, where $A = [a_{ij}]$

For example

$$\begin{bmatrix} 2 & 7 & 9 \\ 7 & 3 & 14 \\ 9 & 14 & 4 \end{bmatrix} \quad \text{is Symmetric Matrix}$$

Antisymmetric or Skew symmetric Matrix :

We consider real matrix A

A square matrix A is called anti symmetric if $\boxed{A = -A^T}$

i.e. matrix elements $a_{ij} = -a_{ji}$ for all values of *i* and *j*, where $A = [a_{ij}]$.

If we put $i = j$ then $a_{ii} = -a_{ii}$ or $2a_{ii} = 0$ or $a_{ii} = 0$. i.e. ***all diagonal elements of the anti symmetric matrix are zero. Trace (sum of diagonal elements of a square matrix) of anti symmetric matrix is zero.***

For example

$$\begin{bmatrix} 0 & -\alpha & \beta \\ \alpha & 0 & -\gamma \\ -\beta & \gamma & 0 \end{bmatrix} \quad \text{is an antisymmetric matrix}$$

5.6 Conjugate of Matrix and Transpose conjugate of Matrix

Conjugate of Matrix:

Conjugate of matrix A is obtained by taking the complex conjugate $(i \to -i)$ of each element of the matrix A conjugate of matrix A is denoted by A^* or $\overline{A}$

If $A = [a_{ij}]$ then $A^* = [a_{ij}^*]$

For example

If $\quad A = \begin{bmatrix} 2+i & 4-5i \\ i & 3 \end{bmatrix}$

Then $\quad A^* = \begin{bmatrix} 2-i & 4+5i \\ -i & 3 \end{bmatrix}$

Important properties of conjugate of a matrix:

(i) $\boxed{\left(A^*\right)^* = A}$

(ii) $\left(A+B\right)^* = A^* + B^*$

(iii) $\boxed{\left(kA\right)^* = k^* A^*}$ where k is a complex number

(iv) $\boxed{\left(AB\right)^* = A^* B^*}$

Transpose of Conjugate (or Conjugate transpose) of a Matrix A :

Transpose conjugate of matrix A is denoted by $A^\dagger$ or A^θ

$A^\dagger$ (read as A dagger)

$$\boxed{A^\dagger = \left(A^*\right)^T = \left(A^T\right)^*}$$

i.e. conjugate transpose of a matrix is the same as the transpose of its conjugate

If $A = \begin{bmatrix} 2 & 3-i \\ 4+i & 5i \end{bmatrix}$

$A^* = \begin{bmatrix} 2 & 3+i \\ 4-i & -5i \end{bmatrix}$

$A^\theta = \left(A^*\right)^T = \begin{bmatrix} 2 & 4-i \\ 3+i & -5i \end{bmatrix}$

Important properties of the transpose conjugate of a matrix

(i) $\boxed{\left(A^\theta\right)^\theta = A}$

(ii) $\left(A+B\right)^\theta = A^\theta + B^\theta$

(iii) $\left(kA\right)^\theta = k^* A^\theta$ Where k is complex number

(iv) $\boxed{\left(AB\right)^\theta = B^\theta A^\theta}$

5.7 Hermitian & Anti Hermitian Matrix

Hermitian Matrix:

A square matrix $A = \left[a_{ij}\right]$ is Hermitian if $\boxed{A = A^\theta}$

i.e. $a_{ij} = a^*_{ji}$ *for all values of i and j*

If we put $i = j$ then $a_{ii} = a^*_{ii}$ for all values of *i*.

i.e. ***all diagonal element of a Hermitian matrix are real numbers***. Thus trace of Hermitian matrix is real number. Determinant of a Hermitian matrix is real. For example ,Hermitian matrix is

$$\begin{bmatrix} 5 & 3-i & 7 \\ 3+i & 0 & i \\ 7 & -i & 2 \end{bmatrix}$$

For real symmetric matrix $a_{ij} = a_{ji}$

i.e. $A^T = A$ (5.3)

$\because a^*_{ji} = a_{ji}$

So $\left(A^T\right)^* = A^T$

i.e. $A^\theta = A^T$ (5.4)

From (5.3) & (5.4)

$A^\theta = A$

i.e. ***real symmetric matrix is Hermitian matrix.***

Skew Hermitian Matrix or Anti Hermitian Matrix :

A square matrix $A = \left[a_{ij}\right]$ is skew Hermitian if $\boxed{A = -A^\theta}$

i.e. $a_{ij} = -a^*_{ji}$ *for all values of i and j*

If we put $i = j$ then $a_{ii} = -a^*_{ii}$

Let $a_{ii} = \alpha + i\beta$ Where α and β are real numbers, then

$(\alpha + i\beta) = -(\alpha + i\beta)^*$

$\alpha + i\beta = -(\alpha - i\beta)$

$2\alpha = 0$ i.e. $\alpha = 0$

So $a_{ii} = i\beta$

Hence every diagonal element of an Anti Hermitian matrix is either zero or a pure imaginary number. Trace of an Anti Hermitian matrix is either purely imaginary or zero. Examples of skew Hermitian matrices are:

$$\begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -3+i \\ 3-i & i \end{bmatrix}, \begin{bmatrix} 4i & -3 & 0 \\ 3 & 0 & -7+i \\ 0 & 7+i & -i \end{bmatrix} \text{ etc.}$$

We can prove that ***real anti symmetric matrix is skew Hermitian matrix. Determinant of skew Hermitian matrix is either zero or purely imaginary number.***

Example 5.3 If Y is Hermitian matrix then prove that $X^{\dagger}YX$ is also Hermitian matrix for every matrix X.

Sol. Since Y is Hermitian matrix

$$Y^{\dagger} = Y$$

Let $Z = X^{\dagger}YX$

$$Z^{\dagger} = \left(X^{\dagger}YX\right)^{\dagger}$$

$$= X^{\dagger}Y^{\dagger}(\mathrm{X}^{\dagger})^{\dagger} \qquad \because (ABC)^{\dagger} = C^{\dagger}B^{\dagger}A^{\dagger}$$

$$= X^{\dagger}Y^{\dagger}\,\mathrm{X} \qquad \because \left(A^{\dagger}\right)^{\dagger} = A$$

$$= X^{\dagger}YX \qquad \because Y \text{ is Hermitian}$$

$$\Rightarrow \quad Z^{\dagger} = Z$$

So $X^{\dagger}YX$ is Hermitian

5.8 Unitary Matrix

A square finite matrix U is unitary if $UU^{\theta} = I$

i.e. $\boxed{U^{\theta}U = I = UU^{\theta}}$

Important properties of Unitary matrices:

(i) ***Determinant of a unitary matrix is of unit modulus.*** Hence unitary matrix is not singular matrix.

Proof **:**

Here $UU^{\theta} = I$

$$\Rightarrow \quad \left|UU^{\theta}\right| = |I|$$

$$\Rightarrow \quad |U|\left|U^{\theta}\right| = 1$$

$$\Rightarrow \quad |U||U^*| = 1 \quad \begin{cases} \because |U^T| = |U| \\ \because |U^\theta| = |(U^T)^*| = |U^*| = |U|^* \end{cases}$$

This shows that modulus of the determinant of a unitary matrix is unity.

(ii) If U is Unitary matrix then

$\boxed{U^\theta = U^{-1}}$ **(*alternate condition of Unitary matrix*)**

Proof. We have $UU^\theta = I$

$$\Rightarrow \quad U^{-1}(UU^\theta) = U^{-1}(I)$$

$$\Rightarrow \quad (U^{-1}U)U^\theta = U^{-1}$$

$$\Rightarrow \quad IU^\theta = U^{-1}$$

$$\therefore U^\theta = U^{-1}$$

(iii) *Real Unitary matrices are orthogonal matrices.*

Proof :

$$U^\theta = (U^*)^T = U^T \quad \{\because A^* = A \text{ for real matrices}\}$$

$$\because UU^\theta = I$$

$$UU^T = I$$

Hence U is also orthogonal matrix.

(iv) The products of two unitary matrices are also unitary matrix.

Proof **:** Let A and B are Unitary matrices then

$AA^\theta = I$ and $BB^\theta = I$

Let $Z = AB$

Here $ZZ^\theta = (AB)(AB)^\theta$

$$= (AB)(B^\theta A^\theta) = A(B^\theta B^\theta)A^\theta$$

$$= A(I)A^\theta = (AI)A^\theta$$

$$= AA^\theta$$

$$\Rightarrow \quad ZZ^\theta = I$$

Thus Z i.e. AB is unitary matrix

5.9 Illustrative Examples

Example 5.4 Prove that Pauli matrix

$$\sigma = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \text{ is Unitary matrix}$$

Sol.
$$\sigma^\theta = \left[\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^*\right]^T = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

We can write

$$\sigma\sigma^\theta = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 - i^2 & 0 + 0 \\ 0 + 0 & -i^2 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 5.5 If A is a Hermitian matrix then prove that matrix e^{iA} is unitary matrix.

Sol. Let $Z = e^{iA}$

$$ZZ^\theta = e^{iA}\left(e^{iA}\right)^\theta = e^{iA}e^{-iA^\theta}$$

$$= e^{iA}e^{-iA} \qquad \{\because A = A^\theta \text{ for Hermitian matrix}\}$$

$$\Rightarrow \qquad ZZ^\theta = I$$

Hence Z i.e. e^{iA} is unitary matrix

5.10 Self Learning Exercise-I

Very Short Answer Type Questions

Q.1 Show that inverse of a unitary matrix is also unitary.

Short Answer Type Questions

Q.2 If A is Hermitian matrix, then prove that iA is anti Hermitian matrix .

Q.3 If X and Y are Hermitian matrices, then prove that

(i) $XY + YX$ is also Hermitian matrix

(ii) $XY - YX$ is anti Hermitian matrix

(iii) $i(XY - YX)$ is Hermitian matrix

Q.4 If matrix $A = a\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ is Unitary ,then find the value of a.

5.11 Eigenvalues and Eigenvectors

Let $A = (a_{ij})$ be an $n \times n$ square matrix and X is a column vector. We consider the vector equation (linear translation) $AX = \lambda X$ (i)

where λ is a scalar.

Here zero vector $X = 0$ is a solution for any value of λ. But solution $X = 0$ is of no importance in practical situations.

Value of λ for which equation $\boxed{AX = \lambda X}$ *has a solution $X \neq 0$ (i.e. a nontrivial solution) is called an* ***eigen value*** *of matrix A.*

Eigenvalue is also known as characteristic value, latent root, proper values. Word 'Eigen' is German and that means 'proper' or 'characteristic'.

Corresponding to each eigenvalue λ, solution $X \neq 0$ is called eigenvector (characteristic vector) of matrix A belonging to that eigenvalue.

Here equation $AX = \lambda X$

$\Rightarrow \quad AX = \lambda IX$ {Where I is unit matrix}

$\Rightarrow \quad AX - \lambda IX = 0$

$\Rightarrow \quad (A - \lambda I)X = 0$

For nontrivial solution

$$\boxed{\det.(A - \lambda I) = 0}$$

This equation in λ is known as characteristic equation and $\det.(A - \lambda I)$ is known as characteristic determinant we can write

$$AX = \lambda X$$

$$\Rightarrow \qquad \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

For nontrivial solution

$$\det(A-\lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix} = 0$$

which shows a polynomial equation of degree n in λ

Operator interpretation of Matrices :

Matrix A can be thought as an operator which operates on column vector X and produces another column vector Y i.e. $AX = Y$

In general X and Y have the different directions. Here X is not an eigenvector. For particular case in which Y has the same direction of X i.e. Y is constant multiple of X or $Y = \lambda X$ where λ is number.

In that case $AX = \lambda X$ and X is known as eigen vector corresponds to eigen value λ

Illustrative Examples

Example 5.6 For given matrix $A = \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}$

Find the eigenvalues and eigenvectors for matrix A.

Sol. We consider equation $AX = \lambda X$ i.e.

$$\begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (5.5)$$

$$3x_1 + 4x_2 = \lambda x_1$$

$$-x_1 - 2x_2 = \lambda x_2$$

We can write above equations as

$$\left.\begin{aligned}(3-\lambda)x_1+4x_2&=0\\ x_1+(2+\lambda)x_2&=0\end{aligned}\right\} \tag{5.6}$$

The system will have nontrivial solution if (we can write directly also following determinant for getting eigenvalues)

$$\begin{vmatrix}3-\lambda & 4\\ 1 & 2+\lambda\end{vmatrix}=0$$

$$\Rightarrow \quad (3-\lambda)(2+\lambda)-4=0$$

$$\Rightarrow \quad 6+3\lambda-2\lambda-\lambda^2-4=0$$

$$\Rightarrow \quad \lambda^2-\lambda-2=0$$

$$\Rightarrow \quad (\lambda+1)(\lambda-2)=0$$

$$\Rightarrow \quad \lambda=-1,2$$

Eigen values are $\lambda_1=-1, \lambda_2=2$

❖ # Eigenvector for $\lambda_1=-1$

Corresponding to $\lambda_1=-1$ eq. (5.6) become

$$4x_1+4x_2=0$$

$$\& \ x_1+x_2=0$$

i.e. $x_1=-x_2$

Then eigenvector is

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1\\ -x_1\end{bmatrix}=x_1\begin{bmatrix}1\\ -1\end{bmatrix} \quad \text{or simply} \quad \begin{bmatrix}1\\ -1\end{bmatrix} \tag{5.7}$$

Any vector which is scalar (constant) multiple of $\begin{bmatrix}1\\ -1\end{bmatrix}$ will be eigenvector of matrix A corresponding to eigenvalue $\lambda=-1$

❖ Eigenvector for $\lambda_2=2$

Using $\lambda=2$ in equation (5.6) we get

$$x_1+4x_2=0$$

$$\& \ x_1+4x_2=0$$

i.e. $x_1=-4x_2$

Then eigenvector is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix} \text{or simply} \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

Any vector which is scalar (constant) multiple of $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$ will be eigenvector of matrix A corresponding to eigenvalue $\lambda = 2$

Note : We can understand this constant multiple of $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$ in following way

From (5.5),we have

$$\begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

From (5.7) $\begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix} x_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \lambda x_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

$$\Rightarrow \underbrace{\begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}}_{Matrix} \underbrace{\begin{bmatrix} -4 \\ 1 \end{bmatrix}}_{Eigenvector} = \underbrace{\lambda}_{\substack{Eigen \\ value}} \underbrace{\begin{bmatrix} -4 \\ 1 \end{bmatrix}}_{Eigenvector}$$

Example 5.7 Find the eigenvalues and normalized eigenvector of the following matrix A

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -2 \\ -1 & -3 & 1 \end{bmatrix}$$

Sol. We consider equation $AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -2 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{5.8}$$

$$\Rightarrow \left. \begin{aligned} x_1 + x_2 + x_3 &= \lambda x_1 \\ x_1 + 2x_2 + 2x_3 &= \lambda x_2 \\ -x_1 - 3x_2 + x_3 &= \lambda x_3 \end{aligned} \right\}$$

$$\Rightarrow \left.\begin{aligned} (1-\lambda)x_1 + x_2 + x_3 = 0 \\ x_1 + (2-\lambda)x_2 - 2x_3 = 0 \\ -x_1 - 3x_2 + (1-\lambda)x_3 = 0 \end{aligned}\right\} \qquad (5.9)$$

The system will have nontrivial solution if

$$\begin{vmatrix} (1-\lambda) & 1 & 1 \\ 1 & (2-\lambda) & -2 \\ -1 & -3 & (1-\lambda) \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)\left[(2-\lambda)(1-\lambda)-6\right] - 1\left[(1-\lambda)-2\right] + 1\left[-3+(2-\lambda)\right] = 0$$

$$\Rightarrow \quad (1-\lambda)(2-\lambda)(1-\lambda) - 6(1-\lambda) - (1-\lambda) + 2 - 2 + (1-\lambda) = 0$$

$$\Rightarrow \quad (1-\lambda)\left[(2-\lambda)(1-\lambda)-6\right] = 0$$

$$\Rightarrow \quad (1-\lambda)\left[\lambda^2 - 3\lambda - 4\right] = 0$$

$$\Rightarrow \quad (1-\lambda)(\lambda+1)(\lambda-4) = 0$$

Eigenvalues are $\lambda = -1, 1, 4$

❖ For Eigenvectors :

Putting $\lambda_1 = -1$ in equation (5.9) we get

$$2x_1 + x_2 + x_3 = 0 \qquad (5.10)$$

$$x_1 + 3x_2 - 2x_3 = 0 \qquad (5.11)$$

$$x_1 - 3x_2 + 2x_3 = 0$$

From equation(5.10)

$$2\left[2x_3 - 3x_2\right] + x_2 + x_3 = 0$$

$$\Rightarrow \quad x_3 = x_2 \qquad (5.12)$$

From (5.10)&(5.12)

$$x_1 = -x_3$$

Thus eigenvector corresponds to eigenvalue $\lambda_1 = -1$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \text{ or simply } \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Normalized eigenvectors i.e. unit eigenvectors have length 1 i.e. sum of the squares of their components is 1.

Such eigenvectors can be found out by dividing each vector by the square root of the sum of the squares of the components.

Thus normalized eigenvector corresponding to $\lambda_1 = -1$ is

$$\frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

{Note : For normalized form of $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, we divide each element by $\sqrt{a^2+b^2+c^2}$ }

We put $\lambda_2 = 1$ in equation(5.9) for getting eigenvector

$$x_2 + x_3 = 0$$
$$x_1 + x_2 - 2x_3 = 0$$
$$-x_1 - 3x_2 = 0$$
$$\text{i.e. } x_1 = -3x_2 \quad \& \quad x_3 = -x_2$$

Thus eigen vector corresponding to eigen value $\lambda_2 = 1$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ x_2 \\ -x_2 \end{bmatrix} = x_2\begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix} \qquad \text{or simply } \begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix}$$

The normalized eigenvector is $\begin{bmatrix} -\frac{3}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \\ \frac{-1}{\sqrt{11}} \end{bmatrix}$

We put $\lambda_3 = 4$ in equation(5.9) for getting eigenvector

$$-3x_1 + x_2 + x_3 = 0 \tag{5.13}$$

$$x_1 + 2x_2 - 2x_3 = 0 \tag{5.14}$$

$$-x_2 - 3x_2 - 3x_3 = 0 \tag{5.15}$$

From (5.15)

$$x_1 = -3(x_2 + x_3)$$

From (5.15) &(5.13)

$$9(x_2 + x_3) + (x_2 + x_3) = 0$$

i.e. $x_2 = -x_3$

From (5.15) $x_1 = 0$

Thus eigenvector corresponding to eigenvalue $\lambda_3 = 4$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \text{or simply} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

The normalized eigen vector is $\begin{bmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

Example 5.8 Obtain the eigen vectors for matrix $A = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$

Sol. We consider equation $AX = \lambda X$ i.e.

$$\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5.16}$$

$$2x_1 + 2x_2 = \lambda x_1$$

$$-2x_1 + 2x_2 = \lambda x_2$$

We can write above equations as

$$\left.\begin{array}{l} (2-\lambda)x_1 + 2x_2 = 0 \\ -2x_1 + (2-\lambda)x_2 = 0 \end{array}\right\} \tag{5.17}$$

The system will have nontrivial solution if

So $$\begin{vmatrix} 2-\lambda & 2 \\ -2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)^2 + 4 = 0$$

$$\Rightarrow \quad (2-\lambda)^2 = -4$$

$$\Rightarrow \quad (2-\lambda)^2 = i^2 4$$

$$\Rightarrow \quad (2-\lambda) = \pm 2i$$

$$\Rightarrow \quad \lambda = 2-2i, 2+2i$$

Eigen values are $\lambda_1 = 2-2i, \lambda_2 = 2+2i$

❖ Eigenvector for $\lambda_1 = 2-2i$

Corresponding to $\lambda_1 = 2-2i$ eq.(5.17) become

$$2i\,x_1 + 2x_2 = 0 \Rightarrow x_2 = -i\,x_1$$

$$\& \ -2x_1 + 2ix_2 = 0 \Rightarrow x_2 = -ix_1 \quad \because i^2 = -1$$

i.e. $x_2 = -ix_1$

Then eigenvector is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -ix_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad \text{or simply} \begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{5.18}$$

Any vector which is scalar (constant) multiple of $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ will be eigenvector of matrix A corresponding to eigenvalue $\lambda_1 = 2-2i$

❖ Eigenvector for $\lambda_2 = 2+2i$

Using $\lambda_2 = 2+2i$ in equation(5.17) we get

$$-2i\,x_1 + 2x_2 = 0 \Rightarrow x_2 = i\,x_1$$

$$\& \ -2x_1 - 2ix_2 = 0 \Rightarrow x_2 = ix_1 \quad \because i^2 = -1$$

i.e. $x_2 = ix_1$

Then eigenvector is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ ix_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ or simply } \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Important properties of Eigenvalues :

(i) The product of eigenvalues of a matrix A is equal to the determinant of the matrix $\boxed{\det A = \lambda_1.\lambda_2.\lambda_3......\lambda_n}$

Where $\lambda_1, \lambda_2, \lambda_3, ...\lambda_n$ are eigenvalues of a matrix. Set of eigen values{ $\lambda_1, \lambda_2, \lambda_3, ...\lambda_n$ } is called spectrum of matrix A.

(ii) The sum of eigenvalues of a matrix is equal to the trace of the matrix i.e. $\boxed{Trace\, A = \lambda_1 + \lambda_2 + ...\lambda_n}$

(iii) Eigenvalues of any matrix A and its transpose A^T are same.

(iv) If eigenvalues of A are $\lambda_1, \lambda_2, \lambda_3, ...\lambda_n$ then

- Eigenvalues of pA are $p\lambda_1, \text{p}\lambda_2, \text{p}\lambda_3, ...\text{p}\lambda_n$, where p is non zero scalar.
- Eigenvalues of A^m are $\lambda_1^m, \lambda_2^m, ...\lambda_n^m$
- Eigenvalues of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, ...\frac{1}{\lambda_n}$

(v) ***The eigenvalues of a diagonal matrix are the elements in the diagonal.***

(vi) ***The eigenvalues of a Hermitian matrix (or real symmetric matrix) are real.***

(vii) ***The eigenvalues of a skew Hermitian matrix (or real skew symmetric matrix) are either zero or pure imaginary numbers.***

(viii) ***The eigenvalues of a unitary (or real orthogonal matrix) are of unit modulus.***

(ix) Every square matrix satisfies its own characteristic equation. This is known as ***Cayley-Hamilton theorem***.

Suppose characteristic polynomial of matrix A is given by

$a_0 + a_1\lambda + a_2\lambda^2 + ...a_n\lambda^n = 0$

Where $a_0, a_1, a_2, ...a_n$ are constants and λ is eigenvalue, then

$a_0 I + a_1 A + a_2 A^2 + ...a_n A^n = 0$

(*Cayley-Hamilton Theorem*)

(x) ***Any two eigenvectors corresponding to two distinct eigenvalues of Hermitian (or real symmetric) matrix are orthogonal.***

(xi) ***Any two eigenvectors corresponding to two distinct eiganvalues of Anti Hermitian (or real asymmetric) matrix are orthogonal.***

(xii) An $n \times n$ matrix B is called similar to A if there is a non singular $n \times n$ matrix P such that $B = P^{-1}AP$

Similar matrices have the same eigenvalues. If an eigenvector of matrix A is X then $Y = P^{-1}X$ will be an eigenvector of B corresponding to the same eigenvalue.

5.12 Diagonalization of a Matrix

In many physical situations it is desirable to reduce the matrix to a diagonal form (non diagonal elements all equal to zero) .Moment of inertia *I* of a rigid body is direct example of this diagonalization process.

Theorem : If a non singular square matrix A of order $n \times n$ has n linearly independent eigenvectors, then a matrix X can be found such that $\boxed{D = X^{-1}AX}$ is a diagonal with the eigenvalues of A as the matrix entries on the main diagonal. Here X is the matrix with these eigenvectors as column vectors.

For example , if distinct eigenvalues of $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ are λ_1, λ_2

and corresponding eigenvectors

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \text{and } X_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

We can write $X = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$

Then $D = X^{-1}AX = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

Now we can say that *A* has been reduced to diagonal form.

We can prove this in following way

Characteristic equation is given by $|A - \lambda I| = 0$ i.e. for eigenvalue λ_1

$$\left.\begin{aligned} (a_1 - \lambda_1)x_1 + b_1 y_1 = 0 \quad &\Rightarrow a_1 x_1 + b_1 y_1 = \lambda_1 x_1 \\ a_2 x_1 + (b_2 - \lambda_1) y_1 = 0 \quad &\Rightarrow a_2 x_1 + b_2 y_1 = \lambda_1 y_1 \end{aligned}\right\} \tag{5.19}$$

Similarly for eigenvalue λ_2

$$\left.\begin{aligned} a_1 x_2 + b_1 y_2 = \lambda_2 x_2 \\ a_2 x_2 + b_2 y_2 = \lambda_2 y_2 \end{aligned}\right\} \tag{5.20}$$

Now $AX = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$

$$= \begin{bmatrix} a_1 x_1 + b_1 y_1 & a_1 x_2 + b_1 y_2 \\ a_2 x_1 + b_2 y_1 & a_2 x_2 + b_2 y_2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 \\ \lambda_1 y_1 & \lambda_2 y_2 \end{bmatrix}$$

$$AX = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$AX = XD$ where $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$\Rightarrow \quad X^{-1}(AX) = X^{-1}(XD)$

$\Rightarrow \quad X^{-1}AX = ID \qquad \because X^{-1}X = I$

$\Rightarrow \quad X^{-1}AX = D$

Illustrative Examples

Example 5.9 If matrix $A = \begin{bmatrix} \alpha & -4 \\ 6 & \beta \end{bmatrix}$ has eigenvalues 4,6.Then find the values of α and β

Sol. Trace=$\alpha + \beta = 4 + 6 = 10$

Det A=$\alpha.\beta + 24 = 4.6$

$\Rightarrow \quad \alpha.\beta = 0$

If $\alpha = 0$, then $\beta = 10$ or

If $\beta = 0$, then $\alpha = 10$

Example 5.10 A matrix of order 2×2 is given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

For this matrix prove that

$\lambda^2 - \lambda.TraceA + Det.A = 0$,where λ is the eigen value of the matrix A.

Sol. Characteristic equation is given by

$$Det(A - \lambda I) = 0$$

$$\begin{vmatrix} (a-\lambda) & b \\ c & (d-\lambda) \end{vmatrix} = 0$$

$\Rightarrow \quad (a-\lambda)(d-\lambda)-bc=0$

$\Rightarrow \quad ad+\lambda^2-\lambda a-\lambda d-bc=0$

$\Rightarrow \quad \lambda^2-\lambda(a+d)+(ad-bc)=0$

$\Rightarrow \quad \lambda^2-\lambda.TraceA+Det.A=0$ Hence Proved

5.13 Self Learning Exercise-II

Very Short Answer Type Questions

Q.1 Find the eigenvalues of the following matrix

$$\begin{bmatrix} 8 & 2 & 4 \\ 0 & 2 & 0 \\ 0 & 6 & 5 \end{bmatrix}$$

Q.2 The eigen values of a Hermitian Matrix are always real. Is this statement true?

Short Answer Type Questions

Q.3 Find the eigenvalues of the following matrix A

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Q.4 The determinant of 3×3 real symmetric matrix is 18 and two of its eigenvalues are 2 and 3 . What is the sum of the eigenvalues ?

5.14 Summary

1. The matrix A is called orthogonal if

   $AA^T = I = A^T A$ or $A^T = A^{-1}$

2. A square matrix $A=\left[a_{ij}\right]$ is Hermitian if $A=A^\theta$

   i.e. $a_{ij}=a^*_{ji}$ for all values of i and j

3. A square matrix $A=\left[a_{ij}\right]$ is skew Hermitian if $A=-A^\theta$

i.e. $a_{ij} = -a^*_{ji}$ for all values of i and j

4. A square finite matrix U is unitary if $UU^\theta = I$

   i.e. $U^\theta U = I = UU^\theta$

5. Value of λ for which equation $AX = \lambda X$ has a solution $X \neq 0$ (i.e. a nontrivial solution) is called an eigen value of matrix A.

   $\det.(A - \lambda I) = 0$

5.15 Glossary

Trace: The sum of the elements in the principal diagonal of a square matrix.

Skew: not symmetrical

Conjugate: having the same real parts and equal magnitudes but opposite signs of imaginary parts.

5.16 Answers to Self Learning Exercises

Answers to Self Learning Exercise-I

Ans.1: For Unitary matrix $U^\theta = U^{-1}$

$$\text{Here}\left(U^{-1}\right)\left(U^{-1}\right)^\theta = U^\theta\left(U^\theta\right)^\theta$$

$$= U^\theta U = I$$

Ans.2: Hint- Show that $(iA)^\dagger = -(iA)$

Ans.4: $\dfrac{1}{\sqrt{2}}$

Answers to Self Learning Exercise-II

Ans.1: $\lambda = 2, 5, 8$

Note: eigen values of triangular matrix are diagonal elements in that matrix.

Ans.2: Yes

Ans.3: $\lambda_1 = \cos\theta + i\sin\theta = e^{i\theta}$ $\quad, \lambda_2 = (\cos\theta - i\sin\theta) = \mathrm{e}^{-i\theta}$

Ans.4: Product of Eigen values=determinant of the Matrix

$\Rightarrow \quad \lambda_1.\lambda_2.\lambda_3 = 18$

$\Rightarrow \quad 2.3.\lambda_3 = 18 \quad \Rightarrow \lambda_3 = 3$

Sum of the eigenvalues

$\lambda_1 + \lambda_2 + \lambda_3 = 2 + 3 + 3 = 8$

5.17 Exercise

Section A: Very Short Answer Type Questions

Q.1 "If the eigenvalues of the anti symmetric matrix A is zero and its determinant is also zero, then each eigenvalue of the matrix A must be zero."Is this statement true?

Q.2 "If one of the eigen value of matrix A is zero, then matrix A must be singular."Is this statement true?

Q.3 **"**A matrix can have real eigen values without being Hermitian".Is this statement true?

Section B : Short Answer Type Questions

Q.4 Show that matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} -1 & 1-i \\ 1+i & 1 \end{bmatrix}$ is Unitary matrix.

Q.5 Show that $\begin{bmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{bmatrix}$ is Unitary matrix

Q.6 Prove that for any square matrix A

(i) $\left(A + A^{\theta}\right)$ is a Hermitian matrix

(ii) $\left(A - A^{\theta}\right)$ is skew Hermitian matrix

(iii) AA^{θ} is a Hermitian matrix

Q.7 Find the eigenvalues of the given matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Q.8 A matrix is given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{bmatrix}$$

Its eigenvalues are $\lambda_1, \lambda_2, \lambda_3$

Find (i) $\lambda_1 + \lambda_2 + \lambda_3$ (ii) $\lambda_1.\lambda_2.\lambda_3$

Section C : Long Answer Type Questions

Q.8 Prove that any complex square matrix can always be expressed as the sum of a Hermitian matrix and a skew Hermitian matrix.

5.18 Answers to Exercise

Ans.1 : No

Ans.2 : Statement is true.

Hint: $\lambda_1.\lambda_2.....\lambda_n = 0$

$\because Det.A = \lambda_1.\lambda_2.....\lambda_n$

$\therefore Det.A = 0$

Thus A is singular Matrix.

Ans.3 : Yes, See example 5.6

Ans.7 : 0,1,2

Ans.8 : (i) Trace= Sum of eigenvalues= $\lambda_1 + \lambda_2 + \lambda_3 = 0 + 0 + 0 = 0$

(ii) Det.A= $\lambda_1.\lambda_2.\lambda_3$

$$= 0(0-2) - 1(0-2) + 0(0-0) = 4$$

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UNIT- 6

Complex Algebra: Cauchy –Riemann Conditions, Cauchy's Integral Theorem

Structure of the Unit

6.0 Objectives

Complex numbers are widely used in modern mathematics and its applications. It turns out that it is convenient to obtain many relationships between real quantities by using complex numbers and functions in intermediate calculations. This chapter is intended to introduce many of the useful results of complex function theory. We shall derive the conditions that a complex function theory. We shall derive the conditions that a complex function $W(z)$ has to satisfy so as to have a unique derivative at the point z. Such a function is said to be analytic at the point z.

6.1 Introduction

In this chapter we develop some of the most powerful and widely useful tools in all of mathematical analysis. These include Cauchy-Riemann conditions and Cauchy's integral theorem. We introduce some elementary function of Z and find their real and imaginary parts. We also discuss their analyticity. We define branch points and branch lines etc.

6.2 Functions of a Complex Variable

"The imaginary numbers are a wonderful flight of God's spirit; they are almost an amphibian between being and not being."

Gotterfied Wilhelm Von Leibniz (1702)

Why complex variables are so important? It will be evident several areas of applications :

1. For many pairs of functions u and v, both u and v satisfy Laplace's equation :

$$\nabla^2 \Psi = \frac{\partial^2 \Psi(x,y)}{\partial x^2} + \frac{\partial^2 \Psi(x,y)}{\partial y^2} = 0$$

Either u or v may be used to describe a two dimensional electrostatic potential. The other function that gives a family of curves orthogonal to those of the first function, may then be used to describe the electric field $\vec{E}$.

In similar case of hydrodynamics : $u \rightarrow$ velocity potential, $v \rightarrow$stream function

u and v : create a co-ordinate system.

2. Second order differential equation – power series

If $f(z)$ at z_0 is given, the behavior of $f(z)$ elsewhere is knowable (analytic continuation).

3. The change of parameter k from real to imaginary, $k \rightarrow ik$ transforms the Helmholtz equation into diffusion equation.

4. Integrals in the Complex plane have a wide variety of useful applications :

(a) Evaluating definite integrals

(b) Inverting power series

(c) Forming infinite products

(d) Stability of oscillating systems

6.3 Complex Algebra

- A complex number is nothing more than an ordered pair of two ordinary(real) numbers (a, b)

We can write this pair as $a + ib$.

- Similarly, a complex variable is an ordered pair of two real variables, $\boxed{z = (x, y) = x + iy}$ All our complex variable analysis can be developed in terms of ordered pairs of numbers (a, b), variables (x, y) and functions $[u(x, y), v(x, y)]$.The i is not necessary but it is convenient.

- Argand Diagram

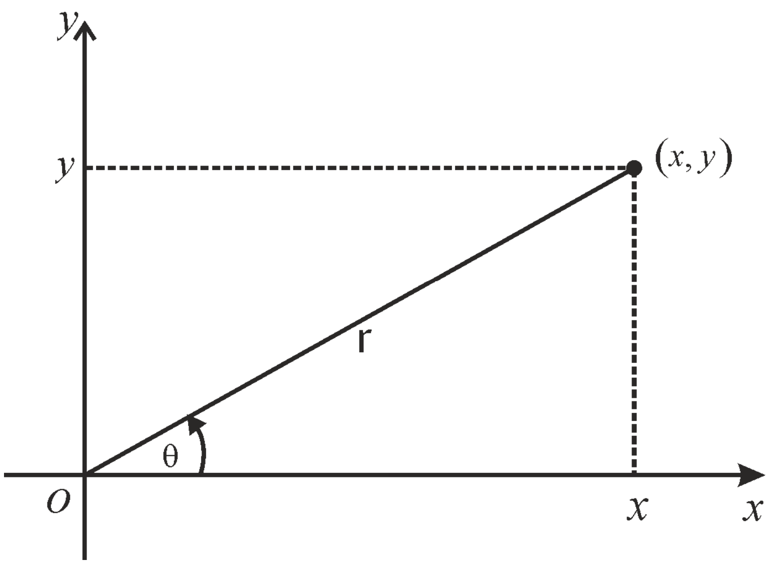

Figure 6.1 Complex Plane

Complex variable $\boxed{z = r[\cos\theta + i\sin\theta]}$

$$\boxed{z = re^{i\theta}}$$

$$\boxed{r = |z|}$$

r is called modules of z

θ is called argument or plane of z .

- **Triangle Inequalities :**

$$\boxed{|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2|}$$

Using Polar form we can show

$$\boxed{|z_1 . z_2| = |z_1| . |z_2|}$$

Also $\boxed{\arg(z_1 . z_2) = \arg z_1 + \arg z_2}$

Complex function $w(z)$ may be resolved into real and complex parts as $w(z) = u(x,y) + i\, v(x,y)$

- The relationship between the independent variable z and dependent variable w is best pictured as a mapping operations :

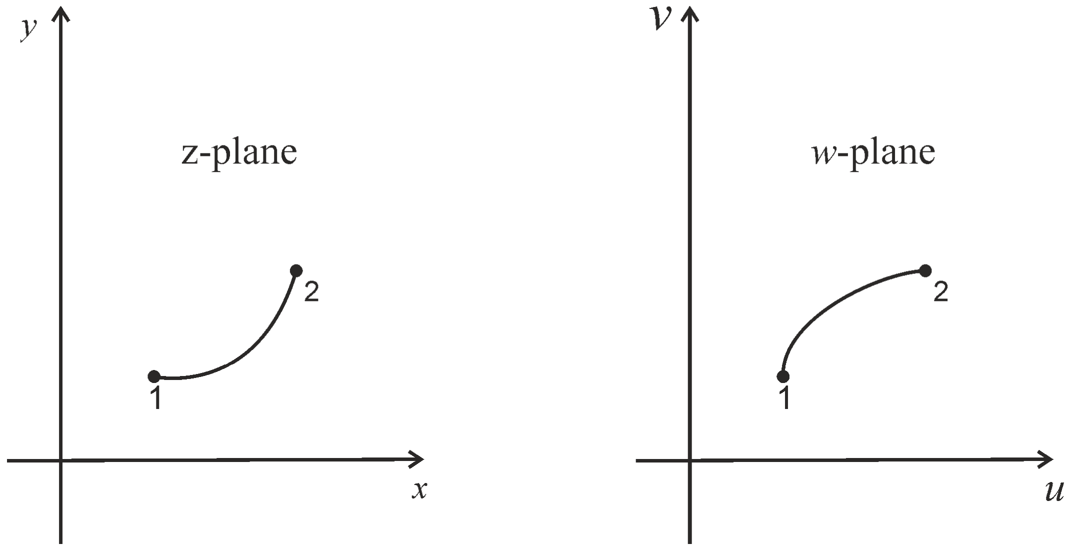

Figure 6.2 The function w(z)=u(x,y)+iv(x,y) maps points in the xy-plane into points in the uv plane

- **Complex Conjugation :**

Complex Conjugation of an expression is change from i to $-i$.

Complex Conjugate Points :

$$\boxed{z = x + iy} \quad \boxed{z^* = x - iy}$$

$$\boxed{zz^* = (x^2 + y^2) = r^2}$$

$$\boxed{(zz^*) = |z|}$$

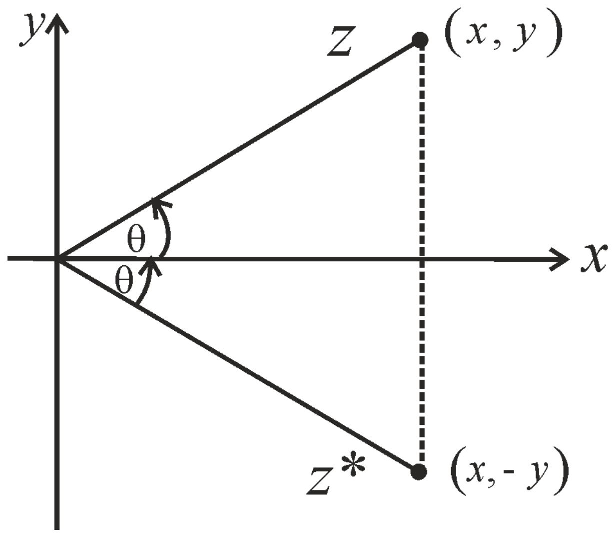

Figure 6.3 Complex conjugate points

- **Functions of a Complex Variable :**

All the elementary functions of real variables may be extended into the complex plane – replacing the real variable x by complex variable z . This is an example of analytic continuation.

- **De Moivre's Formula :**

$$\boxed{e^{i\theta} = (cos\theta + i\,sin\theta)}$$

$$\boxed{\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n}$$

$$\ln z = \ln r\, e^{i\theta}$$
$$= \ln r + i\,\theta \quad \textit{This is not complete}$$

We can add $2n\pi$ in θ with no change in z

$$\ln z = \ln\{r\, e^{i(\theta+2n\pi)}\} = \ln r + i\,(\theta + 2n\pi)$$

$\therefore \ln z$ is a multivalued function having an infinite number of values for a single pair of real values r and θ.

To avoid ambiguity, we usually agree to set $n = 0$ and limit the phase to an interval of length 2π such as $(-\pi, \pi)$. The line in the z-plane that is not crossed the negative real axis in this case , is labeled a cut line. The value of $ln\ z$ with $n = 0$ is called principal value of $ln\ z$.

- **Analytic Functions :** If $f(z)$ is differentiable at $z = z_0$ and in some small region around z_0, we say that $f(z)$ is analytic at $z = z_0$.

If $f(z)$ is analytic everywhere in the (finite) complex plane, we call it an entire function. If $f'(z)$ does not exist at $z = z_0$, then z_0 is labeled a singular point.

6.4 Cauchy-Riemann Conditions

We know proceed to differentiate complex functions of a Complex variables.

The derivative of $f(z)$, like that of a real function, is defined as :

$$\lim_{\delta z \to 0} \frac{f(z + \delta z)}{z + \delta z - z} = \lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z}$$

$$= \frac{df}{dz}\ or\ f'(z)$$

Provided that the limit is independent of the particular approach to the point z.

For real variables we require that the right hand limit ($x \to x_0$ from above) and the left hand limit ($x \to x_0$ from below) be equal for the derivative $\frac{df(x)}{dx}$ to exist at $x = x_0$.

Now with $z(or\ z_0)$ some point in a plane, our requirement that the limit be independent of path.

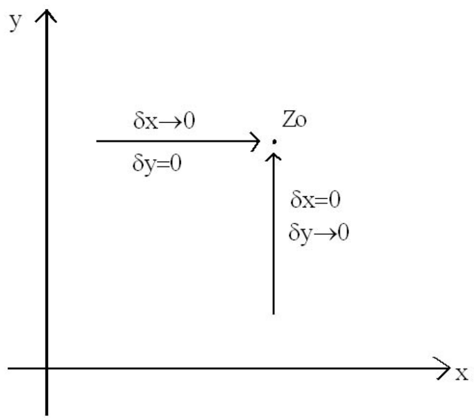

Figure 6.4 Alternate approach to z_0

Consider increments δx and δy of the variables x *and* y respectively Then

$$\delta z = \delta x + i\delta y$$

Also $$\delta f = \delta u + i\delta v$$

So that $$\frac{\delta f}{\delta Z} = \frac{\delta u + i\,\delta v}{\delta x + i\,\delta y}$$

For I-approach ($\delta y = 0, \delta x \to 0$)

Now $$\lim_{\delta Z \to 0} \frac{\delta f}{\delta Z} = \lim_{\delta x \to 0} \left(\frac{\delta u + i\,\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \qquad (1)$$

For II-approach ($\delta x = 0, \delta y \to 0$)

$$\lim_{\delta y \to 0} \left(\frac{\delta u + i\delta v}{i\,\delta y}\right) = -i\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad (2)$$

If we are to have a derivative $\frac{df}{dz}$, these two limits must be identical. Equating real and imaginary parts we obtain

For function $f(z) = u + iv$ to be analytic following conditions are required

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

These are ***Cauchy Riemann conditions***.

C.R Conditions were discovered by Cauchy and used extensively by Riemann in his theory of Analytic functions. These conditions are necessary for $\frac{df}{dz}$ to exist. It is worth noting that the C.R. condition guarantees that the curves $u = c_1$ will be orthogonal to the curves $v = c_2$.

6.5 Illustrative Examples

Example 6.1 If $f(z)\ or\ w(z) = z^2$,then find out real and imaginary part of $f(z)$

Sol. $f(z) = (x + iy)^2$

$= (x^2 - y^2) + i\,2xy$

$$Real\ part\ of\ \ f(z) = u(x, y) = x^2 - y^2$$

$$Imaginary\ part\ of\ \ w(z) = v\,(x, y) = 2xy$$

Example 6.2 $f(z) = x - i\,y$,State whether this function analytic or not analytic.

Sol. $u = x, v = -y$

$$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 0\ ;\ \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = -1$$

CR equations are not satisfied.

Nowhere analytic.$f(z)$is continuous everywhere but nowhere differentiable.

Example 6.3 $f(z) = z\,\bar{z} = x^2 + y^2$

Is this function analytic?

Sol. $u = x^2 + y^2, v = 0$

$$\frac{\partial u}{\partial x} = 2x\ , \frac{\partial u}{\partial y} = 2y\ ;\ \frac{\partial v}{\partial x} = 0\ , \frac{\partial v}{\partial y} = 0$$

are continuous everywhere, However the Cauchy Riemann equation is $2x = 0, 2y = 0$ are satisfied only at the origin. Hence $z = 0$ is the only point at which $f'(z)$ exists and therefore $f(z) = z\bar{z}$ nowhere analytic.

The Elementary functions of z :

The exponential function e^z is of fundamental importance, not only for its own sake but also as a basis for defining all the other transcendental functions. In its definition we seek to preserve as many of the familiar properties of the real exponential function e^x as possible. Specifically we desire that

a. e^z shall be single valued and analytic

b. $\frac{de^z}{dz} = e^z$

c. e^z shall reduce to e^x when $Im(z) = 0$

If we let

$$e^z = u + i\,v \tag{a}$$

The derivative of an analytic function can be written in the form :

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

Now, to satisfy the condition b, we must have :

$$\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = u + iv$$

$$\therefore \frac{\partial u}{\partial x} = u \tag{b}$$

$$\frac{\partial v}{\partial x} = v \tag{c}$$

Eq. (b) will be satisfied if $u = e^x \phi(y)$

Also form (c)

$$\frac{\partial v}{\partial x} = v$$

$$or\quad -\frac{\partial u}{\partial y} = v$$

$$or\quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial v}{\partial y} \text{ (by CR Condition)}$$

$$or\quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial u}{\partial x} = -u \quad from\ (b)$$

Substituting $u = e^x \phi(y)$ in eq. $\left(\frac{\partial^2 u}{\partial y^2} = -u\right)$

$$e^x \phi''(y) = -e^x \phi(y)$$

$$or\quad \phi''(y) = -\phi(y)$$

This is a simple linear differential equation whose solution can be written down at once :

$$\phi(y) = A\cos y + B\sin y$$

$$\therefore u = e^x \phi(y)$$

$$= e^x[A\cos y + B\sin y]$$

$$\text{and } v = -\frac{\partial u}{\partial y} = -e^x[-A\sin y + B\cos y]$$

$$\therefore e^z = u + iv = e^x[(A\cos y + B\sin y) + i\,(A\sin y - B\cos y)]$$

Finally, if this is to reduce to e^x when $y = 0$ as required, we must give

$e^x[A - i\,B]$, which will be true if and only if $A = 1, B = 0.$

Thus we have been led inevitably to the conclusion that if there is a function of z satisfying the conditions(a),(b),(c) then it must be :

$$e^z = e^{x+iy} = e^x[\cos y + i\sin y] \qquad \text{(d)}$$

That this expression does indeed meet our requirements can be checked immediately ; hence we adopt its as the definition of e^z.

mod $e^z = |e^z| = e^x$

and $\arg e^z = y$

From eq.(d)

If $x = 0, y = \theta$ we have

$$\cos\theta + i\sin\theta = e^{i\theta}$$

and thus $r\,[\cos\theta + i\sin\theta] = re^{i\theta}$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$\boxed{e^{-i\theta} = \cos\theta - i\sin\theta}$$

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \ \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

- On the basis of these equations, we extend the definition of *sine* and *cosine* into the complex domain by the formulas:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

From these definitions it is easy to establish the validity of such familiar formulas as :

$$cos^2 z + sin^2 z = 1$$

$$\cos (z_1 \pm z_2) = \cos z_1 \cos z_2 \pm \sin z_1 \sin z_2$$

$$\sin (z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\frac{d}{dt} \cos z = -\sin z$$

$$\frac{d}{dt} \sin z = \cos z$$

$$\cos z = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \frac{e^{-y} e^{ix} + e^{y} e^{-ix}}{2}$$

$$= \cos x \frac{e^{y} + e^{-y}}{2} - i \sin x . \frac{e^{y} - e^{-y}}{2} \equiv \cos x \cos hy - i \sin x \sinh y$$

- The logarithm of z, we define implicitly as the function $w = lnz$ which satisfies the equation:

$$e^{w} = z \qquad (1)$$

Let $w = u + i\,v, z = r\,e^{i\theta}$

$$\therefore\ e^{u+iv} = re^{i\theta}$$

$$e^{u} e^{iv} = re^{i\theta}$$

$$e^{u} = r \ \therefore u = \ln r$$

$$e^{iv} = e^{i\theta} \ \therefore v = \theta$$

$$w = \ln r + i\theta$$

$$w = \ln|z| + i \arg z \quad (2)$$

➢ If we let θ_1 be the principal argument of z, i.e. the particular argument of z which lies in the interval then $-\pi < \theta \leq \pi, then\ (2)$ can be written as

$$ln\ z = \ln|z| + i\,(\theta_1 + 2n\pi), \quad n = 0, \pm 1, \pm 2, \dots \dots \dots$$

Which shows that the logarithmic function is infinitely many valued.

- For any particular value of n a unique branch of the function is determined and the logarithm becomes effectively single valued.
- If $n = 0$, the resulting branch of the logarithmic function is called the principal value.
- For every n, the corresponding branch of $\ln z$ is obviously discontinuous at $z = 0$. Moreover, for each n the corresponding branch is also discontinuous at every point of the negative real axis.

To verify this, we note that if $n = n_0$, the corresponding branch of $\ln z$ is

$$= \ln|z| + i \arg z, where\ (2n_0 - 1)\pi < \arg z \ \leq (2n_0 + 1)\pi$$

Hence if P is an arbitrary point on the negative real axis, the limit of $arg\ z$ as z approaches P through the second quadrant is $(2n_0 + 1)\pi$. While the limit of $arg\ z$ as z approaches P through the third quadrant is $(2n_0 - 1)\pi$.

Since these two values are different, it follows that on any particular branch, $ln\ z$ does not approach a limit as z approaches and arbitrary point on the negative real axis and therefore is discontinuous at every such point.

At all points except the points on the nonpositive real axis each branch of $\ln z$ is continuous and analytic. In fact from the definition.

$$\boxed{\ln z = \ln|z| + i \arg z}$$

$$ln\ z = \frac{1}{2}\ln(x^2 + y^2) + i\ tan^{-1}\frac{y}{x}$$

$=u+iv$

It is easy to verify that the Cauchy-Reimann equations are satisfied everywhere at the origin. Moreover from the preceding discussions, it is clear that

$$u = \frac{1}{2}\ln(x^2 + y^2) \quad , \quad v = \ tan^{-1}\frac{y}{x}$$

are continuous except on the nonpositive real axis.

Hence, by Cauchy Reimann theorem it follows that everywhere except on the nonpositive real axis :

The familiar laws for the logarithms of real quantities all hold for the logarithms of complex quantities in the following sense. If a suitable choice is made among the infinite number of possible values of $\ln(z_1 z_2), \ln\frac{z_1}{z_2}$, $\ln z^n$,then

$$\ln z_1\, z_2 = \ln z_1 + \; ln\, z_2$$

$$\ln\frac{z_1}{z_2} = \ln z_1 - \; ln\, z_2$$

$$\ln z^m = m \ln z$$

For example, to show that $\ln z_1 \;\, z_2 = \ln z_1 + \; ln\, z_2$

Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$. Then

$$\ln z_1 + \; ln\, z_2 \;\; =[\ln r_1 + i\,(\theta_1 + 2n_1\pi)] + [\ln r_2 + i\,(\theta_2 + 2n_2\pi)]$$

$$=[\ln r_1 + \ln r_2] + i[(\theta_1 + \theta_2) + 2\,(n_1 + n_2)\pi]$$

$$= \ln r_1 r_2 + i\,[(\theta_1 + \theta_2) + 2n_3\pi]$$

$$= \ln|z_1 z_2| + i \arg z_1 z_2 = ln z_1 z_2$$

Since $\theta_1 + \theta_2 + 2\,(n_1 + n_2)\pi$ is one of the arguments of $z_1 z_2$

However, the familiar laws of logarithms are not necessarily true if we restrict ourselves to a particular branch of $\ln z$.

Since $\ln z = \ln|z| + i \arg z$

$$\ln[i(-1+i)] = \ln(-1-i) = \ln\sqrt{2} - i\frac{3\pi}{4}$$

$$where \quad \ln i + \ln(-1+i) = i\frac{\pi}{2} + \left(\ln\sqrt{2} + i\frac{3\pi}{4}\right)$$

$$= \ln\sqrt{2} + \; i\frac{5\pi}{4}$$

Clearly, the principal value of $ln\,[i(-1+i)]$ differs from the sum of the principal values of $ln\; i$ and $ln\;(-1+i)$ by $2\pi\; i$.

For principal values, the proper generalization of the familiar laws of logarithms are contained in the following theorem whose proof we shall leave as an exercise.

Note :

$$(1)\ \ln z_1 z_2 = \begin{cases} \ln z_1 + \ln z_2 - 2i\pi\ ; & \pi < \arg + \arg z_2 \le 2\pi \\ \ln z_1 + \ln z_2\ ; & -\pi < \arg z_1 + \arg z_2 \le \pi \\ lnz_1 + \ln z_2 + 2\,i\,\pi\ ; & 2\pi < \arg z_1 + \arg z_2 \le -\pi \end{cases}$$

$$\ln \frac{z_1}{z_2} = \begin{cases} \ln z_1 - \ln z_2 - 2i\pi\ ; & \pi < \arg z_1 - \arg z_2 \le 2\pi \\ \ln z_1 - \ln z_2\ ; & -\pi < \arg z_1 - \arg z_2 \le \pi \\ lnz_1 - \ln z_2 + 2\,i\,\pi\ ; & -2\pi < \arg z_1 - \arg z_2 \le -\pi \end{cases}$$

$$\ln z^m = m \ln z - 2\,k\,i\,\pi$$

m an integer

Where k is the unique integer such that

$$\left(\frac{m}{2\pi} argz - \frac{1}{2}\right) \le k \le \left(\frac{m}{2\pi} \arg z + \frac{1}{2}\right)$$

(2) General powers of z are defined by the formula.

$$z^\alpha = \exp(\alpha \ln z)$$

Since $ln\ z$ is infinitely many valued, so too in z^α, in general specifically.

$$z^\alpha = \exp\ (\alpha \ln z) = \exp\{\alpha\ [\ln|z| + i\ (\theta_1 + 2n\pi)]\}$$
$$= \exp\ (\alpha \ln|z|) e^{\alpha\theta_1} . e^{2n\,\alpha\,\pi\,i}$$

The last factor in the product clearly involves infinitely many different values unless α is a rational number, say $\frac{p}{q}$, in which case, as we saw in our discussion of De Movier's theorem, there are only q distinct values.

Example 6.4 What is the principal value of $(1+i)^{2-i}$

Sol. $z^\alpha = \exp(\alpha \ln z) = e^{lnz^\alpha}$

$$\therefore (1+i)^{2-i} = \exp[(2-i)\ln(1+i)]$$

$$= \exp\left\{(2-i)\left[\ln \sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right)\right]\right\}$$

$$\ln z = \ln|z| + i\ (\theta_1 + 2n\pi)$$

The principal value of this, obtained by taking $n = 0$, is

$$= \exp\left[(2-i)\left(\ln \sqrt{2} + i\frac{\pi}{4}\right)\right]$$

$$= \exp\left[\left(2\ln \sqrt{2} + \frac{\pi}{4}\right) + i\left(-\ln \sqrt{2} + \frac{\pi}{2}\right)\right]$$

$$= \exp\left(\ln 2 + \frac{\pi}{4}\right) + \left[\cos\left(\frac{\pi}{2} - \ln\ \sqrt{2}\right) + i \sin\left(\frac{\pi}{2} - \ln\ \sqrt{2}\right)\right]$$
$$= e^{1.4785}(\sin 0.3466 + i \cos 0.3466)$$
$$= 1.490 + 4.126\, i$$

- **The inverse trigonometric and hyperbolic function**

These functions we define implicitly.

For instance $w = cos^{-1}z$

We define as the value or values of w which satisfies the equation.:

$$z = \cos w = \frac{e^{i\,w} + e^{-i\,w}}{2}$$

From this, by obvious steps, we obtain successively

$$e^{2\,i\,w} - 2ze^{iw} + 1 = 0$$
$$e^{\,i\,w} = z \pm \sqrt{z^2 - 1}$$

and finally, by taking logarithms and solving for w.

$$w = cos^{-1}z = -i \ln(z \pm \sqrt{z^2 - 1})$$

Since the logarithm is infinitely many valued, so too is $cos^{-1}z$.

Similarly, we can obtain the formulas

$$sin^{-1}z = -i \ln(iz \pm \sqrt{1 - z^2})$$
$$tan^{-1}z = \frac{i}{2} \ln \frac{i + z}{i - z}$$
$$cosh^{-1}z = \ln(z \pm \sqrt{z^2 - 1})$$
$$sinh^{-1}z = \ln(z \pm \sqrt{z^2 - 1})$$
$$tanh^{-1}z = \frac{1}{2} \ln \frac{1 + z}{1 - z}$$

From, these after their principal values have been suitably defined by choosing the plus sign preceding the square root and the principal value of the logarithm in each case, the usual differentiation formulas can be obtained without difficulty.

- **Polynomial functions** :

These functions are defined by

$$w = a_0 z^n + a_1 z^{n-1} + a_{n-1} z + a_n = P(z) \qquad (1)$$

Where $a_0 \neq 0, a_1, a_2, \dots \dots \dots \dots, a_n$ are complex constants and n is a positive integer called the degree of the polynomial.

$w = a\,z + b$ *is called a linear transform*...(2)

- **Rational Algebraic Functions** : are defined by

$$w = \frac{P(z)}{Q(z)} \qquad (3) \quad P(z), Q(z) \text{ are polynomials}$$

(Sometimes (3) is called a rational transform)

Special case $w = \frac{a\,z+b}{c\,z+d}$, *where* $ad - bc \neq 0$

is often called a bilinear or fractional linear transform

- $.\, w = e^z = e^{x+iy} = e^x [\cos y + i \sin y]$

If a real and positive : $a^z = e^{z \ln a}$

- Logarithmic Functions : If $z = e^w$, *then we write*

$$w = \ln z$$

$$= \ln r + i\,(\theta + 2k\pi)\,, \qquad k = 0, \pm 1, \pm 2, \dots \dots ..$$

Principal value of $ln\,z$ (or principal branch) of $ln\,z = ln\,r + i\theta$, *where* $0 \leq \theta \leq 2\pi$

However, any other interval of length 2π can be used. e.g

$. -\pi < \theta \leq \pi$ *etc*.

If $z = a^w$

$$\therefore w = log_a z, \text{where } a > 0 \text{ and } a \neq 0, 1.$$

6.6 Self Learning Exercise-I

Section A : Very Short Answer Type Questions

Q.1 Write down the Cauchy Riemann condition in polar form for a function to be analytic?

Q.2 $f(z) = z^2 = (x^2 - y^2) + 2\,i\,xy$

Is this function analytic?

Section B: Short Answer Type Questions

Q.3 Obtain the general value of the log(1+i)+log(1−i)

Q.4 Prove $\overline{z_1 z_2 z_3} = \overline{z_1}\ \overline{z_2}\ \overline{z_3}$

6.7 Branch Points and Branch Lines

Suppose we are given the function $z^{\frac{1}{2}}$

We allow z to make a complete circuit around the origin starting from point A (Counter clockwise) $w = z^{\frac{1}{2}}$

$$z = r\, e^{i\,\theta}$$

$$\therefore w = r^{\frac{1}{2}}\, e^{i\frac{\theta}{2}}$$

at point A, $w = r^{\frac{1}{2}}\, e^{i\frac{\theta_1}{2}}$

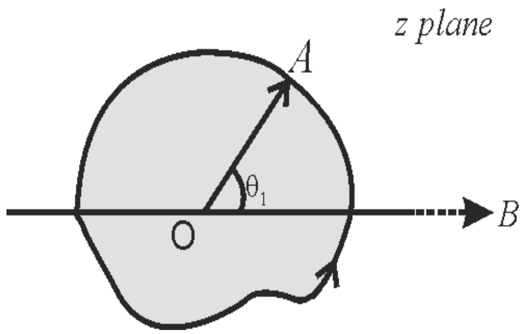

Figure 6.5

After a complete circuit back to A, $\theta = \theta_1 + 2\pi$

$$w = \sqrt{r}\ e^{\frac{i(\theta_1+2\pi)}{2}} = -\sqrt{r}\, e^{i\frac{\theta_1}{2}}$$

Thus we have not achieved the same value of w with which we started.

However, by making a second complete circuit back to A, i.e. $\theta = \theta_1 + 4\pi$,

$$w = \sqrt{r}\, e^{i\frac{(\theta_1+4\pi)}{2}} = \sqrt{r}\, e^{i\frac{\theta_1}{2}}$$

and we do obtain the same value of w with which we started. We can describe the above by stating that if $0 \le \theta < 2\pi$ we are on one branch of the multiple valued function $z^{\frac{1}{2}}$, while if $2\pi \le \theta < 4\pi$ we are on the other branch of the function.

Each branch of the function is single valued .In order to keep the function single valued, we setup an artificial barrier such as OB, where B is at ∞ (although any other line from O can be used) which we agree not to cross. This barrier (drawn heavy in the figure) is called a branch line or branch cut, and point O is called branch point.

It should be noted that a circuit around any point other than $z = 0$ does not lead to different values, thus $z = 0$ is the only finite branch point.

6.8 Illustrative Examples

Example 6.5 Evaluate $\int_{(0,3)}^{(2,4)}[(2y + x^2)\,dx + (3x - y)dy]$ along the parabola $x = 2t, y = t^2 + 3$.

Sol. Integral

$$= \int_0^1 [2(t^2 + 3) + 4t^2]2dt + \int_0^1 [3(2t) - (t^2 + 3)]2dt$$

$$= \left[\left(4\left(\frac{t^3}{3} + 3t\right) + 8\frac{t^3}{3}\right)_0^1 + \left(12\frac{t^2}{2} - 2\frac{t^3}{3} - 6t\right)_0^1\right]$$

$$= \left[4\left(\frac{1}{3} + 3\right) + \frac{8}{3} - 0 + 12 \times \frac{1}{2} - 2 \times \frac{1}{3} - 6\right]$$

$$= \left[4 \times \frac{10}{3} + \frac{8}{3} + 6 - \frac{2}{3} - 6\right]$$

$$= \frac{40}{3} + \frac{8}{3} - \frac{2}{3} = \frac{46}{3}$$

Example6.6 Evaluate $\int_c \bar{z}\, dz \; from \; z = 0 \; to \; z = 4 + 2\,i$ along the curve C given by $z = t^2 + it$

Sol. If $z = 0, \text{then } t = 0$

$$z = 4 + 2\,i, \text{then } t = 2$$

$\int_c \bar{z}\, dz = \int_0^2 (t^2 - it)(2tdt + idt)$

$$= \int_0^2 [2t^3\,dt - it^2dt + tdt] = 10 - \frac{8i}{3}$$

Theorem : Prove that if $f(z)$ is integrable along a curve C having finite length L and if there exists a positive number M such that $|f(z)| \le M$ on C then

$$\left|\int_C f(z)\,dz\right| \le ML$$

Proof : $\int_c f(z)dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\ \Delta z_k$

Now $|\sum_{k=1}^{n} f(\xi_k)\Delta z_k| \le \sum_{k=1}^{n} |f(\xi_k)||\Delta z_k|$

$$\le M \sum_{k=1}^{n} |\Delta z_k|$$

$$\le ML$$

Example 6.7 Prove Green's Theorem in the Plane i.e. prove

$$\oint_c (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Sol. $\because \iint_R \frac{\partial P}{\partial y}\,dx\,dy = \int_e^f \left[\int_{y=y_1(x)}^{y=y_2(x)} \frac{\partial P}{\partial y}\,dy\right] dx$

$$= \int_e^f [P(x,y)]_{y=y_1(x)}^{y=y_2(x)} dx \qquad = \int_e^f [P(x,y_2) - P(x,y_1)]dx$$

$$= -\int_e^f P(x,y_1)dx - \int_f^e P(x,y_2)dx$$

$$= -\oint P\,dx \quad (1)$$

$$similarly \iint_R \frac{\partial Q}{\partial x}\,dx\,dy = \oint Q\,dy \quad (2)$$

$$\therefore Adding\ (1)\ and\ (2):$$

$$\oint Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

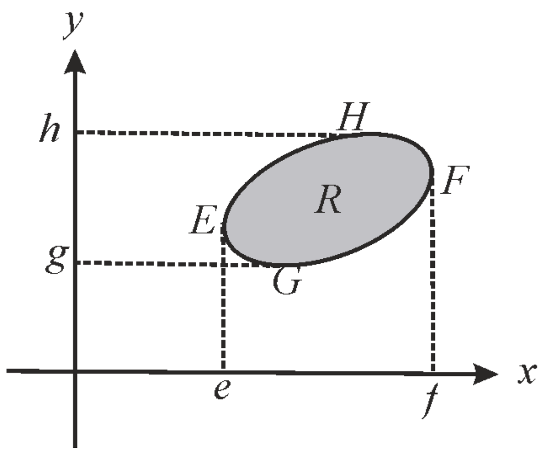

Figure 6.6

6.9 Cauchy's Integral Theorem

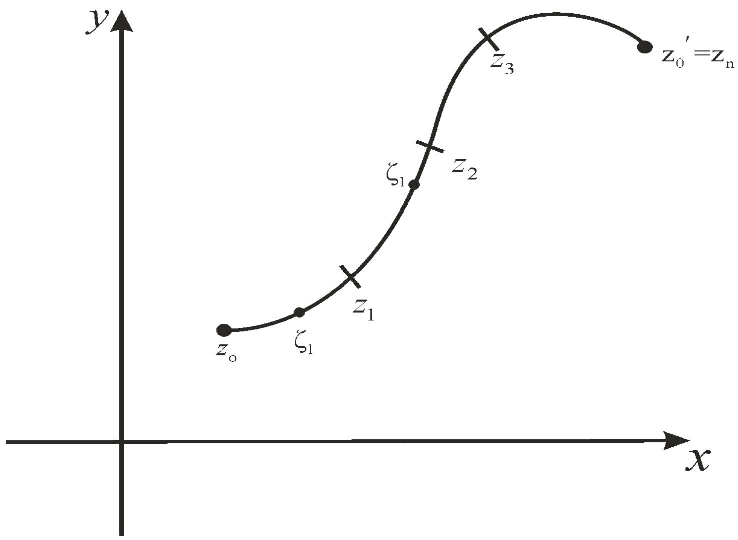

Figure 6.7 Integration path

The integral of a complex variable over a contour in the complex plane may be defined in close analogy to the (Riemann) integral of a real function integrated along the real x-axis.

We divide the contour $z_0 z_0'$ into n intervals by picking $n-1$ intermediate points $z_1, z_2, \ldots\ldots..$, on the contour. Consider the sum

$$S_n = \sum_{j=1}^{n} f(\xi_j)(z_j - z_{j-1})$$

Where ξ_j is a point on the curve between z_j and z_{j-1}. Now let $n \to \infty$ with $|z_j - z_{j-1}| \to 0$ for all j. If the $\lim_{n\to\infty\infty}, S_n$ exists and is independent of the details of choosing the points z_j and ξ_j then

$$\lim_{n\to\infty\infty} \sum_{j=1}^{n} f(\xi_j)(z_j - z_{j-1}) = \int_{Z_0}^{Z_0'} f(z)dz$$

If a function $f(z)$ is analytic (therefore single valued) and its partial derivatives are continuous throughout some simply connected region R (A simply connected region or domain is one in which every closed contour in that region encloses only the points contained in it), the line integral of $f(z)$ around C is zero :

$$\int_c f(z)dz = \oint_c f(z)dz = 0$$

$$\oint_c f(z)dz = \oint_c (u + i\,v)(dx + i\,dy)$$

$$= \oint (u\,dx - v\,dy) + i\int (v\,dx + u\,dy)$$

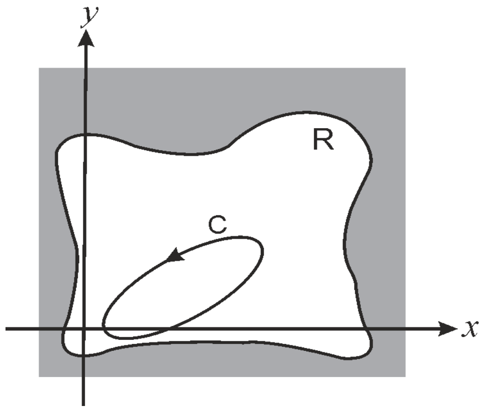

Figure 6.8 A closed contour C within a simply connected region R

Stokes's Theorem Proof : These two line integrals may be converted to surface integrals by Stokes's Theorem, then a procedure that is justified if the partial derivatives are continuous within C.

Using $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} \implies \oint_C \vec{v}\,.\,d\vec{l} = \oint_s \vec{\nabla}\times\vec{v}\,.\,d\vec{a}$

$$\oint_c (v_x dx + v_y dy) = \int\left[\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right] dx\,dy$$

$$\oint_C f(z)dz = \oint (u\,dx - v\,dy) + i\int (v\,dx + u\,dy)$$

$$= \int\left[\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dx\,dy + \int i\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dx\,dy = 0$$

(Using CR condition)

Cauchy-Goursat Proof :

In the Stoke's theorem proof Cauchy's integral theorem, the proof is marred from a theoretical point of view by the need for continuity of the first partial derivatives. Actually, as shown by Goursat, this condition is not essential. An outline of the Goursat proof is as follows :

We subdivide the region inside the contour C into a network of small squares as indicated in figure. Then

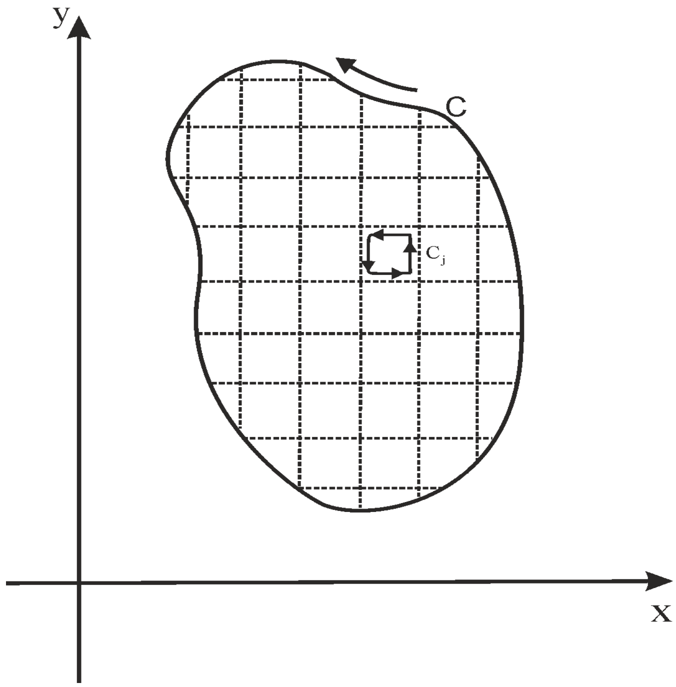

Figure 6.9 Cauchy Goursat contours

$$\oint_{c} f(z)dz = \sum_{j} \oint_{C_j} f(z)dz$$

For $\oint_{C_j} f(z)dz$,We construct the function :

$\delta_j(z\,,z_j) = \frac{f(z)-f(Z_j)}{Z-Z_j} - \left.\frac{df(z)}{dz}\right|_{z=z_j}$,z_j is an interior point of the j^{th} subregion

Approximation to the derivative at $z = z_j$

$\delta_j(z\,,z_j) \sim (z - z_j)$, approaching to zero as the network was made finer we may take

$$|\delta_j(z\,,z_j)| < \varepsilon$$

Where $\mathcal{E}$ is an arbitrary chosen small positive quantity.

Solving for $f(z)$ and integrating around c_j , we obtain

$$\oint_{c_j} f(z)dz = \oint_{c_j} (z - z_j)\delta_j(z, z_j)dz$$

The consequence of the Cauchy integral theorem is that for analytic functions, the line integral is a function only of its end points, independent of the path of integration :

$$\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1) = -\int_{z_2}^{z_1} f(z)dz$$

again exactly like the case of a conservative force.

6.10 Multiply Connected Regions

The original statement of Cauchy Integral Theorem demanded simply connected region.

This restriction may easily be relaxed by the creation of a barrier ,a cut line.

Consider the multiply connected region of figure (a), in which $f(z)$ is not defined for the interior R'

Cauchy's integral theorem is not valid for the contour C, as shown but we can construct a contour C' for which the theorem holds.

We cut from the interior forbidden region R' to the forbidden region exterior to R and then run a new contour C' , as shown (b)

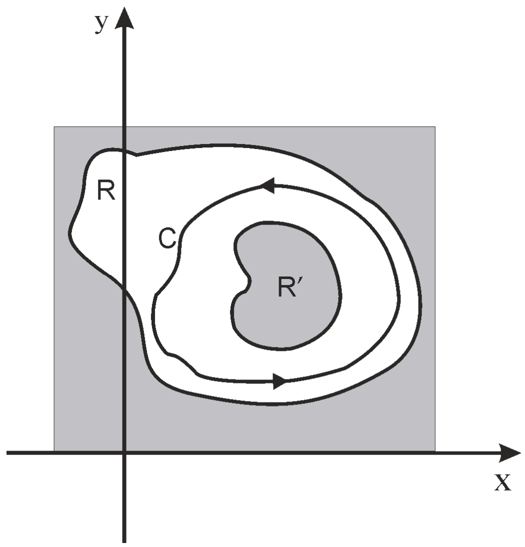

Figure 6.10 A closed contour C in a multiply connected region

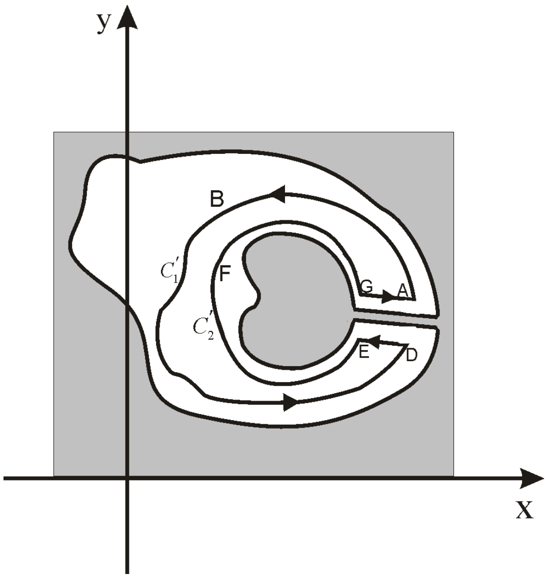

Figure 6.11 Conversion of a multiply connected region into a simply connected region

The new contour C' through ABDEFGA never crosses the cut lines that literally converts R into a simply connected region.

$\because \int_G^A f(z)dz = -\int_E^D f(z)dz,$

$f(z)$ having been continuous across the cut line segments DE as GA. Arbitrarily close together

Then $\oint_{C'}, f(z)dz = \int_{ABD} f(z)dz + \int_{EFG} f(z)dz = 0$

Applying again equation $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1) = -\int_{z_2}^{z_1} f(z)dz$

With $ABD \rightarrow C_1'$ and $EFG \rightarrow -C_2'$ $, we$ $obtain$

$\oint_{C_1'} f(z)dz = \oint_{C_1'} f(z)dz$

In which C_1' and C_1' are noth travesed in the same (Counter clockwise) direction.

It should be emphasized that the cut line here is a matter of mathematical convenience, to permit the application of Cauchy's integral theorem. Since $f(z)$ is analytic in the annular region, it is necessarily single valued and continuous across any such cut line. When we consider branch points our functions will not be single valued and a cutline will be required to make them single valued.

6.11 Self Learning Exercise-II

Section A : Very Short Answer Type Questions

Q.1 Evaluate $\int_{i}^{1}(z+1)^2 dz$

Section B : Short Answer Type Questions

Q.2 Show that for any analytic function $f=u+iv$,the following relation must hold $|\Delta u| = |\Delta v|$

Q.3 What do you mean by Branch points?

6.12 Summary

This chapter summarizes some of the important theorem regarding analytic functions. These are Cauchy-Riemann conditions and Cauchy integral theorem. These are basic to further advance in the theory of functions of complex variables.

6.13 Glossary

Mar **:** to spoil something, making it less good or less enjoyable

Annular **:** Ring-shaped

6.14 Answer to Self Learning Exercises

Answer to Self Learning Exercise-I

Ans.1: $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ **Ans.2:** $Analytic$

Ans.3: log2+4nπi

Answer to Self Learning Exercise-II

Ans.1: $\frac{10-2i}{3}$

6.15 Exercise

Section A : Very Short Answer Type Questions

Q.1 What do you mean by Analytic function?

Q.2 Evaluate $\int_{0}^{\pi i} z\, cos z^2 dz$

Section B : Short Answer Type Questions

Q.3 Derive the necessary and sufficient condition for a function to be analytic.

Section C : Long Answer Type Questions

Q.4 Which of the following are analytic functions of complex variable $z=x+iy$

(i) $|z|$ (ii) Re.z (iii) sinz (iv) logz

Q.5 State and prove Cauchy's integral theorem.

6.16 Answers to Exercise

Ans.2: $-\frac{1}{2}\sin\pi^2$

Ans.4 : (i) Not analytic (ii) Not analytic (iii) analytic (iv) Not analytic

References and Suggested Readings

1. George B. Arfken, Hans J. Weber ,Mathematical Methods for Physics, 5e, Academic Press2001.

2. Satya Prakash ,Mathematical Physics with Classical Mechanics, , Sultan Chand & Sons.1999

3. Erwin Kreyszing, Advance Engineering Mathematics, , Wiley student edition2000

UNIT -7
Cauchy's Integral Formula, Laurent expansion, Analytic Continuation Mapping

Structure of the Unit

7.0 Objectives

This chapter introduces most fundamental formulae in the theory of the functions of complex variables. These include Cauchy's integral formula, Laurent expansion and the Concept of analytic continuation.

7.1 Introduction

In this chapter we deduce a remarkable result concerning analytic function $f(z)$ that is analytic on a closed contour C and within the interior region bounded by C. We shall deduce that the value of an analytic function $f(z)$ is given at an interior point once the values on the boundary C are specified. This result is Cauchy's integral formula. This result guarantees not only the first derivative of $f(z)$ but derivatives of all orders as well. This formula opens up the way for the derivative of Taylor's series.

7.2 Cauchy's Integral Formula

We consider a function $f(z)$ that is analytic on a closed contour C and within the interior region bounded by C. We seek to prove that :

$$\oint_C \frac{f(z)dz}{z-z_0} = 2\pi i\, f(z_0) \tag{7.1}$$

In which z_0 is some point in the interior region bounded by C.

Note : carefully that since z is on the contour C while z_0 is in the interior, $z - z_0 \neq 0$ and the integral (7.1) is well defined.

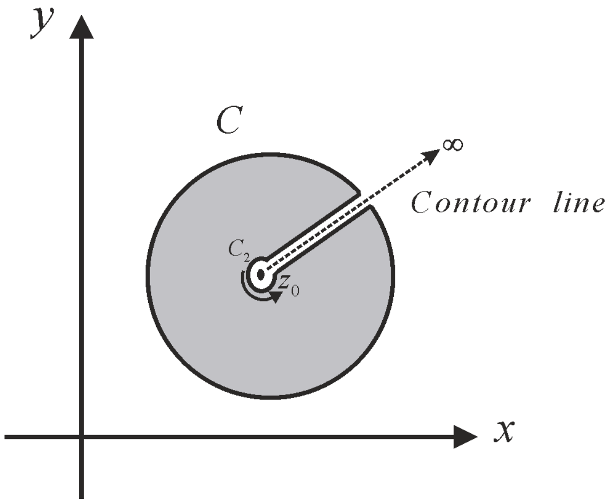

Figure 7.1 Exclusion of singular point

Although $f(z)$ is assumed analytic, the integral $\frac{f(z)}{z-z_0}$ is not analytic at $z = z_0$.

Cauchy's integral theorem applies :

$$\oint_c \frac{f(z)}{z-z_0}\,dz - \oint_{C_2} \frac{f(z)dz}{z-z_0} = 0$$

Where C = original contour

C_2= Circle surrounding the point z_0 transvered in a counter clockwise direction.

Let $z = z_0 + r\, e^{i\theta}$ Here r is small and will eventually be made to approach zero.

$$\therefore \oint_{C_2} \frac{f(z)dz}{z - z_0} = \oint_{C_2} \frac{f\left(z_0 + re^{i\theta}\right)}{re^{i\theta}}\, r\, i\, e^{i\theta}\, d\theta$$

Taking the limit as $r \to 0$, we obtain

$$\oint_{C_2} \frac{f(z)dz}{z - z_0} = i\, f\,(z_0) \int_{C_2} d\theta = 2\,\pi\, i\, f\,(z_0)$$

Since $f(z)$ is analytic and therefore continuous at $z = z_0$.

If z_0 is exterior to C, in this case the entire integrand is analytic and either C, Cauchy's integral theorem applies and the integral vanishes.

$$\boxed{\frac{1}{2\pi i}\oint_C \frac{f(z)dz}{(z - z_0)} = \begin{cases} f(z_0) & ;\ z_0\ interior \\ 0 & ;\ z_0\ exterior \end{cases}} \qquad (7.2)$$

Derivatives :

Cauchy's integral formula may be used to obtain an expression for the derivative of $f(z)$. From (7.1) with $f(z)$ analytic

$$\frac{f(z_0 - \delta z_0) - f(z_0)}{\delta z_0}$$

$$= \frac{1}{2\pi\, i\, \delta z_0}\left(\oint \frac{f(z)}{z - z_0 - \delta z_0}dz - \oint \frac{f(z)}{z - z_0}\, dz\right)$$

Then, by definition of derivative :

$$f'(z_0) = \lim_{\delta z_0 \to 0} \frac{1}{2\pi i \delta z_0}\left(\oint \frac{\delta z_0\, f(z)}{(z - z_0)(z - z_0 - \delta z_0)}\, dz\right)$$

$$\boxed{f'(z) = \frac{1}{2\pi i}\oint \frac{f(z)}{(z - z_0)^2}dz}$$

This technique for constructing derivatives may be repeated :

$$f^{(2)}(z_0) = \frac{2}{2\pi i}\oint \frac{f(z)dz}{(z - z_0)^3}$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)dz}{(z - z_0)^{n+1}} \tag{7.3}$$

That is ,the requirement that $f(z)$ be analytic not only guarantees a first derivative but derivatives of all orders as well.

For example

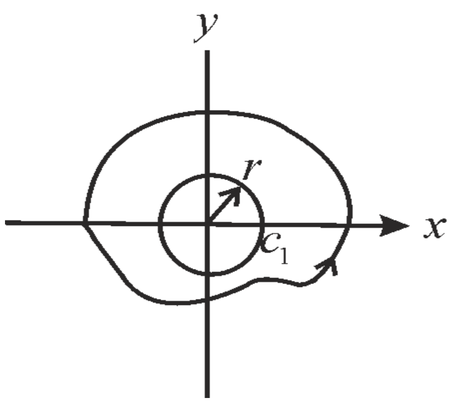

Figure 7.2

$I = \oint_C \frac{dz}{z}$, Where C is a simple closed curve.

The function $w(z) = \frac{1}{z}$ is analytic for any value of z except for $z = 0$. If therefore, the simple closed curve C enclosed the origin, let us draw an arc C_1 of small radius r with center at the origin as shown.

Since $\frac{1}{z}$ is analytic in the region between C_1 and C we have

$\oint_C \frac{dz}{z} = \oint_{C_1} \frac{dz}{z}$, Now on circle

$z = r\, e^{i\theta}, dz = r\, i\, e^{i\theta}\, d\theta$

$$\oint_C \frac{dz}{z} = \oint_{C_1} \frac{dz}{z} = \begin{cases} 0 & \text{If } C \text{ does not enclose origin} \\ 2\pi i & \text{If } C \text{ encloses origin} \end{cases}$$

7.3 Laurent Expansion

The Cauchy integral formula opens up the way for another derivation of Taylor's series, but this time for functions of a complex variables.

Suppose we are trying to expand $f(z)$ about $z = z_0$ and we have $z = z_1$ as the nearest point on the argand diagram for which $f(z)$ is not analytic.

We construct a circle C centred at $z = z_0$ with radius

$|z' - z_0| < |z_1 - z_0|$

Since z_1 was assumed to be the nearest point at which $f(z)$ was not analytic $f(z)$ is necessarily analytic on and within C.

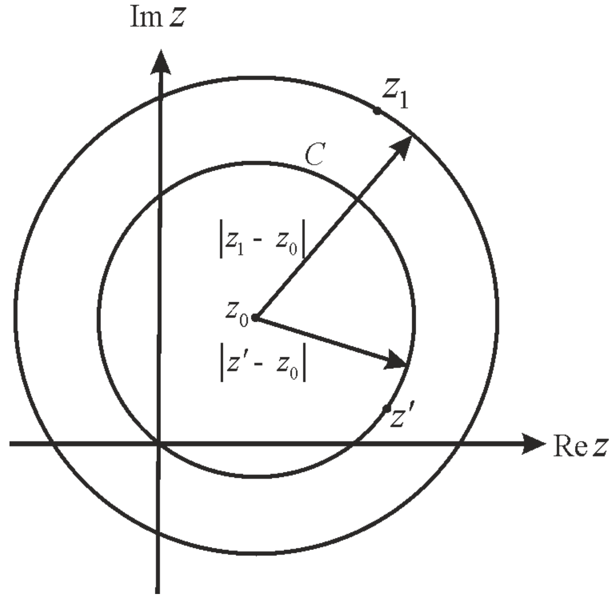

Figure 7.3

From the Cauchy integral formula.

$$f(z) = \frac{1}{2\pi i}\oint_c \frac{f(z')dz'}{(z' - z)}$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{(z'-z_0)-(z-z_0)}$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{(z'-z_0)\left[1-\left(\frac{(z-z_0)}{(z'-z_0)}\right)\right]}$$

Here z' is a point on the contour C and z is any point interior or to C.

It is not rigorously legal to expand the denominator of the integrand by the Binomial theorem, for we have not yet proved the complex variables. Instead we note the identity.

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots = \sum_{n=0}^{\infty} t^n \qquad (7.4)$$

Which may easily be verified by multiplying both sides by $1-t$. The infinite series is convergent for $|t|<1$.

Now for point z interior to C,. $|z-z_o|<|z'-z_0|$, ∴Using (7.4) we get from (7.1)

$$f(z)=\frac{1}{2\pi i}\oint_c \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(z'-z_0)^{n+1}}f(z')dz'$$

Interchanging the order integration and summation

$$f(z)=\frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\oint_c \frac{f(z')dz'}{(z'-z_0)^{n+1}}$$

$$=\frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n.\frac{f^n(z_0)}{n!}2\pi i$$

$$\left(\because \oint\frac{f(z)dz}{(z-z_0)^{n+1}}=\frac{2\pi i\, f^n(z_0)}{n!}\right)$$

$$\boxed{f(z)=\sum_{n=0}^{\infty}(z-z_0)^n.\frac{f^n(z_0)}{n!}}$$

Which is our desired Taylor expansion.

Note that it is based only on the assumption that $f(z)$ is analytic for $|z-z_0|<|z'-z_0|$. Just as for real variable power series, this expansion is unique for a given z_0.

From the Taylor expansion for $f(z)$ a binomial theorem may be derived.

Schwarz Reflection Principle :

From the binomial expansion of $g(z)=(z-x_0)^n$ for integral n it is easy to see that the complex conjugate of the function is the function of the complex conjugate:

$$g^*(z)=(z-x_0)^{n*}=(z^*-x_0)^n=g(z^*)$$

This leads us to the Schwartz reflection principle.

If a function $f(z)$ is (i)analytic over some region including the real axis and (ii)real when z is real, then $f^*(z)=f(z^*)$

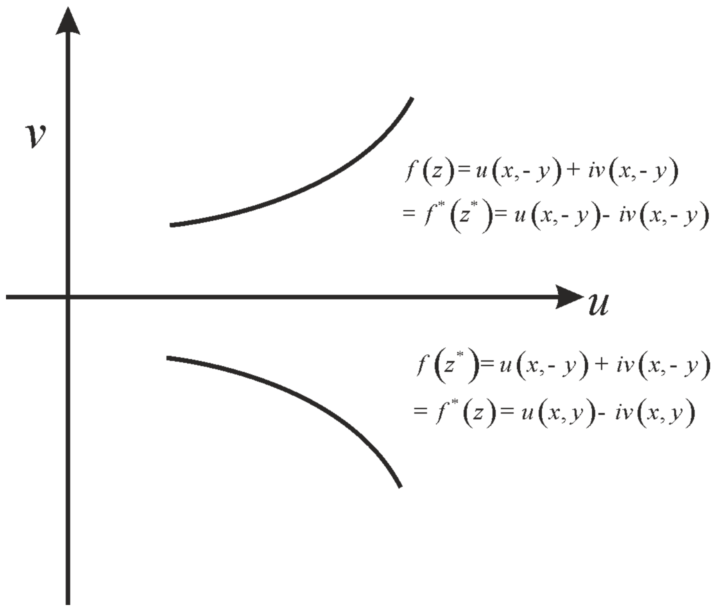

Figure 7.4 Schwartz Reflection

7.4 Analytic Continuation

In the foregoing discussion we assumed that $f(z)$ has an isolated non analytic or singular point $z = z_1$.

For a specific example of this behavior consider

$$f(z) = \frac{1}{1+z}$$

which becomes infinite at $z = -1$. Therefore $f(z)$ is non-analytic at $z_1 = -1$ or $z_1 = -1$ is our singular point.

Using Taylor Expansion formula:

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \frac{f^{(n)}(z_0)}{n!}\;, \quad it\; follows\; that$$

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots \sum_{n=0}^{\infty} (-1)^n z^n \qquad (7.5)$$

$$Convergent\; for\; |z| < 1$$

If we label this circle of convergence C_1 eq.(7.5) holds for $f(z)$ in the interior of C_1, which we label region S_1.

The situation is that $f(z)$ expanded about the origin holds only in S_1 (and on

C_1excluding $z_1 = -1$), but we know from the form of $f(z)$ that it is well defined and analytic everywhere in the complex plane outside S_1.

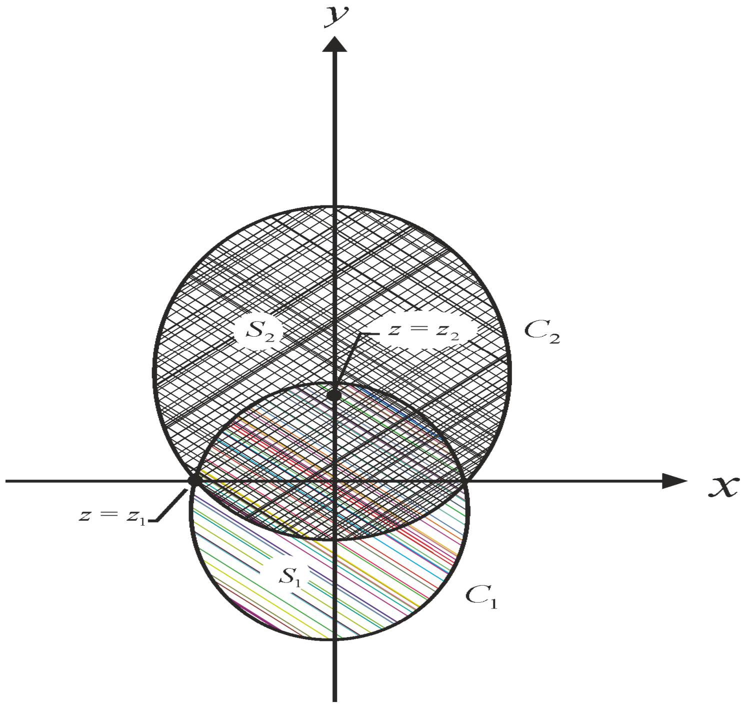

Figure 7.5 Analytic Continuation

Analytic continuation is a process of extending the region in which a function such as the series (7.5) is defined :

For instance, suppose we expand $f(z)$ about the point $z = i$. we have

$$f(z) = \frac{1}{1+z} = \frac{1}{1+i+\ (z-i)}$$

$$= \frac{1}{(1+i)\left(1+\frac{z-i}{1+i}\right)}$$

$$= \frac{1}{1+i}\left(1+\frac{z-i}{1+i}\right)^{-1}$$

$$f(z) = \frac{1}{1+i}\left[1-\frac{z-i}{1+i}+\left(\frac{z-i}{1+i}\right)^2 \dots\dots\dots\dots.\right]$$

Convergent for $|z-i| < |1+i| = \sqrt{2}$

Our circle of convergence is C_2 and the region bounded by C_2 is labeld S_2. Now $f(z)$ is defined for S_2 and extends out further in the complex plane.

This extension is an analytic continuation, and when we have only isolated singular points to contend with, the function can be extended indefinities.

Permanence of Algebraic form :

All elementary functions $e^z, \sin z$ and so on can be extended into the complex plane. For instance, they can be defined by power series expansions such as

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots \ldots . = \sum_{n=0}^{\infty} \frac{z^n}{n!} \text{ for the exponential.}$$

Such definitions agree with the real variable definitions along the real x-axis and literally constitute an analytic continuation of the corresponding real function into the complex plane. This result is often called permanence of the algebraic form.

7.5 Self Learning Exercise-I

Section A : Very Short Answer Type Questions

Q.1 Write down the Cauchy's Integral formula**.**

Section B: Short Answer Type Questions

Q.2 What do you mean by Analytic Continuation

Q.3 Show that $\frac{1}{2\pi i} \oint z^{m-n-1}\, dz = \delta_{m,n}$

(with the contour encircling the origin once counter clockwise)

Q.4 Evaluate $\int_C \frac{z+4}{z^2+2z+5} dz$,where C is the circle $|z+1| = 1$

7.6 Laurent Series

We frequently encounter functions that are analytic in an annular region, say of inner radius r and outer radius R.

Drawing an imaginary contour line to convert our region into a simply connected region, we apply Cauchy's integral formula, and for two circles C_2 and C_1, centered at $z = z_0$ and with radii r_2 or r_1 respectively, where $r < r_2 < r_1 < R$, we have

$$f(z) = \frac{1}{2\pi i} \oint_{c_1} \frac{f(z')dz'}{z' - z} - \frac{1}{2\pi i} \oint_{c_2} \frac{f(z')dz'}{(z' - z)}$$

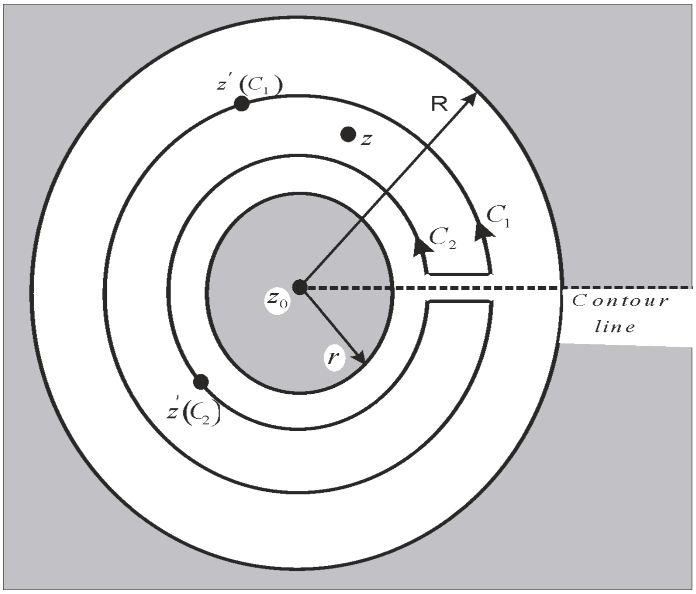

Figure 7.6

We write $z' - z \Longrightarrow (z' - z_0) - (z - z_0)$

Note that for C_1 ; $|z' - z_0| > |z - z_0|$

While for C_2 ; $|z' - z_0| < |z - z_0|$

We find :

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z - z_0)^n \oint_{C_1} \frac{f(z')dz'}{(z' - z_0)^{n+1}} + \frac{1}{2\pi i}\sum_{n=1}^{\infty}(z - z_0)^{-n} \oint_{C_2} (z' - z_0)^{n-1} f(z')dz'$$

The minus sign has been absorbed by the binomial expansion.

Labeling the first series S_1 and the second S_2.

$$S_1 = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z - z_0)^n \oint_{C_1} \frac{f(z')dz'}{(z' - z_0)^{n+1}}$$

Which is the regular Taylor expansion, convergent for $|z - z_0| < |z' - z_0| = r_1$

That is, for all Z interior to the larger circle C_1.

For the second series, we have

$$S_2 = \frac{1}{2\pi i}\sum_{n=1}^{\infty}(z - z_0)^{-n} \oint_{C_2} (z' - z_0)^{n-1} f\,(z')\,dz'$$

Convergent for $|z - z_0| > |z' - z_0| = r_2$, that is, for all z exterior to the smaller circle C_2. Remember C_2 now goes counter clockwise.

The two series may be combined into one series (a Laurent series) by

$$f(z) = \sum_{n=-\infty}^{n=\infty} a_n (z - z_0)^n \quad \text{where } a_n = \frac{1}{2\pi i} \oint \frac{f(z')dz'}{(z' - z_0)}$$

Note :-

Frequently our interest in a function will be restricted to its behavior at the points of some specified part of the z-plane. However before we can undertake discussions of this sort, we must define and explain some of the simpler properties of the sets of points we intend to consider.

Neighborhood of a point z_0, we mean any set consisting of all the points which satisfy an inequality of the form : $|z - z_0| < \epsilon \ \ , \epsilon > 0$

Geometrically speaking a neighborhood of z_0 thus consists of all the points within but not on a circle having z_0 as center.

Important Points :-

- A point z_0 belonging to a set S is said to be an interior point of S if there exists at least one neighborhood of z_0 whose points all belong to S.
- A set each of whose points is an interior point is said to be open.
- A point z_0 not belonging to a set S is said to be exterior to S or an exterior point of S if there exists at least one neighbor hood of z_0 none of whose points belongs to S.
- Intermediate between points interior to S and point exterior to S are the boundary points of S.
- A point z_0 is said to be a boundary point of a set S if every neighborhood of z_0 contains both points belonging to S and points not belonging to S.
- A point z_0 is said to be a limit point of a set if every neighborhood of the point contains at least one point of the set distinct from z_0.
- A set which contains all its boundary points is said to be closed.
- Clearly, a set can be defined to contain some but a not all its boundary points; hence it is clear that set may be neither open nor closed.

➢ If a set S has the property that every pair of its points can be joined by a polygonal line whose points all belong to the set, it is said to be connected.

➢ An open connected set is said to be a domain.

➢ A set consisting of a domain together with none, some, or all its boundary points is called a region.

➢ A connected set S with the property that every simple closed curve which can be drawn in its interior contains only points of S is said to be simply connected.

➢ If it is possible to draw in S at least one simple closed curve whose interior contains one or more points not belonging to S, then S is said to be multiply connected.

➢ If there exists a circle with a center at the origin enclosing all the points of a set S, that is, if these exists a number d such that

$|z| < d \quad for\ all\ z\ in\ S$,then S is said to be bounded.

➢ A set which is not bounded is said to be unbounded.

➢ The set consisting of points between two concentric circles is said to be an annular region or annulus.

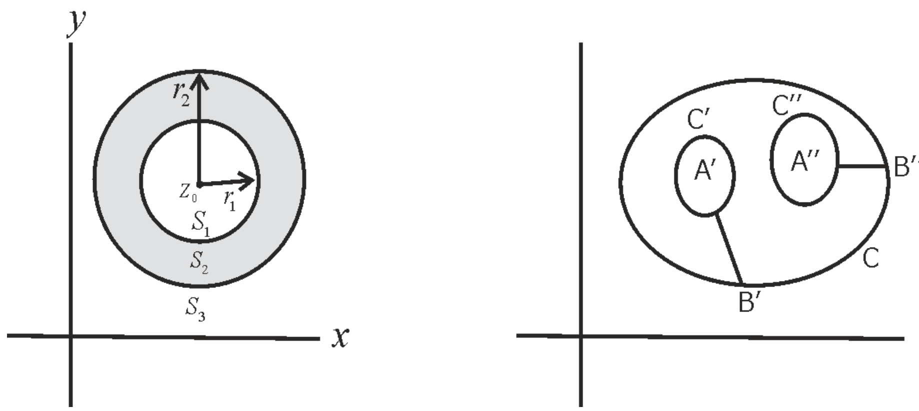

Figure 7.7 (a),(b)

S_1: $|z - z_0| < r_1$

S_2 : $r_1 \leq |z - z_0| < r_2$

S_3: $r_2 \leq |z - z_0|$

➢ S_1 consists of all points interior to circle $|z - z_0| = r_1$. It is bounded and simply connected. Since the points on the boundary circle $|z - z_0| = r_1$ are not included in the definition of S_1 ,the set is open and is therefore a domain.

- The set S_2 consists of all the points in the annulus between the circles $|z - z_0| = r_1$ and $|z - z_0| = r_2$ plus the point on the inner boundary of the annulus but not those on the outer boundary.Since S_2 thus contains some but not all of its boundary points ,it is neither open nor closed and is therefore neither a domain nor a closed region.
- Clearly, there are closed curves in S_2, namely, any curve encircling the inner boundary, which encloses points not belonging to S_2, namely, the point of S_1.Hence S_2 is multiply connected. Obviously S_2 is bounded.

The set S_3 consists of all points on and outside the circle $|z - z_0| = r_2$. It is therefore unbounded, closed and multiply connected.

Example 7.1 . If $f(z) = \frac{(x+y)^2}{x^2+y^2}$, *show that*

$\lim_{x\to 0}\left[\lim_{y\to 0} f(z)\right] = 1$ *and* $\lim_{y\to 0}[\lim_{x\to 0} f(z)] = 1$

But that $\lim_{z\to 0} f(z)$ does not exist.

Sol. Clearly $\lim_{x\to 0}\left[\lim_{y\to 0} \frac{(x+y)^2}{x^2+y^2}\right] = \lim_{x\to 0}[1] = 1$

And $\lim_{y\to 0}\left[\lim_{x\to 0} \frac{(x+y)^2}{x^2+y^2}\right] = \lim_{y\to 0}[1] = 1$

On the other hand, for $\lim_{z\to 0} f(z)$ to exist ,it is necessary that $f(z)$ approach the same value along all paths leading to the origin, and this is not the case, for along the path $y = mx,$ we have

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{(x+y)^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2\,(1+m)^2}{x^2\,[1+m^2]}$$

The limiting value have clearly depends on m, i.e.$f(z)$ approaches different values along different radial lines and hence no limit exists.

Because simply connection regions are in many respects easier to work with then multiply connected regions, it is often desirable to be able to reduce the later to the former through the introduction of auxiliary boundary arcs, or cross cuts, joining boundary curves that were originally disconnected. The modified region is therefore simply connected, as desired.

Mapping :

$$w = f(z) = u\,(x, y) + i\,v\,(x, y)$$

Then for a point in the Z-plane (specific values for x and y) there may correspond specific values for $u(x, y)$ and $v(x, y)$,which then yield a point in the w-plane.

Our purpose is to see how lines and areas map from the z-plane to the w-plane for a number of simple functions.

A. Translation :

$$w = z + z_0 \quad = x + i\,y + x_0 + i\,y_0$$
$$= (x + x_0) + i\,(y + y_0)$$
$$= u + i\,v$$
$$u = (x + x_0)$$
$$v = (y + y_0)$$

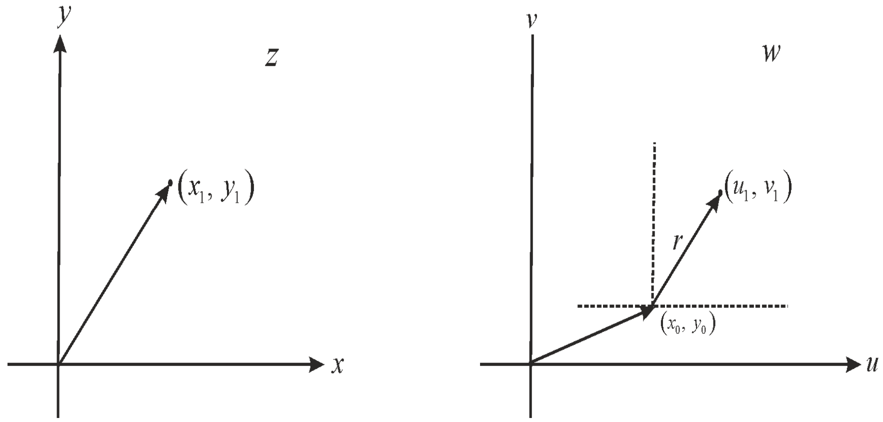

Figure 7.8 Translation

Rotation :

$w = zz_0$ Here it is convenient to return to the polar representation, using

$$w = \rho e^{i\phi} \;, \quad z = r\,e^{i\theta} \;, \quad z_0 = r_0 e^{i\theta_0}$$
$$\therefore \; \rho e^{i\phi} = r\,r_0\,e^{i(\theta+\theta_0)}$$

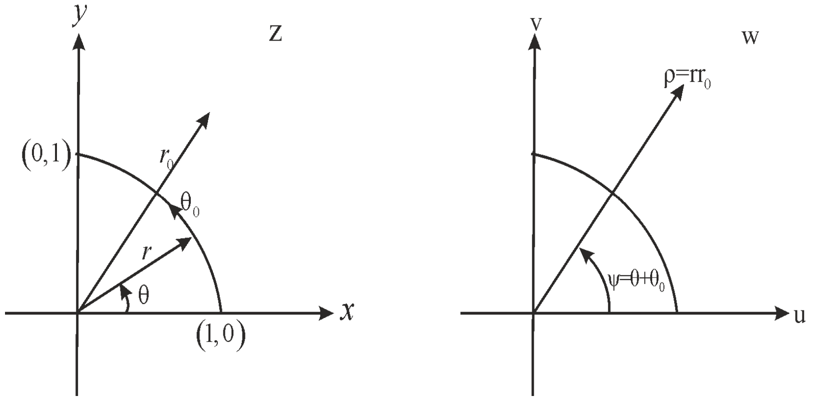

Figure 7.9 Rotation

The modules r has been modified, either expanded or contracted by the factor r_0.

Second, the argument θ has been increased by the additive constant θ_0.This represents the rotation of complex variable through an angle θ_0.

Special case of $z_0 = i$, we have a pure rotation through $\frac{\pi}{2}$ radians.

Inversion :

$$w = \frac{1}{z}$$

$$\rho\, e^{i\phi} = \frac{1}{r\, e^{i\,\theta}} = \frac{1}{r}\, e^{-i\theta}$$

$$\therefore\ \rho = \frac{1}{r}\ ,\ \phi = -\theta \qquad \dots\dots\dots\dots..(A)$$

The first part of (A) shows that inversion clearly. The interior of the unit circle is mapped onto the exterior and vice-versa.

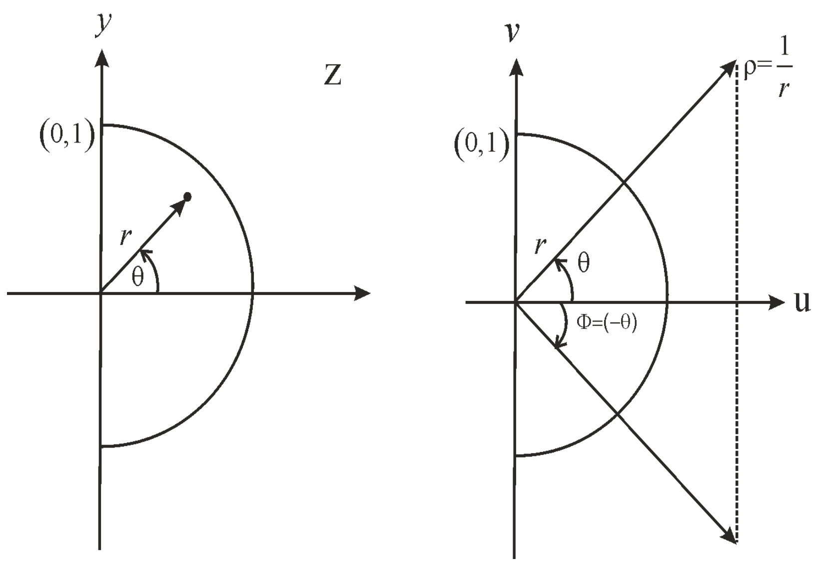

Figure 7.10 Inversion

To see how lines in the z-plane transform into the w-plane. We simply return to the Cartesian form :

$$w = \frac{1}{z}$$

$$u + iv = \frac{1}{x + i\,y} = \frac{x - i\,y}{x^2 + y^2}$$

$$u + iv = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

$$u = \frac{x}{x^2+y^2} \quad , \quad v = -\frac{y}{x^2+y^2}$$

$$\frac{u}{v} = -\frac{x}{y} \quad (1)$$

$$\therefore u = -\frac{y\frac{u}{v}}{y^2\frac{u^2}{v^2} + y^2}$$

$$or \ \frac{y^2\left(\frac{u^2}{v^2}+1\right)}{y} = -\frac{1}{v}$$

$$y = -\frac{v}{u^2+v^2} \quad (2)$$

$$x = \frac{u}{u^2 + v^2} \qquad since \ \frac{u}{v} = -\frac{x}{y}$$

A circle centered at the origin in the z-plane has the form :

$$x^2 + y^2 = r^2$$

$$\frac{u^2 + v^2}{(u^2 + v^2)^2} = r^2$$

$$u^2 + v^2 = \frac{1}{r^2} = \rho^2$$

which describes a circle in the w-plane also centered at the origin.

Using (2),the horizontal line $y = c_1$ transforms into

$$\frac{-v}{u^2 + v^2} = c_1$$

Or $$u^2 + \left(v + \frac{1}{2c_1}\right)^2 = \frac{1}{(2c_1)^2}$$

which describes a circle in the w-plane of radius $\frac{1}{2c_1}$and centred at $u = 0, v = -\frac{1}{2}c_1$.

- Instead of transformations involving one to one correspondence of point involving one to one correspondence of points in the z-plane to the points in the w-plane. Now to illustrate the variety of transformations that are possible and the problems that can arise, we introduce first a two to one correspondence and then a many to one correspondence.

Consider first the transformation :

$$w = z^2$$

$\rho = r^2$ Non linear , $\phi = 2\theta$ phase angle of the argument is doubled

First quadrant of z; $0 \leq \theta < \frac{\pi}{2}$ $\Rightarrow$ upper half plane of w; $0 \leq \phi < \pi$.

Upper half plane of z; $0 \leq \theta < \pi$ $\Rightarrow$ whole plane of w; $0 \leq \phi < 2\pi$.

The lower half plane of Z maps into the already covered entire plane of W, thus covering the w-plane a second time. This is our two to one correspondence, two distinct points in the z-plane, z_0 and $z_0\, e^{i\pi} = -z_0$, corresponding to single point $w = {z_0}^2$ (two to one transform)

In Cartesian representation :

$$w=z^2$$

$$\Rightarrow u + i\, v = (x + i\, y)^2$$

$$= x^2 - y^2 + 2\, i\, x\, y$$

$$\therefore u = x^2 - y^2$$

$$\&\, v = 2xy$$

Hence the lines $u = c_1, v = c_2$ in the w-plane correspond to $x^2 - y^2 = c_1, 2xy = c_2$, rectangular (and orthogonal) hyperbolas in the z-plane.

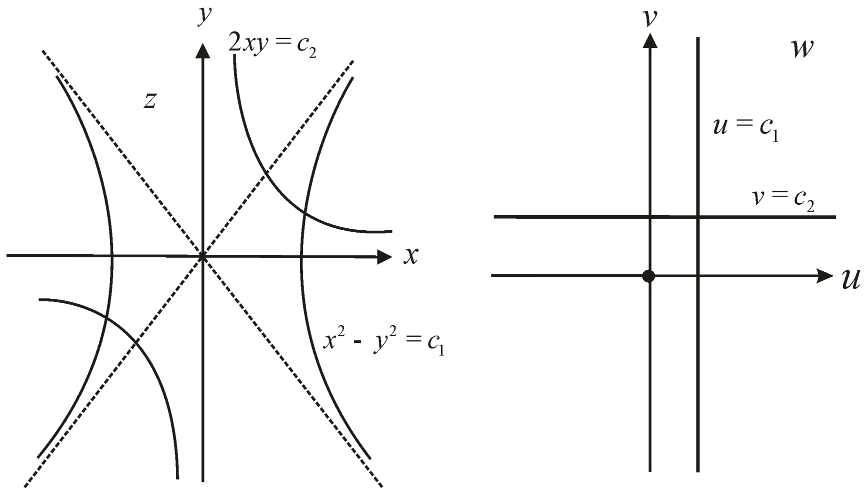

Figure 7.12 Mapping – Hyperbolic coordinates

To every point on the hyperbola $x^2 - y^2 = c_1$ in the right half plane $x > 0$, one point on the line $u = c_1$ corresponds and viceversa.

7.7 Illustrative Examples

Example7.2 Find the Laurent series about the indicated singularity for the following function. Name the singularity in this case and give the region of convergence $\frac{e^{2z}}{(z-1)^3}$; $z = 1$

Sol. $z - 1 = v$

Then . $z = 1 + v$ and

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^{2+2v}}{(v)^3} = \frac{e^2}{(v)^3} e^{2v}$$

$$= \frac{e^2}{(v)^3}\left\{1 + 2v + \frac{(2v)^2}{2!} + \frac{(2v)^3}{3!} + \frac{(2v)^4}{4!} + \cdots.\right\}$$

$$= \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{z-1} + \frac{4e^2}{3} + \frac{2e^2}{3}(z-1) + \cdots$$

$z = 1$ is a pole of the order 3 or triple pole.

7.8 Self Learning Exercise-II

Section A : Very Short Answer Type Questions

Q.1 Write the Laurent series expansion formula.

Section B : Short Answer Type Questions

Q.2 Show that $\oint_c (z - z_0)^n dz = \begin{cases} 2\pi i & n = -1 \\ 0 & n \neq -1 \end{cases}$

Where the contour C encircles the point $z = z_0$ in a positive (counter clockwise) sense. The exponent n is an integer.

Q.3 (a) Prove that the sequence $\left\{\frac{1}{1+nz}\right\}$ is uniformly convergent to zero for all z such that $|z| \geq 2$

(b) Can the region of uniform convergence in part (a) be extended ? Explain.

7.9 Summary

In this chapter we have summarized the derivation of Cauchy's integral formula and its application to the Taylor's expansion. We have studied Schwarz reflection principle and the concept of analytic continuation.

7.10 Glossary

Contour **:** An outline representing or bounding the shape or form of something:

Mapping :An operation that associates each element of a given set (the domain) with one or more elements of a second set (the range).

7.11 Answer to Self Learning Exercises

Answer to Self Learning Exercise-I

Ans.1: We consider a function $f(z)$ that is analytic on a closed contour C and within the interior region bounded by C.

$$\oint_c \frac{f(z)dz}{z - z_0} = 2\pi i f(z_0)$$

In which z_0 is some point in the interior region bounded by C.

Ans.4 : 0

Answer to Self Learning Exercise-II

Ans.1: $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$

Where $a_n = \frac{1}{2\pi i} \oint \frac{f(z')dz'}{(z'-z_0)^{n+1}}$

Ans.3: (b) If δ is any positive number, the largest value of $\left\{\frac{\frac{1}{\varepsilon}-1}{|z|}\right\}$ in $|z| \geq \delta$ occurs for $|z| = \delta$ and is given by $\left\{\frac{\frac{1}{\varepsilon}-1}{\delta}\right\}$.As in part (a),it follows that the sequence converges uniformly to zero for all z such that $|z| \geq \delta$,i.e. in any region which excludes all points in a neighborhood of z=0.

Since δ can be chosen arbitrarily close to zero,it follows that the region of (a) can be extended considerably.

7.12 Exercise

Short Answer Type Questions

Q.1 Evaluate $\oint \frac{dz}{z^2-1}$ where C is the circle $|z| = 2$

Q.2 Expand $f(z) = \frac{1}{(z-3)}$ in a Laurent series valid for

(a)$|z| < 3$ (b) $|z| > 3$

Long Answer Type Question

Q.3 Find the region of the convergence of the series $\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+1)^3 4^n}$

Q.4 If $F_2(z) = \frac{1}{1+i} \sum_{n=0}^{\infty} \left(\frac{z+i}{1+i}\right)^n$ is an analytic continuation of $F_1(z) = \sum_{n=0}^{\infty} (z)^n$, showing graphically the regions of convergence of the series.

7.13 Answers To Exercise

Ans.2: (a) $-\frac{1}{3} - \frac{1}{9}z - \frac{1}{27}z^2 - \frac{1}{81}z^3 - \cdots$

(b) $z^{-1}+3z^{-2}+9z^{-3}+27z^{-4}+\ldots\ldots$

Ans.3: Hint: The given series converges absolutely for $|z+2| \leq 4$.Geometrically this is the set of all points inside and on the circle of radius 4 with center at z=–2, called the circle of convergence .The radius of convergence is equal to 4 .

References and Suggested Readings

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2. Satya Prakash ,Mathematical Physics with Classical Mechanics, , Sultan Chand & Sons.1999
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UNIT-8
Calculus of Residues

Structures of the Unit

8.0 Objectives

In this chapter we define the singular point z_0 of the function $f(z)$ if $f(z)$ is not analytic at $z = z_0$ but is analytic at neighboring points. We deduce Cauchy's residue theorem and use it to evaluate definite integrals.

8.1 Introduction

Definite integrals appear repeatedly in problems of mathematical physics as well as in pure mathematics. Three general techniques are useful in evaluating definite integrals viz. contour integration, conversion to gamma or beta functions, and numerical integration.

The method of contour integration is perhaps the most versatile of these methods, since it is applicable to a wide variety of integrals. In this chapter we introduce the different methods of evaluation of definite integrals.

8.2 Calculus of Residues

Singularities : The Laurent expansion represents a generalization of the Taylor series in the presence of singularities. We define the point z_0 as an isolated singular point of the function $f(z)$ if $f(z)$ is not analytic at $z = z_0$ but is analytic at neighboring points. A function that is analytic throughout the entire finite complex plane except for isolated poles is called "***meromorphic***".

Poles : In the Laurent expansion of $f(z)$ about z_0 :

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad (1)$$

If $a_n = 0$ for $n < -m < 0$ and $a_{-m} \neq 0$, we say that z_0 is a pole of order m. For instance, if $m = 1$, i.e. if $\frac{a_{-1}}{(z-z_0)}$ is the first nonvanishing term in the Laurent series, we have a pole of order one, often called a ***simple pole***.

If on the other hand, the summation continues to $n = -\infty$, the z_0 is a pole of infinite order and is called an essential singularity of $f(z)$.

The essential singularities have many pathological features. For instance, we can show that in any small neighborhood of an essential singularity of $f(z)$ the function $f(z)$ comes arbitrarily close to any (and therefore every) preselected complex quantity w_0 (Due to Picard theorem) Literally the entire w-plane is mapped into the neighborhood of the point z_0. One point of fundamental difference between the a pole of finite order and an essential singularity is that a pole of order m can be removed by multiplying $f(z)$ by $(z - z_0)^m$. This obvious can not be done for an essential singularity.

The behavior of $f(z)$ as $z \to \infty$ is defined in terms of the behavior of $f\left(\frac{1}{t}\right)$ as $t \to 0$.

Consider the function :

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n \, z^{2n+1}}{(2n+1)!}$$

As $z \to \infty$, we replace the z by $\frac{1}{t}$ to obtain

$$sin\left(\frac{1}{t}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! \; t^{2n+1}}$$

Clearly, from the definition, $sin\ z$ has an essential singularity at infinity.

$sin\, z = sin\, iy, = i\, sin\, hy \quad when\, x = 0$, which approaches infinite exponentially as $y \to \infty$.

Branch Points :

There is another sort of singularity that will be important in the latter sections.

Consider $f(z) = z^a$, here a is not an integer.

As z moves around the unit circle from e^0 to $e^{2\pi i}$.

$f(z) = e^{2\pi a i} \neq e^{0.i}$ for nonintegral a.

We have a branch point at the origin another at infinity. The points $e^{0i}$and $e^{2\pi i}$ in the z-plane coincide but these coincident points lead to different values of $f(z)$; that is $f(z)$ is a multivalued function.

The problem is resolved by constructing a cut line joining both branch points so that $f(z)$ will be uniquely specified for a given point in the z-plane.

Note carefully that a function with a branch point and a required cut line will not be continuous across the cut line. In general, there will be a phase difference on opposite sides of this cut line. Hence line integrals on opposite sides of this branch point cut line will not generally cancel each other. Numerous examples of this are given below :

The contour line used to convert a multiply connected region into a simply connected region is completely different. Our function is continuous across the contour line, and no phase difference exists.

We can take following example

Consider the function : $f(z) = (z^2 - 1)^{\frac{1}{2}} = (z+1)^{\frac{1}{2}}\ (z-1)^{\frac{1}{2}}$

The first factor on the right hand side, $(z+1)^{\frac{1}{2}}$ has a branch point at $z = -1$.

The second factor has a branch point at $z = +1$.

To check on the possibility of taking the line segment joining $z = +1$ and $z = -1$ as a cut line, let us follow the phases of these two factors.

as we move along the contour shown in Fig.

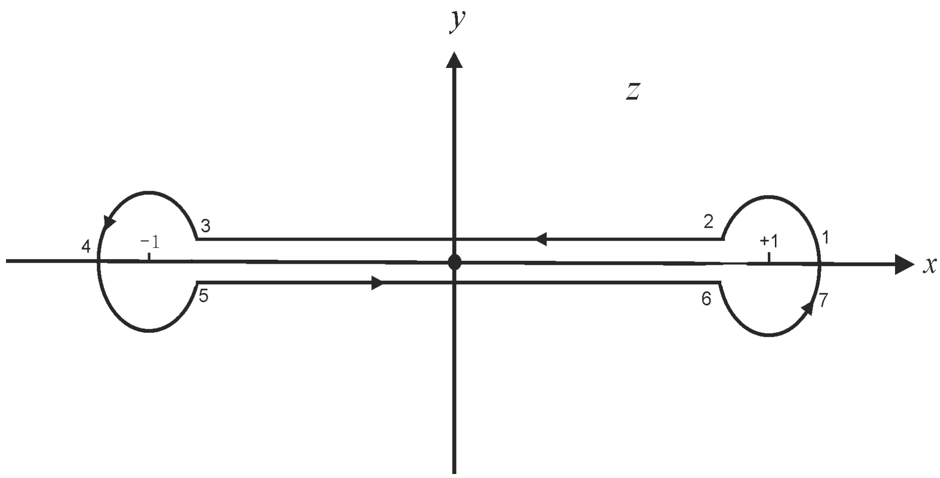

Figure 8.1

For convenience in following the changes of phase $let\, z + 1 = r\, e^{i\theta}$ and $let\ z - 1 = \rho\, e^{i\phi}$

∴phase of $f(z)$ is $\frac{\theta+\phi}{2}$.

We start at point 1 where both $z+1$and $z-1$ have a phase of zero. Moving from point 1 to point 2, ϕ, the phase of $z - 1 = \rho\, e^{i\phi}$ increases by.

π ($z - 1$becomes negative).The phase ϕ then stays constant until the circle is completed, moving from 6 to 7. θ, the phase of $z + 1 = r\, e^{i\theta}$ shows a similar behavior, increasing by 2π as we move from 3 to 5.

The phase of the function $f(z) = (z+1)^{\frac{1}{2}}\, (z-1)^{\frac{1}{2}} = r^{\frac{1}{2}}\, \rho^{\frac{1}{2}} e^{i\frac{(\theta+\phi)}{2}}$ is $\frac{\theta+\phi}{2}$. This is tabulated below:

Phase angle :

Points	θ	ϕ	$\frac{\theta+\phi}{2}$
1	0	0	0
2	0	π	$\frac{\pi}{2}$
3	0	π	$\frac{\pi}{2}$
4	π	π	π
5	2π	π	$\frac{3\pi}{2}$
6	2π	π	$\frac{3\pi}{2}$
7	2π	2π	2π

Two features emerge :

1. The phase at points 5 and 6 is not the same as the phase at point 2 and 3. This behavior can be expected at a branch point cut line.

2. The phase at point 7 exceeds that at point 1 by 2π and the function

$f(z) = (z^2 - 1)^{\frac{1}{2}}$ is therefore single valued for the contour shown, encircling the both branch points.

If we take the x-axis $-1 \leq x \leq 1$ as a cut line, $f(z)$ is uniquely specified.

Alternatively the positive x-axis for $x > 1$ and the negative x-axis for $x < -1$ may be taken as cut lines. The branch points cannot be encircled and the function remains single valued.

Generalizing from this example we have that the phase of a function.

$$f(z) = f_1(z).f_2(z).f_3(z) \ldots.$$

is the algebraic sum of the phase of its individual factors.

$$\arg f(z) = \arg f_1(z) + \arg f_2(z) + \arg f_3(z) + \ldots$$

Where $\arg f_i(z) = \tan^{-1}\left(\frac{v_i}{u_i}\right)$

For the case of a factor of the form : $f_i(z) = (z - z_0)$,the phase corresponds to the phase angle of a two dimensional vector from z_0 to z ,the phase increasing by 2π

as the point $+z_0$ is encircled. Conversely the traversal of any closed loop not encircling z_0 does not change the phase of $z - z_0$.

Note : *Liouville's theorem* : A function that is finite everywhere (bounded) and analytic must be a constant.

8.3 Calculus of Residues : (Residue Theorem)

If the Laurent expansion of a function $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ is integrated term by term using a closed contour that encircles one isolated singular point z_0 once in a counterclockwise sense, we obtain

$$a_n \oint (z - z_0)^n \, dz = \left[a_n \frac{(z - z_0)^{n+1}}{n+1} \right]_{z_1}^{z_1} = 0 \text{ for all } n \neq -1 \quad \text{(i)}$$

However, if $n = -1$

$$a_{-1} \oint (z - z_0)^{-1} dz = a_{-1} \oint \frac{ire^{i\theta} d\theta}{re^{i\theta}} = 2\pi i a_{-1} \quad \text{(ii)}$$

From (i)&(ii)

$$\therefore \frac{1}{2\pi i} \oint f(z) dz = a_{-1}$$

The constant a_{-1}, the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion, is called the residue of f(z) at $z = z_0$.

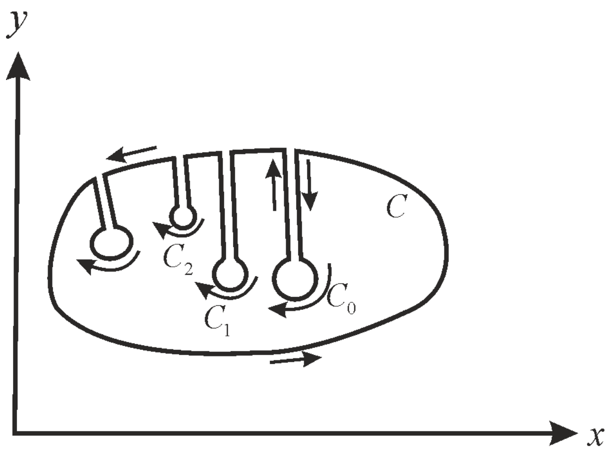

Figure 8.2 Excluding isolated Singularity

A set of isolated singularities can be handled very nicely by deforming our contour as shown in figure8.2. Cauchy's integral theorem leads

The circular integral around any singular point is given by

$$\oint_C f(z)dz + \oint_{C_0} f(z)dz + \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + ... = 0$$

$\therefore \oint_{C_i} f(z)dz = -2\pi i a_{-1z_i}$ (Clockwise assuming a Laurent expansion about the singular point $z = z_i$.

$$\therefore \oint_C f(z)dz = 2\pi i\left[a_{-1z_0} + a_{-1z_1} + a_{-1z_2} + ...\right]$$

$$\boxed{\oint_C f(z)dz = 2\pi i\left[\text{Sum of the enclosed residues}\right]}$$

This is the residue theorem.

The problem of evaluating one or more contour integrals is replaced by the algebraic problem of computing residues at the enclosed singular points.

For example if we take

$$\frac{1}{z(z-1)^2} = \frac{[1+(z-1)]^{-1}}{(z-1)^2}$$

For

$$0 < |z-1| < 1$$

$$\frac{1}{z(z-1)^2} = \frac{1}{(z-1)^2}\left\{1-(z-1)+(z-1)^2-(z-1)^3+...\right\}$$

$$= \frac{1}{(z-1)^2} - \frac{1}{(z-1)} + 1 - (z-1) + ...$$

This function has a pole of order 2 at $z = 1$ and its principal part there is

$$\frac{1}{(z-1)^2} - \frac{1}{(z-1)}$$

In fact if the Laurent expansion of f(z) in the neighborhood of an isolated singular point z = a contains only a finite number of negative powers of (z – a), then z = a is called a pole of f(z).If $(z-a)^{-m}$ is the highest negative power in the expansion, the pole is said to be of order *m* and the sum of all terms containing negative powers namely

$$\frac{a-m}{(z-a)^m}+.......+\frac{a-2}{(z-a)^2}+\frac{a-1}{(z-a)}$$ is called powers of part of f(z) at z = *a*

The Laurent expansion of f(z) in the neighborhood of an isolated singular point z = *a* contains infinitely many negative powers of z−*a* ,then z = *a* is called essential singulars of f(z).

On the other hand $e^{\frac{1}{z}}$ is represented for all values of *z* except z = 0 by the series

$$e^{\frac{1}{z}}=1+\frac{1}{z}+\frac{1}{\lfloor 2 z^2}+\frac{1}{\lfloor 3 z^2}+.....$$

It has an essential ***singularity at the origin***.

8.4 Illustrative Examples

Example 8.1 What is the integral of $f(z)=\dfrac{-3z+4}{z(z-1)(z-2)}$

Around the circle $|z|=\dfrac{3}{2}$?

Sol. Let $f(z)=\dfrac{A}{z}+\dfrac{B}{z-1}+\dfrac{C}{z-2}$

$$\frac{-3z+4}{z(z-1)(z-2)}=\frac{A(z-1)(z-2)+B(z)(z-2)+C(Z)(z-1)}{z(z-1)(z-2)}$$

$$=\frac{A\{z^2-3z+2\}+B(z^2-2z)+C(z^2-z)}{z(z-1)(z-2)}$$

By comparing powers of z

$$-3A-2B-C=-3,\ 2A=4, A+B+C=0$$

Solving this we get $A=2, B=-1, C=-1$

$$f(z)=\frac{A}{z}+\frac{B}{z-1}+\frac{C}{z-2}\equiv\frac{2}{z}-\frac{1}{z-1}-\frac{1}{z-2}$$

In this case, although there are three singular points of function namely, the three first order poles at z = 0, z = 1 and z = 2. Ones z = 0 and z = 1 lie within the path of integration.

Hence the core of the problem is to find the residues of f(z) at these two points

We write

$$f(z) = \frac{2}{z} - \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{2}{z} + (1-z)^{-1} + (2-z)^{-1}$$

$$= \frac{2}{z} + (1 + z + z^2 + \ldots) + 2^{-1}(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \ldots)$$

$$= \frac{2}{z} + (1 + z + z^2 + \ldots) + (\frac{1}{2} + \frac{z}{4} + \frac{z^2}{8} + \ldots)$$

$$= \frac{2}{z} + \frac{3}{2} + \frac{5}{4}z + \frac{9}{8}z^2 + \ldots.$$

Thus the residue of f(z) at z = 0, that is the coefficient of the term $\frac{1}{z}$ in the last expansion is 2. Also in the neighborhood of z =1, we have

$$f(z) = \frac{2}{z} - \frac{1}{z-1} - \frac{1}{z-2}$$

$$f(z) = 2\left[1 + (z-1)\right]^{-1} - \frac{1}{z-1} + \left[1 - (z-1)\right]^{-1}$$

$$= 2[1 - (z-1) + (z-1)^2 + \ldots] - \frac{1}{z-1} + [1 + (z-1) + (z-1)^2 + \ldots.]$$

$$= -\frac{1}{z-1} + 3 - (z-1) + 3(z-1)^2 + \ldots..$$

Hence the residue of f(z) at z = 1 is −1.

Therefore according to the residue theorem

$$\int_c \frac{-3z+4}{z(z-1)(z-2)} dz = 2\pi i[2 + (-1)] = 2\pi i$$

Since the determination of residues by the use of series expansions in the manner just illustrated is often tedious and sometimes very difficult, it is desirable to have a simpler alternative procedure. Such a process is provided by the following considerations. Suppose first that f(z) has a simple or first order, pole at z = 1. It follows that we can write :

$$f(z) = \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + ...$$

If we multiply this identity by (z– a) , we get

$$(z-a)f(z) = a_{-1} + a_0(z-a) + a_1(z-a)^2 + ...$$

Now let $z \to a$, we obtain for the residue

$$\boxed{\text{For first order pole, residue is } a_{-1} = \lim_{z\to a}(z-a)f(z)}$$

If f(z) has a second order pole at z = a, then

$$f(z) = \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{(z-a)} + a_0 + a_1(z-a) + a_2(z-a)^2 ....$$

To obtain the residue a_{-1} we must multiply this identity by $(z-a)^2$, getting

$$(z-a)^2 f(z) = a_{-2} + a_{-1}(z-a) + a_0(z-a)^2 + a_1(z-a)^3 + a_2(z-a)^4 + ...$$

and then differentiate w.r.t. z before we let $z \to a$. The result this time is

$$\boxed{\text{For Second order pole, residue is } a_{-1} = \lim_{z\to a}\frac{d}{dz}\left[(z-a)^2 f(z)\right]}$$

The same procedure can be extended to poles by higher order.

$$\boxed{\text{For pole of order } m\text{, residue is } a_{-1} = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]}$$

8.5 Evaluation of Definite Integrals

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

The calculus of residues is useful in evaluating a wide variety of definite integrals in both physical and purely mathematical problems. We consider, first, integrals of the form :

$$I = \int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta$$ Where *f* is finite for all values of θ.

We also require *f* to be a rational function of $\sin\theta$ and $\cos\theta$ so that it will be single valued. Let

$$\boxed{z = e^{i\theta}, dz = ie^{i\theta}d\theta}$$

$$\therefore d\theta = -\frac{idz}{e^{i\theta}} = -\frac{idz}{z}$$

$$\sin\theta = \frac{z - z^{-1}}{2i}, \cos\theta = \frac{z + z^{-1}}{2}$$

$$\therefore I = -i\oint f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right)\frac{dz}{z}$$

With path of integration the unit circle. By the residue theorem

$I = (-i)2\pi i \sum residues$ within the unit circle.

Note that we evaluate the residues of $\frac{f(z)}{z}$

8.6 Illustrative Examples

Example 8.2 $I = \int_0^{2\pi} \frac{d\theta}{1 + \varepsilon\cos\theta}$ $\qquad |\varepsilon| < 1$

Sol. Let $z = e^{i\theta}$ $\therefore dz = id\theta z$

$$\therefore d\theta = -\frac{idz}{z}$$

$$\therefore I = \oint \frac{-idz}{z\left[1 + \in . \frac{e^{i\theta} + e^{-i\theta}}{2}\right]}$$

$$= \oint_{\substack{Unit \\ circle}} \frac{-idz}{z\left[1 + \in . \frac{z + z^{-1}}{2}\right]} = \oint \frac{-2idz}{z\left[2 + \in . \frac{\left[z^2 + 1\right]}{z}\right]}$$

$$= -2i\oint \frac{dz}{2z + \in (z^2+1)}$$

$$= -\frac{2i}{\in}\oint \frac{dz}{z^2 + 1 + \frac{2z}{\in}}$$

$$I = -\frac{2i}{\in}\oint \frac{dz}{\left(z^2 + \frac{2}{\in} z + 1\right)}$$

The denominator has the roots

$$z^2 + \frac{2}{\in} z + 1 = 0$$

$$z = \frac{-\frac{2}{\in} \pm \sqrt{\frac{4}{\in^2} - 4}}{2}$$

$$= -\frac{1}{\in} \pm \sqrt{\frac{1}{\in^2} - 1}$$

$$z_1 = -\frac{1}{\in} - \frac{1}{\in}\sqrt{1 - \in^2}$$

$$z_2 = -\frac{1}{\in} + \frac{1}{\in}\sqrt{1 - \in^2}$$

As $|\in| < 1,\ z_2$ is within the unit circle and z_1 is outside.

For first order pole at z = z_2, residue is

$$I = -\frac{2i}{\in}\oint \frac{dz}{\left(z^2 + \frac{2}{\in} z + 1\right)}$$

$$I = -\frac{2i}{\in}\oint f(z)dz$$

$$I = -\frac{2i}{\in}\left\{2\pi i\left(Sum\,of\ enclosed\ residues\right)\right\}$$

$$f(z) = \frac{1}{(z - z_2)(z - z_1)}$$

For first order pole, residue is $a_{-1} = \lim_{z \to z_2} (z - z_2) f(z)$

$$a_{-1} = \lim_{z \to z_2} (z - z_2) \frac{1}{(z - z_2)(z - z_1)}$$

$$a_{-1} = \frac{1}{(z_2 - z_1)} = \frac{\in}{2\sqrt{1 - \in^2}}$$

$$\therefore I = -\frac{2i}{\in} \left\{ 2\pi i \left(Sum\, of\, enclosed\, residues \right) \right\}$$

$$\therefore I = -\frac{2i}{\in} \left\{ 2\pi i \left(\frac{\in}{2\sqrt{1 - \in^2}} \right) \right\}$$

$$= \frac{2\pi}{\sqrt{1 - \in^2}}, |\in| < 1$$

Example 8.3 Evaluate

$$\int_0^{2\pi} \frac{\cos 2\theta d\theta}{1 - 2p\cos\theta + p^2} \qquad -1 < p < 1$$

Sol. We note first that by adding and subtracting 2p, the denominator of the integrand can be written in either of two equivalent forms.

$$1 - 2p\cos\theta + p^2 = 1 - 2p + p^2 + 2p - 2p\cos\theta = (1 - p)^2 + 2p(1 - \cos\theta)$$

$$= 1 + p^2 + 2p - 2p - 2p\cos\theta = (1 + p)^2 - 2p(1 + \cos\theta)$$

From the first of these it is clear that if $0 \le p < 1$ the denominator is different from zero for all values of θ and from the second it is clear that if $-1 < p \le 0$, the denominator is also different from zero for all values of θ. Hence if $-1 < p < 1$ the integrand is finite on the closed interval $0 \le \theta \le 2\pi$.

$$\cos 2\theta = \frac{e^{2i\theta} + e^{-2i\theta}}{2} = \frac{z^2 + z^{-2}}{2}$$

and thus the given integral becomes

$$I = \int_C \frac{z^2 + z^{-2}}{2} \cdot \frac{1}{1 - 2p\frac{(z + z^{-1})}{2} + p^2} \cdot \frac{dz}{iz}$$

$$= \oint \frac{z^4 + 1}{2z^2} \cdot \frac{z}{z - pz^2 - p + p^2 z} \cdot \frac{dz}{iz}$$

$$=\oint \frac{(1+z^4)\,\mathrm{dz}}{2iz^2\{(z-p)-pz(z-p)\}}$$

$$=\oint \frac{(1+z^4)\,\mathrm{dz}}{2iz^2\{(z-p)(1-\mathrm{p}z)\}}$$

of the three poles z = 0 ,z = p ,z = 1/p of the integrand, only the first order pole at z = p and the second order pole at z = 0 lie within the unit circle C .

For the residue at the pole z = p, we have

$$\lim_{z\to p}(z-p)f(z) \equiv \lim_{z\to p}(z-p)\frac{(1+z^4)}{2iz^2\{(z-p)(1-\mathrm{p}z)\}}$$

$$=\frac{1+p^4}{2ip^2(1-\mathrm{p}^2)}$$

For the residue at the second order pole z = 0 we have

$$\lim_{z\to 0}\frac{d}{dz}\left[z^2\frac{(1+z^4)}{2iz^2(z-pz^2-p+p^2z)}\right]$$

$$=\lim_{z\to 0}\frac{1}{2i}\left[\frac{(z-pz^2-p+p^2z)4z^3-(1+z^4)(1-2zp+p^2)}{(z-pz^2-p+p^2z)^2}\right]$$

$$=-\frac{1}{2i}\left[\frac{1+p^2}{p^2}\right]$$

$$I=2\pi i(Sum of\ enclosed\ residues)$$

$$\therefore I=2\pi i\left[\frac{1+p^4}{2ip^2(1-p^2)}-\frac{1+p^2}{2ip^2}\right]$$

$$=\frac{2\pi p^2}{1-p^2}$$

Example 8.4 Find the residues of the function

$$w(z)=\frac{e^z}{z^2+a^2} \quad \text{at its poles}$$

Sol. $w(z)=\dfrac{e^z}{(z+ia)(z-ia)}$

Has two simple poles, one at $z=ia$ and another at $z=-ia$

To evaluate the residue at $z = ia$ we write

$$w(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ...$$

$$\therefore a_{-1} = \lim_{z \to z_0} (z - z_0) w(z)$$

$$= \lim_{z \to ia} (z - ia) w(z)$$

$$= \lim_{z \to ia} (z - ia) \frac{e^z}{(z + ia) - (z - ia)} = \frac{e^{ia}}{2ia}$$

Similarly the Residue at z = -ia is $\dfrac{e^{-ia}}{2ia}$

Residues at Simple Poles of $w(z) = \dfrac{F(z)}{G(z)}$ **:**

Frequently it is required to evaluate residues of a function w(z) that has the form

$$w(z) = \frac{F(z)}{G(z)}$$

where G(z) has simple zeros and hence w(z) has simple poles. If $z = z_0$ is a simple pole of w(z), then we have

$$Res.w(z)_{z=z_0} = \lim_{z \to z_0} [(z - z_0)\, w(z)]$$

$$= \lim_{z \to z_0} \left[(z - z_0) \frac{F(z)}{G(z)}\right]$$

Since $z = z_0$ is a simple pole of w(z), we must have $G(z_0) = 0$, so that expression (A) becomes $0/0$. To evaluate we use L' Hospital's rule and obtain

$$Res.w(z)_{z=z_0} = \lim_{z \to z_0} \left[\frac{1.F(z) + (z - z_0)F'(z)}{G'(z)}\right]$$

$$\boxed{\text{Residue of } w(z) \text{ at } z = z_0 \text{ is } \frac{F(z_0)}{G'(z_0)}}$$

For example, we can find the residue of $w(z) = \dfrac{e^{iz}}{z^2 + a^2}$ at the simple pole z=ia

In following way

$$w(z) = \frac{e^{iz}}{(z+ia)(z-ia)} = \frac{F(z)}{G(z)}$$

$$\therefore Res.w(z)\Big|_{z=ia} = \frac{F(z_0)}{G'(z_0)} = \frac{e^{i.ia}}{2ia.1} = \frac{e^{-a}}{2ia}$$

Example 8.5 Find the residue of $w(z) = \frac{ze^z}{(z-a)^3}$ at the third order pole z = a.

Sol. Using the formula

$$a_{-1} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m w(z)\right]_{z=z_0}$$

$$= \frac{1}{2!}\frac{d^2}{dz^2}\left[(\text{z}-\text{a})^3.\frac{ze^z}{(z-a)^3}\right]_{z=a}$$

$$= \frac{1}{2!}\frac{d}{dz}\left[z.e^z + 1.e^z\right]$$

$$= \frac{1}{2!}\left[1.e^z + z.e^z + e^z\right]_{z=a}$$

$$= \frac{1}{2}\left[e^a + a.e^a + e^a\right]$$

$$= \frac{e^a}{2}[2+\text{a}] = \frac{2+a}{2}.e^a = \left(1+\frac{a}{2}\right)e^a$$

8.7 Self Learning Exercise- I

Section A : Very Short Answer Type Questions

Q.1 Write Residue theorem.

Section B: Short Answer Type Questions

Q.2 Determine the poles and residue of the following function

$$f(z) = \frac{2z+1}{z^2-z-2}$$

8.8 Illustrative Examples

Example 8.6 If a > b > 0, prove that

$$I=\int_0^{2\pi}\frac{\sin^2\theta d\theta}{(a+b\cos\theta)}=\frac{2\pi}{b^2}\left[a-\sqrt{a^2+b^2}\right]$$

Sol. Let $e^{i\theta}=z$ $\qquad \therefore \sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}=\frac{z-z^{-1}}{2i}$

$$ie^{i\theta}d\theta=dz$$

$$\therefore d\theta=\frac{1}{i}\frac{dz}{z}$$

$$I=\oint\frac{1}{(2i)^2}\frac{\left(z-\frac{1}{z}\right)^2\frac{1}{i}\frac{dz}{z}}{\left[a+b\left(\frac{z+\frac{1}{z}}{2}\right)\right]}=\oint\frac{1}{(2i)^2}\frac{\left(z^2+\frac{1}{z^2}-2\right)\frac{1}{i}\frac{dz}{z}}{\left[\frac{2az+b(z^2+1)}{2z}\right]}$$

$$I=\oint\frac{1}{(2i)^2}\frac{\left(z^2+\frac{1}{z^2}-2\right)\frac{1}{i}\frac{dz}{z}}{\left[\frac{2az+bz^2+b}{2z}\right]}$$

$$=-i\oint\frac{z^4+1-2z^2}{z^2[2\text{az}+\text{bz}^2+\text{b}]}dz.\left(\frac{1}{-4}\right).2$$

$$=+\frac{i}{2}\oint\frac{(z^2-1)^2dz}{bz^2[\text{z}^2+\frac{2a}{b}\text{z}+1]}$$

$$=\frac{i}{2b}\oint\frac{(z^2-1)^2dz}{z^2[\text{z}^2+\frac{2a}{b}\text{z}+1]} \qquad \text{(i)}$$

For poles $z^2\left(z^2+2\frac{a}{b}z+1\right)=0$

$$=\frac{i}{2b}\oint\frac{(z^2-1)^2dz}{z^2(z-p)(z-q)}$$

For poles $z^2\left(z^2+2\frac{a}{b}z+1\right)=0$

$$\because z = 0 \ \& \ z = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2} - 4}}{2}$$

$$= -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1}$$

$$\therefore p = \frac{-a + \sqrt{a^2 - b^2}}{b}$$

$$q = \frac{-a - \sqrt{a^2 - b^2}}{b}$$

$\because a > b > 0$ it is seen that p is the only simple pole of the integrand inside the unit circle and the origin is a pole of order 2.

We must now compute the residues of {by (i)}

$$S(z) = \frac{(z^2 - 1)^2}{z^2 (z - p)(z - q)}$$

at the poles z = p and z = 0.

The residue at z = p.

$$\underset{z \to p}{\text{Res.}}\, S(z) = \lim_{z \to p} \frac{(z^2 - 1)^2}{z^2 (z - q)} = \frac{(p^2 - 1)^2}{p^2 (p - q)} = \frac{\left(p - \frac{1}{p}\right)^2}{(p - q)}$$

$$= \frac{(p - q)^2}{(p - q)} = p - q \qquad \left(\begin{array}{l} \because pq = 1 \\ \therefore q = \frac{1}{p} \end{array} \right)$$

$$= \frac{2\sqrt{a^2 - b^2}}{b}$$

The residue at the double pole z = 0.

$$\underset{z \to 0}{\text{Res.}}\, S(z) = \frac{d}{dz}\left[z^2 S(z) \right]_{z=0}$$

$$= \lim_{z \to 0} \frac{d}{dz} \left[\frac{z^2 . (\mathrm{z}^2 - 1)^2}{z^2 (z - p)(\mathrm{z} - \mathrm{q})} \right]$$

$$= \lim_{z \to 0} \frac{(z-p)(z-q)2(z^2-1).2z-(z^2-1)^2[2z-(p+q)]}{(z-p)^2(z-q)^2}$$

$$= \frac{(p+q)}{p^2q^2} = \frac{\left[-\frac{2a}{b}\right]}{1^2} = -\frac{2a}{b}$$

$$\therefore I = \frac{i}{2b} 2\pi i \left[-\frac{2a}{b} + \frac{2\sqrt{a^2-b^2}}{b} \right]$$

$$= \frac{2\pi}{b^2}\left[a - \sqrt{a^2-b^2} \right]$$

8.9 Evaluation of Certain Integrals between the Limits $-\infty$ and ∞.

We shall now consider the evaluation of integrals of the type :

$$\int_{-\infty}^{\infty} Q(x)dx = I$$

Where Q(z) is a function that satisfies the following restrictions :

(1) It is analytic in the upper half plane except at a finite number of poles.

(2) It has no poles on the real axis.

(3) $zQ(z) \to 0$ Uniformly as $|z| \to \infty$ for $0 \le \arg z \le x$

(4) When x is real, $xQ(z) \to 0$ as $x \to \pm\infty$ in such a way that $\int_{\infty}^{0} Q(x)dx$ and $\int_{-\infty}^{0} Q(x)dx$ both converge

then $\int_{-\infty}^{\infty} Q(x)dx = 2\pi i \sum R^+$

Where $\sum R^+$ denotes the sum of the residues of $a(z)$ at its pole in the upper half plane.

Proof.

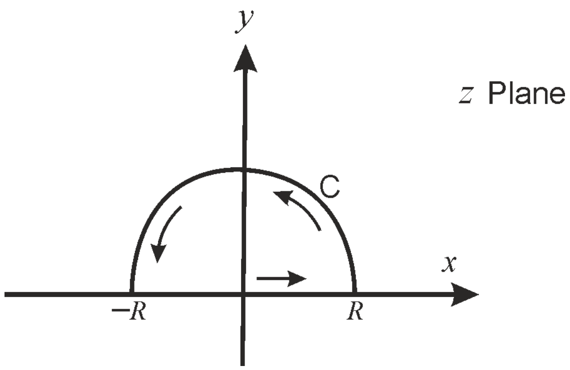

Figure 8.3

Then by Cauchy's residue theorem we have

$$\int_{-R}^{R} Q(x)dx + \int_{C} Q(z)dz = 2\pi i \sum R^{+}$$

Now by condition (3), if R is large enough, we have

$$|zQ(z)| < \in$$

for all points on C and so

$$\left|\int_{c} Q(z)dz\right| = \left|\int_{0}^{\pi} Q(z)izd\theta\right| = \left|\int_{0}^{\pi} Q(z)zd\theta\right| < \in \int_{0}^{\pi} d\theta = \in \pi$$

Hence $R \to \infty$ the integral around C tends to zero and if IV is satisfied we have

$$\int_{-\infty}^{\infty} Q(x)dx = 2\pi i \sum R^{+}$$

Example 8.7 Evaluate $I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Sol. According to theorem $I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i \sum residues$ (upper half plane)

Rewriting the integrand as

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$

We see that there are simple poles (order 1) at $z = i$, and $z = -i$

A simple pole at $z = z_0$ indicates (and in indicated by) a Laurent Expansion of the form

$$f(z)=\frac{a_{-1}}{z-z_0}+a_0+\sum_{n=1}^{\infty}a_n(z-z_0)^n$$

The residue a_{-1} is easily isolated as

$$a_{-1}=(z-z_0)f(z)\Big|_{z=z_0}$$

$\therefore$ Residue at z = i is

$$a_{-1}=(z-i).\frac{1}{(z+i)(z-i)}\Big|_{z=i}=\frac{1}{2i}$$

Similarly residue at $z = -i$ is $-\frac{1}{2i}$

$$\therefore \int_{-\infty}^{\infty}\frac{dx}{1+x^2}=2\pi i\sum R^+=2\pi i\frac{1}{2i}=\pi$$

Here we have used $a_{-1}=\frac{1}{2i}$ for the residue of the one included pole at $z = i$.

Example 8.8 Prove that

$$\int_0^{\infty}\frac{dx}{x^4+a^4}=\frac{\pi}{2\sqrt{2}a^3}$$

Sol. Consider $Q(z)=\frac{1}{z^4+a^4}$

For poles $z^4+a^4=0$ This function has ***simple poles*** at $ae^{\frac{\pi i}{4}}, ae^{\frac{3\pi i}{4}}, ae^{\frac{5\pi i}{4}}, ae^{\frac{7\pi i}{4}}$

Only the first two of these poles are in the upper half plane.

The function Q(z) clearly satisfies the conditions of the theorem.

Therefore $\int_{-\infty}^{\infty}\frac{dx}{x^4+a^4}=2\pi i\sum(\text{Res.at } ae^{\frac{i\pi}{4}} \text{ and } ae^{\frac{3i\pi}{4}})$

$$w(z)=\frac{F(z)}{G(z)}\Rightarrow \text{Res}.w(z)_{z=z_0}=\frac{F(z_0)}{G'(z_0)}$$

$$\therefore \text{Res}.Q(z)_{z=z_0}=\lim_{z\to z_0}\frac{1}{4z^3}$$

$$\therefore \operatorname{Res} Q(z)_{z=ae^{i\pi/4}} = \frac{1}{4a^3} e^{-i\frac{3\pi}{4}}$$

and $$\therefore \operatorname{Res} Q(z)_{z=ae^{i3\pi/4}} = \frac{1}{4a^3} e^{-i\frac{9\pi}{4}}$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4} = 2\pi i \left[\frac{1}{4a^3} e^{-i\frac{3\pi}{4}} + \frac{1}{4a^3} e^{-i\frac{9\pi}{4}} \right]$$

$$= \frac{\pi i}{2a^3} \left[\cos\frac{3\pi}{4} - i \sin\frac{3\pi}{4} + \cos\frac{9\pi}{4} - i \sin\frac{9\pi}{4} \right]$$

$$= \frac{\pi}{\sqrt{2} a^3}$$

Since the function Q(x) is an even function of x, we have

$$\int_{-\infty}^{\infty} \frac{dx}{x^4 + a^4} = 2 \int_{0}^{\infty} \frac{dx}{x^4 + a^4}$$

$$\therefore \int_{0}^{\infty} \frac{dx}{x^4 + a^4} = \frac{\pi}{2\sqrt{2} a^3}$$

8.10 Jordan's Lemma

$I = \int_{-\infty}^{\infty} f(x) e^{iax} dx$ with a real and positive.

This is a Fourier Transform.

We assume the two conditions.

a. F(z) is analytic in the upper half plane except for a finite number of poles.

b. $\lim_{|z| \to \infty} f(z) = 0 \qquad 0 \leq \arg z \leq \pi$

A very useful and important theorem will now be proved. It is usually known as Jordan's Lemma.

Let Q(z) be a function of the complex variable z that satisfies the following conditions.

1. Analytic in the upper half plane except at a finite number of poles.
2. $Q(z) \to 0$ Uniformly as $|z| \to \infty$ for $0 < \arg z < \pi$
3. m is a positive number

Then $\lim_{R\to\infty} \int_c e^{imz} Q(z)dz = 0$

This result is known as ***Jordan's Lemma***

Where C is a semicircle with its center at the origin and radius R.

Proof. For all points on C we have

$$z = \mathrm{Re}^{i\theta} = R(cos\theta + i\sin\theta)$$

$$dz = i\mathrm{Re}^{i\theta}\, d\theta$$

Now $\left|e^{imz}\right| = \left|e^{imR(cos\theta + i\sin\theta)}\right| = \left|e^{-mR\sin\theta}\right|$

By condition 2, if R is sufficiently large, we have for all points on C

$$\left|Q(z)\right| < \delta$$

$$\therefore \left|\int_c Q(z)e^{imz}dz\right| = \left|\int_0^{\pi} Q(z)e^{imz}\,\mathrm{Re}^{i\theta}\, d\theta\right| < \delta\int_0^{\pi} \mathrm{Re}^{-mR\sin\theta} d\theta$$

$$= 2R\delta \int_0^{\pi/2} \mathrm{e}^{-mR\sin\theta} d\theta$$

It can be proved that $\frac{\sin\theta}{\theta}$ decreases steadily from 1 to $\frac{2}{\pi}$ as θincreases from 0 to $\frac{\pi}{2}$ Hence

$$\frac{\sin\theta}{\theta} \geq \frac{2}{\pi} \quad \text{when} \quad 0 \leq \theta \leq \frac{2}{\pi}$$

Therefore

$$\left|\int_c Q(z)e^{imz}dz\right| \leq 2R\delta \int_0^{\pi/2} e^{-2m\left(\frac{R\theta}{\pi}\right)} d\theta$$

$$= \frac{\pi\delta}{m}\left(1 - e^{-mR}\right) < \frac{\pi\delta}{m}$$

From which its follows

$$\lim_{R\to\infty} \int_c e^{imz} Q(z)dz = 0$$

Note : Following type of integrals may be evaluate

$$Q(z) = \frac{N(z)}{D(z)}$$

Where N(z) and D(z) are polynomial and D(z) has no real zeros.

Then if (i) the degree of D(z) exceeds that of N(z) by at least 1 and (ii) m > 0 we have,

$$\int_{-\infty}^{\infty} a(x)e^{imz}dx = 2\pi i \sum R^+$$

Where $\sum R^+$ is sum of the residues of $a(z)e^{imz}$ at its poles in the upper half plane.

To prove this integrate $Q(Z)\text{e}^{imz}$ around the closed contour of figure

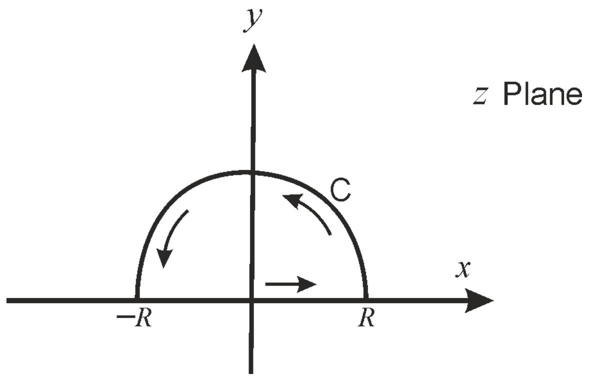

Figure 8.4

We then have

$$\int_{-R}^{R} \text{Q(x)}e^{imx}dx + \int_C Q(z)e^{imz}dz \ = 2\pi i \sum \text{Res.inside the contour}$$

Since $Q(Z)\text{e}^{imz}$ satisfies the conditions of Jordan's Lemma, we have on letting $R \to \infty$, we get the result

$$\int_{-\infty}^{\infty} \text{Q(x)}e^{imz}dx = 2\pi i \sum R^+ \quad (\because \int_C Q(z)e^{imz}dz \to 0 \text{ for infinite arc})$$

Taking the real and imaginary parts

we can evaluate integrals of the type

$$\int_{-\infty}^{\infty} Q(x)\cos mx\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} Q(x)\sin mx\,dx$$

Example 8.9 Show $\displaystyle\int_0^{\infty} \frac{\cos x dx}{x^2+a^2} = \frac{\pi e^{-a}}{2a}$ where $a > 0$

Sol. We consider the function $\dfrac{e^{iz}}{z^2+a^2}$, and since it satisfies the above conditions, we have $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2} dx = 2\pi i \sum R^+$

The only pole of the integrand the upper half plane is at *ia*, the residue there is $\dfrac{e^{-a}}{2ia}$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{ix} dx}{x^2+a^2} = 2\pi i \frac{e^{-a}}{2ia} = \frac{\pi e^{-a}}{a}$$

Taking the real part of e^{ix}, we have

$$\int_{-\infty}^{\infty} \frac{\cos x dx}{x^2+a^2} = \frac{\pi e^{-a}}{a} = 2\int_0^{\infty} \frac{\cos x dx}{x^2+a^2}$$

$$\Rightarrow \quad \int_0^{\infty} \frac{\cos x dx}{x^2+a^2} = \frac{\pi e^{-a}}{2a}$$

8.11 Self Learning Exercise - II

Section A : Very Short Answer Type Questions

Q.1 Find the poles for the following function $\dfrac{\sin z}{z^2}$

Section B : Short Answer Type Questions

Q.2 Prove that $\displaystyle\oint_C \frac{\cosh z}{z^3} dz = \pi i$ if C is the square with vertices at $\pm 2, \pm 2i$

8.12 Summary

We have discussed Cauchy's residue theorem and its application in evaluating definite integrals. We have given several illustrative examples to clarify its use in evaluating the integrals.

8.13 Glossary

Lemma **:** A subsidiary or intermediate theorem in an argument or proof

Singular **:** Relating to or of the nature of singularity.

8.14 Answer to Self Learning Exercises

Answer to Self Learning Exercise-I

$$z = -1, 2; 1/3, 5/3$$

Ans.2:

Answer to Self Learning Exercise-II

Ans.1: z=0,

8.15 Exercise

Section A : Very Short Answer Type Questions

Q.1 Write the statement of Jordan's-Lemma Theorem

Q.2 Find $\int_C \frac{\cos \pi z}{z-1} dz$ where C is the circle $|z| = 3$

Section B : Short Answer Type Questions

Q.3 Evaluate $\int_0^{2\pi} \frac{\sin 3\theta}{5 - 3\cos\theta} d\theta$

Section C : Long Answer Type Questions

Q.4 Find the residues of

(a) $f(z) = \frac{z^2 - 2z}{(z+1)^2 (z^2+4)}$ *and*

(b) $f(z) = e^z \csc^2 z$

At all its poles in the finite plane.

Q.5 Find $\int_0^{\infty} \frac{dx}{x^6+1}$

8.16 Answers to Exercise

Ans. 2 : $-2\pi i$

Ans. 3: 0

Ans. 4(a) Residue at $z = -1$ is $\frac{14}{25}$,Residue at $z = 2i$ is $\frac{7+i}{25}$,

Residue at $z = -2i$ is $\frac{7-i}{25}$

(b) $f(z) = e^z \csc^2 z = \frac{e^z}{\sin^2 z}$ has double poles at

$z = 0, \pm\pi, \pm 2\pi, \ldots i.e. z = m\pi \, where \, m = 0, +1, +2, \ldots$

Ans. 5 : $\frac{2\pi}{3}$

References and Suggested Readings

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UNIT-9
Partial Differential Equations

Structure of the unit

9.0 Objectives

The simplified analysis of physical systems with various assumptions leads to ordinary differential equations, but when more realistic approach is adopted the partial differential equations arise. The formulation of problems in most of the areas of physics leads to partial differential equations. Thus to know about partial differential equations and methods of solving them is of great significance for a physicist. The objective of this unit is to define partial differential equations and to describe the methods for solving them.

9.1 Introduction

The problems in physics involve changing entities which are known as ***variables***. The variables may change with respect to other variables; therefore the rate of change of one variable with respect to another variable is called as ***derivative***.

For example in a problem, velocity $\vec{v}$ (one variable) is changing with respect to time t (another variable), then $\frac{d\vec{v}}{dt}$ is the derivative.

When an equation is written which shows relation between variables and their derivatives, this equation is known as ***differential equation*****.**

When a differential equation involves derivatives with respect to a single independent variable only, it is known as ***ordinary differential equation*****.**

And when a differential equation involves partial derivatives with respect to more than one independent variable, it is known as ***partial differential equation.***

For example $dy = (x + x^2 + e^2)dx$ and $\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 = e^t$ are ordinary differential equations.

Whereas $\frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} = 3x$ (Partial derivatives with respect to x, y) and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$ (Partial derivatives with respect to x, y, z) are partial differential equations.

Thus partial differential equations involve dependence on two or more independent variables. Partial differential equations arise in connection with various physical problems in different areas of physics such as heat transfer, electricity, fluid mechanics, electromagnetic theory, quantum mechanics etc.

9.2 Order of a Partial Differential Equation

The order of a partial different equation is the ***order of the highest derivative involved in that equation***.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The order of the above equation is 2

Whereas for $\left(\frac{\partial u}{\partial x}\right)^2 \left(\frac{\partial u}{\partial z}\right) = 0$ the order is one.

9.3 Degree of a Partial Differential Equation

The degree of Partial differential equations is the degree of the highest derivative in that equation, after removing radicals and fractions from the concerned derivatives in that equation.

$$\sqrt{2}\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial z/\partial y}{\partial z/\partial x}\right) = 1$$

Multiplying $\partial z/\partial x$ on both the sides

$$\sqrt{2}\left(\frac{\partial z}{\partial x}\right)^2 + \partial z/\partial y = \frac{\partial z}{\partial x}$$

Thus this equation is of second degree because the order of the highest derivative $\frac{\partial z}{\partial x}$ is one and the highest degree is 2.

9.4 Linear and Non-Linear Partial Differential Equations

A partial differential equation is known as linear if following conditions are satisfied.

(i) Every dependent variable and each derivative in the equation is present in the first degree only.

(ii) There should not be products of dependent variables and/or derivatives.

A partial differential equation for which the above conditions are not satisfied is called as non-linear partial differential equation.

e.g. $z\left(\frac{\partial z}{\partial x}\right)=3\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$ the above equation is non-linear as derivative $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ occur in second degree and also there is product of z and $\partial z/\partial x$.

Note: The above definitions for order, degree and linearity are true for both ordinary and partial differential equations.

9.5 Illustrative Examples

Example 1 Find the order and degree of the following partial differential equations

(i) $2x^2\frac{\partial p}{\partial x}-3\frac{\partial p}{\partial y}=0$

(ii) $\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=xy$

(iii) $2x^2\frac{\partial^2 z}{\partial x^2}+\left(\frac{\partial z}{\partial x}\right)^2=2\left(\frac{\partial z}{\partial y}\right)$

(iv) $\frac{\partial^2 u}{\partial x^2}=\left(1+\frac{\partial u}{\partial y}\right)^{1/2}$

Sol.

(i) Order is one and degree is also one as the derivative $\frac{\partial p}{\partial x},\frac{\partial p}{\partial y}$ occur with one degree.

(ii) Order is one and degree is two.

(iii) Order is two $\left(\frac{\partial^2 z}{\partial x^2}\right)$ and degree is one.

(iv) On squaring the equation $\frac{\partial^2 u}{\partial x^2}=\left(1+\frac{\partial u}{\partial y}\right)^{1/2}$ to remove radical we get-

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^2=\left(1+\frac{\partial u}{\partial y}\right)$$

Thus for this equation order is two and degree is also two.

Example 2 Check the following partial differential equations for linearity.

(i) $\frac{\partial u}{\partial x} - 3\frac{\partial u}{\partial y} = 0$

(ii) $\frac{\partial p}{\partial x} + 3\frac{\partial p}{\partial y} = p + xy$

(iii) $z\left(\frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial y} = y$

(iv) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = xy$

Sol. Eq. (i) and (ii) are linear but (iii) and (iv) are non-linear as e.q (iii) consist of product of z and $\frac{\partial z}{\partial x}$ e.q. (iv) is of second degree.

9.6 Self Learning Exercise-I

Very Short Answer type Questions

Q.1 Check the order of equation $\left(\frac{\partial \phi}{\partial y}\right)^2 + \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^3} = 2x$

Q.2 Check the degree of equation $\left(\frac{\partial u}{\partial y}\right)^3 + \left(\frac{\partial^2 u}{\partial x^2}\right)^2 = 2$

Q.3 Is equation linear? $\frac{\partial u}{\partial x} = \left(1 + \frac{\partial u}{\partial y}\right)^{1/2}$

Short Answer type Questions

Q.4 Why equation $\frac{\partial z}{\partial x} + z\frac{\partial z}{\partial y} = 5$ is non-linear?

Q.5 Name the dependent and independent variable in equation $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 2x\frac{\partial v}{\partial x}$

Q.6 How many independent variables are present in ordinary differential equation?

9.7 Notations

Let us consider the one of two independent variables x and y and let us assume the dependent variable as z, then usually following notations are used in the study of partial differential equations.

Independent variables x, y

Dependent variables z

$$p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}, r=\frac{\partial^2 z}{\partial x^2}, s=\frac{\partial^2 z}{\partial x \partial y} \text{ and } t=\frac{\partial^2 z}{\partial y^2}$$

If there are n-independent variable such as $x_1, x_2 ....x_n$ and the dependent variable is z then following notations are generally used:

Independent variable $x_1, x_2, x_3 ....x_n$

Dependent variable z

$$p_1=\frac{\partial z}{\partial x_1}, \quad p_2=\frac{\partial z}{\partial x_2}, \quad p_3=\frac{\partial z}{\partial x_3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots p_n=\frac{\partial z}{\partial x_n}$$

Another way of expressing partial differential equations is by making use of suffixes, such as

$$u_x=\frac{\partial u}{\partial x}, u_y=\frac{\partial u}{\partial y}, u_{xy}=\frac{\partial^2 u}{\partial x \partial y}, u_{xx}=\frac{\partial^2 u}{\partial x^2} \text{ and } u_{yy}=\frac{\partial^2 u}{\partial y^2}$$

9.8 Classification of First Order Differential Equation

(i) Linear equation :– The equation is linear if it is of the form $P(x,y)P+Q(x,y)q=R(x,y)z+S(x,y)$

For e.q. $x^2yp+xyq=xy^2z+xy^3$

(ii) Semi-Linear equation:- An equation which is of the form

$P(x,y)p+Q(x,y)q=R(x,y,z)$ is semi-linear

For e.q. $x^2yp+xyq=x^2y^2z^3$

(iii) Quasi-Linear Equation: – An equation which is of the form

$P(x,y,z)p+Q(x,y,z)q=R(x,y,z)$ is quasi-linear

$x^2yzp+xy^2zq=xy+z$

(iv) Non-Liner equation: – An equation apart from above three types is non-linear

For e.q. $p^2+q^2=1$

9.9 Linear Partial Differential Equation of Order One

A quasi-linear Partial differential equation of order one of the form

$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$ is known as ***Lagrange's equation***

Equation of form $\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$ ***are known as Lagrange's auxiliary or subsidiary equations***.

Theorem (without proof): If u(x,y,z) =a and v(x,y,z) =b are two independent solutions of the system of differential equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ then $\phi(u,v) = 0$) is a ***solution*** of Lagrange's equation $Pp + Qq = R$

Lagrange's method of solving $Pp + Qq = R$

(i) Convert the linear Partial differential equation of order one to standard from $Pp + Qq = R$ (1)

(ii) Write the Lagrange's auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ (2)

(iii) Let u(x,y,z)=a and v(x,y,z)=b be the two independent solution of (2) obtained by solving (2).

(iv) The general solution of (1) is then $\phi(u,v) = 0,\ u = \phi(v)$ or $v = \phi(u)$ where ϕ is arbitrary function.

9.10 Illustrative Examples

Example 3 Solve 2p+5q=1

Sol. Given 2p+5q=1 (1)

Here P=2, Q=5, R=1 thus Lagrange's auxiliary equations as

$$\frac{dx}{2} = \frac{dy}{5} = \frac{dz}{1} \quad (2)$$

Taking first two fractions of $\frac{dx}{2} = \frac{dy}{5}$ or 5dx-2dy=0 (3)

Integrating (3)

we have 5x-2y=c_1, where c_1 is an arbitrary constant. (4)

Taking last two fractions of (2)

$$\frac{dy}{5} = \frac{dx}{1} \text{ or dy-5dz=0} \tag{5}$$

Integrating (5) y-5z=c_2 (6)

Where c_2 is an arbitrary constant From (4) and (6) the required general solution is $\phi(5x-2y, y-5z) = 0$, ϕ being an arbitrary function.

Example 4 Solve $\left(y^2 z/x\right)p + xzq = y^2$

Sol. the Lagrange's subsidiary equations for the given equations are

$$\frac{dx}{y^2 z/x} = \frac{dy}{xz} = \frac{dz}{y^2} \tag{1}$$

Taking first two fractions and rearranging

$$x^2 zdx = y^2 zdy \text{ Or } x^2 dx = y^2 dy \tag{2}$$

Multiplying (2) by 3 both the sides

$$3x^2 dx = 3y^2 dx \text{ or } 3x^2 dx = 3y^2 dx \tag{3}$$

Integrating (3) $x^3 - y^3 = c_1$ where c_1 is an arbitrary constant (4)

How taking first and last fractions $\dfrac{dx}{y^2 z/x} = \dfrac{dz}{y^2}$

Rearrange $xdx - zdz$ (5)

Multiplying (5) by 2 both the sides $2xdx - 2zdz$ or $2xdx - 2zdz = 0$ (6)

Integrating (6) $x^2 - z^2 = c_2$, c_2 is one arbitrary constant (7)

From (4) and (7) the general solution is $\phi(x^3 - y^3, x^2 - z^2) = 0$ where ϕ is an arbitrary function.

Example 5 Solve $z(z^2 + xy)(px - qy) = x^4$

Sol. Rearranging the given equation

$$xz(z^2 + xy)p - yz(z^2 + xy)q = x^4 \tag{1}$$

The Lagrange's auxiliary equations for (1) are given by

$$\frac{dx}{xz(z^2 + xy)} = \frac{dy}{-yz(z^2 + xy)} = \frac{dz}{x^4} \tag{2}$$

Taking first two fractions and rearranging

$$\frac{dx}{x}+\frac{dy}{y}=0 \qquad (3)$$

Integrating (3) we have

$$\log x + \log y = \log c_1, \text{ or } xy = c_1 \qquad (4)$$

Using (4) and taking first and last fractions of (2) we have

$$\frac{dx}{xy\left(z^2+c_1\right)}=\frac{dz}{x^4} \text{ or } x^3dx-\left(z^3+c_1z\right)dz=0 \qquad (5)$$

Integrating (5)

$$\frac{x^4}{4}-\frac{z^4}{4}-\frac{c_1z^2}{2}=c_2 \text{ or } x^4-z^4-2c_1z^2=4c_2 \qquad (6)$$

From (4) and (6) the required solution is $\phi\left(xy, x^4-z^4-2c_1z^2\right)=0$ where ϕ is an arbitrary function.

Example 6 Solve $(bz-cy)p+(cx-az)q=ay-bx$

Sol. Writing the Lagrange's auxiliary equations for the given equations

$$\frac{dx}{bz-cy}=\frac{dy}{cx-az}=\frac{dz}{ay-bx} \qquad (1)$$

Taking x,y,z as multipliers of each fraction respectively and adding

$$\frac{xdx+ydy+zdz}{x(bz-cy)+y(cx-az)+z(ay-bx)}=\frac{xdx+ydy+zdz}{0}$$

$$xdx+ydy+zdz=0 \text{ Or } 2xdx+2ydy+2zdz=0$$

Integrating $x^2+y^2+z^2=c_1$, c_1 is as arbitrary constant (2)

Now taking a,b,c as multipliers of each fraction respectively and adding

$$\frac{adx+bdy+cdz}{a(bz-cy)+b(cx-az)+c(ay-bx)}=\frac{adx+bdy+cdz}{0}$$

$$\Rightarrow adx+bdy+cdz=0 \qquad (3)$$

Integrating (3) $\Rightarrow ax+by+cz=c_2$, c_2 is an arbitrary constant (4)

From (2) and (4) the required solution is $\phi\left(x^2+y^2+z^2, ax+by+cz\right)=0$ where ϕ an arbitrary constant.

9.11 Non-Linear Partial Differential Equations of Order One

Let a relation $\phi(x, y, z, a, b) = 0$ be derived from partial differential equations $f(x, y, z, p, r) = 0$

Have x, y, z are variables such that z is dependent on x and y and $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$. a and b are two arbitrary constants.

Complete Integral or complete solution

The solution $\phi(x, y, z, a, b) = 0$ consisting of as many arbitrary constants as the number of independent variables is called the complete integral of equation $f(x, y, z, p, r) = 0$

Particular Integral

If we give particular values to the constants a and b then the solution becomes particular integral

Singular Integral or Singular Solution

The relation formed by eliminating a and b between $\phi(x, y, z, a, b) = 0$, $\frac{d\phi}{da} = 0$, $\frac{d\phi}{db} = 0$ is called the singular integral

(i) Charpit's Method

(General method of solving partial differential equations of order one but of any degree)

Let the partial differential equations be $f(x, y, z, p, r) = 0$ (1)

since z depends on x and y, thus

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy \tag{2}$$

The ***Charpit's auxiliary equations*** are given by

$$\boxed{\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{df}{0}} \tag{3}$$

Where $f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}, f_z = \frac{\partial f}{\partial z}, f_p = \frac{\partial f}{\partial p}, f_q = \frac{\partial f}{\partial q}$,

Select two proper fractions such that a simple relation involving at least one of p and q is found

$$f(p,q) = 0 \qquad (4)$$

The relation (4) is solved along with the given equation to determine p and q. putting these values of p and q in (2) and integrating will give the complete integral of the given equation.

Example 7 Find complete integral of $z = px + qy + p^2 + q^2$

Sol. The given equation $f(x, y, z, p, q) = z - px - qy - p^2 - q^2 = 0$ (1)

The Charpit's auxiliary equation are written as

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{\frac{-\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} \qquad (2)$$

Thus $\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{p(x+2p) + q(y+2q)} = \frac{dx}{x+2p} = \frac{dy}{y+2q}$ (3)

From first fraction $dp = 0 \Rightarrow p = a$ (4)

From second fraction $dq = 0 \Rightarrow q = b$ (5)

Putting $p = a$ and $q = b$ in (1) the complete integral is $z = ax + by + a^2 + b^2$, where a and b are arbitrary constant.

Example 8 Find complete integral of $p^2 - y^2 q = y^2 - x^2$

Sol. The given equation is $f(x, y, z, p, q) = p^2 - y^2 q - y^2 + x^2 = 0$ (1)

Writing the Charpit's auxiliary equations

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

Thus $\frac{dp}{2x} = \frac{dq}{-2qy - 2y} = \frac{dz}{-2p^2 + qy^2} = \frac{dx}{-2p} = \frac{dy}{y^2}$ (2)

Taking first and fourth fraction and rearranging

$$pdp + xdx = 0$$

Integrating $p^2 + x^2 =$ constant

Let constant be a^2 thus $p^2 + x^2 = a^2$ (3)

Solving (1) and (3) and obtaining values of p and q

$$p = (a^2 - x^2)^{1/2} \text{ And } q = \left(\frac{a^2}{y^2} - 1\right) \quad (4)$$

Using (4) in $dz = pdx + qdy$

$$dz = (a^2 - x^2)^{1/2} dx + \left(\frac{a^2}{y^2} - 1\right) dy$$

Integrating $z = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) - \frac{a^2}{y} - y + b$

$$\left[\because \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + cons\tan t\right]$$

Thus $z = \frac{x}{2}(a^2 - x^2)^{1/2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\frac{-a^2}{2} - y + b$

is required complete integral where a,b are arbitrary constants

(ii) Standard Methods

Charpit's method is a general method, however many equations can be reduced to four standard forms and their complete integral can be found by inspection or short methods.

Standard Form I: Equations involving p and q only $f(p,q) = 0$ (1)

The complete integral is given by $z = ax + by + c$ (2)

Where a and b are related by $f(a,b) = 0$, $b = F(a)$ (3)

The equation of standard form I do not have singular solution.

Example 9 Find the complete integral of $p + q = pq$

Sol. The solution is $z = ax + by + c$ according to standard form I provided $a + b = ab$

,

$$\Rightarrow b = \frac{a}{a-1}$$

$\therefore z = ax + \frac{a}{(a-1)}y + c$ is the complete integral

Standard Form II Clairaut equation

A first order partial differential equations is said to be ***Clairaut equation*** if it can be written as

$$z = px + qy + f(p,q)$$

The complete integral is of $z = ax + by + f(a,b)$

Example 10 Find complete integral of $(px + qy - z)^2 = 1 + p^2 + q^2$

Sol. Rewriting the given equation $px + qy = z = \pm\sqrt{1 + p^2 + q^2}$

Or $z = ax + qy \mp \sqrt{1 + p^2 + q^2}$ (1)

The complete integral is given by $z = ax + by \mp \sqrt{1 + a^2 + b^2}$ (2)

Standard Form III – Equation of the form $f(p,q,z) = 0$. To solve equation of the form $f(p,q,z) = 0$ take $u = x + ay$ where is an arbitrary constant. Put $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$ in the equation and solve the resulting ordinary differential equation.

Example 11 Find Complete Integral of $pq = 4z$

Sol. The given equation $pq = 4z$ (1)

is of standard form III i.e $f(p,q,z) = 0$. So therefore taking, $u = x + ay$, where a is an arbitrary constant and putting $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$ in (1)

$$\left(\frac{dz}{du}\right)\left(a\frac{dz}{du}\right) = 4z$$

$$a\left(\frac{dz}{du}\right)^2 = 4z$$

Or $\frac{dz}{du} = \pm 2\sqrt{\frac{z}{a}}$

Or $\pm\sqrt{\frac{a}{z}}dz = 2du$

Or $\pm\frac{\sqrt{a}}{2}\frac{dz}{z^{1/2}} = du$

Integrating $\pm\sqrt{az} = u + b$

Or $az = (u + b)^2$

Or $az = (x + ay + b)^2$ is the complete integral.

Standard Form IV Equation of the form $f_1(x, p) = f_1(y, q)$

Such equation are solved by using $f_1(x, p) = a$ and $f_2(y, q) = a$

Solving p and q and putting their values in $dz = pdx + qdy$

Integrating (2) we obtain the complete integral.

Example 12 Find complete integral of $p - 3x^2 = q^2 - y$

Sol. This equation is of standard form IV thus equating each side to arbitrary constant a, we have

$$p - 3x^2 = a \text{ And } q^2 - y = a$$

$$p = a + 3x^2 \text{ And } q = (a + y)^{1/2}$$

Using these values in $dz = pdx + qdy$

$$dz = (a + 3x^2)dx + (a + y)^{1/2} dy$$

Integrating (3)

$z = ax + x^3 + \dfrac{2}{3}(a + y)^{3/2} + b$ is the complete integral

Note: Charpit's method is a general method for solving equation with two independent variables. It is used mainly when the given equation is not of any standard forms I-IV.

9.12 Method of Separation of Variables

The partial differential equation involves the dependence on two or more independent variables which may be space or time coordinators. By use of method of separation of variable the given equation is separated into differential equations where each differential equation contains only a single variable. For this the given function ϕ which depends on coordinates (x, y, z) is written as $\phi = xyz$ where x is function of x only, Y is fraction of y only and Z is function of z only.

9.13 Illustrative Examples

Example 13 Solve $2x\dfrac{\partial\phi}{\partial x} - 5y\dfrac{\partial\phi}{\partial y} = 0$

Sol. The given equation is $2x\frac{\partial \phi}{\partial x}-5y\frac{\partial \phi}{\partial y}=0$ (1)

Where ϕ depends on x and y. Now writing $\phi=(x,y)$ as product of functions X(x) and Y(y)

Where X is function of x only and y is function of Y only.

$$\phi(x,y)=X(x)Y(y) \quad (2)$$

Putting (2) in (1)

$$2x\frac{\partial (xy)}{\partial x}-5y\frac{\partial (xy)}{\partial y}=0$$

$$2Y\frac{\partial Y}{\partial x}-5X\frac{\partial Y}{\partial y}=0$$

Dividing (3) by XY

$$\frac{2x}{X}\frac{\partial X}{\partial x}-\frac{5y}{Y}\frac{\partial Y}{\partial y}=0$$

$$\frac{2x}{X}\frac{\partial X}{\partial x}=\frac{5y}{Y}\frac{\partial Y}{\partial y} \quad (4)$$

In this equation L.H.S. is function of only and R.H.S. is function of Y only. Equating each side to some constant (say m)

$$\frac{2x}{X}\frac{\partial X}{\partial x}=m$$

$$\frac{5y}{Y}\frac{\partial Y}{\partial y}=m$$

Integrating $\log_e X=\frac{m}{2}\log_e x+\log c_1$

$$\log_e X=\log_e x^{m/2}+\log c_1$$

$$\Rightarrow X=c_1x^{m/2} \quad (7)$$

Similarly solving (6) we have

$$Y=c_2y^{m/5}$$

Therefore solution of given differential equation is

$$\phi=xy=c_1c_2x^{m/2}y^{m/5}$$

Or $\phi = cx^{m/2} y^{m/5}$

Where $c = c_1 c_2$

Example 14 Solve $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$

Sol. The given equation is $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ (1)

Since ϕ is a function of variable x and y therefore we can write

$$\phi(x, y) = X(x)Y(y) \quad (2)$$

Using (2) in (1)

$$\frac{\partial^2(XY)}{\partial x^2} + \frac{\partial^2(XY)}{\partial y^2} = 0 \text{ or } Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0 \quad (3)$$

Dividing (3) by XY

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{-1}{Y}\frac{\partial^2 Y}{\partial y^2} \quad (4)$$

Each side of equation (4) is function of one variable only and therefore we can equate each side to some constant (say k^2)

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k^2 \Rightarrow \frac{\partial^2 X}{\partial x^2} - k^2 X = 0$$

$$-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k^2 \Rightarrow \frac{\partial^2 Y}{\partial y^2} + k^2 y = 0$$

The solution of equation (5) is given as

$$X = A_1 e^{kx} \quad + \quad B_1 e^{-kx} \quad (7)$$

[Ordinary differential equation of order two solutions]

Similarly solving equation (6) gives

$$Y = A_2 \cos ky + B_2 \sin ky$$

Therefore the solution is given by

$$\phi(x, y) = XY = (A_1 e^{kx} + B, e^{-kx}) \times (A_2 \cos ky + B_2 \sin ky)$$

Example 15 Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Sol. This equation is most commonly used and very important differential equation in physics such as in study of gravitational potential, electrostatic potential, magnetic potential, thermal equilibrium, hydro-dynamics etc.

The equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ (1)

is known as ***Laplace's equation***

Here u is a function of x,y,z. by method of ***separation of variables***

$$u = X(x)\ Y(y)\ Z(z) \quad (2)$$

Where X,Y and Z is function of x,y,z only respectively.

Using (2) in (1)

$$YZ\frac{\partial^2 X}{\partial x^2} + ZX\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2} \quad (3)$$

Dividing (3) by XYZ and rearranging

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} - \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} \quad (4)$$

L.H.S. is function of x only whereas R.H.S. is function of y and z.

Equating each side to some constant say m_1^2 we have

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = m_1^2 \text{ and } -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} - \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = m_1^2 \text{ or } \frac{\partial^2 X}{\partial x^2} - m_1^2 X = 0 \quad (5)$$

Solution of (5) is $X = C_1 e^{m_1 x}$ Where C_1 is an arbitrary constant

Now taking other part

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} - m1^2 \quad (6)$$

Here L.H.S. depends on y only and R.H.S. is function of z only therefore, we can equate each side to some constant $m2^2$.

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = m_2^2 \text{ or } \frac{\partial^2 Y}{\partial y^2} - m_2^2 Y = 0 \quad (7)$$

And $-\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} - m_1^2$ or $\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -(m_1^2 + m_2^2)$

$$\therefore \frac{\partial^2 Z}{\partial z^2} - m_3{}^2 Z = 0 \qquad (8)$$

And $m_1{}^2 + m_2{}^2 + m_3{}^2 = 0 \qquad (9)$

Solution of (7) and (8) is

$$y = C_2\, e^{m_2} y \qquad (10)$$

And $Z = C_3 e^{m_3 z} \qquad (11)$

This general solution of Laplace's equation is

$$U = XYZ = C_1 C_2 C_3 \;\; e^{m_1 x} \;\; e^{m_2 y} \;\; e^{m_3 z}$$

$$= C_1 C_2 C_3 \quad e^{m_1 x + m_2 y + m_3 z} \qquad (12)$$

Where m_1, m_2 m_3 are related by (9) and c_1, c_2 ,c_3 are arbitrary constants.

9.14 Self Learning Exercise-II

Very Short Answer type Questions

Q.1 The standard form of equation Pp + Qq = R is known as

Q.2 Equation of form Pp + Qq =R is linear or non-linear?

Q.3 What is equation of form $z = px + qy + f(p,q)$ known as?

Short Answer type Questions

Q.4 What is the complete integral of equation in standard form I $f(p,q) = 0$?

Q.5 What is the complete integral of equation in standard form II $z = px + qy + f(p,q)$?

Q.6 Charpit's method is used for solution of which equations?

9.15 Summary

The unit presents an introduction of partial differential equations. The order, degree, linearity and non-linearity of partial differential equations have been discussed. Method of solutions for linear and non-linear partial differential equations of order one have been described and explained through illustrative examples. The method of separation of variables has been described which is very important in solution of many problems in physics. Many second degree partial differential equations occur in physics such as Laplace equation, wave equation, heat conduction equation, Schrodinger equation etc., which may be solved through method of separation of variables.

9.16 Glossary

Arbitrary: *Mathematics* (Of a constant or other quantity) of unspecified value.

Quasi: Being partly or almost

9.17 Answers to Self-Learning Exercises

Answers to Self-Learning Exercise-I

Ans.1 : Three

Ans.2 : Two

Ans.3 : No

Ans.4 : Because dependent variable z is multiplied with $\frac{\partial z}{\partial y}$

Ans.5 : Dependent variable v, independent variables x, y.

Ans.6 : Single independent variable

Answers to Self-Learning Exercise-II

Ans.1 : Lagrange's equation

Ans.2 : Linear

Ans.3 : Clairaut equation

Ans.4 : $z = ax + by + c, f(a,b) = 0$

Ans.5 : $z = ax + by + f(a,b)$

Ans.6 : Equation of order one but of any degree

9.18 Exercise

Section A (Very Short Answer type Questions)

Q.1 Is equation $x^2 p + y^2 q = 2z$ linear or non-linear?

Q.2 What are equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ known as?

Q.3 What is the order and degree of equation $p^2 + q^2 = 1$ where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$

Q.4 What is the order and degree of equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 1$

Section B (Short Answer type Questions)

Q.5 Define partial differential equation

Q.6 What is the difference between ordinary and partial differential equations?

Q.7 What is the significance of partial differential equations in physics?

Q.8 Give four examples of second order partial differential equations which commonly occur in physics

Q.9 Define complete integral and particular integral.

Q.10 What do you understand by singular solution of a partial differential equation?

Section C (Long Answer type Questions)

Q.11 Find complete integral of $z^2 = pqxy$ using Charpit's method.

Q.12 Solve $yzp + zxq = xy$

Q.13 Solve Helmholtz equation $(\nabla^2 + k^2)u = 0$

Or $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0$ by method of separation of variable

Q.14 Find complete and singular integral of $z = px + qy - 2\sqrt{pq}$

Q.15 Solve $py + qy + pq = 0$

9.19 Answers to Exercise

Ans.1 : Linear

Ans.2 : Lagrange's auxiliary equation

Ans.3 : Order one, degree two

Ans.4 : Order two, degree one

Ans.11 : $z = x^a y^{1/a} b$

Ans.12 : $\phi(x^2 - y^2, x^2 - z^2) = 0$

Ans.13 : $u = XYZX = Ae^{ik_1x} + Be^{-ik_1x},\ Y = A_2 \cos k_2x + B_2 \sin k_2x,$
$Z = A_3 \cos k_3x + B_3 \sin k_3x$

Ans.14: Clairaut equation solution complete integral $z = ax + by - 2\sqrt{ab}$ Singular integral $(x - z)(y - z) = 1$

Ans.15 : Standard Form IV

$$z = \left(\frac{a}{1-a}\right)\frac{x^2}{2} - \frac{ay^2}{2} + \frac{b}{2}$$ Where a, b are arbitrary Constants.

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UNIT-10
Series Solutions- Frobenius Method, Recurrence relation

Structure of the Unit

10.0 Objectives

In this unit we briefly discuss series solution of second order differential equations. We will solve the differential equation by power series method and Frobenius method.

10.1 Introduction

Differential equations may be divided into two large classes

(i) linear equations and,

(ii) nonlinear equations

Nonlinear equations of second and higher orders are rather difficult to solve while linear equations are much simpler in many respects. The linear differential equations play an important role in theoretical physics for example, in connection with mechanical vibrations, electric circuits, networks etc.

A linear differential equations of order *n* is having the form

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots.+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=f(x)$$

where $a_0, a_1, a_2, ...a_n$ and *f(x)* are given functions of independent variable *x* and $a_0 \neq 0$. This equation is linear in y and its derivative.

If n = 2, we have the ***linear differential equation of second order***; this is written in the form

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=F(x)$$

The characteristic feature of this equation is that it is linear in y and its derivatives while *P(x), Q(x)* and *F(x),* may be any given functions of independent variable *x*. The function on the right *F(x)* represents a source (Ex. electrostatic charge) or a driving force (Ex. driven oscillator).

Any equation of second order which cannot be written in above form is said to, be non-linear equations.

For example

$$\frac{d^2y}{dx^2}+4y=e^{-x}\sin x$$

and

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-1)y=0$$

are linear equations, while

$$\frac{d^2y}{dx^2}+y\frac{dy}{dx}=0 \; and \; \frac{d^2y}{dx^2}+\sqrt{y}=0$$

are nonlinear equations.

Linear differential equations of second order play an important role in many differential problems of theoretical physics. It may be noted that some of these equations are very simple because their solutions are elementary functions; while others are more complicated and their solutions are important higher functions as Bessel, Legendre and hypergeometric functions.

If the function F(x) on the right hand of the equation is zero, it is said to be homogeneous and if F(x) is not zero, it is said to be non homogeneous.

The linear homogeneous differential equations with constant coefficient can be solved by algebraic methods and their solutions are elementary functions. On the other hand, in case of linear homogeneous differential equations with variable coefficient, the situation is more complicated and their solutions may be important higher non elementary functions. Bessels, Legendre's and hypergeometric equations are of this type. Since these and other such equations and their solutions play an important role in theoretical physics, we shall now consider a method for solving such equations. The solutions will appear in the form of power series and the method is, therefore, known as power series or series integration method.

$$\boxed{\sum_{m=0}^{\infty} c_m (x-x_0)^m = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 \ldots}$$

where c_0, c_1, c_2 ….. are constants, called the coefficients of the series, x is a variable and x_0 is a constant called the centre.

The expression

$$S_n(x) = c_0 + c_1(x - x_0) + \ldots.. + c_n(x - x_0)^n$$

is called the nth partial sum of the series. Clearly, if we omit the terms of S_n, the remaining expression

$$R_n(x) = c_{n+1}(x-x_0)^{n+1} + c_{n+2}(x-x_0)^{n+2} + \ldots$$

This is known as remainder, after the term $c_n(x)$.

It may happen that for $x=x_0$, the sequence of partial sums $S_1(x)$, $S_2(x)$, …. $S_n(x)$ …. converges, say

$$\lim_{n\to\infty} S_n(x_0) = S(x_0)$$

Then we may say that the series is convergent at $x = x_0$, the number $S(x_0)$ is called the sum of x_0 and we write

$$S(x_0) = \sum_{m=0}^{\infty} c_m (x - x_0)^m$$

If this series is divergent at $x=x_0$, then the series is said to be divergent at $x=x_0$.

10.2 Solution of Second Order Differential Equations with Constant Coefficients

Consider the differential equation

$$\frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = f(x)$$

Let us introduce the symbol of operation

$$D^r = \frac{d^r}{dx^r} i.e.\ D = \frac{d}{dx}\ and\ D^2 \frac{d^2}{dx^2}$$

Then above equation may be expressed in the form :

$$(D^2 + a_1 D + a_2) y = f(x)$$

$$L(D) y = f(x)$$

If $f(x) = 0$, then equation reduces to $L(D) y = 0$

This is called the reduced equation and its solution is called the complementary function denoted by y_C. Then we may specify

$$L(D) y_C = 0$$

The general solution of equation of equation consists of them sum of two parts:

- The complementary function y_C and
- The particular integral y_P; which may be seen as follows :

The particular integral y_P satisfies equation, i.e.

$$L(D) y_P = f(x)$$

Adding above equations; we get

$$L(D)(y_P + y_C) = f(x)$$

$$L(D)(y_P + y_C) = f(x)$$

If we substitute $y = y_C + y_P$

We obtain

L(D)y = f(x)

This proves the proposition that the general solution of a linear differential equation with constant coefficients is the sum of a particular integral y_P and the complementary function $y_{C.}$

If α and β are the roots of auxiliary equation.

$$(D^2 + a_1 D + a_2)y = 0$$

then we may write equation in the form

$$(D-\alpha)(D-\beta)y = f(x)$$

Substituting $(D-\beta)y = u$

then $(D-\alpha)u = f(x)$

$$\frac{du}{dx} - \alpha u = f(x)$$

This is a first order linear equation and solution of this equation with $p(x) = -\alpha$ and $y(x) = u(x)$

$$u = A_1 e^{\alpha x} + e^{\alpha x}\int e^{-\alpha x} f(x)dx$$

$$= e^{\alpha x}[A_1 + \phi(x)]$$

Where

$$\phi(x) = \int_0^x e^{-\alpha x} f(x)dx$$

If we substitute this value of u in equation; we get

$$(D - \beta)y = e^{\alpha x}\,[(A_1 + \Phi(x)]$$

$(D-\beta)y = f(x)$

where $f(x) = e^{\alpha x}[(A_1 + \Phi(x)]$

Equation is again first order linear differential equation; hence its solution is

$$y = A_2 e^{\beta x} + e^{\beta x}\int e^{-\beta x} F(x)dx$$

Substituting value of f(x); we get.

$$y = A_2 e^{\beta x} + e^{\beta x}\int e^{-\beta x} e^{\beta x}[A_1 + \phi(x)]dx$$

$$= A_2 e^{\beta x} + e^{\beta x}\int e^{(\alpha-\beta)x}[A_1 + \phi(x)]dx$$

$$= A_2 e^{\beta x} + e^{\beta x} A_1 \int e^{(\alpha-\beta)x} dx + e^{\beta x}\int e^{(\alpha-\beta)x}\phi(x)dx$$

$$= A_2 e^{\beta x} + \frac{A_1 e^{\beta x}}{\alpha-\beta}.e^{(\alpha-\beta)x} + e^{\beta x}\int e^{(\alpha-\beta)x}\phi(x)dx$$

On changing the meaning of constant A_1; solution may be written as

$$y = A_1 e^{\alpha x} + A_2 e^{\beta x} + e^{\beta x}\int e^{(\alpha-\beta)x}\phi(x)dx$$

In this solution the first two terms represent the complementary function while remaining last term represents the particular integral.

10.3 Self Learning Exercise I

Very Short Answer Type Questions

Q.1 Write down the uses of linear differential equation in theoretical physics.

Q.2 Solve y’-2xy=0

Short Answer Type Questions

Q.3 Define linear differential equation.

Q.4 Define nonlinear differential equation.

10.4 Power Series Solution; Frobenius' Method

The standard form of linear homogeneous differential equation of second order

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \qquad (1)$$

We shall confine our discussion to a solution of (1) in the neighborhood of $x=0$. When coefficients of series $P(x)$ and $Q(x)$ at point $x = a$ i.e. $P(a)$ and $Q(a)$ are finite, the point $x= a$ is called an ***ordinary point*** of (1). When $(x-a)P(x)$ and $(x-a)^2 Q(x)$ remain finite at $x=a$, then the point $x=a$ is called a regular point of (1). Otherwise $x=a$ is called a ***singular point*** of (1). The following method is applicable when $x=0$ is ordinary or regular point of (1).

$$y(x) = x^k = (c_0 + c_1x + c^2x^2 + ... c^mx^m + ...)$$

$$= x^k\sum_{m=0}^{\infty} c_m x^m = \sum_{m=0}^{\infty} c_m x^{k+m}, c \neq 0 \qquad (2)$$

where the exponent k and all the coefficients c_m are undermined. By differentiating (term by term) twice is succession, we obtain

$$\frac{dy}{dx} = \sum_{m=0}^{\infty}(k+m)c_m x^{k+m-1}$$

$$= x^{k-1}[kc_0 + (k+1)c_1x + ... + (k+m)c_mx^m + ...] \qquad (3)$$

$$\frac{d^2y}{dx^2} = \sum_{m=0}^{\infty}(k+m)(k+m-1)c_m x^{k+m-2}$$

$$= x^{k-2}\,[k(k-1)c_0 + [k+1)kc_1x + ... + (k+m)(k+m-1)x^m + ...] \qquad (4)$$

when $x=0$ is a regular point, then $xP(x)$ and $x^2\,Q(x)$ are finite at $x=0$; so we can write

$$xP(x) = p(x) \qquad (5)$$

and

$$x^2Q(x) = q(x) \qquad (6)$$

Substituting $P(x)$ and $Q(x)$ from (5) and (6), (1) takes the form

$$\frac{d^2x}{dx^2} + \frac{p(x)}{x}\frac{dy}{dx} + \frac{q(x)}{x^2}y = 0$$

or $$x^2 \frac{d^2x}{dx^2} + xp(x)\frac{dy}{dx} + q(x)y = 0 \qquad (1')$$

$p(x)$ and $q(x)$ may be expanded as a power series as

$$p(x) = p_0 + p_1x + p_3x^2 + p_mx^m + ...$$
$$q(x) = q_0 + q_1x + q_3x^2 + q_mx^m + ... \qquad (7)$$

Inserting values from (2), (3), (4) and (7) in equation (1'), we obtain

$$x^k[k(k-1)\, c_0 + (k+1)kc_1x + ...+(k+m)(k+m-1)c_mx^m+.....]$$
$$+(p_0+p_1x+p_2x^2+...+p_mx^m+..)\, x^k(kc_0+(k+1)\, c_1x + ...+(k+m)c_mx^m + ...]$$
$$+(q_0+q_1x+q_2x^2+...+q_mx^m+....)\, x^k(c_0+c_1x+c_2x^2+...+c_mx^m+....]= 0 \qquad (8)$$

This is a power series and will be equal to zero if the coefficients of various powers of x are separately zero. Hence equating the coefficients of x^k, x^{k+1},...., x^{k+m},... equal to zero, we obtain a system of equation involving the unknown coefficients c_m viz.

$$[k(k - 1) + p_0k + q_0] = 0 \qquad (9a)$$
$$[(k+1)k + p_0(k+1) + q_0]c_1 + (p_1k + q_1)c_0 = 0 \qquad (9b)$$
$$[(k+2)(k+1)+p_0(k+2)+q_0]c_2+[p_1(k+1)+q_1]c_1+...+(p_2k+q_2)c_0 = 0 \qquad (9c)$$

In general

$$[(k+m)(k+m-1)+p_0(k+m)+q_0]c_m+[p_1(k+m-1)+q1]c_{m-1}+...+(p_mk+q_m)c_0 = 0 \qquad (9d)$$

Since $c_0 \neq 0$, equation (9a) implies

$$k(k-1) + p_0k + q_0 = 0 \text{ i.e. } k^2 + (p_0 -1)k\, q_0 = 0 \qquad (10)$$

This is an important quadratic equation and is called the indicial equation of (1) or (1'). Let its roots be α and β. Equation (9d) gives two sets of coefficients c_m which are determined by α and β respectively. Thus two independent solutions of (1) can be derived.

Let us consider the following three possible cases :

Case (i) The roots of the indicial equation are distinct and do not differ by an integer i.e. $\alpha \neq \beta$, $|\alpha-\beta|$ is not a positive integer.

If we substitute $k = \alpha$ into the system of equations (9) and determine the coefficients c_1, c_2, ... successively, then we obtain a solution.

$$y_1(x) = x^{\alpha}(c_0 + c_1x + c_2x^2 + ... + c_mx^m + ...)$$

If we substitute k $=\beta$ into that system of equations and determine the coefficients c_m successively then we get another independent solution.

$$y_2(x) = x^{\beta}(c_0* + c_1* + c_2* + ... + c_m*x^m + ...)$$

Case (ii). The indicial equation has double root : i.e. $\alpha=\beta$.

The indicial equation (10) may be written as

$$k^2 + (p_0 -1)k + + q_0 = 0$$

Its roots are given by

$$\frac{-(p_0 - 1) \pm \sqrt{(p_0 - 1)^2 - 4q_0}}{2}$$

Obviously, the indicial equation has double root $k=\alpha=\beta$ is the only if $(p_0-1)^2 - 4q_0 = 0$ and then

$$k = \alpha = \beta = \frac{1 - p_0}{2}$$

Then we may determine the first solution $y_1(x)$ as usual

$$y_1(x) = x^{\alpha}(c_0 + c_1x + c_2x^2 + ... + c_mx^m + ...) \qquad (11)$$

To find another solution we apply the method of variation of parameters i.e. we replace the constant c in the solution cy_1 by a function $u(x)$ to be determined such that.

$$y_2(x) = u(x)y_1(x) \qquad (12)$$

is a solution of equation (1).

Differentiating Eq. (12) twice successively, we get.

$$\frac{dy_2}{dx} = \frac{du}{dx}y_1 + u\frac{dy_1}{dx}\ \frac{dy_2}{dx} = \frac{du}{dx}y_1 + u\frac{dy_1}{dx} \qquad (13)$$

$$\frac{d_2 y}{dx^2} = \frac{d^2 u}{dx^2} y_1 + 2\frac{du}{dx}.\frac{dy_1}{dx} + u\frac{d^2 y_1}{dx^2} \tag{14}$$

Substituting values of y and its derivative from (12), (13) and (14) in (1) we get

$$x^2\left(\frac{d^2 u}{dx^2} y_1 + 2\frac{du}{dx}.\frac{dy_1}{dx} + u\frac{d^2 y_1}{dx^2}\right) + xp\left(\frac{du}{dx} y_1 + u\frac{dy_1}{dx}\right) + quy_1 = 0 \tag{15}$$

As y_1 is a solution of (1'), therefore the sum of the terms involving u must be zero i.e.

$$u\left[x^2 \frac{d^2 y_1}{dx^2} + xp\frac{dy_1}{dx} + qy_1\right] = 0$$

Consequently equation (15) reduces to

$$x^2 y_1 \frac{d^2 u}{dx^2} + 2x^2 \frac{dy_1}{dx}\frac{du}{dx} + xpy_1 \frac{du}{dx} = 0$$

Dividing throughout by $x^2 y_1$ and inserting the power series for $p(x)$, we obtain

$$\frac{d^2 u}{dx^2} + \left[2\frac{(dy_1 / dx}{y_1} + \frac{p_0}{x} + p_1 + p_2 x + ...\right]\frac{du}{dx} = 0 \tag{16}$$

Now from equation (11) if follows that

$$\frac{(dy_1 / dx}{y_1} = \frac{x^{\alpha-1}\left[\alpha c_0 + (\alpha+1)c_1 x + ...\right]}{x^{\alpha}[c_0 + c_1 x + ...]}$$

$$= \frac{1}{x}\left[\frac{\alpha c_0 + (\alpha+1)c_1 x + ...}{[c_0 + c_1 x + ...}\right] \quad = \frac{\alpha}{x} + ...$$

In the last expression and in the following the dots denote the terms which are constant and involve positive powers of x.

Hence eqn. (16) can be written as

$$\frac{d^2 u}{dx^2} + \left[\frac{2\alpha + p_0}{x} + ...\right]\frac{du}{dx} = 0 \tag{17}$$

From equation (11)

$$\frac{d^2u/dx^2}{du/dx} = -\frac{1}{x} + \ldots..$$

Integrating we get

$$\log\frac{du}{dx} = -\log x + \ldots$$

$$\frac{du}{dx} = \frac{1}{x}e^{(\ldots)}$$

Expanding the exponential function in powers of x and integrating again, the expression for u will be of the form.

$$u = logx + k_1x + k_2x^2 + \ldots.$$

Substituting this in (12), we get the second solution

$$y_2(x) = (logx + k_1x + k_2x_2 + \ldots..)y_1(x)$$

Using (11) this may be written in the form.

$$y_2(x) = y_1 \log x + x^{\alpha} \sum_{m-0}^{\infty} \lambda_m x^m \tag{18}$$

Case (iii) The roots of the indicial equation differ by an integer i.e., α≠β, |α-β| is a positive integer.

If the roots of the indicial equation (10) differ by an integer, say β=α-l, where l is a positive integer. Then we may determine the first solution $y_1(x)$ corresponding to root α as

$$y_1(x) = x^{\alpha}(c_0 + c_1x + c_2x^2 + \ldots.)$$

To determine the second solution $y_2(x)$ we proceed as in case (ii). The first steps are literally the same and yield equation (17). From elementary algebra we known that in indicial equation (10) the coefficient.

$$(p_0 - l) = -(\alpha+\beta)$$

Here β=α-l; therefore $(p_0 - l) = l - 2\alpha$

From 17,

$$\frac{d^2u}{dx^2}+\left[\frac{l+1}{x}+.....\right]\frac{du}{dx}=0$$

$$\frac{d^2u/dx^2}{du/dx}=-\left[\frac{l+1}{x}+....\right]$$

Integrating we find

$$\log\frac{du}{dx}=-(l+1)\log x+....$$

$$\frac{du}{dx}=x^{-(l+1)}e^{(...)}$$

where dots denote some series involving non-negative powers of *x*.

Expanding the exponential function in powers, of x, we obtain a series of the form

$$\frac{du}{dx}=\frac{1}{x^{l+1}}+\frac{k_1}{x^l}+....+\frac{k_l}{x}+k_{l+1}+k_{l+2}x+...$$

Integrating again we get

$$u=\frac{1}{lx^1}+\frac{k_1}{(l-1)x^{l-1}}-....+k_1\log x+k_{l+1}x+...$$

Substituting this in second solution $y_2(x) = u(x)y1(x)$, we obtain

$$y_2(x)=\left[-\frac{1}{lx^1}-\frac{k_1}{(l-1)x^{l-1}}-...+k_l\log x+k_{l+1}x+...\right]y_1(x)$$

Using (19) and remembering that α-l =β, we get the second solution $y_2(x)$ in the form

$$y_2(x)=k_1y_1(x)\log x+x^{\beta}\sum_{m=0}^{\infty}\lambda_m x^m \tag{20}$$

Conclusively if the indicial equation (10) has roots that differ by an integer, then there are two independent solutions, one corresponding to the larger root (α) is given by (19) while the other corresponding to the smaller root (β) is given by (20).

It may be noted that for a double root of indicial equation (10) the second solution always contains a logarithmic term ; but in the case (iii) where roots differ by an integer, the coefficient k_i may be zero and consequently the logarithmic term may be missing.

10.5 Linear Independence of Solutions

Let y_i be set of functions. These functions are linearly dependent if there exists a relation of the form

$$\sum_i \lambda_i y_i = 0$$

in which not all of the coefficient (λ_i) are zero. On the other hand, if the only solution is $\lambda_i = 0$ for all values of *i,* then the set of functions y_i are said to be ***linearly independent.***

Let us assume that the functions y_i are differentiable as needed. Then differentiating above equation repeatedly, we generate a set of equations

$$\sum_i \lambda_i y_i' = 0, \quad \sum_i \lambda_i y_i'' = 0, \ldots, \quad \sum_i \lambda_i y_i''' = 0$$

where we have used y_i' $(\frac{dy}{dx})$, y_i'' $(\frac{d^2y}{dx^2})$ and so on i.e., the order of derivative is indicated by the number of dashes.

Thus we get a set of homogeneous linear equations in which λ_i are the unknown quantities. This system of equations has a solution $\lambda_i \neq 0$ if and only if the determinant of the coefficients of the λ_i's vanishes. This means.

$$\begin{vmatrix} y_1 & y_2 & \cdots & \cdots & y_n \\ y_1' & y_2' & \cdots & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & \cdots & y_n^{(n-1)} \end{vmatrix}$$

This determinant is called the ***Wronskian***.

- If the Wronskian is not equal to zero, then above equation has no solution other than $\lambda_i = 0$. The set of functions y_i is therefore linearly independent.

- If the Wronskian is zero over the entire range of variable, the functions y_i are linearly dependent over this range.

In particular in the case of second order linear homogeneous equations, the two solutions y_1 *(x)* and y_2 *(x)* are *linearly independent* if the Wronskian of y_1 and y_2 i.e.,

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \neq 0$$

10.6 Recurrence Relation

Example 1 Find the power series solution of linear oscillator equation.

$$\frac{d^2y}{dx^2} + \omega^2 y = 0 \ or \ y'' + \omega^2 y = 0$$

in powers of x.

Sol. The given differential equation is

$$\frac{d^2y}{dx^2} + \omega^2 y = 0 \tag{1}$$

The point x = 0 is a regular point; therefore power series Frobenius' method is applicable. Let the solution be

$$y(x) = x^k = x^k \sum_{m=0}^{\infty} a_m x^m, a_0 \neq 0$$

$$= \sum_{m=0}^{\infty} a_m x^{k+m}$$

Differentiating equation twice we get

$$\frac{dy}{dx} = \sum_{m=0}^{\infty} a_m (k+m) x^{k+m-1}$$

$$\frac{d^2y}{dx^2} = \sum_{m=0}^{\infty} a_m (k+m)(k+m-1) x^{k+m-2}$$

Substituting these values of y, $\frac{d^2y}{dx^2}$ in equation (1), we get.

$$\sum_{m=0}^{\infty} a_m (k+m)(k+m-1)x^{k+m-2} + \omega^2 \sum_{m=0}^{\infty} a_m x^{k+m} = 0 \qquad (2)$$

From the condition of uniqueness of power series, the coefficients of each power of x must vanish individually.

The lowest power of above equation is x^{k-2} for $m = 0$ in the first summation: The requirement that the coefficients of x^{k-2} vanish, yields,

$$a_0 k(k-1) = 0$$

As $a_0 \neq 0$ being the coefficient of lowest non-vanishing terms of the series; hence.

$$k(k-1) = 0$$

This is called the *indicial equation* and gives $k = 0$ or $\mathbf{k} = 1$. Now equating the coefficients of x^{k+r} in equation (2) to zero ($m = \mathbf{r} + 2$) in first summation and in $m = r$ in the second summation (since both summations are independent), we get.

$$a_{r+2}(k+r+2)(k+r+1) + \omega^2 a_r = 0$$

$$a_{r+2} = -\frac{\omega^2}{(k+r+2)(k+r+1} a_r \qquad (3)$$

This is a two term ***recurrence relation***. If a_r is given, then a_{r+2}, a_{r+4} ... etc. are calculated throw above equation.

It is obvious that if we start with a_o, equation leads to the even coefficients a_2, a_4 ... etc. and ignores a_1, a_3. a_5 etc. Since *a,* is arbitrary, it may be set equal to zero and then by above equation

$$a_3 = a_5 = a_7 ... = 0$$

i.e.., all the odd power coefficient vanish.

Case (i) When k = 0; the recurrence relation (3) becomes.

$$a_{r+2} = -\frac{\omega^2}{(r+2)(r+1)} a_r$$

This leads to

$$a_2 = \frac{\omega^2}{1.2} a_0 = -\frac{\omega^2}{2!} a_0$$

$$a_4 = -\frac{\omega^2}{3.4} a_4 = +\frac{\omega^4}{4!} a_0$$

$$a_6 = -\frac{\omega^2}{5.6} a_4 = -\frac{\omega^6}{6!} a_0$$

and so on

... ...

The general term is

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n)!} a_0$$

Substituting these even coefficient in series solution [keeping in mind k = 0], we get the solution.

$$y(x) = a_0 \left[1 - \frac{(\omega x)^2}{2!} + \frac{(\omega x)^4}{4!} + \frac{(\omega x)^6}{6!} + \ldots \right] = a_0 \cos \omega x \tag{4}$$

Case (ii) When k = 1, the recurrence relation (3) gives

$$a_{r+2} = -\frac{\omega^2}{(r+3)(r+2)} a_r$$

Substituting r = 0, 2, 4 successively, we obtain

$$a_2 = -\frac{\omega^2}{3.2} a_0 = -\frac{\omega^2}{3!} a_0$$

$$a_4 = -\frac{\omega^2}{5.4} a_0 = \frac{\omega^4}{5!} a_0$$

$$a_6 = -\frac{\omega^2}{7.6} a_0 = -\frac{\omega^6}{7!} a_0$$

and so on. The General term is

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n+1)!} a_0$$

Substituting these values [keeping in mind k = 1] we get.

$$y(x) = a_0 x\left[1 - \frac{(\omega x)^2}{3!} + \frac{(\omega x)^4}{5!} - \frac{(\omega x)^6}{7!} + \ldots\right]$$

(5)

$$= \frac{a_0}{\omega}\left[(\omega x) - \frac{(\omega x)^3}{3!} + \frac{(\omega x)^5}{5!} - \frac{(\omega x)^7}{7!}\ldots\right] = \frac{a_0}{\omega}\sin \omega x$$

Thus the Frobenius' method gives two solution (4) and (5) of the linear oscillator equation.

Example 2 Solve the differential equation

$$\frac{d^2\psi}{dx^2} + \left(E - x^2\right)\psi = 0$$

such that $\Psi = 0$ as $|x| \to \infty$

Sol. The given differential equation is

$$\frac{d^2\psi}{dx^2} + \left(E - x^2\right)\psi = 0 \qquad (1)$$

This equation often occurs in quantum mechanics.

Substituting

$$\psi(x) = \phi(x)e^{-x^2/2} \qquad (2)$$

$$\frac{d\psi}{dx} = e^{x^{-2}/2}\frac{d\phi}{ex} - \phi.x.e^{-x^2/2}$$

And

$$\frac{d^2\psi}{dx^2} = e^{-x^2/2}\frac{d^2\phi}{dx^2} - 2xe^{-x^2/2}\frac{d\phi}{dx} - \phi e^{-x^2/2} + \phi x^2 e^{-x^2/2}$$

Substituting these values in equation (1), we get.

$$e^{-x^2/2}\left[\frac{d^2\phi}{dx^2} - 2x\frac{d\phi}{dx} + \phi(x^2 - 1) + (E - x^2)\phi\right] = 0$$

Dividing throughout by $e^{-x^2/2}$; we get

$$\frac{d^2\phi}{dx^2} - 2x\frac{d\phi}{dx} + (E-1)\phi = 0$$

Let its series solution in descending powers of x be

$$\phi = \sum_{r=0}^{\infty} a_r x^{k-r} \tag{3}$$

then

$$\frac{d\phi}{dx} = \sum_{r=0}^{\infty} a_r (k-r) x^{k-r-1}$$

and

$$\frac{d^2\phi}{dx^2} = \sum_{r=0}^{\infty} a_r (k-r)(k-r-1) x^{k-r-2}$$

substituting these values; we get

$$\sum_{r=0}^{\infty} a_r (k-r)(k-r-1)x^{k-r-2} - 2x\sum_{r=0}^{\infty} a_r (k-r)x^{k-r-1} + (E-1)\sum_{r} a_r x^{k-r} = 0$$

or

$$\sum_{r=0}^{\infty} a_r (k-r)(k-r-1)x^{k-r-2} + \sum_{r=0}^{\infty} a_r \{(E-1) - 2(k-r)\}x^{k-r} = 0 \tag{4}$$

This equation is an identity; therefore coefficients of various powers of x must vanish.

Let us first equate the coefficients of x^k, the highest power of x to zero (by putting $r=0$ in second term, we get

$$(E - 1 - 2k)a_0 = 0$$

As $a_0 \neq 0$; being the coefficient of the first term of the series; therefore, we must have

$$E - 1 - 2k = 0 = or\ k = \frac{E-1}{2} \tag{5}$$

Now equating to zero the coefficients of x^{k-1} to zero [by putting $r=1$ in second term of (4)]; we get

$$[E-1-2(k-1)]a_1 = 0$$

As $[E-1-2(k-1)] \neq 0$; therefore we must have

$$a_1 = 0 \tag{6}$$

Now equating to zero the coefficient of x^{k-r-2}, the general term in equation (4), we get

$$a_r(k-r)(k-r-1) + [E-1-2(k-r-2)]a_{r+2} = 0$$

$$a_{r+2} = -\frac{a_r(k-r)(k-r-1)}{[E-1-2(k-r-2)]}$$

$$= -\frac{a_r\left(\frac{E-1}{2}-r\right)\left(\frac{E-1}{2}-r-1\right)}{\left[E-1-2\left(\frac{E-1}{2}-r-2\right)\right]} \tag{7}$$

Substituting r = 0, 2, 4, 6 … successively in equation (7); we get

$$a_2 = -\frac{\left(\frac{E-1}{2}\right)\left(\frac{E-3}{2}\right)}{2.2} a_2 = -\frac{(E-1)(E-3)(E-5)(E-7)}{8^2.2.4} a_0$$

$$a_6 = -\frac{\left(\frac{E-9}{2}\right)\left(\frac{E-11}{2}\right)}{2.6} a_4$$

$$= -\frac{(E-1)(E-3)(E-5)(E-7)(E-9)(E-11)}{8^3.2,\ 4,\ 6} a_0$$

and so on

Also since $a_1 = 0$, we have from (7) $a_3 = a_5 = a_7 \ldots = 0$

Substituting this values in (3); we get

$$\phi(x) = a_0\left[x^{(E-1/)/2} - \frac{(E-1)(E-3)}{8.2} x^{(E-5)/2}\right.$$

$$+\frac{(E-1)(E-3)(E-5)(E-7)}{8^2.2.4}x(E-9)/2.....\Bigg] \qquad (9)$$

of ψ in equation (2), the series solution of given differential equation (1) becomes

$$\psi = a_0 e^{-x^2/2}\left[x^{(E-1)/2} - \frac{(E-1)(E-3)}{8.2}x^{(E-5)/2}\right.$$

$$\left.+\frac{(E-1)(E-3)(E-5)(E-7)}{8^2.2.4}x^{(E-9)/2}\right]$$

10.7 Self Learning Exercise II

Very Short Answer Type Questions:

Q.1 Define Wronskian.

Short Answer Type Questions:

Q.2 Find a series solution around x_0=0 for the following differential equation.

$$y'-xy=0$$

Q.3 Find a series solution around x_0=2 for the following differential equation.

$$y'-xy=0$$

10.8 Summary

The unit starts with the introduction of nonlinear and linear differential equations and then solved second order differential equations with constant coefficients. We solved the differential equation by power series method and Frobenius method and understand them with many examples.

10.9 Glossary

Differential equation:

An equation that expresses a relationship between functions and their derivatives.

Recurrence relation:

In mathematics, a recurrence relation is an equation that recursively defines a sequence, once one or more initial terms are given: each further term of the sequence is defined as a function of the preceding terms. The Fibonacci numbers are the archetype of a linear, homogeneous recurrence relation with constant coefficients. The logistic map is another common example.

Dependent equations:

A system of equations that has an infinite number of solutions

Independent equations:

A system of equations is said to be independent if the system has exactly one solution.

10.10 Answer to Self Learning Exercises

Answer to Self Learning Exercise-I

Ans.1 : Mechanical vibrations, electric circuits, networks etc

Ans.2 : $y = a_0 + a_1 x + a_2 x^2 + ...$

Ans.3 : A linear differential equations of order *n* is having the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = f(x)$$

where $a_0, a_1, a_2, \ldots a_n$ and *f(x)* are given functions of independent variable *x* and $a_0 \neq 0$. This equation is linear in y and its derivative.

Ans.4 : Any equation of second order which cannot be written in above form is said to, be non-linear equations. For example

$$\frac{d^2 y}{dx^2} + 4y = e^{-x} \sin x \quad ,$$

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - 1)y = 0$$

are linear equations, while

$$\frac{d^2 y}{dx^2} + y\frac{dy}{dx} = 0 \; and \; \frac{d^2 y}{dx^2} + \sqrt{y} = 0$$

are nonlinear equations.

Answer to Self Learning Exercise-II

Ans.2: $$y(x) = a_0\left\{1+\sum_{k=1}^{\infty}\frac{x^{3k}}{2.3.5.6...(3k-1).3k}\right\}+$$

$$a_1\left\{x+\sum_{k=1}^{\infty}\frac{x^{3k+1}}{2.3.5.6...3k.(3k+1)}\right\}$$

Ans.3: $$y(x) = a_0\left\{1-(x+2)^2+\frac{1}{6}(x+2)^3+...\right\}+$$

$$a_1\left\{(x+2)-\frac{1}{3}(x+2)^3+\frac{1}{12}(x+2)^4+...\right\}$$

10.11 Exercise

Q.1 Use Frobenius method to obtain the general solution of the equation.

$$x^2y''+xy'+(x^2-\frac{9}{4})y=0$$

Q.2 Use the method of Frobenius to solve the homogeneous differentia equation.

$$y''+2xy'+(x^2+2)y=0$$

References and Suggested Readings

1. Mathematical Physics with Classical Mechanics by Satya Prakash, 2014.

2. Mathematical Physics by H.K. Das, 1997.

3. Special Functions and their applications by N.N. Lebedev, R. Silverman, 1973

UNIT-11
Bessel Functions of the First Kind, Generating Function, Orthogonality

Structure of the Unit

11.0 Objectives

In this unit we briefly discuss Bessel equation and their solution. After reading this unit students can solve a special kind of differential equation.

11.1 Introduction

The differential equation

$$\boxed{x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0}$$

is called the ***Bessel's differential equation and particular solutions of this equation are called Bessel's functions of order n.***

We find the Bessel's equation while solving Laplace equation in polar coordinates by the method of separation of variables. This equation has a number of applications in engineering and science.

Bessel's functions are involved in

1. Theory of plane wave
2. Cylindrical and spherical waves
3. Potential theory
4. Elasticity
5. Fluid motion
6. Propagation of waves
7. Planetary motion
8. Oscillatory motion

Bessel's functions are also known as Cylindrical and Spherical function.

11.2 Solution of Bessel Equation

Bessel differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0$$

This equation can also be put in the form

$$\boxed{\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0} \tag{1}$$

The solutions of this equation are called Bessel's functions of order n.

Let the series solution of equation (1) in ascending powers of x may be written as

$$y = x^k \left(a_0 + a_1 x + a_2 x^2 + \ldots\ldots a_2 x^r + \ldots\right) = \sum_{r=0}^{\infty} a \;\; x^{k+r} \tag{2}$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k + r) x^{k+r-1} \tag{3}$$

and $$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1)x^{k+r-2} \tag{4}$$

Substituting these value in (1), we get

$$\sum_{r=0}^{\infty} a_r (k+r)(k+r-1)x^{k+r-2} + \frac{1}{x}\sum_{r=0}^{\infty} a_r (k+r)x^{k+r-1} + \left(1-\frac{n^2}{x^2}\right)\sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

On simplification, we get

$$\sum_{r=0}^{\infty} a_r [(k+r)(k+r-1)+(k+r)-n^2]x^{k+r-2} + \sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

$$\sum_{r=0}^{\infty} a_r [(k+r)(k+r-1+1)-n^2]x^{k+r-2} + \sum_{r=0}^{\infty} a_r x^{k+r} = 0$$

$$\sum_{r=0}^{\infty} a_r [(k+r)^2-n^2]x^{k+r-2} + \sum_{r=0}^{\infty} a_r x^{k+r} = 0 \tag{5}$$

This equation is an identity, therefore the coefficients of various powers of x must be equal to zero.

Equating to zero the coefficients of lowest of x i.e. x^{k-2} in eq (5); we get

$$a_0 (k^2 - n^2) = 0$$

But $a_0 \neq 0$, therefore.

$$k^2 - n^2 = 0 \text{ or } k = \pm n \tag{6}$$

Now equating to zero the coefficient of x^{k-1} in (5), we get

$$a_1 [(k+1)^2 - n^2] = 0$$

But $(k+1)^2 - n^2 \neq 0$ since $k = \pm n$; hence we have $a_1 = 0$.

Again, equating to zero coefficient of general term i.e. x^{k-r} in (5); we get

$$a_{r+2}[(k+r+2)^2 - n^2] + a_r = 0$$

$$a_{r+2}[(k+r+2+n)(k+r+2-n)] + a_r = 0$$

$$a_{r+2} = -\frac{a_r}{[(k+r+2+n)(k+r+2-n)]} \tag{7}$$

Since $a_1 = 0$, therefore, eq (7) gives all odd a coefficient to be zero.

Now for $k = \pm n$; there are two cases :

Case (i) when $k = n$; we have

$$a_{r+2} = -\frac{a_2}{(2n+r+2)(r+2)} \tag{8}$$

Substituting r = 0, 2, 4 … etc. we get

$$a_2 = \frac{a_0}{(2n+2).2} = -\frac{a_0}{2.2.(n+1)} = -\frac{1}{2^2.1!(n+1)}a_0$$

$$a_4 = -\frac{a_0}{(2n+4)4} = +\frac{1}{2^2 2!(n+2)}.\frac{1}{2^2(n+1)}a_0$$

$$= (-1)^2 \frac{1}{2^4 2!(n+1)(n+2)}a_0$$

Similarly,

$$a_6 = -\frac{a_0}{2^6 3!(n+1)(n+2)(n+3)}$$

Substituting $k = n$ and the values of $a_1, a_2, a_3, \ldots a_n$ etc. in equation (2); we get

$$y = a_0 x^n \left[1 + (-1)\frac{x^2}{2^2 1!(n+1)} + (-1)^2 \frac{x^4}{2^4 2!(n+1(n+2))} + \ldots + (-1)^r \frac{x^{2r}}{2^{2r} r!(n+1)(n+2)\ldots(n+r)\ldots}\right] \tag{9}$$

If we substitute

$$a_0 = \frac{1}{2^n \Gamma(n+1)},$$

then the solution of Bessel's equation represented by eq (9) is called Bessel's function of first kind and is symbolized by

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left[1 + (-1)\frac{x^2}{2^2 1!(n+1)} + (-1)^2 \frac{x^4}{2^4 2!(n+1)(n+2)} + \ldots + (-1)^r \frac{x^{2r}}{2^{2r} r!(n+1)(n+2)\ldots(n+r)\ldots}\right]$$

$$= \frac{x^n}{2^n \Gamma(n+1)} \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{2^{2r}!(n+1)(n+2)\ldots(n+r)}$$

$$\boxed{J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{r!(n+r+1)!}\left(\frac{x}{2}\right)^{n+2r}} \qquad (10)$$

Case (ii) When $k = -n$; then the series solution is obtained by substituting $-n$ for n in equation (10) i.e.

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r} \qquad (11)$$

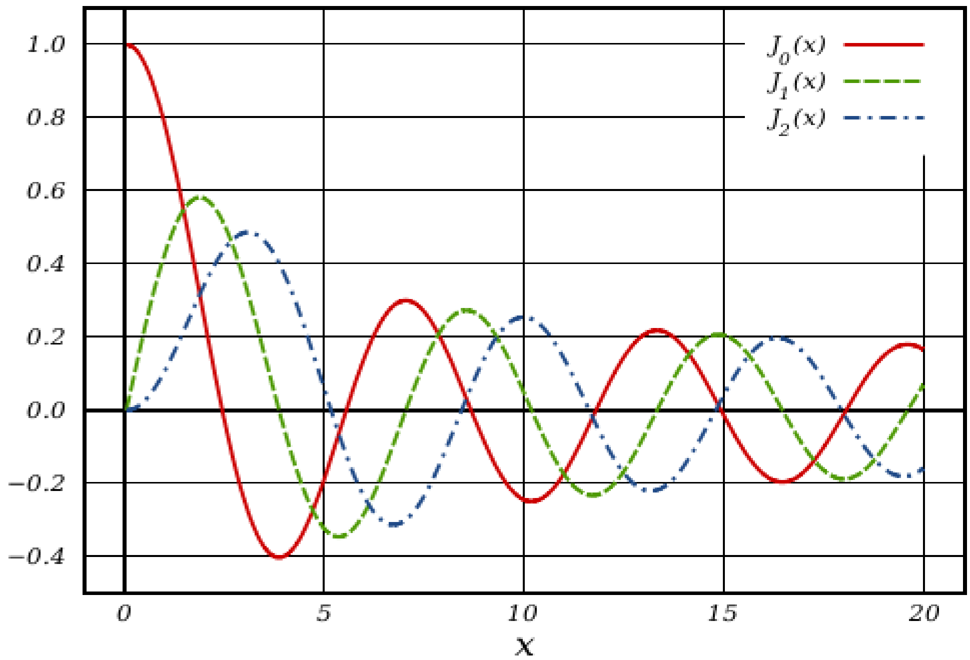
If n=0,

$$J_0(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{r!r)!}\left(\frac{x}{2}\right)^{2r} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \ldots$$

If n=1

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2.4} + \frac{x^5}{2^2.4^2.6} + \ldots$$

We draw the graph of these functions. These functions are oscillatory with period and decreasing amplitude.



Case I. If n is not an integer or zero, then the complete solution of Bessel's equation is

$$\boxed{y = A\,J_n(x) + B\,J_{-n}(x)} \qquad (12)$$

where *A* and *B* are two arbitrary constants.

Case II. If $n=0$, then $y_1=y_2$ and complete solution of Bessel's equation of order zero.

Case III. If *n* is positive integer, then y_2 is not solution of Bessel's equation and y_1 fails to give a solution for negative value of *n*. of order zero.

Example 1 Show that Bessel function $J_n(x)$ is an even function when *n* is even and odd when *n* is odd.

Sol. We know

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{r!(n+r+1)!}\left(\frac{x}{2}\right)^{n+2r}$$

Replacing *x* by –*x*

$$J_n(-x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{r!(n+r+1)!}\left(-\frac{x}{2}\right)^{n+2r}$$

Case I

If n is even then $n+2r$ is even

$$J_n(-x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{r!(n+r+1)!}\left(\frac{x}{2}\right)^{n+2r} = J_n(x)$$

Hence $J_n(x)$ is even function

Case II

If n is odd then $n+2r$ is odd

$$J_n(-x) = -\sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{r!(n+r+1)!}\left(\frac{x}{2}\right)^{n+2r} = -J_n(x)$$

Hence $J_n(x)$ is odd function

Example 2 Show that

$$(i)\ J_{1/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\sin x$$

$$(ii)\ J_{-1/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\cos x$$

$$(iii)\left[J_{1/2}(x)^2\right] + \left[J_{-1/2}(x)\right]^2 = \frac{2}{\pi x}$$

Sol. We know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$= \frac{x^n}{2^n \Gamma n+1}\left[1 - \frac{x^2}{2.2(n+1)} + \frac{x^4}{2.4.2^2(n+1)(n+2)} - \dots\right] \dots.(1)$$

(i) Substituting n = 1/2 in equation (1) we get

$$J_{1/2}(x) = \frac{x^{1/2}}{2^{1/2}\Gamma\left(\frac{3}{2}\right)}\left[1 - \frac{x^2}{2.3} + \frac{x^4}{2.3.4.5} - \dots\right] = \frac{x^{1/2}}{2^{1/2}\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)^x}.\left[x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} \dots.\right]$$

$$= \sqrt{\left(\frac{2}{\pi x}\right)}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right] = \sqrt{\left(\frac{2}{\pi x}\right)} \sin x \qquad (2)$$

(ii) Substituting n =−1/2 in equation (1), we get

$$J_{-1/2}(x) = \frac{x^{-1/2}}{2^{-1/2}\Gamma(1/2)}\left[1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} \dots\right]$$

$$= \frac{x^{-1/2}}{2^{-1/2}\sqrt{\pi}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots\right]$$

$$= \sqrt{\left(\frac{2}{\pi x}\right)} \cos x \qquad (3)$$

(iii) Squaring and adding Eq. (2) and (3); we get

$$(J_{1/2}(x))^2 + (J_{-1/2}(x))^2 = \frac{2}{\pi x}$$

Example 3 Show that

(1) $$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

(2) $$J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(-\sin x - \frac{\cos x}{x}\right)$$

Sol. We know that

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$= \frac{x^n}{2^n \Gamma n+1}\left[1 - \frac{x^2}{2.2(n+1)} + \frac{x^4}{2.4.2^2(n+1)(n+2)} - \ldots\right] \ldots.(1)$$

(1) Substituting $n = 3/2$ in (1), we get

$$J_{3/2}(x) = \frac{x^{3/2}}{2^{3/2}\Gamma\left(\frac{5}{2}\right)}\left[1 - \frac{x^2}{2.5} + \frac{x^4}{2.4.5.7} + \frac{x^6}{2.4.6.7.9} + \ldots\right]$$

$$= \frac{x^{3/2}}{2^{3/2}\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}.\frac{1}{x^2}\left[x^2 - \frac{x^4}{2.5} + \frac{x^6}{2.4.5.7} - \frac{x^8}{2.4.6.5.7.9} + \ldots\right]$$

$$= \frac{1}{2^{-1/2}\sqrt{\pi}}.\frac{1}{x^{1/2}}.\frac{1}{3}\left[x^2 - \frac{x^4}{2.5} + \frac{x^6}{2.4.5.7} - \frac{x^8}{2.4.6.5.7.9} + \ldots\right]$$

$$= \sqrt{\left(\frac{2}{\pi x}\right)}\left[\frac{x^2}{3} - \frac{x^4}{2.3.5} + \frac{x^6}{2.3.4.5.7} - \frac{x^8}{2.3.4.6.7.9} + \ldots\right]$$

$$= \sqrt{\left(\frac{2}{\pi x}\right)}\left[\frac{2x^2}{3!} - \frac{4x^4}{5!} + \frac{6x^6}{7!} - \frac{8x^8}{9!} + \ldots\right]$$

$$= \sqrt{\left(\frac{2}{\pi x}\right)}\left[\frac{(3-1)}{3!}x^2 - \frac{(5-1)}{5!}x^4 + \frac{(7-1)}{7!}x^6 - \frac{(9-1)}{9!}x^8 + \ldots\right]$$

$$= \sqrt{\left(\frac{2}{\pi x}\right)}\left[\left(\frac{1}{2!} - \frac{1}{3!}\right)x^2 - \left(\frac{1}{4!} - \frac{1}{5!}\right)x^4 + \left(\frac{1}{6!} - \frac{1}{7!}\right)x^6 - \frac{1}{8!} - \frac{1}{9!}x^8 + \ldots\right]$$

$$= \sqrt{\left(\frac{2}{\pi x}\right)}\left[\left(\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\right) + \left(-\frac{x^2}{3!} + \frac{x^4}{5} - \frac{x^6}{7} + \ldots\right)\right]$$

{Adding and subtracting 1}

$$J_{3/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)} - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$$

$$= \sqrt{\left(\frac{2}{\pi x}\right)}\left(-\cos x + \frac{1}{x}\sin x\right) = \sqrt{\left(\frac{2}{\pi x}\right)}\left(\frac{\sin x}{x} - \cos x\right) \quad \ldots (2)$$

(2) We have from (1)

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left[1 - \frac{x^2}{2.2(n+1)} + \frac{x^4}{2.4.2^2(n+1)(n+2)}\right]$$

Multiplying the nominator and denominator by (n+1)

$$J_n(x) = \frac{x^n(n+1)}{2^n\Gamma(n+1)}\left[1 - \frac{x^2}{2.2(n+1)} + \frac{x^4}{2.4.2^2(n+1)(n+2)}\right]$$

Substituting n = -3/2

$$J_{-3/2}(x) = \frac{x^{-3/2}\left(-\frac{1}{2}\right)}{2^{-3/2}\Gamma\left(\frac{1}{2}\right)}\left[1 - \frac{x^2}{2} - \frac{x^4}{2.4.1.1} + \ldots\right] = -\sqrt{\left(\frac{2}{\pi x}\right)}.\frac{1}{x}\left[1 + \frac{x^2}{2} - \frac{x^4}{2.4} + \ldots\right]$$

$$= -\sqrt{\left(\frac{2}{\pi x}\right)}\left[\frac{1}{x}\left\{1 + x^2\left(1 - \frac{1}{2}\right) - x^4\left(\frac{1}{6} - \frac{1}{24} + \ldots\right)\right\}\right]$$

$$= \sqrt{\left(\frac{2}{\pi x}\right)}.\left[-\frac{1}{x}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots\right) - \frac{1}{x}\left(x^2 - \frac{x^4}{6} + \ldots\right)\right]$$

$$= \sqrt{\left(\frac{2}{\pi x}\right)}\left[-\frac{1}{x}\left(1 - \frac{x^2}{2!} + \frac{x^4}{24!} - \ldots\right) - \frac{1}{x}\left(x^2 - \frac{x^3}{3!} + \ldots\right)\right]$$

$$= \sqrt{\left(\frac{2}{\pi x}\right)}\left[-\frac{1}{x}\cos x - \sin x\right]$$

11.3 Self Learning Exercise I

Q.1 Write down Bessel equation of order *n*.

Q.2 Bessel's functions are also known as ....

Q.3 Write down applications of Bessel equation.

Q.4 Draw $J_0(x)$

Q.5 Draw $J_1(x)$

Q.6 Draw $J_2(x)$

11.4 Recurrence Formula For Bessel Function

These formulae are very useful in solving the questions.

(1) $$x.J'_n(x) = nJ_n(x) - J_{n+1}(x)$$

Proof. We know that Bessel's function of first kind is

$$J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

Differentiating above equation with respect to x, we get.

$$J_n^{'}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma\,(n+r+1)}(n+2r)\left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}$$

Multiplying both sides by x:

$$xJ_n^{'}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{n+2r}{r!\,\Gamma\,(n+r+1)} \cdot \frac{x}{2}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \sum_{r=0}^{\infty} n.\frac{(-1)^r}{r!\,\Gamma\,(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} + \sum_{r=0}^{\infty} \frac{(-1)^r\,2r}{r!\,\Gamma\,(n+r+1)} \cdot \frac{x}{2}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$= nJ_n(x) + x.\sum_{r=0}^{\infty} \frac{(-1)^r}{(r-1)!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$= nJ_n(x) + x.\sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

Now substituting *r-1 =s*

$$xJ_n^{'} = nJ_n(x) + x.\sum_{s=0}^{\infty} \frac{(-1)^r}{s!\,\Gamma(n+s+2)}\left(\frac{x}{2}\right)^{n+2s+1}$$

$$nJ_n(x) - x\sum_{s=0}^{\infty} \frac{(-1)^r}{s!\,\Gamma\{(n+1)+s+1\}}\left(\frac{x}{2}\right)^{(n+1)+2s}$$

Hence $xJ_n^{'}(x) = nJ_n(x) - xJ_{n+1}(x)$

(2) $xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x)$ (2)

Proof. We have

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{r!(n+r+1)!}\left(\frac{x}{2}\right)^{n+2r}$$

Differentiating above equation with respect of x, we get

$$J_n^{'}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}(n+2r)\left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}$$

Multiplying both sides by *x*; we get.

$$xJ_n^{'}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r-n)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r 2(n+r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} - n\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1} x - nJ_n(x)$$

$$= x\sum_{r=0}^{\infty} (-1)^r \frac{1}{r!(n-1-(r+1))!} \left(\frac{x}{2}\right)^{n-1+2r} - nJ_n(x)$$

$$= xJ_{n-1}(x) - nJ_n(x)$$

Hence

$$xJ_n^{'}(x) = -nJ_n(x) + xJ_{n-1}(x)$$

(3) $$2xJ_n^{'}(x) = -nJ_{n-1} + xJ_{n-1}(x) \qquad (3)$$

Proof. Recurrence relation 1 and 2 are

$$xJ_n^{'}(x) = nJ_n(x) - xJ_{n+1}(x)$$

$$xJ_n^{'}(x) = -nJ_n(x) + xJ_{n-1}(x)$$

Adding above equations, we get

$$2xJ_n^{'}(x) = x[J_{n-1}(x) - J_{n+1}]\ (x)]$$

Hence $2J_n^{'}(x) = J_{n-1}(x) - J_{n+1}(x)$

(4) $2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$

Proof. Recurrence relations 1 and 2 are

$$xJ_n^{'}(x) = nJ_n(x) - xJ_{n+1}(x) \qquad (1)$$

$$xJ_n^{'}(x) = -nJ_n(x) + xJ_{n-1}(x) \qquad (2)$$

Subtracting (2) from (1); we get

$$0 = 2nJ_n(x) - xJ_{n+1}(x) - xJ_{n-1}(x) \qquad (3)$$

$$\therefore \quad 2nJ_n = x[J_{n+1}(x) + J_{n-1}(x)] \qquad (4)$$

(5) $$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$$ (5)

Proof :

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -nx^{-n-1}J_n(x) + x^{-n}J_n^{'}(x) = x^{-n-1}\left[-nJ_n(x) + xJ_x^{'}(x)\right]$$

Using recurrence relation

$$xJ_n^{'}(x) = nJ_n(x) - xJ_{n+1}(x)$$

We have

$$\frac{d}{dx}[x^{-n}J_n(x)] = x^{-n-1}[-nJ_n(x) + nJ_n(x) - xJ_{n+1}(x)] = -x^{-n}J_{n+1}(x)$$

i.e. $$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$$

(6) $$\frac{d}{dx}[x^nJ_n(x)] = x^nJ_{n-1}(x)$$ (6)

Proof:

$$\frac{d}{dx}[x^nJ_n(x)] = nx^{n-1}J_n(x) + x^nJ_n^{'}(x) = x^{n-1}[nJ_n + xJ_n^{'}]$$

$$= x^{n-1}[nJ_n(x) + \left\{-nJ_n(x) + xJ_{n-1}(x)\right\}$$

$= x^{n-1}[xJ_{n-1}(x)]$ (using recurrence relation 2)

$$\frac{d}{dx}[x^nJ_n(x)] = x^nJ_{n-1}(x)$$

11.5 Generating Function For $J_n(x)$

Theorem: When *n* is a positive integer, $J_n(x)$ is the coefficient of z^n in the expansion of $e^{x\left(z-\frac{1}{z}\right)/2}$ in ascending and descending powers of *z* and also $J_{-n}(x)$ or $(-1)^nJ_n^{'}(x)$ is the coefficient of z^{-n} in the above expression i.e.

$$e^{x\left(z-\frac{1}{z}\right)/2} = \sum_{-\infty}^{+\infty} z^nJ_n(x)$$ (1)

Proof : We know that :

$$e^{xz/2} = 1 + \frac{xz}{2} + \frac{1}{2!}\left(\frac{xz}{2}\right)^2 + \ldots\ldots + \frac{1}{n!}\left(\frac{xz}{2}\right)^n + \ldots \tag{2}$$

$$e^{-x/2z} = 1 - \frac{x}{2z} + \frac{1}{2!}\left(\frac{x}{2z}\right)^2 + \ldots. + (-1)^n \frac{1}{n!}\left(\frac{x}{2z}\right)^n + \ldots \tag{3}$$

Hence

$$e^{x\left(z-\frac{1}{z}\right)\Big/2} = e^{xz/2}.e^{-x/2z}$$

$$= \left[1 + \frac{xz}{2} + \frac{1}{2!}\left(\frac{xz}{2z}\right)^2 + \ldots + \frac{1}{n!}\left(\frac{xz}{2}\right)^n \right.$$
$$\left. + \frac{1}{(n+1)!}\left(\frac{xz}{2}\right)^{n+1} + \frac{1}{(n+2)!}\left(\frac{xz}{2}\right)^{n+2} + \ldots\right]$$
$$\times \left[1 - \frac{x}{2z} + \frac{1}{2!}\left(\frac{x}{2z}\right)^2 + \ldots + \frac{(-1)^n}{n!}\left(\frac{x}{2x}\right)^n \right.$$
$$\left. + \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2z}\right)^{n+1} + \frac{(-1)^n}{(n+2)!}\left(\frac{x}{2z}\right)^{n+2} + \ldots\right] \tag{4}$$

Coefficient of z^n in the above product is:

$$\frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)}{(n+2)!2!}\left(\frac{x}{2}\right)^{n+4} + \ldots$$

$$= \frac{(-1)^0}{n!}\left(\frac{x}{2}\right)^n + \frac{(-1)}{1!(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^2}{2!\,(n+2)!}\left(\frac{x}{2}\right)^{n+4}$$
$$+ \ldots. + \frac{(-1)^r}{r!(n+r)!}\left(\frac{x}{2}\right)^{n+2r} + \ldots.$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!(n+r)!}\left(\frac{x}{2}\right)^{n+2r} = \sum_{r=0}^{\infty} \frac{(-1)r}{r!\,\Gamma\,(n+r+1)}\left(\frac{x}{2}\right)^{n+2} = J_n(x) \tag{5}$$

Similarly, coefficient of z^{-n} in the product (4) is

$$\frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n + \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^{n+2}}{2!(n+2)}\left(\frac{x}{2}\right)^{n+4} + \ldots$$

$$= (-1)^n \left[\frac{1}{n!} \left(\frac{x}{2} \right)^n + \frac{(-1)}{(n+1)!} \left(\frac{x}{2} \right)^{n+2} + \frac{(-1)^2}{2!(n+2)!} \left(\frac{x}{2} \right)^{n+4} + \ldots \right]$$

$$= (-1)^n \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!(n+r)!} \left(\frac{x}{2} \right)^{n+2r}$$

$$= (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)} \left(\frac{x}{2} \right)^{n+2r} = (-1)^n J_n(x) = J_{-n}(x) \qquad (6)$$

Since $\boxed{J_{-n}(x) = (-1)^n J_n(x)}$

Combining (5) and (6), we may write

Hence

$$e^{x\left(z-\frac{1}{z}\right)/2} = \sum_{n=-\infty}^{+\infty} z^n J_n(x)$$

Cor.I We have

$$e^{x\left(z-\frac{1}{z}\right)/2} = \sum_{n=-\infty}^{+\infty} z^n J_n(x)$$

$$= J_0(x) + zJ_1(x) + z^{-1}J_{-1}(x) + z^2 J_2(x) + z^{-2}J_{-2}(x) + \ldots + z^n J_n(x) + z^{-n}J_{-n}(x) + \ldots.$$

Using the property $J_{-n}(x) = (-1)^n J_n(x);$ we get

$$e^{x\left(z-\frac{1}{z}\right)/2} = J_0 x + \left(z - \frac{1}{z} \right) J_1(x) + \left(z^2 - \frac{1}{z^2} \right) J_2 x + \ldots$$

$$+ \left[z^n + (-1)^n . \frac{1}{z^n} \right] J^n(x) + \qquad (8)$$

Cor. II. The coefficient of z^0 in the expression (4) is

$$1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} \ldots\ldots \qquad (9)$$

Therefore from equation (7), we get

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \qquad (10)$$

11.6 Orthonormality of Bessel's Functions

If α and β are the roots of the equation $J_n(\mu) = 0$ then :

The condition of orthogonality of Bessel's function over the interval (0,1) with weight function x is:

$$\int_0^1 J_n(\alpha x) J_n(\beta x) x dx = 0 \quad for \ \alpha \neq \beta \tag{1}$$

with the condition of normalization is

$$\int_0^1 x[J_n(\alpha x)]^2 dx = \frac{1}{2} J^2{}_{n+1}(x) \tag{2}$$

Both the above equation represent the condition of **orthonormality** and may be written in the form of a single equation as

$$\int_0^1 J_n(\alpha x) J_n(\beta x) x dx = \frac{1}{2} J^2{}_{n+1}(\alpha) \delta_{\alpha\beta} \tag{3}$$

where $\delta_{\alpha\beta}$ is Kronecker delta symbol.

Proof. We know that $J_n(x)$ is the solution of Bessel's equation

$$\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0 \tag{4}$$

Let us consider two Bessel's function of first kind of order n

$$u = J_n(\alpha x) \text{ and } v = J_n(\beta x) \tag{5}$$

Substituting αx for x and u for y in equation (4)

$$\frac{d^2 u}{d(\alpha x)^2} + \frac{1}{\alpha x}\frac{du}{d(\alpha x)} + \left(1 - \frac{n^2}{(\alpha x)^2}\right) u = 0$$

$$\frac{1}{\alpha^2}\frac{d^2 u}{dx^2} + \frac{1}{\alpha^2 x}\frac{du}{dx} + \left(1 - \frac{n^2}{\alpha^2 x^2}\right) u = 0$$

$$x^2 \frac{d^2 u}{dx^2} + x\frac{du}{dx} + \left(\alpha^2 x^2 - n^2\right) u = 0 \tag{6}$$

Similarly, substitute βx for x and v for y in Bessel's equation (4)

$$x^2 \frac{d^2 v}{dx^2} + x \frac{dv}{dx} + (\beta^2 x^2 - n^2) v = 0 \tag{7}$$

Multiplying equation (6) by $\frac{v}{x}$ and (7) by $\frac{u}{x}$ and subtracting, we get

$$x\left[v \frac{d^2u}{dx^2} - u \frac{d^2v}{dx^2}\right] + \left(v \frac{du}{dx} - u \frac{dv}{dx}\right) + (\alpha^2 - \beta^2) xuv = 0$$

$$\frac{d}{dx}\left[x\left(v \frac{du}{dx} - u \frac{dv}{dx}\right)\right] + (\alpha^2 - \beta^2) xuv = 0$$

$$\frac{d}{dx}\left[x\left\{J_n(\beta x)\frac{d}{dx}J_n(\alpha x) - J_n(\alpha x)\frac{d}{dx}J_n(\beta x)\right\}\right] + (\alpha^2 - \beta^2) x J_n(\alpha x) J_n(\beta x) = 0 \tag{8}$$

Integrating above equation with respect to *x* between the limits 0 and 1, we get

$$\frac{d}{dx}\left[x\left\{J_n(\beta x)\frac{d}{dx}J_n(\alpha x) - J_n(\alpha x)\frac{d}{dx}J_n(\beta x)\right\}\right]_0^1 + \int_0^1 (\alpha^2 - \beta^2) x J_n(\alpha x) J_n(\beta x) = 0 \tag{9}$$

Case (i)

If α and β are different roots of $J_n(\mu) = 0$ when $J_n(\alpha)=0$, $J_n(\beta)=0$ and also $J_n(0)$ are all finite; the first term in equation (9) vanishes for both the limits. Hence equation (9) give

$$\left(\alpha^2 - \beta^2\right)\int x J_n(\alpha x) J_n(\beta x) dx = 0$$

as α≠β; we have the condition of orthogonality

$$\int_0^1 x J_n(\alpha x) J_n(\beta x) dx = 0 \tag{10}$$

Case (ii)

If α=β then equation (9) gives

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)$$

$$= \lim_{\alpha \to \beta} \frac{1}{\beta^2 - \alpha^2}\left[x\left\{J_n(\beta x)\frac{d}{dx}J_n(\alpha x) - J_n(\alpha x)\frac{d}{dx}J_n(\beta x)\right\}\right]_0^1$$

$$\to \frac{0}{0}(\beta \to \alpha) \tag{11}$$

To evaluate this we let $J_n(\alpha) = 0$; but let β approach α as a limit. Then we can use L Hospital's rule to evaluate the right hand side of (11). With $J_n(\alpha)=0$ R.H.S. of (11) is

$$Lim\ \beta \to \alpha\ \frac{\left[xJ_n(\beta x)\frac{d}{dx}J_n(\alpha x)\right]_0^1}{\beta^2-\alpha^2} \quad (12)$$

Recurrence relation (5) for $J_n(x)$ is

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$$

or

$$x^{-n}\frac{d}{dx}J_n(x) - nx^{-n-1}J_n(x) - x^{-n}J_{n+1}(x)$$

Or

$$x\frac{d}{dx}J_n(x) = nJ_n(x) - xJ_{n+1}(x) \quad (13)$$

Replacing x by (αx), we get

$$\alpha x\frac{d}{d(\alpha x)}J_n(\alpha x) = nJ_n(\alpha x) - \alpha xJ_{n+1}(\alpha x)$$

Or

$$x\frac{d}{dx}J_n(\alpha x) = nJ_n(\alpha x) - \alpha xJ_{n+1}(\alpha x) \quad (14)$$

R.H.S. of (11)

$$= \underset{\beta\to\alpha}{Lim}\frac{\left[J_n(\beta x)\left\{nJ_n(\alpha x) - \alpha xJ_{n+1}(\alpha x)\right\}\right]_0^1}{\beta^2-\alpha^2}$$

$$\underset{\beta\to\alpha}{Lim}\frac{\left[-\alpha\pi J_n(\beta x)J_{n+1}(\alpha x)\right]_0^1}{\beta^2-\alpha^2}$$

[using J_n=(0) = 0 for *n = 1, 2, 3 …]*

$$\underset{\beta\to\alpha}{Lim}\frac{-\frac{\partial}{\partial\beta}[\alpha J_n(\beta)J_{n+1}(\alpha)]}{\frac{\partial}{\partial\beta}(\beta^2-\alpha^2)} = \underset{\beta\to\alpha}{Lim}\frac{-\frac{\alpha J_n(\beta)}{\partial\beta}J_{n+1}(\alpha)}{2\beta} \quad (15)$$

Substituting β for x in recurrence relation equation (13); we get

$$\beta \frac{\partial}{\partial \beta} J_n(\beta) = nJ_n(\beta) - \beta J_{n+1}(\beta)$$

i.e.

$$\frac{\partial J_n(\beta)}{\partial \beta} = \frac{1}{\beta}[nJ_n(\beta) - \beta J_{n+1}(\beta)]$$

Substituting this in (15); we get

R.H.S. of (11)

$$= \underset{\beta \to \alpha}{Lim} \frac{\alpha}{2\beta} \frac{1}{\beta}\left[nJ_n(\beta) - \beta J_{n+1}(\beta)\right]J_{n+1}(\alpha)$$

$$= \frac{\alpha}{2\alpha^2}\left[nJ_n(\alpha) - \alpha J_{n+1}(\alpha)\right]J_{n+1}(\alpha)$$

$$= \frac{1}{2} J^2{}_{n+1}(\alpha)$$

using this relation, equation (11) becomes

$$\int_0^1 xJ_n^2(\alpha x)dx = \tfrac{1}{2} J_{n+1}^2(\alpha)$$

This required normalization condition for Bessel's functions.

Combining (10) and (16) we may express orthonormality condition of Bessel's functions as

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \tfrac{1}{2} J_{n+1}^2(\alpha)\delta_{\alpha\beta}$$

11.7 Self Learning Exercise II

Q.1 Write down the condition of orthogonality of Bessel's function

Q.2 Prove Recurrence formulae.

- $x.J'_n(x) = nJ_n(x) - J_{n+1}(x)$
- $xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x)$

11.8 Summary

The unit starts with the introduction of Bessel's equation and its solution. The solution of Bessel's equation explains many differential equations. In this chapter we also understand the properties of Bessel's equation like Recurrence Formulae, Generating Function and Orthonormality of it.

11.9 Glossary

Differential equation:

An equation that expressesa relationship between functions and their derivatives.

Recurrence relation: In mathematics, a recurrence relation is an equation that recursively defines a sequence, once one or more initial terms are given: each further term of the sequence is defined as a function of the preceding terms. The Fibonacci numbers are the archetype of a linear, homogeneous recurrence relation with constant coefficients. The logistic map is another common example.

Spherical coordinates:

(Also called polar coordinates in space, geographical coordinates.) A system of curvilinear coordinates in which the position of a point in space is designated by its distance r from the origin or pole along the radius vector, the angle φ between the radius vector and a vertically directed polar axis called the cone angle or colatitude, and the angle θ between the plane of φ and a fixed meridian plane through the polar axis, called the polar angle or longitude.

Dependent equations:

A system of equations that has an infinite number of solutions

Independent equations:

A system of equations is said to be independent if the system has exactly one solution.

11.10 Answer Self Learning Exercises

Answer Self Learning Exercise-I

Ans.1 : The differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0$$

is called the Bessel's differential equation

Ans.2 : Cylindrical and Spherical function.

Ans.3 : Bessel's functions are involved in

Theory of plane wave, Cylindrical and spherical waves, Potential theory

Elasticity, Fluid motion, Propagation of waves, Planetary motion Oscillatory motion

Ans.4 : If n=0,

$$J_0(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{r!r)!}\left(\frac{x}{2}\right)^{2r} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \ldots$$

Ans.5 : If n=1

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2.4} + \frac{x^5}{2^2.4^2.6} + \ldots$$

Ans.6 : refer section 11.2

Answer Self Learning Exercise-II

Ans.1 : The condition of orthogonality of Bessel's function over the interval (0,1) with weight function x is:

$$\int_0^1 J_n(\alpha x)J_n(\beta x)x\,dx = 0 \quad for\ \alpha \neq \beta$$

11.11 Exercise

Q.1 Show that $x/2 = J_1(x) + 3J_3(x) + 5J_5(x)....$

Q.2 Show that $x \sin x/2 = 2^2 J_2(x) - 4^2 J_4(x) + 6^2 J_6(x)....$

Q.3 Show that $x \cos x/2 = 1^2 J_1(x) - 3^2 J_3(x) + 5^2 J_5(x)....$

References and Suggested Readings

1. Mathematical Physics with Classical Mechanics by Satya Prakash, 2014.

2. Mathematical Physics by H.K. Das, 1997.

3. Special Functions and their applications by N.N.Lebedev, R. Silverman, 1973.

UNIT-12
Bessel Functions of the Second kind

Structure of the Unit

12.0 Objectives

In this unit we briefly discuss Bessel functions of the Second kind, recurrence relations, Wronskian formulas. After reading this unit students can solve a special kind of differential equation.

12.1 Introduction

When n is not an integer, $J_{-n}(x)$ *is distinct from* $J_n(x)$, hence the most general solution of Bessel's equation is

$$\boxed{y = A J_n(x) + B J_{-n}(x)}$$

where A and B are two arbitrary constants.

However, when n is integer and since n appears in the differential equation only as n^2, there is no loss of generality in taking n to be positive integer. Then $J_{-n}(x)$ is not distinct from $J_n(x)$, In this case the denominator of the first n terms of series of $J_{-n}(x)$ *for values of* $r=0,1,2,\dots,(n-1)$ will have gamma function of negative numbers. As the gamma function of negative numbers is always infinite, so

$$\frac{1}{\Gamma(-n+r+1)} = 0$$

thereby indicating that the first n terms of the series for $J_{-n}(x)$ vanish. Therefore, we shall have the terms left for r=n and onwards, i.e.

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{-n+2r}$$

Substituting s = r n, i.e. r = n+s, we get

$$J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^{n+s} \frac{1}{n+s!\,\Gamma(s+1)} \left(\frac{x}{2}\right)^{n+2s}$$

As n+s!=Γ(n+s+1) and Γ(s+1)=s!, we have

$$J_{-n}(x) = (-1)^n \sum_{s=0}^{\infty} (-1)^s \frac{1}{\Gamma(n+s+1)s!} \left(\frac{x}{2}\right)^{n+2s}$$

i.e. $$\boxed{J_{-n}(x) = (-1)^n J_n(x)}$$

Thus, in this case we no longer have two linearly independent solutions of Bessel's equation and an independent second solution must be found.

We have seen that when n is an integer, then $J_n(x)$ are not independent since they are related as $J_n(x) = (-1)^n J_n(x)$. Therefore, it is necessary to find a second solution of Bessel's equation. Let us first consider the simplest case of $n = 0$; for which the two solutions $J_n(x)$ and $J_{-n}(x)$ are identical.

12.2 Bessel functions of the Second kind

The Bessel's equation of order zero $(n = 0)$ is

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0$$

The solution of this equation is $J(x)$**.** As this is a second order equation, it must have two solutions. By *power-series solution method:* the second solution of this equation must be of the form

$$y(x)=J_0(x)\log x+\sum_{m=1}\lambda_m x^m$$

So that

$$\frac{dy}{dx}=J_0(x)\log x+\frac{1}{x}J_0(x)+\sum_{m=1} m\lambda_m x^{m-1}$$

and

$$\frac{d^2y}{dx^2}=J_0(x)\log x+\frac{2}{x}J_0(x)+\sum_{m=1} m(m-1)\,\lambda_m x^{m-2}$$

Substituting these values in Bessel's equation of order zero; we get

$$J_0(x)\log x+\frac{2}{x}J_0(x)-\frac{1}{x^2}J_0(x)+\sum_{m=1} m(m-1)\,\lambda_m x^{m-2}$$

$$+\frac{J_0}{x}log\ x+\frac{1}{x^2}\,J_0(x)+\frac{1}{x}\sum_{m=1} m\,\lambda_m x^{m-1}+J_0(x)\log\ x\ +\sum_{m=1}\lambda_m x^m=0$$

Or

$$log x\left[\frac{d^2\,J_0(x)}{dx^2}+\frac{1}{x}\frac{dJ_0(x)}{dx}+J_0(x)\right]+\frac{2}{x}J_0(x)$$

$$+\sum_{m=1} m(m-1)\,\lambda_m x^{m-2}+\frac{1}{x}\sum_{m=1} m\lambda_m x^{m-1}+\sum_{m=1}\lambda_m x^m=0$$

The first term in bracket is zero, since $J_0(x)$ is the solution of Bessel's equation of order zero; therefore, we have

$$\frac{2}{x}J_0(x)+\sum_{m=1} m(m-1)\,\lambda_m x^{m-2}+\frac{1}{x}\sum_{m=1} m\lambda_m x^{m-1}+\sum_{m=1}\lambda_m x^m=0$$

Or $2\,J_0(x)+\sum_{m=1} m(m-1)\,\lambda_m x^{m-1}+\sum_{m=1} m\lambda_m x^{m-1}+\sum_{m=1}\lambda_m x^{m+1}=0$

or $2\frac{d}{dx}\sum_{m=0}\frac{(-1)^{m}x^{2m}}{2^{2m}(m!)^{2}}+\sum_{m=1}\{m(m-1)+m\}\,\lambda_m x^{m+1}+\sum_{m=1}\lambda_m x^{m+1}=0$

or

$$2\sum_{m=0} \frac{(-1)^{m} 2m\, x^{2m-1}}{2^{2m}(m!)^{2}} + \sum_{m=1} m^2\lambda_m\, x^{m-1} + \sum_{m=1} \lambda_m x^{m+1} = 0$$

or

$$\sum_{m=1} \frac{(-1)^{m}\, x^{2m-1}}{2^{2m-2}\, m!\,(m-1)!} + \sum_{m=1} m^2\lambda_m\, x^{m-1} + \sum_{m=1} \lambda_m x^{m+1} = 0$$

Above Equation is an identity, therefore the coefficients of various powers of x must separately vanish.

Equating coefficients of x^0 to zero i.e. $\lambda_1 = 0$

Equating the coefficients of x^{2p} to zero, we get

$$(2p+1)^2\,\lambda_{2p+1} + \lambda_{2p-1} = 0$$

$$\therefore\ \lambda_{2p+1} = -\frac{1}{(2p+1)^2}\lambda_{2p-1}$$

Since $\lambda_1 = 0$; therefore if we substitute $p = 1,2,3,\ldots$ etc., we get

$$\lambda_1 = \lambda_3 = \lambda_5 = \ldots\ldots = 0$$

Again equating to zero the coefficients of x^{2p+1}, we get

$$\frac{(-1)^{p+1}}{2^{2p}(p+1)!\,p!} + (2p+2)^{2}\lambda_{2p+2} + \lambda_{2p} = 0$$

Substituting

$$p = 0,\ -\frac{1}{1(1!)} + 2^2\,\lambda_2 = 0$$

$$\therefore \lambda_2 = \frac{1}{4} = \frac{1}{2^2(1!)^2}\ 1$$

Substituting $p = 1$, $\quad \frac{1}{2^2 2!.1!} + 4^2\lambda_4 + \lambda_2 = 0$

$$\therefore 16\lambda_4 = -\frac{1}{2^2\,2!.1!} - \lambda_2 = -\frac{1}{2^2 2!.1!} - \frac{1}{4}$$

$$\therefore \lambda_4 = -\frac{1}{2^4\,(2!)^2}\ (1 - \frac{1}{2})$$

…………………

Ingeneral $A_{2m} = \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right) = \frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2}$

$$(\,where\ h_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m})$$

Hence ,the ***Bessel's function of second kind of zero order are given*** as

$$Y_0(x) = J_0(x)\log x + \sum_{m=!}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m}(m\,!)^2} x^{2m}$$

Where

$$h_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$$

Therefore the ***complete solution of Bessel's equation of zero order*** is

$$y = AJ_0(x) + BY_0(x)$$

In the case of Bessel's equation of nth order, the complete solution of Bessel's equation is

$$y = AJ_n(x) + BY_n(x)$$

where

$$\begin{aligned} Y_n(x) = {} & \frac{2}{\pi}\left(log\frac{x}{2} + \gamma\right) \\ & - \frac{1}{\pi}\sum_{m=0}^{\infty} \frac{(-1)^m (\frac{x}{2})^{n+2m}}{m\,!\,(n+m)!}\left(\sum_{k=1}^{n+m} k^{-1} + \sum_{k=1}^{m} k^{-1}\right) \\ & - \frac{1}{\pi}\sum_{m=0}^{n-1} (\frac{x}{2})^{-n+2m} \frac{(n-m-1)!}{m!} \end{aligned}$$

where γ is Euler's constant defined by

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right) = 0.57772157$$

and for $m = 0$ instead of $(\sum_{k=1}^{n+m} k^{-1} + \sum_{k=1}^{m} k^{-1})$ *we write* $\sum_{k=1}^{n} k^{-1}$

The form of Y_n is not the usual standard form. The usual standard form of Bessel function of second kind Y_n(x), also denoted by N_m(x), is obtained by taking the particular linear combination of J_n(x) and J_{-n}(x) as

$$\boxed{Y_n(x) = N_n(x) = \frac{\cos n\pi\ J_n(x) - J_{-n}(x)}{\sin n\pi}}$$

This is also known as ***Neumann's function***. For non-integral n, $Y_n(x)$ clearly satisfies Bessel's equation because it is a linear combination of known solution $J_n(x)$ and $J_{-n}(x)$. However for integral n, we have

$$J_n(x) = (-1)^n J_{-n}(x)$$

So that above equationbecomes indeterminate. Evaluating $Y_n(x)$ by L'Hospital rule for indeterminate forms, we obtain

$$Y_n(x) = \frac{\frac{\mathrm{d}}{\mathrm{dn}}[\cos n\pi\, J_n(x) - J_{\{-n\}}(x)}{\frac{\mathrm{d}}{\mathrm{dn}}\sin n\pi}$$

$$= \frac{-\sin n\pi\, J_n(x) + cosn\pi \frac{dJ_n}{dn} - \frac{dJ_{\{-n\}}(x)}{dn}}{\cos n\pi}$$

$$= \frac{1}{\pi}\left[\frac{dJ_{n(x)}}{dn} - \frac{(-1)^n dJ_n(x)}{dn}\right]$$

A series expansion using $\frac{d}{dn}(x^n) = x^n logx$ gives result $Y_n(x)$. The logarithmic dependence of $Y_n(x)$ verifies the dependence of $J_n(x)$ and $Y_n(x)$. It is seen fro $Y_n(x)$ diverges at least logarithmically. Any boundary condition that requires the solution to be finite at the origin, automatically excludes $Y_n(x)$. Conversely, in the absence of such a requirement $Y_n(x)$ must be considered. Thus we conclude that the most general solution of Bessel's equation for any value of m may be written as

$$Y = A\, J_n(x) + BY_n(x)$$

12.3 Limiting Values of $J_n(x)$ and $Y_n(x)$

A precise analysis shows

$$\lim_{x\to\infty} J_n(x) = \frac{\cos(x - \frac{\pi}{4} - \frac{n\pi}{2})}{\sqrt{\pi x}/2}$$

$$\lim_{x\to\infty} Y_n(x) = \frac{\sin(x - \frac{\pi}{4} - \frac{n\pi}{2})}{\sqrt{\pi x}/2}$$

That is, for large values of argument x, the Bessel functions behave like trigonometric functions of decreasing amplitude.

Also $\lim_{x\to 0} J_n(x) = \frac{x^n}{2^n n!}$

and $\lim_{x\to 0} Y_n(x) = \infty$

Since for small values of x, $Y_n(x)$ is of the order $1/x^n$ of $n\neq 0$ and of the order logx if n=0.

Example 1 Derive Bessel equation from Legendre differential equation.

Sol. The Legendre differential equation is the second-order ordinary differential equation

$$\boxed{\frac{d^2y}{dx^2}\left(1-x^2\right)-2x\frac{dy}{dx}+l\left(l+1\right)y=0}$$

which can be rewritten

$$\frac{d}{dx}\left[\frac{dy}{dx}(1-x^2)\right]+l(l+1)y=0$$

The above form is a special case of the so-called "***associated Legendre differential equation***" corresponding to the case m=0. ***The Legendre differential equation has regular singular points at -1, 1 and ∞.***

If the variable x is replaced by cosθ, then the Legendre differential equation becomes

$$\frac{d^2y}{d\theta^2}+\frac{cos\theta}{sin\theta}\frac{dy}{d\theta}+l(l+1)y=0$$

derived below for the associated $m\neq 0$ case.

Since the Legendre differential equation is a second-order ordinary differential equation, it has two linearly independent solutions. A solution $P_l(x)$ which is regular at finite points is called a Legendre function of the first kind, while a solution $Q_l(x)$ which is singular at ± 1 is called a Legendre function of the second kind. If l is an integer, the function of the first kind reduces to a polynomial known as the Legendre polynomial.

Differentiating m times by Leibnitz theorem

$$(1-x^2)\frac{d^{m+2}y}{dx^{m+2}} + {}^mC_1\,(-2x)\frac{d^{m+1}y}{dx^{m+1}} + {}^mC_2\,(-2)\frac{d^my}{dx^m} -$$
$$2x\frac{d^{m+1}y}{dx^{m+1}} + {}^mC_1\,(2)\frac{d^my}{dx^m} + n(n+1)\frac{d^my}{dx^m} = 0$$

i.e.

$$(1-x^2)\frac{d^{m+2}y}{dx^{m+2}} + (-2mx)\frac{d^{m+1}y}{dx^{m+1}} - \frac{2m(m-1)}{2!}\frac{d^my}{dx^m} - 2x\frac{d^{m+1}y}{dx^{m+1}} - 2m\frac{d^my}{dx^m}$$
$$+ n(n+1)\frac{d^my}{dx^m} = 0$$

Or $(1-x^2)\frac{d^{m+2}y}{dx^{m+2}} - 2(m+1)x\frac{d^{m+1}y}{dx^{m+1}} + [(n+1) - m(m+\quad 1)\frac{d^my}{dx^m} = 0$

Substituting $f = \frac{d^my}{dx^m}$

We get

$$(1-x^2)\frac{d^2f}{dx^2} - 2(m+1)x\frac{dy}{dx} + [n(n+1) - m(m+1)f = 0$$

Now using $g = f(1-x^2)^{\frac{m}{2}}$

i.e. f= $g(1-x^2)^{\frac{-m}{2}}$

$$\frac{df}{dx} = (1-x^2)^{\frac{-m}{2}}\frac{dg}{dx} + mx(1-x^2)^{\frac{m}{2}-1}\,g$$

$$\frac{d^2f}{dx^2} = (1-x^2)^{\frac{-m}{2}}\frac{d^2g}{dx^2} + 2mx(1-x^2)^{\frac{m}{2}-1}\,\frac{dg}{dx}$$
$$+ m\{1+(m+1)x^2\}(1-x^2)^{-\frac{m}{2}-2}.\,g$$

Now

$$(1-x^2)[(1-x^2)^{\frac{-m}{2}}\frac{d^2g}{dx^2} + 2mx(1-x^2)^{\frac{m}{2}-1}\,\frac{dg}{dx}$$
$$+ m\{1+(m+1)x^2\}(1-x^2)^{-\frac{m}{2}-2}.\,g]$$
$$-2(m+1)x\left\{(1-x^2)^{\frac{-m}{2}}\frac{dg}{dx} + mx(1-x^2)^{\frac{m}{2}-1}\,g\right\}$$
$$+ [n(n+1) - m(m+1)]g(1-x^2)^{\frac{-m}{2}} = 0$$

$$(1-x^2)\frac{d^2g}{dx^2} + \{2mx - 2(m+1)x\}\frac{dg}{dx}$$
$$+\left[n(n+1) - m(m+1) - \frac{2m(m+1)x^2}{1-x^2} + \frac{m\{1+(m+1)x^2\}}{1-x^2}\right]g$$
$$= 0$$

Or

$$(1-x^2)\frac{d^2g}{dx^2}+\{2x\}\frac{dg}{dx}+\left[n(n+1)+\frac{m^2}{1-x^2}\right]g=0$$

Now to change independent variable x, we put

$$z=n\sqrt{1-x^2}$$

Or $\quad 1-x^2=\frac{z^2}{n^2}$

$$\frac{dz}{dx}=-\frac{nx}{\sqrt{1-x^2}}$$

$$\frac{dg}{dx}=\frac{dg}{dz}\frac{dz}{dx}=-\frac{nx}{\sqrt{1-x^2}}\frac{dg}{dz}$$

$$\frac{d^2g}{dx^2}=\frac{d}{dx}\left(\frac{dg}{dx}\right)=\frac{d}{dx}\left(-\frac{nx}{\sqrt{1-x^2}}\frac{dg}{dz}\right)$$

$$\frac{dg}{dx}=\frac{dg}{dz}\frac{dz}{dx}=-\frac{nx}{\sqrt{1-x^2}}\frac{dg}{dz}$$

$$=-n\left[\left(1-x^2\right)^{-\frac{1}{2}}\frac{dg}{dz}-\frac{1}{2}\left(1-x^2\right)^{-\frac{3}{2}}(-2x)x\frac{dg}{dz}+x\left(1-x^2\right)^{-\frac{1}{2}}\frac{d^2g}{dz^2}\frac{dg}{dx}\right]$$

$$=-n\left[\left(1-x^2\right)^{-\frac{3}{2}}\left(1-x^2+x^2\right)\frac{dg}{dz}+\left\{x\left(1-x^2\right)^{-\frac{1}{2}}\frac{d^2g}{dz^2}\right\}\left(-\frac{nx}{\sqrt{1-x^2}}\right)\right]$$

$$=-n\left[(1-x^2)^{-\frac{3}{2}}\frac{dg}{dz}+\left\{\frac{d^2g}{dz^2}\right\}\left(-\frac{nx^2}{1-x^2}\right)\right]$$

$$=-\frac{n^2x^2}{1-x^2}\frac{d^2g}{dz^2}-\left[n(1-x^2)^{-\frac{3}{2}}\frac{dg}{dz}\right]$$

Now

$$\left(1-x^2\right)\left\{-\frac{n^2x^2}{1-x^2}\frac{d^2g}{dz^2}-\left[n\left(1-x^2\right)^{-\frac{3}{2}}\frac{dg}{dz}\right]\right\}+\{2x\}\left\{-\frac{nx}{\sqrt{1-x^2}}\frac{dg}{dz}\right\}$$

$$+\left[n\left(n+1\right)+\frac{m^2}{1-x^2}\right]g=0$$

When $n\to\infty$, above equation becomes

$$\frac{d^2g}{dz^2}+\frac{1}{z}\frac{dg}{dz}+\left(1-\frac{m^2}{z^2}\right)g=0$$

$$z^2\frac{d^2g}{dz^2}+z\frac{dg}{dz}+(z^2-m^2)g=0$$

This is required Bessel's equation.

12.4 Self Learning Exercise I

Q.1 Define Bessel equation.

Q.2 Define Legendre differential equation.

Q.3 Write down singular points of Legendre differential equation

Q.4 The behavior of Bessel function

(a) For large value of x

(b) For small value of x

Q.5 Plot Bessel function of the second kind.

12.5 Differential Equation Reducible to Bessel's Equation

In various branches of Physics there occur differential equations which can be reduced to Bessel's equation. Some of such equations are listed below with substitutions to reduce these equations in the form of Bessel's equation for convenience of solution in terms of Bessel's function.

1. $x^2y" + xy' + (\lambda^2x^2 - n^2)y = 0$ $\quad (\lambda x = z)$
2. $x^2y" + xy' + 4(x^2 - n^2)y = 0$ $\quad (x^2 = z)$
3. $4x^2y" + 4xy' + (x - n^2)y = 0$ $\quad (\sqrt{x} = z)$
4. $xy" - y' + xy = 0$ $\quad (y = xu)$
5. $y" + xy = 0$ $\quad [y = u\sqrt{x}, \frac{3}{8}x^{\frac{3}{2}} = z]$
6. $xy" + (1 + 2k)y' + xy = 0$ $\quad (y = \frac{u}{x^k})$
7. $x^2y" + (1 - 2)xy' + n^2(x^{2n} + 1 - n^2 = 0$

   $(y = x^n u, x^n = z)$

Example 2 Starting from the relation

$$e^{x\frac{\left(t-\frac{1}{t}\right)}{2}} = \sum_{n=-\infty}^{+\infty} J_n(n)t^n$$

Hence deduce that

$$J_n(x + y) = \sum_{k=-\infty}^{k=+\infty} J_k(x)J_{n-k}(y)$$

Sol. The generating function for $J_n(x)$ is

$$e^{x\frac{\left(t-\frac{1}{t}\right)}{2}} = n = \sum_{n=-\infty}^{n=+\infty} J_n(x)t^n$$

$$\therefore e^{(x+y)\frac{\left(t-\frac{1}{t}\right)}{2}} = \sum_{n=-\infty}^{+\infty} J_n(x+y)t^n$$

i.e.

$$e^{x\frac{\left(t-\frac{1}{t}\right)}{2}} e^{y\frac{\left(t-\frac{1}{t}\right)}{2}} = \sum_{n=-\infty}^{+\infty} J_n(x+y)t^n$$

i.e.

$$\sum_{k=-\infty}^{+\infty} J_k(x)t^k \sum_{s=-\infty}^{+\infty} J_s(y)t^s = \sum_{n=-\infty}^{+\infty} J_n(x+y)t^n$$

$$\sum_{k=-\infty}^{+\infty} \sum_{s=-\infty}^{+\infty} J_k(x)J_s(y)t^{k+s} = \sum_{n=-\infty}^{+\infty} J_n(x+y)t^n$$

Equating coefficients of t^n on both sides i.e. $k+s=n$

(or $s=n-k$)

we get

$$J_n(x+y) = \sum_{k=-\infty}^{+\infty} J_k(x)J_{n-k}(y)$$

This is the required result

Example 3 Starting from the generating function for $J_n(x)$, find the Jacobi series and hence show that

(i) $\cos x = j_0 - 2J_2 + 2J_4 - \cdots$ ...

and

(ii) $\sin x = 2J_1 - 2J_3 + 2J_5 - \cdots$

Sol. We know that

exp.$\left\{x\left(z-\frac{1}{z}\right)/2\right\} = J_0 + \left(z-\frac{1}{z}\right)J_1 + \left(z^2+\frac{1}{z^2}\right) + \left(z^3+\frac{1}{z^3}\right) + \cdots$

Substituting $z=e^{i\theta}$, we get

$$exp\left\{\frac{x\left(e^{i\theta}-e^{-i\theta}\right)}{2}\right\} = J_0 + \left(e^{i\theta}-e^{-i\theta}\right)J_1 + \left(e^{2i\theta}-e^{-2i\theta}\right)J_2 + \left(e^{3i\theta}-e^{-3i\theta}\right)J_3 + \cdots$$

or

$$\exp.\{xi\ \sin\theta\} = J_0 + (2i\sin\theta)J_1 + (2\cos 2\theta)J_2 + 2i\sin 3\theta\, J_3 + \cdots$$

or

$$\cos(x\sin\theta) + i\sin(x\sin\theta) = (J_0 + 2\cos 2\theta\, J_2 + 2\cos 4\theta\, J_4 + \ldots) + i(2\sin\theta J_1 + 2\sin 3\theta\, J_3 + \cdots)$$

Equating real and imaginary parts, we get

$$\cos(x\sin\theta) = J_0 + 2\cos 2\theta\, J_2 + 2\cos 4\theta\, J_4 + \cdots$$

$$\sin(x\sin\theta) = 2\sin\theta\, J_1 + 2\sin 3\theta\, J_3 + 2\sin 5\theta\, J_5 +$$

Above equations are known as Jacobi series

Substituting $\theta = \frac{\pi}{2}$ in equations (1) and (2) ; we get

$$\cos x = J_0 - 2J_2 + 2J_4 + \cdots$$

$$\sin x = 2J_1 - 2J_3 + 2J_5 - \cdots$$

12.6 Recurrence Relations

These formulae are very useful in solving the questions.

(5) $2Y_n'(x) = Y_{n-1}(x) - Y_{n+1}(x)$

Proof. Recurrence relation 1 and 2 are

$$xY_n'(x) = nY_n(x) - xY_{n+1}(x)$$

$$xY_n'(x) = -nY_n(x) + xY_{n-1}(x)$$

Adding above equations, we get

$$2xY_n'(x) = x[Y_{n-1}(x) - Y_{n+1}(x)]$$

Hence $2Y_n'(x) = Y_{n-1}(x) - Y_{n+1}(x)$

(6) $2nY_n(x) = x[Y_{n-1}(x) + Y_{n+1}(x)]$

Proof. Recurrence relations are

$$xY_n'(x) = nY_n(x) - xY_{n+1}(x) \qquad \ldots (1)$$

$$xY_n'(x) = -nY_n(x) + xY_{n-1}(x) \qquad \ldots (2)$$

Subtracting (2) from (1); we get

$$0 = 2nY_n(x) - xY_{n+1}(x) - xY_{n-1}(x) \qquad \ldots (3)$$

$$\therefore \quad 2nY_n = x[Y_{n+1}(x) + Y_{n-1}(x)] \qquad \ldots (4)$$

12.7 Wronskian Formulas

If $y_1(x)$ and $y_2(x)$ are solutions of a self-adjoint ODE of the form

$$p(x)y'' + q(x)y' + r(x)y = 0,$$

for which q(x) = p'(x), we can use Abel's Theorem to obtain the Wronskian

$$W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x) = Ce^{-\int\frac{p'(x)}{p(x)dx}} = \frac{C}{p(x)}$$

where C is a constant.

By writing the Bessel equation in the form

$$xy'' + y' + (x - \upsilon^2/x)y = 0;$$

so that it is self-adjoint, we obtain, for non-integer υ,

$$J_\upsilon(x)J_{-\upsilon}'(x) - J'_\upsilon(x)J_{-\upsilon}(x) = \frac{A_\upsilon}{x}$$

where A_υ is a constant that depends only on υ, not x.

This constant can be determined by considering any convenient value of x, such as x = 0. Examining the leading terms of the series representations of the Bessel functions, which yield approximations for small x

$$J_\upsilon(x) \approx \frac{1}{\Gamma(\upsilon + 1)}\left(\frac{x}{2}\right)^{\upsilon}$$

$$J'_\upsilon(x) \approx \frac{\upsilon}{2\Gamma(\upsilon + 1)}\left(\frac{x}{2}\right)^{\upsilon-1}$$

$$J_{-\upsilon}(x) \approx \frac{1}{\Gamma(1 - \upsilon)}\left(\frac{x}{2}\right)^{-\upsilon}$$

$$J'_{-\upsilon}(x) \approx \frac{-\upsilon}{2\Gamma(1 - \upsilon)}\left(\frac{x}{2}\right)^{-\upsilon-1}$$

We obtain

$$W(J_\upsilon, J_{-\upsilon})(x) = -\frac{2\upsilon}{x\Gamma(1+\upsilon)(1-\upsilon)} = -\frac{2sin\upsilon\pi}{\pi x}$$

We conclude that

$$A_\upsilon = -\frac{2sin\upsilon\pi}{\pi x}$$

When υ is an integer, we obtain $A_\upsilon = 0$, and therefore the Wronskian is zero. This is expected, since J_n and J_{-n} are linearly dependent when n is an integer.

12.8 Spherical Bessel's Functions

In some physical problems (e.g., Helmholtz equation is spherical polar coordinates when separated, the radial equation) the following equation is encountered

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + [k^2r^2 - n(n+1)]R = 0 \quad ...(1)$$

This equation is not Bessel's equation; but if we substitute

$$R(kr) = \frac{Z(kr)}{(kr)^{1/2}} \quad ...(2)$$

Then equation (1) reduces to Bessel's equation of order $\left(n+\frac{1}{2}\right)$ $i.e.$

$$\boxed{r^2\frac{d^2Z}{dr^2} + r\frac{dZ}{dr} + \left[k^2r^2 - \left(n+\frac{1}{2}\right)^2\right]Z = 0} \quad ...(3a)$$

Or

$$\frac{d^2Z}{dr^2} + \frac{1}{r}\frac{dZ}{dr} + \left[k^2 - \frac{\left(n+\frac{1}{2}\right)^2}{r^2}\right]Z = 0 \quad ...(3b)$$

The solution of this equation is $Z(kr)$. The solution of equation (1), i.e.

$$R(kr) = \frac{Z_{n+1/2}(kr)}{(kr)^{1/2}} \quad ...(4)$$

is called ***Spherical Bessel function*** and because of importance of spherical coordinates this combination often occurs. The spherical Bessel functions are defined in terms of Bessel functions of different kinds as follows:

$$j_n(x) = J_{n+\left(\frac{1}{2}\right)}\sqrt{\frac{\pi}{2x}} \quad (5a)$$

$$y_n(x) = \sqrt{\frac{2}{2\pi}}\,Y_{n+\left(\frac{1}{2}\right)}(x) = (-1)^{n+1}\sqrt{\frac{\pi}{2x}}J_{n-\left(\frac{1}{2}\right)}(x) \quad (5b)$$

The spherical Bessel functions $j_n(x)$ of first kind may be expressed in series form by using series expansion of $J_n(x)$, $i.e.$

i.e.

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

Replacing n by $n + \frac{1}{2}$

$$J_{n+\left(\frac{1}{2}\right)}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r+\left(\frac{1}{2}\right)} \quad \ldots(6)$$

Using Legendre duplication formula

$$\Gamma(z+1)\Gamma\left(z+\frac{3}{2}\right) = 2^{-2z-1}\pi_{\frac{1}{2}}(2z+2) \quad \ldots(7)$$

$$\Gamma\left(n+r+\frac{3}{2}\right) = \frac{2^{-2n-2r-1}\sqrt{\pi}\Gamma(2n+2r+2)}{\Gamma(n+r+1)}$$

$$= \frac{2^{-2n-2r-1}\sqrt{\pi}(2n+2r+1)!}{(n+r)!}$$

Substituting this value in equation (6), we get

$$J_{n+\left(\frac{1}{2}\right)}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\, 2^{2n+2r+1}(n+r)!}{\sqrt{\pi}\,(2n+2r+1)!r!} \left(\frac{x}{2}\right)^{n+2r+\left(\frac{1}{2}\right)} \quad \ldots(8)$$

Substituting this value in (5a); we get

$$J_n(x) = \sqrt{\frac{\pi}{2x}} \sum_{r=0}^{\infty} \frac{(-1)^r\, 2^{2n+2r+1}(n+r)!}{\sqrt{\pi}\,(2n+2r+1)!\,r!} \left(\frac{x}{2}\right)^{n+2r+\left(\frac{1}{2}\right)}$$

$$= 2^n x^n \sum_{r=0}^{\infty} \frac{(-1)^r\,(n+r)!}{r!\Gamma(2n+2r+1)} x^{2r} \quad \ldots(9)$$

Which is series expansion for $J_n(x)$

Now $Y_{n+\left(\frac{1}{2}\right)} = (-1)^{n+1} J_{-n-(1/2)}(x)$ and from definition of $J_n(x)$ (substituting $-n-\frac{1}{2}$ for n); we find that

$$J_{-n-(1/2)} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(r-n-\frac{1}{2})!} \left(\frac{x}{2}\right)^{2r-n-(1/2)} \quad (10)$$

This yields

$$y_n(x) = (-1)^{n+1} \frac{2n\sqrt{\pi}}{x^{n+1}} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(r-n-\frac{1}{2})!} \left(\frac{x}{2}\right)^{2r}$$

Using again Legendre duplication formula, we get

$$y_n(x) = \frac{(-1)^{n+1}}{2^n x^{n+1}} \sum_{r=0}^{\infty} \frac{(-1)^r\,(r-n)!}{r!(2r-2n)!} x^{2r} \quad (11)$$

However, for positive integral values of n, it is awkward to use equation (11) because of the factorials in both numerator and denominator.

These spherical Bessel's functions are closely related to trigonometric functions as may be seen by considering the special case for n=0. Then we have from (9);

$$j_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\, r\,!}{r!(2r+1)!} x^{2r} = \frac{1}{x} \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r+1}}{(2r+1)!} = \frac{\sin x}{x} \quad ..(12)$$

And from(11)

$$y_0(x) = \frac{(-1)}{x} \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r\,)!} x^{2r} = -\frac{\cos x}{x} \quad \ldots(13)$$

12.9 Self Learning Exercise II

Q.1 Prove that

$$J_n(x+y) = \sum_{k=-\infty}^{+\infty} J_k(x) J_{n-k}(y)$$

Q.2 Using $J_n(x)$, find the Jacobi series.

Q.3 Define the Spherical Bessel function of first kind of order n.

Q.4 Define Spherical Bessel function of second kind.

where n is non-integer.

Q.5 Prove Recurrence formulae.

$x.Y'_n(x) = nY_n(x) - Y_{n+1}(x)$

$x Y_n^{'}(x) = -n Y_n(x) + x Y_{n-1}(x)$

12.10 Summary

The unit starts with the introduction of Bessel functions of the Second kind and its solution. The solution of Bessel's equation explains many differential equations. In this chapter we also understand the properties of Bessel's equation like Recurrence Formulae, Wronskian formulas, Spherical Bessel's Functions.

12.11 Glossary

Differential equation

An equation that expresses a relationship between functions and their derivatives.

Recurrence relation:

In mathematics, a recurrence relation is an equation that recursively defines a sequence, once one or more initial terms are given: each further term of the sequence is defined as a function of the preceding terms.

Legendre polynomials:

Legendre polynomials $P_n(x)$ are polynomial of degree n defined as:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n$$

12.12 Answers to Self Learning Exercises

Answers to Self Learning Exercise-I

Ans.1 : The ordinary differential equation of the form

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

Is called Bessel equation of order n, n being a nonnegative real number.

Ans.2 : The Legendre differential equation is the second-order ordinary differential equation

$$\frac{d^2y}{dx^2}(1-x^2) - 2x\frac{dy}{dx} + l(l+1)y = 0$$

which can be rewritten

$$\frac{d}{dx}\left[\frac{dy}{dx}(1-x^2)\right] + l(l+1)y = 0$$

The above form is a special case of the so-called "associated Legendre differential equation" corresponding to the case m=0.

Ans.3 : The Legendre differential equation has regular singular points at -1, 1 and ∞.

Ans.4 : $\lim_{x\to\infty} J_n(x) = \frac{\cos(x-\frac{\pi}{4}-\frac{n\pi}{2})}{\sqrt{\pi x}/2}$

$$\lim_{x\to\infty} Y_n(x) = \frac{\sin(x-\frac{\pi}{4}-\frac{n\pi}{2})}{\sqrt{\pi x}/2}$$

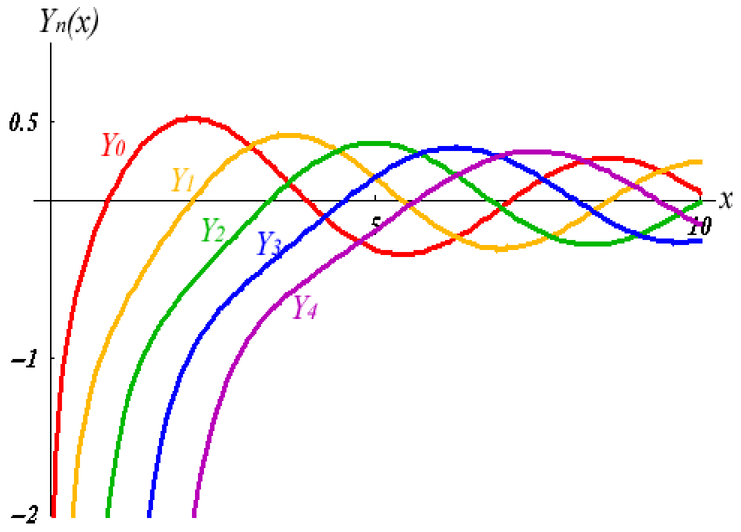
a. for large values of argument x, the Bessel functions behave like trigonometric functions of decreasing amplitude.

Also $\lim_{x \to 0} J_n(x) = \frac{x^n}{2^n n!}$

And $\lim_{x \to 0} Y_n(x) = \infty$

b. for small values of x, $Y_n(x)$ is of the order $1/x^n$ of $n \neq 0$ and of the order logx if n=0.

Ans.5 :



Answers to Self Learning Exercise-II

Ans.1 : See example

Ans.2 : See Example

Ans.3 : $j_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \, r\,!}{r!(2r+1)!} x^{2r} = \frac{1}{x} \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r+1}}{(2r+1)!} = \frac{\sin x}{x}$

Ans.4 : $y_0(x) = \frac{(-1)}{x} \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r\,)!} x^{2r} = -\frac{\cos x}{x}$

Ans.5 : See section Recurrence formula

12.13 Exercise

Q.1 Use the definition of Neumann function $Y_\upsilon(x)$ to show that $Y_\upsilon(0)$ is unbounded for any real υ ($\upsilon \neq 0$).

Q.2 Use the series representation of $Y_n(x)$ to derive the principal asymptotic form for large order, where x is fixed and $n \to \infty$ for Neumann function of integer order n.

$$Y_n(x) \sim -\sqrt{(2}/\pi n)\ (ex/2n)^{-n}$$

Q.3 Use the series representation of $Y_0(x)$ to Find the asymptotic expression for $Y_0(x)$ when $x \rightarrow 0$.

References and Suggested Readings

1. Mathematical Physics with Classical Mechanics by Satya Prakash, 2014.

2. Mathematical Physics by H.K. Das, 1997.

3. Special Functions and their applications by N.N. Lebedev, R. Silverman, 1973.

UNIT-13
Hankel functions, Modified Bessel functions

Structure of the Unit

13.0 Objectives

In this unit we briefly discuss Hankel functions, modified Bessel functions. After reading this unit students can understand third kind of Bessel function.

13.1 Introduction

The standard form of Bessel function of first kind J_n(x),

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{r!(n+r+1)!} \left(\frac{x}{2}\right)^{n+2r}$$

Bessel function of second kind Y_n(x), also denoted by N_n(x), is obtained by taking the particular linear combination of J_n(x) and J_{-n}(x) as

$$Y_n(x) = N_n(x) = \frac{\cos n\pi \ J_n(x) - J_{-n}(x)}{\sin n\pi}$$

This is also known as Neumann's function.

13.2 Bessel's Functions of Third kind : Hankel Functions

In some physical problems there arise complex combinations of Bessel's functions of the first and second kinds so frequently that it has been found convenient to define the new functions known as ***Bessel's functions of third kind or Hankel functions***. The Hankel functions $H_n^{(1)}(x)$ and $H_n^{(2)}(x)$ are defined as

$$\boxed{H_n^{(1)}(x) = J_n(x) + iY_n(x)} \qquad (1)$$

$$\boxed{H_n^{(2)}(x) = J_n(x) - iY_n(x)} \qquad (2)$$

Series expansion of Hankel functions may be obtained by combining definitions of $J_n(x)$ and $Y_n(x)$. Since Hankel functions are linear combinations, with constant coefficients, of $J_n(x)$ and $Y_n(x)$; they satisfy the same recurrence relations.

For large x, i.e. , fixed x>>n

$$H_n^{(1)}(x) = \sqrt{(2}/\pi x) \ e^{i\left(x-\frac{\pi}{4}-\frac{n\pi}{2}\right)}$$

$$H_n^{(2)}(x) = \sqrt{(2}/\pi x) \ e^{-i\left(x-\frac{\pi}{4}-\frac{n\pi}{2}\right)}$$

13.3 Bessel's Integral

Example 1 Show that

(i) $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta$

(ii) $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\theta)\, d\theta$

and hence deduce that

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots \sum_{r=0} \frac{(-1)^r x^{2r}}{(2^r r!)^2}$$

Sol. From Jacobi series we have

$$\cos(x \sin\theta) = J_0 + 2J_2 \cos 2\theta + \cdots + 2J_{2m} \cos 2m\theta + \cdots \quad (1)$$

$$\sin(x \sin\theta) = 2\sin\theta\, J_1 + 2\sin 3\theta\, J_3 + 2\sin 5\theta\, J_5 + \ldots + 2J_{2m+1} \sin(2m + 1)\theta + \cdots \quad (2)$$

(i) Multiplying both sides of (1) by cos(2mθ)and integrating between the limits 0 to μ

$$\int_0^\pi \cos(x\sin\theta)\cos 2m\theta = J_0 \int_0^\pi \cos 2m\theta d\theta + 2J_2 \int_0^\pi \cos 2\theta \cos 2m\theta d\theta + \cdots$$

$$+2J_{2m} \int_0^\pi \cos^2 2m\theta d\theta + \cdots$$

$$= 0 + 0 + \cdots J_{2m} \int_0^\pi (1 + \cos 4\mathrm{m})\, \mathrm{d} + \cdots$$

$$= J_{2m}.\pi$$

$$or\ \int_0^\pi \cos(x\sin\theta)\cos 2m\theta d\theta = \pi.J_{2m} \quad (3)$$

Again, multiplying (1) by $cos(2m + 1)\,\theta$ and integrating between limits 0 and π; we get

$$\int_0^\pi \cos(x\sin\theta)\cos(2m+1)\theta\, d\theta = 0 \quad (4)$$

Now multiplying (2) by sin (2m+1) θ and integrating between the limits 0 and ; we get

$$\int_0^\pi \sin\ (x\sin\theta)\sin\ (2m+1)\theta\, d\theta$$

$$= 0 + 0 \ldots + 2J_{2m+1} \int_0^\pi \sin^2(2m+1)\theta\, d\theta + 0 + \cdots$$

or

$$\int_0^\pi \sin(x\sin\theta)\sin(2m+1)\theta\, d\theta = J_{2m+1} \int_0^\pi \{1 - \cos(2m+1)\,\theta\}\, d\theta$$

$$= \pi J_{2m+1} \quad (5)$$

Again multiplying (2) by sin $2m\theta$ and integrating between the limits 0 and ; we get

$$\int_0^\pi \sin(x\sin\theta)\sin 2m\theta\, d\theta = 0 \quad (6)$$

Now adding (3) and (6); we get

$$\int_0^\pi \cos(2m\theta - x\sin\theta)\ d\theta = \pi J_{2m} \quad (7)$$

adding (4) and (5); we get

$\int_0^{\pi} \cos\{(2m+1)\theta - x\sin\theta\}\, d\theta = \pi J_{2m+1}$ (8)

Equation (7) holds for even integers (2m) and (8) holds for odd integers (2m+1); therefore combining (7) and (8); we have for all integral values of n

$$\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta = \pi J_n$$

i.e.
$$J_n = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta) d\theta \quad (9)$$

(ii) Substituting $\theta = \frac{\pi}{2} + \emptyset$ in equation (1); we get

$$\cos(x\cos\emptyset) = J_0 - 2J_2\cos 2\emptyset + 2J_4\cos 4\emptyset - \cdots$$

$$\therefore \int_0^{\pi} \cos(x\cos\emptyset)\, d\emptyset = J_0\int_0^{\pi} d\emptyset - 2J_2\int_0^{\pi} \cos 2\emptyset\, d\emptyset + \cdots$$

$$= J_{0.\pi} - 0 + 0 \ldots$$

$\therefore J_0 = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\emptyset)\, d\emptyset$ (10)

Deduction: Equation (10) may be expressed as

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi}\left[1 - \frac{x^2\cos^2\emptyset}{2\,!} + \frac{x^4\cos^4\emptyset}{4\,!} - \frac{x^6\cos^6\theta}{6\,!} + \cdots\right] d\emptyset$$

Using the definite integral

$$\int_0^{\pi} \cos^{2r}\emptyset\, d\emptyset = \frac{1.3.5..(2r-1)}{2.4.6\ldots 2r}.\pi;$$

We get

$$J_0(x) = \frac{1}{\pi}\left[\pi\frac{x^2}{2\,!}.\frac{\pi}{2} + \frac{x^4}{4\,!}.\frac{1.3}{2.4}\pi - \frac{x^6}{6\,!}.\frac{1.3.5}{2.4.6}\pi + \cdots\right]$$

$$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots$$

$= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(2)^2\,!} - \frac{x^6}{2^6(3\,!)^2} + \cdots = \sum_{r=0}\frac{(-1)^r x^{2r}}{[2^r r\,!]^2}$ (11)

Example 2 Prove that

$$\int_0^{\pi} e^{-ax}J_0(bx)dx = -\frac{1}{\sqrt{(a^2+b^2)}}; a, b \geq 0$$

Sol. From Jacobi series we have

$\cos(x\sin\theta) = J_0 + 2J_2\cos 2\theta + \cdots + 2J_{2m\cos 2m\theta + \cdots}$ (1)

Integrating above equation with respect to θ between limits 0 to π , we get

$$\int_0^{\pi} \cos(x\, sin\theta)\, d\theta = \int_0^{\pi} J_0\, d\theta + \int_0^{\pi} 2J_2 \cos 2\theta\, d\theta + \cdots + \int_0^{\pi} 2J_{2m} \cos 2m\theta\, d\theta + \cdots$$

`

$$= J_0.\pi + 0 + 0 + \cdots + 0 + \cdots$$

$$\therefore J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin\theta)\, d\theta$$

$$I = \int_0 e^{-ax}\, J_0(bx)dx = \int_0 e^{-ax}\left[\frac{1}{\pi}\int_0^{\pi} cos\,(bx\, sin\,\theta)d\theta\right]dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\int_0 e^{-ax}\cos(bx\, sin\theta)\, dx\right]d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\int_0 e^{-ax}.\frac{e^{ibx\sin\theta +} e^{-ibx\, sin\theta}}{2}dx\right]d\theta$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\int_0 \left\{e^{-(a+ib\sin\theta)x} + e^{-(a-ib\sin\theta x}\right\}dx\right]d\theta$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\frac{e^{-(a+ib\sin\theta)x}}{-(a+ib\sin\theta} + \frac{e^{-(a-ib\sin\theta)x}}{-(a-ib\sin\theta}\right]_0 d\theta$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\frac{1}{a+ib\sin\theta} + \frac{1}{a-ib\sin\theta}\right]d\theta$$

$$= \frac{1}{2\pi}\int_0^{\pi}\frac{2a}{a^2 + b^2\, sin^2\,\theta}d\theta$$

$$= \frac{2a}{2\pi}2\int_0^{\pi/2}\frac{1}{a^2 + b^2\, sin^2\,\theta}d\theta$$

$$= \frac{2a}{\pi}\int_0^{\pi/2}\frac{cosec^2\theta}{a^2\, cosec^2\,\theta + b^2}d\theta$$

$$= \frac{2a}{\pi}\int_0^{\pi/2}\frac{cosec^2\theta\, d\theta}{(a^2+b^2) + a^2 cot^2\theta}$$

$$= \frac{2a}{\pi}\left[\frac{1}{a\sqrt{(a^2+b^2)}}cot^{-1}\frac{a\cot\theta}{\sqrt{(a^2+b^2)}}\right]_0^{\pi/2}\quad for\ a,b \geq 0$$

$$= \frac{2}{\pi\sqrt{(a^2+b^2)}}\left[cot^{-1}0 - cot^{-1}\right]$$

$$= \frac{2}{\pi\sqrt{(a^2+b^2\,)}}\pi/2$$

$$= \frac{1}{\sqrt{(a^2+b^2)}}$$

Example 3

If n > -1, show that

$$\int_0^x x^{-n} J_{n+1}(x)dx = \frac{1}{2^n n\,!} - x^{-n} J_n(x)$$

Sol. We have

$$\frac{d}{dx}[x^{-n} J_n(x)] = x^n J_{n+1}(x)$$

Integrating this between limits 0 and x, we get

$$\int_0^x x^{-n} J_{n+1}(x)dx = -[x^{-n} J_n(x)]_0^x$$

$$= x^n J_n(x) + \lim_{x\to 0} x^{-n} J_{n(x)}$$

But $\lim_{x\to 0} x^{-n} J_n(x) = \lim_{x\to 0} \frac{J_n(x)}{x^n}\left(\frac{0}{0}\ form\right)$

$$\lim_{x\to 0} \frac{\frac{d^n}{dx^n} J_{n(x)}}{n\,!} = \frac{1}{n\,!}.2^n \frac{n\,!}{2n(n+1)}$$

$$= \frac{n\,!}{2^n (n\,!)^2} = \frac{1}{2^n n\,!}$$

∴ Equation (2) gives

$$\int_0^x x^{-n} J_{n+1}(x)\,dx = -x^{-n} J_n(x) + \frac{1}{2^n\,n\,!}$$

13.4 Self Learning Exercise I

Q.1 Show that

$$\int_0^x x^{-n} J_{n-1}(x)dx = x^n J_{n-1}(x)$$

Q.2 Prove that

$$J_n(x) = (-2)^n x^n \frac{d^n}{d(x^2)^n} J_0(x)$$

Q.3 Prove that

$$J_n(x) = \frac{1}{\sqrt{\pi}\left|\left(n+\frac{1}{2}\right)\right.}\left(\frac{x}{2}\right)^n \int_0^x cos\,(x\,sin\emptyset)\,cos^{2n}\emptyset\,d\emptyset$$

Q.4 Write down the standard form of Bessel function of first and second kind.

Q.5 Define Hankel function.

13.5 Spherical Bessel's Functions

The spherical Bessel functions are defined in terms of Bessel functions of different kinds as follows:

$$j_n(x) = J_{n+\left(\frac{1}{2}\right)}\sqrt{\frac{\pi}{2x}}$$

$$y_n(x) = \sqrt{\frac{2}{2\pi}}\, Y_{n+\left(\frac{1}{2}\right)}(x) = (-1)^{n+1}\sqrt{\frac{\pi}{2x}} J_{n-\left(\frac{1}{2}\right)}(x)$$

$$\boxed{h_n^{(1)}(x) = \sqrt{\frac{\pi}{2x}} H^{(1)}_{n+\left(\frac{1}{2}\right)}(x) = j_n(x) + iy_n(x)}$$

$$\boxed{h_n^{(2)}(x) = \sqrt{\frac{\pi}{2x}} H^{(2)}_{n+\left(\frac{1}{2}\right)}(x) = j_n(x) - iy_n(x)}$$

Spherical Bessel's functions are closely related to trigonometric functions as may be seen by considering the special case for n=0.

$$j_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\ r\ !}{r!\,(2r+1)!} x^{2r} = \frac{1}{x}\sum_{r=0}^{\infty}(-1)^r \frac{x^{2r+1}}{(2r+1)!} = \frac{\sin x}{x}$$

And

$$y_0(x) = \frac{(-1)}{x}\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r\,)!}x^{2r} = -\frac{\cos x}{x}$$

Spherical Hankel functions

$$h_0{}^{(1)}(x) = \frac{1}{x}(\sin x + i\cos x) = \frac{1}{x}e^{ix}$$

$$h_0{}^{(2)}(x) = \frac{1}{x}(\sin x + i\cos x) = \frac{1}{x}e^{-ix}$$

Recurrence Relations

The recurrence relations for spherical Bessel functions may be obtained from known recurrence relations for $j_0(x)$. The recurrence relation for $j_n(x)$ is

$$j_{n-1+}j_{n+1} = \frac{2n}{x} j_n(x)$$

Replacing n by $n + \frac{1}{2}$; we get

$$J_{n-(1/2)} + J_{n-(3/2)} = \frac{(2n+1)}{x} J_{n+(1/2)}(x)$$

Multiplying both sides by $\sqrt{\left(\frac{\pi}{2x}\right)}$; we get

$$\sqrt{\left(\frac{\pi}{2x}\right)}\, J_{n-(1/2)} + \sqrt{\left(\frac{\pi}{2x}\right)}\, J_{n-(3/2)} = \frac{(2n+1)}{x} \sqrt{\left(\frac{\pi}{2x}\right)} .\, J_{n+(1/2)}(x)$$

and therefore the recurrence relation for spherical function takes the form

$$J_{n-1}(x) + J_{n+1}(x) = \frac{(2n+1)}{x} J_n(x)$$

Similarly, in recurrence relation III for $J_n(x)$ and multiplying with $\sqrt{\left(\frac{\pi}{2x}\right)}$; we get

$$nj_{n-1}(x) - (n+1)j_{n+1}(x) = (2n+1)j_n'(x)$$

Similarly, from recurrence relation; we get

$$\frac{d}{dx}\left[x^{n+2} j_n(x)\right] = x^{n+1} j_{n-1}(x)$$

$$\frac{d}{dx}\left[x^{-n} j_n(x)\right] = -x^{-n} j_{n+1}(x)$$

Similar recurrence relations hold for $y_n(x),\ h_n{}^{(1)}(x)$ and $h_n{}^{(2)}(x)$

Substituting n=0 in above equation; we get

$$\frac{d}{dx}\left[j_0(x)\right] = -j_1(x)$$

$$\therefore\ j_1(x) = -\frac{d}{dx}\left[j_0(x)\right]$$

$$= -\frac{d}{dx}\left[\frac{\sin x}{x}\right] = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

Substituting n=1; we get

$$\frac{d}{dx}\left[\frac{j_1(x)}{x}\right] = -j_2(x)$$

$$\therefore\ j_2(x) = -\frac{d}{dx}\left[\frac{\sin x}{x^3} - \frac{\cos x}{x^2}\right]$$

$$= \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x$$

Similarly, if we use recurrence relation for $y_n(x)$: we get

$$y_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$$

$$y_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x$$

By mathematical induction, we have in general

$$j_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right)$$

$$j_n(x) = -(-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right)$$

Obviously, the spherical Bessel's functions $j_n(x)$ and $y_2(x)$ may always be expressed as $sin\, x$ and $cos\, x$ with coefficients that are polynomials involving negative powers of x. For spherical Hankel functions

$$h_n{}^{(1)}(x) = i(-1)^n\, x^n \left(\frac{1}{x}.\frac{d}{dx}\right)^n \left(\frac{e^{ix}}{x}\right)$$

$$h_n{}^{(2)}(x) = i(-1)^n\, x^n \left(\frac{1}{x}.\frac{d}{dx}\right)^n \left(\frac{e^{-ix}}{x}\right)$$

13.6 Modified Bessel's Functions

a) First Kind: $I_n(x)$ in the solution to the modified Bessel's equation is referred to as a modified Bessel function of the first kind.

b) Second Kind $K_n(x)$ in the solution to the modified Bessel's equation is referred to as a modified Bessel function of the second kind or sometimes the Weber function or the Neumann function.

In physical problems the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(-1 - \frac{n^2}{x^2}\right)y = 0$$

…(1)

Occurs frequently. This is not quite Bessel's equation; but it may be put in the form of Bessel's equation by the substitution of $x = -it$, so that $dx = i\, dt, i.e.,$

$$\frac{dt}{dx} = \frac{1}{i} = i$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = i\frac{dy}{dt}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(i\frac{dy}{dt}\right).i = \frac{d^2y}{dt^2}$$

Substituting these values (1); we get

$$-\frac{d^2y}{dt^2}+\frac{1}{(-it)}i\frac{dy}{dt}\left[-1-\frac{n^2}{(-it)^2}\right]y=0$$

or

$$\frac{d^2y}{dt^2}+\frac{1}{t}\frac{dy}{dt}+\left(1-\frac{n^2}{t^2}\right)y=0$$

(2)

This is Bessel's equation whose solution is

$$y(t)=J_n(t)J_n(ix)\left[since\ t=-\tfrac{1}{i}x=ix\right] \quad (3)$$

The equation (1) which has been reduced to Bessel equation (2) is called modified Bessel equation.

We have

$$J_n(t)=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,(n+r)!}\left(\frac{t}{2}\right)^{n+2r}$$

$$\therefore \quad J_n(ix)=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,(n+r)!}\left(\frac{ix}{2}\right)^{n+1r}$$

$$=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,(n+r)!}(i)^{n+2r}\left(\frac{x}{2}\right)^{n+2r}$$

$$=i^n\sum_{r=1}^{\infty}\frac{(-1)^r(i^2)}{r!\,(n+r)!}\left(\frac{x}{r}\right)^{n+2r}$$

$$=i^n\sum_{r=1}^{\infty}\frac{(-1)^{2r}}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r}$$

(Since $i^2=-1$)

$$\therefore \quad i^{-n}J_n(ix)=\sum_{r=0}^{\infty}\frac{1}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r}$$

(Since $(-1)^{2r}=1$)

The function $i^{-n}J_n(ix)$ is denoted by $I_n(x)$, i.e.,

$$I_n(x)=i^{-n}J_n(ix)=\sum_{r=0}^{\infty}\frac{1}{r!(n+r)!}\left(\frac{x}{2}\right)^{n+2r} \quad (4)$$

Choosing the normalization so that $I_n(x)$ is defined by equation (4); then $I_n(x)$ is a real function and being the solution of modified Bessel equation (1) is called the modified Bessel function of first kind of order n. Often $I_n(x)$ is written in the form

$$I_n(x) = e^{-\frac{in\pi}{2}} J_n\left(xe^{\frac{i\pi}{2}}\right) \quad (5)$$

Another fundamental solution to equation (1) is known as modified Bessel function of second kind and is defined as

$$K_n(x) = \frac{\pi/2}{\sin n\pi}\left[I_{-n}(x) - I_n(x)\right] \quad (6)$$

where n is non-integer.

The general solution of equation (1) is then

$$y = AI_n(x) + BK_n(x) \quad (7)$$

Where n is non- integer. A and B are arbitrary constants.

In contrast to the Bessel function $J_n(x)$ and $Y_n(x)$, the functions $I_n(x)$ and $K_n(x)$ are not of oscillating type; but they behave like exponential functions.

For large value of x, we have

$$I_0(x) = \frac{e^x}{\sqrt{2\pi x}} \; and \; K_0(x) = \sqrt{\left(\frac{\pi}{2x}\right)^{e-x}}$$

And for small values of x; we have

$I_0(x) \approx 1$ and $K_0(x) \simeq -\log\frac{x}{2}$

13.7 Self Learning Exercise II

Q.1 Define the modified Bessel function of first kind of order n.

Q.2 Define modified Bessel function of second kind

$$K_n(x) = \frac{\pi/2}{\sin n\pi}[I_{-n}(x) - I_n(x)]$$

where n is non-integer.

Q.3 Plot $J_{\frac{1}{3}}, J_{\frac{4}{3}}, J_{\frac{7}{3}}$, 0 < x < 20.

Q.4 Define Spherical Hankel functions

Q.5 Write down recurrence relation for $y_n(x), J_n(x), h_n(x)$

13.8 Summary

The unit starts with the introduction of Bessel functions of the third kind and its solution. The solution of Bessel's equation explains many differential equations. In this chapter we also understand the Bessel's Integral, Spherical Bessel's Functions and Modified Bessel's Functions.

13.9 Glossary

Differential equation:

An equation that expresses a relationship between functions and their derivatives.

Recurrence relation:

In mathematics, a recurrence relation is an equation that recursively defines a sequence, once one or more initial terms are given: each further term of the sequence is defined as a function of the preceding terms. The Fibonacci numbers are the archetype of a linear, homogeneous recurrence relation with constant coefficients. The logistic map is another common example.

Legendre polynomials:

$P_n(x)$ are polynomial of degree n defined as:

$$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}\left(x^2 - 1\right)^n$$

13.10 Answer to Self Learning Exercises

Answer to Self Learning Exercise-I

Ans. 1: From recurrence relation, we have

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(n)$$

integrating this equation with respect of x between limits 0 and x, we get

$$[x^n J_n(x)]_0^x = \int_0^x x^n J_{n-1}(x)dx$$

Or

$$x^n J_n(x) - 0 = \int_0^x x^n J_{n-1}(x)dx$$

Hence we get

$$\int_0^x x^n J_{n-1}(x)dx = x^n J_n(x)$$

Ans. 2: Bessel's equation of zeroth order is

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$$

The solution of this equation is $J_0(x)$.

Changing the independent variable from x to t by the relation $x^2 = t$;

$$\therefore \frac{dt}{dx} = 2x$$

So that

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = 2x\frac{dy}{dt} = 2\sqrt{t\frac{dy}{dt}}$$

And

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2\sqrt{t\ \frac{dy}{dt}}\right)$$

$$= \frac{d}{dt}\left(2\sqrt{t\frac{dy}{dx}}\right)\frac{dt}{dx}$$

$$= \left(2\sqrt{t}\frac{d^2y}{dt^2} + \frac{1}{\sqrt{t}}\frac{dy}{dt}\right).2\sqrt{t}$$

$$= 4t\frac{dy^2}{dt^2} + 2\frac{dy}{dt}$$

Substituting these values in equation (1); we get

$$4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + \frac{1}{\sqrt{t}}2\sqrt{t}\frac{dy}{dt} + y = 0$$

$$4t\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 0$$

Differentiating above equation n times w.r. to t by Leibnitz's theorem, we get

$$4\left[t\frac{d^{n+2}}{dt^{n+2}} + {}^nc_1.1.\frac{d^{n+1}y}{dt^{n+1}}\right] + 4\frac{d^{n+1}y}{dt^{n+1}} + \frac{d^nt}{dt_n} = 0$$

Or

$$4t\frac{d^{n+2}y}{dt^{n+2}}4(n+1)\frac{d^{n+1}y}{dt^{n+1}} + \frac{d^ny}{dt^n} = 0$$

Substituting $\emptyset = \frac{d^ny}{dt^n} = \frac{d^nJ_0}{dt^n}$, equation becomes

$$4t\frac{d^2\emptyset}{dt^2} + 4(n+1)\frac{d\emptyset}{dt} + \emptyset = 0$$

…|(a)

As $J_n(x)$ is the solution of Bessel's equation of n^{th}order

$$\frac{d^2y\prime}{dx^2}+\frac{1}{x}\frac{dy\prime}{dx}+\left(1-\frac{n^2}{x^2}\right)y'=0$$

Substituting $y'=x^n z$;so that

$$\frac{dy'}{dx}=x^n\frac{dz}{dx}+n\,x^{n-1}z$$

$$\frac{d^2y'}{dx^2}=x^n\frac{d^2z}{dx^2}+2n\,x^{n-1}\frac{dz}{dx}+n(n-1)x^{n-2}z=0$$

we get

$$x^n\frac{d^2z}{dx^2}-2n\,x^{n-1}\,z+\frac{1}{x}\left(x^n\frac{dz}{dx}+n\,x^{n-1}z\right)+\left(1-\frac{n^2}{x^2}\right)x^2z=0$$

Or

$$x^n\frac{d^2z}{dx^2}+(2n+1)x^{n-1}\frac{dz}{dx}x^nz=0$$

Or

$$\frac{d^2z}{dx^2}+\frac{(2n+1)}{x}\frac{dz}{dx}+z=0$$

Substituting $x^2=t$, equation, becomes

$$4t\frac{d^2z}{dt^2}+4(n+1)\frac{dz}{dt}+z=\ 0$$

…(b)

Comparing equations (a) and (b); we get

$$z=\emptyset=\frac{d^nJ_0(x)}{dt^n}=\frac{d^nJ_0(x)}{d(x^2)^n}$$

But y= $x^n z$

Hence $J_n(x)=Cx^n\frac{d^nJ_0(x)}{d(x)^2}$

Where C is a constant to be determines.

As

$$J_0(x)=\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{(2^r r\,!)^2}$$

$$\therefore\ \frac{d^nJ_0(x)}{d(x^2)^n}=\frac{d^n}{d(x^2)^n}\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{(2^r r\,!)^2}$$

$$=\frac{d^n}{d(x^2)^n}\sum_{r=0}^{\infty}\frac{(-1)^{n+r}(x^2)^{n+r}}{[2^{n+r}(n+r)\,!]^2}$$

[Since all those terms in which index of x is less than 2n will vanish on differentiation n times with respect to x^2]

$$= \sum_{R=0}^{\infty} \frac{(-1)^{n+r}(n+r-1)\dots\dots(r+1)(x^2)^r}{2^{2n+2r}(n+r\,!)^2}$$

$$= \sum_{r=0}^{\infty} (-1)^{(n+r)} \frac{(n+r)\,!}{r\,!} \frac{x^{2r}}{2^{2n+2r}\{(n+r)\,!\}^2}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^{n+r} x^{2r}}{2^{2n+2r}(n+r)\,!\; r\,!}$$

$$\therefore\; J_n(x) = Cx^n \sum_{r=0}^{\infty} \frac{(-1)^{n+r} x^{2r}}{2^{2n+2r}(n+r)\,!\,!} = C\frac{(-1)^n}{2^n} \sum_{r=0}^{\infty} (-1)^{r\cdot} \frac{1}{r\,!\;(n+r)\,!} \left(\frac{x}{2}\right)^{n+2r}$$

$$= C\frac{(-1)^n}{2^n} J_n(x)$$

$$C = \frac{2^n}{(-1)^n} = (-2)^n$$

Hence

$$J_n(x) = (-2)^n x^n \frac{d^n J_0(x)}{d(x^2)^n}$$

Ans. 3: From trigonometric expansion of cosine, we now that

$$\cos(x \sin \emptyset) = 1 - \frac{x^2 sin^2\emptyset}{2\,!} + \frac{x^4 sin^4\emptyset}{4\,!} + \cdots + (-1)^r \frac{x^{2r} sin^{2r}\emptyset}{(2r\,!)}$$

$$\sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{(2r)\,!} sin^{2r}\emptyset$$

Hence

$$\int_0^x cos(x \sin \emptyset)\, cos^{2n}\emptyset\, d\emptyset$$

$$= \int_0^{\pi} \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{(2r\,!)} sin^{2r}\emptyset\, cos^{2n}\emptyset\, d\emptyset$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{(2r\,!)} . 2 \int_0^{\pi/2} sin \;\; 2r\,\emptyset\, cos^{2n}\,\emptyset\;\; d\emptyset$$

$$= \sum_{r=0}^{\infty} 2.(-1)^r \frac{x^{2r}}{(2r\,!)} . \frac{\left\lceil\left(\frac{2r+1}{2}\right) \left\lceil\left(\frac{2n+i}{2}\right)\right.\right.}{2\left\lceil\left(\frac{2n+2r+2}{2}\right)\right.}$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{(2r\,!)} \frac{\left\lceil\left(\frac{2n+1}{2}\right) \left\{\frac{2r-1}{2} . \frac{2r-3}{2} \dots \frac{3}{2} . \frac{1}{2} \left\lceil\frac{1}{2}\right.\right\}\right.}{2\lceil(n+r+1)}$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{(2r)!} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+r+1)} \cdot \frac{1}{2^r} \frac{2r!\, \Gamma\frac{1}{2}}{2r.(2r-2)(2r-4)\dots 4.2}$$

$$= \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{2r} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+r+1)} \frac{\sqrt{\pi}}{r!}$$

$$\therefore \frac{1}{\sqrt{\pi \Gamma\left(n+\frac{1}{2}\right)}} \left(\frac{x}{2}\right)^n \int_0^{\pi} cos(x\, sin\, \emptyset) cos^{2n}\emptyset\, d\emptyset$$

$$= \frac{1}{\sqrt{z \Gamma\left(n+\frac{1}{2}\right)}} \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{2r} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+r+1)} \frac{\sqrt{\pi}}{r!}$$

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} = J_n(x)$$

$$\therefore J_n(x) = \frac{1}{\sqrt{\pi} \Gamma\left(n+\frac{1}{2}\right)} \left(\frac{x}{2}\right)^n \int_0^{x} cos(x\, sin\, \emptyset) \cos^{2n} \emptyset\, d\emptyset$$

Ans. 4: The standard form of Bessel function of first kind J_n(x),

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{r!(n+r+1)!} \left(\frac{x}{2}\right)^{n+2r}$$

Bessel function of second kind Y_n(x) is obtained by taking the particular linear combination of J_n(x) and J_{-n}(x) as

$$Y_n(x) = N_n(x) = \frac{\cos n\pi\ J_n(x) - J_{-n}(x)}{\sin n\pi}$$

Ans. 5: The Hankel functions $H_n^{(1)}(x)$ and $H_n^{(2)}(x)$ are defined as

$$H_n^{(1)}(x) = J_n(x) + i\, Y_n(x)$$
$$H_n^{(2)}(x) = J_n(x) - i\, Y_n(x)$$

Answer to Self Learning Exercise-II

Ans.1: The modified Bessel function of first kind of order n.

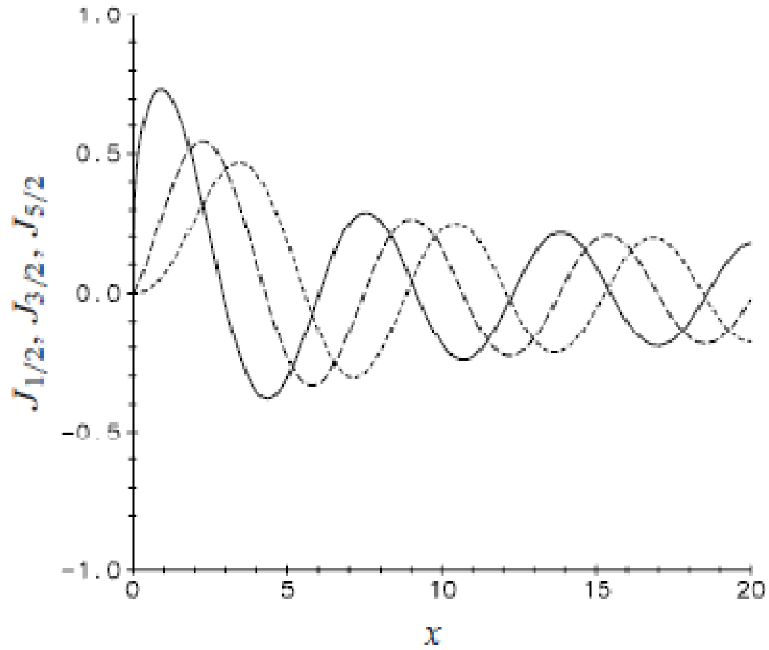
$$I_n(x) = e^{-\frac{in\pi}{2}} J_n(xe^{i\pi/2})$$

Ans. 2: Modified Bessel function of second kind

$$K_n(x) = \frac{\pi/2}{\sin n\pi} [I_{-n}(x) - I_n(x)]$$

where n is non-integer.

Ans. 3 :



Ans. 4: $h_0^{(1)}(x) = \frac{1}{x}(\sin x + i\cos x) = \frac{1}{x}e^{ix}$

$$h_0^{(2)}(x) = \frac{1}{x}(\sin x + i\cos x) = \frac{1}{x}e^{-ix}$$

Ans. 5 : $y_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}$

$$y_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x$$

$$j_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right)$$

$$j_n(x) = -(-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right)$$

$$h_n^{(1)}(x) = i(-1)^n x^n \left(\frac{1}{x}.\frac{d}{dx}\right)^n \left(\frac{e^{ix}}{x}\right)$$

$$h_n^{(2)}(x) = i(-1)^n x^n \left(\frac{1}{x}.\frac{d}{dx}\right)^n \left(\frac{e^{-ix}}{x}\right)$$

13.11 Exercise

Q.1 Plot modified Bessel functions.

Q.2 Prove orthogonality of spherical Bessel's function.

Q.3 Use the formulas for derivatives and recurrence relations to show that:

$$J_0(x) = -J_1(x)$$

$$J_2(x) = \frac{1}{2}[J_1(x) - J_3(x)]$$

$$\int x^v J_{v-1}(x)dx = x^v J_v(x) + C$$

$$\int x^{-v} J_{v+1}(x)dx = -x^{-v} J_v(x) + C$$

$$\int J_{v+1}(x)dx = J_{v-1}(x)dx - 2J_v(x) + C$$

there C denotes an arbitrary constant.

Q.4 To evaluate

1. $\int J_5(x)dx$
2. $\int J_3(x)dx$
3. $\int x^3 J_0(x)dx$
4. $\int J_5(x)dx$

References and Suggested Readings

1. Mathematical Physics with Classical Mechanics by Satya Prakash, 2014.

2. Mathematical Physics by H.K. Das, 1997.

3. Special Functions and their applications by N.N. Lebedev, R. Silverman, 1973.

UNIT-14
Legendre Polynomials

Structure of the Unit

14.0 Objectives

After reading this unit, Student can understand Legendre polynomial & its properties. Student can able to use Legendre polynomial & its properties in different physical problems.

14.1 Introduction

It is not always possible to obtain the closed form of solution to a differential equation in whish dependent variables are functions of x. Therefore it requires obtaining the solutions of such series in terms of infinite series. In this chapter we will solve some differential equations in terms of typical infinite series, which can be put in terms of some function of special characters. These functions are called as special function. These are also called as power series solutions.

14.2 Legendre Differential Equation

The differential equation, frequently occurring in the magneto hydrodynamic problems, of the type

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0$$

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{dy}{dx}\right)+n\left(n+1\right)y=0$$

is known as ***Legendre's equation***. It's solution is called as ***Legendre polynomial*** & denoted by $P_n(x)$.

14.3 Solution of Legendre Equation

The differential equation of the form

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dy}{dx}\right]+n\left(n+1\right)y=0$$

or $$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0 \qquad (1)$$

is called Legendre's differential equation where n is any real number. This equation can be solved in series of ascending or descending powers of x, but the solution of (1) in descending power of x is more important, so we apply the Frobenius method in descending power of x. Assume that the solution of (1) is

$$y=\sum_{r=0}^{\infty}A_r\,x^{m-r}\,,\ A_0\neq 0 \qquad (2)$$

Now using (2) in (1), we get

$$\left(1-x^2\right)\sum_{r=0}^{\infty}A_r\left(m-r\right)\left(m-r-1\right)x^{m-r-2}$$

$$-2x\sum_{r=0}^{\infty} A_r (m-r)x^{m-r-1} + n(n+1)\sum_{r=0}^{\infty} A_r\, x^{m-r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} A_r (m-r)(m-r-1)x^{m-r-2} - \sum_{r=0}^{\infty} A_r (m-r)(m-r-1)x^{m-r}$$

$$-2\sum_{r=0}^{\infty} A_r (m-r)x^{m-r} + n(n+1)\sum_{r=0}^{\infty} A_r\, x^{m-r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} A_r (m-r)(m-r-1)x^{m-r-2}$$

$$-\sum_{r=0}^{\infty} A_r \left[(m-r)(m-r-1) + 2(m-r) - n(n+1)\right]x^{m-r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} A_r (m-r)(m-r-1)x^{m-r-2} \sum_{r=0}^{\infty} A_r \left[(m-r)(m-r+1) - n^2 - n\right]x^{m-r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} A_r (m-r)(m-r-1)x^{m-r-2} - \sum_{r=0}^{\infty} A_r \left[(m-r)^2 - n^2 + (m-r-n)\right]x^{m-r} = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} A_r (m-r)(m-r-1)x^{m-r-2} - \sum_{r=0}^{\infty} A_r \left[(m-r-n)(m-r+n+1)\right]x^{m-r} = 0 \qquad (3)$$

Equating to zero the coefficient of highest power of x (i.e. x^m), we get the indicial equation is

$$A_0 (m-n)(m+n+1) = 0, \qquad A_0 \neq 0$$

$$\Rightarrow \qquad m = n, \quad -(n+1) \qquad (4)$$

This shows that the roots are distinct. The difference of the roots is $(2n+1)$ and is assumed to be a non-integer; hence two independent solutions can be obtained corresponding to the roots.

Equating to zero the coefficient of the general term, i.e. x^{m-r}, we obtain the recurrence relation is

$$A_{r-2}(m-r+2)(m-r+1) - A_r (m-r-n)(m-r+n+1) = 0$$

$$\Rightarrow \qquad A_r = \frac{(m-r+2)(m-r+1)}{(m-r-n)(m-r+n+1)} A_{r-2} \qquad (5)$$

Here, A_1 is to be evaluated. It can be done by equating to zero the next lower power of x, i.e. x^{m-1}, which gives

$$A_1 (m-1-n)(m+n) = 0$$

$\Rightarrow \qquad A_1 = 0$ [$\because$ neither $m - 1 - n = 0$ nor $m + n = 0$ by virtue of (4)]

$\therefore \qquad$ From (5), it follows that

$$A_3 = A_5 = A_7 = ... = 0$$

and
$$A_2 = \frac{m(m-1)}{(m-n-2)(m+n-1)} A_0,$$

$$A_4 = \frac{(m-2)(m-3)}{(m-n-4)(m+n-3)} A_2$$

$$= \frac{m(m-1)(m-2)(m-3)}{(m-n-2)(m-n-4)(m+n-1)(m+n-3)} A_0$$

and so on.

Now substituting the above values in (2), we get

$$y = A_0 x^m \left[1 + \frac{m(m-1)}{(m-n-2)(m+n-1)} x^{-2} + \frac{m(m-1)(m-2)(m-3)}{(m-n-2)(m-n-4)(m+n-1)(m+n-3)} x^{-4} + ... \right] \qquad (6)$$

When $m = n$, we get one solution of (1) as

$$y = A_0\, x^n \left[1 - \frac{n(n-1)}{2(2n-1)} x^{-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} x^{-4} - ... \right]$$

Now, if we take $A_0 = \dfrac{1.3.5.....(2n-1)}{n!}$, then the above solution is denoted by $P_n(x)$, i.e.

$$P_n(x) = \frac{1.3.5.....(2n-1)}{n!} \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)} x^{n-4} - ..... \right] \qquad (7)$$

$P_x(x)$ is called the Legendre's function of first kind. When n is a positive integer, the series in (7) terminator and therefore $P_x(x)$ is also called Legendre polynomial of degree n. We can also write

$$P_x(x) = \sum_{r=0}^{N} \frac{(-1)^r}{r!} \frac{(2n-2r)!}{2^n (n-2r)!(n-r)!} x^{n-2r}$$

Where
$$N = \begin{cases} n/2 \text{ if } n \text{ is even} \\ (n-1)/2, \text{ if } n \text{ is odd} \end{cases} \qquad (8)$$

Further, when $m = -(n+1)$ the other solution of (1) is

$$y = A_0\, x^{-n-1}\left[1 + \frac{(n+1)(n+2)}{2(2n+3)} x^{-2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)} x^{-4} + \ldots\right]$$

Now if we take $A_0 = \dfrac{n!}{1.3.4.....(2n+1)}$, then the above solution is denoted by $Q_n(x)$, i.e.

$$Q_n(x) = \frac{n!}{1.3.5.....(2n+1)}\left\{x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)} x^{-n-5} + \ldots\right\} \quad (9)$$

$Q_n(x)$ is called Legendre's function of second kind. $Q_n(x)$ is an infinite (or non terminating) series as n is positive integer.

Hence, the complete solution of (1) given by

$$y = C_1 P_n(x) + C_2 Q_n(x)$$

Where C_1 and C_2 are arbitrary constants.

On putting $n = 0$, 1, 2, … in (7), we get

$$\boxed{P_0(x) = 1},$$

$$\boxed{P_1(x) = x},$$

$$\boxed{P_2(x) = \frac{3x^2 - 1}{2}},$$

$$\boxed{P_3(x) = \frac{5x^3 - 3x}{2}},$$

$$\boxed{P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)}, \text{ etc.}$$

14.4 Generating Function

To show that $P_n(x)$ is the coefficient of t^n in the expansion of $\left(1 - 2xt + t^2\right)^{-1/2}$ in ascending powers of t, i.e.

$$\boxed{\left(1-2xt+t^2\right)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n\left(x\right)}$$

Proof. Since

$$\left(1-z\right)^{-1/2} = 1+\frac{1}{2}z+\frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}z^2+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}z^3+.....$$

Therefore, we write

$$\left[1-\left(2x-t\right)t\right]^{-1/2} = 1+\frac{1}{2}t\left(2x-t\right)+\frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}t^2\left(2x-t\right)^2+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}t^3\left(2x-t\right)^3+.....$$

$$= 1+\frac{1}{2}t\left(2x-t\right)+\frac{1.3}{2.4}t^2\left(2x-t\right)^2+\frac{1.3.5}{2.4.6}t^3\left(2x-t\right)^3+...$$

$$.....+\frac{1.3.5.....\left(2n-3\right)}{2.4.6.....\left(2n-2\right)}t^{n-1}\left(2x-t\right)^n+\frac{1.3.5.....\left(2n-3\right)}{2.4.6.....2n}t^n\left(2x-t\right)^n$$

The coefficient of t^n in the term $\frac{1.3.5.....\left(2n-1\right)}{2.4.6.....2n}t^n\left(2x-t\right)^n$ is

$$= \frac{1.3.5.....\left(2n-1\right)}{2.4.6.....2n}\left(2x\right)^n$$

$$= \frac{1.3.5.....\left(2n-1\right)}{n!}.x^n$$

Also, the coefficient of t^n in the term $\frac{1.3.5.....\left(2n-3\right)}{2.4.6.....\left(2n-2\right)}t^{n-1}\left(2x-t\right)^{n-1}$ is

$$= \frac{1.3.5.....\left(2n-3\right)}{2.4.6.....\left(2n-2\right)}\left[{}^{n-1}C_1\left(2x\right)^{n-2}\right]$$

$$= \frac{1.3.5.....\left(2n-3\right)\left(n-1\right)}{2.4.6.....\left(2n-2\right)}2^{n-2}x^{n-2}$$

$$= \frac{1.3.5.....\left(2n-3\right)\left(2n-1\right)\left(2n\right)\left(n-1\right)}{2.4.6.....\left(2n-2\right)\left(2n\right)\left(n-1\right)}2^{n-2}x^{n-2}$$

$$= \frac{1.3.5.....\left(2n-3\right)\left(2n-1\right)\left(2n\right)\left(n-1\right)}{2^n\,n!\left(2n-1\right)}2^{n-2}x^{n-2}$$

$$= \frac{1.3.5.....\left(2n-3\right)\left(2n-1\right)}{n!}\cdot\frac{n\left(n-1\right)}{2\left(2n-1\right)}x^{n-2}$$

Again, the coefficient of t^n in the term $= \frac{1.3.5.....(2n-5)}{2.4.6.....(2n-4)} t^{n-2}(2x-t)^{n-2}$ is

$$= \frac{1.3.5.....(2n-1)}{n!} \cdot \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} x^{n-4}$$

Proceeding in similar manner, we see that the coefficient of t^n in the expansion of $\left(1-2xt+t^2\right)^{-1/2}$ is

$$= \frac{1.3.5.....(2nm-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} x^{n-4} - .....\right]$$

$$= P_n(x)$$

Thus, it can be observed that $P_1(x)$, $P_2(x)$, ….., $P_n(x)$ etc. are the coefficients of t, t^2, ….., t^n etc. respectively.

Hence, we have

$$\left(1-2xt+t^2\right)^{-1/2} = 1 + tP_1(x) + t^2 P_2(x) + ... + t^n P_n(x) + ...$$

or $$\left(1-2xt+t^2\right)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$$

The function $\left(1-2xt+t^2\right)^{-1/2}$ ***is called the generating function of the Legendre polynomials*** $P_n(x)$**.**

14.5 Rodrigue's Formula

To prove that

$$\boxed{P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n}\left(x^2-1\right)^n}$$

Proof: Let $y = \left(x^2-1\right)^n$

which on differentiation gives

$$\frac{dy}{dx} = n\left(x^2-1\right)^{n-1}.2x$$

or $$\left(x^2-1\right)\frac{dy}{dx} = 2nxy$$

Now differentiating the above equation with respect to x, $(n+1)$ times by using

Leibnitz theorem, we get

$$\left(x^2-1\right)\frac{d^{n+2}y}{dx^{n+2}} + 2\left(n+1\right)x\,\frac{d^{n+1}y}{dx^{n+1}} + n\left(n+1\right)\frac{d^n y}{dx^n}$$

$$= 2n\left[x\frac{d^{n+1}y}{dx^{n+1}} + \left(n+1\right)\frac{d^n y}{dx^n}.1\right]$$

Leibnitz theorem for the n^{th} derivative of product of two functions of x states that

$$D^n(uv) = \left(D^n u\right)v + {}^nC_1\left(D^{n-1}\right)(Dv) + {}^nC_2\left(D^{n-2}u\right)\left(D^2 v\right) + ...+ u\left(D^n v\right)$$

Where D^n stands for $\left.\frac{d^n}{dx^n}\right]$

or $$\left(x^2-1\right)\frac{d^{n+2}y}{dx^{n+2}} + 2x\frac{d^{n+1}y}{dx^{n+1}} - n(n+1)\frac{d^n y}{dx^n} = 0$$

or $$\left(1-x^2\right)\frac{d^{n+2}y}{dx^{n+2}} - 2x\frac{d^{n+1}y}{dx^{n+1}} + n(n+1)\frac{d^n y}{dx^n} = 0$$

Here, on substituting $z = \frac{d^n y}{dx^n}$, we get

$$\left(1-x^2\right)\frac{d^2 z}{dx^2} - 2x\frac{dz}{dx} + n\left(n+1\right)z = 0 \tag{1}$$

Which is Legendre's equation and its solution is

$$z = c\,P_n\left(x\right)$$

or $$\frac{d^n y}{dx^n} = c\,P_n\left(x\right)$$

(2)

Now on taking $x = 1$, we have

$$c = \left[\frac{d^n y}{dx^n}\right]_{x=1} \qquad \left[\because\ P_n(1) = 1\right] \tag{3}$$

Since $$y = \left(x^2-1\right)^n = (x+1)^n(x-1)^n$$

Differentiating n times by using Leibnitz theorem, we obtain

$$\frac{d^n y}{dx^n} = (x-1)^n \left\{ \frac{d^n}{dx^n}(x+1)^n \right\} + {}^nC_1 \left\{ \frac{d^{n-1}}{dx^{n-1}}(x+1)^n \right\} n(x-1)^{n-1}$$

$$+ \ldots.. + (x+1)^n \left\{ \frac{d^n}{dx^n}(x-1)^n \right\}$$

or $$\frac{d^n y}{dx^n} = (x-1)^n \, n! + n . \frac{n!}{1}(x+1) . n(x-1)^{n-1} + \ldots..$$

$$+ n . n!(x+1)^{n-1} . \frac{n}{1}(x-1) + (x+1)^n \; n!$$

Putting $x = 1$, we have

$$\left[\frac{d^n y}{dx^n} \right]_{x=1} = 2^n n! = c \qquad \text{[using (3)]} \qquad (4)$$

Now, from (2) and (4) we arrive at the required result

$$P_n(x) = \frac{1}{c} \frac{d^n y}{dx^n}$$

or $$P_n(x) = \frac{1}{2^n \, n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n$$

This is called Rodrigue's formula.

Put $n = 0$ in Rodrigue's formula

$$P_0(x) = 1$$

Put $n = 1$ in Rodrigue's formula

$$P_1(x) = \frac{1}{2!1!} \frac{d}{dx}\left(x^2 - 1\right) = \frac{1}{2} \times (2x) = x$$

$\Rightarrow P_1(x) = x$

Similarly $$P_2(x) = \frac{3x^2 - 1}{2}$$

$$P_3(x) = \frac{5x^3 - 3x}{2}, \qquad P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

From above results

$$1 = P_0(x), \qquad x = P_1(x)$$

$$P_2(x)=\frac{3x^2-1}{2}\Rightarrow 2P_2(x)=3x^2-1\Rightarrow x^2=\frac{2P_2+1}{2}$$

$$\Rightarrow x^2=\frac{2P_2+P_0}{2}$$ (Putting $1=P_0(x)$)

In similar way

$$x^3=\frac{2P_3+3P_1}{5}, \qquad x^4=\frac{1}{35}(8P_4+20P_2+7P_0)$$

14.6 Illustrative Examples

Example 1 Express $f(x)=x^4+3x^4-x^2+5x-2$ in terms of Legendre Polynomials.

Sol. As found above $1=P_0$, $x=P_1$, $x^2=\frac{2P_2+P_0}{3}$

$$x^3=\frac{2P_1+3P_1}{5}$$

and as $P_4=\frac{1}{8}(35x^4-30x^2+3)$

so $x^4=\frac{1}{35}\{8P_4+20P_2+7P_0\}$

Substituting these values in $f(x)$,

$$f(x)=\frac{8}{35}P_4(x)+\frac{6}{5}P_3(x)-\frac{2}{21}P_2(x)-\frac{224}{105}P_0(x)$$

Example 2 Express $f(x)=x^3-5x^2+x+2$, in terms of Legendre Polynomial.

Sol. Using $P_0=1$, $x=P_1$, $P_2=\frac{2P_2+P_0}{3}$ and $x^3=\frac{2P_3+3P_1}{5}$

$$f(x)=\frac{1}{5}(2P_3+3P_1)-\frac{5}{3}(2P_2+P_0)+P_1+2P_0$$

$$=\frac{2}{5}P_3(x)-\frac{10}{3}P_2(x)+\frac{1}{3}P_0(x)$$

14.7 Self Learning Exercise

Section – A (Very Short Answer type Questions)

Q.1 Write Legendre Differential equation.

Q.2 Write generating function for $P_n(x)$.

Section – B (Short Answer type Questions)

Q.3 Express $f(x)=5x^3-x^2+3x-4$ in term of Legendre polynomial.

Q.4 Prove that $(1-x^2)(1-2xt+t^2)^{-3/2}=\sum_{n=0}^{\infty}(2n+1)P_n(x)t^n$

Section – C (Long Answer type Questions)

Q.5 Solve Legendre equation $(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$

Q.6 Prove that $(1-2xt+t^2)^{-1/2}$ is generating function for Legendre Polynomial.

14.8 Recurrence Relations for Legendre Polynomial

(I) $(2n+1)xP_n=(n+1)P_{n+1}+nP_{n-1}$ **or** $(n+1)P_{n+1}=(2n+1)x\,P_n-n\,P_{n-1}$

Proof : By generating formula of the Legendre's function,

$$(1-2xz+z^2)^{-1/2}=\sum_{n=0}^{\infty}z^nP_n \tag{1}$$

Differentiating both sides of (1) w.r.t. z,

$$-\frac{1}{2}(1-2xz+z^2)^{-3/2}(-2x+2z)=\sum_{n=0}^{\infty}nz^{n-1}P_n \tag{2}$$

Multiplying (2) throughout by $(1-2xz+z^2)$,

$$(x-z)(1-2xz+z^2)^{-1/2}=(1-2xz+z^2)\sum_{n=0}^{\infty}nz^{n-1}P_n$$

or $$(x-z)\sum_{n=0}^{\infty}z^nP_n=(1-2xz+z^2)\sum_{n=0}^{\infty}nz^{n-1}P_n \quad \text{[by (1)]}$$

or $$x\sum_{n=0}^{\infty}z^nP_n-\sum_{n=0}^{\infty}z^{n+1}P_n=\sum_{n=0}^{\infty}nz^{n-1}P_n-2x-\sum_{n=0}^{\infty}nz^nP_n+\sum_{n=0}^{\infty}nz^{n+1}P_n \tag{3}$$

Comparing the coefficient of z^n on both the sides, we get,

$$xP_n-P_{n-1}=(n+1)P_{n+1}-2xn\,P_n\,\_(n-1)P_{n-1} \quad \text{[Note]}$$

By transporting the required formula is

$$(2n+1)x\,P_n=(n+1)P_{n+1}+n\,P_{n-1} \tag{4}$$

Another form :

Replacing n by $(n-1)$ in (4), we get the following useful form

$$\boldsymbol{n\, P_n = (2n-1)x\, P_{n-1} - (n-1)P_{n-2}} \tag{5}$$

(II) $\boldsymbol{n\, P_n = xP_n' - P_{n-1}'}$

Proof : By generating formula of the Legendre's function

$$\left(1-2xz+z^2\right)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \tag{1}$$

Differentiating (1) w.r.t. x and simplifying

$$(x-z)\left(1-2xz+z^2\right)^{-3/2} = \sum_{n=0}^{\infty} nz^{n-1} P_n(x) \tag{2}$$

Again differentiating (1) w.r.t. x, and simplifying,

$$z\left(1-2xz+z^2\right)^{-3/2} = \sum_{n=0}^{\infty} z^n P_n'$$

where dash $(')$ denote the differentiation w.r.t. x

Multiplying both sides by $(x-z)$,

$$z(x-z)\left(1-2xz+z^2\right)^{-3/2} = (x-z)\sum_{n=0}^{\infty} z^n P_n' \tag{3}$$

Now by (2) and (3),

$$z\sum_{n=0}^{\infty} nz^{n-1} P_n = (x-z)\sum_{n=0}^{\infty} z^n P_n'$$

or, $$\sum_{n=0}^{\infty} P_n nz^n = x\sum_{n=0}^{\infty} P_n' z^n - \sum_{n=0}^{\infty} P_n' z^{n+1}$$

Comparing the coefficients of z^n on both sides, the required results in obtained.

(III) $\boldsymbol{(2n+1)P_n = P_{n+1}' - P_{n-1}'}$

Proof : By recurrence relation LR I,

$$(2n+1)x\, P_n = (n+1)P_{n+1} + n\, P_{n-1} \tag{1}$$

Differentiating (1) w.r.t. x

$$(2n+1)x\, P_n' + (2n+1)P_n = (n+1)P_{n+1}' + n\, P_{n-1}' \tag{2}$$

Again by recurrence relation II,

$$x P_n' = n P_n + P_{n-1}' \qquad (3)$$

Using (3) in (2), we have

$$(2n+1)(nP_n + P_{n-1}') + (2n+1)P_n = (n+1)P_{n+1}' + nP_{n-1}'$$

or, $$(2n+1)(n+1)P_n = (n+1)P_{n+1}' - (n+1)P_{n-1}'$$

or, $$(2n+1)P_n = P_{n+1}' - P_{n-1}'$$

Deduction. Directly by integration,

$$c + \int P_n dx = \frac{1}{(2n+1)}(P_{n+1} - P_{n-1}),$$

c is the constant of integration.

(IV) $\boldsymbol{(n+1)P_n = P_{n+1}' - xP_n'}$

Proof : By recurrence relation LR II and LR III,

$$nP_n = xP_n' - P_{n-1}' \qquad (1)$$

and $$(2n+1)P_n = P_{n+1}' - P_{n-1}' \qquad (2)$$

$$(2)-(1) \Rightarrow (n+1)P_n = P_{n+1}' - xP_n'$$

(V) $\boldsymbol{(1-x^2)P_n' = n(P_{n-1} - xP_n)}$

Proof : Multiplying recurrence formula LR II throughout by x,

$$xnP_n = x^2 P_n' - xP_{n-1}' \qquad (1)$$

Replacing n by $(n-1)$ in recurrence formula IV,

$$nP_{n-1} = P_n' - xP_{n-1}' \qquad (2)$$

$$(2)-(1) \Rightarrow n(P_{n-1} - xP_n) = (1-x^2)P_n'$$

(VI) $\boldsymbol{(1-x^2)P_n' = (n+1)(xP_n - P_{n+1})}$

Proof : The recurrence formula LR I is

$$(2n+1)x P_n = (n+1)P_{n+1} + n P_{n-1} \qquad (1)$$

or, $$[(n+1)+n]x P_n = (n+1)P_{n+1} + n P_{n-1}$$

or, $$(n+1)[x P_n - P_{n+1}] = n[P_{n-1} - xP_n] \qquad (2)$$

again by recurrence formula LR V,

$$(1-x^2)P_n' = n[P_{n-1} - xP_n] \qquad (3)$$

The required result is obtained by (2) and (3).

(VII) Beltrami's results

$$\mathbf{(2n+1)(x^2-1)P_n' = n(n+1)(P_{n+1}-P_{n-1})}$$

Proof : By recurrence formula LR V,

$$(1-x^2)P_n' = n(P_{n-1}-n) \qquad (1)$$

and by LR VI, $(1-x^2)P_n' = (n+1)(n-P_{n+1})$ (2)

Multiplying (1) by $(n+1)$ and (2) by n and adding, we get

$$(n+1)(1-x^2)P_n' + n(1-x^2)P_n'$$

$$= n(n+1)P_{n-1} - n(n+1)P_{n+1}$$

or, $(2n+1)(1-x^2)P_n' = n(n+1)(P_{n-1}-P_{n+1})$

or, $(2n+1)(x^2-1)P_n' = n(n+1)(P_{n+1}-P_{n-1})$

14.9 Illustrative Examples

Example 3 Prove that

(i) $P_n(1)=1$

(ii) $P_n(-1)=(-1)^n$

(iii) $P_n(-x)=(-1)^n P_n(x)$

Sol. (i) We know that $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$

Putting $x = 1$, we get $(1-2t+t^2)^{-1/2} = (1-t)^{-1} = \sum_{n=0}^{\infty} t^n P_n(1)$

$$\Rightarrow \quad 1+t+t^2+\ldots+t^n+\ldots = \sum_{n=0}^{\infty} t^n P_n(1)$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} t^n P_n(1)$$

Equating the coefficients of t^n, we get $P_n(1)=1$

(ii) Again if we put $x = -1$, we have

$$(1+2t+t^2)^{-1/2} = (1+t)^{-1} = \sum_{n=0}^{\infty} t^n P_n(-1)$$

$$\Rightarrow \quad 1-t+t^2-t^3+\ldots+(-1)^n t^n \ldots = \sum_{n=0}^{\infty} t^n P_n(-1)$$

$$\Rightarrow \quad \sum_{n=0}^{\infty}(-1)^n t^n = \sum_{n=0}^{\infty} t^n P_n(-1)$$

Now equating the coefficients of t^n, we get

$$P_n(-1) = (-1)^n$$

(iii) We have $\left(1-2xt+t^2\right)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$

$$\therefore \quad \left(1+2xt+t^2\right)^{-1/2} = \left[1-2x(-t)+(-t)^2\right]^{-1/2}$$

$$= \sum_{n=0}^{\infty}(-t)^n P_n(x)$$

$$= \sum_{n=0}^{\infty}(-1)^n t^n P_n(x) \quad (1)$$

Since $\left(1+2xt+t^2\right)^{-1/2} = \left[1-2(-x)t+t^2\right]^{-1/2}$

$$= \sum_{n=0}^{\infty} t^n P_n(-x) \quad (2)$$

Equating the coefficients of t^n, we get

$$\boxed{P_n(-x) = (-1)^n P_n(x)}$$

Example 4 Show that

(i) $(2n+1)x\, P_n(x) = (n+1)P_{n+1}(x) + n\, P_{n-1}(x)$

(ii) $\displaystyle\int_{-1}^{1} x\, P_n(x) P_{n-1}(x)\, dx = \frac{2n}{(2n-1)(2n+1)}$

(iii) $\displaystyle\int_{-1}^{1} x^2\, P_{n+1}(x) P_{n-1}(x)\, dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$

Sol. **(i)**We know that

$$\left(1-2xt+t^2\right)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$$

Differentiating both sides with respect to t, we get

$$\frac{-1}{2}\left(1-2xt+t^2\right)^{-3/2}(-2x+2t)=\sum_{n=0}^{\infty}P_n(x).nt^{n-1}$$

or $\quad (x-t)\left(1-2xt+t^{2x-}\right)^{-1/2}=\left(1-2xt+t^2\right)\sum_{n=0}^{\infty}n\,P_n(x)t^{n-1}$

or $\quad (x-t)\sum_{n=0}^{\infty}P_n(x)t^n=\left(1-2xt+t^{2x-}\right)\sum_{n=0}^{\infty}n\,P_n(x)t^{n-1}$

or $\quad (x-t)\left\{P_0(x)+t\,P_1(x)+\ldots+P_{n-1}(x)t^{n-1}+P_n(x)t^n\ldots..\right\}=\left(1-2xt+t^2\right)$

$\left\{P_1(x)+2P_2(x)t+\ldots+(n-1)P_{n-1}(x)t^{n-2}+nP_n(x)t^{n-1}+(n+1)P_{n+1}(x)t^n\ldots..\right\}$

Now equating the coefficients of t^n from both the sides, we get

$$xP_n(x)-P_{n-1}(x)=(n+1)P_{n+1}(x)-2nxP_n(x)+(n-1)P_{n-1}(x)$$

or $\quad (2n+1)xP_n=(n+1)P_{n+1}(x)+nP_{n-1}(x) \qquad (2)$

(ii) Replacing n by $(n-1)$ in the above equation (2), we have

$$(2n-1)xP_{n-1}=nP_n(x)+(n-1)P_{n-2}(x)$$

Multiplying the above equation by $P_n(x)$ on both the sides and integrating with respect to x between the limits – 1 to 1, we get

$$(2n+1)\int_{-1}^{1}x\,P_n(x)P_{n-1}(x)dx=n\int_{-1}^{1}P_n^2(x)dx+(n-1)\int_{-1}^{1}P_n(x)P_{n-2}(x)dx$$

$$=n\frac{2}{2n+1}$$

(other integral becomes zero by virtue of orthogonality property of the Legendre's function)

$$\therefore \quad \int_{-1}^{1}xP_n(x)P_{n-1}(x)dx=\frac{2}{(2n-1)(2n+1)}$$

(iii) Again replacing n by $(n-1)$ and $(n+1)$ in the equation (2), we have

$$(2n-1)xP_{n-1}(x)=nP_n(x)+(n-1)P_{n-2}(x) \qquad (3)$$

$$(2n+3)xP_{n+1}(x)=(n+2)P_{n+2}(x)+(n+1)P_n(x) \qquad (4)$$

Multiplying the above equations (3) and (4) and integrating with respect to x between the limits -1 to 1, we have

$$(2n-1)(2n+3)\int_{-1}^{1}x^2P_{n+1}(x)P_{n-1}(x)dx=n(n+1)\int_{-1}^{1}P_n^2(x)dx$$

(using orthogonality of Legendre's function)

$$= n(n+1).\frac{2}{2n+1}$$

$$\therefore \qquad \int_{-1}^{1} x^2 P_{n+1}(x) P_{n-1}(x) dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$$

Example 5 Prove that

(i) $$P_{2m}(0) = (-1)^m \frac{1.3.5....(2m-1)}{2.4.6..2n} = (-1)^m \frac{2m!}{2^{2m}(m!)^2}$$

(ii) $$P_{2m+1}(0) = 0$$

Sol. **(i)**We have $\sum_{n=0}^{\infty} P_n(x) t^n \left(1 - 2xt + t^2\right)^{-1/2}$

Putting $x = 0$, we get

$$\sum_{n=0}^{\infty} P_n(0) t^n \left(1 + t^2\right)^{-1/2} = 1 - \frac{1}{2}t^2 + \frac{1.3}{2.4}t^4 + ... + (1-)^r \frac{1.3.5.....(2r-1)}{2.4.6.....2r} t^{2r} + ...$$

Comparing the coefficients of t^{2m} on both the sides, we get

$$P_{2m}(0) = (-1)^m \frac{1.3.5....(2m-1)}{2.4.6.2m}$$

[$\because$ all the power of t on R.H.S. are even]

$$= (-1)^m \frac{1.3.5...(2m-1)2.4.6...2m}{2.4.6.2m - 2.4.6....2m}$$

$$= (-1)^m \frac{1.2.3.4.5.6...(2m-1)(2m)}{2^m m! 2.4.6...2m}$$

$$= (-1)^m \frac{2m!}{2^m m!(2.1)(2.2)(2.3).....2gm}$$

$$= (-1)^m \frac{2m!}{2^m m! 2^m m!} = \frac{(-1)^m 2m!}{2^{2m}(m!)^2}$$

(ii) $$P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r 2n! - 2r! x^{n-2r}}{2^n r! n! - r! n! - 2r}$$

where $N = \frac{n}{2}$, when n is even and $N = \frac{n-1}{2}$ when n is odd.

Putting $n = 2m + 1$, the last term will be $N = m$ and the last term in $P_{2m+1}(0) = 0$ when $x = 0$.

Example 6 If $m \neq n$, then show that $I_{mn} = \int_{-1}^{1}\left(1-x^2\right)P_n' P_m' \, dx = 0$

Sol. Integrating by part taking $\left(1-x^2\right)P_n'$ as first function

$$I_{mn} = \underbrace{\left[\left(1-x^2\right)P_n' P_m(x)\right]_{-1}^{1}}_{=0} - \int_{-1}^{1}\left(\left(1-x^2\right)P_n'' - 2x P_n'\right)P_m \, dx$$

But $\left(1-x^2\right)P_n'' - 2x P_n' = -n(n+1)P_n$

$$= -(n)(n+1)\int_{-1}^{1} P_n P_m \, dx$$

As $n \neq m$ so it is $= 0$.

Example 7 Prove that

$$\frac{1-z^2}{\left(1-2xz+z^2\right)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)P_n(x)z^n$$

Sol. The generating formula of the Legendre's function

$$\left(1-2xz+z^2\right)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n \tag{1}$$

Differentiating both sides of (1) w.r.t. z, we get

$$-\frac{1}{2}\left(1-2xz+z^2\right)^{-3/2}(-2x+2z) = \sum_{n=0}^{\infty} n z^{n-1} P_n$$

Multiplying throughout by $2z$,

$$2z(x-z)\left(1-2xz+z^2\right)^{-3/2} = \sum_{n=0}^{\infty} 2n z^n P_n \tag{2}$$

$$(1)+(2) \Rightarrow \frac{1}{\left(1-2xz+z^2\right)^{-1/2}} + \frac{2z(x-z)}{\left(1-2xz+z^2\right)^{-3/2}} = \sum_{n=0}^{\infty} z^n P_n + \sum_{n=0}^{\infty} 2n z^n P_n$$

or $$\frac{1-2xz+z^2+2z(x-z)}{\left(1-2xz+z^2\right)^{-3/2}} = \sum_{n=0}^{\infty}(2n+1)z^n P_n$$

or $$\frac{1-z^2}{\left(1-2xz+z^2\right)^{-3/2}} = \sum_{n=0}^{\infty}(2n+1)z^n P_n$$ Hence Proved.

Example 8 Prove that

$$\frac{1+z}{z\sqrt{\left(1-2xz+z^2\right)}}-\frac{1}{z}=\sum_{n=0}^{\infty}\left(P_n+P_{n+1}\right)z^n$$

Sol. LHS $=\frac{1}{z}\left(1-2xz+z^2\right)^{-1/2}+\left(1-2xz+z^2\right)^{-1/2}-\frac{1}{z}$

$$=\frac{1}{z}\sum_{n=0}^{\infty}z^nP_n(x)+\sum_{n=0}^{\infty}z^nP_n(x)-\frac{1}{z}\text{ [By generating function]} \qquad (1)$$

$$\sum_{n=0}^{\infty}z^nP_n=P_0+zP_1+z^2P_2+z^3P_3+.....+z^{n+1}P_{n+1}+...$$

$$=1+z\left(P_1+zP_2+z^2P_3+...+z^nP_{n+1}+...\right) \qquad [\because P_0=1]$$

$$=1+z\sum_{n=0}^{\infty}z^nP_{n+1}$$

Using (2) in (1), we get

$$\text{LHS} =\frac{1}{z}\left[1+z\sum_{n=0}^{\infty}z^nP_{n+1}\right]+\sum_{n=0}^{\infty}P_n-\frac{1}{z}$$

$$=\sum_{n=0}^{\infty}z^nP_{n+1}+\sum_{n=0}^{\infty}z^nP_n$$

$$=\sum_{n=0}^{\infty}\left(P_n+P_{n+1}\right)z^n=\text{ RHS}$$ **Hence Proved.**

Example 9 Prove that

$$\int_{-1}^{1}\left(1-x^2\right)P_m'P_n'\,dx=\begin{cases}0,\ m\neq n; & m,n\in N\\ \dfrac{2n(n+1)}{2n+1} & m=n\end{cases}$$

Sol. Case I. When $m\neq n$

Integrating by parts taking P_n as the second function in the given integral, we obtain

$$\int_{-1}^{1}\underbrace{\left(1-x^2\right)P_m'}_{I}\underbrace{P_n'\,dx}_{II}=\left[\left(1-x^2\right)P_m'\,P_n\right]_{-1}^{1}-\int_{-1}^{1}P_n\frac{d}{dx}\left\{\left(1-x^2\right)P_m'\right\}dx$$

$$=0-\int_{-1}^{1}\left[\left(1-x^2\right)P_m''-2xP_m'\right]P_n\,dx$$

$$= 0 - \int_{-1}^{1} [- m(m+1). P_m] P_n dx$$

[by Legendre's eq.] $\left[(1-x^2)P_m'' - 2xP_m' + m(m+1)P_m = 0\right]$

$$= m(m+1)\int_{-1}^{1} P_m \, P_n \, dx$$

$$= m(m+1)\begin{cases} 0, & \because n \neq m \\ \dfrac{2}{2m+1}, & when \;\; n = m \end{cases}$$ [By orthogonality]

Therefore $$\int_{-1}^{1} (1-x^2) P_m' \, P_n' \, dx = \begin{cases} 0, & when \, n \neq m \\ \dfrac{2m(m+1)}{2m+1}, & when \;\; n = m \end{cases}$$

Example 10 Prove that all the roots of $P_n(x) = 0$ are distinct,

Sol. Suppose all the roots of $P_n(x) = 0$ are not different, then at least two roots must be equal.

Let the equal root be α, then by the theory of equations,

$$P_n(\alpha) = 0 \qquad \text{and} \qquad P_n'(\alpha) = 0 \tag{1}$$

Since $P_n(\alpha)$ is the solution of Legendre's equation, therefore

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0 \tag{2}$$

Differentiate (2) r times w.r.t. x with Leibnitz theorem, and simplifying, we obtain

$$(1-x^2)D^{r+2}[P_n(x)] - 2x({}^rC_1 + 1)D^{r+1}[P_n(x)]$$
$$- \left[2.\,{}^rC_2 + 2\,{}^rC_1 - n(n+1)\right]D^r[P_n(x)] = 0 \tag{3}$$

Now putting $r = 0$ and $x = \alpha$ in (3) and using (1), we get

$$(1-\alpha^2)P_n''(\alpha) - 0 - 0 = 0 \qquad \Rightarrow \qquad P_n''(\alpha) = 0 \tag{4}$$

Again putting $r = 1$ and $x = \alpha$ in (3) and using (1) and (4), we have

$$(1-\alpha^2)P_n'''(\alpha) - 0 - 0 = 0 \qquad \Rightarrow \qquad P_n'''(\alpha) = 0 \tag{5}$$

similar replacing $r = 2,3,.....(n-3)(n-2)$ in (3) and simplifying as above, we obtain

$$P_n'''(\alpha) = 0 = P_n^{(4)}(\alpha) = ...= P_n^{(n)}(\alpha) \tag{6}$$

But $$P_n(x) = \frac{1.3.......(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}\,.\,x^{n-2} + ...\right]$$

$$\therefore D^n[P_n(x)] = \frac{1.3.......(2n-1)}{n!} \cdot (n)! = 1.3......(2n-1) \neq 0$$

$$\Rightarrow D^n[P_n(\alpha)] \neq 0 \tag{7}$$

Example 11 Prove that

$$\int_{-1}^{1} x^m P_n(x)dx = \begin{cases} 0, & m < n, \quad n \in Z \\ \dfrac{2^{n+1}(n!)^2}{(2n+1)}, & m = n \end{cases}$$

Sol. By Rodrigues formula, the given integral

$$\int_{-1}^{1} x^m P_n(x)dx = \int_{-1}^{1} x^m \left[\frac{1}{2^n (n)! \, dx^n}\left(x^2-1\right)\right]dx$$

$$= \frac{1}{2^n.(n)!}\int_{-1}^{1} x_I^m D^n\left(x_{II}^2 - 1\right)^n dx, \quad D \equiv d/dx$$

$$= \frac{1}{2^n.(n)!}\left[\left\{x^m D^{n-1}\left(x^2-1\right)^n\right\}_{-1}^{1} - \int_{-1}^{1} m x^{m-1} D^{n-1}\left(x^2-1\right)^n dx\right]$$

[Integrating by parts]

$$= \frac{(-1).m}{2^n(n)!}\int_{-1}^{1} x_I^{m-1} D^{n-1}\left(x_{II}^2-1\right)^n dx,$$

$$= \frac{(-1).m(m-1)}{2^n(n)!}\int_{-1}^{1} x^{m-2} D^{n-2}\left(x^2-1\right)^n dx,$$

[Again integrating by parts]

$$= \frac{(-1)^m.(m)!}{2^n(n)!}\int_{-1}^{1} x^0 D^{n-m}\left(x^2-1\right)^n dx$$

[$(m-2)$ times, when $m < n$] (1)

$$= \frac{(-1)^m.(m)!}{2^n(n)!}\left[D^{n-m-1}\left(x^2-1\right)^n\right]_{-1}^{1}$$

$= 0$ [because $n > m+1$]

Part II. When $m = n$, then by (1).

$$\int_{-1}^{1} x^n P_n(x)dx = \frac{(-1)^n}{2^n}\int_{-1}^{1}\left(x^2-1\right)^n dx = \frac{(-1)^{2n}}{2^n}\int_{-1}^{1}\left(1-x^2\right)^n dx$$

$$= \frac{2}{2^n}\int_{-1}^{1}\left(1-x^2\right)^n dx \qquad \text{[integrand is even function]}$$

$$= \frac{1}{2^{n-1}}\int_{0}^{\pi/2} cos^{2n+1}\theta\, d\theta \qquad \text{[on taking } x = sin\ \theta \text{]}$$

$$= \frac{1}{2^{n-1}} \cdot \frac{\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{2n+3}{2}\right)} \qquad \text{[By Gamma Formula]}$$

$$= \frac{1}{2^n} \cdot \frac{n!\sqrt{\pi}}{\frac{2n+1}{2}\cdot\frac{2n-1}{2}\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}} \qquad [\because \Gamma(n+1)=n]$$

$$= \frac{n!}{2^n}\cdot\frac{2^{n+1}}{(2n+1)(2n-1)........3.1.}$$

$$= (2n!)\frac{(2n)(2n-2).......4.2.}{(2n+1)(2n)(2n-1)(2n-2).......4.3.2.1.}$$

$$= (2n!)\frac{(2n)[2.(n-1)]...(2.2)(2.1)}{(2n+1)!}$$

$$= (2n!)\frac{2^n(n)!}{(2n)!} = \frac{2^{n+1}(n!)^2}{(2n+1)!}$$ **Hence Proved.**

Example 12 Prove that :

$$\int_{-1}^{1} P_n(x)dx = \begin{cases} 2, & n=0 \\ 0, & n \in N \end{cases}$$

Sol. Case I. when $n = 0$

We know that $P_0(x) = 1$

Therefore, $\int_{-1}^{1} P_0(x)dx = \int_{-1}^{1} 1.dx = [x]_{-1}^{1} = 2$

Case II. when $n \in N$: By Rodrigues formula

$$P_n(x) = \frac{1}{2^n(n!)}D^n\left(x^2-1\right), \qquad D \equiv d/dx$$

Integrating both sides w.r.t. x between the limits -1 and 1,

$$\int_{-1}^{1} P_n(x)dx = \frac{1}{2^n(n!)} \int_{-1}^{1} D^n.(x^2-1)^n dx$$

$$= \frac{1}{2^n(n!)} \left\{ D^{n-1}(x^2-1)^n \right\}_{-1}^{1} \tag{1}$$

Now $D^{n-1}\left\{(x^2-1)^n\right\} = D^{n-1}\left\{(x+1)^n(x-1)^n\right\}$

Using Leibnitz theorem in RHS.

$$= (x+1)^n D^{n-1}\left\{(x-1)^n\right\} + (n-1)n.(x+1)^{n-1} D^{n-2}(x-1)^n + \ldots$$

$$+ (x-1)^n D^{n-1}\left\{(x+1)^n\right\}$$

$$= (x+1)^n \frac{(n)!}{1!}(x-1) + n(n-1)(x+1)^{n-1}.\frac{(n)!}{2!}(x-1)^2 + \ldots$$

$$+ (x-1)^n (n)!(x+1) \tag{2}$$

$(x-1)$ and $(x+1)$ exist in each terms of (2), therefore for $x = -1$ or 1, the RHS of $(2) = 0$.

Therefore $\left\{ D^{n-1}(x^2-1)^n \right\}_{-1}^{1} = 0$

Hence by (1), $\int_{-1}^{1} P_n(x)dx = 0$.

14.10 Orthogonal Property for Legendre Polynomial

The Legendre polynomial $P_n(x)$ satisfy the following orthogonal property

$$\int_{-1}^{1} P_m(x).P_n(x)dx = \begin{cases} 0 & , m \neq n \\ \dfrac{2}{2n+1} & , m = n \end{cases}$$

$$\boxed{\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn}}$$

Proof : Case I $(m \neq n)$

Let the Legendre polynomials $P_m(x)$ and $P_n(x)$ satisfy the differential equations

$$(1-x^2)P_m'' - 2xP_m' + m(m+1)P_m = 0 \tag{1}$$

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0 \tag{2}$$

Multiplying (1) by $P_n(x)$ and (2) by $P_m(x)$ and subtracting we obtain

$$\left(1-x^2\right)\left[P_m''P_n - P_n''P_m\right] - 2x\left[P_m'P_n - P_n'P_m\right] + \{m(m+1) - n(n+1)\}P_m \cdot P_n = 0$$

Combining the first two terms, we obtain

$$\frac{d}{dx}\left[\left(1-x^2\right)\left(P_m'P_n - P_n'P_m\right)\right] + m(m+1) - n(n+1)P_mP_n = 0$$

$$\Rightarrow \qquad (m-n)\,(m+n+1)P_mP_n = \frac{d}{dx}\left[\left(1-x^2\right)\left(P_m'P_n - P_n'P_m\right)\right]$$

Integrating from -1 to 1 we obtain

$$(m-1)\,(m+n+1)\int_{-1}^{1} P_m(x).\,P_n(x)dx = \left[\left(1-x^2\right)\left(P_m' - P_n - P_n'P_m\right)\right]_{-1}^{1} = 0$$

$$\Rightarrow \qquad \int_{-1}^{1} P_m(x).\,P_n(x)dx = 0$$

Case II $(m = n)$

This part can be proved using Rodrigue's formula or using generating function. Using the generating function we have

$$\left(1-2xt+t^2\right)^{-1/2} = \sum_{n=0}^{\infty} P_n(x).t^n \qquad (1)$$

Sparing both sides and then integrating w.r.t. x from -1 to 1 we obtain

$$\int_{-1}^{1}\frac{dx}{1-2xt+t^2} = \int_{-1}^{1}\left[\Sigma P_n(x)t^n\right]^2 dx \qquad (2)$$

Now $$\int_{-1}^{1}\frac{dx}{1-2xt+t^2} = \left[\frac{1\text{n}\left(1-2xt+t^2\right)}{-2t}\right]_{-1}^{1}$$

$$= -\frac{1}{2t}\left[1\text{n}\left(1-2t+t^2\right) - 1\text{n}\left(1-2t+t2^2\right)\right]$$

$$= -\frac{1}{2t}\left[1\text{n}\,(1-t)^2 - 1\text{n}\,(1+t)^2\right]$$

$$= \frac{1}{t}\left[1\text{n}\,(1-t) - 1\text{n}\,(1+t)\right]$$

$$= \frac{1}{t}\left[1\text{n}\,(1+t) - 1\text{n}\,(1-t)\right]$$

$$=\frac{1}{t}\left[\left(t-\frac{t^2}{2}+\frac{t^3}{3}\right)-...\right]-\left(-t-\frac{t^2}{2}-\frac{t^3}{3}-...\right)$$

$$=2\left[1+\frac{t^2}{3}+\frac{t^4}{5}+...+\frac{t^{2n}}{2n+1}+...\right] \qquad (3)$$

Also $\int_{-1}^{1}\left[\Sigma P_n(x).t^n\right]^2 dx=\int_{-1}^{1}\left[\Sigma P_n(x).t^n\right]\left[\Sigma P_n(x).t^n\right]dx$

$$=\sum_{n=0}^{\infty}\int_{-1}^{1}P_n^2(x).t^{2n}dx \qquad \text{[using the 1}^{\text{st}}\text{ part]}$$

$$=\sum_{n=0}^{\infty}t^{2n}\int_{-1}^{1}P_n^2(x)dx \qquad (4)$$

From (2) using (3) and (4) we obtain

$$2\left[1+\frac{t^2}{3}+\frac{t^4}{5}+...+\frac{t^{2n}}{2n+1}+...\right]=\sum_{n=0}^{\infty}t^{2n}\int_{-1}^{1}P_n^2(x)dx$$

Equating the coefficients of t^{2n} from both sides we obtain

$$\int_{-1}^{1}P_n^2(x)dx=\frac{2}{2n+1}$$

Example 13 Prove that

$$\int_{-1}^{1}\frac{P_n(x)}{\sqrt{(1-2xh+h^2)}}dx=\frac{2h^n}{2n+1}$$

Sol. We have

$$\int_{-1}^{1}\frac{P_n(x)}{\sqrt{(1-2xh+h^2)}}dx=\int_{-1}^{1}P_n(x)\left[\sum_{m=1}^{\infty}P_m(x)h^m\right]dx$$

$$=\int_{-1}^{1}P_n(x).P_n(x)h^n dx \quad \text{[Other terms vanish due to orthogonality]}$$

$$=h^n\int_{-1}^{1}P_n^2(x)dx=\frac{2h^n}{2n+1}$$

Example 14 Show that $(2n+1)(x^2-1)P_n' = n(n+1)(P_{n+1}-P_{n-1})$ and hence prove that

$$\int_{-1}^{1}(x^2-1)P_{n+1}P_n'\,dx = \frac{2n(n+1)}{(2n+1)(2n+3)}$$

Sol. We know that

$$(1-x^2)P_n' = n(P_{n-1}-xP_n) \qquad \text{[Recurrence relations]} \qquad (1)$$

and $$(1-x^2)P_n' = (n+1)(xP_n - P_{n+1}) \qquad (2)$$

multiplying (1) by $(n+1)$ and (2) by n and then on adding we get

$$(n+1)(1-x^2)P_n' + n(1-x^2)P_n' = n(n+1)P_{n-1} - n(n+1)P_{n+1}$$

or $$(2n+1)(1-x^2)P_n' + n(n+1)(P_{n-1}-P_{n+1})$$

or $$(2n+1)(x^2-1)P_n' + n(n+1)(P_{n+1}-P_{n-1})$$

or $$(x^2-1)P_n' = \frac{n(n+1)}{2n+1}(P_{n+1}-P_{n-1})$$

Multiplying both sides by P_{n+1} and integrating w.r.t. x from -1 to 1, we have

$$\int_{-1}^{1}(x^2-1)P_n'P_{n+1}\,dx = \frac{n(n+1)}{(2n+1)}\int_{-1}^{1}P_{n+1}(P_{n+1}-P_{n-1})dx$$

$$= \frac{n(n+1)}{(2n+1)}\left[\int_{-1}^{1}P_{n+1}^2dx - \int_{-1}^{1}P_{n+1}P_{n-1}dx\right]$$

$$= \frac{n(n+1)}{(2n+1)}\left[\frac{2}{2(n+1)+1} - 0\right]$$

$$\left[\because \int_{-1}^{1}P_mP_ndx = 0 \text{ if } m \neq n\right]$$

14.11 Summary

In this unit Legendre polynomial & its properties are discussed. Solution of Legendre differential equation is explained. Generating function & Rodrigue formula for obtaining Legendre polynomial are discussed. Recurrence relations & orthogonal property of Legendre polynomial are explained.

14.12 Glossary

Polynomial : An expression in integer power of variable.

Differential equation : An equation involving independent variable, dependent variable & its derivatives.

Generating function : A function when expanded gives values in coefficients of powers of variables.

Recurrence relation : A relation by which we can find next value by using previous value.

Orthogonality :Is a property in which curves intersect each other at right angle.

14.13 Answer to Self Learning Exercise

Ans.1: $\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0$

Ans.2: $\left(1-2xt+t^2\right)^{-1/2}$

Ans.3: $\frac{1}{3}\left[6P_3(x)-2P_2(x)+18P_1(x)-13P_0(x)\right]$

14.14 Exercise

Section A (Very Short Answer type Questions)

Q.1 State Rodrigue's formula for Legendre polynomial.

Q.2 Write the value of $P_2(x)$

Q.3 Write the formula for Legendre polynomial

Q.4 Write Legendre differential equation.

Q.5 State orthogonal property for Legendre polynomial.

Section B (Short Answer type Questions)

Q.6 Using Rodrigue's formula. Evaluate $P_5(x)$

Expend the polynomial in a series of Legendre polynomial (Q. 7-8)

Q.7 $x^4+3x^3-x^2+5x-2$

Q.8 $1+x-x^2$

Q.9 Prove that $$\frac{(1-t^2)}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)P_n(x)t^n$$

Q.10 Prove that $$\int_{-1}^{1} x^2 P_{n+1}P_{n-1}\,dx = \frac{2n(n+1)}{(4x^2-1)(2n+3)}$$

Section C (Long Answer type Questions)

Q.11 State & Prove Rodrigue's formula.

Q.12 Prove that

(i) $(n+1)P_{n+1} = (2n+1)x\,P_n - x\,P_{n-1}$

(ii) $(2n+1)P_n = P'_{n+1} - P'_{n-1}$

Q.13 Prove that

(i) $(2n+1)(x^2-1)P'_n = n(n+1)(P_{n+1} - P_{n-1})$

(ii) $n\,P_n = x\,P'_n - P'_{n-1}$

Q.14 State & prove Orthogonal property for Legendre polynomial.

Q.15 Prove that

(i) $$\int_{-1}^{1}(1-x^2)P'_m P'_n\,dx = \begin{cases} 0 & ,\ m \neq n, \quad m,n \in N \\ \dfrac{2n(n+1)}{(2n+1)} & m = n \end{cases}$$

(ii) $$\int_{-1}^{1}(x^2-1)P_{n+1}P'_n\,dx = \frac{2n(n+1)}{(2n+1)(2n+3)}$$

14.15 Answers to Exercise

Ans.1 : $P_n(x) = \dfrac{1}{n!\,2^n}\dfrac{d^n}{dx^n}(x^2-1)^n$

Ans.2 : $P_2(x) = \dfrac{3x^2-1}{2}$

Ans.3 : $P_n(x) = \displaystyle\sum_{r=0}^{N}\frac{(-1)^r(2n-2r)!}{r!\,2^n(n-2r)!(n-r)!}\,x^{n-2r}$ $\quad N = \begin{cases} \dfrac{n}{2} & \text{if is even} \\ \dfrac{n-1}{2} & \text{if is odd} \end{cases}$

Ans.4 : $\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$

Ans.5 : $\int_{-1}^{1} P_n(x)P_m(x)dx = \begin{cases} 0 & , m \neq n \\ \frac{2}{2n+1} & , m = n \end{cases}$

Ans.6 : $\frac{1}{8}\left(63x^5-70x^3+15\right)$

Ans.7 : $\frac{8}{35}P_4(x)+\frac{6}{5}P_3(x)-\frac{2}{21}P_2(x)+\frac{34}{5}P_1(x)-\frac{224}{105}P_0(x)$

Ans.8 : $\frac{2}{3}P_0(x)+P_1(x)-\frac{2}{3}P_2(x)$

References and Suggested Readings

1. Mathematical Physics with Classical Mechanics, Satya Prakash, Sultan Chand & Sons.

2. Advance Engineering Mathematics, Erwin Kreyszing, Wiley student edition

3. Mathematical Physics by H.K. Das, 1997.

UNIT -15

Hermite functions, Hermite Polynomials

Structure of the Unit

15.0 Objectives

In this unit we are going to discuss about another type of polynomial i.e. Hermite polynomials and its properties.

After going through this unit you will be able to learn

- Hermite differential equation
- Rodrigues' formula for Hermite polynomial $H_n\ (x)$ and Recurrence Relations

- Generating function for the $H_n(x)$
- Orthogonality relation for Hermite polynomials
- The Harmonic oscillator energy Eigenfunctions

15.1 Introduction

In the treatment of the harmonic oscillator in quantum mechanics, Hermite's differential equation arises and which is defined as

$$\boxed{y'' - 2xy' + 2ny = 0} \qquad (1)$$

Where $y'' = \frac{d^2y}{dx^2}$, $y' = \frac{dy}{dx}$. n is a real number. For n is a non-negative integer, i.e. n = 0,1,2,3,..., the classical set of solutions of Hermite's differential equation are often referred to as ***Hermite Polynomials*** $H_n(x)$**.** These polynomials are useful in solving physical problems using algebraic and analytic methods. Hermite polynomials are defined by

$$\boxed{H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}} \qquad (2)$$

Let $y = \sum_0^\infty a_r x^{m+r}$ (3)

$$y' = \sum (m+r) a_r x^{m+r-1} \qquad (4)$$

$$y'' = \sum (m+r)(m+r-1) a_r x^{m+r-2} \qquad (5)$$

Putting equation (2)-(4) in (1) we get

$$\sum (m+r)(m+r-1) a_r x^{m+r-2} - 2\sum (m+r) a_r x^{m+r} + 2n \sum a_r x^{m+r} = 0$$

$$\sum (m+r)(m+r-1) a_r x^{m+r-2} + \sum (2n-2m-2r) a_r x^{m+r} = 0$$

Comparing coefficient of x^{m-2} , m(m-2)a_0 = 0

Since $a_0 \neq 0$, m = 0 or 1

Comparing coefficient of x^{m-1} , m(m+1)a_1 = 0 or a_1 = 0

Finally, comparing coefficient of x^{m+r},

(m+r+2)(m+r+1) a_{r+2}+2(n-m-r) a_r = 0

$$\therefore a_{r+2} = -\frac{2(n-m-r)}{(m+r+2)(m+r+1)} a_r$$

Case I: *m*=0.

$$\therefore a_{r+2} = \frac{2(r-n)}{(r+2)(r+1)} a_r$$

$$\therefore a_2 = -\frac{2n}{2!} a_0 ;$$

$$a_4 = \frac{2(2-n)}{4.3}\left(\frac{-2n}{2!} a_0\right) = \frac{2^2 n(n-2)}{4!} a_0 ;$$

$$a_6 = \frac{2(4-n)n(n-2)}{6.5.6} a_0$$

$$= -\frac{2^3 n(n-2)(n-4)}{6!} a_0$$

Now, $y = \sum a_r x^r = a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \cdots$

$$\therefore y = a_0[1 - \frac{2n}{2!} x^2 + \frac{2^2 n(n-2)}{4!} x^4 - \frac{2^3 n(n-2)(n-4)}{6!} x^6 + \cdots]$$

Case II : *m* =1.

$$a_{r+2} = \frac{2(1+r-n)}{(r+3)(r+2)} a_r$$

$$\therefore a_2 = \frac{2(1-n)}{6} a_0 = -\frac{2(n-1)}{6} a_0 ;$$

$$a_4 = \frac{2(3-n)}{4.5}\frac{2(1-n)}{6} a_0 = \frac{2^2 (n-1)(n-3)}{5!} a_0$$

$$\therefore y = a_0 x[1 - \frac{2(n-1)}{3!} x^2 + \frac{2^2(n-1)(n-3)}{5!} x^4 + \cdots]$$

Hence we can write the solution for m = 0 and m =1 in more general form

$$y = a_0[1 - \frac{2n}{2!} x^2 + \frac{2^2 n(n-2)}{4!} x^4 - \frac{2^3 n(n-2)(n-4)}{6!} x^6 + \cdots + \frac{(-2)^r n(n-2)(n-4)\ldots(n-2r+2)}{2r!} x^{2r} + \cdots] \tag{6}$$

$$y = a_0 x[1 - \frac{2(n-1)}{3!} x^2 + \frac{2^2(n-1)(n-3)}{5!} x^4 + \cdots + \frac{(-2)^r (n-1)(n-3)\ldots(n-2r+1)}{(2r+1)!} x^{2r} + \cdots] \tag{7}$$

and the general solution of Hermite equation is the superposition of (6) and (7) .

15.2 Rodrigues' formula for Hermite Polynomial $H_n(x)$

The Hermite polynomials are given by the formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

If we put $n = 0,1,2, \ldots$ in

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^2 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

Proof : Let $p = e^{-x^2}$, then

$D\text{p} + 2x\text{p} = 0,$ Where $D = \frac{d}{dx}$

Differentiating (n+1) times by the Leibnitz' rule, we get

$$D^{n+2}\text{p} + 2xD^{n+1}p + 2(n+1)D^n\text{p} = 0$$

Writing $y = (-1)^n D^n p$

$$D^2\text{y} + 2xDy + 2(n+1)y = 0 \tag{8}$$

Substitute $u = e^{x^2}\text{y}$ then

$Du = e^{x^2}(2xy + Dy)$And

$$\text{D}^2\text{u} = \text{e}^{\text{x}^2}(\text{D}^2\text{y} + 4\text{xDy} + 4\text{x}^2\text{y} + 2\text{y})$$

Hence by equation (8), we get

$$D^2\text{u} - 2xDu + 2nu = 0$$

Which indicates that

$$u = (-1)^n\, e^{x^2}\, D^n e^{-x^2}$$

is a polynomial solution of Hermite equation.

15.3 Recurrence Relations

Hermite polynomials satisfy the following relations :-

1. $H_n'(x) = 2xH_n(x) - H_{n+1}(x)$

Proof: We know the Rodrigues' formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

Differentiating this with respect to , we get

$$H_n'(x) = (-1)^n 2x\, e^{x^2} \frac{d^n}{dx^n} e^{-x^2} + (-1)^n e^{x^2} \frac{d^{n+1}}{dx^{n+1}} e^{-x^2} \tag{9}$$

Which gives us

$$H_n'(x) = 2xH_n(x) - H_{n+1}(x) \tag{10}$$

2. $H_n''(x) = 2H_n(x) + 2xH_n'(x) - H_{n+1}'(x)$

Proof: Again differentiating equation (9) with respect to x , we get

$$\begin{aligned} H_n''(x) = {} & 2(-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} + (-1)^n 4x^2 e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \\ & + (-1)^n 2x\, e^{x^2} \frac{d^{n+1}}{dx^{n+1}} e^{-x^2} + (-1)^n 2x\, e^{x^2} \frac{d^{n+1}}{dx^{n+1}} e^{-x^2} \\ & + (-1)^n e^{x^2} \frac{d^{n+2}}{dx^{n+2}} e^{-x^2} \end{aligned}$$

From this, we get

$$H_n''(x) = 2H_n(x) + 2xH_n'(x) - H_{n+1}'(x) \tag{11}$$

3. $H_{n+1}'(x) = 2(n+1)H_n(x)$

Proof: We know that Hermite function $H_n(x)$ satisfies the Hermite's equation

$$H_n''(x) - 2xH_n'(x) + 2n)H_n(x) = 0 \tag{12}$$

Putting the value of $H_n''(x)$ from equation (11) to equation (12), we get

$$2H_n(x) + 2xH_n'(x) - H_{n+1}'(x) - 2xH_n'(x) + 2nH_n'(x) = 0$$

Which gives us the result

$$H_{n+1}'(x) = 2(n+1)H_n(x) \tag{13}$$

4. $H_{n+2}(x) = 2xH_{n+1}(x) - 2(n+1)H_n(x)$

Proof: Replacing n by n+1 in equation (10) we get

$$H'_{n+1}(x) = 2xH_{n+1}(x) - H_{n+2}(x)$$

Putting the value of $H'_{n+1}(x)$ from equation (13), we get

$$2(n+1)H_n(x) = 2xH_{n+1}(x) - H_{n+2}(x)$$

Then we get the following result

$$H_{n+2}(x) = 2xH_{n+1}(x) - 2(n+1)H_n(x)$$

From this relation, we can get the higher polynomials.

15.4 Generating function for the $H_n(x)$

The generating function for the Hermite polynomial is given by

$$g(x,t) = e^{2xt-t^2} = e^{\{x^2-(t-x)^2\}} = \sum_{n=0}^{n=\infty} \frac{t^n H_n(x)}{n!}$$

Differentiating $g(x,t) = e^{x^2-(t-x)^2}$ with respect to t

$$\frac{\partial g}{\partial t} = (2x-2t)g$$

$$\frac{\partial g}{\partial t}\Big|_{t=0} = 2x = H_1(x) = -e^{x^2}\frac{d}{dx}e^{-x^2}$$

Where $H_1(x)$ is defined by the Rodrigues' formula.

Now to establish the general term

$$(-1)^{n+1}e^{x^2}\frac{d^{n+1}}{dx^{n+1}}e^{-x^2} = (-1)^{n+1}e^{x^2}\frac{d}{dx}\frac{d^n}{dx^n}e^{-x^2}$$

$$= (-1)^{n+1}\left[\frac{d}{dx}e^{x^2} - 2xe^{x^2}\right]\frac{d^n}{dx^n}e^{-x^2}$$

$$= -\frac{d}{dx}H_n(x) + 2xH_n(x)$$

Using recurrence relation, we get

$$(-1)^{n+1}e^{x^2}\frac{d^{n+1}}{dx^{n+1}}e^{-x^2} = -\frac{d}{dx}H_n(x) + 2xH_n(x)$$

$$= H_{n+1}(x)$$

In general, n^{th} differential of function $g(x,t)$ at t = 0

$$H_n(x) = \frac{\partial^n (e^{-t^2+2tx})}{\partial t^n}_{t=0}$$

$$= (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

So the function $g(x,t)$ is known as the ***generating function*** of Hermite polynomials. Using the generating function, we can also prove the recurrence reation:

$$H_n'(x) = 2nH_{n-1}(x)$$

Proof: We know that

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n H_n(x)}{n!}$$

Differentiating above equation with respect to x, we get

$$2t\, e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n H_n{}'(x)}{n!}$$

or
$$2t \sum_{n=0}^{\infty} \frac{t^n H_n(x)}{n!} = \sum_{n=0}^{\infty} \frac{t^n H_n{}'(x)}{n!}$$

Now equating the coefficient of t^n , it follows that

$$H_n'(x) = 2nH_{n-1}(x)$$

And
$$H_n''(x) = 2nH'_{n-1}(x)$$

15.5 Illustrative Examples

Example 15.1 Calculate the first three Hermite polynomials by using the generating function.

Sol. We know that generating function for Hermite polynomial is given by

$$g(x,t) = e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n H_n(x)}{n!}$$

Expanding the left-hand side of the above equation and combining equal power of t yields

$$1 + 2xt + (2x^2 - 1)t^2 + \text{higher degree}$$

Comparing the coefficients of powers of t in both sides, we get

$H_0(x) = 1,$

$H_1(x) = 2x,$

$H_2(x) = 4x^2 - 2$

These are the first three Hermite polynomials calculated by using the generating function.

Example15.2 Calculate the third Hermite polynomial by using the recurrence relation.

Sol. We know that recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

Setting n=1 , we get

$$H_2(x) = 2xH_1(x) - 2H_0(x)$$

But we know that

$H_0(x) = 1,$

$H_1(x) = 2x$

Putting the values of $H_0(x)$ and $H_1(x)$, we get

$$H_2(x) = 2x.2x - 2$$
$$= 4x^2 - 2$$

The same result for the third Hermite polynomial can also be obtained by using the generating function.

15.6 Orthogonality Relation for Hermite Polynomials

We know that Hermite function satisfies the Hermite differential equation

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$$

This equation can be written as

$$e^{x^2}\frac{d}{dx}(e^{-x^2}H_n'(x)) + 2nH_n(x) = 0 \quad (14)$$

Changing the index n by m, we can write

$$e^{x^2}\frac{d}{dx}(e^{-x^2}H_m'(x)) + 2mH_m(x) = 0 \quad (15)$$

Multiplying equation (15) by $H_n(x)$ and equation (14) by $H_m(x)$, we get the following equations

$$H_n(x)e^{x^2}\frac{d}{dx}(e^{-x^2}H_m'(x)) + 2mH_n(x)H_m(x) = 0 \tag{16}$$

And

$$H_m(x)e^{x^2}\frac{d}{dx}(e^{-x^2}H_n'(x)) + 2nH_m(x)H_n(x) = 0 \tag{17}$$

Subtracting equation (16) and (17), we get

$$2(m-n)e^{-x^2}H_m(x)H_n(x) = H_m(x)e^{x^2}\frac{d}{dx}\left(e^{-x^2}H_n'(x)\right) - H_n(x)e^{x^2}\frac{d}{dx}\left(e^{-x^2}H_m'(x)\right)$$

Integrating the above equation from $-\infty$ to ∞, we get

$$2(m-n)\int_{-\infty}^{\infty} e^{-x^2}H_m(x)H_n(x)\,dx$$

$$= \int_{-\infty}^{\infty}\frac{d}{dx}\left(e^{-x^2}H_m(x)H_n'(x) - H_n(x)H_m'(x)\right)dx = 0$$

Which gives

$$\boxed{\int_{-\infty}^{\infty} e^{(-x^2)}H_m(x)H_n(x)dx = 0 \quad for\, m \neq n}$$

This is known as the ***orthogonality relation of Hermite polynomials***.

Let us define

$$I_{m,n} = \int_{-\infty}^{\infty} e^{-x^2}H_m(x)H_n(x)\,dx$$

Then

$$I_{n-1,n+1} = \int_{-\infty}^{\infty} e^{-x^2}H_{n-1}(x)H_{n+1}(x)\,dx$$

We know the recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}H_{n-1}(x)(2xH_n(x) - 2nH_{n-1}(x))dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} 2xe^{-x^2}H_n(x)H_{n-1}(x)dx = 2nI_{n-1,n-1}(x)$$

Putting the value of Hermite polynomial $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$, we get

$$-\int_{-\infty}^{\infty} 2xe^{x^2}\frac{d^n}{dx^n}e^{-x^2}\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}dx = 2nI_{n-1,n-1}(x)$$

$$\Rightarrow -\int_{-\infty}^{\infty} e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \frac{d^n}{dx^n} e^{-x^2} dx - \int_{-\infty}^{\infty} \frac{d^n}{dx^n} e^{-x^2} \frac{d}{dx}\left[e^{x^2} \frac{d^{n-1}}{dx^{n-1}} e^{-x^2}\right] = 2nI_{n-1,n-1}(x)$$

Which gives us

$$I_{n,n}(x) = 2^n n!\, I_{0,0}$$

Where

$$I_{0,0} = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Then

$$I_{n,n}(x) = 2^n n!\,\sqrt{\pi}$$

So the ***orthonormality relation*** can be written as

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = 2^n n! \sqrt{\pi}\, \delta_{mn}}$$

15.7 Recurrence relation for the Harmonic Oscillator Energy Eigenfunctions

The harmonic oscillator energy eigenfunctions $\psi_n(x)$ satisfy the recurrence relation

$$\alpha x \psi_n(x) = \sqrt{\frac{n+1}{2}}\psi_{n+1}(x) + \sqrt{\frac{n}{2}}\psi_{n-1}(x) \quad .... \qquad (18)$$

Where $\alpha = \sqrt{\frac{m\omega}{\hbar}}$.

We can prove this in following manner

The energy eigenfunction for harmonic oscillator is defined as

$$\boxed{\psi_n(x) = N_n H_n(\alpha x) e^{-\alpha^2 x^2/2}}$$

$$n = 0,1,2, ...$$

The recurrence relation for the Hermite polynomial is

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

Then

$$\alpha x\psi_n(x) = \alpha x N_n H_n(\alpha x)e^{-\alpha^2x^2/2}$$

$$= N_n\left[\tfrac{1}{2}H_{n+1}(\alpha x) + nH_{n-1}(\alpha x)\right]e^{-\alpha^2x^2/2}$$

$$= \frac{N_n}{2N_{n+1}}N_{n+1}H_{n+1}(\alpha x)e^{-\alpha^2x^2/2} + \frac{nN_n}{N_{n-1}}N_{n-1}H_{n-1}(\alpha x)e^{-\alpha^2x^2/2}$$

Which gives us

$$\alpha x\psi_n(x) = \sqrt{\frac{n+1}{2}}\psi_{n+1}(x) + \sqrt{\frac{n}{2}}\psi_{n-1}(x)$$

15.8 Illustrative Examples

Example 15.3 Using the recurrence relation calculate $\psi_3(x)$.

Sol. The recurrence relation for the energy eigenfunction for harmonic oscillator is

$$\psi_{n+1}(x) = \sqrt{\tfrac{2}{n+1}}\left[\alpha x\psi_n(x) - \sqrt{\tfrac{n}{2}}\psi_{n-1}(x)\right]$$

Setting n = 2, we get

$$\psi_3(x) = \sqrt{\tfrac{2}{3}}[\alpha x\psi_2(x) - \psi_1(x)]$$

$$= \sqrt{\tfrac{2}{3}}\left[\alpha x\left(\alpha x\psi_1(x) - \sqrt{\tfrac{1}{2}}\psi_0(x)\right) - \psi_1(x)\right]$$

$$= \sqrt{\tfrac{2}{3}}(\alpha^2x^2 - 1)\psi_1(x) - \tfrac{\alpha x}{\sqrt{3}}\psi_0(x)$$

By knowing the wave functions $\psi_0(x)$ and $\psi_1(x)$ we can the value of ψ_3.

Example15.4 Prove that

$$H_{2n}(0) = (-1)^n\frac{(2n)!}{n!} \quad ; H_{2n+1}(0) = 0$$

Sol. Using the Generating function ,we have

$$\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^4 = e^{2tx-t^2} \tag{1}$$

Replacing *x* by 0 in equation(1),We have

$$\sum_{n=0}^{\infty}\frac{H_n(0)}{n!}t^4 = e^{-t^2} = \sum_{n=0}^{\infty}\frac{(-t^2)^n}{n!}$$

Or $$\sum_{n=0}^{\infty}\frac{H_n(0)}{n!}t^4 = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^{2n} \tag{2}$$

Equating coefficients of t^{2n} on both sides of (2) ,we have

$$\frac{H_{2n}(0)}{\lfloor 2n} = \frac{(-1)^n}{\lfloor n}$$

Or $$\boxed{H_{2n}(0) = \frac{(-1)^n \lfloor 2n}{\lfloor n}}$$

The right hand side of equation (2) does not contain odd powers of t .Then equating coefficients of t^{2n+1} on the both side of equation (2) gives

$$\frac{H_{2n+1}(0)}{\lfloor 2n+1} = 0$$

So $\boxed{H_{2n+1}(0) = 0}$

Example 15.5 Express $H(x) = x^4 + 2x^3 + 2x^2 - x - 3$ in terms of Hermite's polynomials.

Sol. We know that

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

And $H_4(x) = 16x^4 - 48x^2 + 12$

From these,we have

$$x^4 = \frac{1}{16}H_4(x) + 3x^2 - \frac{3}{4} \tag{1}$$

$$x^3 = \frac{1}{8}H_3(x) + \frac{3x}{2} \tag{2}$$

$$x^2 = \frac{1}{4}H_2(x) + \frac{1}{2} \tag{3}$$

$$x = \frac{1}{2} H_1(x) \qquad (4)$$

$$1 = H_0(x) \qquad (5)$$

$$\therefore H(x) = \frac{1}{16} H_4(x) + 3x^2 - \frac{3}{4} + 2x^2 - x - 3 \quad \text{by(1)}$$

$$= \frac{1}{16} H_4(x) + 2x^3 + 5x^2 - x - \frac{15}{4}$$

By using(2)

$$H(x) = \frac{1}{16} H_4(x) + 2\left[\frac{1}{8} H_3(x) + \frac{3}{2} x\right] + 5x^2 - x - \frac{15}{4}$$

$$= \frac{1}{16} H_4(x) + \frac{1}{4} H_3(x) + 5x^2 + 2x - \frac{15}{4}$$

By using(3)

$$H(x) = \frac{1}{16} H_4(x) + \frac{1}{4} H_3(x) + 5\left[\frac{1}{4} H_2(x) + \frac{1}{2}\right] + 2x - \frac{15}{4}$$

$$= \frac{1}{16} H_4(x) + \frac{1}{4} H_3(x) + \frac{5}{4} H_2(x) + 2x - \frac{5}{4}$$

By using(4)

$$H(x) = \frac{1}{16} H_4(x) + \frac{1}{4} H_3(x) + \frac{5}{4} H_2(x) + H_1(x) - \frac{5}{4} H_0(x)$$

The above equation represents the expression of $H(x)$ in terms of Hermite's polynomials.

Example 15.6 Express $H(x) = 5x^2 + 2x$ in terms of Hermite's polynomials.

Sol. We know that

$$H_0(x) = 1, \; H_1(x) = 2x \text{ and } H_2(x) = 4x^2 - 2$$

From these we have

$$x^2 = \frac{1}{4} H_2(x) + \frac{1}{2}$$

$$x = \frac{1}{2} H_1(x) \text{ then}$$

$$H(x) = 5x^2 + 2x$$

$$H(x) = 5\left(\frac{1}{4}H_2(x) + \frac{1}{2}\right) + 2.\frac{1}{2}H_1(x)$$

$$= \frac{5}{4}H_2(x) + \frac{5}{2} + H_1(x)$$

$$= \frac{5}{4}H_2(x) + H_1(x) + \frac{5}{2}$$

Example 15.7 Prove that

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} \left[H_n(x)\right]^2 dx = \left(\sqrt{\pi}\right) 2^n \lfloor n \left(n + \frac{1}{2}\right)$$

Sol. From the recurrence relation ,we know that

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

Or $xH_n(x) = nH_{n-1}(x) + \frac{1}{2}H_{n+1}(x)$ (1)

Or $x^2H_n(x) = nxH_{n-1}(x) + \frac{x}{2}H_{n+1}(x)$ (2)

Replacing n by $n-1$ and $n+1$ successively in eq(1).

We get

$$xH_{n-1}(x) = (n-1)H_{n-2}(x) + \frac{1}{2}H_n(x) \quad (3)$$

and $xH_{n+1}(x) = (n+1)H_n(x) + \frac{1}{2}H_{n+2}(x)$ (4)

Using(3) and (4),(2) becomes

$$x^2H_n(x) = n\left[(n-1)H_{n-2}(x) + \frac{1}{2}H_n(x)\right] + \frac{1}{2}\left[(n+1)H_n(x) + \frac{1}{2}H_{n+2}(x)\right]$$

$$= n(n-1)H_{n-2}(x) + \frac{1}{4}H_{n+2}(x) + \left(n + \frac{1}{2}\right)H_n(x) \quad (5)$$

Multiplying both sides of (5) by $e^{-x^2}H_n(x)$ and then integrating with respect to x from $-\infty$ to ∞ ,we get

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} \{H_n(x)\}^2 dx = n(n-1)\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_{n-2}(x)dx$$

$$+\frac{1}{4}\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_{n+2}(x)dx + \left(n+\frac{1}{2}\right)\int_{-\infty}^{\infty} e^{-x^2} \{H_n(x)\}^2 dx$$

$$= 0 + 0 + \left(n+\frac{1}{2}\right)\sqrt{\pi}\ 2^n \lfloor n$$

$$\because \int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x)dx = \left(\sqrt{\pi}\right) 2^n \lfloor n\ \delta_{mn}$$

$$\therefore \int_{-\infty}^{\infty} x^2 e^{-x^2} \left[H_n(x)\right]^2 dx = \left(n+\frac{1}{2}\right)\left(\sqrt{\pi}\right) 2^n \lfloor n$$

Example 15.8 Show that

$$\sum_{k=0}^{n} \frac{H_k(x)H_k(y)}{2^k \lfloor k} = \frac{H_{n+1}(y)H_n(x) - H_{n+1}(x)H_n(y)}{2^{n+1}\lfloor n (y-x)}$$

Sol. From the recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

Or $xH_n(x) = nH_{n-1}(x) + \frac{1}{2}H_{n+1}(x)$ (1)

Replacing x by y in equation(1),we get

$$yH_n(y) = nH_{n-1}(y) + \frac{1}{2}H_{n+1}(y) \tag{2}$$

Multiplying equation (2) by $H_n(x)$ and equation (1) by $H_n(y)$ then subtracting, we get

$$\begin{aligned}(y-x)H_n(x)H_n(y) = &\frac{1}{2}\left[H_{n+1}(y)H_n(x) - H_{n+1}(x)H_n(y)\right] \\ &- 2\left[H_{n-1}(x)H_n(y) - H_{n-1}(y)H_n(x)\right]\end{aligned} \tag{3}$$

Putting $n = 0,1,2,...(n-1), n$ in eq(3),we have

$$(y-x)H_0(x)H_0(y) = \frac{1}{2}\left[H_1(y)H_0(x) - H_1(x)H_0(y)\right] - 0 \tag{4}$$

$$(y-x)H_1(x)H_1(y)=\frac{1}{2}\left[H_2(y)H_1(x)-H_2(x)H_1(y)\right] \\ -\left[H_0(x)H_1(y)-H_0(y)H_1(x)\right] \tag{5}$$

$$(y-x)H_2(x)H_2(y)=\frac{1}{2}\left[H_3(y)H_2(x)-H_3(x)H_2(y)\right] \\ -2\left[H_1(x)H_2(y)-H_1(y)H_2(x)\right] \tag{6}$$

$$(y-x)H_{n-1}(x)H_{n-1}(y)=\frac{1}{2}\left[H_n(y)H_{n-1}(x)-H_n(x)H_{n-1}(y)\right] \\ -2\left[H_{n-1}(x)H_{n-1}(y)-H_{n-2}(y)H_{n-1}(x)\right] \tag{7}$$

$$(y-x)H_n(x)H_n(y)=\frac{1}{2}\left[H_{n+1}(y)H_n(x)-H_{n+1}(x)H_n(y)\right] \\ -2\left[H_{n-1}(x)H_n(y)-H_{n-1}(y)H_n(x)\right] \tag{8}$$

Multiplying (4),(5),(6),(7),(8) by $1,\frac{1}{2.\lfloor\underline{1}},\frac{1}{2^2\lfloor\underline{2}.},\frac{1}{2^3.\lfloor\underline{3}},\ldots\ldots,\frac{1}{2^{n-1}\lfloor\underline{n-1}.},\frac{1}{2^n.\lfloor\underline{n}}$

Respectively and adding ,we get

$$(y-x)\sum_{k=0}^{n}\frac{H_k(x)H_k(y)}{2^k\lfloor\underline{k}}=\frac{H_{n+1}(y)H_n(x)-H_{n+1}(x)H_n(y)}{2^{n+1}\lfloor\underline{n}}$$

Or $$\sum_{k=0}^{n}\frac{H_k(x)H_k(y)}{2^k\lfloor\underline{k}}=\frac{H_{n+1}(y)H_n(x)-H_{n+1}(x)H_n(y)}{2^{n+1}\lfloor\underline{n}(y-x)}$$

Example 15.9 Prove that $H_n(-x)=(-1)^n H_n(x)$

Sol. We have $H_n(x)=\sum_{s=0}^{\frac{n}{2}}\frac{(-1)^s n!(2x)^{n-2s}}{(n-2s)!s!}$

$$H_n(-x)=\sum_{s=0}^{\frac{n}{2}}\frac{(-1)^s n!(-2x)^{n-2s}}{(n-2s)!s!}$$

$$=\sum_{s=0}^{\frac{n}{2}}\frac{(-1)^s(-1)^{n-2s} n!(2x)^{n-2s}}{(n-2s)!s!}$$

$$=(-1)^n\sum_{s=0}^{\frac{n}{2}}\frac{(-1)^s n!(2x)^{n-2s}}{(n-2s)!s!}$$

$$= (-1)^n H_n(x)$$

Thus $\boxed{H_n(x) = (-1)^n H_n(x)}$

Example 15.10 Prove that for $m < n$,

$$\frac{d^m}{dx^m} H_n(x) = \frac{2^m n!}{(n-m)!} H_{n-m}(x)$$

Sol.

$$\frac{d^m}{dx^m} H_n(x) = \frac{2^m n!}{(n-m)!} H_{n-m}(x)$$

After differentiation *m* times partially with respect to *x*, we have

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{d^m}{dx^m} H_n(x) = \frac{d^m}{dx^m}\left(e^{-z^2+2zx}\right)$$

$$= (2z)^m e^{-z^2+2zx}$$

$$= (2z)^m \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n$$

$$= (2)^m \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^{m+n}$$

Putting $m+n=s$ or $n=s-m$ on right side we get

$$\sum_{n=s}^{\infty} \frac{z^n}{n!} \frac{d^m}{dx^m} H_n(x) = 2^m \sum_{s=m}^{\infty} \frac{H_{s-m}(x)}{(s-m)!} z^s$$

Comparing the powers of z^n on either side, we get

$$\frac{1}{n!} \frac{d^m}{dx^m} H_n(x) = 2^m \frac{H_{n-m}(x)}{(n-m)!}$$

$$\Rightarrow \quad \frac{d^m}{dx^m} H_n(x) = 2^m \frac{n!}{(n-m)!} H_{n-m}(x)$$

Hence Proved

15.9 Self Learning Exercise

Q.1 Write one example from quantum mechanics where Hermite polynomials appear.

Q. 2 Write the value of Hermite polynomial $H_2(x)$.

Q. 3 Write the value of Hermite polynomial $H_1(0)$.

Q. 4 Write the value of Hermite polynomial $H_4(x)$.

Q.5 By using Rodrigues' formula for Hermite polynomials prove the following Recurrence relation

$$H_n''(x) = 2H_n(x) + 2xH_n'(x) - H_{n+1}'(x)$$

Q.6 By using Rodrigues' formula for Hermite polynomials prove the following Recurrence relation

(i) $H'_{n+1}(x) = 2(n+1)H_n(x)$

(ii) $H_{n+2}(x) = 2xH_{n+1}(x) - 2(n+1)H_n(x)$

15.10 Summary

We started this topic with the introduction of Hermite differential equation which appears in the treatment of Harmonic oscillator in quantum mechanics. We defined the recurrence relations for the Hermite polynomials by which we can get the higher degree polynomial. We also defined the generating function for the Hermite polynomial which can generate the Hermite polynomial of any degree. We ended up with application of recurrence relation for the harmonic oscillator energy eigenfunctions.

15.11 Glossary

Eigen : Proper; characteristic

Polynomial: An expression of more than two algebraic terms, especially the sum of several terms that contain different powers of the same variable(s).

Recurrence: the fact of happening again

Generate: to cause something to exist

Hermite: Charles Hermite (December 24, 1822 – January 14, 1901) was a French mathematician who did research on number theory, invariant theory, orthogonal polynomials, elliptic functions, and algebra.Hermite polynomials, Hermite interpolation, Hermite normal form, Hermitian operators are named in his honor

15.12 Answers of Self Learning Exercise

Ans. 1: In the treatment of Harmonic oscillator

Ans. 2: $4x^2 - 2$

Ans. 3: $H_1(x) = 2x$

$\Rightarrow H_1(0) = 2.0 = 0$

Ans. 4: $H_4(x) = 16x^4 - 48x^2 + 12$

15.13 Exercise

Section A-Very Short Answer Type Questions

Q.1 Write down the Rodrigues' formula for Hermite polynomial $H_n\ (x)$

Q.2 What do you mean by the ***generating function*** of Hermite polynomials.

Section B-Short Answer type Questions

Q.3 Using the recurrence relation calculate the Hermite polynomial $H_4(x)$.

Q.4 Show that for the energy eigenfunction of harmonic oscillator

$$\psi_5(x) = \sqrt{\frac{2}{5}}\left[\alpha x \psi_4(x) - \sqrt{2}\psi_3(x)\right]$$

Q.5 Using the recurrence relation calculate $H_3'(x)$.

Q.6 By using Rodrigues' formula for Hermite polynomials prove the following Recurrence relations

$$H_n'(x) = 2xH_n(x) - H_{n+1}(x)$$

Section C-Long Answer type Questions

Q.7 Show that the uncertainty relation

$$\boxed{\Delta x . \Delta p = (n + \frac{1}{2})\hbar}$$

holds for the harmonic oscillator energy eigenstates.

15.14 Answers to Exercise

Ans. 1: $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$

Ans. 2: The generating function for the Hermite polynomial is given by

$$g(x,t) = e^{2xt-t^2} = e^{\{x^2-(t-x)^2\}} = \sum_{n=0}^{n=\infty} \frac{t^n H_n(x)}{n!}$$

References and Suggested Readings

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UNIT- 16
Fourier Series

Structure of theUnit

16.0 Objectives

The objective of this unit is to introduce the concept of Fourier series and their applications in various problems of Physics. As Fourier series is a powerful tool for analysis of periodic functions, the unit aims at describing the methods for expressing piecewise continuous periodic functions through Fourier series expansion.

16.1 Introduction

In practice we encounter many signals which are periodic in nature. For example

-During communications and transmitting signals, periodic signals are used in modulator.

-Problems involving vibrations or oscillations- water waves, electromagnetic waves and fields, sound waves, alternating electrical current, voltage etc.

-Periodic signals are used in power supplies.

-Periodicity in crystal structure and x-ray crystallography.

The simple periodic signal with definite frequency is expressed mathematically by the sine or cosine function. This definite (single) frequency is known as fundamental frequency. But many times the signal is not pure i.e. it contains a number of frequencies (harmonics) which are integer time the fundamental frequency along with the fundamental frequency. Such a signal is complicated periodic function.

This complicated function may be mathematically expressed as an infinite series of terms including fundamental frequency and harmonics and the series is known as Fourier series.

16.2 Definition of Fourier Series

A Fourier series may be defined as an expansion of any complex periodic function f(x) in a series of series or cosines such as

$$\boxed{f(x) = a_0 + \sum_{n=1}^{n=\infty} a_n \cos nx + \sum_{n=1}^{n=\infty} b_n \sin nx} \tag{1}$$

The above equation is valid if the function f(x) satisfies the following two conditions known as ***Dirichlet conditions*** :

(i) The function has only a finite number of extreme values i.e. maxima and minima.

(ii) The function has finite discontinuities in finite number in one oscillation.

These conditions are sufficient but not necessary because some functions which do not satisfy these conditions may be expressed by Fourier series.

The function $f(x)$ is defined in interval $(-\pi, \pi)$ and has period 2π as $sin\ x$ and $cos\ x$ have period of 2π and also $sin\ n\ x$ and $\cos n\ x$

$$\because \sin n\,(x + 2\pi) = \sin n\,x$$
$$\cos n\,(x + 2\pi) = \cos n\,x$$

16.3 Evaluation of Coefficients of Fourier Series

In equation (1) $a_0, a_n\ and\ b_n$ are called the coefficients of the Fourier series.

- **Value of coefficient a_0 :**

Let us integrate equation (1) on both the sides between the limits $-\pi\ and\ \pi.$

$$\int_{-\pi}^{\pi} f(x)dx = a_0 \int_{-\pi}^{\pi} dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos n\,x + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} b_n \sin n\,x \qquad (2)$$

It reduces to

$$\int_{-\pi}^{\pi} f(x)dx = a_0 \int_{-\pi}^{\pi} dx + 0 + 0$$

$$\{As\ all\ other\ integrales\ vanish\}$$

Or $\int_{-\pi}^{\pi} f(x)dx = a_0\, 2\pi$

$$\boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx} \qquad (3)$$

- **Value of coefficient a_n :**

Multiply equation (i) by $cos\ m\ x$ on both the sides and integrating between the limits $-\pi\ and\ \pi.$

$$\int_{-\pi}^{\pi} f(x)\cos m\,x\,dx = a_0 \int_{-\pi}^{\pi} \cos m\,x\,dx$$
$$+ \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos m\,x \cos n\,x\,dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \cos m\,x \sin n\,x\,dx \tag{4}$$

By ***orthogonal property*** of sine and cosine function we have

$$\boxed{\int_{-\pi}^{\pi} \sin mx \cos nx dx = 0} \tag{5}$$

$$for\ m, n\ \neq 0\ \boxed{\int_{-\pi}^{\pi} \sin mx \sin nx dx = \int_{-\pi}^{\pi} \cos mx \cos nx dx = \pi\delta_{mn}} \tag{6}$$

$$\begin{cases} \delta_{mn} = 0 & for\ m \neq n \\ \delta_{mn} = 1 & for\ m = n \end{cases}$$

Using (5) and (6) in (4) we have

$$\int_{-\pi}^{\pi} f(x)\cos m\,x\,dx = 0 + a_n\,\pi\,\delta_{mn} + 0$$

$$\int_{-\pi}^{\pi} f(x)\cos m\,x\,dx = a_m\,\pi \qquad [for\ \ m = n]$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos m\,x\,dx$$

Replacing m by n, we have

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx} \tag{7}$$

- **Value of coefficient $\boldsymbol{b_n}$:**

Multiply (1) on both the sides by $sin\ m\ x$ and integrate between the limits $-\pi\ and\ \pi.$

$$\int_{-\pi}^{\pi} f(x)\sin m\,x\,dx = a_0 \int_{-\pi}^{\pi} \sin m\,x\,dx$$
$$+ \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \sin m\,x \cos n\,x\,dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin m\,x \sin n\,x\,dx$$
$$= 0 + 0 + b_n\,\pi\,\delta_{mn} \qquad [Using\ 5\ and\ 6]$$

$$= b_m \pi$$

$$\therefore b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin m\,x\,dx$$

Replacing m by n

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx} \qquad (8)$$

Thus the values of the coefficients are given by eq (3), (7) and (8).

16.4 Even and Odd Functions

We can write any function $f(x)$ as

$$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} \qquad (1)$$

$$or \qquad f(x) = f_e(x) + f_0(x) \qquad (2)$$

$$where \qquad f_e(x) = \frac{f(x)+f(-x)}{2} \qquad (3)$$

$$and\ f_0(x) = \frac{f(x)-f(-x)}{2} \qquad (4)$$

From (3) we can see that

$$f_e(-x) = \frac{f(-x)+f(x)}{2} = f_e(x)$$

$$or\ f_e(-x) = f_e(x) \qquad (5)$$

Thus the functions which satisfy equation (5) are known as ***even function*** for which the functions remain same on replacing $x\ by - x$.

For example $cos\ (-x)\ =\ cos\ x$

Thus $cos\ x$ is an even function

From (4) we can see that

$$f_0(x) = \frac{f(-x)+f(x)}{2} = -\left\{\frac{f(x)-f(-x)}{2}\right\}$$

$$or \quad f_o(-x) = -f_0(x) \qquad (6)$$

Thus the functions which satisfy equation (6) are known as ***odd functions***.

For example $sin\ (-x)\ =\ -sin\ x$

Thus $sin\ x$ is an odd function.

From (2) we can see that we can always split each function into some of even and odd functions but if $f(x)$ is a periodic function, then $f_e(x)$ and $f_0(x)$ should also be periodic functions.

If functions $f(x)$ is an even function then $f(-x) = f(x)$

$$\therefore\ f_0(x) = \frac{f(x) - f(x)}{2} = 0 \tag{7}$$

Therefore $f(x) = f_e(x)$ (8)

Thus even functions when represented by Fourier series will not contain odd function terms i.e. sinnx series.

The Fourier series for an even function is

$$f(x) = f_e(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\, x \tag{9}$$

Similarly if $f(x)$ is an odd function then $f(-x) = -f(x)$

Therefore $f_e(x) = 0$ (10)

Thus Fourier series for an odd function will contain series of sinnx.

The Fourier series for an odd function is

$$f(x) = f_0(x) = \sum_{n=2}^{\infty} b_n \sin n\, x \tag{11}$$

The graph of an even function is symmetrical about $x = 0$.

The area of the curve of even function form $-\pi$ to π is twice the area under the curve from 0 to π i.e.

$$\int_{-\pi}^{\pi} f(x)dx = 2\int_{0}^{\pi} f(x)dx$$

Therefore for even function

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)dx = \frac{1}{\pi}\int_{0}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos n\, x\ dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos n\, x\, dx$$

$$and \quad b_n = 0$$

The graph of an odd function is asymmetrical with respect to $x = 0$. The area under the curve from $-\pi$ to π for an odd function is zero.

$$\therefore \int_{-\pi}^{+\pi} f(x)dx = 0$$

$$\therefore\ a_0 = 0, \qquad a_n = 0$$

$$and\ \ b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin n\, x\, dx$$

Thus Fourier series of a periodic function may be written as

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\, x + b_n \sin n\, x)$$

Periodic function = constant + even periodic function f_e + odd periodic function f_0.

This function is periodic in phase angle x as.

$$f\ (x + 2\pi) = a_0 + \sum_{n=1}^{\infty} \{a_n \cos n(x + 2\pi) + b_n \sin n(x + 2\pi)\}$$

$$= a_0 + \sum_{n=1}^{\infty} \{a_n \cos n\, x + b_n \sin n\, x\}$$

If $x = t$ and T is time period then Fourier series is written as

$$f\ (t) = a_0 + \sum_{n=1}^{\infty} \left\{a_n \cos n \frac{2\pi}{T} t + b_n \sin n \frac{2\pi}{T} t\right\}$$

$$= a_0 + \sum_{n=1}^{\infty} \{a_n \cos n\, \omega\, t + b_n \sin n\, \omega\, t\}$$

16.5 Dirichlet's Conditions

The Dirichlet's Theorem states that if a function $f(x)$ is well defined and bounded in the interval $-\pi < x < \pi$ and has only a finite number of maxima and minima, has finite number of points of discontinuities and satisfies periodicity condition

$$f(x + 2\pi) = f(x)$$

then the function may be expanded in Fourier series.

Thus Fourier series expansion is valid for following conditions.

(i) Function should be well defined, single valued in interval $(-\pi,\pi)$

(ii) Function should be continuous or may have only finite number of points of infinite discontinuities and only a finite number of maxima and minima.

(iii) The function is integrable in the interval $(-\pi,\pi)$ and series is integrable term by term and series is uniformly convergent in the interval.

16.6 Self-Learning Exercise I

Very Short Answer Type Questions

Q.1 What are the conditions imposed on function for expansion by Fourier series known as?

Q.2 If Fourier series of a function contain only sine terms then what is the function known as?

Short Answer Type Questions

Q.3 What is an even function?

Q.4 Function f(x) = x sin x is even or odd in range $(-\pi,\pi)$?

16.7 Applications of Fourier Series

Fourier series decomposes any periodic function or signal into the sum of sine and cosine functions (or complex exponentials). Fourier series was originally developed to solve the heat equation but it has been applied to a wide variety of mathematical and physical problems. Fourier series has its applications in heat transfer, electrical engineering, vibration analysis, acoustics, optics, signal processing, image processing, mechanical engineering, quantum mechanics etc. Since Fourier series is a sum of multiple sines and cosines it is easily differentiated and integrated which often simplifies the analysis of functions. In particular the fields of electronics, quantum mechanics and electrodynamics all make use of Fourier series. Other very useful methods for fields of Digital Signal processing and spectral Analysis are fast Fourier transform and discrete Fourier transform which are based on Fourier series. Thus Fourier series finds applications in Harmonic analysis, spectrum analyzer, Lock-in-amplifier, in solutions of partial differential equations, radio and communication etc.

16.8 Illustrative Examples

Example 1 Express output voltage in a Half wave Rectifier by Fourier series.

Sol. The output signal of a half-wave rectifier with real value is known.

Mathematically it is expressed as

V = 0 : $-\pi \le \omega t \le 0$

V = $V_0 \sin \omega t$: $0 \le \omega t \le \pi$

Taking $f(x) = \text{v} \quad and\ x = \omega t$

$$f(x) = 0 \qquad -\pi \le x \le 0$$

$$f(x) = V_0 \sin x \qquad 0 \le x \le \pi$$

This function cannot be expressed as even or odd therefore Fourier series for this function is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\,x + b_n \sin n\,x)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx \right]$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{0} 0\ dx + \int_{0}^{\pi} V_0 \sin x\ dx \right]$$

$$= \frac{1}{2\pi} [0 + V_0 \,(-\cos x)_0^{\pi}]$$

$$\therefore\ a_0 = \frac{V_0}{\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\ (x) \cos n\,x\ dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} f\ (x) \cos n\,x\ dx + \int_{0}^{\pi} f\ (x) \cos n\,x\ dx \right]$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0. \cos n\,x\ dx + \int_{0}^{\pi} V_0 \sin x\ \cos n\,x\ dx \right]$$

$$= \frac{V_0}{\pi}\left[\frac{1}{2}\int_0^{\pi}\{\sin(1+n)x + \sin(1-n)x\}\, dx\right]$$

$$= \frac{V_0}{2\pi}\left[\frac{-\cos(1+n)x}{(1+n)}\bigg|_0^{\pi} + \frac{-\cos(1-n)x}{1-n}\bigg|_0^{\pi}\right]$$

$$= \frac{V_0}{2\pi}\left[\frac{1-\cos(1+n)\pi}{(1+n)} + \frac{1-\cos(1-n)\pi}{(1-n)}\right]$$

The second term on R.H.S. is indeterminate at n=1, 3, 5 and on evaluation it comes out to be zero.

$$a_n = 0 \qquad\qquad n = 1, 3, 5 \dots \dots$$

$$a_n = \frac{2V_0}{\pi(1-n^2)} \qquad\qquad n = 2, 4, 6 \dots \dots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin n\, x\, dx$$

$$= \frac{1}{\pi}\{\int_{-\pi}^{0} f(x)\sin n\, x\, dx + \int_0^{\pi} f(x)\sin n\, x\, dx\}$$

$$= \frac{V_0}{2\pi}\{\cos(n-1)x - \cos(n+1)x\}\, dx$$

Solving we have

$$b_1 = \frac{V_0}{2\pi} \times \pi = \frac{V_0}{2}$$

$$b_n = 0\ for\ other\ values$$

Thus Fourier series is

$$f(x) = \frac{V_0}{\pi} + \sum_{n=even}^{\infty} \frac{2V_0}{\pi(1-n^2)}\cos n\, x \; + \; b_1 \sin x$$

$$f(x) = \frac{V_0}{\pi} + \frac{V_0}{2}\sin \omega t - \frac{2V_0}{\pi}\left\{\frac{\cos 2\ \omega t}{3} + \frac{\cos 4\ \omega t}{15} + \dots \dots ..\right\}$$

Example 2 Express saw tooth wave by Fourier series. The function for saw tooth wave is $f(x) = x$ in interval $-\pi < x \le \pi$.

Sol. The function for saw tooth wave is represented as

$$f(x) = x \qquad -\pi < x \le \pi$$

It can be seen that function is a odd function as

$$f(-x) = -x = -f(x)$$

Therefore Fourier series for this function will consist of only sine terms.

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin n\,x$$

Where

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin n\,x\,dx$$

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin n\,x\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\sin n\,x\,dx$$

$$= \frac{2}{\pi}\left[\frac{-x\cos n\,x}{n} + \frac{\sin n\,x}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi\cos n\,\pi}{n} - 0\right]$$

$$= \frac{2}{\pi} \times \frac{(-\pi)}{n}(-1)^n \qquad as \cos n\,\pi = (-1)^n$$

$$\therefore b_n = \frac{2}{n}(-1)^{n+1}$$

$$and\ f(x) = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin n\,x$$

$$or\ f(x) = 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots \ldots..\right]$$

Example 3 Express the following function by Fourier series in interval $(-\pi, \pi)$.

$$f(x) = 0\ when -\pi < x \le 0$$

$$= \frac{\pi\,x}{4}\ when\ 0 < x \le \pi$$

and hence prove $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \ldots \ldots ..$

Sol. Let

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\, x + \sum_{n=1}^{\infty} b_n \sin n\, x \qquad (1)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left[\int_{\pi}^{0} f(x)dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)dx \right]$$

$$= \frac{1}{2\pi}\left[0 + \int_0^{\pi} \frac{\pi x}{4} dx\right] = \frac{1}{2\pi} \cdot \frac{\pi}{4}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi^2}{16} \qquad (2)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos n\, x\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos n\, x\, dx + \int_0^{\pi} f(x)\cos n\, x\, dx\right]$$

$$= \frac{1}{\pi}\left[0 + \int_0^{\pi}\frac{\pi x}{4}\cos n\, x\, dx\right] = \frac{1}{4}\int_0^{\pi} x\cos n\, x\, dx$$

$$= \frac{1}{4}\left[\frac{x\sin n\, x}{n} + \frac{\cos n\, x}{n^2}\right]_0^{\pi} = \frac{1}{4}\left[\frac{\cos n\,\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{1}{4n^2}[\cos n\,\pi - 1] = \frac{(-1)^n - 1}{4n^2} \qquad (3)$$

$\because$ $\boxed{\cos n\pi = (-1)^n}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin n\, x\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin x\, dx + \int_0^{\pi} f(x)\sin n\, x\, dx\right]$$

$$= \frac{1}{\pi}\left[0 + \int_0^{\pi}\frac{\pi x}{4}\sin n\, x\, dx\right]$$

Integrating

$$b_n = -\frac{(-1)^n\pi}{4\, n} \qquad (4)$$

Using (2), (3) and (4) in (1)

$$f(x) = \frac{\pi^2}{16} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{4n^2}\cos n\, x - \frac{(-1)^n\pi}{4n}\sin n\, x\right] \qquad (5)$$

$$= \frac{\pi^2}{16} + \left(-\frac{1}{2}\cos x + \frac{\pi}{4}\sin x\right) - \frac{\pi}{4.2}\sin 2\,x - \frac{1}{2.3^2}\cos 3\,x + \frac{\pi}{4.3}\sin 3x \quad (6)$$

Putting $x = \pi \ in\ (6)$

$$f(x) = \frac{\pi^2}{16} + \frac{1}{2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots..\right)$$

$$But\ \ f(\pi) = \frac{1}{2}\left[f\ (-\pi + 0) +\ f(\pi - 0)\right] = \frac{1}{2}\left[0 + \left(\frac{\pi x}{4}\right)_{x=\pi}\right] = \frac{\pi^2}{8}$$

$$\therefore \frac{\pi^2}{8} = \frac{\pi^2}{16} + \frac{1}{2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots..\right)$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \qquad hence\ proved$$

16.9 Complex form of Fourier Series

Fourier series can be expressed in complex form by using

$$\boxed{\cos nx = \frac{1}{2}\left(e^{inx} + e^{-inx}\right)}$$

and

$$\boxed{\sin nx = \frac{1}{2}\left(e^{inx} - e^{-inx}\right)}$$

In

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\,x + \sum_{n=1}^{\infty} b_n \sin n\,x$$

$$\therefore f(x) = a_0 + \sum_{n=1}^{\infty} a_n \frac{\left(e^{inx} + e^{-inx}\right)}{2} + \sum_{n=1}^{\infty} b_n \frac{\left(e^{inx} - e^{-inx}\right)}{2i}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[\frac{a_n - ib_n}{2}e^{inx} + \frac{(a_n + ib_n)}{2}e^{-inx}\right] \qquad (1)$$

As we know

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad and \quad b_n$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx\,dx \qquad (2)$$

$$\therefore\ a_n \pm\ ib_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ (\cos nx \pm i \sin nx)dx \qquad (3)$$

Keeping $-n$ instead of n

$$a_{-n} +\ ib -_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ (\cos nx - i \sin nx)dx$$

$$\therefore\ a_{-n} +\ ib_{-n} = a_n - ib_n \qquad \ldots\ldots(4)$$

Keeping in last term of RHS of (1) $-n$ instead of n

$$f(x) =\ a_0 + \sum_{n=1}^{\infty} \frac{(a_n - ib_n)}{2} e^{inx} + \sum_{n=-1}^{\infty} \frac{(a_{-n} + ib_{-n})}{2}\ e^{inx}$$

Using (4)

$$f(x) =\ a_0 + \sum_{n=1}^{\infty} \frac{(a_n - ib_n)}{2} e^{inx} + \sum_{n=-1}^{-\infty} \frac{(a_n - ib_n)}{2}\ e^{inx}$$

Let

$$C_n = \frac{a_n - ib_n}{2} \quad , \qquad C_0 = a_0$$

$$f(x) =\ C_0 + \sum_{n=1}^{\infty} C_n\ e^{inx} + \sum_{n=-1}^{-\infty} C_n\ e^{inx}$$

$$= \sum_{n=0}^{\infty} C_n\ e^{inx} + \sum_{n=-1}^{-\infty} C_n\ e^{inx}$$

$$\boxed{f(x) = \sum_{-\infty}^{+\infty} C_n e^{inx}} \qquad \ldots.(5)$$

Here

$$C_n = \frac{a_n - ib_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)(\cos nx - i \sin nx)dx$$

$$\boxed{C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx} \qquad \ldots\ldots(6)$$

Equation (5) is required ***complex form of Fourier series***.

16.10 Fourier Series in interval (0, T)

The Fourier series of a periodic piecewise continuous wave function $f(t)$ with time period T is given by

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t \qquad (1)$$

Constant a_0 : To evaluate constant a_0 we integrate (1) with respect to t between the limits 0 to T and obtain

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt \qquad (2)$$

Constant a_n : To evaluate a_n multiply (1) by $cos\ n\ \omega t$ and integrate between limits 0 to T

$$\int_0^T f(t) \cos n\ \omega t\ \ dt = \ a_n \int_0^T cos^2\ n\ \omega t\ dt + 0$$

Solving

$$a_n = \frac{2}{T}\int_0^T f(t) \cos n\omega t\, dt \qquad (3)$$

b_n: To evaluate b_n multiply (1) by $sin\ n\ \omega t$ and integrate between the limits 0 to T.

$$\int_0^T f(t) \sin n\ \omega t\ dt = \ b_n \int_0^T sin^2\ n\ \omega t\ dt + 0$$

$$= \frac{b_n}{2} T$$

$$b_n = \frac{2}{T}\int_0^T f(t) \sin n\omega t\, dt \qquad (4)$$

The complex form of Fourier series can also be written for function $f(t)$ in interval (0, T)

As

$$f(T)=\sum_{n=-\infty}^{\infty} C_n e^{in\omega t} \quad or \quad \sum_{n=-\infty}^{\infty} C_n e^{\frac{in2\pi t}{T}} \tag{5}$$

Where C_n is given by

$$C_n=\frac{1}{T}\int_0^T f(t)e^{-in\omega t}dt=\frac{1}{T}\int_0^T f(t)e^{\frac{-in2\pi t}{T}}dt \tag{6}$$

$$\left[\because\ \omega=\frac{2\pi}{T}\right]$$

16.11 Internal Change for Fourier expansion from $(-\pi,\pi)$ to (-*l*, *l*)

Let the periodic function has period 2*l*. Let us consider.

$$z=\frac{\pi x}{l} \quad or \quad x=\frac{lz}{\pi}$$

Then Fourier series is written as

$$f(x)=f\left(\frac{lz}{\pi}\right)=a_0+\sum_{n=1}^{\infty}a_n\cos nz+\sum_{n=1}^{\infty}b_n\sin nz$$

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l} \tag{1}$$

This is the Fourier series and the coefficients are given by

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f\left(\frac{lz}{\pi}\right)dz \quad = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)d\left(\frac{\pi x}{l}\right)$$

$$a_0=\frac{1}{2l}\int_{-l}^{l}f(x)dx$$

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}dx$$

16.12 Fourier Half Range Series

Interval $(0, \pi)$

Within the range $(0, \pi)$ or $(0, l)$ both series and cosines form mathematically complete sets. This means we can expand any function within this range in terms of either sines or cosines depending on nature of $f(x)$.

Fourier Cosine Series :

The cosine representation of a function $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

Where

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Fourier Sine Series :

It is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Interval $(0, l)$

Fourier cosine series in interval $(0, l)$ is written as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where

$$a_0 = \frac{1}{l}\int_0^{l} f(x)dx$$

and

$$a_n = \frac{2}{l}\int_0^l f(x)\ \cos\frac{n\pi x}{l}\,dx$$

Fourier sine series is written as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\ dx$$

16.13 Some Important Points

1. Every periodic function may be decomposed into a sum of one or more cosine and or sine terms of selected frequency dependent on the original function.
2. If $f(x)$ is piecewise continuous the definite integrals exist and fourier coefficient can be evaluated but if $f(x)$ is not piecewise continuous then we cannot find Fourier coefficient with surity as some of the integrals may be improper which are divergent.
3. The function $f(x)$ and its Fourier series are only equal to each other if and whenever $f(x)$ is continuous.
4. The constant term (a_0) in a Fourier series represents the average value of the function $f(x)$ over its entire domain.

16.14 Illustrative Examples

Example 4 Find a Fourier series for $f(x) = x\ ,-2 < x < 2$

$$f(x+4) = f(x)$$

Sol. Here T = 2 l = 4 hence l = 2

As the function f(x) is an odd function it would contain sine terms.

$$b_n = \frac{1}{2}\int_{-2}^{2} x\ \sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left\{\left[\frac{-2x}{n\pi} cos \frac{n\pi x}{2}\right]_{-2}^{2} - \frac{2}{n\pi}\int_{-2}^{2} cos \frac{n\pi x}{2} dx\right\}$$

$$= \frac{-4}{n\pi} cosn\pi$$

If n = even $b_n = \frac{-4}{n\pi}$, if n is odd $b_n = \frac{4}{n\pi}$

$$\therefore b_n = \frac{(-1)^{n+1}4}{n\pi}$$

Thus

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} sin \frac{n\pi x}{2}$$

Example 5 Write the Fourier series for a square wave.

defined by $f(t) = V_o \quad ; \quad 0 < t < \frac{T}{2}$

$$= -V_0 \quad ; \quad \frac{T}{2} < t < T$$

Sol. The Fourier series is given by

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t \quad (1)$$

The function is an odd function hence contain sine terms only in Fourier expansion.

$$b_n = \frac{2}{T}\int_0^T f(t) \sin(n\omega t) dt$$

$$= \frac{2}{T}\left[\int_0^{T/2} V_0 \sin n\omega t \, dt + \int_{T/2}^{T} (-V_0) \sin n\omega t \, dt\right]$$

$$= \frac{2V_o}{T}\left[\left\{\frac{-\cos n\omega t}{n\omega}\right\}_0^{T/2} + \left\{\frac{\cos n\omega t}{n\omega}\right\}_{T/2}^{T}\right]$$

$$= \frac{2V_o}{n\omega T}\left[\left\{-\cos n\frac{2\pi t}{T}\right\}_0^{T/2} + \left\{\cos n\frac{2\pi t}{T}\right\}_{T/2}^{T}\right]$$

$$= \frac{2V_o}{n\omega T}[-\cos n\pi + 1 + \cos 2n\pi - \cos n\pi]$$

$$= \frac{2V_o}{n\omega T}[2 - 2\cos n\pi] \qquad [\because \cos 2n\pi = 1]$$

$$= \frac{2V_o \times 2}{n \times 2\pi}[1 - \cos n\pi] \qquad [\because \omega T = 2\pi]$$

$$= \frac{2V_o}{n\pi}(1 - \cos n\pi)$$

When n is even , $\cos n\pi = +1$ therefore $b_n = 0$

When n is odd, $\cos n\pi = -1$ therefore $b_n = \frac{4V_0}{n\pi}$

Therefore the Fourier series for square wave is.

$$f(t) = \sum_n b_n \sin n\omega t \qquad when\ n = 1,3,5 \ldots.$$

$$= \frac{4V_o}{\pi}\left[\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \cdots \ldots \ldots\right]$$

16.15 Self-Learning Exercise II

Very Short Answer Type Questions

Q.1 If function is odd the Fourier half range series will consist of which terms?

Q.2 If function is even which Fourier coefficient is zero?

Short Answer Type Questions

Q.3 What is a periodic function?

Q.4 The formulae for evaluating a_o, a_n and b_n are known as?

Q.5 What is the significance of coefficient a_o?

Q.6 When is the function and its Fourier series representation equal to each other?

16.16 Summary

This unit presents the introduction to Fourier series and its applications. Fourier series is a powerful fool for representing any periodic functions as a sum of sines and cosines. The full range Fourier series has been defined and evaluation of Fourier coefficients has been specified. Even and odd functions are defined and Fourier series for them has been described. Half range Fourier series, Change of interval and complex form of Fourier series has been explained. Fourier series finds its applications in solution of differential equations, signal analysis, data analysis, communications etc.

16.17 Glossary

Periodic: occurring at intervals

Digital : (of signals or data) expressed as series of the digits 0 and 1,

16.18 Answers to Self Learning Exercises

Answers to Self Learning Exercise-I

Ans.1: Dirichlet's conditions

Ans.2: Odd Function

Ans.3: $f(-x) = f(x)$ for an even function

Ans.4: even function

Answers to Self Learning Exercise-II

Ans.1: sine terms

Ans.2: b_n

Ans.3: A function is periodic with period $T > 0$ if for all x, $f(x + T) = f(x)$ and T is the least of such values.

Ans.4: Euler-Fourier Formulae

Ans.5: It represents average value of function over entire domain.

Ans.6: Whenever function is continuous.

16.19 Exercise

Section A: Very Short Answer Type Questions

Q.1 Function x^2 when expanded by Fourier series will contain which terms in interval $(-\pi, \pi)$

Q.2 Function $x^3 \sin x$ is even or odd in interval $(-\pi, \pi)$?

Q.3 Fourier series is used to represent which kind of functions?

Section B: Short Answer Type Questions

Q.4 Define Fourier series and Fourier coefficient for interval $(-\pi, \pi)$.

Q.5 State the Dirichlets conditions for Fourier series expansion.

Q.6 Define even and odd functions.

Q.7 What do you understand by Fourier Half Range series?

Q.8 Write the complex form of Fourier series.

Section C: Long Answer Type Questions

Q.9 A function $g(x)$ is defined by

$$g(x) = \begin{cases} \frac{x}{L} \; if \; o \leq x < L/2 \\ 1 - \frac{x}{L} if \; L/2 \leq x < L \end{cases}$$

By expanding $g(x)$ as a fourier sine series show that

$$g(x) = \sum_{n\,odd} \frac{4(-1)^{\frac{n-1}{2}}}{n^2\pi^2} \sin\frac{n\pi x}{L}$$

Q.10 A function $f(x)$ is defined by

$$f(x) = \begin{cases} 1 & 0 \leq x < L/2 \\ 0 & L/2 \leq x < L \end{cases}$$

Expand $f(x)$ as a fourier cosine series and show that $a_n = 0$ if n is even and if n is odd

$$a_n = \frac{2}{n\pi}(-1)^{\frac{(n-1)}{2}}$$

Write the cosine series in $0 \leq x < L$ and deduce

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$$

Q.11 Find Fourier series expansion of $f(x) = e^x$ in the interval $(-\pi, \pi)$.

Q.12 Represent function $f(x) = x^2$ in the interval $(0, 2\pi)$by Fourier series and show that

$$x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

Q.13 Expand the function $f(x) = x^2$ as Fourier series in $(-\pi, \pi)$ and hence deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \ldots \ldots \ldots = \frac{\pi^2}{6}$

Q.14 Express function $f(x) = |x|$ in interval (-L, L) by Fourier series expansion and show

$$f(x) = \frac{L}{2} - \frac{4L}{\pi^2}\left[\frac{cos\frac{\pi x}{L}}{1^2} + \frac{cos\frac{3\pi x}{L}}{3^2} + \ \ldots \ldots . .\right]$$

6.20 Answers to Exercise

Ans.1 : Cosine

Ans.2 : Even

Ans.3 : Piecewise continuous periodic

Ans.7: $e^x = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} - \left(\frac{1}{2}\cos x - \frac{1}{5}\cos 2x + \cdots .\right) + \frac{1}{2}\left(\sin x - \frac{2}{5}\sin 2x + \cdots . .\right)\right]$

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UNIT- 17
Integral Transforms

Structure of the Unit

17.0 Objectives

The aim of the chapter is to introduce the readers to the concept and importance of integral transforms. Various transform such as Laplace, Fourier and Hankel transforms have been discussed. These transforms have wide applicability in Physics and other areas of science. In the chapter Laplace, Fourier and Hankel transforms are defined and their properties are presented. Methods for finding these transforms have been shown through examples to familiarize readers with fundamental concepts.

17.1 Introduction

Integral transforms constitute a very powerful tool for the solution of various problems in science and especially physics. Laplace transforms provide a very convenient method of solution of Linear constant-coefficient differential equations. Likewise Fourier transforms are widely applied to areas such as optics, astronomy, quantum mechanics, crystallography, electrical, electronics and biomedical engineering etc. Laplace, Fourier and Hankel transforms have applications in various boundary value problems. Hankel transform may be considered as the Fourier transform of a Bessel function expansion.

17.2 Integral Transforms

A general linear integral transformation of a function is expressed by following equation:

$$g(s) = Tf(t) = \int_a^b f(t)k(\mathrm{s},t)dt \qquad (1)$$

The function g(s) is called the integral transform of f(t) though kernel $k(\mathrm{s},t)$. The kernel $k(\mathrm{s},t)$ is a prescribed function of parameter s and variable t. There are different integral transforms depending on the type of kernel $k(\mathrm{s},t)$ used and the range of integration. For example

$$g(s)=\int_{-\infty}^{\infty} f(t)e^{-ist}dt \qquad \text{(Fourier Transform)}$$

$$g(s)=\int_{0}^{\infty} f(t)e^{-st}dt \qquad \text{(Laplace Transform)}$$

$$g(s)=\int_{0}^{\infty} f(t)\,\mathrm{J}_n(st)dt \qquad \text{(Hankel Transform)}$$

$$g(s)=\int_{0}^{\infty} f(t)\,\mathrm{t}^{s-1}\,dt \qquad \text{(Mellin Transform)}$$

All integral transforms have the following linearity properties.

$$I(f+g)=I(f)+I(g)$$

$$I(cf) = cI(f) \quad \text{for constant c.}$$

17.3 The Laplace Transform

The Laplace transform of a function f(t) is given by:

$$\boxed{\mathrm{F(s)}=L\{f(t)\}=\int_{0}^{\infty} f(t)e^{-st}dt}$$

where the symbol L indicates the Laplace transform of the function. The Laplace transform is said to exist when and only when the integral $\int_{0}^{\infty} f(t)e^{-st}dt$ converges for some value of s. Here s is a parameter which may be real or complex.

17.4 Illustrative Examples

Example 1 Find the Laplace transform of following

(i) $f(t)=1$ (ii) $f(t)=t$

Sol. **(i)** $f(t)=1$

$$\mathrm{F(s)}=L\{1\}=\int_{0}^{\infty} e^{-st}dt=\left[\frac{e^{-st}}{-s}\right]_0^{\infty}=\frac{1}{s}$$

$$\boxed{L\{1\}=\frac{1}{s}}$$

(ii) $f(t)=t$

$$F(s) = L\{t\} = \int_0^\infty e^{-st}.t dt$$

$$= \left[\frac{e^{-st}}{-s}t\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} dt = 0 + \frac{1}{s}\int_0^\infty e^{-st} dt = \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s^2}$$

$$\boxed{L\{t\} = \frac{1}{s^2}}$$

Example 2 Find the Laplace transform of

(i) c (constant) (ii) $t^n,\ n \geq 0$ (iii) $\sqrt{t}$ (iv) kt

Sol. **(i)** $f(t) = c(constant)$

$$F(s) = L\{c\} = \int_0^\infty e^{-st} c dt = c\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{c}{s}$$

$$\boxed{L\{c\} = \frac{c}{s}}$$

(ii) $f(t) = t^n$

$$F(s) = L\{t^n\} = \int_0^\infty e^{-st} t^n dt$$

$$= \left[\frac{e^{-st}}{-s}.t^n\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}.t^{n-1} dt \qquad \text{(Integration by parts)}$$

$$= 0 + \frac{n}{s}\left[\frac{e^{-st}}{-s}.t^{n-1}\right]_0^\infty + \frac{n(n-1)}{s^2}\int_0^\infty e^{-st}.t^{n-2} dt$$

(again Integrating by parts)

The first two term vanish and repeating integration by parts

$$= \frac{n(n-1)(n-2)}{s^3}\int_0^\infty e^{-st}.t^{n-3} dt$$

Like this repeating integration by parts

$$= \frac{n(n-1)(n-2)...........3.2.1}{s^n}\int_0^\infty e^{-st}.\mathrm{t}^0\,dt$$

$$\frac{\lfloor n}{s^n}.\frac{1}{s} = \frac{n!}{s^{n+1}}$$

Alternative

$$\mathrm{F(s)} = L\left\{t^n\right\} = \int_0^\infty e^{-st}t^n dt = \int_0^\infty e^{-st}t^{(n+1)-1}dt$$

$$= \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}} \qquad \left\{\because \int_0^\infty e^{-sx}x^{n-1}dx = \frac{\Gamma n}{s^n}\right.$$

$$\boxed{L\left\{t^n\right\} = \frac{n!}{s^{n+1}}}$$

(iii) $f(t) = \sqrt{t} = t^{1/2}$

$$\mathrm{F(s)} = \int_0^\infty t^{1/2}e^{-st}dt = \int_0^\infty t^{\left(\frac{3}{2}-1\right)}e^{-st}dt$$

$$= \frac{\Gamma\frac{3}{2}}{s^{\frac{3}{2}}} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$$

(iv) $f(t) = kt$

$$\mathrm{F(s)} = \int_0^\infty e^{-st}ktdt = k\int_0^\infty e^{-st}tdt = \frac{k}{s^2}$$

Example 3 Find the transform of

(i) e^{kt} (ii) e^{-kt} (iii) $\sin kt$

(iv) $\cos kt$ (v) $\sinh kt$ (vi) $\cosh kt$

Sol. **(i)** $e^{kt} \Rightarrow f(\mathrm{t}) = \mathrm{e}^{kt}$

$$\mathrm{F(s)} = L\left\{e^{kt}\right\} = \int_0^\infty e^{kt}e^{-st}dt = \int_0^\infty e^{(k-s)t}dt$$

$$= \int_0^\infty e^{-(s-k)\mathrm{t}}dt = \left[\frac{e^{-(s-k)t}}{-(s-k)}\right]_0^\infty = \frac{1}{s-k}$$

$$\boxed{L\left\{e^{kt}\right\} = \frac{1}{s-k}}$$

(ii) $e^{-kt} \quad \Rightarrow f(t) = e^{-kt}$

$$\text{F(s)} = L\left\{e^{-kt}\right\} = \int_0^\infty e^{-kt} e^{-st} dt = \int_0^\infty e^{-(s+k)t} dt = \frac{1}{s+k}$$

$$\boxed{L\left\{e^{-kt}\right\} = \frac{1}{s+k}}$$

(iii) $\sin kt \quad \Rightarrow f(t) = \sin kt$

$$\text{F(s)} = L\left\{\sin kt\right\} = \int_0^\infty e^{-st} \sin kt dt = \int_0^\infty e^{-st} \left(\frac{e^{ikt} - e^{-ikt}}{2i}\right) dt$$

$$= \frac{1}{2i} \int_0^\infty \left[e^{-(s-ik)t} - e^{-(s+ik)t}\right] dt$$

$$= \frac{1}{2i}\left[\frac{1}{s-ik} - \frac{1}{s+ik}\right] = \frac{k}{s^2+k^2}$$

$$\boxed{L\left\{\sin kt\right\} = \frac{k}{s^2+k^2}}$$

(iv) $\cos kt$

$$\text{F(s)} = L\left\{\cos kt\right\} = \int_0^\infty e^{-st} \cos kt dt = \int_0^\infty e^{-st} \left(\frac{e^{ikt} + e^{-ikt}}{2}\right) dt$$

$$= \frac{1}{2i} \int_0^\infty \left[e^{-(s-ik)t} + e^{-(s+ik)t}\right] dt$$

$$= \frac{1}{2i}\left[\frac{1}{s-ik} + \frac{1}{s+ik}\right] = \frac{s}{s^2+k^2}$$

$$\boxed{L\left\{\cos kt\right\} = \frac{s}{s^2+k^2}}$$

(v) $\sinh kt$

$$\text{F(s)} = L\left\{\sinh kt\right\} = \int_0^\infty e^{-st} \sinh kt dt =$$

$$= \int_0^\infty e^{-st}\left(\frac{e^{kt}-e^{-kt}}{2}\right)dt = \frac{1}{2}\int_0^\infty \left(e^{-(s-k)t} - e^{-(s+k)t}\right)dt$$

$$= \frac{1}{2}\left[\frac{1}{s-k} - \frac{1}{s+k}\right] = \frac{k}{s^2-k^2}$$

(vi) $\cosh kt$

$$F(s) = L\{\cosh kt\} = \int_0^\infty e^{-st}\left(\frac{e^{kt}+e^{-kt}}{2}\right)dt$$

$$= \frac{1}{2}\int_0^\infty \left(e^{-(s-k)t} + e^{-(s+k)t}\right)dt$$

$$= \frac{1}{2}\left[\frac{1}{s-k} + \frac{1}{s+k}\right] = \frac{s}{s^2-k^2}$$

Example 4 Find Laplace transform of

(i) $\sin^2 t$ (ii) $e^{kt}\sin\omega t$

Sol. **(i)** $\sin^2 t$

$$F(s) = L\{\sin^2 t\} = L\left[\frac{1}{2}(1-\cos 2t)\right]$$

$$= \frac{1}{2}\left[L(1) - L(\cos 2t)\right] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

(ii) $e^{kt}\sin wt$

$$F(s) = L\{\mathrm{e}^{kt}\sin\omega\,\mathrm{t}\} = \int_0^\infty e^{kt}\sin\omega t\; e^{-st}dt = \int_0^\infty e^{kt}\left(\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\right)e^{-st}dt$$

$$= \frac{1}{2i}\int_0^\infty \left[e^{-(\mathrm{s-k-i}\,\omega)\mathrm{t}} e^{-(s-k+i\omega)t}\right]dt$$

$$= \frac{1}{2i}\left[\frac{1}{s-k-i\omega} - \frac{1}{s-k+i\omega}\right] = \frac{\omega}{(s-k)^2+\omega^2}$$

17.5 Sufficient Conductions for Existence of Laplace Transforms

(i) The function f(t) should be piecewise or sectionally continuous in every finite interval.

(ii) The function should be of exponential order.

A function f(t) is said to be the function of exponential order m as $t \to \infty$,if for a given positive integer m there exists real constant M > 0 such that

$$\left|e^{-mt} f(t)\right| < M \ or \ \left|f(t)\right| \le Me^{mt} \text{ for every } t \ge 0$$

17.6 Some Properties of Laplace Transform

(1) Linearity – For every pair of constants k_1 and k_2 the Laplace transform of the linear combination of any number of function satisfies.

$$\boxed{L\{k_1 f_1(t) + k_2 f_2(t)\} = k_1 L\{f_1(t)\} + k_2 L\{f_2(t)\}}$$

The proof is simple and left as an exercise for readers.

(2) Change of scale – If F(s) is the Laplace transform of f(t), the Laplace transform of f(kt) is $\frac{1}{k} F\left(\frac{S}{k}\right)$

Proof. $Lf(kt) = \int_0^\infty e^{-st} f(kt) dt$

Putting kt = u

$$= \frac{1}{k} \int_0^\infty e^{-su/k} f(\text{u}) du = \frac{1}{k} F\left(\frac{s}{k}\right)$$

$$\boxed{L\{f(kt)\} = \frac{1}{k} F\left(\frac{s}{k}\right)}$$

(3) Shifting or translation properties –

(i) First translation or shifting property – If F(s) is Laplace transform of f(t) then Laplace transform of $e^{kt} f(t)$ will be $F(\text{s}-\text{k})$.

Proof. $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(\text{t}) dt$

$$L\left\{e^{kt} f(t)\right\} = \int_0^\infty e^{-st} e^{kt} f(\text{t}) dt = \int_0^\infty e^{-(s-k)t} f(\text{t}) dt$$

$= \mathrm{F(s-k)}$

Similarly it can be shown that $\boxed{\mathrm{F(s+k)} = L\left\{e^{-kt} f(t)\right\}}$

(ii) Second translation or shifting property (Heaviside shifting theorem) -

If a function is defined by

$$g(t) = \begin{cases} 0 \text{ if } 0<t<k \\ f(t-k) \text{ if } t > k \end{cases}$$

then the Laplace transform of g(t) is $e^{-ks} F(s)$, where $F(s)$ is Laplace transform of $f(t)$.

Proof. $L\{g(t)\} = \int_0^\infty e^{-st}\, \mathrm{g(t)}dt = \int_0^k e^{-st} g(\mathrm{t})dt + \int_k^\infty e^{-st}\, \mathrm{g(t)}dt$

$= \int_k^\infty e^{-st}\, \mathrm{f(t-k)}dt$ (using Property of given function)

$= e^{-sk} \int_0^\infty e^{-su}\, \mathrm{f(u)}du$ (Putting t - k = u)

$= e^{-sk} F(s)$

(4) Derivative of Laplace transform -

If F(s) is Laplace transform of f(t) then $\boxed{\mathrm{F'(s)} = \frac{dF}{ds} = L\{-tf(t)\}}$ and in general

$\boxed{\mathrm{F}^{(n)}\mathrm{(s)} = (-1)^n L\left\{t^n f(t)\right\}}$

Proof. $F(s) = \int_0^\infty e^{-st} f(\mathrm{t})dt$

differentiating both the sides with respect to s

$$\mathrm{F'(s)} = \frac{dF}{ds} = \int_0^\infty (-t)e^{-st} f(\mathrm{t})dt = \int_0^\infty e^{-st}\{-tf(t)\}dt$$

carrying out the process of differentiation n times we obtain.

$$\mathrm{F}^{(n)}\mathrm{(s)} = \frac{d^n F(s)}{ds^n} = L\left\{(-1)^n t^n f(t)\right\} = (-1)^n L\left\{t^n f(t)\right\}$$

(5) Laplace transform of derivatives -

Let f(t) be a continuous differentiable function with a sectionally continuous derivative f′(t). If f(t) and $\frac{df(t)}{dt}$ are Laplace transformable, then the Laplace transform of derivative $\frac{df(t)}{dt}$ is given by

$$L\left\{\frac{df(t)}{dt}\right\} = sL\{f(t)\} - f(0) = sF(s) - f(0)$$

Where f(0) is the value of f(t) at t = 0 and

$\mathrm{F(s)} = L\{f(t)\}$

Proof. $L\left\{\frac{df(t)}{dt}\right\} = \int_0^\infty e^{-st}\left\{\frac{df(t)}{dt}\right\}dt$

By integration by Parts

$$L\left\{\frac{df(t)}{dt}\right\} = \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$

$$= -f(0) + sL\{f(t)\} = sF(s) - f(0)$$

$$\boxed{L\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0)}$$

This theorem is useful in solving differential equations with constant coefficients and it is applicable for n^{th} derivatives.

General formula for the Laplace transform of the n^{th} derivative $f^n(t)$ is

$$L\{\mathrm{f}^n\mathrm{(t)}\} = \mathrm{s}^n F(s) - s^{n-1} f(0) - \mathrm{s}^{n-2} f'(0)\ldots\ldots\ldots\ldots\mathrm{sf}^{n-2}(0) - \mathrm{f}^{n-1}(0)$$

(6) Laplace transform of Integral -

$\mathrm{F(s)} = L\{\mathrm{f(t)}\}$ then $L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$

Proof. Let $g(t) = \int_0^t f(t)dt$

then $g'(t) = \frac{d}{dt}\int_0^t f(t)dt = f(t)$

By Laplace transform of derivative we have

$$L\{g'(t)\} = sL\{g(t)\} - g(0)$$

$$= sL\{g(t)\} - 0$$

$$= \frac{1}{s}L\{f(t)\} = L\{g(t)\}$$

Hence

$$L\left\{\int_0^t f(t)dt\right\} = \frac{F(s)}{s}$$

* (i) If $F(s) = L\{f(t)\}$ then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$

(ii) If $F(s) = L\{f(t)\}$ then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)ds$

provided the integral exists

(7) Laplace transform of periodic functions -

If f(t) is a periodic function with the period T i.e. $f(t+T) = f(t)$

Then $L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)dt}{1-e^{-sT}}$

Proof. $L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

$$= \int_0^T e^{-st} f(t)dt + \int_T^{2T} e^{-st} f(t)dt + \ldots\ldots\ldots \int_{nT}^{(n+1)T} e^{-st} f(t)dt\ldots..$$

$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t)dt$$

Putting t = u + nT i.e. dt = du

$$L\{f(t)\} = \sum_{n=0}^{\infty} \int_0^T e^{-s(u+nT)} f(u+nT)du$$

$$= \sum_{n=0}^{\infty} \int_0^T e^{-sTn} e^{-su} f(u+nT)du$$

$$= \sum_{n=0}^{\infty} e^{-nsT} \int_0^T e^{-su} f(u) du$$

$$= \left(1 + e^{-sT} + e^{-2sT} + ...\right) \int_0^T e^{-su} f(\text{u}) \, \text{du}$$

$$= \frac{1}{(1-e^{-sT})} \int_0^T e^{-su} f(\text{u}) \, \text{du}$$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(\text{t}) \, \text{dt} \quad \text{[replacing u by t]}$$

Example 5 Find Laplace transform of $\sinh mt \sin mt$

Sol. $\text{f(t)} = \text{sinhmt} \sin mt = \frac{e^{mt} - e^{-mt}}{2} . \frac{e^{imt} - e^{-imt}}{2i}$

$$= \frac{1}{4i} \left[e^{mt(1+i)} + e^{-mt(1+i)} - e^{mt(1-i)} - e^{-mt(1-i)} \right]$$

Putting $m(1+i) = r$ and $m(1-i) = \text{q}$

$$L\{\text{sinhmt sinmt}\} = L\left\{ \frac{1}{4i}\left(e^{rt} + e^{-rt} - e^{-qt} - e^{qt} \right) \right\}$$

$$= \frac{1}{4i}\left[L\left\{e^{rt}\right\} + L\left\{e^{-rt}\right\} - L\left\{e^{-qt}\right\} - L\left\{e^{qt}\right\} \right]$$

As $L\left\{e^{rt}\right\} = \frac{1}{s-r}$ and $L\left\{e^{-rt}\right\} = \frac{1}{s+r}$

$$\therefore \quad = \frac{1}{4i}\left[\frac{1}{s-r} + \frac{1}{s+r} - \frac{1}{s-q} - \frac{1}{s+q} \right]$$

$$= \frac{1}{4i}\left[\frac{2s}{s^2-r^2} - \frac{2s}{s^2-q^2} \right] = -\frac{is(\text{r}^2-\text{q}^2)}{2(s^2+r^2)(s^2-q^2)}$$

Putting $r^2 = 2im^2, q^2 = -2im^2, -i(r^2-q^2) = 4m^2$

$$(\text{s}^2-\text{r}^2)(\text{s}^2-\text{q}^2) = \text{s}^4 + 4m^4$$

$$L(\text{sinhmt sinmt}) = \frac{4m^2 s}{2(\text{s}^4 + 4\text{m}^4)} = \frac{2m^2 S}{s^4 + 4m^4}$$

Example 6 Find Laplace transform of a square wave

$$f(t) = V \; for \; 0 < t < \frac{T}{2}$$

$$= 0 \, for \frac{T}{2} < t < T$$

and $f(t+T) = f(\text{t})$

Sol. For periodic functions $Lf(t) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$

$$= \frac{1}{1-e^{-sT}} \left[\int_0^{T/2} e^{-st} \, \text{Vdt} + 0 \right]$$

$$= \frac{V}{1-e^{-sT}} \left[\frac{e^{-st}}{-s} \right]_0^{T/2} \qquad = \frac{V\left(1-e^{-sT/2}\right)}{s(1-e^{-sT})}$$

Example 7 Find Laplace transform of $f(t) = 6\sin 2t - 2e^{-2t} \cos 4t$

Sol. $F(s) = L\{f(t)\} = L\{6\sin 2t\} - 2L\{\text{e}^{-2t}\cos 4\,\text{t}\}$

Using linearity property

$$\text{L}\{\sin 2\,\text{t}\} = \frac{2}{s^2+4}$$

$$\text{L}\{e^{-2t} \cos 4\,\text{t}\} = \frac{s-2}{(s-2)^2+16}$$

$$F(s) = 6\frac{2}{s^2+4} - 2\frac{s-2}{(s-2)^2+16}$$

$$= \frac{12}{s^2+4} - \frac{2s-4}{(s-2)^2+16}$$

Example 8 Find Laplace transform of $t^2 e^{-2t}$

Sol. We know $L\{\text{t}^n \text{f(t)}\} = (-1)^n \frac{d^n}{ds^n} F(s)$

$$\therefore L\{\text{t}^2\text{e}^{-2t}\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s+2}\right) \qquad = \frac{2!}{(s+2)^3}$$

17.7 Self Learning Exercise-I

Q.1 What is Laplace transform of $\sin 3t$?

Q.2 Find the Laplace transform of $t^3 e^{2t}$.

Q.3 Find Laplace transform of $\frac{\sin t}{t}$.

Q.4 What is Laplace transform of $7t^3$?

Q.5 What is Laplace transform of $e^{-2t}\cos t$?

17.8 Fourier Transform

A general Fourier Transform is defined by

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-ist} dt$$

which is based on kernel e^{-ist} and its real and imaginary parts. Fourier Integral transform can be defined from Fourier integral.

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} d\omega \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

This is Fourier integral

It can be rewritten as

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} d\omega \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} F(\omega) d\omega$$

Here $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$

Or $F(s) = \mathcal{F}\{\mathrm{f(t)}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ist} dt$

is the Fourier integral transform.

The inverse Fourier transform is expressed as

$$\mathcal{F}^{-1}\{\text{F(s)}\} = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} F(s) ds$$

If we define

$$\boxed{F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ist} dt} \qquad \ldots(1)$$

then $\boxed{f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{ist} ds}$ …(2)

Eq.(1)&(2)constitute ***Fourier Integral Theorem.***

Here F(s) is known as the Fourier transform of the f(t) i.e. $F(s) = \mathcal{F}\{\text{f(t)}\}$

Here f(t) is known as the inverse Fourier transform of the F(s) i.e. $f(t) = \mathcal{F}^{-1}\{F(s)\}$

Alternative

There is no way why the factors e^{-ist} *or* e^{ist} can not be interchanged i.e. we could have defined $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt$ …(3)

then $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} F(s) ds$ …(4)

However, we will follow the eq.(1) &(2) in this chapter

Alternative

If we define $F(s) = \int_{-\infty}^{\infty} f(t) e^{-ist} dt$ then

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ist} F(s) ds$$

17.9 Fourier Sine Transforms

The Fourier transform can be written as

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\{\cos(\text{st}) - \text{i} \ \sin(\text{st})\} \text{dt}$$

If f(t) is an odd function of t i.e.

$$f(-t) = -f(t)$$

$$\mathrm{F(s)} = -\mathrm{iF}_s(s) = -i\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} f(t)\sin st\ dt$$

$$\boxed{\mathrm{F}_s(s) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\sin st\, dt}$$

or $\boxed{\mathrm{F}_s(\omega) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\sin \omega t dt}$

Here $\mathrm{F}_s(\omega)$ is infinite Fourier sine transform of odd function f(t).

The inverse Fourier sine transform

$$\boxed{f(t) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_s(\omega)\sin \omega t d\omega}$$

If the function f(t) is non-vanishing only in the interval $0 < t < \pi$.

$$\mathrm{F}_s(\omega) = \sqrt{\frac{2}{\pi}}\int_{0}^{\pi} f(t)\sin \omega t dt$$

$$\mathrm{F}_s(\mathrm{n}) = \sqrt{\frac{2}{\pi}}\int_{0}^{\pi} f(t)\sin \mathrm{n}\, t dt$$

In interval (0, l) finite Fourier sine transform is

$$\mathrm{F}_s(n) = \frac{\sqrt{2\pi}}{l}\int_{0}^{\pi} f(x)\sin\frac{n\pi x}{l}dx$$

17.10 Fourier Cosine Transform

If f(t) is an even function then the infinite Fourier cosine transform is written as

$$\boxed{\mathrm{F}_c(\mathrm{s}) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(t)\cos(\mathrm{st})\,\mathrm{dt}}$$

$$\boxed{f(t) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_c(\omega)\cos \omega t\, d\omega}$$

The finite Fourier cosine transform is written as

$$F_c(n) = \frac{\sqrt{2\pi}}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

By putting $t = \frac{\pi x}{l}$ in $F_c(n) = \sqrt{\frac{2}{\pi}}\int_0^{\pi} f(t)\cos n\,tdt$

17.11 Complex Fourier Transform

For $-\infty < x < \infty$ the complex Fourier transform of a function f(x) is expressed as

$$F(n) = \int_{-\infty}^{\infty} f(x)e^{-inx}dx$$

Here e^{inx} is kernel of the transform

The inverse is $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(n)e^{inx}dx$

17.12 Some Theorems and Properties of Fourier Transform

(1) **Linearity** – If $f(t) = a_1 f_1(t) + a_2 f_2(t) + ...$

then Fourier transform of f(t) is

$F(s) = a_1 F_1(s) + a_2 F_2(s) + ...$

Where $F_1(s)$, $F_2(s)$, ……. are Fourier transforms of $f_1(t)$, $f_2(t)$, ……..

(2) **Change of scale or similarity theorem :**

If F(s) is Fourier transform of f(t) then ***Fourier transform*** of f(at) is $\frac{1}{a}F\left(\frac{s}{a}\right)$

(3) **Shifting –** If F(s) is Fourier transform of f(t) then the Fourier transform of $f(t \pm a)$ is $e^{\pm isa}F(s)$

(4) **Conjugate Theorem** – If F(s) is Fourier transform of f(t) then the Fourier transform of complex conjugate of f(t) is $F^*(-s)$.

(5) **Modulation Theorem –** Fourier transform of $f(t)\cos at$ is

$\frac{1}{2}\left[\mathrm{F}(s-a)+F(s+a)\right]$ Where F(s) in Fourier transform of f(t).

(6) Convolution Theorem or Faltung Theorem

If f(x) and g(x) are two functions for $-\infty < x < \infty$ then convolution $f*g$ is defined as

$$f*g=\int_{-\infty}^{\infty} f(n)g(x-n)dn$$

and the Fourier transform of the convolution is the product of their Fourier transforms.

(7) Parseval's Theorem – If F(s) is Fourier transform of f(t) then

$$\boxed{\int_{-\infty}^{\infty} \left|f(t)\right|^2 dt=\int_{-\infty}^{\infty} \left|F(s)\right|^2 ds}$$

(8) Fourier transform of the square modulus of a function is $\frac{1}{\sqrt{2\pi}}$ times the self convolution of its Fourier transform.

If $g(t)=f(t)f*(t)=\left|f(t)\right|^2$

Then $\mathcal{F}\{\mathrm{g(t)}\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s')F*(s'-s)ds'$

Where F(s) is Fourier transform of f(t)

(9) Derivative of Fourier transform

$$\boxed{\frac{d^nF(s)}{ds^n}=(-i)^n\mathcal{F}\{\mathrm{t}^n\mathrm{f(t)}\}}$$

or $\frac{d^nF(s)}{ds^n}=\frac{(-i)^n}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^n f(t)e^{-ist}dt$ (1)

is the n^{th} derivative of Fourier transform of f(t).

The inverse Fourier transform of (1) is

$$\boxed{\mathcal{F}^{-1}\left\{\frac{d^nF(s)}{ds^n}\right\}=(-it)^n f(t)} \quad (2)$$

(10) Fourier transform of derivative

$$\mathrm{F}_n(s) = (is)^n F(s)$$

Here F(s) is Fourier transform of f(t) and $\mathrm{F}_n(s)$ is Fourier transform of n^{th} derivative of function f(t).

(11) Fourier sine and cosine transforms of derivatives

The Fourier sine and cosine transforms of a function f(t) are

$$\boxed{\mathrm{F}_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin\omega\,\mathrm{tdt}} \quad (1)$$

$$\boxed{\mathrm{F}_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos\omega\,\mathrm{tdt}} \quad (2)$$

Here function f(t) is well behaved such that it and its derivatives approach zero as $t \to \infty$

Fourier sine transform of first derivative $\frac{df}{dt}$ is

$$\mathrm{F}_{1s}(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{df}{dt}\sin\omega\,\mathrm{tdt}$$

$$= \sqrt{\frac{2}{\pi}}\left[f(t)\sin\omega t\right]_0^\infty - \sqrt{\frac{2}{\pi}}.\omega\int_0^\infty f(t)\cos\omega\,\mathrm{tdt}$$

The first term is zero as $f(t) \to 0$ for $t \to \infty$.

$$\therefore \mathrm{F}_{1s}(\omega) = -\omega\,\mathrm{F}_c(\omega) \quad (3)$$

The Fourier cosine transform of first derivative of f(t) is

$$\mathrm{F}_{1c}(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{df}{dt}\cos\omega\,\mathrm{tdt}$$

$$= \sqrt{\frac{2}{\pi}}\left[f(t)\cos\omega t\right]_0^\infty - \sqrt{\frac{2}{\pi}}.\omega\int_0^\infty f(t)\sin\omega\,\mathrm{tdt}$$

$$= -\sqrt{\frac{2}{\pi}}f(0) + \omega F_s(\omega) = \omega F_s(\omega) - \sqrt{\frac{2}{\pi}}f(0) \quad (4)$$

Now Fourier transform of second derivative of f(t) is

$$F_{2s}(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{d^2 f}{dt^2}\sin\omega\, tdt = \sqrt{\frac{2}{\pi}}\left[\frac{df}{dt}\sin\omega t\right]_0^\infty - \sqrt{\frac{2}{\pi}}\omega\int_0^\infty \frac{df}{dt}\cos\omega\, tdt$$

The first term becomes zero as $t \to \infty$.

$$\therefore F_{2s}(\omega) = -\omega F_{1c}(\omega) = \sqrt{\frac{2}{\pi}}\omega f(0) - \omega^2 F_s(\omega) \qquad (5)$$

Like this the Fourier cosine transform of second derivative of f(t) is

$$F_{2c}(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{d^2 f}{dt^2}\cos\omega\, tdt = \sqrt{\frac{2}{\pi}}\left[\frac{df}{dt}\cos\omega t\right]_0^\infty + \sqrt{\frac{2}{\pi}}\omega\int_0^\infty \frac{df}{dt}\sin\omega\, tdt$$

$$= -\sqrt{\frac{2}{\pi}}f'(0) + \omega F_{1s}(\omega) = -\sqrt{\frac{2}{\pi}}f'(0) - \omega^2 F_c(\omega)$$

(12) Fourier transforms of two and three variable functions

If f(x, y) is a function of two variables then Fourier transform of f(x, y is

$$F(u,v) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i(ux+vy)}dxdy \qquad (1)$$

Fourier inverse transform in

$$f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{i(ux+vy)}dudv \qquad (2)$$

If f(x, y, z) is a function of three variables then Fourier transform of f(x, y, z) is

$$F(u,v,w) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)e^{-i(ux+vy+wz)}dxdydz \qquad (3)$$

Fourier inverse transform is

$$f(x,y,z) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v,w)e^{i(ux+vy+wz)}dudvdw \qquad (4)$$

17.13 Illustrative Examples

Example 9 Find the Fourier transform of $e^{-2\alpha\pi|t|}$

Sol. $F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-2\pi\alpha|t|}e^{-ist}dt$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{2\pi\alpha t} e^{-ist} + \int_{-\infty}^{\infty} e^{-2\pi\alpha t} e^{-ist} dt \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[\frac{e^{t(2\pi\alpha - is)}}{(2\pi\alpha - is)} \right]_{-\infty}^{0} + \left[\frac{e^{-t(2\pi\alpha + is)}}{(2\pi\alpha + is)} \right]_{0}^{\infty} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{(2\pi\alpha - is)} + \frac{1}{(2\pi\alpha + is)} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2\pi\alpha + is + 2\pi\alpha - is}{4\pi^2\alpha^2 + s^2} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{4\pi\alpha}{(4\pi^2\alpha^2 + s^2)}$$

Example 10 Find the Fourier transform of the Gaussian distribution function $f(x) = Ne^{-\alpha x^2}$ where α and N are constants

Sol. $$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Ne^{-\alpha x^2} e^{-i\omega x} dx = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(\alpha x^2 + i\omega x)} dx$$

$$= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha\left\{x^2 + \frac{i\omega}{\alpha}x + \left(\frac{i\omega}{2\alpha}\right)^2\right\}} e^{\alpha\left(\frac{i\omega}{2\alpha}\right)^2} dx$$

$$= \frac{N}{\sqrt{2\pi}} e^{-\frac{\omega^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-\alpha\left(x + \frac{i\omega}{2\alpha}\right)^2} dx$$

$$= \frac{N}{\sqrt{2\pi}} e^{-\frac{\omega^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-\alpha m^2} dm \qquad \text{(Putting } x + \frac{i\omega}{2\alpha} = m\text{)}$$

$$= \frac{N}{\sqrt{2\pi}} e^{-\frac{\omega^2}{4\alpha}} \sqrt{\frac{\pi}{\alpha}} \qquad \left[\because \int_{-\infty}^{\infty} e^{-\alpha m^2} dm = \sqrt{\frac{\pi}{\alpha}}\right]$$

$$= \frac{N}{2\alpha} e^{-\omega^2/4\alpha}$$

Example 11 Find the Fourier sine transformation of e^{-x}

Sol. $F_s(\omega) = \int_0^\infty e^{-x} \sin \omega x dx$

$$F_s(\omega) = \left[\frac{e^{-x}}{1+\omega^2} (-\sin \omega x - \omega \cos \omega x) \right]_0^\infty = \frac{\omega}{1+\omega^2}$$

Example 12 Find Fourier complex transform of

$$f(x) = \begin{cases} 1 - x^2, |x| < 1 \\ 0, |x| < 1 \end{cases}$$

Sol. $f(\omega) = \int_{-1}^{1} \left(1 - x^2\right) e^{-i\omega x} dx$

$$= \left[(1 - \mathrm{x}^2) \frac{e^{-i\omega x}}{-i\omega} + \frac{2}{-i\omega} \int_{-1}^{1} x e^{-i\omega x} dx \right]$$

$$= 0 + \frac{2}{-i\omega} \left[\frac{x e^{-i\omega x}}{-i\omega} \right]_{-1}^{1} - \frac{2}{(-i\omega)^2} \int_{-1}^{1} e^{-i\omega x} dx$$

$$= \frac{2}{-\omega^2} \left[e^{i\omega} + e^{-i\omega} \right] + \frac{2}{\omega^2} \left[\frac{e^{-i\omega x}}{-i\omega} \right]_{-1}^{1}$$

$$= \frac{2}{-\omega^2} \left[e^{i\omega} + e^{-i\omega} \right] + \frac{2}{i\omega^3} \left[e^{i\omega} - e^{-i\omega} \right]$$

$$= -\frac{4}{\omega^2} \cos \omega + \frac{4}{\omega^3} \sin \omega = -\frac{4}{\omega^3} (\omega \cos \omega - \sin \omega)$$

17.14 Hankel Transform

If $\mathrm{J}_n(px)$ be the Bessel function of the first kind of order n, then the Hankel transform of a function f(x) is the interval $0 < \mathrm{x} < \infty$ is expressed as

$$F(p) = \int_0^\infty f(x).x \mathrm{J}_n(px) dx$$

Here $\mathrm{xJ}_n(px)$ is the kernel of the transform.

17.15 Some Points Related to Hankel Transform

(i) Inversion formula

If F(p) be the Hankel transform of the function f(x) for $-\infty < x < \infty$ i.e.

$$H\{f(x)\} = f(p) = \int_{-\infty}^{\infty} f(x).xJ_n(px)dx$$

then

$$f(x) = \int_0^{\infty} f(p).pJ_n(px)dp$$ is the inversion relation.

(ii) Parseval Theorem

If F(p) and G(p) are Hankel transforms of f(x) and g(x) respectively then

$$\int_0^{\infty} xf(x)g(x)dx = \int_0^{\infty} pF(p)G(p)dp$$

(iii) Linearity Theorem

$$H\{c_1f(x) + c_2g(x)\} = c_1 H\{f(x)\} + c_2 H\{g(x)\}$$

(iv) Hankel transform of the derivatives of function

The first derivative of Hankel transform of order n of the function f(x) is given by

$$F_n'(p) = -p\left[\frac{n+1}{2n}F_{n-1}(p) - \frac{n-1}{2n}F_{n+1}(p)\right]$$

Similarly the second derivative is

$$F_n''(p) = \frac{p^4}{4}\left[\frac{n+1}{n-1}F_{n-2}(p) - 2\frac{n^2-3}{n^2-1}F_n(p) + \frac{n-1}{n+1}F_{n+2}(p)\right]$$

Here $F_{n-1}(p), F_{n+1}(p), F_{n-2}(p), F_n(p)$ *and* $F_{n+2}(p)$ are Hankel transforms of function f(x) of order n-1, n+1, n-2 and n+2, respectively.

(v) Finite Hankel Transforms

If f(r)is a function which satisfies Dirichlet's conditions in the interval (0, a) then its finite Hankel transform is expressed as

$$F(p_i) = \int_0^{a} rf(r)J_n(p_i.r)dr$$

where p_i is a positive root of the transcendental equation $J_n(p_i.a) = 0$.

17.16 Some Relations for Bessel functions of First kind

While finding Hankel transform the following relations for Bessel's function are frequently used.

Recurrence relation

$$xJ_n' = nJ_n - xJ_{n+1} = xJ_{n-1} - nJ_n \tag{1}$$

$$2J_n' = J_{n-1} - J_{n+1} \tag{2}$$

$$2nJ_n = x\left(J_{n-1} + J_{n+1}\right) \tag{3}$$

$$\frac{d}{dx}(x^{-n}J_n) = -x^{-n}J_{n+1} \tag{4}$$

$$\frac{d}{dx}(x^{n}J_n) = x^{n}J_{n-1} \tag{5}$$

$$\int_0^u x^n J_{n-1}(\mathrm{x})\,\mathrm{dx} = \left[x^n J_n(x)\right]_0^u = u^n J_n(u), n > 0 \tag{6}$$

$$\int_0^a rJ_0(pr)dr = \frac{a}{p}J_1(ap) \tag{7}$$

$$\int_0^a r\left(a^2 - r^2\right)J_0(pr)dr = \frac{4a}{p^3}J_1(pa) - \frac{2a^2}{p^2}J_0(pa) \tag{8}$$

$$\int_0^\infty e^{-ax}J_0(px)dx = \left[a^2 + p^2\right]^{-1/2} \tag{9}$$

$$\int_0^\infty e^{-ax}J_1(px)dx = \frac{1}{p} - \frac{1}{p\sqrt{a^2 + p^2}} \tag{10}$$

$$\int_0^\infty xe^{-ax}J_0(px)dx = a(a^2 + p^2)^{-3/2} \tag{11}$$

$$\int_0^\infty xe^{-ax}J_1(px)dx = p(a^2 + p^2)^{-3/2} \tag{12}$$

$$\int_0^\infty e^{-ax}J_1(px)\frac{dx}{x} = \frac{(a^2 + p^2)^{1/2} - a}{p} \tag{13}$$

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{\lfloor 1 \, 2^2 (n+1)} + \frac{x^4}{\lfloor 2 \, 2^2 . 4^2 (n+1)(n+2)} \right] \qquad (14)$$

17.17 Illustrative Examples

Example 13 Find Hankel transform of $\frac{e^{-ax}}{x}$ with kernel $xJ_0(px)$

Sol. Were $f(x) = \frac{e^{-ax}}{x}$ hence

$$\text{H}\{\text{f(x)}\} = \int_0^\infty \frac{e^{-ax}}{x} . xJ_0(px)dx$$

$$= \int_0^\infty e^{-ax} J_0(px)dx$$

Using (9) of previous section

$$\text{H}\{\text{f(x)}\} = (\text{a}^2 + \text{p}^2)^{-1/2}$$

Example 14 Find Hankel transform of following function

$$f(x) = \begin{cases} 1, o < x < a, n = 0 \\ 0, x > a, n = 0 \end{cases}$$

Taking $xJ_0(px)$ as the kernel

Sol. $$\text{H}\{\text{f(x)}\} = \int_0^\infty \text{f(x)}.xJ_0(px)dx$$

$$= \int_0^a 1.xJ_0(px)dx + \int_a^\infty 0.xJ_0(px)dx$$

$$= \int_0^a xJ_0(px)dx$$

Using relation (7) of previous section

$$\text{H}\{\text{f(x)}\} = F(p) = \frac{a}{p} J_1.(ap)$$

Example 15 Find $\text{H}^{-1}[\text{p}^{-2}\text{e}^{-ap}]$ for n = 1.

Sol. Using inversion formula

$$H^{-1}[p^{-2}e^{-ap}] = \int_0^{\infty} p^{-2}e^{-ap} . p\, J_1(px)\, dp$$

$$= \int_0^{\infty} e^{-ap} J_1(px) \frac{dp}{p}$$

Using (13) of previous section

$$H^{-1}[p^{-2}e^{-ap}] = \frac{\left(a^2 + x^2\right)^{1/2} - a}{x}$$

Example 16 Find Hankel transform of $\frac{df}{dx}$ when $f(x) = \frac{e^{-ax}}{x}$ for n = 1.

Sol. $H\left\{\frac{df}{dx}\right\} = \int_0^{\infty} x \frac{df}{dx} J_1(px)dx \quad = -pF_0(p)$

Using (iv) of previous section

$$= -p\int_0^{\infty} xf(x)J_0(px)\,dx = -p\int_0^{\infty} e^{-ax} J_0(px)\,dx$$

Using (9) of previous section

$$H\left\{\frac{df}{dx}\right\} = \frac{-p}{(a^2+p^2)^{1/2}}$$

17.18 Inversion Theorem

The inversion theorem states that if F(s) is Laplace transform of f(t) where f(t) is of exponential order and has a continuous derivative than

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s) ds, t > 0$$

Proof. Let $\psi(t)$ be a function of t with $\int_{-\infty}^{\infty} \psi(t)dt$ being absolutely convergent, then the Fourier's integral is written as

$$\psi(t) = \frac{1}{\pi}\int_0^{\infty} dx \int_{-\infty}^{\infty} \psi(u) \cos x(t-u) du \qquad (1)$$

The equation (1) can also be expressed in parts in terms of $\cos x(t-u)$ and $\sin x(t-u)$ as cosine is an even function and sine is an odd function of x. Therefore

$$\psi(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}\psi(u)\cos x(t-u)du \tag{2}$$

and $$0=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}\psi(u)\sin x(t-u)du \tag{3}$$

Expanding

$$0=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}\psi(u)\left[\cos xt\cos xu-\sin xt\sin xu\right]du \tag{4}$$

and $$0=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}\psi(u)\left[\sin xt\cos xu-\cos xt\sin xu\right]du \tag{5}$$

Multiply (5) by i and add to (4) we obtain

$$\psi(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}\psi(u)\left[\cos xt\cos xu+\sin xt\sin xu\right]+i\left[\sin xt\cos xu-\cos xt\sin xu\right]du$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}\psi(u)\left[(\cos xt+i\sin xt)(\cos xu-i\sin xu\right]du$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}\psi(u)e^{ixt}e^{-ixu}du$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ixt}dx\int_{-\infty}^{\infty}\psi(u)e^{ixu}du$$

Let $\psi(t)=e^{-\gamma t}f(t)$ for $t>0$ and $\psi(t)=0$ for $t<0$ then for $t>0$.

$$e^{-\gamma t}f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ixt}dx\int_{-\infty}^{\infty}e^{-\gamma u}f(u)e^{-ixu}du, \gamma>t$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ixt}dx\int_{-\infty}^{\infty}e^{-(\gamma+ix)u}f(u)du$$

Thus $$e^{-\gamma t}f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ixt}dxF(\gamma+ix) \tag{6}$$

[Here $F(\gamma+ix)$ is Fourier transform]

Using $\gamma + \text{ix} = \text{s}$ and $\text{ds} = i\,\text{dx}$ in (6) we have

$$e^{-\gamma t} f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{(s-\gamma)t} F(\text{s})\,\text{ds} = \frac{e^{-\gamma t}}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(\text{s})\,\text{ds}$$

Therefore $$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(\text{s})\,\text{ds}$$

Hence proved

17.19 Self Learning Exercise -II

Section A (Very Short Answer type Questions)

Q.1 What is the kernel of Fourier transform?

Q.2 What is the kernel of Hankel transform?

Q.3 For which functions Fourier cosine transform is used?

Section B (Short answer type Questions)

Q.4 State Convolution Theorem.

Q.5 State Parseval's theorem for Fourier transform.

Q.6 State Parseval theorem for Hankel transform.

17.20 Summary

The unit described the three types of integral transforms i.e. Laplace transform, Fourier transform and Hankel Transform. Some important properties and theorems related to these transforms have been outlined. Methods of estimation of these transforms have been shown through examples.

17.21 Glossary

Periodic: occurring at intervals

Convergent:(Of a series) approaching a definite limit as more of its terms are added.

17.22 Answer to Self Learning Exercises

Answer to Self Learning Exercise-I

Ans.1 : $\frac{3}{s^2+9}$ **Ans.2 :** $\frac{3!}{(s-2)^4}$

Ans.3 : $\cot^{-1} s$ **Ans.4 :** $\frac{42}{s^4}$

Ans.5 : $\frac{s-2}{(s-2)^2+1}$

Answer to Self Learning Exercise-II

Ans.1 : e^{-ist} **Ans.2 :** $xJ_n(px)$

Ans.3 : Even

17.23 Exercise

Section A (Very Short Answer type Questions)

Q.1 Find Laplace transform of e^{-at}.

Q.2 What is the Fourier transform of complex conjugate of f(t)?

Q.3 What is the kind of function if Fourier transform of a real function is real?

Q.4 If F(s) is the Laplace transform of f(t), what is the Laplace transform of f(kt)?

Q.5 If F(s) is the Laplace transform of f(t) then what is the Laplace transform of e^{-kt} f(t)?

Section B (Short answer type Questions)

Q.6 Define integral transform.

Q.7 State and prove linearity property of Laplace transform.

Q.8 State Inversion theorem.

Q.9 Write the expression for finite Hankel transform.

Section C (Long answer type Questions)

Q.10 Find complex Fourier transform of $F(x) = e^{-a|x|}$ where a > 0 and x lies in the range (-∞, ∞)

Q.11 Describe Fourier sine and cosine transforms.

Q.12 Explain Laplace transform of derivatives.

Q.13 Find finite Fourier cosine transform of the function $F(x) = x^2$, x lies in range (0,4).

17.24 Answers to Exercise

Ans.1 : $\frac{1}{s+a}$

Ans.2 : F*(-s)

Ans.3 : Even

Ans.4 : $\frac{1}{k}F(\frac{s}{k})$

Ans.10 : $\sqrt{\frac{2}{\pi}}\frac{a}{a^2+\omega^2}$

Ans.13 : $\frac{128}{n^2\pi^2}cosn\pi$

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UNIT – 18
Non-homogeneous Equation, Green's Function, The Gamma Function

Structure of the Unit

18.0 Objectives

A mathematician George Green (1793-1841) was from England, has given contributions significantly to electricity and magnetism, fluid mechanics and partial differential equations. The importance of potential functions was recognized first by Green, that was published in 1828 as an essay on electricity and magnetism. In this paper, the functions, called Green's functions as a means of solving boundary value problem and the integral transformation theorems of which Green's theorem in the plane is a particular case, was introduced.

The famous mathematician L. Euler (1729) introduced gamma function as a natural extension of the factorial operation n! from positive integers n to real and even complex values of the argument. The gamma functions are useful in mathematics, the exact sciences and engineering. The incomplete gamma function (special case of gamma function) is used in solid state physics and statistics.

18.1 Introduction

Many physical problems involve second order differential equations. Some of these applications contain homogeneous equation but the more general case is the non-homogeneous equation. ***Laplace's equation*** $\nabla^2\phi = 0$ ***is an example of homogeneous equation and Poisson's equation*** $\nabla^2\phi = \rho$ ***is an example of non-homogeneous equation.*** The general solution for non-homogeneous equation contain a solution y_h of its homogeneous equation.

Consider the differential equation of ***non-homogeneous equation***

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y(x) = r(x)$$

Its general solution $y(x) = y_h(x) + y_p(x)$

The Green's function method is a powerful method for solving non-homogeneous differential equations.

18.2 Non-homogenous Equation

Consider the second order linear differential equation of the form:

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y(x) = G(x) \tag{1}$$

where P, Q, R and G are continuous functions. ***This equation is called non-homogeneous equation, if*** $G(x) \neq 0$***. Equation (1) is said to be homogeneous if*** $G(x) = 0$***.***

Again, consider the non-homogeneous Boundary Value Problem (BVP)

$$P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y(x)=G(x) \qquad a\le x\le b \tag{2}$$

with boundary conditions

$$a_1y(a)+a_2y'(a)=c_1$$
$$b_1y(b)+b_2y'(b)=c_2 \tag{3}$$

where P, Q, R and G are continuous functions and a_i,b_i,c_i , i = 1, 2 are real constants. Also a_1 and a_2 are not both zero simultaneously. Similarly b_1 and b_2 are not both zero simultaneously.

The BVP is said to be homogeneous if G = 0 and $c_1=c_2=0$.

18.3 Definition: Green's function

Our goal of this chapter is to construct Green's function and using this, we solve a non-homogeneous BVP.

Consider the non-homogeneous BVP

$$L[y]+f(x)=0,\ \ a\le x\le b \qquad \ldots\ldots(1)$$

where L is differential operator defined as $L[y]=(py')'+qy$, where $p(\neq 0), p'$ and q are real valued continuous functions on $[a,b]$. The problem has following boundary conditions:

$$a_1y(a)+a_2y'(a)=0 \tag{2}$$

$$b_1y(b)+b_2y'(b)=0 \tag{3}$$

where a_1,a_2,b_1,b_2 are constants. Also, at least one of a_1,a_2 and atleast one of b_1,b_2 are non-zero.

Definition:- A function $G(x,s)$ defined on $[a,b]\times[a,b]$ is said to be Green's function for $L[y]=0$, if for a given s,

$$G(x,s)=\begin{cases} G_1(x,s) & , \textit{if}\ \ x<s \\ G_2(x,s) & , \textit{if}\ \ x>s \end{cases}$$

where G_1 and G_2 be such that

(1) G_1 satisfies the boundary condition (2) at $x=a$ and $L[G_1]=0$ for $x<s$.

(2) G_2 satisfies the boundary condition (3) at $x=b$ and $L[G_2]=0$ for $x>s$.

(3) The function $G(x,s)$ is continuous at $x=s$.

(4) The function $\frac{\partial G}{\partial x}$ has a jump discontinuity at $x=s$, and $\left[\frac{\partial G_2}{\partial x}-\frac{\partial G_1}{\partial x}\right]_{x=s}=-\frac{1}{p(s)}$

18.4 Construction of Green's Function

Let $y_1(x)$ and $y_2(x)$ are linearly independent solutions of $L[y]=0$ on (a,b).

Let $G_1=c_1y_1(x)$ and $G_2=c_2y_2(x)$, where c_1 and c_2 are constants.

Again, let, $$G(x,s)=\begin{cases} c_1y_1(x) & , \text{ if } x\le s \\ c_2y_2(x) & , \text{ iff } x\ge s \end{cases} \tag{4}$$

We choose c_1 and c_2 such that

$$c_2y_2(s)-c_1y_1(s)=0 \tag{4a}$$

$$c_2y_2'(s)-c_1y_1'(s)=-\frac{1}{p(s)} \tag{4b}$$

In starting, we have assumed that y_1 and y_2 are linearly independent solutions of $L[y]=0$. Therefore,

$$L[y_1]=0 \qquad \text{and} \qquad L[y_2]=0$$

$$\Rightarrow \quad (py_1')'+qy_1=0 \tag{5}$$

$$\text{and} \quad (py_2')'+qy_2=0 \tag{6}$$

Multiplying (5) by y_2 and multiplying (6) by y_1, we have

$y_2(py_1')'+qy_1y_2=0$ …….(7) and $y_1(py_2')'+qy_1y_2=0$ (8)

Using (7) and (8), we get

$$y_1(py_2')'-y_2(py_1')'=0$$

$$\Rightarrow \quad \frac{d}{dx}\left[p\left(y_1y_2'-y_1'y_2\right)\right]=0$$

$$\Rightarrow \quad p(x)\left[y_1(x)y_2'(x)-y_1'(x)y_2(x)\right]=c\,, \quad \forall\, x\in[a,b] \tag{9}$$

where c is non-zero constant due to the Wronskian term

$\boxed{w(x)=\left[y_1(x)y_2'(x)-y_1'(x)y_2(x)\right]}$, which must be non-zero as y_1 and y_2 are linearly independent solutions of $L[y]=0$.

Putting $x=s$ in equation (9), we get

$$y_1(s)y_2'(s)-y_1'(s)y_2(s)=\frac{c}{p(s)}\,, \qquad c\neq 0 \tag{10}$$

Using equation (4b) and (10), we get

$$c_2=+\frac{y_1(s)}{c},\quad c_1=-\frac{y_2(s)}{c}$$

Using these values of c_1 and c_2 in equation (4), we get,

$$G(x,s)=\begin{cases}-\dfrac{y_1(x)y_2(s)}{c}\,, & if \quad x\leq s\\ -\dfrac{y_1(s)y_2(x)}{c}\,, & if \quad x\geq s\end{cases}$$

$$\Rightarrow \quad G(x,s)=\begin{cases}-\dfrac{y_1(x)y_2(s)}{p(x)w(x)}\,, & if \quad x\leq s\\ -\dfrac{y_1(s)y_2(x)}{p(x)w(x)}\,, & if \quad x\geq s\end{cases} \tag{11}$$

which is Green's function.

Theorem 1:- Consider the non-homogeneous BVP:

$$L[y]+f(x)=0, \quad a\leq x\leq b$$

$$\text{with} \quad a_1y(a)+a_2y'(a)=0$$

$$b_1y(b)+b_2y'(b)=0 \tag{12}$$

where a_1,a_2,b_1 and b_2 fulfill the conditions defined above in equation (1), (2) and (3).

Then, $y(x)$ is a solution of equation (12) if and only if

$$y(x)=\int_a^b G(x,s)f(s)ds$$

where $G(x,s)$ is a Green's function defined in equation (11).

18.5 Illustrative Examples

Example-1: Solve $\frac{d^2y}{dx^2} = f(x)$, $0 \le x \le 1$ with Boundary conditions, $y(0) = y(1) = 0$.

Sol. Compare this with $L[y] + f(x) = 0$

$$\Rightarrow \ (py')' + q(y) + f(x) = 0$$

$$\Rightarrow \ py'' + p'y' + qy + f(x) = 0$$

We get, $p(x) = 1$, $q(x) = 0$, $a = 0$, $b = 1$.

Let $y_1(x)$ be a function which satisfy $\frac{d^2y}{dx^2} = 0$ with $y(0) = 0$. Then

$$\frac{d^2y_1}{dx^2} = 0 \quad \Rightarrow \quad y_1(x) = c_1x + c_2$$

Using $y(0) = 0$, we get $c_2 = 0$. Then,

$$y_1(x) = c_1x$$

Let $y_2(x)$ be a function which satisfy $\frac{d^2y}{dx^2} = 0$ with $y(1) = 0$. Then,

$$\frac{d^2y_2}{dx^2} = 0 \quad \Rightarrow \quad y_2(x) = c_3x + c_4$$

Using $y(1) = 0$, $c_4 = -c_3$. Then,

$$y_2(x) = c_3(x-1)$$

Now, the Wronskian $w(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$

$$= (c_1x)(c_3) - c_1c_3(x-1)$$

$$= c_1c_3$$

The Green's function is defined by

$$G(x,s) = \begin{cases} -\dfrac{y_1(x)y_2(s)}{p(x)w(x)}, & if \quad x \le s \\ -\dfrac{y_1(s)y_2(x)}{p(x)w(x)}, & if \quad x \ge s \end{cases}$$

$$= \begin{cases} -\dfrac{c_1 x c_3 (s-1)}{1\ c_1 c_3} & , \ if \quad x \le s \\ -\dfrac{c_1 s c_3 (x-1)}{1\ c_1 c_3} & , \ if \quad x \ge s \end{cases}$$

$$= \begin{cases} x(1-s) & , \ if \quad x \le s \\ s(1-x) & , \ if \quad x \ge s \end{cases}$$

Hence, the solution of given problem is given by

$$y(x) = \int_0^1 G(x,s)\left[-f(x)\right] ds$$

$$\Rightarrow \qquad y(x) = -\int_0^1 G(x,s) f(s) ds$$

Example-2 Solve $\dfrac{d^2y}{dx^2} + k^2 y = f(x) \quad , \quad 0 \le x \le L$ with $y(0) = 0 \quad , \ y(L) = 0$.

Sol. Let $y_1(x)$ be a function which satisfy $\dfrac{d^2y}{dx^2} + k^2 y = 0$ with $y(0) = 0$. Then,

$$\frac{d^2y_1}{dx^2} + k^2 y_1(x) = 0 \qquad , \quad y_1(0) = 0$$

$$\Rightarrow \qquad y_1(x) = c_1 \cos kx + c_2 \sin kx$$

using $y_1(0) = 0$, we get $c_1 = 0$, so

$$y_1(x) = c_2 \sin kx$$

Let $y_2(x)$ be the solution of $\dfrac{d^2y}{dx^2} + k^2 y = 0$ with $y(L) = 0$. Then,

$$\frac{d^2y_2}{dx^2} + k^2 y_2 = 0 \quad \text{with} \quad y_2(L) = 0$$

$$\Rightarrow \quad y_2(x) = c_3 \cos kx + c_4 \sin kx$$

Using $y_2(L) = 0$ we get $\quad c_3 = -c_4 \dfrac{\sin kL}{\cos kL}$

So, $y_2(x) = -c_4 \dfrac{\sin kL}{\cos kL} \cos kx + c_4 \sin kx$

$$= \frac{c_4}{\cos kL}\left[-\cos kx \sin kL + \sin kx \cos kL\right]$$

$$\Rightarrow \qquad y_2(x) = \frac{c_4 \sin k(x-L)}{\cos kL}$$

Now, the ***Wronskian***, $w(x) = y_1(x)\,y_2'(x) - y_1'(x)\,y_2(x)$

$$= c_2 \sin kx\left[\frac{c_4 k\cos(x-L)k}{cos\,kL}\right] - c_2 k\cos kx.\left[\frac{c_4 \sin k(x-L)}{\cos kL}\right]$$

$$= \frac{c_2 c_4 k}{\cos kL}\left[\sin k(x-x+L)\right]$$

$$= c_2 c_4 k\frac{\sin kL}{\cos kL}$$

So, the Green's function $G(x,s) = \begin{cases} -\dfrac{y_1(x)\,y_2(s)}{p(x)\,w(x)} & ,\ if \quad x \le s \\[2ex] -\dfrac{y_1(s)\,y_2(x)}{p(x)\,w(x)} & ,\ if \quad x \ge s \end{cases}$

$$= \begin{cases} -\dfrac{c_2\sin(kx)\ c_4\sin k(s-L)}{1.c_2 c_4 k\dfrac{\sin kL}{\cos kL}\cos kL} & ,\ if \quad x \le s \\[4ex] -\dfrac{c_2\sin ks\ c_4\sin k(x-L)}{1.c_2 c_4 k\dfrac{\sin kL}{\cos kL}\cos kL} & ,\ if \quad x \ge s \end{cases}$$

$$= \begin{cases} -\dfrac{\sin kx\sin(s-L)k}{k\sin kL} & ,\ if \quad x \le s \\[2ex] -\dfrac{\sin ks\sin(x-L)k}{k\sin kL} & ,\ if \quad x \ge s \end{cases}$$

The solution of the given problem is

$$y(x) = -\int_0^L G(x,s)\,f(s)\,ds$$

18.6 Self Learning Exercise - I

Very Short Answer Type Questions:

Q.1 Identify the BVP $y'' + y = 0$ with $y(0) = y(1) = 1$ whether it is homogeneous or non-homogenous.

Q.2 For what value of c, the BVP $y'' + y' + y = 0$ with $y(0) = y(2) = c$, is homogeneous.

Q.3 Define Green's function.

Short Answer Type Questions:

Q.4 Solve $\frac{d^2y}{dx^2} = f(x)$ with $y(0)=0, y(L)=0$ using Green's function method.

Q.5 Find the Green's function for BVP $\frac{d^2y}{dx^2} = f(x)$, $0 \le x \le 1$ with $y(0)=\alpha$, $y'(1)=\beta$.

Q.6 Find the Green's function for BVP $-y'' = f(x)$ with $y(0)=0$, $y(1)+y'(1)=0$

18.7 Dirac-delta Function

The one-dimensional Dirac-delta function is defined by the following property as

$$\boxed{\delta(x) = 0 \quad at\, x \neq 0}$$

$$\boxed{f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)\,dx} \tag{1}$$

where $f(x)$ is any well behaved function and the integration includes the origin.

If $f(x)=1$, then

$$f(0) = \int_{-\infty}^{\infty} 1.\delta(x)\,dx = 1$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \delta(x)\,dx = 1} \text{ [Special case of equation(1)]}$$

The value $\delta(x)$ at $x=0$ is so large that its integral is equal to unity.

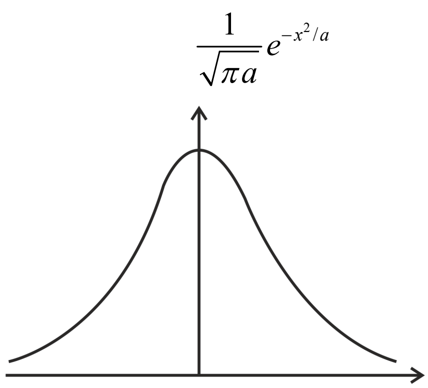
The function $\delta(x)$ is not analytic but it can be obtained as a limiting case of either analytic continuous or piesewise continuous functions. Some of the possible representations are given by

$$\delta(x) = \lim_{a\to\infty} \frac{1}{\sqrt{\pi a}} e^{-x^2/a}$$

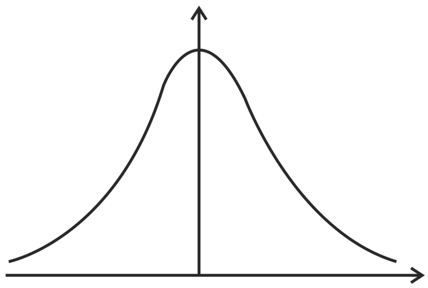
$$\delta(x) = \lim_{a\to\infty} \frac{1}{\pi} \cdot \frac{a}{a^2+x^2}$$

and $\delta(x) = \lim_{a\to\infty} \frac{\sin ax}{\pi x} = \lim_{a\to\infty} \frac{1}{2\pi} \int_{-a}^{a} e^{ixt}\,dt$

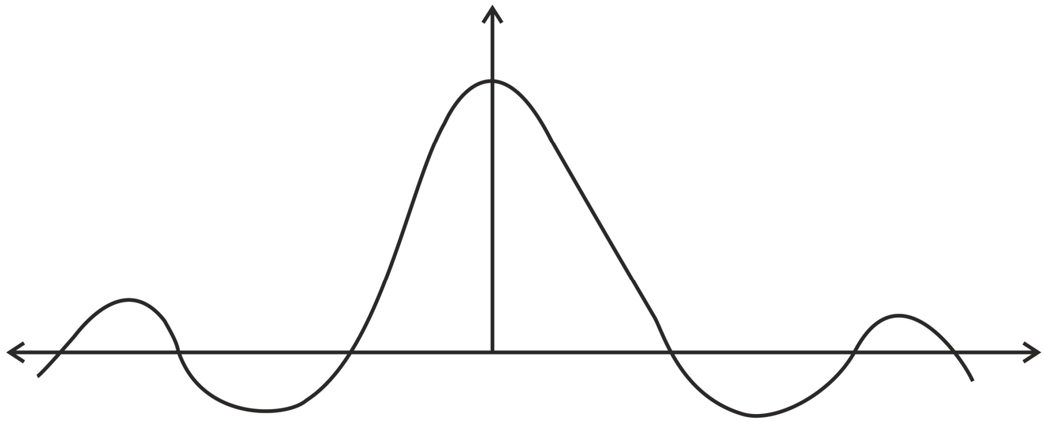
And graphical representations *f(x)* versus *x* are given by



$f(x)=\frac{1}{\sqrt{\pi a}}e^{-x^2/a}$



$f(x)=\frac{1}{\pi}\frac{a}{a^2+x^2}$



$f(x)=\frac{\sin ax}{\pi x}$

Also, $\delta(x)$ is ***symmetric about*** $x=0$ as the idea can be taken from these graphs.

If we shift the singularity at $x=x'$, then the Dirac-delta function may be written as $\delta(x-x')$ and

$$\boxed{\int_{-\infty}^{\infty} f(x)\delta(x-x')dx=f(x')}$$

and $\boxed{\delta(x-x')=\delta(x'-x)}$ (Symmetry)

18.8 Solution of Poisson Equation by Green's function method

The electrostatic potential ϕ satisfies Poisson's non-homogeneous equation in the presence of charges ρ and is given by

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0} \qquad \text{(SI system)} \tag{1}$$

and Laplace's homogeneous equation

$$\nabla^2\phi = 0 \tag{2}$$

where there is no electric charge (i.e. $\rho = 0$).

If the charges are point charges q_i, then

$$\phi = \frac{1}{4\pi \epsilon_0}\sum_i \frac{q_i}{r_i} \tag{3}$$

Using Coulomb's law for the force between two point charges q_1 and q_2, a superposition of single point charge solution is given by

$$\boldsymbol{F} = \frac{q_1 q_2 \hat{\boldsymbol{r}}}{4\pi \epsilon_0 r^2} \tag{4}$$

By replacement of the discrete point charges with a smeared-out distributed charge, charge density ρ, equation (3) gives.

$$\phi(r=0) = \frac{1}{4\pi \epsilon_0}\int \frac{\rho(\boldsymbol{r})}{r} d\tau \tag{5}$$

For the potential at $\boldsymbol{r} = \boldsymbol{r_1}$ away from the origin and the charge at $\boldsymbol{r} = \boldsymbol{r_2}$, we have,

$$\boxed{\phi(\boldsymbol{r}_1) = \frac{1}{4\pi \epsilon_0}\int \frac{\rho(\boldsymbol{r}_2)}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} d\tau_2} \tag{6}$$

Now, we need a Green's function G to satisfy Poisson's equation with a point source at the point $\boldsymbol{r}_2$, is given by

$$\boxed{\nabla^2 G = -\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)} \tag{7}$$

Clearly, G is the potential at $\boldsymbol{r}_1$ corresponding to a unit source (ϵ_0) at $\boldsymbol{r}_2$. Therefore, using Green's theorem

$$\int\left(\phi\nabla^2 G - G\nabla^2\phi\right) d\tau_2 = \int\left(\phi\nabla G - G\nabla\phi\right).d\boldsymbol{\sigma} \tag{8}$$

Taking the volume so large that the surface integral vanishes, leaving

$$\int \phi\, \nabla^2 G\, d\tau_2 = \int G\, \nabla^2\phi\, d\tau_2 \tag{9}$$

or by substituting in equation (1) and (7), we get

$$-\int \phi(r_2)\delta(r_1-r_2)\, d\tau_2 = -\int \frac{G(r_1,r_2)\rho(r_2)}{\in_0} d\tau_2 \qquad (10)$$

Using the Dirac-delta function property, we get

$$\phi(r_1) = \frac{1}{\in_0}\int G(r_1,r_2)\rho(r_2)\, d\tau_2 \qquad (11)$$

By Gauss's law, we know that

$$\boxed{\int \nabla^2\left(\frac{1}{r}\right) d\tau = \begin{cases} 0 & , \textit{ if volume doesn't include the origin} \\ -4\pi & , \textit{ if volume includes the origin} \end{cases}} \qquad (12)$$

Also, we know that

$$\boxed{\nabla^2\left(\frac{1}{4\pi r}\right) = -\delta(\mathbf{r}) \quad , \nabla^2\left(\frac{1}{4\pi r_{12}}\right) = -\delta(\mathbf{r}_1-\mathbf{r}_2)} \qquad (13)$$

where $\boxed{r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|}$

corresponding to a shift of the electrostatic charge from the origin to position $\mathbf{r} = \mathbf{r}_2$
.

Also, $\delta(\mathbf{r}_1 - \mathbf{r}_2) = 0 \quad$ if $r_1 \neq r_2$

Using equation (7) and (13), we have

$$\boxed{G(\mathbf{r}_1,\mathbf{r}_2) = \frac{1}{4\pi|\mathbf{r}_1-\mathbf{r}_2|}}$$

Therefore, the solution of the Poisson's equation, is given by

$$\phi(\mathbf{r}_1) = \frac{1}{4\pi \in_0}\int \frac{\rho(\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|} d\tau_2$$

18.9 Symmetry of Green's Function

A three-dimensional version of the self-adjoint eigenvalue equation, is given by

$$\nabla.\left[p(\mathbf{r})\nabla G(\mathbf{r},\mathbf{r}_1)\right] + \lambda q(\mathbf{r})G(\mathbf{r},\mathbf{r}_1) = -\delta(\mathbf{r}-\mathbf{r}_1) \qquad .....(1)$$

Corresponding to a mathematical point source of $\mathbf{r} = \mathbf{r}_1$ here the function $p(\mathbf{r})$ and $q(\mathbf{r})$ are well behaved but otherwise arbitrary functions of $\mathbf{r}$.

If the source point is $\mathbf{r}_2$, then Green's function satisfies the equation.

$$\nabla\left[p(\mathbf{r})\nabla G(\mathbf{r},\mathbf{r}_2)\right]+\lambda q(\mathbf{r})G(\mathbf{r},\mathbf{r}_2)=-\delta(\mathbf{r}-\mathbf{r}_2) \qquad (2)$$

Multiplying equation (2) by G($\mathbf{r}$, $\mathbf{r}_1$) and (1) by G($\mathbf{r}$, $\mathbf{r}_2$) and then subtracting, we get,

$$G(\mathbf{r},\mathbf{r}_2)\nabla.\left[p(\mathbf{r})\nabla G(\mathbf{r},\mathbf{r}_1)\right]-G(\mathbf{r},\mathbf{r}_1)\nabla.\left[p(\mathbf{r})\nabla G(\mathbf{r},\mathbf{r}_2)\right]$$
$$=-G(\mathbf{r},\mathbf{r}_2)\delta(\mathbf{r}-\mathbf{r}_1)+G(\mathbf{r},\mathbf{r}_1)\delta(\mathbf{r}-\mathbf{r}_2)$$

Integrating over whole volume, we get

$$\int_V\left[G(\mathbf{r},\mathbf{r}_2)\nabla.\{p(\mathbf{r})\nabla G(\mathbf{r},\mathbf{r}_1)\}-G(\mathbf{r},\mathbf{r}_1)\nabla.\{p(\mathbf{r})\nabla G(\mathbf{r},\mathbf{r}_2)\}\right]d\tau$$
$$=+\int_V\left[G(\mathbf{r},\mathbf{r}_1)\delta(\mathbf{r}-\mathbf{r}_2)-G(\mathbf{r},\mathbf{r}_2)\delta(\mathbf{r}-\mathbf{r}_1)\right]d\tau$$

where $d\tau$ is the small element of the volume. Using property of Dirac-delta function on RHS and Green's theorem to change volume integral into surface integral on LHS, we get.

$$\int_S\left[G(\mathbf{r},\mathbf{r}_2)p(\mathbf{r})\nabla G(\mathbf{r},\mathbf{r}_1)-G(\mathbf{r},\mathbf{r}_1)p(\mathbf{r})\nabla G(\mathbf{r},\mathbf{r}_2)\right].d\mathbf{s}$$
$$=G(\mathbf{r}_2,\mathbf{r}_1)-G(\mathbf{r}_1,\mathbf{r}_2)$$

Under the requirement that Green's function $G(\mathbf{r},\mathbf{r}_1)$ and $G(\mathbf{r},\mathbf{r}_2)$ have the same values over the surface S and their normal derivatives have the same values over the surface S or that the Green's functions vanish over the surface S the surface integral vanishes and $G(\mathbf{r}_1,\mathbf{r}_2)=G(\mathbf{r}_2,\mathbf{r}_1)$ showing the ***symmetry*** of Green's function.

18.10 The Gamma Function

The Gamma function is useful in some physical problems such as the normalization of Coulomb wave functions and the computation of probabilities in statistical mechanics. It is also useful in developing other functions that have direct physical applications.

There are the following types of definitions of the Gamma function:

(1) In form of Infinite Limit (Euler):

The first definition, named after Euler, is

$$\boxed{\Gamma z=\lim_{n\to\infty}\frac{1.2.3.........n}{z(z+1)(z+2)...........(z+n)}n^z}, \quad z\neq 0,-1,-2,-3,....$$

where z may be real or complex.

Replacing z with $z+1$, we get

$$\Gamma(z+1) = \lim_{n\to\infty} \frac{1.2.3.........n}{(z+1)(z+2)(z+3)..........(z+n+1)} n^{z+1}$$

$$\Rightarrow \qquad \Gamma(z+1) = \lim_{n\to\infty} \frac{nz}{z+n+1}\; \frac{1.2.3........n}{z(z+1)(z+2)...........(z+n)} n^{z}$$

$$\Rightarrow \qquad \boxed{\Gamma(z+1) = z\Gamma(z)}$$

which is the basic functional relation for the gamma function and also a difference equation. The Gamma function is one of a general class of functions that do not satisfy any differential equation with rational coefficients as well as the gamma function doesn't satisfy either the hypergeometric differential equation or the confluent hypergeometric equation.

In particular,

$$\Gamma(1) = \lim_{n\to\infty} \frac{1.2.3.......n}{1.2.3..........n.(n+1)} n = 1$$

$$\boxed{\Gamma(1) = 1}$$

$$\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1$$

$$\Gamma(3) = \Gamma(2+1) = 2\Gamma(2) = 2.1 = 2$$

$$\Gamma(4) = \Gamma(3+1) = 3\Gamma(3) = 3.2.1 = 6$$

$$\boxed{\Gamma(n) = (n-1).........3.2.1 = (n-1)!}$$

(2) In form of Definite Integral (Euler):

The following definite integral represents the gamma function, also called the ***Euler integral***, is given by

$$\boxed{\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt}, \qquad \operatorname{Re}(z) > 0 \tag{1}$$

This form is found in some physical problems and some other forms which found in physical problems, are

$$\Gamma(z) = 2\int_0^\infty e^{-t^2} t^{2z-1} dt, \qquad \operatorname{Re}(z) > 0 \tag{2}$$

$$\Gamma z = \int_0^1 \left[1n\left(\frac{1}{t}\right) \right]^{z-1} dt\ , \quad \text{Re}(z) > 0 \tag{3}$$

Put $z = \frac{1}{2}$ in equation (1), the integral is called Gauss Error Integral and we have

$$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

Equivalence of these two definitions:

For showing this, consider the function of two variables

$$F(z,n) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt\ , \quad \text{Re}(z) > 0 \tag{4}$$

where n being a positive integer as $\lim\limits_{n\to\infty}\left(1 - \frac{t}{n}\right)^n = e^{-t}$ (5)

Taking $n \to \infty$ in equation (4), we get

$$\lim_{n\to\infty} F(z,n) = F(z,\infty) = \int_0^\infty e^{-t} t^{z-1} dt = \Gamma z \tag{6}$$

Again, put $u = \frac{t}{n}$ in equation (4), we get

$$F(z,n) = n^z \int_0^1 (1-u)^n u^{z-1} du \tag{7}$$

Integrating by part , we get

$$\frac{F(z,n)}{n^z} = (1-u)^n \left(\frac{u^z}{z}\right)_0^1 + \frac{n}{z} \int_0^1 (1-u)^{n-1} u^z du \tag{8}$$

Repeating this with the integrated part vanishing at both end points each time, we get

$$F(z,n) = n^z \frac{n(n-1).......1}{z(z+1)...........(z+n+1)} \int_0^1 u^{z+n-1} du$$

$$= \frac{1.2.3...n}{z(z+1)(z+2)...(z+n)} n^z \tag{9}$$

Which is the form of infinite limit (Euler) for the gamma function. Therefore,

$$\lim_{n\to\infty} F(z,n) = F(z,\infty) = \Gamma z$$

(3) In form of Infinite Product (Weierstrass):

The another definition of the gamma function, which is called a Weierstrass form, is given by

$$\frac{1}{\Gamma z} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n} \tag{10}$$

where γ is the usual Euler-Mascheroni constant and $\gamma = 0.577216\ldots\ldots\ldots\ldots$

This infinite product can be used to develop the reflection identity. This can be found by

$$\Gamma z = \lim_{n\to\infty} \frac{1.2.3.......n}{z(z+1)........(z+n)} n^z = \lim_{n\to\infty} \prod_{m=1}^{n} \left(1 + \frac{z}{m}\right)^{-1} n^z \tag{11}$$

Inverting equation (11) and using $n^{-z} = e^{(-1nn)z}$

We get

$$\frac{1}{\Gamma z} = z \lim_{n\to\infty} e^{(-1n\,n)z} \prod_{m=1}^{n} \left(1 + \frac{z}{m}\right) \tag{12}$$

Multiplying and dividing by

$$\exp\left[\left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right) z\right] = \prod_{m=1}^{n} e^{z/m} \tag{13}$$

We obtain

$$\frac{1}{\Gamma z} = z \left\{ \lim_{n\to\infty} \exp\left[\left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - ln\, n\right) z\right]\right\} \times \left[\lim_{n\to\infty} \prod_{m=1}^{n} \left(1 + \frac{z}{m}\right) e^{-z/m}\right] \tag{14}$$

The infinite series in the exponent converges and defines γ, the Euler-Mascheroni constant. Hence, equation (10) follows.

In probability theory, the gamma distribution is given by

$$f(x) = \begin{cases} \frac{1}{\beta^{\alpha} \Gamma x} x^{\alpha-1} e^{-x/\beta} & , x > 0 \\ 0 & , x \le 0 \end{cases} \tag{15}$$

The constant $\left[\beta^{\alpha} \Gamma \alpha\right]^{-1}$ is choosen so that the total probability will be unity.

Factorial Notation :

According to Jeffreys and others, the -1 of z-1 exponent in our second definition is a continual nuisance. The equation (4) can be re-written as

$$\int_0^{\infty} e^{-t}\, t^{z}\, dt \equiv z!, \qquad \text{Re}(z) > 1 \qquad (16)$$

for factorial notation $z!$. We may still encounter Gauss's notation $\prod(z)$ for the factorial function

$$\prod(z) = z! \qquad (17)$$

The Γ notation is due to Legendre. The factorial function of Equation (16), is related to the gamma function

$$\Gamma(z) = (z-1)! \quad \Rightarrow \quad \boxed{\Gamma(z+1) = z!} \qquad (18)$$

If $z = n$, a positive integer, we have seen that $z! = n! = 1.2.3...n$

It should be noted that the factorial function defined by equation (16) and (18) are no longer limited to positive integral values of the argument. The difference equation, then becomes,

$$(z-1)! = \frac{z!}{z}$$

This shows that $\boxed{0! = 1}$ and $n! = \pm\infty$ ***for n, a negative integer.***

Some Important results of Gamma functions:

(1) $$\boxed{\int_0^{\infty} x^{n-1} e^{-ax}\, dx = \frac{\Gamma(n)}{a^n}}$$

(2) $$\boxed{\int_0^{\pi/2} \sin^m\theta \cos^n\theta\, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}}$$

(3) Legendre's Duplication formula:

$$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m), \qquad m \in Z$$

(4) $$\boxed{\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}} = \pi \operatorname{cosec} n\pi$$

18.11 Illustrative Examples

Example 3: Evaluate

(i) $\int_0^\infty x^5 \bar{e}^{x}\, dx$ (ii) $\int_0^\infty x^6 \bar{e}^{2x}\, dx$

(iii) $\int_0^\infty \sqrt{x}\, \bar{e}^{x^3}\, dx$

Sol. (i) $\int_0^\infty x^5 \bar{e}^{x}\, dx = \int_0^\infty x^{6-1} \bar{e}^{x}\, dx = \overline{|6} = 5! = 120$

(ii) Put $2x = y$, then

$$\int_0^\infty x^6 \bar{e}^{2x}\, dx = \int_0^\infty \left(\frac{y}{2}\right)^6 \bar{e}^{y} \left(\frac{dy}{2}\right)$$

$$= \frac{1}{128} \int_0^\infty y^{7-1} \bar{e}^{y}\, dy = \frac{1}{128} \overline{|7} = \frac{6!}{128} = \frac{45}{8}$$

(iii) Put, $x^3 = y$ then

$$\int_0^\infty \sqrt{x}\, \bar{e}^{x^3}\, dx = \int_0^\infty \sqrt{\left(y^{1/3}\right)}\, \bar{e}^{y} \frac{dy}{3y^{2/3}}$$

$$= \frac{1}{3} \int_0^\infty y^{-1/2}\, \bar{e}^{y}\, dy$$

$$= \frac{1}{3} \int_0^\infty y^{\frac{1}{2}-1}\, \bar{e}^{y}\, dy$$

$$= \frac{1}{3} \overline{\left|\frac{1}{2}\right.} = \frac{\sqrt{\pi}}{3}$$

Example 4 Prove that $\int_0^\infty \frac{x}{1+x^6}\, dx = \frac{\pi}{3\sqrt{3}}$

Sol. Put $x^3 = \tan\theta \Rightarrow x = \tan^{1/3}\theta$, then LHS becomes

$$I = \int_0^{\pi/2} \frac{\tan^{1/3}\theta}{1+\tan^2\theta} \cdot \frac{1}{3} \tan^{-2/3}\theta \sec^2\theta\, d\theta \quad \text{(say)}$$

$$= \frac{1}{3} \int_0^{\pi/2} \tan^{-1/3}\theta\, d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} \sin^{-1/3}\theta \cos^{1/3}\theta\, d\theta$$

$$= \frac{1}{3} \frac{\boxed{1/3}\ \boxed{2/3}}{2\boxed{1}} = \frac{1}{6} \frac{\pi}{\sin\frac{\pi}{3}}$$

$$= \frac{1}{6} \cdot \frac{2\pi}{\sqrt{3}} = \frac{\pi}{3\sqrt{3}}$$

Example 5 Prove that $\int_0^\infty \cos\left(x^2\right) dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

Sol. Put $x^2 = y \Rightarrow x = y^{1/2}$, then

$$I = \int_0^\infty \cos y . \frac{1}{2} y^{-1/2} dy = \frac{1}{2} \int_0^\infty y^{1/2 - 1} \cos y \, dy$$

$$= \frac{1}{2} \frac{\boxed{1/2}}{(1)^{1/2}} \cos\left(\frac{1}{2} . \frac{\pi}{2}\right) = \frac{1}{2}\sqrt{\pi} \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

18.12 Self Learning Exercise – II

(Very Short Answer Type Questions):-

Q.1 Define the Gamma function in terms of the definite integral form.

Q.2 Define the Gamma function in terms of the infinite limit form.

Q.3 What is relation between factorial and the Gamma function.

(Short Answer Type Questions)

Q.4 Prove that $\frac{\left\{\boxed{1/3}\right\}^2}{\boxed{1/6}} = \frac{\sqrt{\pi} . 2^{1/3}}{\sqrt{3}}$

Q.5 Evaluate $\int_0^1 \sqrt{1 - x^4} \, dx$

Q.6 Prove that $\int_0^{\pi/2} \tan^m x \, dx = \left(\frac{\pi}{2}\right) \sec\left(\frac{m\pi}{2}\right)$

18.13 Summary

The unit starts with construction of the Green's function and then using it in solving non-homogeneous boundary value problem. The example of non-homogeneous BVP can be Poisson's Equation with boundary condition which we solved by Green's function technique. Many practical problems involves non-homogeneous BVP, which can be solved by Green's function technique.

Further, we define the gamma function in various forms, and then some results based on it. Many physical problems involve the gamma function form that can be solved

18.14 Glossary

Homogeneous – all of the same or similar kind or nature

Differential – the result of mathematical differentiation

Equivalence – a state of being essentially equal or equivalent.

Symmetry – an attribute of a shape or balance among the parts of something.

Gamma – the 3rd letter of the Greek alphabet

18.15 Answers to Self Learning Exercises

Answers to Self Learning Exercise – I

Ans.1 : Non-homogeneous

Ans.2 : $c = 0$

Ans.4 : $y(x) = -\int_0^L f(s)G(x,s)\,ds$, where $G(x,s) = \begin{cases} \dfrac{x(L-s)}{L} & , \ if\ x \le s \\ \dfrac{s(L-x)}{L} & , \ if\ x \ge s \end{cases}$

Ans.5 : $G(x,s) = \begin{cases} x & , \quad 0 \le x \le s \\ s & , \quad s \le x \le 1 \end{cases}$

Ans.6 : $G(x,s) = \begin{cases} \dfrac{1}{2}s(x-2) & , \quad 0 \le x \le s \\ \dfrac{1}{2}x(s-2) & , \quad s \le x \le 1 \end{cases}$

Answers to Self Learning Exercise – II

Ans.1 : $\lceil z = \int_0^\infty \bar{e}^t\, t^{z-1} dt$, $\qquad \mathrm{Re}(z) > 0$

Ans.2 : $\lceil z = \lim_{n \to \infty} \dfrac{1.2.3.....n}{z(z+1)(z+2).....(z+n)} n^z$, $\qquad z \ne 0$, -1, -2, -3….

Ans.3 : $\Gamma z = (z-1)!$, $z \neq 0$, -1, -2, -3, …

Ans.5 : $\dfrac{\sqrt{\pi}\,\Gamma\left(\frac{1}{4}\right)}{6\,\Gamma\left(\frac{3}{4}\right)}$

18.16 Exercise

Section – A (Very Short Answer Type Questions)

Q.1 Define the Green's function.

Q.2 Write the solution of the non-homogeneous BVP in terms of the Green's function.

Q.3 Define the non-homogeneous equation (differential equation).

Q.4 What is the relation between Beta and the Gamma function.

Q.5 Evaluate $\Gamma\left(\frac{7}{2}\right)$

Section – B (Short Answer Type Questions)

Q.6 Evaluate (i) $\int_0^1 x^2 (1-x)^3\, dx$ (ii) $\int_0^1 \sqrt{\dfrac{1-x}{x}}\, dx$

Q.7 Evaluate $\int_0^2 x^4 \left(4-x^2\right)^{\frac{1}{2}} dx$

Q.8 Prove $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{a\cos^4\theta + b\sin^4\theta}} = \frac{\left\{\Gamma\left(\frac{1}{4}\right)\right\}^2}{4(ab)^{\frac{1}{4}}\sqrt{\pi}}$

Q.9 Prove the symmetry property of the Green's function.

Q.10 Prove the theorem-1.

Section – C (Long Answer Type Questions)

Q.11 Construct the Green's function for the non-homogeneous BVP.

Q.12 Find the solution of the Poisson's equation using Green's function.

Q.13 Define the various definitions of the Gamma function and prove the equivalency of the definite integral form and infinite limit form.

Q.14 Prove that

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\ldots\Gamma\left(\frac{n-1}{n}\right)=\frac{(2\pi)^{\frac{n-1}{2}}}{n^{1/2}}, \qquad n\in Z,\ n>1$$

18.17 Answers to Exercise

Ans.1 : See Section 18.3

Ans.2 : See the theorem-1

Ans.3 : See section 18.2

Ans.4 : $B(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Ans.5 : $\Gamma\left(\frac{7}{2}\right)=\Gamma\left(\frac{5}{2}+1\right)=\frac{5}{2}\Gamma\left(\frac{5}{2}\right)=\frac{5}{2}\Gamma\left(\frac{3}{2}+1\right)=\frac{5}{2}.\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{5}{2}.\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)=\frac{5}{3}.\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right)=\frac{5\sqrt{\pi}}{4}$

Ans.6 : (i) $\frac{1}{60}$ (ii) $\frac{\pi}{2}$

Ans.7 : 2π

References and Suggested Readings

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2. Mathematical Physics with Classical Mechanics, Satya Prakash, Sultan Chand & Sons.

UNIT-19
Introduction to Computers

Structure of the Unit

19.0 Objectives

The aim of the Unit is to introduce the readers to the fundamentals of computer system, operating systems such as DOS and windows and familiarize them to the word processing, graphics, spreadsheet and database packages. The present era is of information technology where computers are being used in every sphere, therefore basic understanding of computer has become a necessity for everyone. This unit intends to provide a fundamental knowledge of computers to the readers.

19.1 Introduction

Computers were originally developed for performing faster calculations and therefore the name 'computer' was given, which implies a machine that computes or calculates. But today the computers have evolved and are now widely used for various purposes apart from calculating. Today a computer is mainly a data processor which operates on data and is used to store, process and retrieve data for wide variety of applications. Computers have become fundamental in exchange of information and therefore they have penetrated everywhere i.e. banks, railways, recruitments, public sector organizations, scientific institutes, educational institutes homes etc. The popularity of computers can be attributed to the fact that they have evolved as a highly powerful and useful tool with high degree of accuracy, speed and versatility.

19.2 Development of Computers

The development of computers has been divided into five generations. Charles Babbage is considered to be the father of computer and Dr. J.V. Neumann influenced the development of modern computer by introducing stored programs in computers.

i) First Generation (1942-1955)- These were made of vacuum tubes, used electromagnetic relay memory and punched cards secondary storage. The instructions were in machine and assembly languages. These were bulky, costly and required constant maintenance. Examples – ENIAC, EDVAC, EDSAC, UNIVACI, IBM 701

ii) Second Generation (955-1964)- These consisted of transistors, magnetic core memory, magnetic tapes and disks secondary storage. Batch operating system and high level programming languages were used with these. Use of transistors made these computers more reliable, smaller and less expensive, but still the commercial production was difficult.

Examples- Honey well 400, IBM 7030, CDC 1604, UNIVAC LARC

iii) Third Generation (1964-1975)- These computers used Integrated circuits (ICs) with SSI and MSI technology. Larger magnetic core memory and larger capacity magnetic disks and tapes were used with these. During this period high level programming languages were standardized and time sharing operating system were used. Use of ICs made these computers faster, smaller, cheaper and easier to produce commercially.

Examples- IBM 360/370, PDP-8, PDP-11, CDC -6600

iv) Fourth Generation (1975-1989)- This period experienced development of microprocessors with ICs involving Very Large Scale Integration (VLSI) technology. Microprocessor contains all circuits for arithmetic, logical and control functions on a single chip, this lead to development of personal computers. Further developments took place in high-speed computer networks (LAN, WAN), operating systems such as MS-DOS, MS-Windows, Tropical User Interface, Programming languages, PC-based applications and network based applications. Semiconductor memories took place of magnetic core memories.

Examples- IBM PC, Apple II, TRS-80, VAX9000, CRAY etc.

v) Fifth Generation (1989-Present)- Ultra Large Scale Integration (ULSI) technology in fifth generation computer led to the increase in speed and reduction in size of microprocessors. Thus very powerful and compact computers were produced commercially at reduced costs. Optical disks became popular, there has been revolution in information through ***Word Wide Web*** (WWW).Various application areas as virtual library, electronic commerce, distance learning, vertical classrooms, multimedia etc. evolved.

Example- IBM notebook, Pentium PCs, PARAM 10000, Sun, workstations etc.

19.3 Classification of Computers

Computers are classified on the basis of

(1) Purpose (2) Technology used (3) Size and storage capacity

(1).Purpose-

(i) General purpose computers- computers which are used commonly in offices, institutes etc for commercial, educational, general application are known as general purpose computers.

(ii) Special purpose computer**-** These are the computer which are designed to perform some specific function such as scientific, weather forecasting, space applications, medical use, research application etc.

(2) Technology

(i) Analog Computer- These computers are designed for measuring physical quantities like temperature, pressure, current, voltage etc. These store data of physical quantities and perform computations on these. These computers are mainly used for scientific and engineering applications.

(ii) Digital Computers- These represent and store data in discrete quantities or numbers. These are binary digits (0 and 1) for data processing. Almost all computers used presently are digital computers for general purpose.

(iii) Hybrid Computers- These employ the technology of both analog and digital computers. These have analog to digital converters and digital to analog convertors. These are mainly used in robotics and computer aided manufacturing.

(3) Size and Storage Capacity-

(i) Micro Computers- These are the smallest computers designed for one person use. These have microprocessor, storage and input-output elements.

(ii) Mini computers- These are designed to handle multiple users simultaneously. They provide more storage capacity and communication link between users.

(iii) Mainframe computers- These are large computers with many powerful peripheral devices.

(iv) Super Computers- These have very high speed and are very large. These are mainly used for high end scientific applications.

19.4 Computer Structure

The internal architecture of computers is usually different for different models but the basic organization is same for all computers. The basic hardware components of a computer system are

(1) Central Processing Unit (CPU)

(2) Memory

(3) Input/Output devices

(1) Central Processing Unit-

It is the brain of the computer. All the major calculations, computations, execution of instructions are being carried out in central processing unit. The central processing unit consists of control unit and ***Arithmetic-Logic Unit*** (ALU). It performs mainly three functions

- Interfacing with main memory through bus
- Controlling each operation through control unit
- Performing arithmetic and logical operations through ALU

The processor cosmists of several special purpose registers. These registers are high speed memory units which are used to hold information temporarily during transfer of information between various parts of the computer. A register which can store 8-bits is referred to as an 8-bit register, which is the length of the register. Presently most of the CPU today usually have 32 bit or 64-bit registers. The larger the word size (i.e. number of bits) the speed of the computer increases proportionally.

The commonly used registers with processor of computers are-

(i) Memory Address Register- It holds the address of the active memory location.

(ii) Memory Buffer Register- It is used to hold information while it is transferred to and from memory.

(iii) Accumulator Register- It holds the data to be operated in ALU, intermediate results of processing and the results from ALU before transfer to main memory through memory buffer register. Thus it accumulates data and results.

(iv) Program control Register- It holds the address of the instruction which is to be executed next in the sequence.

(v) Instruction Register- The instruction which is to be executed currently is stored in it. When instruction gets stored its address part and operation part are separated and sent to memory address register and control unit respectively.

(vi) Input/Output Register- The transfer of data and information from input devices and to output devices in done through this registers.

Along with registers a control unit has decoder which decoder thee instructions and interprets. Thus the control of the execution of an instruction is done by control unit with help of registers and decoder. It does not perform any actual processing of data but controls and coordinates the various components of a computer system. It takes instructions from a program in main memory, interprets them, sends signals to the appropriate unit for execution of the task and communicates with input/output devices.

The Arithmetic and Logic Unit (ALU) of the processor is responsible for the actual execution of the set of instructions during data processing. Arithmetic operations such as addition, subtraction, multiplication, division etc or logical operations such as lesser than, greater than, equal to etc. are performed in ALU. Thus whenever control unit comes across an instruction related to arithmetic or logical operation it sends the instructions to ALU. The ALU consists of many registers and circuits which help it perform the required operation and processing of data.

The speed of the execution of an instruction is linked directly to the built-in clock of the computer. This clock speed is expressed in MHz or GHz (Megahertz or Gigahertz). The higher the clock frequency, the faster is the processor.

The Processors are mainly classified as-

1. CISC Processors- Processors with large instruction set with variable length instruction are known as complex Instruction set computer processor.

2. RISC Processor- To develop less complex and less expensive computers, the instructions were reduced and therefore reduced instruction set computer architecture was developed.

3. EPIC Processor- It breaks the sequential nature of processing for operation to be performed in parallel. For this computer extracts information in parallel and describes it to processor, thus uses Explicitly Parallel Instruction Computing.

4. Multicore processors- These processors have multiple core instead of a single core. Such processors can handle more work in parallel and are more energy-efficient but requires multi threaded soft ware.

(2) Memory-

The memory unit of a computer stores the data, information and instruction before and after processing. The memory unit of a computer consists of three parts.

(i) Main memory or primary or internal memory

(ii) Secondary memory (secondary storage device)

(iii) Cache memory.

(i) Main memory- It is internal or built in memory of the computer. It stores instruction and data temporarily for execution. It consists of collection of integrated circuits for storage of data and information for CPU so that the processing speed is increased as it fetches and retrieves data very quickly as compared to secondary storage devices.

The memory capacity of a computer system is usually referred to in kilobytes (KB) where ***1 KB = 1024 bytes***, megabytes (MB) which is 1, 048,567 bytes ($\sim 10^6$) or gigabytes (GB) which is 1,073,741,824 bytes ($\sim 10^9$).

The main memory is of following types:

a) ***Read only Memory (ROM)-*** It is a non-volatile memory chip in which data is stored permanently and can only be read. The instructions are not lost or erased when computer is switched off. Computer manufacturers store important micro programs in ROM which cannot be modified by users.

Manufacturer –Programmed ROM- It is a chip in which data is burnt by the manufacturer of an electronic equipment and is supplied by manufacturers. For example printer manufacturer may store printer without software in ROM chip of each printer.

User-Programmed ROM- It is commonly known as Programmable Read-Only memory (PROM) as users can record the program or information in it. But once the information is recorded it cannot be changed.

Erasable Programmable Read Only Memory- Here it is possible to erase the information stored in it and it can be reprogrammed for storage of new information. The information on EPROM may be erased through use of ultra violet light or by using high voltage electric pulses, and therefore these are known as ultraviolet EPROM (UVEPROM) or Electrically EPROM (EEPROM), respectively. EEPROM are also known as flash memory.

b) ***Random Access Memory (RAM)-*** it is the component of memory which is temporary in nature and is erased when the computer is switched off. It is a read/write type of memory and can be read and written by the user. It is of two types static RAM and Dynamic RAM. Static RAM stores data as long as power is supplied to the chip whereas Dynamic RAM retains data only for a limited time (almost 10ns) after which data is lost.

c) Complementary metal oxide semiconductor memory- It is used to store the system configuration data, time and some other important data. During booting when computer is switched on the Basic Input Output System (BIOS) matches and checks the information on CMOS memory with peripheral devices.

2) Secondary Memory- The primary memory is important for maintaining the speed of processing data but it is volatile in nature and has limited capacity. The cost is also higher as compared to secondary storage elements. Thus when more data is to be stored, secondary or auxiliary memory is used. Secondary storage has low operating speeds and lower cost. These are used for storage of large amount of data permanently and can be transferred to primary storage for processing requirements. The secondary storage devices are classified as

a) Sequential Access Device and (b) Direct Access Devices

a) Sequential Access Devices – These devices employ the accessing of information in sequential or serial manner i.e. the information is retrieved in the same sequence in which it is stored. Example- magnetic tape.

b) Direct Access Devices- Through these the information can be accessed directly or randomly. Example- Magnetic disks, Optical disks and memory storage devices.

Magnetic disks are floppy disks and hard disks (zip disk, Disk pack and Winchester Disk) Optical Disks are available as CD-ROM, WORM (CD-R), CD-RW and DVD. Memory Storage devices are available as Flash drives (Pen drives) and memory cards.

3) Cache Memory- Though primary memory helps in reducing the memory-processor speed mismatch but still the CPU is almost 10 times faster than the main memory. Thus to improve the overall performance of the processor it becomes essential to minimize the memory-processor speed mismatch. For this a small memory which is extremely fast is placed between the CPU and primary memory.

This high speed memory is known as cache memory and is used to store data temporarily during processing. This aids in improving the processing speed.

(3) Input/Output devices-

These are the devices with which computer communicates with the users. These are also known as peripheral devices.

An input device accepts data from the outside environment and transfers it to computer's primary storage in a form understandable by computer.

An output device accepts data from the computer converts it to a form understandable by users and supplies it to users.

The main types of input devices are

1) Keyboard devices- These use set of keys mounted on a keyboard connected to a computer system for transfer of data.

2) Paint and Draw devices- These are mainly used to create graphic elements such as lines, curves, shapes etc. on computer screen. These allow freehand movement and faster response. Example- mouse, trackball, joystick, electronic pen, touch screen etc.

3) Data Scanning devices- These allow direct entry of data from source document to computer system. Example- Image scanner, Optical character recognition, optical mark reader, Bar code reader, magnetic ink character recognition etc.

4) Digitizer – These are used to digitize pictures, drawings etc for storage in computers. These are used in computer Aided Design (CAD) and Geographical Information System (GIS).

5) Electronic Card Reader- It is used to read the data encoded with the help of computer and transfers data to computer for processing Example- ATM Card

6) Speech Recognition devices- Through these data is transferred to the computer through speaking.

7)Vision-input system**-** A computer with vision input device consists of a digital camera which transfers data in form of image to computer in digital form.

The main types of output devices are-

1) Monitors- Monitors are just like TV screens and display the output of the computer. Monitors are mainly two types ***CRT (Cathode Ray Tube***) monitor and ***LCD (Liquid Crystal Display)*** monitors.

2) Printers- Through printers the output can be obtained on paper in form of hard copy. There are varieties of printers such as dot-matrix, inkjet, laser etc.

3) Plotters – For high precision graphic output in hard copy, plotters are used by architects, engineers etc.

4) Screen image projector - The output from computer can be projected on a large screen through the help of projector , such system is used during presentation lectures etc. so that a large number of people can view the information simultaneously.

5) Voice response system – These system enable the computer to communicate with the users through audio output.

19.5 Self Learning Exercise -I

Very Short Answer Type Questions

Q.1 What is called as brain of computer?

Q.2 What is gigabyte?

Q.3 What is the memory between CPU and primary memory knows as?

Short Answer Type Questions

Q.4 Define Ram.

Q.5 What is function of control unit?

Q.6 What does EPROM means?

19.6 Operating System

It is a set of programs or system software which manages the resources of computer, facilitates its operation and provides an interface to the users to operate the computer system.

The major functions of an operating system is to manage processes, file, memory, input/output devices, secondary storage, provides security of data, allocates data and resources, facilitates communication and detects errors or faults

Some popular operating systems are MS- DOS, WINDOWS, UNIX, LINUX etc.

19.7 Disk Operating System (DOS)

Microsoft disk operating system (MS–DOS) is a ***single user operating system***; it is a 16 bit operating system. It does not support multiuser or multitasking. It was most popular operating system for personal computers in 1980's. It provides way to store information, process data and coordinate computer components. It instructs computer to read data stored on disk. DOS translates the command issued by the user in the format that is understandable by the computer and instructs the computer to function accordingly. It translates the result and converts the error message in the format for the user to understand. DOS can be loaded into the PC from hard disk; loading DOS is known as booting up the pc.

When the computer is switched on, the program in read only memory (ROM) known as basic input output system (BIOS) reads programs and data i.e. operating system and loads it into memory (RAM) This process is known as bootstrapping the operating system. Once loaded takes charge of the computer, handles user interaction and executes application programs. DOS booting involves reading following files namely IO.SYS. MSDOS.SYS and COMMAND.com

The main functions of DOS are to manage disk files and allocate system resources according to the requirement. It provides vital features to control hardware device such as keyboard, screen, disk devices, printers, modems and programs.

Information or data is stored on a disk in form of a file. When storing a file, a unique name is given to the file. Different files are identified by their extensions such as EXE, COM, BAT are executable files, extensions TXT, DOC, BAK, BAS represent text file. A directory is a named section of the storage of the disk which is used for storing files. The directory serves the purpose of sharing the files in an organized manner.

Any instruction given to the computer to perform a specific task is called a command. DOS has several commands which are mainly classified as two types 1) internal 2) External.

Internal commands are built-in commands stored in command interpreter file example DATE, TIME, DIR, CLS etc.

External commands are separate files that are kept in disk until required. For example FORMAT, CHKDSK, MOVE, PRINT etc.

19.8 Windows

Microsoft windows operating system was developed by Microsoft to overcome the drawbacks of the MS DOS. The main features of Microsoft Windows are

- It is a graphical user interface and therefore it is easy to use and learn for beginners
- It is a single user multitasking operating system. More than one program can be run at a time different program can be viewed on different windows of the monitor screen.
- All the program of Microsoft windows follow a standard way of working hence it is easier to use different programs of MS Windows.
- It is faster, reliable and more compatible than DOS.
- With Windows the user need not learn or type any command. He can click on with the help of mouse and perform operations.

Microsoft Windows successfully addressed the limitation of MS DOS and presented user friendly interface. Windows are rightly named as it allows user to work on several windows simultaneously Microsoft released the first version of Windows in 1985. Windows 3.0 was released by Microsoft in 1990. It was a 32 bit operating system with advanced graphics. It included program manager, file manager and print manager, further developments in windows led to better feature for performance, power, reliability, security, enhanced multimedia capabilities, easy installation etc. Various versions have been released- Window 95, Windows 98, Windows 2000, Windows XP , Windows Professional and Windows Vista.

Microsoft Windows NT is a multi user timesharing operating system. It can be used for powerful workstation networks and database servers. It included support for multiprocessor architecture. It was developed for users and had a rich application programming interface which made it easy to run high end engineering and scientific applications.

19.9 Word Processing Package

A word processing package enables to create, edit, view, format, store and print documents through a computer system. The package makes use of hardware and software for creating a document. The package allows entering text, editing text, formatting text, inserting mathematical symbols and equations, inserting picture and graphics, changing page setup, saving, retrieving, deleting document, checking spelling and grammar.

The word processing software is one of the most significant application package of Windows. Drafts, letters, reports, write ups articles etc. can be created with the help of work processing software. Earlier Word Star was popularly used but presently the most commonly used work processing package is Microsoft Word. It is a component of Microsoft office system. The main features of MS Word are

- It is easy to learn and use for general users.
- It's feature such as page setup, symbols, font, spell check, table, bullets etc. allow users to create document with accuracy.
- The text file generated by MS Word is. Doc or .docx.
- This file can be used with other applications and such files can be easily attached with emails for transfer of information.

19.10 Graphics Package

A graphics package allows a user to create edit, format view, store, retrieve and print graphs drawings pictures etc. Some commonly supported features in modern graphic packages are drawing designs, printing drawing and pictures, creating graphics and chats, importing and exporting images and graphics objects etc. Examples of these packages are Coral Draw, Auto CAD etc.

The feature of drawing design enable users to draw objects of various shapes such as lines, circles ellipses, rectangular, arcs etc. Various features are provided for moving, selecting, rotating, cropping, resizing etc. Options are provided for filling the objects with colours and also to insert textures and effects in the objects as per requirements.

In vector graphics a design is composed of pattern of lines, point circle or other geometric shapes with x, y coordinates. In raster graphics an image consists of pattern of dots called pixels. This image is mapped of screen based on binary bits so it is called bit mapped image. The resolution of image or picture improves with more numbers of pixels. Vector graphics usually occupy less space than raster graphics, but raster graphics provide more flexibility and creativity to the images and drawings.

Usually print software comes with a clip art library of stored images which can be used to drag and drop objects in creating pictures or images. The graphics software also provides an option for importing pictures and editing them as per requirement. The feature of screen capture allows users to take a snapshot of screen and convert it into image. The screen is captured as a bitmapped image and stored in computer it can be inserted into any document when required.

19.11 Spread Sheet Package

Spread Sheet is a computer application that allows creation of computerized ledger. Just like a paper worksheet or manual ledger, spreadsheet consists of multiple cells forming grid having rows and columns each forming grid having rows and columns. Each spreadsheet cell may contain text, number or formula.

VisiCalc is considered as the original electronic spreadsheet. Lotus 1-2-3 was the most popular spreadsheet in use when DOS was used as operating system commonly. Presently Excel is the most popular choice. Microsoft Excel dominates the commercial spreadsheet market.

Spread sheet package has made the preparation of worksheets for financial purpose, research purpose, keeping stocks, tracking records and analysis etc. a lot easier as compared to manual efforts. Further the speeds and accuracy also increase with computerized worksheets. Today spreadsheet package is commonly used in business accountancy, investments, personal records, institutional records, educational organizations etc.

A spread sheet consists of rows and columns. Rows are designated by numbers and columns are identified by alphabets. The intersection of a row and column is known as cell. Data in form of number, letter or formula is stored in cell.

Cell is identified by its address for example D 6, which shows that column is D and row is 6.

Spreadsheet package provides a set of commands such as move, delete, format, save, copy, insert etc. Charts can be created in the spread sheet through the use of the numerical data stored in the spreadsheet. Different types of charts such as line bar, pie, venn diagram, hologram etc can be created and edited. Formulae can also be inserted in the spreadsheet and calculations can be performed in the worksheet in a much easier way and less time consuming manner.

There are various database functions available in spreadsheet Microsoft Excel. These assist in extracting information from database. The database functions are DAVERAGE, DCOUNT, DGET, DMIN, DPRODUCT, DSTDEV, DSUM, DVAR etc.

19.12 Database Management System (DBMS)

Database is a software program which enables the users to organize a large volume of data. It is used to store, delete, update and retrieve data. Some of the database management systems available in the market are Sybase, Microsoft SQL server, Oracle RDBMS, MySQL etc.

The database management system offer advantage of storage of large volume of data in a systematic manner. The unique data field is assigned a primary key which helps in identification of data and reduces data redundancy. There systems are not language dependent and the tables can be edited conveniently. The database can be used by number of users simultaneously. The database management system also offers data security, data consistency and ease of retrieval.

The common examples of commercial application of database management system are inventory, personnel record, grading, banks etc.

Users interact with database systems through query languages. The query languages perform two main tasks – one is defining of the data structure and other is data manipulation in a speedily manner.

The main components of a DBMS are

- the memory manager
- the query processor

- the transaction manager.

The query processor converts a user query into instructions a DBMS can process. The memory manager obtains data from the database that satisfies the queries compiled by the query processor. Finally the transaction manager ensures that the execution of transaction satisfies the ACID (atomicity, consistency, isolation and durability) properties.

19.13 Self Learning Exercise-II

Very Short Answer Type Questions

Q.1 What is the extension of MS Word file?

Q.2 What is the intersection of a row and column in a spreadsheet known as?

Q.3 What is the pattern of dots in an image known as?

Short Answer Type Questions

Q.4 Which software is used for preparing reports and letters?

Q.5 What is GUI?

Q.6 What is DBMS?

19.14 Summary

The unit described in brief the development, classification and structure of computers. The functions of various component like central processing unit, memory and input-output derives has been outlined. Operating system has been defined. MS-DOS and MS-Windows features have been described. Various software packages such as word processing, graphics, spreadsheet and database management system has been introduced.

19.15 Glossary

Computer – A machine that allows user to store and process information in a fast accurate manner.

Data – Representation of information in a formalized manner suitable for communication, interpretation and processing.

Hardware – The physical equipment required to create, use, manipulate and store electronic data.

Software – The set of instructions that enables operating a computer, processing of data and carrying out specific tasks.

Peripheral Device – Any piece of equipment in a computer system which is not actually inside the computer.

IC – Integrated circuit

Process – A systematic series of action of computer carried out to manipulate data.

Programming Language – A set of rules, vocabulary and syntax used to instruct the computer to perform certain tasks.

Computer Program – A set of instructions that are used by a computer to carry out a process.

System Software – It includes operating system and all the utilities that enable a computer to function.

Application Software – Programs that users use to carry out certain tasks.

Binary number system – It uses 0 and 1 to represent values.

Bit or Binary Digit – A digit in the binary number system. It is the smallest unit of information.

Byte – A combination of bits that represent one character. It is composed of 8 bits.

Bus – The channel that allows the different parts of computer to communicate with each other.

Back up – To copy a computer file or collection of files to a second medium.

Virus – A program that is planted in one computer and then transferred with intention of corrupting or wiping out information in recipient computer.

Local Area Network (LAN) – A computer network located in a small area such as building or campus.

Wide Area Network (WAN) – A computer network that covers a large area.

Intranet – An internal computer network that belongs to an organization and is accessible to its members only.

Internet – A collection of computer networks that are linked together to exchange data and information.

World Wide Web – Network system which allows users to browse through information round the world available on computers.

19.16 Answers to Self Learning Exercises

Answers to Self Learning Exercise-I

Ans.1: CPU

Ans.2: 1,073, 741, 824 bytes

Ans.3: Cache memory

Answers to Self Learning Exercise-II

Ans.1: doc or .docx

Ans.2: cell

Ans.3: Pixel

Ans.4: MS-Word

Ans.5: Graphical User Interface

Ans.6: DataBase Management System

19.17 Exercise

Section A:Very Short Answer Type Questions

Q.1 How many bits form a byte?

Q.2 What are computers used to measure physical quantities known as?

Q.3 The results of arithmetic and logical operation are stored in which register?

Q.4 Which output device is preferred for high precision hard copy output?

Q.5 Users interact with database system through which language?

Section B: Short answer type Questions

Q.6 What is primary memory?

Q.7 What does CISC and RISC stand for?

Q.8 What are input devices? Give examples.

Q.9 What are the advantages of MS-Windows over MS-DOS?

Q.10 Where are spreadsheets packages commonly used?

Section C: Long Answer type Questions

Q.11 Describe the components of Central Processing Unit.

Q.12 Differentiate between primary and secondary memory of computer.

Q.13 What is an operating system and what is its function? Describe features of MS-Windows?

Q.14 What is Graphics software and where is it used? List some features of graphics package?

Q.15 Describe the features and uses of word-processing packages.

19.18 Answers to Exercise

Ans.1: 8

Ans.2: Analog

Ans.3: Accumulator

Ans.4: Plotter

Ans.5: Query Language

References and Suggested Readings

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UNIT-20
Finite difference,Least square curve fitting

Structure of the Unit

20.0 Objectives

After gone through this unit learner will aware of various numerical methods of interpolation. He will be able to fit a desire curve from given set of data of an observation. He can approximate a value within an interval.

20.1 Introduction

In this unit finite differences & several interpolation formulas discussed. In the topic curve fitting, using principle of least square fitting of various curves is explained. Fitting of lines, parabolas & general polynomial from given data is explained.

20.2 Finite Differences

Forward finite differences

First finite forward differences $\Delta f(x)$ is defined as

$$\Delta f(x) = f(x+h) - f(x)$$

2nd finite difference $\Delta f(x)$ is defined as

$$\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$$

$$= \{f(x+2h) - f(x+h) - \{fx+h - f(x)\}$$

$$= f(x+2h) - 2f(x+h) + f9(x)$$

So in general the nth finite difference $\Delta^n f(x)$ is defined as

$$\Delta^n f(x) = \Delta^{n-1}\{\Delta f(x)\} = \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x)$$

Writing the finite differences in the form of a table, a model finite difference table is as given below :

In this table the difference Δy_0 is written in space between y_0 and y_1 in the next column.

The same difference table acts as the forward differenced table when read form above to below and as the backward difference table when read form below successively in upwards direction.

For given 5 observations (x_0, y_0), (x_1, y_1) ....(x_4, y_4)

the5th difference $\Delta^5 y_0$ will be = 0

Difference Table

x	$y=f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
x_0	y_0					
		$\Delta y_0 = y_1 - y_0$ $= \nabla y_1$				
x_1	y_1		$\Delta^2 y_0$ $= \nabla^2 y_1$			
		$\Delta y_1 = y_2 - y_1$ $= \nabla y_2$		$\Delta^3 y_0$ $= \nabla^3 y_2$		
x_2	y_2		$\Delta^2 y_1$ $= \nabla^2 y_2$		$\Delta^4 y_0$ $= \nabla^4 y_4$	
		$\Delta y_2 = y_3 - y_2$ $= \nabla y_3$		$\Delta^3 y_1$ $= \nabla^3 y_4$		$\Delta^5 y_0 = 0$ $= \nabla^5 y_4 = 0$
x_3	y_3		$\Delta^3 y_2$ $= \nabla^2 y_4$			
		$\Delta y_3 = y_4 - y_2$ $= \nabla y_4$				
x_4	y_4					

Reading Backward Differences from the table.

The difference $f(a+nh) - f(a+(n-1)h) = \nabla f(a+nh)$ is called back difference

So $\nabla f(a+nh) = y_n - y_{n-1}$ for $n = 1, 2, ...$ The back differences

are $y_1 - y_0,\ y_2 - y_{1,} .....y_n - y_{n-1}$

are denoted by $\nabla y_{1,}\ \ \nabla y_{2,} ......,\ \nabla y_n$

respectively, and are called first second etc. backward differences where ∇ ,is the backward difference operator. Similarly we can define higher order differences i.e.

$$\nabla y_n = y_n - y_{n-1}$$

$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$$

$$\nabla^3 y_n = \nabla y_n - \nabla y_{n-1}$$

We again remind that the same table when read form above acts as a forward difference table and when read form below it acts as back difference table.

Difference Table

x	$y = f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
x_0	y_0					
		$y_1 - y_0 = \nabla y_1$				
x_1	y_1		$\Delta^2 y_0 = \nabla^2 y_2$			
		$y_2 - y_1 = \nabla y_2$		$\Delta^3 y_0 = \nabla^3 y_3$		
x_2	y_2		$\nabla^2 y_3$			
					$\nabla^4 y_4 = \Delta^4 y_0$	$\Delta^5 y_0 = 0$ $= \nabla^5 y_4$
				$\nabla^3 y_4$		
x_3	y_3	∇y_3	$\nabla^2 y_4$			
		∇y_4				
x_4	y_4					

As already mentioned the difference table when read form below gives the back differences i.e.

so $\nabla y_4, \Delta^2 y_4,$ lie along ascending lines.

The difference table when read form top in descending direction gives the forward difference i.e.

$\nabla y_1, \Delta^2 y_2,$ Lie along descending lines.

20.3 Newton – Gregory Forward Interpolation Polynomial

Given a set of $(n+1)$ data

$(x_0, y_0), (x_1 = x_0 + h, y_1) (x_2 = x_0 + 2h, y_2).....(x_n = nh, y_n)$

Let a polynomial $P_n(x) = f(x)$, where

$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + ...+ a_n(x - x_0)(x - x_1).....(x - x_{n-1})$ ..(1)

Be fitted to these data

Putting $x = x_0$ in (1), we get

$$f(x_0) = a_0 + 0 + 0$$

Next Putting $x = x_1 = x_0 + h$, we get

$$f(x_0 + h) = a_0 + a_1 = f(x_0) + a_1$$

So $a_1 = f(x_0 = h) - f(x_0) = \dfrac{\Delta f(x_0)}{h}$

Similarly, we get $a_2 = \dfrac{\Delta^2 f(x_0)}{2h^2}$....... and $a_n = \dfrac{\Delta^n f(x_0)}{h^n\ n}$

Also putting $x = x_0 + hu$ so that $x - x_0 = hu$

$$x_1 - x_0 = h(u-1).......,(x_n - x_0) = h^n\ ^n\ f(x)$$

Putting these values in (1), we get

$$f(x) = f(x_0) + \frac{u}{\lfloor 1} \Delta f(x_0) + \frac{u(u-1)}{\lfloor 2} \Delta^2 f(x_0) + ... + \frac{u^{(n)}}{\lfloor n} \Delta^n f(x_0)$$

This is known as the Newton – Gregory's forward difference interpolation formula.

20.4 Newton-Gregory backward difference interpolation formula

Let $P_n(x) = f(x_n + hu)$ be a nth degree polynomial expressed as

$$f(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}).....a_n(x - x_n)(x - x_{n-1}).....(x - x_1)$$

To be fitted to the set of $(n+1)$ data.

$\left(x_{0,}y_0\right),\left(x_1+h,y_1\right),\left(x_2-x_0\ +2h,y_2\right)......\left(x_n=x_0\ +nh\,,y_n\right)$

Putting $x=x_n$ in (1)

$$f\left(x_n\right)=a_0$$

Putting $x=x_{n-1}$ in (1)

$$f\left(x_{n-1}\right)=f\left(x_n\right)+a_1\left(x_{0-1}-x_n\right)$$

So $$a_1=\frac{f\left(x_{n-1}\right)-f\left(x_n\right)}{x_{n-1}-x_n}=\frac{-\nabla f\left(x_n\right)}{-h}$$

$$=\frac{\nabla f\left(x_n\right)}{h\ 1}\text{ or }=\frac{\nabla y_n}{h\ 1}$$

Similarly on putting $x=x_{n-2}$ in (1) we get

$$a_2=\frac{\nabla^2}{h^2\ \ 2}f\left(x_n\right)\text{or }\frac{\nabla^2 y_n}{h^2\ 2}....$$

Proceeding in their way $a_n=\dfrac{1}{h^n\ \ n}\ \nabla^n y_n$

Putting $x=x_n+hu,$ we get

$$f\left(x\right)=f\left(x_n+hu\right)=f\left(x_n\right)+\frac{u}{\lfloor 1}\nabla f\left(x_n\right)+....\frac{u\left(u+1\right)}{\lfloor 2}\nabla^2\ f\left(x_n\right)....\frac{\left(u+n-1\right)}{\lfloor n}\nabla^n f\left(x_n\right)$$

20.5 Stirling's formula

In order to interpolate the value of y corresponding some x lying near the centre of the interval $\left(x_0,x_0+nh\right)$, is assumed as origin at some central value which may be $x_0+\dfrac{n}{2}h$ or $x_0+\dfrac{n-1}{2}h$ or $x_0\ +\ \dfrac{n+1}{2}h$ which ever is integral. Relative to this origin denoted by 0 the values prior to it are denoted by $-1,-2,-3......$ and those after it as + 1, + 2………..

The difference table is then designated as follows :							
x_0	relative to original at $x_0\ +3h$	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$

x_0	-3	y_{-3}					
			Δy_{-3}				
$x_0 + h$		y_{-2}					
	-2			$\Delta^2 y_{-3}$			
			Δy_{-2}		$\Delta^3 y_{-3}$		
$x_0 + 2h$		y_{-1}					
	-1			$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$	
			Δy_{-1}		$\Delta^3 y_{-2}$		
$x_0 + 3h$		y_0					$\Delta^5 y_{-4}$
				$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$	
$x_0 + 4h$	0		Δy_0		$\Delta^3 y_{-1}$		
$x_0 + 4h$		y_1		$\Delta^2 y_0$			
			Δy_1				
$x_0 + 6h$	1	y_2					
	2	y_3					
	3						

Origin is at $x_0 + 3h$.

Let $x = x_0 + 3h + hu$

Then the Stirling's central difference formula which is average of Gauss's forward and back difference formula is given below is expressed as.

$$y_u = y_0 + \frac{u}{\lfloor 1}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{\lfloor 2}\Delta^2 y_{-1}$$

$$+\frac{u\left(u^2-1\right)}{\lfloor 3}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{u^2\left(u^2-1\right)}{\lfloor 4}\Delta^4 y_{-2} + \ldots..$$

Gauss's forward difference formula is

$$f\left(x_0 + hu\right) = f\left(0\right) + \frac{u}{\underline{|1}}\Delta f\left(0\right) + \frac{u\left(u^{-1}\right)}{\underline{|2}}\Delta^2 f\left(-1\right) + \frac{\left(u+1\right)u\left(u-1\right)}{\underline{|3}}\Delta^3 f\left(-1\right) + ...$$

and back difference formula is

$$f\left(x_0 + hu\right) = f\left(0\right) + \frac{u}{\underline{|1}}\Delta f\left(-1\right) + \frac{\left(u+1\right)u}{\underline{|2}}\Delta^2 f\left(-1\right) + \frac{\left(u+1\right)u\left(u-1\right)}{\underline{|3}}\Delta^3 f\left(-2\right) + ...$$

The upper arrow in wavy form corresponds to Gauss's back difference formula and lower arrow in wavy form to Gauss's forward different formula. The arrows marked in the above table, starting from y_0 will help in writing the above formula. The upward arrows diverge to $\Delta^2 y_{-1}$ and so on. The term in the stirling's formula with coefficient $\frac{u}{1}$ etc. will be

Multiplied by their averages. So the 2nd term is $\frac{u}{1}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right)$

At second difference. both the arrows converge to $\Delta^2 y_{-1}$ so the corresponding term is $= \frac{u\left(u-1\right)}{2} + \frac{u\left(u+1\right)}{2}\Delta^2 y_{-1}$

$$= \frac{u^2}{2}\Delta^2 y_{-1}$$

And so on in this way.

20.6 Illustrative Examples

Example 1 Find the cubic polynomial which take the following values

x	:	0	1	2	3
$f(x)$	:	0	1	2	3

Hence or otherwise find $f\left(4\right)$

Sol . we know that

$$f\left(a+hu\right) = f\left(a\right) + {}^uC_1\Delta f\left(a\right) + {}^uC_2\Delta^2 f\left(a\right)$$

Substituting $a = 0, h = 1, and \quad u = x$

$$\therefore f\left(x\right) = f\left(0\right) + x f\left(0\right) + \frac{x\left(x-1\right)}{\underline{|2}}\Delta^2 f\left(0\right) + \frac{x\left(x-1\right)\left(x-2\right)}{\underline{|3}} \times \Delta^3 f\left(0\right)$$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	i			
		-1		
1	0		2	
		1		6
2	1		8	
		9		$\therefore\ y_4 - 27 = 6$
3	10		$(y_4 - 19)$	
		$(y_4 - 10)$		
4	y_4			

or $\quad f(x) = 1 - x + (x^2 - x) + (x^3 - 3x^2 + 2x)$ (putting values)

$f(x) = x^3 - 2x^2 + 1$The polynomial is cubic, therefore $y_4 - 27 = 6 \Rightarrow y_4 = 33$

(Third difference should be constant and equal to 6)

Example 2 In an examination the number of candidates who obtained marks between certain limits were as follows :

Marks :	0-19	20-39	40-59	60-79	80-99
No of candidates	41	62	65	50	17

Sol. : from given data the cumulative frequency table is given below :

Marks : (x)	19	39	59	79	99
No of candidates	41	103	168	218	235

Difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
19	41				
		62			
39	103		3		
		65		-18	
59	168		-15		0
		50		-18	
79	218		-33		
		17			
99	235				

Here we have $a = 19,\ \ h = 20,\ \ a + uh = 70$

$\therefore \qquad 19 \times u \times 20 = 70 \Rightarrow u = 2.55$

Now by Newton's forward interpolation formula.

$$f(a+uh) = f(a) + u\Delta f\ (a) + \frac{u\ (u-1)}{2!}\ \Delta^2 f(a) + \frac{u\ (u-1)(u-2)}{3!}\ \Delta^3 f(a)$$

Or $f(70) = 41 + 2.55 \times 62 + \dfrac{2.55 \times 1.55}{2} \times 3 + \dfrac{2.55 \times 1.55 \times 0.55}{6} \times (-18)$

$= 41 + 158.10 + 5.39 - 6.52$

$= 198.51 = 199\ (approx.)$

Example3. The population of a town in the decennial census were as under. Estimate the population for the year 1925.

years x:	1891	1901	1911	1921	1931
Population y:	46	66	81	93	101

(in thousands)

Sol. From the given data, the difference table is as given below :

Year x	**Population** y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	46				
		20			
1901	66		−5		
		15		2	
1911	81		−3		−3
		12		−1	
1921	93		−4		
		8			
1931	101				

Here h = 10, a = 1891, a + nh = 1931, (a + nh) + uh =1925 (given) or

$1931 + \mu \times 10 = 1925 \qquad \Rightarrow u = \dfrac{1925 - 1931}{10} = -0.6$

Now using Newton's backward interpolation formula.

$$y(a+nh+hu) = f(a+nh)+\frac{\mu}{11}\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh)+......$$

$$y1925 = 101+(-0.6)\times 8+\frac{(-0.6)(0.4)}{2!}\times(-4) + \frac{(-0.6)(0.4)(1.4)}{3!}\times(-1)$$

$$= 101 - 4.8 + 0.48 + 0.056 = 96 \quad 96.6352(approx.)$$ **Ans.**

Example 4. Use Stirling's formula to find $y_{28,}$ and $y'_{30,}$ given

$y_{20} =$ 49225, $y_{25} = 48316$ $y_{30} = 47236$

$y_{35} =$ 45926, $y_{40} = 44306$

Sol .: Difference table is

x	$\frac{x-30}{5}=u$	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
20	–2	49225				
			–909			
25	–1	48316		–171		
			–1080		–69	
30	0	47236		–230		–11
			–1310		–80	
35	1	45926		–310		
			–1620			
40	2	44306				

$x = 30+5u$

$$f(x) = f(0)+u\frac{\Delta f(0)+\Delta f(-1)}{2} + \frac{u^2}{2}\Delta^2 f(-1)$$

$$+\frac{4(u^2-1)}{6}\left\{\frac{\Delta f(-1)+\Delta^3 f(-2)}{2}\right\} + u^2\frac{u^2-1}{24}\Delta^4 f(-2)$$

For $x = 28,\ u = \frac{28-30}{5} = -0.4$

$$f(28) = 47236 - 0.4\frac{(-1310-1080)}{2} + \frac{(0.4)^2}{2}(-230)\frac{-0.4(+0.16-1)}{6}\frac{(-80-5a)}{2}$$

$= 47236+478 - 18.4 - 3.9 + 0.1176 =$ **47692** Approx

$$y_u = u_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{4^2}{2}\,\Delta^2 y - 1 + \frac{4^3 - 4}{6}\left(\frac{\Delta^3 y - 1 + \Delta^3 y - 2}{2}\right)$$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx} = \frac{1}{x}\left[\frac{\Delta y_0 + \Delta_{y-1}}{2}\; u\Delta^2 y - 1 + \frac{3u^2 - 1}{6}\left(\frac{\Delta^2 y - 1 + \Delta^3 y - 2}{2}\right)\right]$$

$$30 = x = 30 + 54 \text{ so } 4 = 0 = \frac{1}{h}\left[\frac{-1310 - 1080}{2}\right] = -239$$

Example 5 Use Stirling's formula to find u_{32} from following table.

$U_{20} = 14.035$, $U_{25} = 13.674$, $U_{30} = 13.257$

$U_{35} = 12.734$, $U_{40} = 12.089$, $U_{45} = 11.309$

Sol : Forward difference table for the given data is.

x	$\frac{x-30}{5}$	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	
20	–2	14.035					
			–.361				
25	–1	13.674		–0.056			
			–0.417		–0.050		
30	0	13.257		–0.106		–0.034	
			–0.523		–0.016		–0.021
35	1	12.724		–0.122		–0.013	
			–0.645		–0.013		
40	2	12.089		–0.125			
			–0.780				
45	3	11.309					

For x=32,   32 $=x_0 + hu = 30 + 54$,so   u= 0.4

By stirling formula

$$f(x) = f(0) + \frac{u}{(1)!}\left(\frac{\Delta f_{-1} + \Delta f_0}{2}\right) + \frac{u^2}{2}\,\Delta^2 f(1)$$

$$+\frac{u(u^2-1)}{6}\left(\frac{\Delta^3 f_{-1} + \Delta^3 f_{-2}}{2}\right) + \frac{u^2(u^2-1)}{24}\,\Delta^4 f_{-2}$$

$$=13.257 + 0.4\left(\frac{-0.417 - 0.523}{2}\right) + \frac{0.16}{2}(-0.106)$$

$$+(0.4)^2\frac{(0.16-1)}{6}\left\{\frac{-0.5-0.016}{2}\right\}$$

$$= 13.257 - 0.188 - 0.00848 = 13.061$$

Example 6 Use Gauss forward formula to find $y_{3.75}$ from the following table

x	*2.5*	*3.0*	*3.5*	*4.0*	*4.5*	*5.0*
y	*24.145*	*22.043*	*20.225*	*18.644*	*17.262*	*16.047*

Sol : Taking 3.5 as origin $(h=5)$. the value of u corresponding to 3.75 is

$$u = \frac{3.75-3.5}{.5} = 0.5$$

Difference Table

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$	$\Delta^5 y_u$
2.5	−2	24.145					
			−2.102				
3.0	−1	22.043		.248			
			−1.818		−.047		
3.5 …..	0	20.225…………………		.237……………………		.009……	
			−1.581		−.038		−.003
4.0	1	18.644		.199		.006	
			−1.382		−.032		
4.5	2	17.262		.167			
			−1.215				
5.0	3	16.047					

Gauss forward formula is : -

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{6}\Delta^6 y_{-1}$$

$$+\frac{(u+1)\,u\,(u-1)\,(u-2)}{24}\Delta^4 y_{-2} + \frac{(u+2)\,(u+1)\,(u-2)}{120}\Delta^5 y_{-2} + ... \qquad .(1)$$

Replacing $u = .5$ and desired values form the difference table in the formula (1), we get

$$y_{1/2} = 20.225 + 5(-1.581) + \frac{5(-5)}{2}(.237) + \frac{(1.5)(.5)(-.5)}{120}(-.003)$$

$$+ \frac{(1.5)(.5)(-.5)(-1.5)}{24}(.009) + \frac{(2.5)(1.5)(.5)(-.5)(-1.5)}{120}(-.003)$$

$$= 20.225 - 0.7905 - 0.029625 + 0.002375 + 0.00210938 - 0.0000352$$

or , $y_{3.75} = 19.40$

Example 7.interpolate by means of Gauss's Backward formula the population for the year 1976, form the following table :

Year :	1931	1941	1951	1961	1971	1981
Population (in lacs) :	12	15	20	27	39	52

Sol :. According to the question $h = 10$ and taking 1961as origin,

For the year 1976, $u = \dfrac{1976 - 1961}{10} = 1.5$

Difference table

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$	$\Delta^5 y_u$
1931	−3	12					
			3				
1941	−2	15		2			
			5		0		
1951	−1	20		2		3	
			7		3		−10
1961	0	27		5		−7	
			12		−4		
1971	1	39		1			
			13				
1981	2	52					

Gauss backward formulas is :

$$y_u = y_0 + {}^{u}C_1 \Delta y_{-1} + {}^{u+1}C_2 \Delta^2 y_{-1} + {}^{u+1}C_3 \Delta^3 y_{-2} + {}^{u+2}C_4 \Delta^4 y_{-2} + {}^{u+2}C_5 \Delta^5 y_{-3} + ...$$

$$\Rightarrow y_{1.5} = 2.7 + 1.5\times 7 + \frac{2.5\times 1.5}{2}\times 5 + \frac{2.5\times 1.5\times .5}{6}\times 3$$

$$+ \frac{3.5\times 2.5\times 1.5\ .5}{24}(-7) + \frac{(3.5)(2.5)(1.5)(-.5)}{120}(-10)$$

$= 27 + 10.5 + 9.375 + 0.9375 - 1.9140625 - 0.2734375$

$= 46.171875$ lacs (apporx).

20.7 Self Learning Exercise

Q.1 The following table gives the sales of a firm for the last five years. Estimate the sale for the year 1951

Years	1946	1948	1950	1952	1954
Sales (in thousands)	40	43	48	52	57

Q.2 The ordinates of the normal curve are given by the following table

x :	0.0	0.2	0.4	0.6	0.8
y :	0.3989	0.3910	0.3683	0.3332	0.2897

Find : (a) y (0.25) (b) y (0.62)

Q.3 From the following tale find the number of students who obtained marks between 40 and 45:

Marks obtained	No. of students
30-40	31
40-50	42
50-60	51
60-70	35
70-80	31

Q.4 Given the following date:

x :	10^0	20^0	30^0	40^0	50^0	60^0	70^0	80^0
y :	0.9848	0.9397	0.8660	0.5660	0.6428	0.500	0.342	0.1737

Evaluate : $(a)\ y(25^0)$ $(b)\ y(32^0)$ $(c)\ y(73^0)$

Q.5 A second degree polynomial passes through $(0.1),(1.3)$, (2.7) and (3.13). find the polynomial

Q.6 Find $f(1.5)$ from the following table:

x :	1	2	3	4	5	6	7	8
y :	1	8	27	64	125	216	343	512

Q.7 Use Stirling's formula to comput $u_{12.2}$ from the following data :

x:	10	11	12	13	14
$10^5 u_x$	23967	28060	31788	35209	38368

Q.8 Given :

$\theta = 0^0$	5^0	10^0	15^0	20^0	25^0	30^0
$\tan\theta = 0.0000$	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Q.9 Use Stirling formula to find y_{35} from the following data :

$y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, $y_{50} = 243$.

Q.10 Interpolate by means of Gauss's Backward formula the population for the year 1936, from the following table :

Year	1901	1911	1921	1931	1941	1951
Population (in thousands)	12	15	20	27	39	52

20.8 Principle of Least Square

Principle of least square is the most systematic procedures to fit a unique curve through given data. It states "Curve of best fit for a given data is that for which sum of square of deviation is minimum".

Suppose we have to fit the curve $y = f(x)$ for given set of data $(x_{i,} y_i)$ (I = 1, 2 …..n) At $x = x_i$ the experimental value of the ordinate is y_i & the corresponding value on fitting curve is $f(x)$.

If e_i be the error of Approximation at $x = x_i$ then we have

$$e_i = y_i - f(x_i)$$

e_i may be (–ve) or (+ve) so by giving equal weightage to each residuals consider

$$S = \sum_{i=1}^{n} (y_i - f(x_i))^2$$

Now according to principal of least square curve of best fit is that for which S is minimum.

20.9 Fitting a straight line

Let $\{(x_{i,} y_i) / i = 1, 2, ..., m\}$ be a set of observations. We have to fit a straight line $y = a + bx$ (2)

and corresponding observed value is y_i. Let e_i be the error at $x = x_i$, then

$$e_i = y_i - Y_{i,} \qquad \text{i = 1, 2, ………, m}$$

Or $\quad e_i = y_i - (a + bx_i), \qquad$ i = 1, 2, ………, m

The sum of squares S (say) of this error is given by

$$S = \sum_{i=1}^{n} \left[y_i - (a + bx_i) \right]^2 \tag{3}$$

Least squares principle requires that S be minimum. Form(3), it is clear that S depends on a and b, that is, S is a function of a and b. Thus, we have to find the value of a and b so that S become minimum. By the theory of maxima-minima,the necessary conditions for S to be minimum are

$$\frac{\partial S}{\partial a} = 0 = \frac{\partial S}{\partial b},$$

Form (3), we have

$$-\sum_{i=1}^{n} 2\left[y_i - (a + bx_i) \right] = 0$$

and $\quad -\sum_{i=1}^{n} 2x_i \left[y_i - (a + bx_i) \right] = 0$

On simplification of these two equations, we have

$$\sum_{i=1}^{n} y_i = na + b \sum_{i=1}^{n} x_i \qquad \left(\therefore \sum_{i=1}^{n} a = ma \right)$$

Or $\quad \sum y_i = na + b \sum x_i \qquad (4)$

and $\quad \sum_{i=1}^{n} x_i y_i = a \sum_{i=1}^{n} x_i + b \sum_{i}^{n} x_i^2 \qquad (5)$

Equation (4) & (5) are said to be normal equations. We can determine values of a & b by solving these equations.

20.10 Fitting of a Parabola

Let $y = a + bx + cx^2$ be a parabola to be fitted for the data (x_i, y_i)

(i =1,2,3….n) the error at $x = x_i$ is

$$e_i = y_i - f(x_i) = e_i = y_i - \left(a + bx_i + cx_i^2 \right)$$

Let $S = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} \left(y_i - \left(a + bx_i + cx_i^2 \right) \right)$

Now by principal of least square for best fit S is minimum i.e.

$$\frac{\partial S}{\partial a}=0, \frac{\partial S}{\partial b}=0 \ \& \ \frac{\partial S}{\partial c}=0$$

$$\sum_{i=1}^{n} 2\left(y_i-\left(a+bx_i+cx_i^2\right)\right)=0$$

$$\sum_{i=1}^{n} 2\left(y_i-\left(a+bx_i+cx_i^2\right)\right)x_i=0$$

$$\sum_{i=1}^{n} 2\left(y_i-\left(a+bx_i+cx_i^2\right)\right)x_i^2=0$$

On solving

$$\sum_{i=1}^{n} y_i=an+b\sum_{i=1}^{n} x_i+c\sum_{i=1}^{n} x_i^2$$

$$\sum_{i=1}^{n} x_iy_i=a\sum_{i=1}^{n} x_i+b\sum_{i=1}^{n} x_i^3$$

$$\sum_{i=1}^{n} x_i^2y_i=a\sum_{i=1}^{n} x_i^2+b\sum_{i=1}^{n} x_i^3+c\sum_{i=1}^{n} x_i^4$$

Note : (1) For fitting parabola $y=a+bx^2$

Normal equations are

$$\sum_{i=1}^{n} y_i=an+b\sum_{i=1}^{n} x_i^2$$

& $$\sum_{i=1}^{n} x_i^2y_i=a\sum_{i=1}^{n} x_i^2+b\sum_{i=1}^{n} x_i^4$$

(2) For fitting parabola $y=ax+bx^2$

Normal equations are

$$\sum_{i=1}^{n} x_iy_i=a\sum_{i=1}^{n} x_i^2+b\sum_{i=1}^{n} x_i^3$$

& $$\sum_{i=1}^{n} x_i^2y_i=a\sum_{i=1}^{n} x_i^3+b\sum_{i=1}^{n} x_i^4$$

20.11 Fitting of General Polynomial

We can fit a general polynomial of degree n by using principal of least square let polynomial be

$$Y=a_o+a_1x+a_2x^2+......+a_nx^n$$

Which is to be filled to given data (x_i, y_i), i=1,2,3,…….n then

$$S=\sum_{i=1}^{n}(y_i-Y_i)$$

Where $Y_i=Y(x_i)$

$$=a_0+a_1x_i+a_{2i}^2+......+a_nx_i^n$$

so $$S=\sum_{i}^{n}\left[y_i-\left(a_0+a_1x_i+a_2x_i^2+.....+a_ns_i^n\right)\right]^2$$

For S to be minimum, we must have

$$\frac{\partial s}{\partial a_0}=0,\frac{\partial s}{\partial a_1}=0,......,\frac{\partial s}{\partial a_n}=0,$$

That is, $\frac{\partial s}{\partial a_0}=\sum_{i=1}^{n}-2\left[y_i-\left(a_0+a_1x_i+a_2+a_2x_i^2+......+a_nx_i^n\right)\right]=0$,

$$\frac{\partial s}{\partial a_1}=\sum_{i=1}^{n}-2x_i\left[y_i-\left(a_0+a_1x_i^2+.....+a_nx_i^n\right)\right]=0,$$

$$\frac{\partial S}{\partial a_2}=\sum_{i=1}^{n}-2x_i\left[y_i\left(a_o+a_1x_i^2+a_2x_i^2+..+a_nx_i^n\right)\right]=0,$$

…. …. ….

$$\frac{\partial S}{\partial a_n}=\sum_{i=1}^{n}-2x_i^n\left[y_i-\left(a_o+a_1x_i+a_2x_i^2+..+a_nx_i^n\right)\right]=0,$$

Simplifying above (n+1) equations, we get following normal equations,

$$\sum y_i=a_on+a_1\sum x_i+a_2\sum x_i^2+....+a_n\sum x_i^n$$

$$\sum x_iy_i=a_o\sum x_i+a_1\sum x_i^2+a_2\sum x_i^3+...+a_n\sum x_i^{n+1},$$

$$\sum x_i^2y_i=a_o\sum x_i^2+a_1\sum x_i^3+a_2\sum x_i^4+...+a_n\sum x_i^{n+2},$$

…………………………………………………………………………………..

$$\sum x_i^ny_i=a_o\sum x_i^2+a_1\sum x_i^2+a_1\sum x_i^{n+1}+a_2\sum x_i^{n+2}+...+a_n\sum x_i^{n+2},$$

These (n+1) equations can be solved for (n+1) unknowns $a_o,a_1,a_2......a_n$,

20.12 Illustrative Examples

Example 8: Using the method of least-squares find a straight line that fits the following data:

x	71	68	73	69	67	65	66	67
y	69	72	70	70	68	67	68	64

Also find the value of y at x = 68.5.

Solution : Let the required straight line be

$$y=a+bx \qquad \text{…(i)}$$

The normal equations are

$$\sum y_i = na + b\sum x_i \qquad \ldots(ii)$$

and $$\sum x_i y_i = a\sum x_i + b\sum x_i^2 \qquad \ldots(iii)$$

Now, to get the values of $\sum y_i, \sum x_i, b\sum x_i y_i$ and $\sum x_i^2$, , we construct following table:

I	x_i	y_i	$x_i y_i$	x_i^2
1	71	69	4899	5041
2	68	72	4896	4624
3	73	70	5110	5329
4	69	70	4830	4761
5	67	68	4556	4489
6	65	67	4355	4225
7	66	68	4488	4356
8	67	64	4288	4489
Sum	546	548	37422	37314

Hence, $\sum x_i = 546, \sum y_i = 548$

$$\sum x_i y_i = 37422, \sum x_i^2 = 37314$$

and total number of given data m =8,

substituting these values in (ii) and (iii), we get

548 = 8a + 546b

37422 = 546a + 37314 b

Solving these two equations for a and b, we get

$a = 39.545484$ and $b = 0.424242$

Thus, the required straight line is

$$y = 39.545484 + 0.424242x$$

Now at x = 68.5, value of y is given by

$$y = 39.545484 + 0.424242 \times 683.5$$

$$= 68.606061$$

Example 9 : Fit a straight line to the given data

x	1	2	3	4	5	6
y	2.6	2.7	2.9	3.025	3.2	3.367

Also find value of y at x = 5.5.

Solution : Let the required straight lne be

$$y = a + bx$$

then, the normal equation are

$$\sum y_i = na + b \sum x_i$$

and $$\sum x_i y_i = a \sum x_i + b \sum x_i^2$$

Now, from the given data, we have following table :

I	x_i	y_i	$x_i y_i$	x_i^2
1	1	2.6	2.6	1
2	2	2.7	5.4	4
3	3	2.9	8.7	9
4	4	3.025	12.1	16
5	5	3.2	16	25
6	6	3.367	20.202	36
Sum	21	17.792	65.002	91

Hence, $\sum x_i = 21, \sum y_i = 17.792$

$$\sum x_i y_i = 65.002, \sum x_i^2 = 91$$

and m = 6

then, the normal equations become

$$17.792 = 6a + 21b$$

and $$65.002 + 21a + 91b$$

Solving these equations, we get

$$a = 2.419333, \; b = 0.156$$

Hence, required straight line is given by the equation

$$y = 2.419333 + 0.156x$$

Now at $x = 5.5,$, value of y is given by

$$y = 2.419333 + 0.156 \times 5.5$$

$= 3.277333$

Example 10 : Fit a curve of the form $y = ax + bx^2$ to the given data:

x	1	1.5	2	2.5	3	3.5	4
y	1.1	1.95	3.2	5	8.1	11.9	16.4

Solution : Equation of the required curve is

$y = ax + bx^2$ which can be written as

$$\frac{y}{x} = a + bx \qquad \text{....(i)}$$

Let $\frac{y}{x} = Y$, then the above equation becomes

$$Y = a + bx \qquad \text{....(ii)}$$

Normal equation for this curve are given by

$$\sum Y_i = na + b\sum x_i$$

and $\sum x_i Y_i = a\sum x_i + b\sum x_i^2$

Using given data, we construct following table:

I	x	y	$Y = \frac{y}{x}$	xY	x^2
1	1	1.1	1.1	1.1	1
2	1.5	1.95	1.3	1.95	2.25
3	2	3.2	1.6	3.2	4
4	2.5	5	2.0	5	6.25
5	3	8.1	2.7	8.12	9
6	3.5	11.9	3.4	11.9	12.25
7	4	16.4	4.1	16.4	16
Sum	17.5	-	16.2	47.65	50.75

From the table we have

$$\sum x_i = 17.5, \sum y_i = 16.2,$$

$$\sum x_i y_i = 47.65, \sum x_i^2 = 50.75$$

and m = 7

substituting these values in normal equations, we get

$$16.2 = 7a \;+17.5b$$

and $\quad 47.65 = 1.5a \;+50.75b,$

Solving these equations, we get

$$a = 80.239827, b = 1.021429$$

Thus, from (ii) we have,Now at $Y = \frac{y}{x}$ we have

$y = 0.239287x + 1.021429x^2$ which is the required equation.

Example 11 : Fit a second degree polynomial to given data:

x	-4	-3	-2	-1	0	1	2	3	4
y	-5	-1	0	1	3	4	4	3	2

Solution : Let the required equation of the curve be

$$y = a + bx = cx^2$$

Normal equation for this curve are

$$\sum y_i = na + b\sum x_i + c\sum x_i^2$$
$$\sum x_i y_i = a\sum x_i + b\sum x_i^2 \;+ c\sum x_i^3$$
$$\sum x_i^2 y_i = a\sum x_i^2 + b\sum x_i^3 \;+ c\sum x_i^4$$

From the given data we construct following table :

i	x_i	y_i	$x_i y_i$	x_i^2	$x_i^2 y$	x_i^3	x_i^4
1	-4	-5	20	16	-80	-64	256
2	-3	-1	3	9	-9	-27	81
3	-2	0	0	4	0	-8	16
4	-1	1	-1	1	1	-1	1
5	0	3	0	0	0	0	0
6	1	4	4	1	4	1	1
7	2	4	8	4	16	8	16
8	3	3	9	9	27	27	81
9	4	2	8	16	32	64	256
Sum	0	11	51	60	-9	0	708

Thus, $\sum x_i = 0, \sum x_i = 11, \sum x_i y_i = 51$

$\sum x_i^2 = 60, \sum x_i^2 y_i = -9, \sum x_i^3 = 0,$

$\sum x_i^4 = 708 \ \ and \ \ n = 9$

Substituting above values in normal equations, we get

$11 = 9a + b.0 + 60c$ or $11 = 9a + 60c$

$51 = a.0 + b.60 + c.0$ or $51 = 60b,$

$-9 = a.60 = b.0 + c.708$ or $-9 = 60a + 708c$

Solving above equations for a, b and c, we get

$A = 3.004329, \ b = 0.85, \ c = -0.267316$

So, the required equation is given by

$Y = 3.004329 + 0.85x - 0.26731x^2$

Example 12 : Population of a city in different years are given in the following table :

x	1970	1980	1990	2000	2010
y (in thousands)	1450	1600	1850	2150	2500

Fit a parabola to the given data, using least squares principle. Also estimate the population of the city in 2005.

Solution : Since the magnitude of given data is large and values of x are given at equal intervals, therefore we reduce it by shift of origin and scale. Let $x_0 = 1990$ be origin of x-values and $y_0 = 1850$ be origin of y-values.

Then, let $X = \dfrac{x-1990}{10} \quad and \quad Y = \dfrac{y-1850}{50}$ ....(i)

Let required curve be $y = a + bx + cx^2$,

after change of orogin and scale, it will be

$Y = a + bX + cX^2$ ....(ii)

Now, we construct following table:

x	X	y	Y	XY	X^2	X^2Y	X^3	X^4
1970	-2	1450	-8	16	4	-32	-8	16
1980	-1	1600	-5	5	1	-5	-1	1
1990	0	1850	0	0	0	0	0	0
2000	1	2150	6	6	1	6	1	1
2010	2	2500	13	26	4	52	8	16
Sum	0	-	6	53	10	21	0	34

Normal equations, in new variables, will be

$$\sum Y_i = na + b\sum X_i + c\sum X_i^2$$

$$\sum X_i Y_i = a\sum X_i + b\sum X_i^2 + c\sum X_i^3$$

$$\sum X_i^2 Y_i = a\sum X_i^2 + b\sum X_i^3 + c\sum X_i^4$$

From the table, we have

$$\sum X_i = 0,\ \sum Y_i = 6, \sum X_i Y_i = 53$$

$$\sum X_i^2 = 10,\ \sum X_i^2 Y_i = 21, \sum X_i^3 = 0$$

$$\sum X_i^4 = 34 \quad and \quad n = 5,$$

Substituting above values in normal equations, we have

$$6 = 5a + b.0 + c.10$$

or $\quad 6 = 5a + 10c$

$$53 = a.0 + b.10 + c.0$$

or $\quad 53 = 10b,$

and $\quad 21 = a.10 + b.0 + c.34$

or $\quad 21 = 10a + 34c$

Solving above equations for a, b and c, we get

$$a = -0.08714,\ b = 5.3,\ c = 0.642857$$

Now from (ii), we have

$$Y = -0.08714, + 5.3x + 0.642857x^2 \qquad \text{.....(iii)}$$

From (i), we have

$$\frac{y-1850}{50} = -\ 0.085714 + 5.3\left(\frac{x-1990}{10}\right) = 0.642857\left(\frac{x-1990}{10}\right)^2$$

On simplification, we have

$$Y = 1222008.286 - 1252.7854\,x + 0.3214285x^2$$

Which is the required equation of parabola. Now, in the year 2005, population of the city will be given by

$$Y = 1222008.286 - 1252.78543(2005) + 0.3214285(2005)^2$$

$$= 2324.10456$$

$\approx$ 2324 thousands, approximately

20.13 Self Learning Exercise-II

Q.1 Derive normal equations for fitting parabola $y = a + bx^2$.

Q.2 Derive normal equations for fitting parabola $y = ax + bx^2$.

20.14 Summary

In this unit operators for interpolation with equal interval are discussed, various methods for interpolation with equal interval viz Newton forward, Newton Backward & Stirling central difference formulas are discussed. In other part of unit principle of least square for fitting of curves is discussed, fitting of straight line, parabola, fitting of polynomial are discussed.

20.15 Glossary

Interpolation :To approximate a value between given values.

Forward difference: Difference of next value & that value.

Backward difference: Difference of that value & previous value.

Curve Fitting :Fitting a curve from given set of data.

Least square :For which sum of square of values is minimum.

20.16 Answers to Self Learning Exercises

Answers to Self Learning Exercise-II

Ans.1: Normal equations are

$$\sum_{i=1}^{n} y_i = an + b\sum_{i=1}^{n} x_i^2$$

$$\sum_{i=1}^{n} x_i^2 y_i = a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i^4$$

Ans.2: Normal equations are

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i^3$$

$$\sum_{i=1}^{n} x_i^2 y_i = a\sum_{i=1}^{n} x_i^3 + b\sum_{i=1}^{n} x_i^4$$

20.17 Exercise

Q.1 Fit a curve of the form $y = ax = bx^2$ to the given data :

x	1	2	3	4	5	6
y	2..6	5.4	8.7	12.1	16	20.2

Q.2 Fit a straight line to the following data :

x	1	2	3	4	5	8
y	2.4	3	3.6	4	5	6

Q.3 Compute the constants α and γ^{β} such that the curve $y = \alpha\gamma^{\beta x}$ fits the given data:

x	1	2	3	4	5	6
y	151	100	61	50	20	8

Q.4 Fit a curve of the form $y = ax^b$ to the given below :

x	2	4	7	10	20	40	60	80
y	43	25	18	13	8	5	3	2

Q.5 Fit the curve $p = V^r = k$ to the data given in the table :

P	0.5	1	1.5	2	2.5	3
V	1.62	1	0.75	0.62	0.52	0.46

Q.6 Fit the curve $y = ae^{bx}$ to the following data :

x	2	4	6	8	10
y	4.077	11.084	30.128	81.897	222.62

Also estimate y at $x = 7$

Q.7 Fit a second degree polynomiasl to the following data, taking x as independent variable:

x	1	1.5	2	2.5	3	3.5	4
y	1.1	1.3	1.6	2.0	2.7	3.4	3.1

Q.8 Fit a second degree parabola to the given data:

x	1929	1930	1931	1932	1934	1935	1936	1937
y	352	356	357	358	360	361	360	359

Q.9 Obtain a least-squares quadratic approximation to the function $y(x) = \sqrt{x}$ on $[0,1]$ with respect to the weight function $w(x) = 1$

Q.10 The temperature θ and length l of a heated rod are given below. Establish a relation between θ and l of the form $l = a + b\theta$ using least-squares principle.

$\theta\left({}^{0}C\right)$	20	30	40	50	60	70
l (mm)	800.3	800.4	800.6	800.7	800.9	800.10

20.18 Answers to Exercise

Ans.1: $y = 2.41973 + 0.15589x^3$

Ans.2: $y = 1.76 + 0.506x, \ y = 3.747 \ at \ x = 3.5$

Ans.3: $a = 309 \ , \gamma^{\beta} = 5754$

Ans.4: $a = 4.36 \ , b = -0.7975$

Ans.5: $r = 1.4224, k = 0.9970$

Ans.6: $a = 1.499, \ b = 0.5 \ c = 49.6401$

Ans.7: $y = 1.0368 - 0.1932x + 0.2429x^2$

Ans.8: $y = -1010135 + 1044.67x - 0.27x^2$

Ans.9: $y = \frac{1}{35}\left(6 + 48x - 20x^2\right))$

Ans.10: a = 800, b = 0.0146

References and Suggested Readings

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2. C.F. Gerald, P.O. Wheatley, Applied Numerical Analysis, Addison-Wesley, 1998.

3. Kuldeep Singh Gehlot, 'Mathematical Methods for Numerical Analysis and Optimization ', 1st edition, College Book House, (Pvt) Ltd, Jaipur (2007).

UNIT -21

Numerical Solution of Nonlinear Equations

Structure of the Unit

21.0 Objectives

In this chapter, we find numerical solution of non-linear equation of the form $f(x)=0$. For this, we discuss two methods named Newton Raphson method and Secant method. We also discuss the convergence of these two methods which depends on the choosen root that how closed the root is choosen to the original root. Instead of this, we discuss the system of linear equations of the form $AX=B$ and their solution. For this, we learn two methods named Gauss Elimination method and matrix inversion method.

21.1 Introduction

The non-linear equations arise in various problems such as projectile motion, pipeline flow, pipe-pump system and many physical problems.

Any polynomial with degree one, is called linear equation. The equation, which is not linear, is called non-linear equation. For example, x-1 =0 is linear and $x^2+x+1=0$ is non-linear.

21.2 Rate of Convergence

We consider that the initial approximation to the root is sufficiently close to the desired root, then the rate of convergence can be defined as following.

Definition: An iterative method is said to be of order p or has the rate of convergence p, if p is the largest positive real number for which there exists a finite constant $C \neq 0$ such that

$$|\in_{k+1}| \leq C\,|\in_k|^p$$

where$\in_k = x_k - \xi$ is the error in thekth iterate.

The constant C is called the asymptotic error constant and usually depends on derivatives of $f(x)$ at $x=\xi$.

21.3 Newton Raphson Method

For roots with different multiplicity, Newton Raphson Method is given by different formulae.

(i) ***For Simple roots**:*

For simple root, the nth iteration root of the equation $f(x) = 0$ is given by

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \qquad n = 0, 1, 2, 3, \ldots.$$

(ii) ***For multiple roots**:*

(a). When multiplicity is given:

In this method, we use the following formula

$$\boxed{x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}}$$

where m is multiplicity of root.

(b). When multiplicity is not given:

Let $x = a$ be root of $f(x) = 0$ of multiplicity m.So, the function *f (x)*can be written as

$$f(x) = (x - a)^m g(x)$$

$$f'(x) = m(x - a)^{m-1} g(x) + (x - a)^m g'(x)$$

$$= (x - a)^{m-1}[mg(x) + (x - a)g'(x)]$$

$$\therefore \frac{f(x)}{f'(x)} = (x - a)h(x)$$

Let $\Phi(x) = \frac{f(x)}{f'(x)}$

NR Method is given by

$$x_{n+1} = x_n - \frac{\Phi(x_n)}{\Phi'(x_n)}$$

$$= x_n - \frac{f(x)/f'(x)}{\frac{f'(x_n)f'(x_n) - f(x_n)f''(x_n)}{\{f'(x_n)\}^2}}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{[f'(x_n)]^2 - f(x_n)f''(x_n)}} \qquad n = 0, 1, 2, 3, \ldots\ldots$$

(iii) ***Newton Raphson Method for pth root**:*

How to find the nth root of a real number N ? For this consider it be *x.* Then

$$x = N^{1/p}$$

$$or \ \ x^p = N$$

$$or \ x^p - N = 0$$

$$Let \ f(x) = x^p - N$$

Newton Raphson Method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$or \quad x_{n+1} = x_n - \frac{x_n^p - N}{p x_n^{p-1}}$$

$$or \quad x_{n+1} = \frac{p x_n^p - x_n^p + N}{p x_n^{p-1}}$$

$$\boxed{x_{n+1} = \frac{(p-1)x_n^p + N}{p x_n^{p-1}}}$$

(a) *For square root i.e.* $\sqrt{N} = N^{1/2}$: In this case $p = 2$ and

$$\boxed{x_{n+1} = \frac{x_n^2 + N}{2x_n}}$$

(b) *For cubic root i.e.* $N^{\frac{1}{3}}$: In this case $p = 3$ and

$$\boxed{x_{n+1} = \frac{2x_n^2 + N}{3x_n^2}}$$

(c)For inverse of a number i.e. $\frac{1}{N}$: *In this case* $p = -1$and

$$x_{n+1} = \frac{-2\left(\frac{1}{x_n}\right) + N}{-1/_{x_n^2}} = x_n(2 - N x_n)$$

21.4 Rate of Convergence of Newton Raphson method

The NR Method for solution of $f(x) = 0$ is

$$x_{n+1} = x - \frac{f(x_n)}{f'(x_n)} \qquad (1)$$

Let ξ be a simple root of given equation then $f(\xi) = 0$ and $\epsilon_n = x_n - \xi$= Error in nth iteration.

Equation (1) becomes

$$\xi + \epsilon_{n+1} = \xi + \epsilon_n - \frac{f(\xi + \epsilon_n)}{f'(\xi + \epsilon_n)}$$

$$or \quad \epsilon_{n+1} = \epsilon_n - \frac{f(\xi) + \epsilon_n f'(\xi) + \frac{\epsilon_n^2}{2} f''(\xi) + \cdots}{f'(\xi) + \epsilon_n f''(\xi) + \frac{\epsilon_n^2}{2} f'''(\xi) + \cdots}$$

$$= \epsilon_n - \frac{\epsilon_n + \frac{\epsilon_n^2 f''(\xi)}{2 f'(\xi)} + \cdots}{1 + \epsilon_n \frac{f''(\xi)}{f'(\xi)} + \frac{\epsilon_n^2 f'''(\xi)}{2 f'(\xi)} + \cdots}$$

$$or \ \epsilon_{n+1} = \epsilon_n - \left\{\epsilon_n + \frac{\epsilon_n^2}{2}\frac{f''(\xi)}{f'(\xi)} + \ldots\right\}\left\{1 + \frac{\epsilon_n f''(\xi)}{f'(\xi)} + \frac{\epsilon_n^2}{2}\frac{f'''(\xi)}{f'(\xi)} + \ldots\right\}^{-1}$$

$$= \epsilon_n - \epsilon_n + \epsilon_n^2 \frac{f''(\xi)}{f'(\xi)} - \frac{\epsilon_n^2}{2}\frac{f''(\xi)}{f'(\xi)} + 0(\epsilon_n^3)$$

$$= \frac{1}{2}\frac{f''(\xi)}{f'(\xi)} \epsilon_n^2 + 0(\epsilon_n^3)$$

$$\epsilon_{n+1} = C\, \epsilon_n^2$$

where$C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$ and neglecting ϵ_n^3 and higher terms.

Hence Newton Raphson Method has second order convergence or quadratic convergence.

21.5 Illustrative Examples

Example 1: Solve $x^4 - x - 10 = 0$ by NewtonRaphson method.

Sol. Let $f(x) = x^4 - x - 10$

$$f'(x) = 4x^3 - 1$$

Now $\quad f(0) = -10$

$f(1) = -10$

$f(2) = 4$

Therefore, $a$root of $f(x) = 0$ lie between 1 and 2 as $f(1) f(2) < 0$.

Let $\quad x_0 = 2$

$f(x_0) = 4$

$\therefore f'(x_0) = 31$

By Newton Raphson method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.87097 \qquad (\text{for} n = 1)$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.85578$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$=1.85558$$

$n = 3$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 1.85558$$

Hence a root of given equation is 1.85558 correct upto 5 decimals.

Example 2: Solve $x log_{10}x = 1.2$ by Newton Raphson method.

Sol. Let $f(x) = xlog_{10}x - 1.2 = 0$

$$f'(x) = log_{10}x + x.\frac{1}{x}.log_{10}e$$

$$= log_{10}x + 0.43429$$

$$f(1) = -1.2$$

$$f(2) = -0.59794$$

$$f(3) = 0.23136$$

$x_0 = 3$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(x_n log_{10}x_n - 1.2)}{log_{10}x_n + .43429}$$

$$= \frac{.43429\,x_n + 1.2}{log_{10}x_n + .43429}$$

For $n = 1,2,3 \ldots$

$$x_1 = 2.74615$$

$$x_2 = 2.74065$$

$$x_3 = 2.74065$$

Example 3: Find double root of equation $x^3 - x^2 - x + 1 = 0$ by Newton-Raphson Method. starting with0.8. (multiplicity is given as 2)

Sol. Let $f(x) = x^3 - x^2 - x + 1$

$$f'(x) = 3x^2 - 2x - 1$$

Here $x_0 = 0.8$

Newton Raphson method for a root with multiplicity 2 is

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}, \qquad n = 0,1,2, \ldots.$$

$$= x_n - \frac{2(x_n^3 - x_n^2 - x_n + 1)}{3x_n^2 - 2x_n - 1}$$

$$= \frac{x_n^2 + x_n - 2}{3x_n^2 - 2x_n - 1}$$

For $n = 0,1,2$

$$x_1 = \frac{x_0^3 + x_0 - 2}{3x_0^2 - 2x_0 - 1} = 1.011765$$

$$x_2 = 1.00003$$

$$x_3 = 1$$

Hence 1 is root of $f(x) = 0$ with multiplicity 2.

Example 4: Find the multiple root of

$$27x^5 + 27x^4 + 36x^3 + 28x^2 + 9x + 1 = 0$$

by Newton Raphson method. (multiplicity is not unknown)

Sol. Let $f(x) = 27x^5 + 27x^4 + 36x^3 + 28x^2 + 9x + 1$

$$\therefore f'(x) = 135x^4 + 108x^3 + 108x^2 + 56x + 9$$

$$f''(x) = 540x^3 + 324x^2 + 216x + 56$$

Here multiplicity of root is not given. Therefore multiple root of $f(x) = 0$ can be considered as simple root of $\Phi(x) = 0, where$

$$\Phi(x) = \frac{f(x)}{f'(x)}$$

$$\Phi(0) = \frac{1}{9} = 0.11111$$

$$\Phi(-1) = -0.18182$$

Multiple root of $f(x) = 0$ lies between 0 and -1. Take $x_0 = 0$.

Newton Raphson Method for multiple root is given by

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{\{f'(x_n)\}^2 - f(x_n)f''(x_n)}$$

For $n = 0,1,2$

$$x_1 = -0.36$$

$$f(x_1) = -0.000578$$

$f'(x_1) = 0.065434$

$f''(x_1) = -4.963840$

$\therefore x_2 = -0.333224$

$f(x_2) = -0.333115$

$f'(x_2) = 0.000001$

$f''(x_2) = 0.019677$

$x_3 = -0.333173$

Example 5 Find $20^{1/2}$.

Sol. Here N = 20 and $\sqrt{16} = 4$, $\sqrt{25} = 5$

Take $x_0 = 4.5$

Newton Raphson method for square root is

$$x_{n+1} = \frac{x_n^2 + N}{3x_n}, \qquad n = 0,1,2,3 \ldots.$$

$$x_{n+1} = \frac{x_n^2 + 20}{2x_n}$$

For n = 0,1,2,3

$$x_1 = \frac{x_0^2 + 20}{2x_0} = \frac{(4.5)^2 + 20}{2(4.5)} = \frac{(4.5)^2 + 20}{9}$$

$$= 4.47222$$

$$x_2 = \frac{x_1^2 + 20}{2x_1} = \frac{(4.47222)^2 + 20}{2(4.47222)} = 4.472136$$

$n = 2$

$$x_3 = \frac{x_2^2 + 20}{2x_2} = 4.472136$$

$\therefore \sqrt{20} = 4.472136$

Example 6: $Find$ $18^{1/3}$.

Sol.

Here $N = 18$

$(8)^{1/3} = 2, \quad (27)^{1/3} = 3$

$Take\ x_0 = 2.5$

Newton Raphson Method for cubic root is

$$x_{n+1} = \frac{2x_n^3 + N}{3x_n^2}$$

For n = 0, 1, 2

$$x_1 = \frac{2x_0^2+N}{3x_0^2} = 2.626667$$

$$x_2 = 2.620755$$

$$x_3 = 2.620741$$

21.6 Secant Method

The Newton Raphson method requires the evaluation of derivatives of the function and this is not always possible, particularly in the case of functions arising in practical problems. In the secant method, the derivative at x, is approximated by the formula

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

which can be written as

$$f_i' = \frac{f_i - f_{i-1}}{x_i - x_{i-1}}$$

where $f_i = f(x_i)$. Hence, the NewtonRaphson formula becomes

$$x_{i+1} = x_i - \frac{(x_i - x_{i-1})f_i}{f_i - f_{i-1}} = \frac{x_{i-1}f_i - x_i f_{i-1}}{f_i - f_{i-1}}$$

$$\boxed{x_{i+1} = \frac{x_{i-1}f_i - x_i f_{i-1}}{f_i - f_{i-1}}}$$

It should be noted that this formula requires two initial approximations to the root.

21.7 Rate of Convergence of Secant Method

We assume that ξ is a simple root of $f(x) = 0$.

Substituting$x_k = \xi + \epsilon_k$ $in\ (1)\ we\ obtain$

$$\epsilon_{k+1} = \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1}) f(\xi + \epsilon_k)}{f(\xi + \epsilon_k) - f(\xi + \epsilon_{k-1})} \qquad (2)$$

Expanding $f(\xi + \epsilon_k)$and $f(\xi + \epsilon_{k-1})$ in Taylor's series about the point ξ and noting that $f(\xi) = 0$.

$$\epsilon_{k+1} = \epsilon_k - \frac{(\epsilon_k - \epsilon_{k-1})\left[\epsilon_k\, f'(\xi) + \frac{1}{2}\epsilon_k^2\, f''(\xi) + \cdots ...\right]}{(\epsilon_k - \epsilon_{k-1}) f'(\xi) + \frac{1}{2}(\epsilon_k^2 - \epsilon_{k-1}^2) f''(\xi) + \cdots ..}$$

$$=\in_k - \left[\in_k + \frac{1}{2}\in_k^2 \frac{f''(\xi)}{f'(\xi)} + \cdots\right]\left[1 + \frac{1}{2}(\in_{k-1} + \in_k)\frac{f''(\xi)}{f'(\xi)} + \cdots\right]^{-1}$$

$$\in_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)} \in_k \in_{k-1} + O(\in_k^2 \in_{k-1} + \in_k \in_{k-1}^2)$$

$$\in_{k-1} = C \in_k \in_{k-1} \tag{3}$$

where$C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$ and higher powers of $\in_k$ are neglected.

The relation of the form (3) is called the error equation. Keeping in view the definition of the rate of convergence, we seek a relation of the from.

$$\in_{k+1} = A \in_k^p \tag{4}$$

where A and p are to be determined.

From (4) we have

$$\in_k = A \in_{k-1}^p \quad or \quad \in_{k-1} = A^{-1/p} \in_k^{1/p}$$

Substituting the values of $\in_{k+1}$and $\in_{k-1}$in (3), we get

$$\in_k^p = CA^{-(1+\frac{1}{p})} \in_k^{1+1/p} \tag{5}$$

Equating the powers of $\in_k$on both sides, we get

$$p = 1 + \frac{1}{p}$$

which gives $p = \frac{1}{2}(1 \pm \sqrt{5})$

Neglecting the minus sign, we find the rate of convergence for the secant method is $p = 1.618$.

From (5), we have $A = C^{p/(p+1)}$

21.8 Illustrative Examples

Example 7: Solve the equation $4\sin x + x^2 = 0$. Find the root of equation by Secant method.

Sol. Let $f(x) = 4\sin x + x^2 = 0$

$$f(-1) = -2.36588, \; f(-2) = 0.362810$$
$$f(-1)f(1) < 0$$

So, a root lies between -1 & -2 and let

$$x_0 = -1 \;\; , \;\; x_1 = -2$$

Now

$$x_2 = \frac{(-1)(0.362810) - (-2)(-2.36588)}{0.362810 + 2.36588}$$

$$= \frac{-5.09457}{2.728690} = -1.867039$$

$$f(x_2) = -0.339926$$

Now $\quad x_1 = -2 \qquad x_2 = 1.867039$

$$f(x_1) = 0.362817 \quad , \qquad f(x_2) = 0.334926$$

$$x_3 = \frac{1.357232}{-0.702726} = -1.93138$$

$$f(x_3) = -0.01269$$

Now

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$= \frac{-0.633002}{0.327326} = -1.93384$$

$$f(x_4) = 0.00045$$

Now

$$x_5 = \frac{x_3\, f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)}$$

$$= \frac{-0.025452}{0.013162} = -1.93375$$

$$f(x_5) = -0.00002$$

Now

$$x_6 = \frac{x_4 f(x_5) - x_5 f(x_4)}{f(x_5) - f(x_4)}$$

$$= \frac{0.001135}{-0.000587} = -1.93375$$

Thus, the approximate value to the root is -1.93375 correct up to five decimals.

Example 8: Determine p, q and r so that the order of iteration method.

$$x_{n+1} = px_n + \frac{aq}{x_n^2} + \frac{ra^2}{x_n^5}$$

becomes as high as possible for root of $\xi = a^{1/3}$.

Sol. Here $\quad \xi = a^{1/3} \qquad => \xi^3 = a$

That is, we have found cube root of a, using $\in_n = x_n - \xi$

We have $\xi + \in_{n+1} = p(\xi + \in_n) + \frac{aq}{(\xi+\in_n)^2} + \frac{ra^2}{(\xi+\in_n)^5}$

$$= p(\xi + \in_n) + aq\,(\xi + \in_n)^{-2} + ra^2(\xi + \in_n)^{-5}$$

$$= p(\xi + \in_n) + \frac{aq}{\xi^2}\left(1 + \frac{\in_n}{\xi}\right)^{-2} + \frac{ra^2}{\xi\ 5}\left(1 + \frac{\in_n}{\xi}\right)^{-5}$$

$$= p(\xi + \in_n) + \frac{aq}{\xi^2}\left(1 - \frac{2\in_n}{\xi} + \frac{3\in_n{}^2}{\xi^2} - \frac{4\in_n{}^3}{\xi^3} + \cdots..\right)$$

$$+ \frac{rq^2}{\xi^5}\left(1 - \frac{5\in_n}{\xi} + \frac{15\in_n{}^2}{\xi^2} - \frac{35\in_n{}^3}{\xi^3} + \cdots..\right)$$

$$\in_{n+1} + \xi = p(\xi + \in_n) + q\,\xi\left(1 - \frac{2\in_n}{\xi} + \frac{3\in_n{}^2}{\xi^2} - \frac{4\in_n{}^3}{\xi^3} + \cdots..\right)$$

$$+ r\,\xi\left(1 - \frac{5\in_n}{\xi} + \frac{15\in_n{}^2}{\xi^2} - \frac{35\in_n{}^3}{\xi^3} + \cdots..\right)$$

$$\in_{n+1} = \xi\,[-1 + p + q + r] + \in_n\,(p - 2q - 5r) + \frac{\in_n^2}{\xi}(3q + 15r)$$

$$+ \frac{\in_n^3}{\xi^2}(-4q - 35r)$$

If we take

$$-1 + p + q + r = 0 \tag{1}$$

$$p - 2q - 5r = 0 \tag{2}$$

$$3q + 15\,r = 0 \tag{3}$$

By (1) $\quad p = +1 - q - r$

By(2) $\quad 1 - q - r - 2q - 5r = 0$

$=> -3q - 6r = -1 \qquad => 3q + 6r = 1$

$$=> 3q + 6r = 1$$

$$3q + 15r = 0$$

$$-9r = 1$$

$$r = -\frac{1}{9}$$

$$-3q = -1 + r^2\left(\frac{1}{9}\right)$$

$$-3q = \frac{-3-2}{3} \qquad => q = +\frac{5}{9}$$

$$\therefore p = 1 - \frac{5}{9} + \frac{1}{9} \qquad => \qquad p = \frac{9-5+1}{9} = \frac{5}{9}$$

We have $p = q = \frac{5}{9}$, $r = \frac{1}{9}$

Eq.(1) becomes $\in_{n+1} = \frac{5}{3\xi^2} \in_n^3 + \cdots$

$$\in_{n+1} = \frac{5}{3\xi^2} \in_n^3$$

$$= C \in_n^3$$

Hence given method has order 3.

21.9 System of Linear Equations

Consider the system of n linear equations in n unknowns :

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \qquad (1)$$

The matrix form of the system (1) is

$$AX = B \qquad (2)$$

where

$$A = \begin{bmatrix} a_{11}a_{12} & \cdots & \cdots & a_{1n} \\ a_{21}a_{22} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots \cdots & \cdots \\ \cdots & \cdots & \cdots \cdots & \cdots \\ a_{n1}a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{bmatrix} \quad and \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_n \end{bmatrix}$$

21.10 Matrix Inversion Method

Suppose A is non-singular, that is det$A \neq 0$.Then A^{-1}exists. Therefore, pre-multiplying (2) by $A^{-1}, we\ get$

$$A^{-1}\,AX = A^{-1}\,B$$
$$X = A^{-1}\,B$$

Using this formula , all unknowns can be determined.

21.11 Illustrative Examples

Example 9 Solve the system of linear equations

$$x + y + 2z = 1$$
$$x + 2y + 3z = 1$$
$$2x + 3y + z = 2$$

Sol. We have

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix} = -4 \neq 0$$

Also

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 7 & -5 & 1 \\ -5 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Hence

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \frac{1}{4}\begin{bmatrix} 7 & -5 & 1 \\ -5 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, $x = 1, y = 0, z = 0$

Example 10 Solve the system of linear equations

$$3x_1 + 6x_2 + x_3 = 1$$
$$x_1 + 2x_2 - x_3 = 2$$
$$3x_1 + 3x_2 + 2x_3 = 3$$

Sol. The coefficient matrix is given by

$$A = \begin{bmatrix} 3 & 6 & 1 \\ 1 & 2 & -1 \\ 3 & 3 & 2 \end{bmatrix}$$

$Thus \quad X = A^{-1}b$

$$=> \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\frac{1}{12}\begin{bmatrix} 7 & -9 & -8 \\ -5 & 3 & 4 \\ -3 & 9 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\frac{1}{12}\begin{bmatrix} -35 \\ 13 \\ 15 \end{bmatrix}$$

$$x_1 = \frac{35}{12}, x_2 = -\frac{13}{12}, x_3 = -\frac{15}{12}$$

21.12 Gauss Elimination Method

We consider the system (3 x 3 system)

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \qquad (1)$$

First stage of elimination, we multiply the first row in (1) by a_{21}/a_{11} and a_{31}/a_{11} respectively and subtract from the second and third rows. We get

$$a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 = b_2^{(2)} \qquad (2)$$
$$a_{32}^{(2)}x_2 + a_{33}^{(2)}x_3 = b_3^{(2)}$$

where

$$a_{22}^{(2)} = a_{22} - \frac{a_{21}}{a_{11}}a_{12}, \quad a_{23}^{(2)} = a_{23} - \frac{a_{21}}{a_{11}}a_{13}$$
$$a_{32}^{(2)} = a_{32} - \frac{a_{31}}{a_{11}}a_{12}, \quad a_{33}^{(2)} = a_{33} - \frac{a_{31}}{a_{11}}a_{13}$$
$$b_2^{(2)} = b_2 - \frac{a_{21}}{a_{11}}b_1, \quad b_3^{(2)} = b_3 - \frac{a_{31}}{a_{11}}b_1$$

Second stage of elimination, we multiply the first row in (2) by $a_{32}^{(2)}/a_{22}^{(2)}$ and subtract from the second row in (2). We get

$$a_{33}^{(3)}x_3 = b_3^{(3)} \qquad (3)$$

where

$$a_{33}^{(3)} = a_{33}^{(2)} - \frac{a_{32}^{(2)}}{a_{22}^{(2)}}a_{23}^{(2)}, \quad b_3^{(3)} = b_3^{(2)} - \frac{a_{32}^{(2)}}{a_{22}^{(2)}}b_2^{(2)}$$

We collect the first equation from each stage, i.e., from (1), (2) and (3) we get

$$a_{11}^{(1)}x_1 + a_{12}^{(1)}x_2 + a_{13}^{(1)}x_3 = b_1^{(1)}$$
$$a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 = b_2^{(2)} \qquad (4)$$
$$a_{33}^{(3)}x_3 = b_3^{(3)}$$

where

$$a_{ij}^{(1)} = a_{ij}, \quad b_i^{(1)} = b_i, \quad i,j = 1,2,3$$

The system (4) is an upper triangular system and solving this is called back substitution method. Thus,

$$[A|b] - - - - \frac{Gauss}{Elimination} - - - \rightarrow [U|c\,] \qquad (5)$$

where $[A|b]$ is the augmented matrix. The elements $a_{11}^{(1)}, a_{22}^{(2)}$ and $a_{33}^{(3)}$ which have been assumed to be non-zero are known as *pivot elements*. The elimination procedure described above to determine the variables(unknowns) is called the ***Gauss elimination method***. We may also make the pivot as 1 before elimination, at each step. At the end of the elimination procedure, we produce 1 at each of the positions of the diagonal elements.

We now solve the system (1) in n unknowns by performing the Gauss elimination on the augmented matrix $[A|b]$.

The elimination is performed in (n — 1) steps, k = 1, 2, ... , n — 1. In the elimination process, if any one of the pivot elements $a_{11}^{(1)}, a_{22}^{(2)}, \dots\dots, a_{nn}^{(n)}$ vanishes or becomes very small compared to other elements in that column, then we attempt to rearrange the remaining rows so as to obtain a non-vanishing pivot or to avoid the multiplication by a large number. This strategy is called ***pivoting***. The pivoting is of the following two types.

Partial Pivoting

In the first stage of elimination, the first column is searched for the largest element in magnitude and brought as the first pivot by interchanging the first equation with the equation having the largest element in magnitude. In the second elimination stage, the second column is searched for the largest element in magnitude among the n-1 elements leaving the first element, and this element is brought as the second pivot by an interchange of the second equation with the equation having the largest element in magnitude. This procedure is continued until we arrive at the equations (5).

Complete Pivoting

We search the matrix A for the largest element in magnitude and bring it as the first pivot. This requires not only an interchange of equations but also an interchange of the position of the variables.

21.13 Illustrative Examples

Example 11 Solve the system of linear equations

$$x_1 + 10x_2 + x_3 = 12$$
$$10x_1 + x_2 + x_3 = 12$$
$$x_1 + x_2 + 10x_3 = 12$$

Sol. Rearranging the system of linear equations

$$10x_1 + x_2 + x_3 = 12 \quad (1)$$

$$x_1 + 10x_2 + x_3 = 12 \quad (2)$$

$$x_1 + x_2 + 10x_3 = 12 \quad (3)$$

$$[A \mid b] = \left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 1 & 10 & 1 & 12 \\ 1 & 1 & 10 & 12 \end{array}\right]$$

$$R_2 \to R_2 - \frac{1}{10}R_1, \quad R_3 \to R_3 - \frac{1}{10}R_1$$

$$= \left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 0 & 99/10 & 9/10 & 108/10 \\ 0 & 9/10 & 99/10 & 108/10 \end{array}\right]$$

$$R_2 \to \frac{10}{99} R_2$$

$$= \left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 0 & 1 & \frac{1}{11} & \frac{12}{11} \\ 0 & \frac{9}{10} & \frac{99}{10} & \frac{108}{11} \end{array}\right]$$

$$R_2 \to R_3 - \frac{9}{10} R_2$$

$$= \left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 0 & 1 & \frac{1}{11} & \frac{12}{11} \\ 0 & 0 & \frac{108}{11} & \frac{108}{11} \end{array}\right]$$

by back substitution, we get

$$x_3 = 1, x_2 = 1, x_1 = 1$$

Example 12 Solve the equations

$$10x_1 - x_2 + 2x_3 = 4$$
$$x_1 + 10x_2 - x_3 = 3$$
$$2x_1 + 3x_2 + 20x_3 = 7$$

using the Gauss elimination method.

Sol. As seen, the system doesn't need pivoting. We get, after the first elimination stage

$$10x_1 - x_2 + 2x_3 = 4$$
$$\frac{101}{10}x_2 - \frac{12}{10}x_3 = \frac{26}{10}$$
$$\frac{32}{10}x_2 + \frac{196}{10}x_3 = \frac{62}{10}$$

second elimination stage

$$10x_1 - x_2 + 2x_3 = 4$$
$$\frac{101}{10}x_2 - \frac{12}{10}x_3 = \frac{26}{10}$$
$$\frac{20180}{1010}x_3 = \frac{5430}{1010}$$

Using back substitution, we get

$$x_3 = 0.269, \quad x_2 = 0.289 \quad and \quad x_1 = 0.375$$

Example 13 Find the solution of thesystem of equations

$$x_1 + x_2 + x_3 = 6$$
$$3x_1 + 3x_2 + 4x_3 = 20$$
$$2x_1 + x_2 + 3x_3 = 13$$

Sol. In the first step we eliminate x_1 from the last two equations and obtain

$$x_1 + x_2 + x_3 = 6$$
$$x_3 = 2$$
$$-x_2 + x_3 = 1$$

Here, the pivot in the second equation is zero and so we cannot proceed as usual. We interchange the equations 2 and 3 before the second step. We obtain the upper triangular system

$$x_1 + x_2 + x_3 = 6$$
$$-x_2 + x_3 = 1$$
$$x_3 = 2$$

which has the solution

$$x_1 = 3, \quad x_2 = 1 \quad and \quad x_3 = 2$$

Example 14: Solve the system of linear equations

$$x_1 + 2x_2 + 3x_3 = 14$$

$$2x_1 + 5x_2 + 2x_3 = 18$$
$$3x_1 + x_2 + 5x_3 = 20$$

Sol. Given system can be written as

$$3x_1 + x_2 + 5x_3 = 20 \qquad (1)$$
$$2x_1 + 5x_2 + 2x_3 = 18 \qquad (2)$$
$$x_1 + 2x_2 + 3x_3 = 14 \qquad (3)$$

$$[A\,|b] = \begin{pmatrix} 3 & 1 & 5 & | & 20 \\ 2 & 5 & 2 & | & 18 \\ 1 & 2 & 3 & | & 14 \end{pmatrix}$$

$$R_2 \to R_2 - \frac{2}{3}R_1, \quad R_3 \to R_3 - \frac{1}{3}R_1$$

$$= \begin{pmatrix} 3 & 1 & 5 & | & 20 \\ 0 & \frac{13}{3} & \frac{-4}{3} & | & \frac{14}{3} \\ 0 & \frac{5}{3} & \frac{4}{3} & | & \frac{22}{3} \end{pmatrix}$$

$$R_3 \to R_3 - \frac{5}{13}R_2$$

$$= \begin{pmatrix} 3 & 1 & 5 & | & 20 \\ 0 & \frac{13}{3} & \frac{-4}{3} & | & \frac{14}{3} \\ 0 & 0 & \frac{24}{3} & | & \frac{72}{3} \end{pmatrix}$$

By back substitution, we get $x_3 = 3,\ x_2 = 2\ ,\ x_1 = 1$

21.14 Self Learning Exercise

Very Short Answer type Questions

Q.1 Define the rate of convergence for any iterative method.

Q.2 Write the order of convergence for Newton Raphson method.

Q.3 Write the order of convergence for Secant method.

Short Answer type questions

Q.4 Solve the non-linear equation $x^4 - x - 10 = 0$ by Newton Raphson method.

Q.5 Solve the following system of linear equation

$$4x_1 + x_2 + x_3 = 4$$
$$x_1 + 4x_2 - 2x_3 = 4$$
$$3x_1 + 2x_2 - 4x_3 = 6$$

by Gauss Elimination method with partial pivoting.

Q.6 Find the positive root of the equation $x^2 - 5x + 2 = 0$, correct to 4 decimal places, using Newton-Raphson method with $x_0 = 0.5$

21.15 Answers to Self Learning Exercise

Ans.2 : Quadratic convergence i.e. order 2

Ans.3 : 1.618

Ans.4 : 1.856

Ans.5 : $x_1 = -\frac{1}{2}, \quad x_2 = \frac{1}{2}, \quad x_3 = 1$

Ans.6 : 0.4384

21.16 Summary

In this chapter, we discussed about the numerical solution of non-linear equation by Newton-Raphson method and Secant method. Then, we discussed about the solution of the system of linear equation by Matrix Inversion method and Gauss Elimination method.

21.17 Glossary

Elimination: The removal of a variable from an equation, typically by substituting another which is shown by another equation to be equivalent:

Convergent: (Of a series) approaching a definite limit as more of its terms are added.

21.18 Exercise

Very Short Answer Type Questions

Q.1 Write the formula of Newton Raphson method for determining the square root of 15.

Q.2 Write the Newton-Raphson formula for determining the cubic root of the number N.

Q.3 Write the Newton Raphson formula for determining the multiple root with multiplicity m for non-linear equation.

Short Answer Type Questions

Q.4 Find the positive root of the equation $x^2 + x - 1 = 0$ correct to 3 decimal places, using Newton Raphson method with $x_0 = 0$.

Q.5 Find the cubic root of 15 by Newton Raphson method.

Q.6 Find the numerical solution of the system of linear equations $x - y + 4z = 16$, $3x + 2y + z = 18$ and $x + 4y - 2z = 20$ correct to 3 decimal places, using Gauss Elimination method without partial pivoting.

Long Answer Type Questions

Q.7 Derive the rate of convergence for

(i) Newton Raphson Method (ii) Secant method

Q.8 Solve the system

$$2x + y + z = 10$$

$$3x + 2y + 3z = 18$$

$$x + 4y + 9z = 16$$

by (i) Gauss Elimination method and (ii) Matrix inversion method

Q.9 Solve the system

$$2x + 2y + z = 12$$

$$3x + 2y + 2z = 8$$

$$5x + 10y - 8z = 10$$

by (i) Gauss Elimination Method (ii) Matrix Inversion Method

21.19 Answers to Exercise

Ans.1 : $x_{n+1} = \dfrac{x_n^2 + 15}{2x_n}$ **Ans.2 :** $x_{n+1} = \dfrac{2x_n^3 + N}{3x_n^2}$

Ans.3 : $x_{n+1} = x_n - \frac{mf(x_n)}{f'(x_n)}$ **Ans.4 :** 0.618

Ans.5 : 2.4662 **Ans.6 :** $x = 0.64, \quad y = 5.44, \quad z = 5.2$

Ans.8 : $x = 7, \quad y = -9, \quad z = 5$

Ans.9 : $x = -12.75, \quad y = 14.375, \quad z = 8.75$

References and Suggested Readings

1. S.R.K. Iyenger. R.K. Jain, 'Numerical Methods', New Age International Publishers (INDIA).

2. C.F. Gerald, P.O. Wheatley, Applied Numerical Analysis, Addison-Wesley, 1998.

UNIT- 22
Numerical Differentiation

Structure of the Unit

22.0 Objectives

Numerical differentiation and integration methods are frequently used in computational physics. In this chapter we will look at ways of derivation of numerical differentiation formulas and their errors types. After completing this chapter student should be able to obtain numerical approximations to the first and second derivatives of certain functions and error analysis in context of truncation and round-off error.

22.1 Introduction

Numerical Differentiation is the process by which we can find the approximate value of the derivative of a function at a given value of the independent variable. The problem of numerical differentiation is mainly solved by the method based on interpolation formulae and other based on operator formulae. Numerical differentiation is used when we approximate the derivative of a function at a specific point. Approximation of derivatives is used to reduce the differential equation to a form that can be easily solved than the original differential equation.

22.2 Approximation of Derivatives

In this section we learn how to approximate the derivative by using finite differences. The idea is very simple: the derivative of a function f(x) is defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

22.2.1 First Order Approximation

(a) Forward difference approximation

The approximation can be analyzed by considering the Taylor's series expansion of a function f(x) is given by

$$f(x+h) = f(x) + h\, f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots \tag{2}$$

where h is understood to be very small

Solving the equation (1) for $f'(x)$ we have

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2!} f''(x) - \frac{h^2}{3!} f'''(x) + \ldots \tag{3}$$

If the series is truncated at the second derivative, there exists a value *c* such that lies in [x, x+h], so that

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2!} f''(c) \qquad (4)$$

where the reminder term shows the approximation error for derivative

$$\boxed{E(f,h) = -\frac{h}{2!} f''(c) = O_{forward}(h)} \qquad (5)$$

is the ***truncation error or first order accurate***. We see that this approximation is first-order accurate because the first term is dominating in the expression O(h) for small h. So, the ***forward difference*** approximation of the first derivative $f'(x)$ is

$$\boxed{f'(x) \approx \frac{f(x+h) - f(x)}{h}} \qquad (7)$$

And a *true error* for the forward difference formula is

$$T_{central}(x = x_0) = \left| f'(x) - \frac{f(x+h) - f(x)}{h} \right|_{x=x_0}$$

(b) Backward difference approximation

In similar way, one can write the Taylor series expansion of a function f(x) about x to determine f(x−h) by replacing h by –h.

$$\boxed{f(x-h) = f(x) - h f'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + \dots} \qquad (8)$$

Solving the equation (8) for $f'(x)$ we have

$$f'(x) = \frac{f(x) - f(x-h)}{h} + \frac{h}{2!} f''(x) - \frac{h^2}{3!} f^3(x) + \frac{h^3}{4!} f^4(x) \dots \qquad (9)$$

If the series is truncated at the second derivative, there exists a value c such that lies in [x-h, x], so that

$$f'(x) = \frac{f(x) - f(x-h)}{h} + \frac{h}{2!} f''(c) \qquad (10)$$

where the reminder term shows the approximation error for derivative

$$\boxed{E(f,h) = \frac{h}{2!} f''(c) = O_{backward}(h)} \qquad (11)$$

is the ***truncation error or first order accurate***. We see that this approximation is first-order accurate because the dominate term in the truncation error is $O_{back}(h)$ for small h. So, the ***backward difference*** approximation of the first derivative $f'(x)$ is

$$\boxed{f'(x) \approx \frac{f(x) - f(x-h)}{h}} \tag{13}$$

And a *True error* for the backward difference formula is

$$T_{central}(x = x_0) = \left| f'(x) - \frac{f(x) - f(x-h)}{h} \right|_{x=x_0}$$

(c) Central difference approximation

We can derive a more accurate estimate of the derivative by using the forward and backward Taylor series expansion about x.

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f^3(x) + \frac{h^4}{4!} f^4(x) \ldots \tag{14}$$

$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f^3(x) + \frac{h^4}{4!} f^4(x) \ldots \tag{15}$$

Subtracting the eqn (15) from (14), we see that the terms involving even powers of h cancel out, leaving terms are

$$f(x+h) - f(x-h) = 2\left[h f'(x) + \frac{h^3}{3!} f^3(x) + \frac{h^5}{5!} f^5(x) \ldots \right]$$

or, solving for $f'(x)$ we have

$$f(x+h) - f(x-h) = 2\left[h f'(x) + \frac{h^3}{3!} f^3(x) + \frac{h^5}{5!} f^5(x) \ldots \right]$$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \left[-\frac{h^2}{3!} f^3(x) - \frac{h^4}{5!} f^5(x) \ldots \right]$$

If the series is truncated at the second derivative, there exists a value c such that lies in [x-h, x+h], so that

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{3!} f^3(c)$$

Where the reminder term shows the approximation error for derivative

$$\boxed{E(f,h) = -\frac{h^2}{3!} f^3(c) = O_{center}(h^2)\ldots} \quad (16)$$

is truncation error or ***first order accurate***. We see that the center difference approximation is second-order accurate because the dominate term in its truncation error is $O_{center}(h^2)$ for small h. The center difference approximation is more accurate than the forward difference due to its smaller truncation error. So, the ***central difference*** approximation of the first derivative $f'(x)$ is

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} \quad (17)$$

And a ***true error*** for the forward difference formula is

$$T_{central}(x = x_0) = \left| f'(x) - \frac{f(x+h) - f(x-h)}{2h} \right|_{x=x_0} \quad (18)$$

Note: Error reduced if forward and backward difference approximations are combined.

22.2.2 Error Analysis

By same proceeding we can obtain the higher order approximation of derivatives with finite difference. Let us next discuss about the round-off error. Such errors arise due to limitation of the finite word in computers. Suppose $f(x_0 - h)$ and $f(x_0 + h)$ are approximated by the numerical values y_{-1} and y_1 and e_{-1} and e_1 are the associated round – off error then

$$f'(x_0) = \frac{y_1 - y_{-1}}{2h} + E(f,h)$$

where

$$E(f,h) = E_{round}(f,h) + E_{trunc}(f,h)$$

$$= \frac{e_1 - e_{-1}}{2h} - \frac{h^2 f^3(c)}{6} \quad (18)$$

Hence the total error in numerical differentiation formula is truncation (approximation) plus round-off. If

$$|e_1| \le \varepsilon, |e_{-1}| \le \varepsilon \qquad \text{and}$$

$$M = \underset{x \in [a,b]}{Max} |f^3(x)|$$

then $$\left|E(f,h)\right| \le \frac{\varepsilon}{2h} + \frac{M h^2}{6} \tag{19}$$

Concept of Total Error (optimal choice of h)

The optimal choice of h represents a compromise between the conflicting requirements of minimizing the round-off error, the first term which requires a large value of h, and minimizing the approximation error, the second term, which requires a small value of 'h'. To reduce the truncation error, we need to reduce h.

But as h is reduced, the round-off error grows. The value of h that minimizes the error term in above eqn.

$$h = \left(\frac{3\varepsilon}{M}\right)^{1/3} \tag{20}$$

Now consider an example to understand the concept of total error

Example 1 Let $f(x) = \sin x$

(a) Use formula (18) with step sizes h= 0.0001, 0.001, 0.01and 0.1 and calculate approximations for $f'(0.5)$.

(b) Calculate the optimal value of 'h'.

Sol. **(a)** $f'(x) = \cos x$ then

true value of cos(0.5) = 0.8775825 (7 decimal place)

from the approximation formula (18)

$$f'(0.5) \approx \frac{f(0.5+h) - f(0.5-h)}{2h}$$

Gives us different values of approximation for the different h (as shown in table)

Step Size	Approximation $f'(0.5)$	Error = True−Approx
0.1	0.8761205	1.4620619×10^{-3}
0.01	0.8775700	1.2561920×10^{-5}
0.001	0.8775906	-8.0380800×10^{-6}
0.0001	4.8279724	− 3.9503898

(b) We can use the bound

$$M = \left|f^3(x)\right| \le \left|-\cos(x)\right| \le 1$$

and the values of *f* are given to seven decimal places, we will assume that the round-off error is bounded by $\varepsilon = 0.5 \times 10^{-7}$(machine epsilon). The optimal value of h can be easily calculated:

$$h = \left(\frac{3\varepsilon}{M}\right)^{1/3} = \left(\frac{3 \times 0.5 \times 10^{-7}}{1}\right)^{1/3} = 0.0053133$$

the step size 0.001is closer to the optimal value 0.0053133 and it gives the best approximation to $f'(0.5)$ among the four choices.

22. 3 Illustrative Examples

Example 2 Let $f(x) = \sin x + \cos x$.

Calculate approximations for $f'(0)$ by all three difference formula with h=0.1, 0.01 and 0.001 and compute also

(a) Compute an upper bound for each approximation error.

(b) Compute the true error.

Sol. Let be given $f(x) = \sin x + \cos x$, then

$$f'(x) = \cos x - \sin x \quad ; \quad f'(0) = 1$$

forward difference approximation

$$f'_{forward}(0) = \frac{f(h) - f(0)}{h} = \frac{\sin(h) + \cos(h) - 1}{h}$$

backward difference approximation

$$f'_{backward}(0) = \frac{f(0) - f(-h)}{h} = \frac{1 - \left[-\sin(h) + \cos(h)\right]}{h} = \frac{1 + \sin(h) - \cos(h)}{h}$$

central difference approximation

$$f'_{central}(0) = \frac{f(h) - f(-h)}{2h} = \frac{\left[\sin(h) + \cos(h)\right] - \left[-\sin(h) + \cos(h)\right]}{2h} = \frac{\sin(h)}{h}$$

Also,

$$f''(x) = -\sin x - \cos x$$

$f''(0) = -1$ and

$f'''(x) = -\cos x + \sin x$

$f'''(0) = -1$

An approximation error is

$$R_{forward} = \frac{h}{2!}\left|f''(c)\right| \le \frac{h}{2!}\left|f''(0)\right| \le \frac{h}{2!}\left|(-1)\right| \le \frac{h}{2}$$

Similarly, one can obtain

$$R_{backward} = \frac{h}{2!}\left|f''(c)\right| \le \frac{h}{2!}\left|f''(0)\right| \le \frac{h}{2!}\left|(-1)\right| \le \frac{h}{2}$$ and

$$R_{central} = \frac{h^2}{3!}\left|f''(c)\right| \le \frac{h^2}{3!}\left|f''(0)\right| \le \frac{h^2}{3!}\left|(-1)\right| \le \frac{h^2}{6}$$

Table for forward difference

h	Approximations $\frac{\sin(h)+\cos(h)-1}{h}$	True Error $\left\|1-\frac{\sin(h)+\cos(h)-1}{h}\right\|$	Approx. Error $\le \frac{h}{2}$
0.1	0.0174380	0.9825619	0.05
0.01	0.0174517	0.9825482	0.005
0.001	0.0174531	0.9825469	0.0005

Table for backward difference

h	Approximations $\frac{1+\sin(h)-\cos(h)}{h}$	True Error $\left\|1-\frac{1+\sin(h)-\cos(h)}{h}\right\|_{x=x_0}$	Approx. Error $\le \frac{h}{2}$
0.1	0.0174685	0.9825315	0.05
0.01	0.0174548	0.9825452	0.005
0.001	0.0174534	0.9825465	0.0005

Table for central difference

h	Approximations $\frac{\sin(h)}{h}$	True Error $\left\|1-\frac{\sin(h)}{h}\right\|_{x=x_0}$	Approx. Error $\le \frac{h^2}{6}$
0.1	0.0174532	0.9825467	1.67×10^{-3}
0.01	0.0174532	0.9825467	1.67×10^{-5}
0.001	0.0174532	0.9825467	1.67×10^{-7}

Example 3 Derive the expression for the Second-Order Approximation of first Derivative on forward difference.

Sol. The Taylor series expansion of the function f(x) can be written as

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f^3(x) + \dots \tag{1}$$

solving this equation for $f'(x)$ we have

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2!} f''(x) - \frac{h^2}{3!} f^3(x) + \dots \tag{2}$$

If we truncate the Taylor series (Eq. 1) at the third term (at the second derivative), the result will be as follows:

Now the forward difference approximation of the second derivative is given by

$$f''(x) = \frac{f'(x+h) - f'(x)}{h} \tag{3}$$

But $\quad f'(x+h) = \frac{f(x+2h) - f(x+h)}{h} \quad$ and $\quad f'(x) = \frac{f(x+h) - f(x)}{h}$

Putting these equations in eqn.3 will give the following results

$$f''(x) = \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$$

Substituting this eqn in (2) we get

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2!}\left[\frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}\right] - \frac{h^2}{3!} f^3(x) + \dots$$

$$f'(x) = \frac{2f(x+h) - 2f(x) - f(x+2h) + 2f(x+h) - f(x)}{2h} - \frac{h^2}{3!} f^3(x) + \ldots$$

$$f'(x) = \frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h} - \frac{h^2}{3!} f^3(x) + \ldots$$

$$f'(x) = \frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h} + E(f, h)$$

Where the reminder term shows the approximation error for derivative

$$E(f, h) = -\frac{h^2}{3!} f^3(x) - \frac{h^3}{4!} f^4(x) \ldots$$

and $$\boxed{O_{forward}(h^2) = -\frac{h^2}{3!} f'''(x)}$$

is truncation error or *second order accurate*. Finally, one can write the second-order approximation of first derivative $f'(x)$ is

$$f'(x) \approx \frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h}$$

22.4 Self Learning Exercises-I

Very Short Answer Type Questions

Q.1 What is first order accurate in forward difference approximation?

Q.2 What is first order accurate in backward difference approximation?

Q.3 What is first order accurate in central difference approximation?

Short Answer Type Questions

Q.4 Why the center difference approximation is more accurate than the forward difference?

Q.5 Use the backward difference formula to approximate the first derivative of the function f(x)= sinx at $x_0 = \pi$ using h=0.1.

Q.6 Use the backward difference formula to approximate the first derivative of the function f(x)= cosx at $x_0 = \pi$ using h=0.01.

22.5 Numerical Differentiation

In numerical differentiation It is essential required the proper selection of interpolation formulae to solve the numerical differentiation problems.

Selection of formula

(a)If the values of the argument are equally spaced, the formula is represented by Newton's Gregory formula to determine the numerical differentiation as desired.

(b)If we want to find the derivative of function at a point near beginning of end of a set of tabular value, then we use Newton's Gregory forward (backward) formula as desired.

(c)If the values of argument are not equally spaced, we shall use Newton's divided formula or Lagrange's formula to represent the function.

22.5.1 Method based on Interpolation formula

In this method, the formula is represented by an interpolation formula and then differentiating many times as required. For exp. let us consider ***Newton's Gregory formula,***

$$f(u) = f(0) + u\Delta f(0) + \frac{u(u-1)}{2!}\Delta^2 f(0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(0) + \dots \tag{1}$$

Where $u = \frac{x - x_0}{h}$ (2)

$$\therefore \quad \frac{df(u)}{dx} = \frac{df(u)}{du}\frac{du}{dx} = \frac{1}{h}\frac{df(u)}{du} \tag{3}$$

Now differentiating eqn (1) w.r.t u and applying (3), we get

$$\frac{df(u)}{dx} = \frac{1}{h}\frac{df(u)}{du} = \frac{1}{h}\left[\Delta f(0) + \frac{(2u-1)}{2!}\Delta^2 f(0) + \frac{3u^2 - 6u + 2}{3!}\Delta^3 f(0) + \dots\right] \tag{4}$$

Once again differentiating above eqn (1) w.r.t 'u', we get

$$\frac{d^2 f(u)}{dx^2} = \frac{d}{dx}\frac{df(u)}{dx} = \frac{d}{du}\left(\frac{df(u)}{dx}\right)\frac{du}{dx} = \frac{1}{h}\frac{d}{du}\left(\frac{df(u)}{dx}\right) = \frac{1}{h^2}\left[\Delta^2 f(0) + (u-1)\Delta^3 f(0) + \dots\right] \tag{5}$$

Similarly, one can determine the higher order derivative as desired.

Special Case: Numerical differentiation at the particular value $x = x_0$

Putting $x = x_0$ and $u = 0$ in eqn (4) and (5)

$$\left(\frac{df(u)}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta f(0) + \frac{1}{2!}\Delta^2 f(0) + \frac{2}{3!}\Delta^3 f(0) + \ldots\right] \qquad (6)$$

and $$\left(\frac{d^2 f(u)}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 f(0) + \Delta^3 f(0) + \ldots\right] \qquad (7)$$

Note: The same process can be used to other interpolation formulae such as Lagrange's, Sterling's and Bessel's etc.

22.6 Illustrative Examples

Example 4 Determine the first (dy/dx) and second derivative (d^2y/dx^2) of function tabulated below at the point x=0.35.

x	0.25	0.50	0.75	1.00	1.25	1.50
f(x)	− 0.4219	− 0.1250	− 0.0156	0.0000	0.0156	0.1250

Sol. By inspection of the question, the value of argument is equally spaced and required derivative are at a point near the beginning of the table. In this case, we shall use Newton's Gregory forward interpolation formula.

The finite difference table is as under:

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0.25	− 0.4219					
		0.2969				
0.50	− 0.1250		− 0.1875			
		0.1094		0.0937		
0.75	− 0.0156		− 0.0938		0.0001	
		0.0156		0.0938		− 0.0001
1.00	0.0000		0.0000		0.0000	
		0.0156		0.0938		
1.25	0.0156		0.0938			
		0.1094				
1.50	0.1250					

Newton's Gregory forward interpolation formula is

$$f(x) = f(a+uh) = f(a) + u\,\Delta f(a) + \frac{(u^2 - u)}{2!}\Delta^2 f(a) + \frac{(u^3 - 3u^2 + 2u)}{3!}\Delta^3 f(a)$$
$$+ \frac{(u^4 - 6u^3 + 11u^2 - 6u)}{4!}\Delta^4 f(a) + \frac{(u^5 - 10u^4 + 35u^3 - 50u^2 + 256u)}{5!}\Delta^5 f(a) + \ldots$$

where, $u = \frac{x-a}{h}$

Differentiating with respect to x, we have

$$\frac{df(x)}{dx} = \frac{df(x)}{du}\frac{du}{dx} = \frac{df(x)}{du}\frac{d}{dx}\left(\frac{x-a}{h}\right) = \frac{1}{h}\frac{df(x)}{du}$$

$$\frac{d}{dx}f(x) = \frac{1}{h}\frac{df(x)}{du} = \frac{1}{h}\left[\Delta f(a) + \frac{(2u-1)}{2}\Delta^2 f(a) + \frac{(3u^2 - 6u + 2)}{6}\Delta^3 f(a)\right.$$
$$\left. + \frac{(4u^3 - 18u^2 + 22u - 6)}{24}\Delta^4 f(a) + \frac{(5u^4 - 40u^3 + 105u^2 - 100u + 256)}{120}\Delta^5 f(a) + \ldots\right]$$

The second derivative of f(x)

$$\frac{d^2 f(x)}{dx^2} = \frac{d}{du}\left(\frac{df(x)}{dx}\right)\frac{du}{dx} = \frac{1}{h}\frac{d}{du}\left(\frac{df(x)}{dx}\right)$$

$$\frac{d^2 f(x)}{dx^2} = \frac{1}{h^2}\frac{d}{du}\left[\Delta f(a) + \frac{(2u-1)}{2}\Delta^2 f(a) + \frac{(3u^2 - 6u + 2)}{6}\Delta^3 f(a)\right.$$
$$\left. + \frac{(4u^3 - 18u^2 + 22u - 6)}{24}\Delta^4 f(a) + \frac{(5u^4 - 40u^3 + 105u^2 - 100u + 256)}{120}\Delta^5 f(a) + \ldots\right]$$

$$\frac{d^2 f(x)}{dx^2} = \frac{1}{h^2}\left[\Delta^2 f(a) + (u-1)\Delta^3 f(a) + \frac{(6u^2 - 18u + 11)}{12}\Delta^4 f(a)\right.$$
$$\left. + \frac{(4u^3 - 24u^2 + 42u - 20)}{24}\Delta^5 f(a) + \ldots\right]$$

Here a=0.25, and h=0.25. At x=0.35, $u = \frac{x-a}{h} = \frac{0.35 - 0.25}{0.25} = 0.4$

Therefore, the first derivative

$$f'(0.35) = \frac{1}{0.25}\left[0.2969 + (-0.1)(-0.1875) + (0.0133)\ (0.0937) + \ldots\right.$$
$$\left. + (0.0073)\ (0.0001) + (1.9197)\ (-0.0001)\right]$$

$$f'(0.35) = \frac{1}{0.25}\left[0.2969 + (0.0188) + (0.0012) + (0.0000) - (0.0002)\right]$$

$$f'(0.35) = \frac{1}{0.25}\left[(0.3169) - (0.0002)\right] = \frac{0.3167}{0.25} = 1.2668$$

In similar way, the second derivative

$$f''(0.35) = \frac{1}{(0.25)^2}\left[(-0.1875) + (-0.6)(0.0937) + (0.3967)\,(0.0001) + \ldots \right.$$
$$\left. + (-0.2827)\,(-0.0001)\right]$$

$$f'(0.35) = \frac{1}{0.0625}\left[-0.1875 - (0.0562) + (0.0000) + (0.0000)\right] = -\frac{0.2437}{0.0625} = -3.8992$$

Example 5 Determine the first (dy/dx) and second derivative (d^2y/dx^2) of function tabulated below at the point x=1.1.

x	0.2	0.4	0.6	0.8	1.0	1.2
f(x)	0.968	0.904	0.856	0.872	1.000	1.288

Sol. The value of argument is equally spaced and required derivative are at a point near the end of the table. So we shell use of the central difference formula. In this case, we shall use Newton's backward formula.

The finite difference table is as under:

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0.2	0.968			
		− 0.064		
0.4	0.904		0.016	
		− 0.048		0.048
0.6	0.856		0.064	
		0.016		0.048
0.8	0.872		0.112	
		0.128		0.048
1.0	1.000		0.160	
		0.288		
1.2	1.288			

Newton's backward formula is

$$f(x) = f(x_n + uh) = f(x_n) + u\Delta f(x_n) + \frac{(u^2+u)}{2!}\Delta^2 f(x_n) + \frac{(u^3+3u^2+2u)}{3!}\Delta^3 f(x_n) + \dots$$

where, $u = \dfrac{x - x_n}{h}$

Differentiating with respect to x, we have

$$\frac{df(x)}{dx} = \frac{df(x)}{du}\frac{du}{dx} = \frac{df(x)}{du}\frac{d}{dx}\left(\frac{x-x_n}{h}\right) = \frac{1}{h}\frac{df(x)}{du}$$

$$\frac{d}{dx}f(x) = \frac{1}{h}\frac{df(x)}{du} = \frac{1}{h}\left[\Delta f(x_n) + \frac{(2u+1)}{2}\Delta^2 f(x_n) + \frac{(3u^2+6u+2)}{6}\Delta^3 f(x_n) + \dots\right]$$

The second derivative of f(x)

$$\frac{d^2 f(x)}{dx^2} = \frac{d}{du}\left(\frac{df(x)}{dx}\right)\frac{du}{dx} = \frac{1}{h}\frac{d}{du}\left(\frac{df(x)}{dx}\right)$$

$$\frac{d^2 f(x)}{dx^2} = \frac{1}{h^2}\left[\Delta^2 f(x_n) + (u+1)\Delta^3 f(x_n)\right]$$

Here x_n= 1.2, and h= 0.2. At x=1.1, $u = \dfrac{x - x_n}{h} = \dfrac{1.1 - 1.2}{0.2} = -0.5$

Therefore, the first derivative

$$f'(1.1) = \frac{1}{0.2}[(0.288) + (0.000)(0.160) + (-0.042)(0.048)] = 5[0.288 - 0.002] = 1.43$$

In similar way, the second derivative

$$f''(1.1) = \frac{1}{(0.2)^2}[(0.160) + (0.5)(0.048)] = 25(0.184) = 4.6$$

Example 6 Determine the first (dy/dx) and second derivative (d^2y/dx^2) of function tabulated below at x=2.

x	1	1.5	2	2.5	3	3.5
f(x)	0	0.25	1	2.25	4	6.25

Sol. By inspection of the question the value of argument is equally spaced and required derivative are at a point near the middle of the table. So we shell use of the central difference formula. In this case, we have used Gauss's forward interpolation formula.

Here new variant $u = \frac{x - x_m}{h}$

for x=2, u=0.

The finite difference table is as under:

u	x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
− 2	1	0			
			0.25		
− 1	1.5	0.25		0.5	
			0.75		0
0	2	1		0.5	
			1.25		0
1	2.5	2.25		0.5	
			1.75		0
2	3	4		0.5	
			2.25		
3	3.5	6.25			

Gauss's interpolation formula with new variant 'u' is;

$$f(x) = f(x_m + uh) = f(0) + u\Delta f(0) + \frac{(u^2 - u)}{2!}\Delta^2 f(-1) + \frac{(u^3 - u)}{3!}\Delta^3 f(-1)$$

$$+ \frac{(u^4 - 2u^3 - u^2 - 2u)}{4!}\Delta^4 f(-2) + \ldots$$

first derivative of

$$\frac{df(x)}{dx} = \frac{df(x)}{du}\frac{du}{dx} = \frac{df(x)}{du}\frac{d}{dx}\left(\frac{x - x_m}{h}\right) = \frac{1}{h}\frac{df(x)}{du}$$

Now differentiating (1) w.r.t u and using u=0 and h=0.5, we get

$$\frac{df(u)}{du} = \frac{1}{h}\left[\Delta f(0) + \frac{(2u-1)}{2}\Delta^2 f(-1) + \frac{(3u^2-1)}{6}\Delta^3 f(-1) + \frac{(4u^3 - 6u^2 - 2u - 2)}{24}\Delta^4 f(-2) + \ldots\right]$$

$$\left(\frac{df(u)}{du}\right)_{x=2} = \frac{1}{0.5}\left[1.25 + \frac{(-1)}{2}(0.5) + 0\right] = 2$$

Similarly,

$$\frac{d^2f(x)}{dx^2} = \frac{d}{du}\left(\frac{df(x)}{dx}\right)\frac{du}{dx} = \frac{1}{h}\frac{d}{du}\left(\frac{df(x)}{dx}\right)$$

$$= \frac{1}{h^2}\left[\Delta^2 f(-1) + u\Delta^3 f(-1) + \frac{(12u^2 - 12u - 2)}{24}\Delta^4 f(-2)\right]$$

Therefore, $$\left(\frac{d^2f(u)}{du^2}\right)_{x=2} = \frac{1}{0.5^2}(0.5) = 2$$

Example 7 Determine the first (dy/dx) and second derivative (d^2y/dx^2) of function tabulated below at the point x=11.

x	2	4	9	13	16	21
f(x)	13	81	811	2367	4353	9703

Sol. The value of argument is unequally spaced. In this case, we shall use Newton's divided difference formula.

The divided difference table is as under:

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	13			
		34		
4	81		16	
		146		1
9	811		27	
		389		1
13	2367		39	
		662		1
16	4353		51	
		1070		
21	9703			

Newton's divided difference formula is

$$f(x) = f(a) + (x-a)\Delta f(a) + (x-a)(x-b)\Delta^2 f(a) + (x-a)(x-b)(x-c)\Delta^3 f(a) + \dots$$

$$f(x) = f(a) + (x-a)\Delta f(a) + \{x^2 - (a+b)x + ab\}\Delta^2 f(a)$$
$$+ \{x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc\}\Delta^3 f(a) + \dots$$

Differentiating w.r.t x

$$f'(x) = \Delta f(a) + \{2x - (a+b)\}\Delta^2 f(a) + \{3x^2 - 2(a+b+c)x + (ab+ac+bc)\}\Delta^3 f(a) + \dots$$

The second derivative of f(x)

$$f''(x) = 2\Delta^2 f(a) + \{6x - 2(a+b+c)\}\Delta^3 f(a) + \dots$$

Putting x=11, a=2,b=4,c=9 and the values of differences from the table, we get

$$f'(11) = 34 + (16)(16) + (95)(1) = 34 + 256 + 95 = 385$$

and

$$f''(x) = 2(16) + (36)(1) = 68$$

22.7 Method based on Operator

Approximate expression for the central difference derivative

(a) First difference derivative

As we know that

$$\delta y_n = y_{n+1/2} - y_{n-1/2} \tag{1}$$

and
$$\mu y_n = \frac{1}{2}[y_{n+1/2} + y_{n-1/2}] \tag{2}$$

Therefore,
$$\mu\delta y_n = \mu[y_{n+1/2} - y_{n-1/2}] \tag{3}$$

Now
$$\mu y_{n+1/2} = \frac{1}{2}[y_{n+1/2+1/2} + y_{n+1/2-1/2}] = \frac{1}{2}[y_{n+1} + y_n]$$

And
$$\mu y_{n-1/2} = \frac{1}{2}[y_{n-1/2+1/2} + y_{n-1/2-1/2}] = \frac{1}{2}[y_n + y_{n-1}]$$

Substituting these values in eqn. (3), we get

$$\therefore \quad \mu\delta y_n = \mu[y_{n+1/2} - y_{n-1/2}] = \frac{1}{2}[y_{n+1} + y_n] - \frac{1}{2}[y_n + y_{n-1}]$$
$$= \frac{1}{2}[y_{n+1} - y_{n-1}] \tag{4}$$

$$= \frac{1}{2}[e^{hD} - e^{-hD}]y_n \qquad [\because y_{n+1} = e^{hD}y_n]$$

$$= [\sinh D]y_n$$

or $$\mu\delta = \sinh D = hD + \frac{h^3D^3}{3} + \ldots.$$

$$\mu\delta \approx hD \qquad \text{(neglecting higher order terms)}$$

Now, $$hD = \frac{1}{2}[y_{n+1} - y_{n-1}] \qquad \text{(approx)}$$

or $$\boxed{Dy_n = \frac{d}{dx}(y_n) = \frac{1}{2h}[y_{n+1} - y_{n-1}]} \qquad (5)$$

This is called as first derivative formula for central difference derivative.

(b) Second difference derivative

Now $$\delta^2 y_n = \delta(\delta y_n) = \delta[y_{n+1/2} - y_{n-1/2}]$$

$$= \delta y_{n+1/2} - \delta y_{n-1/2}$$

$$= (y_{n+1/2+1/2} - y_{n+1/2-1/2}) - (y_{n-1/2+1/2} - y_{n-1/2-1/2})$$

$$[\because \quad \delta y_{n+1/2} = (y_{n+1/2+1/2} - y_{n+1/2-1/2}) \quad \text{and} \quad \delta y_{n-1/2} = (y_{n-1/2+1/2} - y_{n-1/2-1/2})]$$

$$\delta^2 y_n = (y_{n+1} - y_n) - (y_n - y_{n-1}) = y_{n+1} + y_{n-1} - 2y_n$$

$$= (e^{hD} + e^{-hD} - 2)y_n = 2(\cosh D - 1)y_n$$

$$= 2\left[\left(1 + \frac{1}{2}h^2D^2 + \ldots\right) - 1\right]y_n = (h^2D^2 + \ldots)y_n$$

$$\delta^2 y_n = h^2D^2y_n \quad \text{approx} \qquad \text{(neglecting higher order terms)}$$

Therefore $$\delta^2 y_n = h^2D^2y_n = (y_{n+1} + y_{n-1} - 2y_n)$$

or $$\boxed{D^2y_n = \frac{d^2}{dx^2}(y_n) = \frac{1}{h^2}(y_{n+1} + y_{n-1} - 2y_n)} \qquad (6)$$

This is called as second derivative formula for central difference derivative.

(c) Third difference derivative

Once again $$h^3D^3y_n = hD(h^2D^2y_n) = hD\delta^2 y_n$$

$$= \mu\delta\left(y_{n+1} + y_{n-1} - 2y_n\right) \qquad \left[\because \mu\delta \approx hD\right]$$

$$= \frac{1}{2}\left(y_{n+1+1} - y_{n+1-1}\right) + \frac{1}{2}\left(y_{n-1+1} - y_{n-1-1}\right) + \left(y_{n+1} - y_{n-1}\right)$$

$$\left[\because \quad \mu\delta y_n = \frac{1}{2}\left(y_{n+1} - y_{n-1}\right)\right]$$

$$= \frac{1}{2}\left(y_{n+2} - y_n\right) + \frac{1}{2}\left(y_n - y_{n-2}\right) - \left(y_{n+1} - y_{n-1}\right)$$

$$= \frac{1}{2}\left(y_{n+2} - 2y_{n+1} + 2y_{n-1} - y_{n-2}\right)$$

$$\boxed{D^3 y_n = \frac{d^3}{dx^3}(y_n) = \frac{1}{2h^3}\left(y_{n+2} - 2y_{n+1} + 2y_{n-1} - y_{n-2}\right)} \qquad (7)$$

This is called as third derivative formula for central difference derivative.

Symbolic derivation of derivatives of a function

Let the function y=f(x) and $x=x_0+hu$

where h= difference of interval

Therefore $\quad y=f(x_0+hu) =E^u y_0$ (8)

where $y_0=y(x_0)$

Now, differentiating (8) w.r.t x at x_0

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \left[\frac{d}{dx}(E^u y_0)\right]_{x=x_0} = \left[\frac{d}{du}(E^u y_0)\frac{du}{dx}\right]_{x=x_0}$$

$$= \frac{1}{h}\left[\frac{d}{du}(E^u y_0)\right]_{u=0} = \frac{1}{h}\left[\left(\log E\right)(E^u y_0)\right]_{u=0}$$

$$= \frac{1}{h}\left(\log E\right)y_0 = \frac{1}{h}\left[\log\left(1+\Delta\right)\right]y_0$$

from logarithmic expansion

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\right]y_0$$

$$= \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right]$$

similarly,

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\log(1+\Delta)\right]^2 y_0$$

by logarithmic expansion and taking square root we get

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \ldots\right]$$

In general form

$$\left(\frac{d^ny}{dx^n}\right)_{x=x_0} = \frac{1}{h^n}\left[\log(1+\Delta)\right]^n y_0$$

Example 8 Prove that

$$y' = \frac{1}{h}\left[\Delta y - \frac{1}{2}\Delta^2 y + \frac{1}{3}\Delta^3 y - \frac{1}{4}\Delta^4 y + \ldots\right]$$

and $$y'' = \frac{1}{h^2}\left[\Delta^2 y + \Delta^3 y + \frac{11}{12}\Delta^4 y + \ldots\right]$$

Sol. As we know that $E = e^{hD}$

$$\therefore \quad 1 + \Delta = e^{hD}$$

or $$D = \frac{1}{h}\log(1+\Delta)$$

therefore $$Dy = \frac{1}{h}\log(1+\Delta)y$$

$$= \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\right]y$$

$$\therefore \quad y' = \frac{1}{h}\left[\Delta y - \frac{1}{2}\Delta^2 y + \frac{1}{3}\Delta^3 y - \frac{1}{4}\Delta^4 y + \ldots\right]$$

Similarly we know that

$$e^{-hD} = 1 - \nabla$$

$$\therefore \quad D = -\frac{1}{h}\log(1-\nabla)$$

Now $$y'' = D^2 y = \left\{-\frac{1}{h}\log(1-\nabla)\right\}^2 y$$

$$= \frac{1}{h^2}\left[\Delta + \frac{1}{2}\Delta^2 + \frac{11}{12}\Delta^3 + ...\right]^2 y$$

$$= \frac{1}{h^2}\left[\Delta y + \frac{1}{2}\Delta^2 y + \frac{11}{12}\Delta^3 y + ...\right]^2$$

22.8 Self Learning Exercises-II

Very Short Answer Type Questions

Q.1 Which formula is used to determine the numerical differentiation while the values of the argument are equally spaced?

Q.2 Which formula is used to determine the numerical differentiation at a point near beginning of end of a set of tabular value?

Q.3 Which formula is used to determine the numerical differentiation while the values of the argument are not equally spaced?

Short Type Answer Type Questions

Q.4 Find the first (dy/dx) derivative of the function y = sinx tabulated below at the point x=0.95.

x	0.7	0.8	0.9	1	1.1
y	0.644218	0.717356	0.783327	0.841471	0.891207

Q.5 Find the first (dy/dx) derivative of the function y = cosx tabulated below at the point x=0.75.

x	0.7	0.8	0.9	1	1.1
y	0.764842	0.696707	0.621610	0.540302	0.453596

Q.6 Prove that

$$\frac{d}{dx}\left(y_x h\right) = \frac{1}{h}\left(y_{x+h} - y_{x-h}\right) + \frac{1}{2h}\left(y_{x+2h} - y_{x-2h}\right) + \frac{1}{3h}\left(y_{x+3h} - y_{x+3h}\right)$$

22.9 Summary

● Numerical Differentiation is the process by which we can find the approximate value of the derivative of a function at a given value of the independent variable.

●**First order approximation and associated truncation error**

(a) Forward difference approximation

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$

$$E(f,h) = -\frac{h}{2!} f''(c)$$

(b) Backward difference approximation

$$f'(x) \approx \frac{f(x)-f(x-h)}{h}$$

$$E(f,h) = \frac{h}{2!} f''(c)$$

(c) Central difference approximation

$$f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$$

$$E(f,h) = -\frac{h^2}{3!} f^3(c)$$

● the total error in numerical differentiation formula is given by sum of the truncation (approximation) and round-off error.

● In numerical differentiation

(a) If the values of the argument are equally spaced, the formula is represented by Newton's Gregory formula to determine the numerical differentiation as desired.

(b) If we want to find the derivative of function at a point near beginning of end of a set of tabular value, then we use Newton's Gregory forward (backward) formula as desired.

(c) If the values of argument are not equally spaced, we shall use Newton's divided formula or Lagrange's formula to represent the function.

22.10 Glossary

Truncate: Shorten (something) by cutting off the top or the end

Interpolate: Insert (an intermediate value or term)into a series by estimating or calculating it from surrounding known values

22.11 Answers to Self Learning Exercises

Answers to Self Learning Exercise -I

Ans.1: $-\frac{h}{2!}f''(c)$

Ans.2: $\frac{h}{2!}f''(c)$

Ans.3: $-\frac{h^2}{3!}f^3(c)$

22.12 Exercises

Section-A: (Very Short Answer Type Questions)

Q.1 Define round-off error.

Q.2 Define the approximation of a function.

Q.3 What is truncation error order for the first derivative of forward difference formula?

Q.4 What is truncation error order for the first derivative of central difference formula?

Q.5 What is truncation error order for the first derivative of backward difference formula?

Section-B :(Short Answer Type Questions)

Q.6 Let $f(x) = \sin x + \cos x$.

Calculate approximations for $f'(0)$ by all three difference formula with =0.1, 0.01 and 0.001 and compute also

(a) Compute an upper bound for each approximation error.

(b) Compute the true error.

Q.7 How many terms are required in calculation of the approximate $^{0.5}$ (1.648721...) correct to four decimal places after rounding?

Q.8 Let $f(x) = \cos x$

(a) Use formula (18) with step sizes h= 0.0001, 0.001, 0.01and 0.1 and calculate approximations for $f'(0.5)$.

(b) Calculate the optimal value of 'h'.

Q.9 Use the central difference formula to approximate the derivative of the function f(x) = cosx at $x_0 = \pi$ using h=0.01.

Q.10 Use the Second-Order approximation of first derivative on forward difference formula to approximate the derivative of the function f(x) = sinx at $x_0 = \pi$ using h=0.1.

Section C: (Long Answer Type Questions)

Q.11 Consider $f(x) = e^x$ and evaluate $f'(1)$ using h=0.01 for the forward, backward and central approximation. Which is the best approximation?

Q.12 Determine the first (dy/dx) and second derivative (d^2y/dx^2) of function tabulated below at the point x=10.

x	2	4	9	13	16	21
f(x)	13	81	811	2367	4353	9703

Q.13 Find the first (dy/dx) derivative of the function y = cosx tabulated below at the point x=0.87.

x	0.7	0.8	0.9	1	1.1
y	0.764842	0.696707	0.621610	0.540302	0.453596

Q.14 Determine the first (dy/dx) derivative of the function tabulated below at the points x=0.1.0.6 and 1.1 by using backward difference, forward difference and central difference approximation. Give comments.

x	0.2	0.4	0.6	0.8	1.0	1.2
f(x)	0.968	0.904	0.856	0.872	1.000	1.288

Q.15 Assuming Bessel's interpolation formula, prove that

$$\frac{d}{dx}(y_x) = \Delta y_{x-1/2} - \frac{1}{24}\Delta^3 y_{x-3/2} + \ldots$$

References and Suggested Readings

1. Thomas Finny, 'Calculus and Analytic Geometry', 9st edition, Pearson Education, Asia (2001).
2. John.H. Mathews, 'Numerical Methods Using MATLAB', Third edition, Prentice Hall, Upper Saddle River, NJ 07458 (1999).
3. Kuldeep Singh Gehlot, 'Mathematical Methods for Numerical Analysis and Optimization ', 1st edition, College Book House, (Pvt) Ltd, Jaipur (2007).

UNIT-23
Numerical integration

Structure of the Unit

23.0 Objectives

This chapter explores study methods for approximating the integral of a function over a given interval and for determining the error associated with the rules. We know that every function cannot be easily solved analytically. This type of integral can be easily solved by numerical methods.

Numerical integration has always been useful in many areas of science and economics to evaluate distribution functions and other quantities.

23.1 Introduction

Numerical integration is the study of how the numerical value of an integral can be found. This method in general is known as numerical *quadrature*, which refers to finding a square whose area is the same as the area under a curve. It is one of the important topics of numerical analysis. Our main interest is to find out the process of approximating a definite integral from values of the integrand.

The basic quadrature methods can be categorized in two main classes:

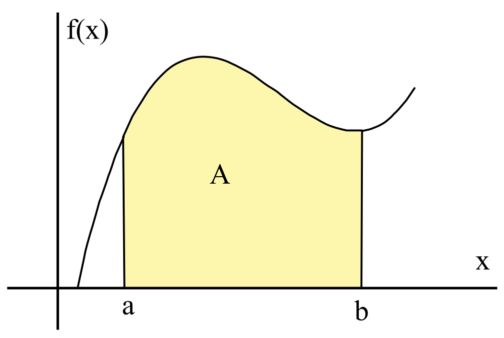
1. The methods that are based on data points which are equally spaced: these are Newton cotes formulas, the midpoint rule, the trapezoid rule and Simpson rule.

2. The methods that are based on data points which are not equally spaced: these are Gaussian quadrature formulas.

23.1.1 Definition of numerical integration

By definition, the integral of some function f(x) between the limits a and b may be thought of as the area A between the curve at the x-axis and is written mathematically as (shown in Figure)

$$A = \int_a^b f(x)\,dx \qquad (1)$$



In general, a numerical integration is the basic idea of the approximation of a definite integration by a "weighted" sum of function values at discrete points within the interval limit. thus

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i) + E_t \qquad (2)$$

where w_i is the weighted factor which depends on the schemes used in the integration, $f(x_i)$ is the function value evaluated at the given point x_i (node) and E_t is the truncation error.

23.1.2 Newton-Cotes Integration Formulas

According to this idea a complicated function or a tabulated data can be replaced by an approximating (interpolating) function.

$$I = \int_a^b f(x)\,dx \approx \int_a^b P_n(x)\,dx \qquad (2)$$

where $P_n(x)$ is an n^{th} order interpolating polynomial.

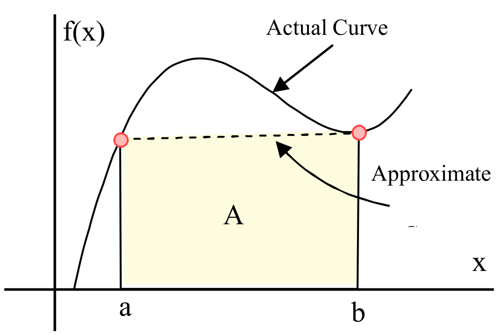
$$\int_a^b P_n(x)dx \approx \int_a^b \left(a_0 + a_1 x + \ldots + a_n x^n\right)dx$$

$$\int_a^b P_n(x)dx \approx a_0(b-a) + a_1\frac{(b^2-a^2)}{2} + \ldots + a_n\frac{(b^{n+1}-a^{n+1})}{n+1}$$

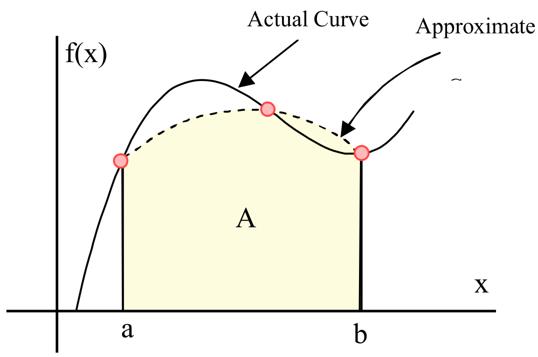
In general, for the given nodes $\{x_0, x_1, \ldots x_n\}$ we can use the interpolation polynomial formula to obtain integration.

$$\int_a^b f(x)\,dx = \underbrace{\int_a^b P_n(x)\,dx}_{\text{The Approximation}} + \underbrace{\int_a^b E_n(x)\,dx}_{\text{The Error Term}} \qquad (3)$$

1st order Polynomial



2nd order Polynomial



This integral is can be determined numerically by dividing the domain [a; b] into n equally spaced. Different choices for *m*'s lead to different formulas as follow (see table below):

Trapezoid method (*First order polynomials are used*)

$$\int_a^b f(x)\,dx \approx \int_a^b (a_0 + a_1 x)\,dx$$

Simpson's 1/3 rule (*Second order polynomials are used*)

$$\int_a^b f(x)\,dx \approx \int_a^b (a_0 + a_1 x + a_2 x^2)\,dx$$

m	Polynomial	Formula	Error
1	Linear	Trapezoid	$O(h^2)$
2	Quadratic	Simpson's 1/3	$O(h^4)$
3	Cubic	Simpson's 3/8	$O(h^4)$

23.2. The Trapezoidal Rule

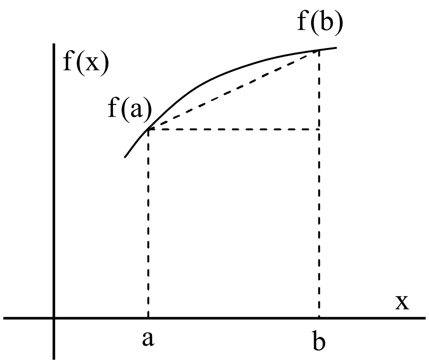


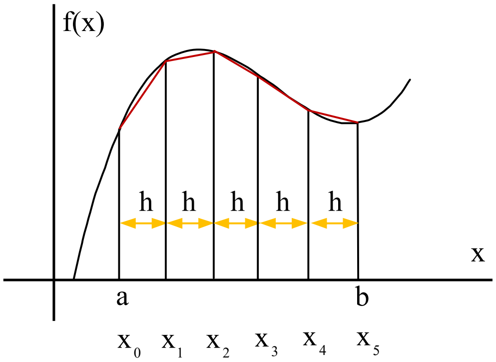
Fig. A Sketch for calculating area under the curve f(x).

The area under the curve $y = F(x)$ is the sum of a triangle and a rectangular. This area A can be easily calculated as

$$I = f(a)\,(b-a) + \frac{\big(f(b) - f(a)\big)(b-a)}{2}$$

$$= h\,\frac{\big(f(b) - f(a)\big)}{2} = \text{Width} \times \text{Average height} \qquad (1)$$

where, we have used h=b-a. For the exact result of numerical integration the f(x) should be a linear function. This is known as ***Trapezoidal Rule***. If f(x) is quadratic or higher order curve then we should find a better way of calculating the numerical integration. For more precession, we can divide whole area in small segments of trapezoidal. For example, we consider an area which is divided into five trapezoids



Now we find the sums of the area of these five trapezoids from a to b. therefore,

$$A = \frac{b-a}{n}\left[\frac{f(x_0)+f(x_1)}{2} + \frac{f(x_1)+f(x_2)}{2} + \ldots + \frac{f(x_4)+f(x_5)}{2}\right]$$

where, $\boxed{h = \frac{b-a}{n}}$ (n=5 for five trapezoids)

$$A = \frac{b-a}{2n}\left[\{f(x_0)+f(x_1)\} + \{f(x_1)+f(x_2)\} + \ldots + \{f(x_4)+f(x_5)\}\right]$$

Finely, we get

$$A = \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+2f(x_4)+f(x_5)\right]$$

In general,

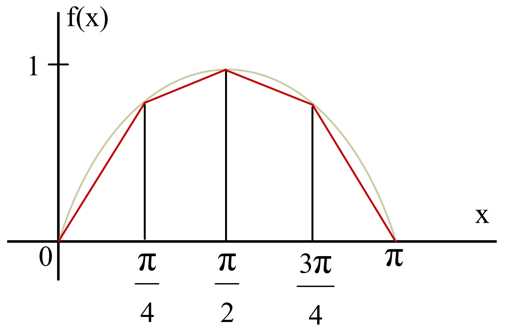
$$\boxed{A = \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)\right]}$$

$$A = \frac{h}{2}\left[f(x_0) + 2\sum_{k=1}^{k=n-1} f(x_n) + f(x_n)\right]$$

$$\approx \lim_{n\to\infty}\int_a^b f(x)\,dx$$

As the bigger n you use, the more accuracy in area will be.

Example 1 Let be f(x) = sinx. Calculate the approximate area A by using the Trapezoidal Rule as shown in figure below.



Sol. we find the numerical integral $A = \int_0^{\pi} \sin dx$

$$h = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4} \qquad \text{(since} \quad n = 4)$$

then, using the trapezoidal rule

$$A = \frac{\pi}{2(4)}\left[f(0) + 2f\left(\frac{\pi}{4}\right) + 2f\left(\frac{\pi}{2}\right) + 2f\left(\frac{3\pi}{4}\right) + f(\pi)\right]$$

$$= \frac{\pi}{8}\left[0 + 2\frac{\sqrt{2}}{2} + 2(1) + 2\left(\frac{\sqrt{2}}{2}\right) + 0\right] \approx 1.896$$

23.2.1 Derivation of the Trapezoidal Rule using Newton-Gregory Formula

Suppose f(x) takes the values $y_0, y_1, y_2, ..., y_n$ for $x_0, x_1, x_2, ..., x_n$. The interval (a,b) is divided into n equal parts of width h, so that

$$a = x_0,\ x_1 = x_0 + h,\ x_2 = x_0 + 2h,\ ...,\ x_n = x_0 + nh = b$$

Then $$I = \int_a^b y\, dx = \int_{x_0}^{x_0+nh} y_x\, dx = \int_0^n \left(y_{x_0+nh}\right) h du \qquad (1)$$

where $$u = \frac{x - x_0}{h} \Rightarrow dx = h\, du$$

By Newton-Gregory Formula of f(x)

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots + \frac{u(u-1)\ldots(u-n+1)}{n!}\Delta^n y_0 + R_n$$

where, $R_n = h^{n+1}\dfrac{u(u-1)\ldots(u-n+1)}{n+1!}y^{n+1}(\xi)$

and $\Delta^n y_0$ is the n^{th} forward difference.

Now, using *Newton cotes* integration formula, we get

$$I = \int_a^b \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots + \frac{u(u-1)\ldots(u-n+1)}{n!}\Delta^n y_0 + R_n \right] dx$$

change integration limits from x to u, then

$$dx = h\,du, \qquad \int_{x=a}^{x=b} dx \rightarrow \int_{u=0}^{u=n} h\,du$$

$$I = \int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots + \frac{u(u-1)\ldots(u-n+1)}{n!}\Delta^n y_0 + R_n \right] h du$$

$$I = h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots + \frac{u(u-1)\ldots(u-n+1)}{n!}\Delta^n y_0 \right] du$$

$$+ h^{n+2}\int_0^n \left[\frac{u(u-1)\ldots(u-n+1)}{n+1!} y^{n+1}(\xi) \right] du \qquad \xi \in [a, b] \qquad (2)$$

Above equation is called as ***Newton quadrature formula***, which is also known as quadrature formula.

For n = 1, we have only one interval (x_0, x_1) such that $a = x_0$ and $b = x_1$ and then the above integration formula gives trapezoidal rule.

Now setting n=1(Polynomial of the first degree in x or a straight line), we get

$$I = \underbrace{h\int_0^1 [y_0 + u\Delta y_0]\,du}_{1^{st}\text{ order}} + \underbrace{h^3\int_0^1 \left[\frac{u(u-1)}{2!} y''(\xi) \right] du}_{\text{Remainder term}}$$

$$I = h\left[y_0 + \Delta y_0 \frac{u^2}{2} \right]_0^1 + h^3\left[\frac{y''(\xi)}{2}\left(\frac{u^3}{3} - \frac{u^2}{2} \right) \right]_0^1$$

where $\Delta y_0 = y_1 - y_0$

$$I = \int_{x_0}^{x_0+h} y\,dx = \int_{x_0}^{x_1} y\,dx = \underbrace{h\left(\frac{y_0 + y_1}{2}\right)}_{\text{Trapezoid Rule}} \underbrace{- \frac{h^3}{12} y''(\xi)}_{\text{Truncation Error}(E_t)} \qquad \xi \in [x_0, x_1] \qquad (3)$$

where ξ is somewhere between x_0 and x_1 .Trapezoidal rule is first order accurate. It can integrate linear polynomials exactly. Similarly, for subsequent intervals,

$$I = \int_{x_0+h}^{x_0+2h} y\,dx = \int_{x_1}^{x_2} y\,dx = h\left(\frac{y_1 + y_2}{2}\right)$$

$$I = \int_{x_0+2h}^{x_0+3h} y\,dx = \int_{x_2}^{x_3} y\,dx = h\left(\frac{y_2 + y_3}{2}\right)$$

$$\ldots$$

$$I = \int_{x_0+(n-1)h}^{x_0+nh} y\,dx = \int_{x_{n-1}}^{x_n} y\,dx = h\left(\frac{y_{n-1} + y_n}{2}\right)$$

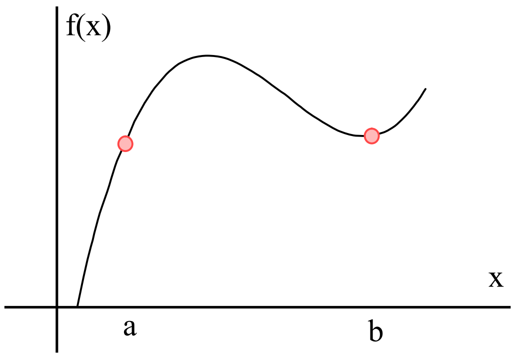
By adding all above equations, we obtain

$$\boxed{I = \int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[y_0 + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right) + y_n\right]}$$

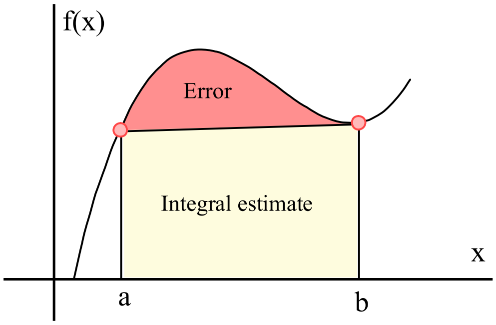
This is known as ***Trapezoidal rule***.

23.3Illustrative Examples

Example 2 show the estimated integral (area) region and error area region according to trapezoidal rule in given figure below.



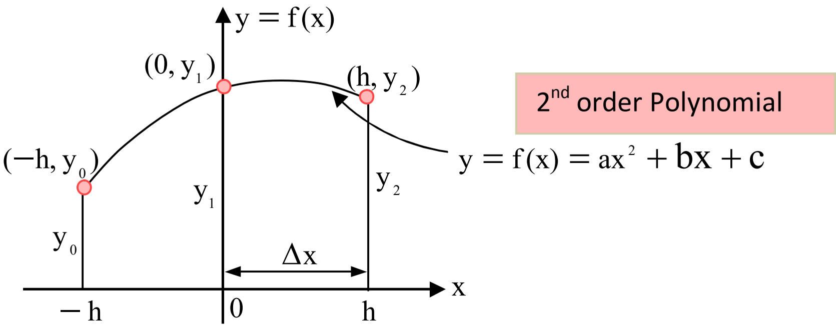
Sol.



23.4. The Simpson's1/3 Rule

More accurate estimate of an integral can be found if a high-order polynomial is used to connect the points. In Simpson's rule it is approximated the value of a definite integral by using quadratic (second order) polynomials. In Simpson's method the straight line is replaced by parabolas.

Let's first derive a formula for the area under a parabola of equation $y = ax^2 + bx + c$ (2nd order Polynomial) passing through the three points $(-h, y_0)$, $(0, y_0)$ and (h, y_2). Then, the definition of integration, the value of integration can be found the area between the limits –h to +h.



$$A = \int_{-h}^{h} (ax^2 + bx + c)\,dx$$

$$= \left(\frac{ax^3}{3} + \frac{bx^2}{2} + cx \right)\Bigg|_{-h}^{h} = \frac{2ah^3}{3} + 2ch$$

$$= \frac{h}{3}(2ah^2 + 6c)$$

Since the points $(-h, y_0)$, $(0, y_1)$, (h, y_2) lie on the parabola, they satisfy the equation $y = ax^2 + bx + c$. Therefore,

$$y_0 = ah^2 - bh + c$$
$$y_1 = c$$
$$y_2 = ah^2 + bh + c$$

we find that

$$y_0 + 4y_1 + y_2 = \left(ah^2 - bh + c\right) + 4c + \left(ah^2 + bh + c\right) = 2ah^2 + 6c$$

Therefore, the area under the parabola is

$$A = \frac{h}{3}(y_0 + 4y_1 + y_2) = \frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$$
$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

Now we consider the definite integral $\int_a^b f(x)\,dx$

We assume that f(x) is continuous on interval [a,b] and we equally divide into an even number of subintervals.

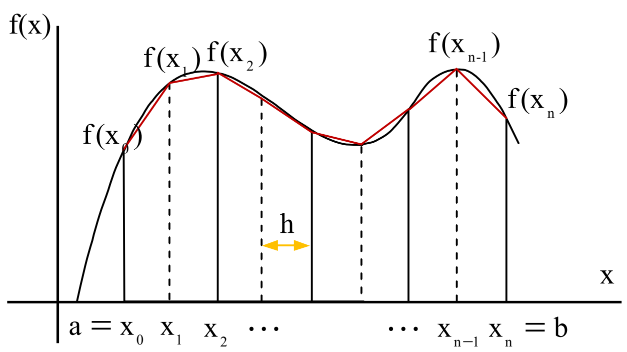
$$\Delta x = h = \frac{b-a}{n}$$

using the n+1 points

$$x_0 = a, \quad x_1 = a + h, \quad x_1 = a + 2h \quad \ldots, \quad x_n = a + nh = b$$

We can determine the value of f(x) at these points.

$$y_0 = f(x_0), \quad y_1 = f(x_1), \quad y_2 = f(x_2) \quad \ldots, y_n = f(x_n)$$



One can determine the integral by adding the areas under the parabolic arcs through these successive points.

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\Big[\{f(x_0)+4f(x_1)+f(x_2)\}+\{f(x_2)+4f(x_3)+f(x_4)\}+\ldots+\{f(x_{n-2})+4f(x_{n-1})+f(x_n)\}\Big]$$

$$\approx \frac{h}{3}\Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\ldots+4f(x_{n-1})+f(x_n)\Big]$$

By simplifying, we obtain

$$=\frac{h}{3}\left[f(x_0)+4\sum_{i=1,3,5\ldots}^{n-1} f(x_i)+2\sum_{i=2,4,6\ldots}^{n-2} f(x_j)+f(x_n)\right]$$

Example 3 Derive the Simpson's 1/3 Rule using Newton-Gregory Formula.

Sol. By using eqn.2 of previous section

$$I = h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots + \frac{u(u-1)\ldots(u-n+1)}{n!}\Delta^n y_0\right] du$$

$$h^{n+2}\int_0^n \left[\frac{u(u-1)\ldots(u-n+1)}{n+1!} y^{n+1}(\xi)\right] du$$

For n = 2, we have two subintervals $[x_0, x_1]$ and $[x_1, x_2]$ of equal width h such that $a= x_0$ and $b = x_2$ and then the above integration formula becomes. Setting n=2(Polynomial of the second degree in x or a parabola), we get

$$I = \int_{x_0}^{x_0+2h} y\,dx = \int_{x_0}^{x_2} y\,dx = \underbrace{h\int_0^2 \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0\right] du}_{2^{nd}\ order} + \underbrace{h^3\int_0^2 \left[\frac{u(u-1)(u-2)}{3!} y''(\xi)\right] du}_{Remainder\ term}$$

$$x_0 < \xi < x_2 \quad \text{or} \quad \xi \in [x_0, x_2]$$

Therefore integral term,

$$I = \int_{x_0}^{x_2} y\,dx = h\int_0^2 \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0\right] du$$

$$I = \int_{x_0}^{x_2} y\,dx = h\left[uy_0 + \frac{u^2}{2}(y_1 - y_0) + \frac{1}{2}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)(y_2 - 2y_1 + y_0)\right]_0^2$$

$$\left[\because \quad \Delta y_0 = (y_1 - y_0) \quad \text{and} \quad \Delta^2 y_0 = (y_2 - 2y_1 + y_0)\right]$$

$$I = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

This result is a part of the Simpson's rule. Similarly, for the next intervals,

$$I = \int_{x_0+2h}^{x_0+4h} y\,dx = \int_{x_2}^{x_4} y\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

$$I = \int_{x_0+4h}^{x_0+6h} y\,dx = \int_{x_4}^{x_6} y\,dx = \frac{h}{3}(y_4 + 4y_5 + y_6)$$

$$\ldots$$

$$I = \int_{x_0+(n-2)h}^{x_0+nh} y\,dx = \int_{x_{n-2}}^{x_n} y\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

By adding all above equations, we obtain

$$\boxed{I = \int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[y_0 + 4\left(y_1 + y_3 + \ldots + y_{n-1}\right) + 2\left(y_2 + y_4 + \ldots + y_{n-2}\right) + y_n\right]}$$

This is known as ***Simpson's 1/3 rule***.

23.5 Self Learning Exercise-I

Very Short Answer Type Questions

Q.1 Define the Trapezoidal Rule.

Q.2 What is highest order of polynomial integrand for which Simpson's 1/3 rule of integration is exact.

Q.3 What is the order of error in Trapezoidal Rule.

Short Answer Type Questions

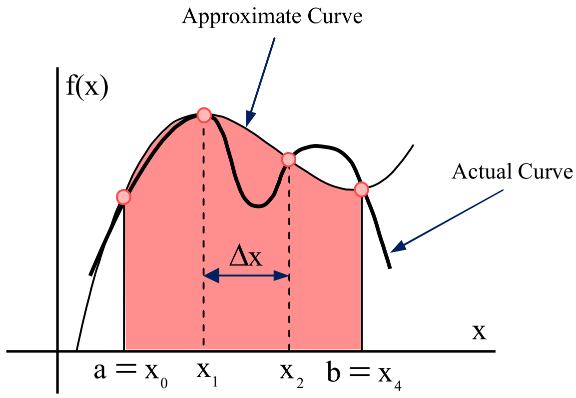
Q.4 Integrate the function $f(x) = e^x$ from a=0 to b=2 using the Trapezoidal rule.

Q.5 Integrate the function $f(x) = x + x^2$ from a=0 to b=10 using the Trapezoidal rule in steps of h=1.

Q.6 Integrate the function $f(x) = (1 + x)^2$ from a=1 to b=5 using the Simpson's 1/3 rule with h=0.5.

23.6. The Simpson's 3/8 Rule

Simpson's 3/8 rule can be obtained by using $y = f(x) = a_0 + a_1x + a_2x^2 + a_4x^3$ fourth-order polynomials to fit four points. Integration over the four points (three intervals) simplifies to:



$$I = \int_a^b f(x)\,dx \approx \int_a^b P_n(x)\,dx \qquad (1)$$

$$\boxed{I = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]}$$

23.7. The Weddle's Rule

Weddle's rule can be obtained by using $y = f(x) = a_0 + a_1 x + a_2 x^2 + a_4 x^3 + a_5 x^4 + a_6 x^5$ sixth-order polynomials to fit sixth points. Integration over the six points (five intervals) simplifies to:

$$\boxed{I = \int_a^b f(x)\,dx \cong \frac{3h}{10}\left[f_0 + 5f_1 + f_2 + 6f_3 + f_4 + 5f_5 + f_6\right]}$$

Where n = 6, $\boxed{h = \frac{b-a}{6}}$

Example 4 Compute the value of the definite integral $\int_0^2 x^3\,dx$ by

(a) Trapezoidal rule

(b) Simpson's 1/3 Rule

(c) Simpson's 3/8 Rule

with using 10 intervals, after computing the true value of the integral, compare the errors in the all four cases.

Sol. In this case divide the range of integration into ten equal parts by taking h=0.2 and then the values of the function are calculated for each point of sub-division which is as under:

	x	$y_x = x^3$
x_0	0.0	0.0000
x_0+h	0.2	0.0080
x_0+2h	0.4	0.0640
x_0+3h	0.6	0.2160
x_0+4h	0.8	0.5120
x_0+5h	1.0	1.0000
x_0+6h	1.2	1.7280
x_0+7h	1.4	2.7440
x_0+8h	1.6	4.0960
x_0+9h	1.8	5.8320
x_0+10h	2.0	8.0000

(a) we have given

$$n=10,\ h=0.2,\ \text{and}\ y_x=x^3$$

By trapezoidal rule,

$$\int_{x_0}^{x_0+nh} y_x\,dx = \frac{h}{2}\left[f(x_0) + 2\sum_{k=1}^{k=n-1} f(x_n) + f(x_n) \right]$$

Substituting the values of h, y_0, y_1,…..,y_n in the above formula , we obtain

$$\int_0^2 x^3 dx = \frac{0.2}{2}\left[0 + 2\left(0.008 + \ldots.. + 5.832\right) + 8\right] = 0.1\left[2(16.2) + 8\right]$$
$$= 4.040$$

(b) By Simpson's 1/3 Rule

$$\int_{x_0}^{x_0+nh} y_x\,dx = \frac{h}{3}\left[f(x_0) + 4\sum_{i=1,3,5\ldots}^{n-1} f(x_i) + 2\sum_{i=2,4,6\ldots}^{n-2} f(x_j) + f(x_n) \right]$$

$$\int_0^2 x^3 dx = \frac{0.2}{3}\Big[0 + 4\big(0.008 + 0.2160 + 1 + 2.744 + 5.832\big)$$
$$+ 2\big(0.064 + 0.512 + 1.728 + 4.096\big) + 8\Big]$$
$$= 4.000$$

(c) By Simpson's 3/8 Rule

$$\int_0^2 x^3 dx = \frac{3h}{8}\Big[y_0 + 3\big(y_1 + y_3 + \ldots\big) + 2\big(y_3 + y_6 + \ldots\big) + y_n\Big]$$

$$\int_0^2 x^3 dx = \frac{3(0.2)}{8}\Big[0 + 3\big(0.008 + 0.2160 + 1 + 2.744 + 5.832\big)$$
$$+ 2\big(0.216 + 1.728 + 5.832\big) + 8\Big]$$
$$= 3.971$$

Now the value of the integral is

$$I = \int_0^2 x^3 dx = \left[\frac{x^4}{4}\right]_0^2 = 4$$

Therefore the errors are

By Trapezoidal rule = - 0.040

By Simpson's 1/3 Rule = 0.000

By Simpson's 3/8 Rule = 0.029

23.8. Runge-Kutta Methods

In numerical analysis, the Runge – Kutta methods are a series of numerical methods for the approximation solutions of differential equations involving initial value problems. In order to understand the fundamental concepts, let us start with the equation

$$y_{i+1} = y_i + \varphi(x_i, y_i, h)h \qquad (1)$$

where $\varphi(x_i, y_i, h)$ is called an increment function which represents the slope from i to i+1. The increment function can be written in general form as

$$\varphi = a_1 k_1 + a_2 k_2 + \ldots + a_n k_n \qquad (2)$$

$$k_1 = f(x_i, y_i) = \text{Slope at the begining of the first interval}$$

$$k_2 = f(x_i + p_1 h\ ,\ y_i + q_{11} k_1 h)$$

$$k_3 = f(x_i + p_2 h\ ,\ y_i + q_{21}k_1 h + q_{22}k_2 h)$$

$$\vdots$$

$$k_n = f(x_i + p_i h\ ,\ y_i + q_{n-1,1}k_1 h + \ldots + q_{n-1,n-1}k_{n-1} h)$$

Here a_i, $p_{i,}$ q_{ij} constants are chosen to match Taylor's series expansion. By Taylor expansion

23.8.1. Runge-Kutta 1st order Method

We can get the first order version of equation (2) by using n=1

$$y_{i+1} = y_i + (a_1 k_1)h$$

where $k_1 = f(x_i, y_i)$, therefore

$$y_{i+1} = y_i + a_1 f(x_i, y_i)h \tag{3}$$

We can use this equation only when we know the constant a_1. To determine the constant a_1, we use that the first-order Taylor series for y_{i+1} in terms of y_i and $f_i(x_i, y_i)$.

$$y_{i+1} = y_i + f(x_i, y_i)h \tag{4}$$

Now, comparing the same term in equations (3) and (4), we get

$$a_1=1$$

thus the equation (3) becomes

$$\boxed{y_{i+1} = y_i + f(x_i, y_i)h}$$

The result of this equation is same as that of ***Euler's method.***

23.8.2. Runge -Kutta 2nd order Method

Similarly, we can get the second order version of equation (2) by using n=2

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h \tag{5}$$

where

$$k_1 = f(x_i, y_i) \tag{6}$$

and

$$k_2 = f(x_i + p_1 h\ ,\ y_i + q_{11}k_1 h) \tag{7}$$

therefore

Now, we use a Taylor series to expand eqn. (7). The Taylor series for a two – variable function can be defined as

$$f(x+r, r+s) = f(x, y) + r\frac{\partial f}{\partial x} + s\frac{\partial f}{\partial y} + \dots$$

Using this equation to expand eqn. (7) gives

$$k_2 = f(x_i + p_1 h\ ,\ y_i + q_{11} k_1 h) = f(x_i, y_i) + p_1 h\frac{\partial f}{\partial x} + q_{11} k_1 h\frac{\partial f}{\partial y} + O(h^2) \qquad (8)$$

Putting eqn. (6) and (8) in (5), we get

$$y_{i+1} = y_i + a_1 h\, f(x_i, y_i) + a_2 h\, f(x_i, y_i) + a_2 p_1 h^2 \frac{\partial f}{\partial x} + a_2 q_{11} h^2 f(x_i, y_i)\frac{\partial f}{\partial y} + O(h^3) \qquad (9)$$

To determine the constant a_1 a_2, p_1 and q_{11}, we use that the second -order Taylor series for y_{i+1} in terms of y_i and $f_i(x_i, y_i)$ same as above.

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{f'(x_i, y_i)}{2!}h^2 + O(h^3)$$

where $f'(x_i, y_i)$ can be determined by chain-rule differentiation

$$f'(x_i, y_i) = \frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y}\frac{dy}{dx}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f$$

Thus

$$y_{i+1} = y_i + f(x_i, y_i)h + \left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f(x_i, y_i)\right]\frac{h^2}{2!} + O(h^3) \qquad (10)$$

Comparing eqn. (9) and (10), we obtain

$$\left.\begin{aligned} a_1 + a_2 &= 1 \\ a_2 p_1 &= \frac{1}{2} \\ a_2 q_{11} &= \frac{1}{2} \end{aligned}\right\} \text{3equtions contains 4 constants} \qquad (11)$$

Here, we have three simultaneous equations with four constants, therefore, we must assume a value of one of the unknown constant to determine for the other three.

Method.1. If we assume a_2=1/2, then eqn. 11. can be solved for a_2=1/2 and p_1= q_{11}= 1. By putting of these parameters in equation (5), second order Runga – Kutta method becomes:

$$\boxed{y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2)h} \tag{12}$$

where

$$k_1 = f(x_i, y_i) \tag{12a}$$

$$k_2 = f(x_i + h\ ,\ y_i + k_1 h) \tag{12b}$$

This is ***Heun's method*** with single corrector.

Method.2. If we assume a_2=1, then eqn. 11. can be solved for a_1= 0 and p_1= q_{11}= 1/2. By putting of these parameters in equation (5), second order Runga – Kutta method becomes:

$$\boxed{y_{i+1} = y_i + k_2 h} \tag{13}$$

where

$$k_1 = f(x_i, y_i) \tag{13a}$$

$$k_2 = f\left(x_i + \frac{1}{2}h\ ,\ y_i + \frac{1}{2}k_1 h\right) \tag{13b}$$

This is ***the midpoint method***.

Method.3. If we chose a_2=2/3, then eqn. 11. can be easily solved for a_2= 1/3 and p_1= q_{11}= 3/4. By putting of these parameters in equation (5), second order Runga – Kutta method becomes:

$$\boxed{y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h} \tag{14}$$

where

$$k_1 = f(x_i, y_i) \tag{14a}$$

$$k_2 = f\left(x_i + \frac{3}{4}h\ ,\ y_i + \frac{3}{4}k_1 h\right) \tag{14b}$$

This is ***Ralston's method***.

23.8.3. Runge -Kutta 3rd order Method

For n=3, a third order derivation similar to first and second order can be obtained with six equation and eight unknown constants, therefore

$$\boxed{y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_2 + k_3)h} \qquad (15)$$

where

$$k_1 = f(x_i, y_i) \qquad (15a)$$

$$k_2 = f\left(x_i + \frac{1}{2}h\ ,\ y_i + \frac{1}{2}k_1 h\right) \qquad (15b)$$

$$k_3 = f\left(x_i + \frac{1}{2}h\ ,\ y_i - k_1 h + 2k_2 h\right) \qquad (15c)$$

Example 5 Use second order Runge – Kutta method.2 (midpoint method). to solve the ordinary differential equation $\frac{dy}{dx} = x + y$, with initial condition y(0) = 2 in steps of 0.1.

Sol. Here x_0= 0, y_0= 2, h=0.1and f(x,y)=x+y giving us

$$k_1 = f(x_0, y_0) = 0 + 2 = 2$$

$$k_2 = f\left(x_0 + \frac{1}{2}h\ ,\ y_0 + \frac{1}{2}k_1 h\right) = \left(x_0 + \frac{1}{2}h\right) + \left(y_0 + \frac{1}{2}k_1 h\right)$$

$$= 0 + 0.05 + 2 + 0.1 = 2.15$$

therefore

$$y_1 = y_0 + k_2 h = 2 + (2.15)(0.1) = 2.215$$

23.9. Self Learning Exercises-II

Very short Answer type Questions

Q.1 Write the Simpson's 1/3 and 3/8 rules.

Q.2 Define the Weddle's rule.

Q.3 What is the order of error in Weddle's rule?

Short Answer type Questions

Q.4 Integrate the function $f(x) = x^2$ from a=0 to b=5 using the Simpson's 3/8 rule.

Q.5 Integrate the function $f(x) = (1+x)^2$ from a=0 to b=10 using the Weddle's rule in steps of h=1.

Q.6 Solve the following problem numerically from x = 0 to 2:

$$\frac{dy}{dx} = x - y \qquad y(0) = 2$$

Use the third-order R-K method with a step size of 0.5.

23.10. Summary

Trapezoidal Rule: The trapezoidal rule is based on a first-order approximation (i.e., correspond to the first order polynomial) of the area of a function between two points:

$$I = \int_{x_1}^{x_2} f(x)\,dx = \frac{x_2 - x_2}{2}[f(x_1) + f(x_2)]$$

Simpson's 1/3 Rule: This method is the second-order polynomial approximation of the function. For equally spaced points, the integral of the function between points x_0 and x_2 is

$$I = \int_{x_0}^{x_2} f(x)\,dx = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

Simpson's 3/8 Rule: This method is based on a fourth-order polynomial approximation of the function.

$$I = \frac{3\Delta x}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

Weddle's Rule: This rule is the sixth-order $y = f(x) = a_0 + a_1x + a_2x^2 + a_4x^3 + a_5x^4 + a_6x^5$ polynomial approximation of the function.

Runge -Kutta Methods: Runge -Kutta Methods are a series of numerical methods for the approximation solutions of differential equations involving initial value problems.

Runge-Kutta 1st order Method (Euler's Method): The first order method corresponds to n=1

$$y_{i+1} = y_i + (a_1 k_1)h$$

Runge -Kutta 2nd order Method

Similarly, we can get the second order method corresponds to n=2

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

23.11. Glossary

Trapezoid: A quadrilateral with one pair of sides parallel

Interpolate: Insert (an intermediate value or term)into a series

by estimating or calculating it from surrounding known values.

23.12. Answers to Self Learning Exercises

Answers to Self Learning Exercise -I

Ans.1: $I = \int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[y_0 + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right) + y_n\right]$

This is known as ***Trapezoidal rule***. $h = \frac{b-a}{n}$

Answers to Self Learning Exercise -II

Ans.1: By adding all above equations, we obtain

$$I = \int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[y_0 + 4\left(y_1 + y_3 + \ldots + y_{n-1}\right) + 2\left(y_2 + y_4 + \ldots + y_{n-2}\right) + y_n\right]$$

This is known as ***Simpson's 1/3 rule***.

23.13. Exercise

Section-A (Very Short Answer Type Questions)

Q.1 State the composite Trapezoidal rule.

Q.2 State the composite Simpson's 1/3 and 3/8 rules.

Q.3 What is the order of error in Simpson's rule?

Q.4 What is the order of error in Weddle's rule?

Q.5 State the Runge- Kutta method.

Section-B (Short Answer Type Questions)

Q.6 Use Trapezoidal rule to evaluate $\int_{-2}^{2} y\, dx$ from the values of x and y tabulated as under:

x	–2	–1.5	–1.0	–0.5	0.0	0.5	1.0	1.5	2.0
y	4	2.25	1	0.25	0	0.25	1	2.25	4

Q.7 Use Simpson's rule to evaluate $\int_{0}^{5} e^{x}\, dx$ from the values of x and y tabulated as under:

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
e^x	1	1.65	2.72	4.48	7.39	12.18	20.09	33.12	54.60

and compare with the exact value.

Q.8 Find the value of the following integral by using Weddle's rule:

$$\int_{0}^{5} \frac{1}{1+x^2}\, dx$$

Q.9 Find the value of the following integral by using Trapezoidal rule:

$$\int_{0}^{\pi/2} \sin dx$$

Q.10 Find the value of the following integral by using Simpson's rule:

$$\int_{0}^{\pi/2} \cos dx$$

Section C (Long Answer Type Questions)

Q.11 Approximate the following integral using composite Trapezoidal rule with the values of n=8

$$\int_{-2}^{2} \frac{1}{1+x}\, dx$$

Q.12 Approximate the following integral using composite Simpson's rule with the values of n=6

$$\int_{2}^{2} xe^{x}\, dx$$

Q.13 Find the value of the following integral $\int_{0}^{6} \log_e x\, dx$ by

(a) Trapezoidal rule

(b) Simpson's 1/3 rule

(c) Simpson's 3/8 rule

(d) Weddle's rule

and compare errors in the four cases.

Q.14 Use the Runge - Kutta 3rd order method to solve the equation $\frac{dy}{dx} = xy$ with initial condition y (0) = 2 from x=0.5 to x=1, when h=0.5.

Q.15 Using the Ralston's (Runge – Kutta 2nd order) method to solve the equation $\frac{dy}{dx} = x^2 + y^2$, y (1) = 2 at x=0.4 with h=0.2.

References and Suggested Readings

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