MA/MSc MT-10

Vardhaman Mahaveer Open University, Kota

Mathematical Programming

Vardhaman Mahaveer Open University, Kota

Mathematical Programming

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PREFACE

The present book entitled **"Mathematical Programming"** has been designed so as to cover the unit-wise syllabus of M.A./MSc. Mathematics-10 course for M.A./ M.Sc. (Final) students of Vardhaman Mahaveer Open University, Kota. It can also be used for competitive examinations. The basic principles and theory have been explained in a simple, concise and lucid manner. Adequate number of illustrative examples and exercises have also been included to enable the students to grasp the subject easily. The units have been written by various experts in the field. The unit writers have consulted various standard books on the subject and they are thankful to the authors of these reference books.

Unit - 1
Hyperplane and Convex Function

Structure of the Unit

1.10 Objective

After studying this unit you will be able to understand a hyperplane in Euclidean space E^n and its use in the solution of linear programming problems. You will also be introduced about a convex function defined on a convex set.

1.1 Introduction

A linear relation in two unknowns (variables) represents a straight line in two dimensional space. A linear relation in three variables represents a plane in three dimensional space. On generalization what is represented by linear equation in n unknowns, is called hyperplane in n dimensional space E^n. It plays an important role in the theory of linear programming. In the last of the unit, concept of convex function is introduced which has importance in the study of non linear programming problems.

1.2 Some Important Definitions

(i) Set of points :- A linear equation in x_1, x_2 i.e. the equation of the form $a_1 x_1 + a_2 x_2 = b$ represents a line in E^2. Similarly a linear equation in x_1, x_2, x_3 i.e. $a_1x_1 + a_2x_2 + a_3x_3 = b$ or $\alpha\overline{X} = b$, where $\alpha = (a_1, a_2, a_3)$ and $\overline{X} = [x_1, x_2, x_3]$ represents a plane in E^3. Both of these can be considered as the sets of points as follows :

$$S_1 = \{(x_1, x_2) : a_1x_1 + a_2x_2 = b\} \text{ and}$$

$$S_2 = \{(x_1, x_2, x_3) : a_1x_1 + a_2x_2 + a_3x_3 = b\}$$

Similarly, the set

$$S = \{(x_1, x_2, x_3 .... x_n) : a_1x_1 + a_2x_2 + ... + a_nx_n = b\}$$

is defined in n-dimensional space E^n.

(ii) Line and line segment : The line joining two points X_1 and $X_2 \in E^n$ is the set of points given by

$$S_L = \{X : X \in E^n \text{ and } X = \lambda X_1 + (1-\lambda) X_2, \lambda \in R\}$$

and the line segment joining two points X_1 and X_2 is the set

$$S = \{X : X \in E^n \text{ and } X = \lambda X_1 + (1-\lambda) X_2, 0 \le \lambda \le 1\}$$

(iii) Hyperplane : The equation $c_1x_1 + c_2x_2 + c_3x_3 + ... + c_nx_2 = z$ or $\overline{c}\overline{X} = z$ defines a hyperplane in n-dimensional space E^n. Here not all c_i's are zero simultenuously.

In this by putting different values of c_i's and z, we can get different hyperplanes. Further a hyperplane is a set of points $X \in E^n$ satisfying $\overline{C}\overline{X} = Z$. Thus the set

$H = \{X : \overline{C}\overline{X} = Z\}$ is called a hyperplane.

The vector $\overline{C}$ is called a vector normal to the hyperplane and $\hat{C} = \pm \dfrac{C}{|C|}$ is called unit normal.

Note : (i) If $z = 0$, then $CX = 0$, then the hyperplane is said to pass through the origin.

(ii) Two hyperplanes $C_1X_1 = Z_1$ and $C_2X_2 = Z_2$ are said to be parallel, if they have the same unit normals i.e. if $\hat{c}_1 = \lambda\,\hat{c}_2$ for some λ, λ being non-zero scalar.

(iv) Neighbourhood of a point : A subset N of E^n is said to be an $\in$-neighbourhood ($\in$-nbd) of the point $X_0 \in E^n$ s.t.

$$N = \left\{X : X \in E^n, |X - X_0| < \in\right\}$$

being a small positive number.

(v) Interior and boundary points : A point X_0 is said to be the interior point of the set S if there exists at least one $\in -nbd$ of the point X_0 which is wholly contained in the set S. On the otherhand, a point X_0 is said to be the boundary point of the set $S \le E^n$ if every $\in$-nbd of X_0 contains at least one point not belonging to S and atleast one point belonging to S.

(vi) Closed and open sets : A subset $S \subseteq E^n$ is said to be closed if every boundary point of S belongs to it. On the otherhand, a subset $S \subseteq E^n$ is said to be an open set if it contains only interior points.

A hyperplane divides the whole space E^n into two half spaces, known as closed half spaces given by

$H_1 = \left\{X : X \in E^n, CX \ge Z\right\}$, and $H_2 = \left\{X : X \in E^n, CX \le Z\right\}$

Also, a hyperplane divides the whole space E^n into three mutually disjoint sets given by

$S_1 = \left\{X : X \in E^n, CX > Z\right\}$, $S_2 = \left\{X : X \in E^n, CX = Z\right\}$

and $S_3 = \left\{X : X \in E^n, CX < Z\right\}$. Here S_1 and S_3 are called open half spaces.

Note : The objective function and coustraint equations of the l.p.p. represents hyperplanes and each constraint $(sign \le, \ge)$ is a closed half space produced by the hyperplane given by the contraint by taking '=' sign in place of $\ge$ or $\le$.

(vii) Convex set : A set of points $S \subset E^n$ is said to be convex if the line segment joining any two points of S lies wholly in the set S. In otherwords, a set S is said convex if for any two points $X_1, X_2 \in S$, $\lambda X_1 + (1-\lambda)X_2 \in S$, where $0 \le \lambda \le 1$.

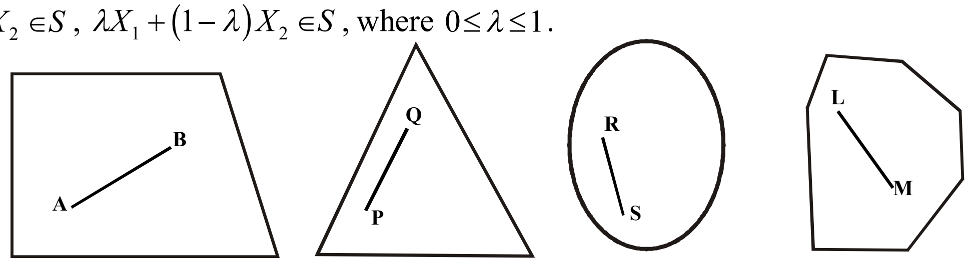
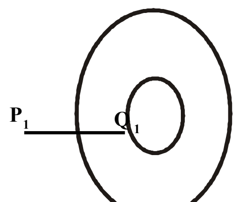


Fig 1.1 (a) Convex Sets



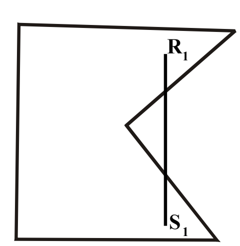


Fig 1.1 (b) Non-Convex Sets

(vii) Extreme point : A point X_0 of a convex set S is said to be an extreme point if it does not lie on the line segment of any two points, different from X, in the set.

The vertices of a polygon and every point on the circumference of the circle is the extreme point of the convex set of the points on and within the polygon or circle.

1.3 Some Theorems

Theorem 1.1 : A hyperplane is a closed set.

Proof : Let the point set $H = \{X : X \in E^n, CX = Z_0\}$ be a hyperplane. To show that it is a closed set, we take a boundary point X_0 of H and prove that $X_0 \in H$.

Contrary, we suppose that X_0 $X_0 \notin H$, then either $CX_0 > Z_0$ or $CX_0 < Z_0$.

Let $CX_0 = Z_1 < Z_0$. Now $CX_0 = C(X_0 + X - X_0) = CX_0 + C(X - X_0)$

$\because \; C(X - X_0) \leq |C(X - X_0)|$

$CX \leq Z_1 + |C(X - X_0)|$

$\Rightarrow \quad CX \leq Z_1 + |C||X - X_0|$ ...(1)

Now consider $\in -$ nbcd of X_0 i.e. $|X - X_0| < \in$, where

$\in = \dfrac{Z_0 - Z_1}{2|C|}$, than (1) implies that

$$CX < Z_1 + \frac{Z_0 - Z_1}{2} = \frac{Z_1 + Z_0}{2} < Z_0.$$

If shows that $\in -$ nbd of X_0 contains no point of the hyperplane H, which is the contradiction as X_0 is a boundary point.

$\Rightarrow \quad C X_0 \not< Z_0$

Similary we can show that $C X_0 \not> Z_0$.

$\Rightarrow$ Only $CX_0 = Z_0$ is possible.

$\Rightarrow$ X_0 is the point in hyperplane

$\Rightarrow$ $X_0 \in H$

$\Rightarrow$ H is a closed set.

In a similar way, one may prove that closed half spaces are also closed sets and open half spaces are open sets.

Theorem 1.2 : A hyperplane is a convex set.

Proof : Let $H=\{X : X \in E^n, CX = Z\}$ be a hyperplane in E^n and X_1, X_2 be two points of H, then $CX_1 = Z$ and $CX_2 = Z$. Now, if $X_3 = \lambda X_1 + (1-\lambda)X_2, 0 \le \lambda \le 1$, then

$$CX_3 = C\{\lambda X_1 + (1-\lambda)X_2\} = \lambda CX_1 + (1-\lambda)CX_2 = \lambda Z + (1-\lambda)Z = Z$$

i.e. X_3 satisfies $CX = Z$

Hence $X_3 = \lambda X_1 + (1-\lambda)X_2 \in H$ and therefore by difinition H is a convex set.

Theorem 1.3 : The closed half spaces $H_1 = \{X : CX \ge Z\}$ and $H_2 = \{X : CX \le Z\}$ are convex sets.

Proof : Let X_1, X_2 be two points of H_1, then $CX_1 \ge Z, CX_2 \ge Z$. Now if $0 \le \lambda \le 1$, then

$$C[\lambda X_1 + (1-\lambda)X_2] = \lambda CX_1 + (1-\lambda)CX_2 \ge \lambda Z + (1-\lambda)Z$$

$$\ge Z$$

$\Rightarrow \quad \lambda X_1 + (1-\lambda)X_2 \in H_1$

$\Rightarrow \quad H_1$ is a convex set.

Similarly, it can be shown that H_2 is also a convex set.

Theorem 1.4 A point y in space either belongs to a given closed convex set X or there exists a hyperplane through y so that whole of the X lies in one open half space produced by that hyperplane.

Proof: The proof is clear for two and three dimensions. In E^2, the situation is show in figure 1.2.

Suppose $y \notin X$ and $w \in X$ be the point closest to y i.e. the distance of w from y is minimum.

Thus $|w - y| = \min_{u \in X} |u - y|$

$\Rightarrow \quad |w - y| \le |u - y|, \ \forall u \in X \quad \quad ...(1)$

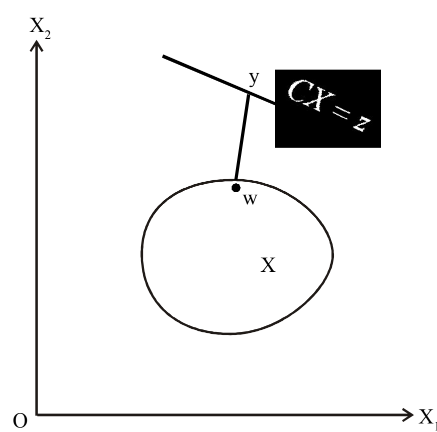


Figure 1.2

Such a point w always exists and unique as the set X is closed. To prove uniqueness, let w_1 and w_2 be two points of X with the some minimum distance. Than

$$\left|\frac{1}{2}(w_1 + w_2) - y\right| = \frac{1}{2}|(w_1 - y) + (w_2 - y)| \le \frac{1}{2}(|w_1 - y| + |w_2 - y|)$$

$\because \quad |w_1 - y| = |w_2 - y|$, we get

$$\left|\frac{1}{2}(w_1 + w_2) - y\right| \leq |w_1 - y| = |w_2 - y|$$

Thus we have obtained a point $\frac{1}{2}(w_1 + w_2) \in X$ (as X is convex) which is nearer to y then, w_1 and w_2. This contradicts that w_1 and w_2 are closest to y. Hence w is unique.

To prove that whole of X lies in one half of closed space : Let u is an arbitrary point of X and X is convex set, we have

$$[\lambda u + (1-\lambda)w] \in X, \ 0 \leq \lambda \leq 1$$

From (1) $|\lambda u - (1-\lambda)w - y|^2 \geq |w - y|^2, \ 0 \leq \lambda \leq 1$

$$\Rightarrow \quad |(w-y) + \lambda(u-w)|^2 \geq |w-y|^2$$

$$\Rightarrow \quad |w-y|^2 + 2\lambda(w-y)^1(u-w) + \lambda^2|u-w|^2 \geq |w-y|^2$$

$$\Rightarrow \quad 2\lambda(w-y)^1(u-w) + \lambda^2|u-w|^2 \geq 0$$

Taking $\lambda > 0$, we get

$$2(w-y)^1(u-w) + \lambda|u-w|^2 \geq 0$$

Taking $\lambda \to 0$,, we get

$$(w-y)^1(u-w) \geq 0$$

$$\Rightarrow \quad (w-y)^1.[(u-y)-(w-y)] \geq 0$$

$$\Rightarrow \quad (w-y)^1.(u-y) \geq |w-y|^2$$

But $|w-y|^2 > 0$ as $w \in X$ and $y \notin X$

$$\Rightarrow \quad (w-y)^1.(u-y) = 0$$

$$\Rightarrow \quad (w-y)^1.u > (w-y)^1.y \qquad ..(2)$$

If we take $C = (w-y)'$ and $z = (w-y)'.y$, then $CX = z$ is a hyperplane through y as

$$c\,y = (w-y)'.y = z$$

and satisfies $cu > c\,y = z, \ \forall u \in X$

Thus we have found a hyperplane through y and X lies in one open half space produced by this hyperplane. Such a plane is called separating hyperplane.

1.4 Supporting Hyperplane

A hyperplane $cx = z$ is said to be a supporting hyperplane at a boundary point w of a convex set X if

(i) $cw = z$ i.e. the hyperplane passes through w.

(ii) $cu \geq z$ or $cu \leq z \;\; \forall\, u \in X$ i.e. the whole of X lies in one half closed space produced by $cx = z$.

Theorem 1.5 The optimal hyperplane in a L.P.P. is a supporting hyperplane to the convex set of feasible solutions.

Proof: Suppose, we have a L.P.P. as

$$Max\, Z = cx$$

$$\text{s.t. } AX \leq b,\; X \geq 0$$

we know that the set of all feasible solutions to L.P.P. is a convex set and the objective function is a hyperplane. We move this hyperplane parallel to itself over the convex set of feasible solutions (feasible region) until z is made as large as possible so that the hyperplane contains at least one point of the feasible region. Note that the hyperplane corresponding to higher values of z will contain a point of feasible region. This is a hyperplane corresponding to the optimum (maximum) value of z. This is known as optimal hyperplane.

To prove that no point of this hyperplane is an interior point of convex set. For this, suppose that $CX = Z_0$ is the optimal hyperplane and X_0 is an interior point of X on this hyperplane. Since X_0 is an interior point of the set X, there exists $\in > 0$ s.t. $\in$-neighbourhood of X_0 wholly lies in the set X. Thus the point $X_1 = X_0 + \frac{\in}{2}\left(\frac{c}{|c|}\right)$ is in X and $z_1 = cx_1 = cx_0 + \frac{\in}{2}\frac{\vec{c}.\vec{c}}{|c|} = z_0 + \frac{\in}{2}\frac{|\vec{c}|^2}{|c|} = z_0 + \frac{\in}{2}|c|$

$\therefore \quad z_1 = CX_1 > z_0$ as $\frac{\in}{2}|c|$ is posivtive.

Thus we have obtained a point $X_1 \in X$ which gives higher values of objective function than z_0 (the maximum value) which is a contradiction as z_0 is the optimal value. Therefore, X_0 is not an interior point, but boundary point of X. Thus $CX = Z_0$ is a hyperplane containing a boundary point of x. Also if $u \in X$ is any point then $cu = z \leq z_0$ (as z_0 is maximum). Hence X lies in one closed half space produced by the hyperplane $CX = Z_0$. Therefore $CX = Z_0$ is the supporting hyperplane at x_0.

Theorem 1.6 Every supporting hyperlane of a closed convex set which is bounded from below contains at least one extreme point of the set.

Proof : Let $CX = Z_0$ be a supporting hyperplane at x_0 to the closed convex set X, bounded from below. Let T be the intersection of X and the hyperplane $S = \{x\,;\, cx = z_0\}$.

It is clear that atleast $x_0 \in T$ showing that T is not empty. Now we shall prove this theorem by showing that T has an extreme point and the extreme point of T are also the extreme point of X. Then hyperplane will clearly contain at least one extreme point of X.

Let $t \in T$ be an extreme point of T; then by definition there do not exist x_1 and $x_2 \in T$. s.t.

$$t = \lambda x_1 + (1-\lambda)x_2 ,\ 0 < \lambda < 1,\ x_1 \neq x_2$$

Now suppose T is not an extreme point of $x(t \in T \Rightarrow t \in x)$. Then $\exists\, x_1, x_2 \in X$ such that $t = \lambda x_1 + (1-\lambda)x_2$, $0 < \lambda < 1$. Since $cx = z_0$ is a supporting hyperplane, $cx_1 \geq z_0$ and $cx_2 \geq z_0$. Also $\bar{t} \in T$ lies on the hyperplane, we must have $\bar{c}\,\bar{t} = z_0$.

But $c\,t = c(\lambda x_1 + (1-\lambda)x_2) = \lambda c x_1 + (1-\lambda)cx_2$

This is equal to z_0 if and only if $cx_1 = z_0$ and $cx_2 = z_0$ as $\lambda > 0$, $(1-\lambda) > 0$. Hence x_1, x_2 also lies on the hyperplane and hence belonging to T. Thus we have obtained two points x_1 and x_2 of T s.t.

$$t = \lambda x_1 + (1-\lambda)x_2,\quad 0 < \lambda < 1$$

This is a contradiction as t is an extreame point of the set T. Hence t is also extreme point of X.

Now it is to show that there exists an extreme point of T. For this, we shall actually find an extreme point. Select that point (vector) of T for which the first component is minimum. Such a point will exist because T is bounded from below as X is bounded from below.

If this point is not unique, i.e. the first component has no unique minimum, then out of these points (for which first component is minimum select the point with the second component minimum. Still the point is not unique, then select the point out of these for which third component is minimum and continue this process until the unique point is obtained. This unique point is an extreme of the set T. For, if this point say t* is not an extreme point of the set T, then $\exists$ $t_1, t_2 \in T$ s.t. $t^* = \lambda t_1 + (1-\lambda)t_2$, $0 < \lambda < 1$, $t_1 \neq t_2$.

Suppose t^* is determined on taking the k^{th} component minimum. If t_{k_1}, t_{k_2} are the components of t_1 and t_2 respectively, then k^{th} component of t^* is given by $t_k^* = \lambda\, t_{k_1} + (1-\lambda)t_{k_2}$, $0 < \lambda < 1$

Now also $t_i^* = \lambda\, t_{i_1} + (1-\lambda)\, t_{i_2}$, $0 < \lambda < 1$ $\qquad (i \leq k-1)$

If $t_{i_1} \neq t_{i_2}$, say $t_{i_1} > t_{i_2}$, the we get

$$t_i^* > \lambda\, t_{i_2} + (1-\lambda)t_{i_2} = t_{i_2}$$

which is a contadiction as t_i^* is minimum. Hence $t_{i_1} \not> t_{i_2}$ similarly $t_{i_1} \not< t_{i_2}$. Hence $t_{i_1} = t_{i_2}$ and hence

$$t_i^* = \lambda\, t_{i_1} + (1-\lambda)\, t_{i_1} = t_{i_1} = t_{i_2}$$

Now, for $t_k^* = \lambda\, t_{k_1} + (1-\lambda)\, t_{k_2}$ to be true we should have $t_k^* = t_{k_1} = t_{k_2}$, otherwise a above t_k^* will be greater than either t_{k_1} and t_{k_2}.

Hence the points t_1 and t_2 also have the same minimum k^{th} component. But with this minimum value of k^{th} component, there is only one point. Thus we get a contradiction.

Therefore $t*$ cannot be written as convex combination of two different points. Hence it is an extreme point.

Example 1.1 A hyperplane is given by the equation

$$3x_1 + 2x_2 + 4x_3 + 7x_4 = 8.$$

Find in which half spaces do the points $(-6,1,7,2)$ and $(1,2,4,1)$ lie.

Solution : Putting $x_1 = -6, x_2 = 1, x_3 = 7, x_4 = 2$ in the L.H.S. of the given equation, we get

L.H.S. = $3.(-6) + 2.1 + 4.7 + 7.2 = 26 > 8 = R.H.S$

$\Rightarrow$ Point $(-6,1,7,2)$ lies in the open half space

$$3x_1 + 2x_2 + 4x_3 + 7x_4 > 8.$$

Similarly substituting $(1,2,-4,1)$, we get

L.H.S. = $3.1 + 2.2 + 4.(-4) + 7.1 = -2 < 8 = R.H.S.$

$\Rightarrow$ Point $(1,2,-4,1)$ lies in the open half space $3x_1 + 2x_2 + 4x_3 + 7x_4 < 8$.

1.5 Self Learning Exercise I

1. Define hyperplane.
2. What are the closed and open sets?
3. Define convex set.
4. Define extreme point.
5. Define supporting hyperplane.

1.6 Convex Function

A function $f(x)$ defined on a convex set $S \subset E^n$ is said to be convex function if for any two points X_1 and X_2 in S and for all λ, $0 \le \lambda \le 1$

$$f[\lambda X_1 + (1-\lambda) X_2] \le \lambda f(X_1) + (1-\lambda) f(x_2)$$

and the function $f(x)$ is said to be strictly convex if for any two different points X_1 and X_2 in S and $0 < \lambda < 1$

$$f[\lambda X_1 + (1-\lambda) X_2] \le \lambda f(X_1) + (1-\lambda) f(X_2)$$

A function $f(X)$ is said to be concave (or strictly concave) if $-f(X)$ is convex (strictly convex).

Geometrical Meaning :

The single variable function $f(x)$ is strictly convex if the line segment joining two point $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the curve of $f(x)$ lies above the curve (figure 1.3). Similarly single variable function $g(x)$ is strictly concave if the line segment joining two points $(x_1, g(x_1))$ and $(x_2, g(x_2))$ on the curve of $g(x)$ lies below the curve (figure 1.4)

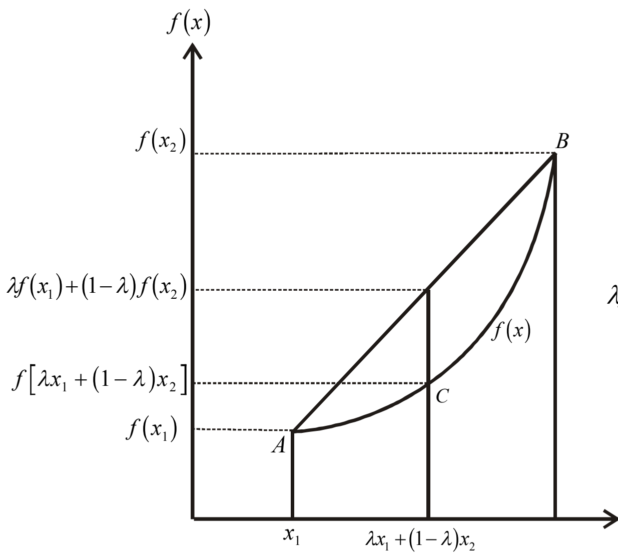


Figure 1.3

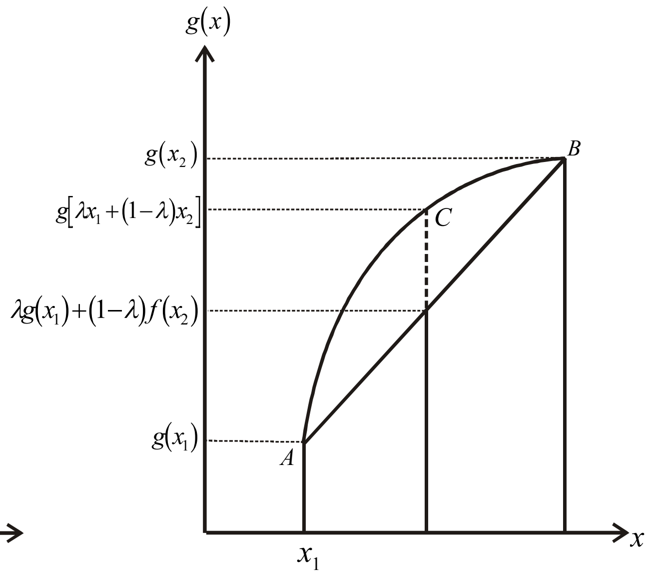


Figure 1.4

From figure 1.3 it is observed that for all $0<\lambda<1$

$$f[\lambda x_1+(1-\lambda)x_2]<\lambda f(x_1)+(1-\lambda)f(x_2),$$

and from figure 1.4 for all $0<\lambda<1$, we get

$$g[\lambda x_1+(1-\lambda)x_2]>\lambda g(x_1)+(1-\lambda)g(x_2).$$

Note : A linear function is convex as well as concave but not strictly convex or strictly concave as shown in following theorems.

1.7 Some Theorems on Convex Function

Theorem 1.7 : A linear function $Z=CX=f(x)(Say)$, $X\in R^n$

Suppose X_1 and X_2 be two points of R^n

Now $f[\lambda X_1+(1-\lambda)X_2] \quad = C[\lambda X_1+(1-\lambda)X_2]$

$0\le\lambda\le1 \quad = \lambda C X_1+(1-\lambda)C X_2$

$= \lambda f(X_1)+(1-\lambda)f(X_2)$

$\Rightarrow f[\lambda X_1+(1-\lambda)X_2]\le\lambda f(x_1)+(1-\lambda)f(x_2)$

$0\le\lambda\le1$

and $f\left[\lambda X_1+(1-\lambda)X_2\right]\geq\lambda f\left(X_1\right)+(1-\lambda)f\left(x_2\right)$

$\therefore\ f(X)=CX$ is a convex function as well as concave. Here strict inequaltiy is not implied.

So $f(x)$ is neither strictly convex nor strictly concave.

Theorem 1.8 The sum of convex functions is convex and if atleast one of the functions is strictly convex, then the sum is strictly convex.

Proof : Let $f_1, f_2, f_3, \ldots f_m$ be m convex functions defined on the convex set $S\subset E^n$. Let $f=f_1+f_2+f_3+\ldots+f_m$ be the sum function defined on the same set S.

Let X_1, X_2 be two points of S and $o\leq\lambda\leq 1$. Then

$$f\left[\lambda X_1+(1-\lambda)X_2\right]=\sum_{i=1}^{m} f_i\left[\lambda x_1+(1-\lambda)X_2\right]$$

$$\leq\sum_{i=1}^{m}\left[\lambda f_i\left(X_1\right)+(1-\lambda)f_i\left(X_2\right)\right]$$

[since f_i is convex $\forall\, i=1,2,\ldots.,m$]

$$\leq\lambda\sum_{i=1}^{m} f_i\left(X_1\right)+(1-\lambda)\sum_{i=1}^{m} f_i\left(X_2\right)$$

$$\leq\lambda f\left(X_1\right)+(1-\lambda)f\left(X_2\right)$$

$\Rightarrow$ The function $f=f_1+f_2+\ldots.+f_m$ is convex function on S.

If atleast one function say f_k, $1\leq k\leq n$ is strictly convex then for $0<\lambda<1$, $f_k\left[\lambda X_1+(1-\lambda)X_2\right]<\lambda f_k\left(X_1\right)+(1-\lambda)f_k\left(x_2\right)$ using it we get

$$f\left[\lambda X_1+(1-\lambda)X_2\right]<\lambda f\left(X_1\right)+(1-\lambda)f\left(X_2\right)$$

$$\forall\ X_1, X_2\in S \text{ and } 0<\lambda<1$$

Hence, f is strictly convex if atleast one of the function is strictly convex.

Theorem 1.9 The sum of concave functions is concave and if atleast one of the functions is strictly concave, than the sum is strictly concave.

Proof : The proof of above theorm can be done in the same manner as of theorem 1.8.

1.8 Illustrative Examples

Example 1.2 Show that $f(x)=x^2$ is a convex function.

Proof : Here $f(x)=x^2$, let $0\leq\lambda\leq 1$

$$f\left[\lambda x_1+(1-\lambda)x_2\right]-\lambda f\left(x_1\right)-(1-\lambda)f\left(x_2\right)$$

$$=\left[\lambda x_1+(1-\lambda)x_2\right]^2-\lambda x_1^2-(1-\lambda)x_2^2$$

$$= -\left[(\lambda-\lambda^2)x_1^2+\left[(1-\lambda)-(1-\lambda)^2\right]x_2^2-2\lambda(1-\lambda)x_1x_2\right]$$

$$= -\left[(\lambda-\lambda^2)(x_1-x_2)^2\right]\le 0\left[as\, 0\le\lambda\le 1,\ \lambda^2\le\lambda,(x_1-x_2)^2\ge 0\right]$$

$\Rightarrow$ $f\left[\lambda x_1+(1-\lambda)x_2\right]\le\lambda f(x_1)+(1-\lambda)f(x_2)\Rightarrow f(x)=x^2$ is a convex function.

Example 1.3 Prove that $f(x)=\dfrac{1}{x}$ is strictly convex for $x>0$ and strictly concave for $x<0$.

Sol. Here $f(x)=\dfrac{1}{x}$

$$f\left[\lambda x_1+(1-\lambda)x_2\right]-\lambda f(x_1)-(1-\lambda)f(x_2)$$

$$= \frac{1}{\lambda x_1+(1-\lambda)x_2}-\frac{\lambda}{x_1}-\frac{1-\lambda}{x_2}$$

$$= \frac{(\lambda^2-\lambda)(x_1-x_2)^2}{x_1x_2\left[\lambda x_1+(1-\lambda)x_2\right]}$$

for $0<\lambda<1,\lambda^2<\lambda$ and for $x_1\ne x_2,(x_1-x_2)^2>0$

for $(x_1,x_2)>0$ and for $(x_1,x_2)<0, x_1x_2>0$

Also for $(x_1,x_2)>0,\lambda x_1+(1-\lambda)x_2>0$ and for $(x_1,x_2)<0,\lambda x_1+(1-\lambda)x_2<0$

Hance $\dfrac{(\lambda^2-\lambda)(x_1-x_2)^2}{x_1x_2\left[\lambda x_1+(1-\lambda)x_2\right]}<0$ for all $x_1,x_2>0$

and $\dfrac{(\lambda^2-\lambda)(x_1-x_2)^2}{x_1x_2\left[\lambda x_1+(1-\lambda)x_2\right]}>0$ for all $x_1,x_2>0$

$\Rightarrow$ $f\left[\lambda x_1+(1-\lambda)x_2\right]<\lambda f(x_1)+(1-\lambda)f(x_2),\ \forall x_1,x_2>0$

and $f\left[\lambda x_1+(1-\lambda)x_2\right]>\lambda f(x_1)+(1-\lambda)f(x_2),\ \forall x_1,x_2<0$

Thus $f(x)=\dfrac{1}{x}$ is strictly convex for $x>0$ and strictly concave for $x<0$.

Example 1.4 Show that $f(x)=\begin{cases}0 & for\quad x\le b\\ a(x-b) & \text{for}\quad x>b\end{cases}$ (Here a > 0) is a convex function.

Sol. : Here $f(x)$ is a constant function for $x\le b$ and is a linear function for $x>b$. The curve of the function is shown below by dark line.

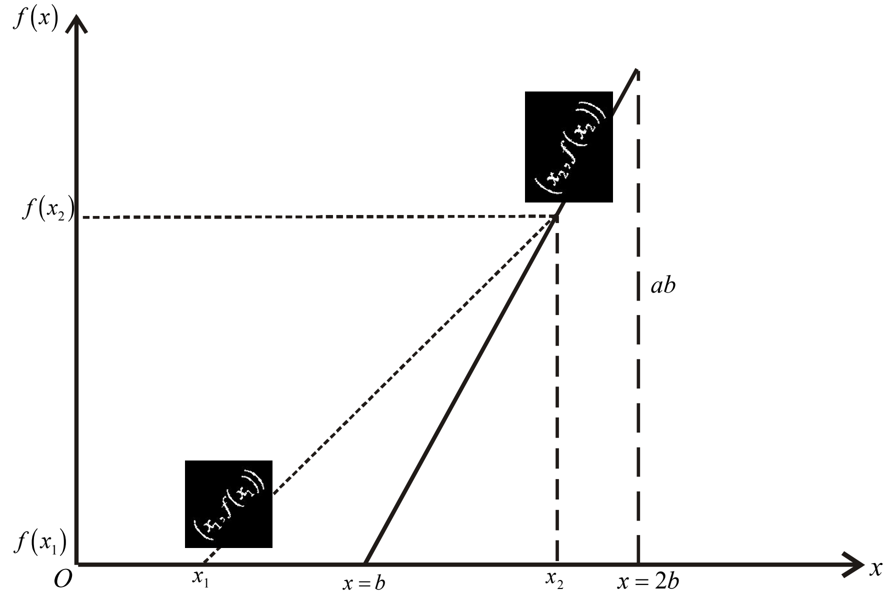


Figure : 1.5

It is clear from above figure 1.5 that for any two points x_1, x_2 of the domain, the line segment joining two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is above the curve of $f(x)$ for $x_1 < x < x_2$ i.e.

$$f[\lambda x_1 + (1-\lambda)x_2] \leq \lambda f(x_1) + (1-\lambda) f(x_2), \quad 0 \leq \lambda \leq 1$$

Hence the function $f(x)$ is a convex function.

Example 1.5 If $f(x)$ is continuous, $f(x) \geq 0, -\infty < x < \infty$ then the function $\phi(x) = \int_x^{\infty} (y-x) f(y) dy$ is a convex function provided the integral converges.

Sol. : Let x_1 and x_2 be two points of the domain of $\phi(x); x_1 < x_2$ and $x_3 = \lambda x_1 + (1-\lambda)x_2, 0 \leq \lambda \leq 1$, then we have to show that $\phi(x_3) \leq \lambda\, \phi(x_1) + (1-\lambda)\, \phi(x_2)$

We have $\phi(x_3) = \int_{x_3}^{\infty} \left[y - \{\lambda x_1 + (1-\lambda)x_2\} \right] f(y) dy$

$$= \lambda \int_{x_3}^{\infty} (y - x_1) f_1(y) dy + (1-\lambda) \int_{x_3}^{\infty} (y - x_2) f(y) dy$$

$$= \lambda \left[\int_{x_3}^{x_1} (y - x) f(y) dy + \int_{x_1}^{\infty} (y - x_1) f(y) dy \right.$$

$$\left. +(1-\lambda) \left[\int_{x_3}^{x_2} (y - x_2) f(y) dy + \int_{x_2}^{\infty} (y - x_2) f(y) dy \right] \right.$$

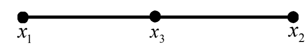
$$\leq \lambda \left[\int_{x_1}^{\infty} (y - x_1) f(y) dy - \int_{x_1}^{x_3} (y - x_1) f(y) dy \right]$$

$$+(1-\lambda)\left[\int_{x_3}^{x_2}(y-x_2)f(y)dy+\int_{x_2}^{\infty}(y-x_2)f(y)dy\right]$$

$\because \int_{x_3}^{x_2}(y-x_2)f(y)dy \le 0$ as $(y-x_2)\le 0, f(y)\ge 0$

and $-\int_{x_1}^{x_3}(y-x_1)f(y)dy \le 0$ as $(y-x_1)\ge 0 \; f(y)\ge 0$

$\therefore \; \phi(x_3)\le \lambda\,\phi(x_1)+(1-\lambda)\,\phi(x_2)$ for $0\le\lambda\le 1$



Hence $\phi(x)$ is a convex function.

Figure : 1.6

1.9 Quadratic form

A quadratic form in variables $x_1, x_2, x_3, ..., x_n$ is a function of these variables which is defined as

$Q(X)=\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\, x_i\, x_j$, where a_{ij} are constants.

If $A=[a_{ij}]=$ a square matrix of order $n\times n$ and $X=[x_1,x_2,...x_n]^T$, then we have.

$Q(X)=X^T A\,X$ or $X'\,A\,X$

Here the square matrix A can always be written as symmetric matrix because the coifficient of $x_i x_j$ is $a_{ij}+a_{ji}$ and if A is not symmetric matrix, we can construct a new matrix B with the property

$$b_{ij}=b_{ji}=\frac{a_{ij}+a_{ji}}{2}$$

$X^T B\,X = X^T AX$ (since $a_{ij}+a_{ji}=b_{ji}+b_{ij}$)

Clearly, B is a symmetric matrix, so A can always be assumed a symmetric matrix i.e. in future we shall always assume matrix associated with a quadratie form is symmetric

1.10 Positive and Negativeness of Quadratic form

A quadric form $Q(X)$ is said to be :

(i) **Positive definite**, if $Q(X)>0$ for all X, except X = 0

(ii) **Positive semi definite**, if $Q(X)\ge 0$ for all X and $\exists$ some $X\ne 0$ for which $Q(X)=0$.

(iii) **Negative definite** ; if $-Q(X)$ is positive definite.

(iv) **Negative semi definite** ; if $-Q(X)$ is positive semi definite.

(v) **Indefinite** ; if $Q(X)>0$ for some X and $Q(X)<0$ for some other X.

Examples :

(i) $Q(X)=(x_1,x_2)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=x_1^2+x_2^2$ is positive definite

(ii) $Q(X)=(x_1,x_2)\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=(x_1-x_2)^2$ is positive sami definite

(iii) $Q(X)=(x_1,x_2)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ is indefinite.

There are several tests to determine the character of the given quadric form. One of these tests is eigen value test. In this test we find the values of the roots of the characteristic equation $|A-\lambda I|=0$. This equation is a polynomial equation of degree n in λ. Since A is symmetric, so all the roots of this equation i.e. the n values of λ (called eigen values) are real. If

(i) All the n values of λ are positive, then $X'AX$ is positive definite.

(ii) Some values of λ are positive and remaining are zero then the quadratic form $X'AX$ is positive semi definite.

(iii) All the n values of λ are negative, $X'AX$ is negative definite.

(iv) Some values of λ are negative and remaining are zero then $X'AX$ is negative semi definite.

(v) Some values of λ are positive, other's are negative then $X'AX$ is indefinite.

Another test : If all the successive principal minors of A are > 0, then $X'AX$ is positive definite and if all the successive principal minors of $(-A)$ are >0, then $X'AX$ is negative definite.

1.11 Illustrative Examples

Example 1.6 Test the nature of quadratic form $Q(X)=X'AX$

where $A=\begin{bmatrix}3 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 1\end{bmatrix}$, $X=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

Sol. : Characteristic equation $|A-\lambda I|=0$

$$\Rightarrow \quad \begin{vmatrix}3-\lambda & 0 & 0\\ 0 & -2-\lambda & 0\\ 0 & 0 & 1-\lambda\end{vmatrix}=0$$

$$\Rightarrow \quad (3-\lambda)(-2-\lambda)(1-\lambda)=0$$

$\Rightarrow \quad \lambda = 3, -2, 1$

Since two eigen values are positive, one in negative so $Q(X)$ is indefinite.

Example 1.7 Show that $f(x) = 2x_1^2 + x_2^2$ is a convex function over R^2.

Sol. : $f(X)$ is a quadratic form, so in matrix form it can be written as

$$f(X) = (x_1\ x_2)\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Here $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $\qquad |A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)(1-\lambda) = 0$$

$$\Rightarrow \quad \lambda = 2, 1$$

$\Rightarrow$ All the two eigen values are positive, therefore $f(x)$ is positive definite. A positive definitive quadratic form is strictly convex function so $f(x)$ is a convex function over R^2. It is clear from the following theorem.

1.12 Theorems on Quadratic form and Convex Function

Theorem 1.10 A positive semi definite quadratic form $f(X) = X^T AX$ is a convex function over R^n.

Proof : Suppose x_1, x_2 be two points of R^n, then for $0 \le \lambda \le 1$

$$f(\lambda x_1 + (1-\lambda)X_2) = f[X_2 + \lambda(X_1 - X_2)]$$

$$= [X_2 + \lambda(X_1 - X_2)]^T A[X_2 + \lambda(X_1 - X_2)]$$

$$= X_2^T AX_2 + \lambda X_2^T A(X_1 - X_2) + \lambda(X_1 - X_2)^T A X_2$$

$$+ \lambda^2 (X_1 - X_2)^T A(X_1 - X_2)$$

$$\left[\because \left[X_2^T A(X_1 - X_2)\right]^T = X_2^T A(X_1 - X_2)\right.$$

$$\left. = (X_1 - X_2)^T A^T X_2 \right]$$

$$= X_2^T AX_2 + 2\lambda X_2^T A(X_1 - X_2) + \lambda^2 (X_1 - X_2)^T A(X_1 - X_2)$$

$$\leq X_2^T AX_2 + 2\lambda\, X_2^T A(X_1 - X_2) + \lambda(X_1 - X_2)^T A(X_1 - X_2)$$

($\because\ 0 \leq \lambda \leq 1$ so $\lambda^2 \leq \lambda, f(X)$ is positive semi definite)

$$\leq X_2^T AX_2 + 2\lambda\, X_2^T AX_1 - 2\lambda\, X_2^T AX_2 + \lambda X_1^T AX_1 - \lambda X_1^T AX_2$$

$$- \lambda X_2^T AX_1 + X_2^T AX_2$$

$$\leq \lambda\, X_1^T AX_1 + (1-\lambda) X_2^T\, AX_2 \qquad \left[\because \left[X_1^T AX_2\right]^T = X_1^T AX_2\right]$$

$$\leq \lambda\, f(X_1) + (1-\lambda) f(x_2)$$

Thus $f(X) = X^T AX$ is a convex function.

Theorem 1.11 A positive definite quadratic form $f(X) = X^T AX$ is a strictly convex function over R^n.

Proof : $\because\ f(X) = X^T AX$ is positive definite quadratic form so $0 < \lambda < 1 \Rightarrow \lambda^2 < \lambda$ and

$$\lambda^2 (X_1 - X_2)^T A(X_1 - X_2) < \lambda(X_1 - X_2)^T A(X_1 - X_2)$$

using this in the proof of above theorem, we get

$$f\left[\lambda X_1 + (1-\lambda) X_2\right] < \lambda f(X_1) + (1-\lambda) f(X_2)$$

$\Rightarrow$ $f(X)$ is strictly convex function over R^n.

Theorem 1.12 A negative definite (negative semi definite) quadratic form $f(X) = X^T AX$ is a strictly concave (concave) function over R^n.

Proof : $\because\ 0 < \lambda < 1 \Rightarrow \lambda^2 < \lambda$ and $f(x)$ is negative definite

$\Rightarrow$ $\lambda^2 (X_1 - X_2)^T A(X_1 - X_2) > \lambda(X_1 - X_2)^T A(X_1 - X_2)$ and $0 \leq \lambda \leq 1, \lambda^2 \leq \lambda$, $f(x)$ is negative semi definite

$$\Rightarrow \qquad \lambda^2 (X_1 - X_2)^T A(X_1 - X_2) \geq \lambda (X_1 - X_2)^T A(X_1 - X_2)$$

using it in the proof of theorem 1.10 we get that $f(x)$ is strictly concave (concave) function over R^n.

1.13 Self Learning Exercise-II

1. Define convex function.
2. Define quadratic form.
3. What is convexity of quadratic form?
4. What is the relation between convexity and cocavity of a function?
5. What is Eigen values test for the positive and negativeness of quadratic form?

6. Write principal minor test for positive and negativeness of quadratic form.

7. Write geometric meaning of convex and concave functions.

8. Write the quadratic form whose associated matrix is $\begin{bmatrix} 1 & 3 & 5 \\ 3 & 6 & -3 \\ 5 & -3 & 14 \end{bmatrix}$

1.14 Summary

In this unit, the concepts of set of points on the line in E^2 and on the plane in E^3 are generalised to n-dimensional space E^n. We call it as hyperplane. A hyperplane is a separating hyperplane if whole of sets lies in one half of space produced by hyperplane. A separating hyperplane is called supporting hyperplane if it passes through a point of S. The optimal hyperplane of a L.P.P is a supporting hyperplane of a convex set of feasible solution. In the second part of the unit a convex or concave function is defined on convex set and discussed its properties. In the quadratic form and its relation with convex function have been studied.

1.15 Answers to Self-Learning Exercise-I

1. $\oint 1.2(iii)$

2. $\oint 1.2(vi)$

3. $\oint 1.2(vii)$

4. $\oint 1.2(viii)$

5. $\oint 1.4$

1.16 Answers to Self-Learning Exercise-II

1. $\oint 1.6$

2. $\oint 1.9$

3. Theorem 1.10

4. Theorem 1.12

5. $\oint 1.10$

6. $\oint 1.10$ Another test

7. $\oint 1.6$

8. $\oint 1.9$

1.17 Exercises

1. Show that a hyperplane is a closed set

2. Prove that the optimal hyperplane in a l.p.p. is a supporting hyperplane to the convex set of feasible solutions.

3. If $f(x)$ is a convex function over the non-negative orthart of E^n, then show that

$S = \{X : f(x) \leq b, X \geq 0\}$ is a convex set.

4. Show that $f(x) = \begin{cases} b(x-\alpha) & b<0, \quad x<\alpha \\ 0 & x \geq \alpha \end{cases}$ is a convex set for all x.

5. Show that $f(x) = \begin{cases} a(x-\alpha), & a>0, \ x \geq \alpha \\ b(x-\alpha), & b<0, x \leq \alpha \end{cases}$ is a convex function

6. Prove that $f(x) = CX + X^T DX$ is strictly convex iff $X^T DX$ is positive definite.

7. Show that $f(x_1, x_2) = x_1 . x_2$ is not a convex set over E^2.

8. Show that a linear function is convex as well as concave.

9. Show that following function are convex.

(i) $f(x) = |x|$ (ii) $f(x) = e^x$

□□□

Unit - 2
Revised Simplex Method

Structure of the Unit

2.0 Objective

A linear programming problem with m constraints and n variable is defined as :

Max. $Z = c_1x_1 + c_2x_2 + ...+ c_nx_n$

s.t. $a_{11}x_1 + a_{12}x_2 + ...+ a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + ...+ a_{2n}x_n = b_2$

....

....

$a_{m1}x_1 + a_{m2}x_2 + ...+ a_{mn}x_n = b_m$

$x_1, x_2, x_3 ....x_n \geq 0$

In the under graduate classes we have studied simplex method to solve these types of problems. For computer programming purposes, our objective is to find a method which use less entries and operations then simplex method. The revised simplex method fulfills this objective.

2.1 Introduction

In the simplex method if $B = (\beta_1, \beta_2, ...\beta_m)$ be the basis of l.p.p., $X_B = \left(x_{\beta_1}, x_{\beta_2}, ...x_{\beta_m}\right)$ the corresponding B.F.S. and $C_B = \left(C_{B_1}, C_{B_2}, ...C_{B_m}\right)$, corresponding price vectors, then we have

(a) $\beta_1 x_{B_1} + \beta_2 x_{B_2} + \ldots . \beta_m x_{B_m} = b$

i.e. $B\overline{X}_B = b$ or $\overline{X}_B = B^{-1}b$

(b) $\beta_1 y_{ij} + \beta_2 y_2 j + \ldots\ldots + \beta_m y_{mj} = \alpha_j$

i.e. $By_j = \alpha_j$ or $y_j = B^{-1}\alpha_j$, in particular $y_j = B^{-1}B_j = e_j$

(c) $z = C_{B_1} x_{B_1} + C_{B_2} x_{B_2} + \ldots\ldots + C_{B_m} x_{B_m}$

$i.e.\ z = C_B X_B = C_B B^{-1} b\ \ as\ X_B = B^{-1}b$

(d) $z_j - c_j = C_{B_1} y_{1j} + C_{B_2} y_{2j} + \ldots . + C_{B_m} y_{m_j} - C_j$

$$= C_B y_j - C_j = C_B B^{-1} \alpha_j - C_j \qquad \ldots(1)$$

In the simplex procedure we get the following important fact :

Not all the elements of simplex tableau used in calculation at any iteration. Suppose that, at the beginning of an iteration, the inverse B^{-1} of the current basis is known. This leads to a direction calculation of $z_j - c_j$, corresponding solution of the problem and the value of the objective function with the help of (1). The different steps in calculating the next iteration may then be realised as follows :

(i) Calculate $y_k = B^{-1}\alpha_k$. If $y_k \le 0$, there is no finite optimum solution exists. If atleast one element of y_k is > 0 the application of exit criterion (calculation of Min $\theta = Min \frac{x_{Bi}}{y_{ik}} y_{ik} > 0$) of the simplex method will determine the vector B_l to be removed from the present basis.

(ii) Calculate the inverse of new basis i.e. $(B')^{-1}$ (Obtained by replacing β_l by α_k in B) with the help of old inverse of the basis i.e. B^{-1}.

(iii) Calculate the new values of $z_j - c_j$ with the help of (1) and the basis inverse $(B')^{-1}$.

(iv) Calculate the new solution and the new value of the objective function with the help of (1) and $(B')^{-1}$

From above remarks, it follows that to apply the simplex method it is sufficient to transform the inverse of the basis (So as to get the inverse of the new basis) and to calculate from inverse only, the necessary guantities, $z_j - c_j, y_k$, value of the objective function and the solution of the problem. The revised simplex method uses this principle.

2.2 Revised Simplex Method (Standard Form - I)

Consider an l.p.p. as Max $Z = CX,$ subject to $AX = b, X \ge 0$. In the revised simplex method, the objective function is treated as an additional coustraints, which inereases the number of coustraint by one. Instead of cousidering the problem in the above form, we consider the problem here as to maximise z subject to

$$AX = b$$

and $\quad z - CX = 0, X \geq 0 \qquad ...(2)$

which can be written in the expanded form as

$$a_{11}x_1 + a_{12}x_2 + ...+ a_{1n}\, x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + ...+ a_{2n}\, x_n = b_2$$

$$z - C_1x_1 - C_2x_2 - \qquad - C_nx_n = 0 \qquad ...(3)$$

$$x_j \geq 0, \;\; j = 1, 2, 3, ..., n$$

The system (2) or (3) can also be written as

$$\begin{bmatrix} 0 & A \\ 1 & -C \end{bmatrix} \begin{bmatrix} Z \\ X \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} \qquad ...(4)$$

Equations (2), (3) and (4) are referred to as standard form I of the problem for the revised simplex method. In this form an identify matrix in available is the original l.p.p. without using artificial variables.

In the standard form I, corresponding to each activity vector α_j of A we can define a new $(m+1)$ component column vector given by $\alpha_j^{(1)} = [\alpha_j, C_j], j = 1, 2, ..., n$

Also for vectors of basis, we have $\beta_i^{(1)} = [\beta_i, C_{B_i}]$ and corresponding to b, we can define $(m+1)$ component vector

$$b^{(1)} = [b, 0].$$

Note that in (3) the column corresponding to Z is the $(m+1)$ component unit vector, i.e. e_{m+1}.

Basis and Inverse of the Basis :

A basis matrix for the set of equations (3) will be of order (m+1). Actually we are in need of a basic feasible solution of the equations (3) with one of the basic variable as *Z* which is unrestricted in sign and the other m basic variables $x_{B_i} \geq 0$ such that *Z* is as large as possible. We always keep the column e_{m+1}. corresponding to z in the $(m+1)^{th}$ column of the basis matrix.

Let B_1 be the basis matrix of order $(m+1)$ and containing e_{m+1}, so that

$$B_1 = \left(\beta_1^{(1)}, \beta_2^{(1)}, ...\beta_m^{(1)}, e_{m+1}\right)$$

$$= \begin{pmatrix} \beta_1 \;\; \beta_2 .... \beta_m & 0 \\ -C_{B_1} - C_{B_2} ....- C_{B_m} & 1 \end{pmatrix} \qquad ...(5)$$

Since B_1 is basis matrix, the vectors $\beta_1^{(1)}, \beta_2^{(1)}, ... \beta_m^{(1)}, e_{m+1}$ are linearly independent. So a subset $\beta_1^{(1)}, \beta_2^{(1)}, ... \beta_m^{(1)}$ are also linearly independent and hence the vectors $\beta_1, \beta_2, ... \beta_m$ will also be linearly independent and therefore these can be considered as to form basis matrix for $A\overline{X} = b$, i.e. for the original problem. Hence representation (5) can be written as

$$B_1 = \begin{bmatrix} B & 0 \\ -C_B & 1 \end{bmatrix},$$

where $B = (\beta_1, \beta_2, ... \beta_m)$ is the basis for the system $AX = b$. Thus every basis matrix of the revised problem can be written in the form of the basis matrix B of $AX = b$. To proceed in revised simplex method, we need inverse of the basis. We find the inverse of B_1 by partitioned method.

Let $B_1^{-1} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, then $\begin{bmatrix} B & 0 \\ -C_B & 1 \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

i.e. $\begin{bmatrix} B\alpha & B\beta \\ -C_B\alpha + \delta & -C_B\beta + \delta \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

which gives $\alpha = B^{-1}, \ \beta = 0, \ \gamma = C_B B^{-1}, \ \delta = 1$

$\therefore \quad B_1^{-1} = \begin{bmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{bmatrix}.$

Now consider the product of B_1^{-1} and any $\alpha_j^{(1)}$, we get

$$B_1^{-1}\alpha_j^{(1)} = \begin{bmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{bmatrix}\begin{bmatrix} \alpha_j \\ -c_j \end{bmatrix} = \begin{bmatrix} B^{-1}\alpha_j \\ C_B B^{-1}\alpha_j - C_j \end{bmatrix} = \begin{bmatrix} y_i \\ z_j - c_j \end{bmatrix}. \quad ...(6)$$

The first m components of the product are the m components of y_j and the last i.e. $(m+1)^{th}$ component in the product is $z_j - c_j$ which is required for the procedure of optimization.

Now we consider the product of B_1^{-1} with $b^{(1)}$, we get

$$X_B^{(1)} = B_1^{-1}b^{(1)} = \begin{bmatrix} B^{-1} & 0 \\ C_B B^{-1} & 1 \end{bmatrix}\begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} B^{-1}b \\ C_B B^{-1}b \end{bmatrix} = \begin{bmatrix} X_B \\ Z \end{bmatrix} \quad ...(7)$$

The first m components of $\overline{X}_B^{(1)}$ are the elements of the basic feasible solution of the original l.p.p. and the last i.e. $(m+1)^{th}$ component is the value of the objective function of the problem. It gives the reason for treating objective function as one extra constraint.

Computational Proceedure for Standard Form-I :-

In the standard form-I, the identity matrix is present in A without using artificial variables. For revised simplex method initially we have the basis matrix

$$B_1 = \begin{bmatrix} B & 0 \\ -\overline{C}_B & 1 \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ -C_B & 1 \end{bmatrix} \qquad (B = I_m)$$

$$\therefore \qquad B_1^{-1} = \begin{bmatrix} B^{-1} & 0 \\ \overline{C}_B B^{-1} & 1 \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ \overline{C}_B & 1 \end{bmatrix} \qquad ...(8)$$

Further, if the columns from A constituting I_m, i.e. the initial basis of $A\overline{X} = b$, correspond to slack or surplus variables, then $C_B = 0$.

The initial basic solution in revised simplex method is given by

$$X_B^{(1)} = \begin{bmatrix} I_m & 0 \\ \overline{C}_B & 1 \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ \overline{C}_B b \end{bmatrix},$$

and it is feasible because the first m components are the elements of $b \geq 0$ and the $(m+1)^{th}$ component, i.e. z can be of any sign. We now have a B.F.S. of (3) and also the inverse of the corresponding basis matrix.

To improve a B.F.S. we compute $z_j - c_j$ corresponding to every $\alpha_j^{(1)}$ not in the basis B_1 by taking inner multiplication of $(m+1)^{th}$ row of B_1^{-1} with each $\alpha_j^{(1)}$.

If $\min_i \left\{ (z_j - c_j) \middle| z_j - c_j < 0 \right\} = z_k - c_k$, then vector $\alpha_k^{(1)}$ is taken a vector to enter into basis. Now we wish to determine a vector from old basis to be deleted, for this we find $\theta = \underset{y_{ik} > 0}{Min} \left\{ \frac{x_{Bi}}{y_{ik}} \right\}$ and y_k is determined as $y_k^{(1)} = B_1^{-1} \alpha_k^{(1)} = (y_k, z_k - c_k)$. Let $\theta = \underset{y_{ik} > 0}{Min} \left\{ \frac{x_{Bi}}{y_{ik}} \right\} = \frac{x_{\infty}}{y_{lk}}$, we remove l^{th} column of B_1 i.e. β_l. At this stage it must be remembered that we wish to have z always in the basis, therefore the $(m+1)^{th}$ column of B_1 is never be a candidate for removal.

After obtaining the vector to enter and to leave the basis we are now ready to perform the transformation to obtain the new basis inverse and the new solution. In this method B_1^{-1} gives all necessary information at each iteration. Hence we transform only B_1^{-1}. Let the now inverse is denoted by B_1^{-1}. The elements of new inverse and new improved solution will be obtained by transforming the elements of B_1^{-1} and X_B. The solution thus obtained will be improved. Repeating this process interatively unless we get all $z_j - c_j \geq 0$ (as in the simplex method) we can get the optimal basic feasible solution, if it exists.

Tableau form of the revised simplex method standard form-I

Variables in B.F.S	Solution $X_B^{(1)}$	B_1^{-1}					$y_k^{(1)} = B_1^{-1}\alpha_k^{(1)}$	$\frac{X_{Bi}}{y_{ik}}$, $y_{ik} > 0$
		$\gamma_1^{(1)}$	$\gamma_2^{(1)}$	......	$\gamma_m^{(1)}$	$\gamma_{m+1}^{(1)}$		
x_1	x_{B_1}	......	......	....	...	....	y_{1k}	....
x_2	x_{B_2}	......	......	....	...	....	y_{2k}	....
......	......						......	......
......	......						......	......
......	......						......	......
x_m	x_{B_m}	......	......	....	...	....	y_{mk}	....
z	z	......	......	....	...	....	$z_k - c_k$	$\theta = Min \frac{X_{Bi}}{y_{ik}}$

Here $\gamma_1^{(1)} \gamma_2^{(1)} ......\gamma_{m+1}^{(1)}$ are the respective columns of the inverse of basis B_1^{-1}. In the column $X_B^{(1)}$ we write values of the variables. In the first table $B_1 = I_m, B_1^{-1} = I_m, X_B^{(1)} = b^{(1)}$ and $\gamma_{m+1}^{(1)} = e_{m+1}$.

2.3 Revised Simplex Algorithm (Standard Form - I)

Step 1 : If the problem is in minimization, write it into maximization form.

Step2 : Write the given l.p.p. in standard form I for revised simplex method i.e. write the objective function as one coustraint.

Step 3 : Write the initial basis B_1 and its inverse B_1^{-1} by using (8).

Step 4 : Calculate the initial B.F.S. $X_B^{(1)} = B_1^{-1} b^{(1)}$

Step 5 : Calculate $z_j - c_j$ for all vectors which are not in the basis. For this, multiply the last row of B_1^{-1} with corresponding column $\alpha_j^{(1)}$. If atleast one of the $z_j - c_j < 0$ then select the entering vector with $\min(z_j - c_j)$. Let it be $z_k - c_k$, then take $\alpha_k^{(1)}$ as the entering vector for the basis.

Step 6 : Calculate $y_k^{(1)} = B_1^{-1}\alpha_k^{(1)}$ and prepare the revised simplex tableau as shown above. Calculate the last column of the tableau i.e. the column of $\frac{x_{Bi}}{y_{ik}}, y_{ik} > 0$.

Select the minimum $\left(\frac{x_{Bi}}{y_{ik}}\right)$, if this minimum occurs in the r^{th} row, then delet the r^{th} vector of the basis.

Step 7 : Form the new basis by introducting $\alpha_k^{(1)}$ and deleting $\beta_r^{(1)}$ (r^{th} vector of the basis). Form the next revised simplex tableau using transformations

$$\bar{y}_{ij} = y_{ij} - \frac{y_{rj}}{y_{rk}} y_{ik}\ ,\ \bar{y}_{rj} = \frac{y_{lj}}{y_{rk}}$$

Step 8 : Repeat the steps 5,6, 7 iteratively until we get an optimal solution or there is an indication for unbounded solution.

2.4 Illustrative Examples

Example 2.1 : Solve the following linear programming problem by revised simplex method :

$$\begin{aligned} \text{Max} \quad & z = 2x_1 + x_2 \\ \text{St.} \quad & 3x_1 + 4x_2 \le 6 \\ & 6x_1 + x_2 \le 3 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Solution : Introducing slack variables x_3 and x_4 the problem can be written as :

$$\begin{aligned} \text{Max} \quad & z = 2x_1 + x_2 + 0.x_3 + 0.x_4 \\ \text{s.t.} \quad & 3x_1 + 4x_2 + 1.x_3 + 0x_4 = 6 \\ & 6x_1 + x_2 + 0.x_3 + 1x_4 = 3 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

Since there are two equations and two slack variables x_3, x_4 yield two unit vectors for the basis of $A\bar{X} = b$, so the basis with identity matrix is available without using artificial variables. The problem is in standard form I is as :

Find z such that

$$\begin{aligned} & 3x_1 + 4x_2 + x_3 + 0.x_4 = 6 \\ & 6x_1 + x_2 + 0x_3 + x_4 = 3 \\ & z - 2x_1 - x_2 - 0x_3 - 0x_4 = 0 \end{aligned}$$

or

$$\begin{matrix} e_3 & \alpha_1^{(1)} & \alpha_2^{(1)} & \alpha_3^{(1)} & \alpha_4^{(1)} \end{matrix} \quad X^{(1)} \quad b^{(1)}$$

$$\begin{bmatrix} 0 & 3 & 4 & 1 & 0 \\ 0 & 6 & 1 & 0 & 1 \\ 1 & -2 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix},\ x_1, x_2, x_3, x_4 \ge 0$$

Here $\quad B_1 =$ first basis $= \begin{bmatrix} I_2 & 0 \\ -\overline{C}_B & 1 \end{bmatrix}$, therefore $B_1^{-1} = \begin{bmatrix} I_2 & 0 \\ \overline{C}_B & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

as $\overline{C}_B = (0,0)$, price vector corresponding the slack variables x_3, x_4 and I_2 is a basis matrix of original problem.

Now calculate $B_1^{-1} b^{(1)}$ and put in $X_B^{(1)}$ column of revised simplex table. Then multiply $(m+1)^{th}$ row i.e. 3rd row of B_1^{-1} with every $\alpha_j^{(1)}$ not in basis B_1 i.e. with $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ to get $z_j - c_j$. Thus

$$z_1 - c_1 = (0,0,1)\begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix} = (-2),\quad z_2 - c_2 = (0,0,1)\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} = (-1)$$

$\because\ z_j - c_j \ngeq 0\ \forall\, j$, therefore the BFS under test is not optimal. Now $Z_k - C_k =$ Min $(Z_j - C_j) = Min\{-2, -1\} = -2(Z_1 - C_1)$, hance to improve the BFS we introduce the vector $\alpha_1^{(1)}$ into the basis. To determine the departing vector form old basis multiplying $\alpha_1^{(1)}$ with B_1^{-1} to get $y_1^{(1)}$ and write in the before last column of the table and then calculate $\theta = \underset{i}{Min}\left\{\frac{X_{Bi}}{y_{ik}}, y_{ik} > 0\right\}$ for first m elements of $y_1^{(1)}$ which gives $\theta = \frac{1}{2}$, corresponding to x_4. So vector $\alpha_4^{(1)}$ will be deleted and $\alpha_1^{(1)}$ will be introduced.

Revised Simplex Table - 1

Variables in B.F.S	Solution $X_B^{(1)}$	B_1^{-1}			$y_1^{(1)}$	$\frac{x_{Bi}}{y_{ik}},\ y_{ik} > 0$
		$\gamma_1^{(1)}$	$\gamma_2^{(1)}$	$\gamma_3^{(1)}$		
x_3	6	1	0	0	3	$\frac{6}{3} = 2$
x_4	3	0	1	0	6	$\frac{3}{6} = \frac{1}{2}$ →
z	0	0	0	1	-2 ↓	$\theta = Min \frac{X_{Bi}}{y_{ik}} = \frac{1}{2}$

The new basis is $\left(\alpha_3^{(1)}, \alpha_1^{(1)}, e_3\right)$

Now transform this table by the transformation used in simplex method to get the next table.

Revised Simplex Table - 2

Variables in B.F.S	Solution $X_B^{(1)}$	B_1^{-1}			$y_2^{(1)}$	$\frac{x_{Bi}}{y_{ik}}$, $y_{ik} > 0$
		$\gamma_1^{(1)}$	$\gamma_2^{(1)}$	$\gamma_3^{(1)}$		
x_3	$\frac{9}{2}$	1	$-\frac{1}{2}$	0	$\frac{7}{2}$	$\frac{9}{7}$ $\rightarrow$
x_1	$\frac{1}{2}$	0	$\frac{1}{6}$	0	$\frac{1}{6}$	3
z	1	0	$\frac{1}{3}$	1	$-\frac{2}{3}$ $\downarrow$	$\therefore\ \theta = \frac{9}{7}$

Now proceeding in the same manner, we get

$$z_2 - c_2 = \left(0, \frac{1}{3}, 1\right)\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} = -\frac{2}{3}, z_4 - c_4 = \left(0, \frac{1}{3}, 1\right)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{3}$$

$\because$ $z_j - c_j \not\geq 0$, $\forall j$, therefore above B.F.S. is not optimal.

min $\left(z_j - c_j\right) = -\frac{2}{3}$ (for $\alpha_2^{(1)}$), so the vector $\alpha_2^{(1)}$ will be introduced in the basis.

Now $y_2^{(1)} = B_1^{-1}\alpha_2^{(1)} = \left(\frac{7}{2}, \frac{1}{6}, -\frac{2}{3}\right)$, $\therefore\ \theta = \frac{9}{7}\left(\alpha_3^{(1)}\right)$

Therefore $\alpha_3^{(1)}$ will be replaced by $\alpha_2^{(1)}$.

Revised Simplex Table - 3

Variables in B.F.S	Solution $X_B^{(1)}$	B_1^{-1}				
		$\gamma_1^{(1)}$	$\gamma_2^{(1)}$	$\gamma_3^{(1)}$		
x_2	$\frac{9}{7}$	$\frac{2}{7}$	$-\frac{1}{7}$	0		
x_1	$\frac{2}{7}$	$-\frac{1}{21}$	$\frac{4}{21}$	0		
z	$\frac{13}{7}$	$\frac{4}{21}$	$\frac{5}{21}$	1		

For non basis variables

$$z_3 - c_3 = \left(\frac{4}{21}, \frac{5}{21}, 1\right)\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{4}{21}$$

$$z_4 - c_4 = \left(\frac{4}{21}, \frac{5}{21}, 1\right)\begin{bmatrix}0\\1\\0\end{bmatrix} = \frac{5}{21}$$

$\because$ $z_j - c_j \geq 0$, $\forall\, j$, therefore above BFS in optional. The optimal solution is

$$x_1 = \frac{2}{7},\ x_2 = \frac{9}{7} \quad Max\, z = \frac{13}{7},$$

Example 2 : Solve the following l.p.p. using revised simplex method :

Max z $3x_1 + 6x_2 + 2x_3$

S.t $3x_1 + 4x_2 + x_3 \leq 2$

$x_1 + 3x_2 + 2x_3 \leq 1$

$x_1, x_2, x_3 \geq 0$

Solution : Introducing slack variable x_4, x_5 and making objective function as an additional third constraint the problem can be written into standard form-I for revised simplex method as :

$$3x_1 + 4x_2 + x_3 + x_4 + 0x_5 = 2$$

$$x_1 + 3x_2 + 2x_3 + 0x_4 + x_5 = 1$$

$$z - 3x_1 - 6x_2 - 2x_3 - 0x_4 - 0x_5 = 0$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

or

$$\begin{array}{c} \\ e_3\ \ \alpha_1^{(1)}\ \ \alpha_2^{(1)}\ \ \alpha_3^{(1)}\ \ \alpha_4^{(1)}\ \ \alpha_5^{(1)} \\ \begin{bmatrix} 0 & 3 & 4 & 1 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 & 1 \\ 1 & -3 & -6 & -2 & 0 & 0 \end{bmatrix}\end{array} \overset{X_B^{(1)}}{\begin{bmatrix} Z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}} = \overset{b^{(1)}}{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}$$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Here initial Basis $B_1 = \begin{bmatrix} I_2 & 0 \\ -c_B & 1 \end{bmatrix}$

$$\therefore \qquad B_1^{-1} = \begin{bmatrix} I_2 & 0 \\ c_B & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Initial BFS $\overline{X}_B^{(1)} = B_1^{-1} b^{(1)} = b^{(1)} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

For non basic variables x_1, x_2, x_3 we have

$$z_1 - c_1 = (0,0,1)\begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix} = -3$$

$$z_2 - c_2 = (0,0,1)\begin{bmatrix} 4 \\ 3 \\ -6 \end{bmatrix} = -6$$

$$z_3 - c_3 = (0,0,1)\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = -2$$

Since $z_j - c_j \not\geq 0, \forall\, j$, therefore above BFS is not optimal. Min $(z_j - c_j) = -6$ (for $\alpha_2^{(1)}$), hence to improve above B.F.S. we take $\alpha_2^{(1)}$ as introducing vector.

Now $\quad y_2^{(1)} = B_1^{-1}\alpha_2^{(1)} = \alpha_2^{(1)} = \begin{bmatrix} 4 \\ 3 \\ -6 \end{bmatrix}$

Revised Simplex Table - 1

Variables in B.F.S	Solution $X_B^{(1)}$	B_1^{-1} $\gamma_1^{(1)}$	$\gamma_2^{(1)}$	$\gamma_3^{(1)}$	$\gamma_2^{(1)}$	$\frac{X_{Bi}}{y_{ik}},\ y_{ik} > 0$
x_4	2	1	0	0	4	$\frac{2}{4}$
x_5	1	0	1	0	$\boxed{3}$	$\frac{1}{3}$
z	0	0	0	1	-6	$\theta = Min\frac{x_{Bi}}{y_{ik}} = \frac{1}{3}$

The departing vector is $\alpha_5^{(1)}$. key element = 3

Revised Simplex Table - 2

Variables in B.F.S	Solution $X_B^{(1)}$	B_1^{-1} $\gamma_1^{(1)}$	$\gamma_2^{(1)}$	$\gamma_3^{(1)}$	$y_2^{(1)}$	$\frac{X_{Bi}}{y_{ik}}$, $y_{ik} > 0$
x_4	$\frac{2}{3}$	1	$\frac{-4}{3}$	0	$\frac{5}{3}$	$\frac{2/3}{5/3} = \frac{2}{5}$ →
x_2	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1/3}{1/3} = 1$
z	2	0	2	1	-1	$Min \frac{x_{Bi}}{y_{ik}} = \frac{2}{5}$

For non basis variables

$$z_1 - c_1 = (0,2,1)\begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix} = -1$$

$$z_3 - c_3 = (0,2,1)\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = 2$$

$$z_5 - c_5 = (0,2,1)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 2$$

$\because z_j - c_j \not\geq 0$, $\forall\, j$, therefore above BFS is not optimal. $Min(z_j - c_j) = -1$ (for $\alpha_1^{(1)}$) so to improve BFS we introduce $\alpha_1^{(1)}$ into the basis. Now

$$y_1^{(1)} = B_1^{-1}\alpha_1^{(1)} = \begin{bmatrix} 1 & \frac{-4}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{1}{3} \\ -1 \end{bmatrix}$$

and we take $\alpha_4^{(1)}$ as departing vector.

Revised Simplex Table - 3

Variables in B.F.S	Solution $\overline{X}_B^{(1)}$	B_1^{-1}				
		$\gamma_1^{(1)}$	$\gamma_2^{(1)}$	$\gamma_3^{(1)}$		
x_1	$\frac{2}{5}$	$\frac{3}{5}$	$-\frac{4}{5}$	0		
x_2	$\frac{1}{5}$	$-\frac{1}{5}$	$\frac{3}{5}$	0		
z	$\frac{12}{5}$	$\frac{3}{5}$	$\frac{6}{5}$	1		

For non basis variables $z_3 - c_3 = \left(\frac{3}{5}, \frac{6}{5}, 1\right)\begin{bmatrix}1\\2\\-2\end{bmatrix} = \frac{2}{5}$

$$z_4 - c_4 = \begin{pmatrix}\frac{3}{5} & \frac{6}{5} & 1\end{pmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{3}{5}$$

$$z_5 - c_5 = \begin{pmatrix}\frac{3}{5} & \frac{6}{5} & 1\end{pmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \frac{6}{5}$$

$z_j - c_j \geq 0, \forall j$, therefore above BFS is optimal.

Optimal solution is $x_1 = \frac{2}{5}, x_2 = \frac{1}{5}, Max\, z = \frac{12}{5}$

Example 3 : Solve the following l.p.p. using revised simplex method.

Max $z = 3x_1 + x_2 + 2x_3 + 7x_4$

st. $2x_1 + 3x_2 - x_3 + 4x_4 \leq 40$

$-2x_1 + 2x_2 + 5x_3 - x_4 \leq 35$

$x_1 + x_2 - 2x_3 + 3x_4 \leq 100$

$x_1 \geq 2, x_2 \geq 1, x_3 \geq 3, x_4 \geq 4$

Solution : Substituting $x_1 - 2 = u_1, x_2 - 1 = u_2, x_3 - 3 = u_3, x_4 - 4 = u_4$ the given problem reduces to

$$Max\ z^* = z - 41 = 3u_1 + u_2 + 2u_3 + 7u_4$$

$$\text{s.t.} \quad 2u_1 + 3u_2 - u_3 + 4u_4 \le 20$$

$$-2u_1 + 2u_2 + 5u_3 + u_4 \le 26$$

$$u_1 + u_2 - 2u_3 + 3u_4 \le 91$$

$$u_1, u_2, u_3 u_4 \ge 0$$

Introducing slack variables u_5, u_6, u_7 the problem in standard form-I can be written as

Find z^* such that

$$\begin{array}{llll} 2u_1 + 3u_2 - u_3 + 4u_4 + u_5 & & & = 20 \\ -2u_1 + 2u_2 + 5u_3 - u_4 & + u_6 & & = 26 \\ u_1 + u_2 - 2u_3 + 3u_4 & & + u_7 & = 91 \\ -3u_1 - u_2 - 2u_3 - 7u_4 & & + z^* & = 0 \end{array}$$

or

$$\begin{array}{c} \alpha_1^{(1)}\ \alpha_2^{(1)}\ \alpha_3^{(1)}\ \alpha_4^{(1)}\ \alpha_5^{(1)}\ \alpha_6^{(1)}\ \alpha_7^{(1)}\ e_4 \\ \begin{bmatrix} 2 & 3 & -1 & 4 & 1 & 0 & 0 & 0 \\ -2 & 2 & 5 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & -2 & 3 & 0 & 0 & 1 & 0 \\ -3 & -1 & -2 & -7 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \overset{X_B^{(1)}}{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ z^* \end{bmatrix}} = \overset{b^{(1)}}{\begin{bmatrix} 20 \\ 26 \\ 91 \\ 0 \end{bmatrix}}$$

$$u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ u_6,\ u_7 \ge 0$$

Initial Basis $B_1 = \begin{bmatrix} I_3 & 0 \\ -C_B & 1 \end{bmatrix}$ *where* $C_B = (0,0,0)$ *as* I_3 *corresponds to slack variables*

$$\therefore \quad B_1^{-1} = \begin{bmatrix} I_3 & 0 \\ C_B & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Initial BFS is given by

$$X_B^{(1)} = B_1^{-1} b^{(1)} = \begin{bmatrix} b \\ 0 \end{bmatrix} = [20,\ 26,\ 91,\ 0]$$

For non basic variables

$$z_j - c_j = \text{(last row of } B_1^{-1}\text{)}.\ \alpha_j^{(1)} \qquad j = 1,2,3,4,5$$

$$z_1 - c_1 = (0,0,0,1)\begin{bmatrix} 2 \\ -2 \\ 1 \\ -3 \end{bmatrix} = -3, \text{ similarly } z_2 - c_2 = -1$$

$$z_3 - c_3 = -1,\ z_4 - c_4 = -7$$

Since $z_j - c_j \ngeq 0, \forall\, j$, therefore above BFS is not optimal. Min$(z_j - c_j) = -7$ (for $\alpha_4^{(1)}$). Hence to find improved BFS. we use $\alpha_4^{(1)}$ as entering vector. Now we calculate

$$y_4^{(1)} = B_1^{-1}\,\alpha_4^{(1)} = I_4 . \alpha_4^{(1)} = \alpha_4^{(1)} = \begin{bmatrix} 4 \\ -1 \\ 3 \\ -7 \end{bmatrix}$$

Revised Simplex Table-1

Variables in B.F.S	Solution $X_B^{(1)}$	B_1^{-1} $\gamma_1^{(1)}$	$\gamma_2^{(1)}$	$\gamma_3^{(1)}$	$\gamma_4^{(1)}$	$\gamma_4^{(1)}$	$\frac{X_{Bi}}{y_{ik}},\ y_{ik} > 0$
u_5	20	1	0	0	0	4	$\frac{20}{4} = 5$ $\rightarrow$
u_6	26	0	1	0	0	-1	....
u_7	91	0	0	1	0	3	$\frac{91}{3}$
$z*$	0	0	0	0	1	-7 $\downarrow$	$\theta = Min\frac{x_{Bi}}{y_{ik}} = 5$

We take $\alpha_5^{(1)}$ as departing vector. The improved BFS can be found an follows :

Revised Simplex Table - 2

Variables in B.F.S	Solution $X_B^{(1)}$	B_1^{-1} $\gamma_1^{(1)}$	$\gamma_2^{(1)}$	$\gamma_3^{(1)}$	$\gamma_4^{(1)}$	$\gamma_3^{(1)}$	$\frac{X_{Bi}}{y_{ik}},\ y_{ik}>0$
u_4	5	$\frac{1}{4}$	0	0	0	$\frac{-1}{4}$	....
u_6	31	$\frac{1}{4}$	1	0	0	$\frac{19}{4}$	$\frac{124}{9}$ $\rightarrow$
u_7	76	$-\frac{3}{4}$	0	1	0	$\frac{-5}{4}$	....
z^*	35	$\frac{7}{4}$	0	0	1	$\frac{-15}{4}$ $\downarrow$	$\theta = Min\frac{x_{Bi}}{y_{ik}} = \frac{124}{9}$

For non basis vectors, calculate $z_j - c_j$, $j = 1,2,3,5$

$$z_1 - c_1 = \text{(last row of } B_1^{-1}\text{)}\ \alpha_1^{(1)} = \left(\frac{7}{4},0,0,1\right)\alpha_1^{(1)} = \frac{1}{2}$$

$$z_2 - c_2 = \left(\frac{7}{4},0,0,1\right)\alpha_2^{(1)} = \frac{17}{4}$$

$$z_3 - c_3 = \left(\frac{7}{4},0,0,1\right)\alpha_3^{(1)} = -\frac{15}{4}$$

$$z_3 - c_5 = \left(\frac{7}{4}\quad 0\quad 0\quad 1\right)\alpha_6^{(1)} = \frac{7}{4}$$

Min. $\left(z_j - c_j\right) = -\frac{15}{4}\left(\alpha_3^{(1)}\right)$, therefore $\alpha_3^{(1)}$ is entering vector.

Now $y_3^{(1)} = B_1^{-1}\alpha_3^{(1)}$ and write in the tableau 2

Thus the improved basic feasible solution is :

Revised Simplex Table -3

Variables in B.F.S	Solution $X_B^{(1)}$	B_1^{-1} $\gamma_1^{(1)}$	$\gamma_2^{(1)}$	$\gamma_3^{(1)}$	$\gamma_4^{(1)}$	$y_1^{(1)}$	$\frac{X_{Bi}}{y_{ik}},\ y_{ik}>0$
u_4	$\frac{126}{19}$	$\frac{5}{19}$	$\frac{1}{19}$	0	0	$\frac{8}{19}$	$\frac{126}{8}$ $\rightarrow$
u_3	$\frac{124}{19}$	$\frac{1}{19}$	$\frac{4}{19}$	0	0	$\frac{-6}{19}$	....
u_7	$\frac{1599}{19}$	$\frac{-13}{19}$	$\frac{5}{19}$	1	0	$-\frac{17}{19}$	....
z^*	$\frac{1130}{19}$	$\frac{37}{19}$	$\frac{15}{19}$	0	1	$\downarrow$ $\frac{-13}{19}$	$\theta = Min\frac{x_{Bi}}{y_{ik}} = \frac{126}{8}$

For non basis variables, compute $z_j - c_j$, $j = 1,2,5,6$

$$z_1 - c_1 = -\frac{13}{19},\ z_2 - c_2 = \frac{122}{19},\ z_5 - c_5 = \frac{37}{19},\quad z_6 - c_6 = \frac{15}{19}$$

From here we again get the entering vector $\alpha_1^{(1)}$ and $z_1 - c_1 < 0$ and is minimum. Calculate

$$y_1^{(1)} = B_1^{-1}\alpha_1^{(1)} = \left[\frac{8}{19}, -\frac{6}{9}, -\frac{17}{19}, -\frac{13}{19}\right]$$

We take $\alpha_4^{(1)}$ as departing vector. The new BFS becomes as :

Revised Simplex Table - 4

Variables in B.F.S	Solution $X_B^{(1)}$	B_1^{-1} $\gamma_1^{(1)}$	$\gamma_2^{(1)}$	$\gamma_3^{(1)}$	$\gamma_4^{(1)}$		
u_1	$\frac{63}{4}$	$\frac{5}{8}$	$\frac{1}{8}$	0	0		
u_3	$\frac{23}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0		
u_7	$\frac{393}{4}$	$-\frac{1}{8}$	$\frac{3}{8}$	1	0		
z^*	$\frac{281}{4}$	$\frac{19}{8}$	$\frac{7}{8}$	0	1		

For non basis variables, calculate $z_j - c_j, j = 2,4,5,6$

$$z_2 - c_2 = \text{ (last row of } B_1^{-1})\,\alpha_2^{(1)} = \left(\frac{19}{8}, \frac{7}{8}, 0, 1\right)\ \alpha_2^{(1)} = \frac{63}{8}$$

Similarly $z_4 - c_4 = \frac{13}{8}$, $z_5 - c_5 = \frac{19}{8}$, $z_6 - c_6 = \frac{7}{8}$

since $z_j - c_j \geq 0, \forall\, j$, the present solution is optimal. Hence optimal solution is

$$u_1 = \frac{63}{4}, u_2 = 0, u_3 = \frac{23}{2}, u_4 = 0, u_5 = 0$$

$$z^* = \frac{281}{4}$$

$\therefore$ The optimal solution of the given problem is $x_1 = u_1 + 2 = \frac{71}{4}$, $x_2 = y_2 + 1 = 1$,

$u_3 + 3 = \frac{29}{2}$, $x_4 = y_4 + \phi = \phi$

Max $z = z^* + 41 = \frac{445}{4}$

2.5 Revised Simplex Method (Standard Form - II)

This form is used when the l.p.p. does not have any basis matrix as identity matrix. For simplification we suppose that the initial basis matrix does not contain any positive unit vector, i.e. the original problem does not give the first basis without use of artifical variables. Therefore we are assuming here that the basis of the original problem contains all the artificial vectors $\alpha_{1a}, \alpha_{2a}, \ldots, \alpha_{ma}$ corresponding to the artificial variables $x_{1a}, x_{2a}, \ldots x_{ma}$ introduced in the first, second,......, and m^{th} constraint, respcetively. Now, we solve the problem by two phase method for the removal of artificial variable and so we consider one more objective function Z_a, known as artificial objective function which is as

Max. $Z_a = -x_1a - x_2a - \ldots\ldots - x_{ma}$

As there are two objective functions, we have to consider the problem in the revised form with $(m+2)$ contraints. So the problem in standard form-II of the revised method is written below:

$$\left.\begin{array}{ll}
a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n + x_{1a} & = b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n + x_{2a} & = b_2 \\
\ldots\ldots\ \ \ldots\ldots\ \ \ldots\ldots\ \ \ldots\ldots & \ldots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n + x_{ma} & = b_m \\
z - c_1x_1 - c_2x_2 - \ldots\ldots - c_n x_n & = 0 \\
z_a \qquad\qquad + x_{1a} + x_{2a} + \ldots + x_{ma} & = 0
\end{array}\right\}$$

$x_j \geq 0, x_{ia} \geq 0$, $j = 1, 2, \ldots\ldots$,m. ...(9)

Basis and Its Inverse in Standard Form - II :

In the above problem the number of constraints is $(m+2)$. So to handle the problem we get a basis matrix of order $(m+2)$. Two vectors out of $(m+2)$ are corresponding to two objective functions z and Z_a and are denoted by e_{m+1}, e_{m+2} and remaining m are corresponding to the m artificial variables introduced one in each of the constraint.

Now the problem in matrix form can be written as :

$$\begin{array}{c} \begin{array}{cccccccc} e_{m+2} & e_{m+1} & \alpha_1^{(2)} & \alpha_2^{(2)} & \alpha_n^{(2)} & \alpha_{1a}^{(2)} & \alpha_{2a}^{2} & \alpha_{ma}^{(a)} \end{array} \\ \begin{bmatrix} 0 & 0 & a_{11} & a_{12} \ldots\ldots a_{1n} & 1 & 0 \ldots\ldots 0 \\ 0 & 0 & a_{21} & a_{22} \ldots\ldots a_{2n} & 0 & 1 \ldots\ldots 0 \\ & & & & & \\ 0 & 0 & a_{m1} & a_{m2} \ldots\ldots a_{mn} & 0 & 0 \ldots\ldots 1 \\ 0 & 1 & -c_1 & -c_2 \ldots\ldots -c_n & 0 & 0 \ldots\ldots 0 \\ 1 & 0 & 0 & 0 \ \ldots\ldots 0 & 1 & 1 \ldots\ldots 1 \end{bmatrix} \end{array} \overset{X_B^{(2)}}{\begin{bmatrix} Z_a \\ Z \\ x_1 \\ \\ x_n \\ x_{1a} \\ \\ x_{ma} \end{bmatrix}} = \overset{b^{(2)}}{\begin{bmatrix} b_1 \\ b_2 \\ \\ b_m \\ 0 \\ 0 \end{bmatrix}} \qquad \ldots(10)$$

The basis matrix given in (10) can be represented as

$$B_2 = \begin{bmatrix} \alpha_1^{(2)} & \alpha_2^{(2)} & & \alpha_m^{(2)} & e_{m+1} & e_{m+2} \\ 1 & 0 & \dots\dots\dots & 0 & 0 & 0 \\ 0 & 1 & \dots\dots\dots & 0 & 0 & 0 \\ & & & & & \\ & & & & & \\ 0 & 0 & \dots\dots\dots & 1 & 0 & 0 \\ 0 & 0 & \dots\dots & 0 & 1 & 0 \\ 1 & 1 & \dots\dots & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I_m & 0 & 0 \\ 0 & 1 & 0 \\ 1_m & 0 & 1 \end{bmatrix} = \begin{bmatrix} B & 0 & 0 \\ -C_B & 1 & 0 \\ -C_{Ba} & 0 & 1 \end{bmatrix} \quad \dots(11)$$

If we write $[C_B, C_{Ba}] = C_B^{(2)}$ then frm (11) we have

$$B_2 = \begin{bmatrix} B & 0 \\ -C_B^{(2)} & I_2 \end{bmatrix}$$

By partitioned method, the inverse of above basis matrix is given by

$$B_2^{-1} = \begin{bmatrix} B^{-1} & 0 \\ C_B^{(2)} & B^{-1} I_2 \end{bmatrix} = \begin{bmatrix} B^{-1} & 0 & 0 \\ C_B B^{-1} & 1 & 0 \\ C_{Ba} B^{-1} & 0 & 1 \end{bmatrix} \quad \dots(12)$$

Here are some properties of B_2^{-1}

(i) $$B_2^{-1}\alpha_j^{(2)} = \begin{bmatrix} B^{-1} & 0 & 0 \\ C_B \; B^{-1} & 1 & 0 \\ C_{B_a} B^{-1} & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_j \\ -C_j \\ 0 \end{bmatrix} = \begin{bmatrix} B^{-1}\alpha_j \\ C_B B^{-1}\alpha_j - C_j \\ C_{Ba} B^{-1}\alpha_j - 0 \end{bmatrix} = \begin{bmatrix} y_j \\ z_j - C_j \\ z_{ja} - 0 \end{bmatrix} \quad \dots(13)$$

(ii) $$B_2^{-1} b^{(2)} = \begin{bmatrix} B^{-1} & 0 & 0 \\ C_B B^{-1} & 1 & 0 \\ C_{Ba} B^{-1} & 0 & 1 \end{bmatrix} \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} B^{-1} b \\ C_B B^{-1} b \\ C_{Ba} B^{-1} b \end{bmatrix} = \begin{bmatrix} X_B \\ Z \\ Z_a \end{bmatrix} \quad \dots(14)$$

From above it is clear that if $(m+2)^{th}$ row of B_2^{-1} is multiplied with $b^{(2)}$, we get the artificial objective function. If $(m+1)^{th}$ row is multiplied, we get the value of the objective function of the original problem and if first m rows of B_2^{-1} is multiplied with $b^{(2)}$, we get the solution of the original problem.

Computational Procedure of the Standard Form - II :

We know that the column vector corresponding to any variable x_j in (10) is

$\alpha_j^{(2)} = [\alpha_j, -c_j, 0], j = 1, 2, ..., n$ for legitimate vectors and $\alpha_{ja}^{(2)} = [\alpha_{ja}, 0, 1], i = 1, 2, ..., m$ for artificial vectors.

The vector corresponding to z is e_{m+1}, a unit vector, and for z_a it is e_{m+2}, another unit vector represented in the second and first column of (10). Now the inverse of the basis of (10), as calculated previously, is

$$B_2^{-1} = \begin{bmatrix} B^{-1} & 0 & 0 \\ C_B B^{-1} & 1 & 0 \\ C_{Ba} B^{-1} & 0 & 1 \end{bmatrix} = \begin{bmatrix} I_m & 0 & 0 \\ C_B & 1 & 0 \\ C_{Ba} & 0 & 1 \end{bmatrix} \qquad [\because \text{ initially } B = I_m]$$

So it is very easy to get the inverse of B_2, as we know that C_B is the price vector of those legitimate variables which are present in the basis and C_{Ba} the artificial price vector.

To start with the computation we start with phase I for removal of artificial variables from the basis. As soon as artificial variables are removed, we proceed for phase-II.

During the phase I neither the variable z nor z_a may be considered as a candidate for removal from the basis. Moreover, neither of these variables is constrained to be non-negative. If the maximum of z_a is strictly negative, the original problem has no solution. Further if the maximum in phase I is zero and no artificial vector is present in the basis we proceed to phase-II.

Phase I of the Revised Problem :

To start with the phase-I, we need first of all the first basis feasible solution. We get it as $\overline{X}_B^{(2)} = B_2^{-1} b^{(2)}$. After getting initial BFS of the problem, we want to improve it i.e. to make $\max z_a = 0$ and for this we want $z_{ja}^{(2)} - C_{ja}^{(2)}$ which is obtained by multiplying $(m+2)^{th}$ row of B_2^{-1} with $\alpha_{ja}^{(2)}$ If $\max z_a = 0$, the phase-I ends and if, $\max z_a < 0$. take $z_{ka} - c_{ka} = min\{z_{ka} - c_{ja} < 0\}$ then $\alpha_k^{(2)}$ is taken as entering vcetor. Now select $\theta = \min\left\{\frac{x_{Bi}}{y_{ik}}, y_{ik} > 0\right\}$ and corresponding vector is eliminated from the old basis, where x_{Bi} are the elements of X_B and y_{ik} are the elements of y_k. To get y_k, as discussed earlier we multiply $\alpha_k^{(2)}$ with B_2^{-1}, the first m elements will result y_k.

Let $\theta = \underset{i}{Min} \frac{X_{Bi}}{y_{ik}} = \frac{X_{Bl}}{y_{lk}}$, then l^{th} vector of the basis will be eleminated. Now we transform the table for the first improved solution containing $\alpha_k^{(2)}$ in place of l^{th} vector of the basis by method used in standard form-I or in the simplex method and proceed in this way unless z_a i.e. the atificial objective function is maximised. If maximum of z_a is zero and none of the artificial variable present in the basis, then proceed phase-II after eleminating $(m+2)^{th}$ row of the tableau. If maximum of z_a is zero but atleast one of the artificial variable is present at the zero level even then, we proceed to Phase-II with the care that in the further process the artificial variable should never become positive. The best way in these case is that in the first step of phase-II eliminate the artificial variable at zero level, in case of tie consider one by one. If maximum of z_a in strictly negative, the original problem has no BFS and no need of further procedure.

Phase - II :

As soon as phase I ends with max $z_a = 0$ remove $(m+2)^{th}$ row of B_2^{-1} and the column corresponding to Z_a. The reason being that inphase - II we deal with the original objective function and so the prices of all artificial variables become zero.

Now proceed exactly in the same way as stadard form-I.

2.6 Illustrative Examples

Example 2.4 : Solve the following l.p.p. by standard form-II of revised simplex method :

$$2x_1 + 5x_2 \geq 6$$

$$x_1 + x_2 \geq 2,\ x_1, x_2 \geq 0$$

$$\text{Min.} \quad z = x_1 + 2x_2$$

Solution : Introducing surplus variables x_3, x_4, the problem can be written as :

$$2x_1 + 5x_2 - x_3 + 0x_4 = 6$$

$$x_1 + x_2 + 0x_3 - x_4 = 2,\ x_1 x_2, x_3, x_4 \geq 0$$

$$\text{Max} \quad z = -x_1 - 2x_2 + 0x_3 + 0x_4$$

Since, there is no basic feasible solution having identity matrix as basis matrix, so we introduce artificial variables x_5, x_6 the problem in standard form-II of revised simplex method becomes as

$$2x_1 + 5x_2 - x_3 + 0x_4 + x_5 + 0x_6 = 6$$

$$x_1 + x_2 + 0x_3 - x_4 + 0x_5 + x_6 = 2$$

$$z + x_1 + 2x_2 - 0x_3 - 0x_4 \qquad = 0$$

$$z_a \qquad + x_5 + x_6 = 0$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

where the artificial objective function is

Maximize $z_a = -x_5 - x_6$

or

$$\begin{array}{cccccccc} e_4 & e_3 & \alpha_1^{(2)} & \alpha_2^{(2)} & \alpha_3^{(2)} & \alpha_4^{(2)} & \alpha_5^{(2)} & \alpha_6^{(2)} \end{array}$$

$$\begin{bmatrix} 0 & 0 & 2 & 5 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \underset{X_B^{(2)}}{\begin{bmatrix} z_a \\ z \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}} = \underset{b^{(2)}}{\begin{bmatrix} 6 \\ 2 \\ 0 \\ 0 \end{bmatrix}}$$

$$x_j \geq 0, j = 1, 2 \ldots . 6$$

The initial basis is $B_2 = \begin{bmatrix} I_2 & 0 & 0 \\ -C_B & 1 & 0 \\ -C_{Ba} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

New $B_2^{-1} = \begin{bmatrix} I_2 & 0 & 0 \\ -C_B & 1 & 0 \\ -C_{Ba} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ as $C_B = (0,0)$ correspondig to z

and $C_{Ba} = (-1, -1)$ corresponding to z_a

Initial BFS $X_B^{(2)} = B_2^{-1} b^{(2)}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 0 \\ -8 \end{bmatrix}$$

For non basis vectors

$$z_1 - c_1 = \left[last\ \ row\ \ of\ \ B_2^{-1} \right] \alpha_1^{(2)} = (-1, -1, 0, 1) \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = -3$$

$$z_2 - c_2 = (-1,-1,0,1)\begin{bmatrix}5\\1\\2\\0\end{bmatrix} = -6,\quad z_3 - c_3 = (-1,-1,0,1)\begin{bmatrix}-1\\0\\0\\0\end{bmatrix} = 1$$

$$z_4 - c_4 = (-1,-1,0,1)\begin{bmatrix}0\\-1\\0\\0\end{bmatrix} = 1$$

Since $z_j - c_j \not\geq 0$, therefore above BFS is not optimal i.e. max $z_a \neq 0$. $Min(z_j - c_j) = -6$ (for $\alpha_2^{(2)}$), so $\alpha_2^{(2)}$ is taken as intering vector. Now $y_2^{(2)} = B_2^{-1}\alpha_2^{(2)}$

$$= \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\-1 & -1 & 0 & 1\end{bmatrix}\begin{bmatrix}5\\1\\2\\0\end{bmatrix} = \begin{bmatrix}5\\1\\2\\-6\end{bmatrix}$$

Revised Simplex Table - 1 : Phase - I

Variables in B.F.S	Solution $X_B^{(2)}$	B_2^{-1} $\gamma_1^{(2)}$	$\gamma_2^{(2)}$	$\gamma_3^{(2)}$	$\gamma_4^{(2)}$	$y_2^{(1)}$	$\frac{x_{Bi}}{y_{ik}}$, $y_{ik} > 0$
x_5	6	1	0	0	0	5	$\frac{6}{5}$ →
x_6	2	0	1	0	0	1	$\frac{2}{1}$
z	0	0	0	1	0	2	–
z_a	–8	–1	–1	0	1	–6 ↓	$\theta = Min\frac{x_{Bi}}{y_{ik}} = \frac{6}{5}$

The vector departing from the basis is x_5.

Now transform the table using transformations as standard form-I.

Revised Simplex Table - 2 : Phases - I

Variables in B.F.S	Solution $X_B^{(2)}$	B_1^{-1} $\gamma_1^{(2)}$	$\gamma_2^{(2)}$	$\gamma_3^{(2)}$	$\gamma_4^{(2)}$	$y_1^{(2)}$	$\frac{x_{Bi}}{y_{ik}}, \; y_{ik} > 0$
x_2	$\frac{6}{5}$	$\frac{1}{5}$	0	0	0	$\frac{2}{5}$	$\frac{6/5}{2/5} = 3$
x_6	$\frac{4}{5}$	$-\frac{1}{5}$	1	0	0	$\frac{3}{5}$	$\frac{4/5}{3/5} = \frac{4}{3}$ →
z	$-\frac{12}{5}$	$-\frac{2}{5}$	0	1	0	$\frac{1}{5}$	....
z_a	$-\frac{4}{5}$	$\frac{1}{5}$	-1	0	1	$-\frac{3}{5}$ ↓	$\theta = Min \frac{x_{Bi}}{y_{ik}} = \frac{4}{3}$

For non basis variables

$$z_1 - c_1 = \left(\frac{1}{5}, -1, 0, 1\right)\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = -\frac{3}{5}$$

$$z_3 - c_3 = \left(\frac{1}{5}, -1, 0, 1\right)\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -\frac{1}{5}, \quad z_4 - c_4 = \left(\frac{1}{5}, -1, 0, 1\right)\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = 1$$

$\because$ $\quad z_j - c_j \ngeq 0 \;\forall j$ so the BFS is not optimal, $\min(z_j - c_j) = -\frac{3}{5} \; \left(f_x \;\alpha_1^{(1)}\right)$

so we take $\alpha_1^{(2)}$ as entering vector,

Now $\quad y_1^{(2)} = B_2^{-1}\,\alpha_1^{(2)} = \begin{bmatrix} \frac{1}{5} & 0 & 0 & 0 \\ -\frac{1}{5} & 1 & 0 & 0 \\ -\frac{2}{5} & 0 & 1 & 0 \\ \frac{1}{5} & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{3}{5} \\ \frac{1}{5} \\ -\frac{3}{5} \end{bmatrix}$

We take $\alpha_{\in}^{(2)}$ as departing vector.

Revised Simplex Table - 3 : Phase - I

Variables in B.F.S	Solution $X_B^{(2)}$	B_1^{-1} $\gamma_1^{(2)}$	$\gamma_2^{(2)}$	$\gamma_3^{(2)}$	$\gamma_4^{(2)}$		
x_2	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	0	0		
x_1	$\frac{4}{3}$	$-\frac{1}{3}$	$\frac{5}{3}$	0	0		
z	$-\frac{8}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	1	0		
z_a	0	0	0	0	1		

Since Max $z_a = 0$ as no artificial variable present in the basis, hence Phase-I ends. Now we go in phase-II.

Revised Simplex Table - I : Phase - II

Variables in B.F.S	Solution $X_B^{(1)}$	B_1^{-1} $\gamma_1^{(1)}$	$\gamma_2^{(1)}$	$\gamma_3^{(1)}$			
x_2	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	0			
x_1	$\frac{4}{3}$	$-\frac{1}{3}$	$\frac{5}{3}$	0			
z	$-\frac{8}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	1			

For non basic variable

$$z_3 - c_3 = \left(-\frac{1}{3}, -\frac{1}{3}, 1\right)\begin{bmatrix}-1\\0\\0\end{bmatrix} = \frac{1}{3}$$

$$z_4 - c_4 = \left(-\frac{1}{3}, -\frac{1}{3}, 1\right)\begin{bmatrix}0\\-1\\0\end{bmatrix} = \frac{1}{3}$$

$\because z_j - c_j \geq 0, \forall j$ so above BFS is optimal. Optimal solution is $x_1 = \frac{4}{3}, x_2 = \frac{2}{3}$

Max $\quad z = -\frac{8}{3}$

or $\quad Min\, z = \frac{8}{3}$

Example 5 : Solve the following l.p.p. with the help of revised simplex method but without use of artificial variables :

$$\text{Max.} \quad z = 2x_1 - 6x_2$$

$$\text{s.t.} \quad x_1 - 3x_2 \leq 6$$

$$2x_1 + 4x_2 \geq 8$$

$$-x_1 + 3x_2 \leq 6, \; x_1, x_2 \geq 0$$

Solution : Since we have to solve the problem with the help of revised simplex method but with use of artificial variables i.e. we have to apply standard form-I of the revised simplex method which is as follow:

Find z as large as possible s.t.

$$\begin{aligned} x_1 - 3x_2 + x_3 \qquad\qquad &= 6 \\ 2x_1 + 4x_2 \qquad - x_4 \qquad &= 8 \\ -x_1 + 3x_2 \qquad\qquad + x_5 &= 6 \\ z - 2x_1 + 6x_2 \qquad\qquad &= 0 \end{aligned} \qquad \text{...(15)}$$

Here three unit vectors corresponding to x_3, x_5 and z are available. But the basis of problem (15) is of order 4. If there is no restriction we would have to introduce artificial variable in the second row but as we have not to introduce any artificial variable so we can consider any of the remaining vectors for the fourth vector of the basis. For simplicity we consider the negative unit vector corresponding to x_4 . Hence the basis will become

$$\begin{matrix} & \alpha_3^{(1)} & \alpha_4^{(1)} & \alpha_5^{(1)} & e_4 \end{matrix}$$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S & O \\ O & I_2 \end{bmatrix}, \text{ where } S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Hence $B_1^{-1} = \begin{bmatrix} S^{-1} & 0 \\ 0 & I_2 \end{bmatrix}$. But $S^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$\therefore \quad B_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Initial BFS $X_B^{(1)} = B_1^{-1} b^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 6 \\ 0 \end{bmatrix}$

for non basic variables

$$z_1 - c_1 = (0,0,0,1) \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} = -2$$

$$z_2 - c_2 = (0,0,0,1) \begin{bmatrix} -3 \\ 4 \\ 3 \\ 6 \end{bmatrix} = 6$$

Revised Simplex Table - 1

Variables in B.F.S	Solution $X_B^{(1)}$	B_1^{-1}				$y_1^{(1)}$	$\frac{x_{Bi}}{y_{ik}}$, $y_{ik} > 0$
		$\gamma_1^{(1)}$	$\gamma_2^{(1)}$	$\gamma_3^{(1)}$	$\gamma_4^{(1)}$		
x_3	6	1	0	0	0	1	$\frac{6}{1} = 6$
x_4	-8	0	-1	0	0	-2	$\frac{-8}{-2} = 4$
x_5	6	0	0	1	0	-1	....
z	0	0	0	0	1	-2	$Min \frac{x_{Bi}}{y_{ik}} = 4$

$\because$ $z_j - c_j \not\geq 0 \ \forall j$, therefore above BFS is not optimal. $Min(z_j - c_j) = -2$ (for $\alpha_1^{(1)}$) so we take $\alpha_1^{(1)}$ as entering vector. Now

$$y_1^{(1)} = B_1^{-1} \alpha_1^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \\ -2 \end{bmatrix}$$

As in this case, we get a non feasible solution, we select θ as

$$\theta = \min\left\{ \underset{y_{ik}>0}{Min} \frac{x_{Bi}}{y_{ik}},\ \underset{\substack{x_{Bi}<0 \\ y_{ik}<0}}{Min} \frac{x_{Bi}}{y_{ik}} \right\} = 4 \quad (\text{for } \alpha_4^{(1)})$$

We take $\alpha_4^{(1)}$ as departing vector

Revised Simplex Table - 2

Variables in B.F.S	Solution $X_B^{(1)}$	B_1^{-1}					
		$\gamma_1^{(1)}$	$\gamma_2^{(1)}$	$\gamma_3^{(1)}$	$\gamma_4^{(1)}$		
x_1	6	1	0	0	0		
x_4	4	2	−1	0	0		
x_5	12	1	0	1	0		
z	12	2	0	0	1		

For non basis vectors

$$z_2 - c_2 = (2,0,0,1)\begin{bmatrix} -3 \\ 4 \\ 3 \\ 6 \end{bmatrix} = 0$$

$$z_3 - c_3 = (2,0,0,1)\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2$$

$\because z_j - c_j \geq 0$, $\forall\ j$, therefore above BFS is optimal. Thus optimal solution is

$x_1 = 6,\ x_2 = 0$

$\max z = 12$

2.7 Self-Learning Exercise - 1

1. In which l.p.p. the standard form-I of revised simplex method used?
2. In which l.p.p. the standard form-II of revised simplex method used?
3. What are artificial variables and when they are used?
4. What is artificial objective function?

2.8 Exercise

1. Solve the following l.p.p. using revised simplex method

$$x_1 + x_2 \le 3$$

$$x_1 + 2x_2 \le 5$$

$$3x_1 + x_2 \le 6,\ x_1, x_2 \ge 0$$

Ans. $x_1 = 0, x_2 = \frac{5}{2}$ $\quad Max\ z = 5$

2. Sole the following l.p.p. using revised simplex method

Max. $z = 3x_1 + 2x_2 + 5x_3$

s.t. $x_1 + 2x_2 + x_3 \le 430$

$$-3x_1 - 2x_3 \ge -460$$

$$x_1 + 4x_2 \le 420$$

$$x_1, x_2 x_3 \ge 0$$

Ans. $x_1 = 0, x_2 = 100, x_3 = 230$, Max $z = 1350$

Solve the following linear programming problem using standard form-I or II of revised simplex method :

3. Maximize $z = x_1 + x_2 + 3x_3$

s.t. $3x_1 + 2x_2 + x_3 \le 3$

$$2x_1 + x_2 + 2x_3 \le 2$$

$$x_1, x_2, x_3 \le 0$$

Ans. $x_1 = 0, x_2 = 0, x_3 = 1,\ Max\ z = 3$

4. Min. $z = x_1 + x_2$

s.t. $x_1 + 2x_2 \ge 7$

$$4x_1 + x_2 \ge 6$$

$$x_1, x_2 \ge 0$$

Ans. $x_1 = \frac{5}{7},\ x_2 = \frac{22}{7},\ Min\ z = \frac{27}{7}$

5. Max $z = 6x_1 - 2x_2 - 3x_3$

s.t. $2x_1 - x_2 + 2x_3 \le 2$

$x_1 \quad + 4x_3 \le 4$

$x_1, x_2, x_3 \ge 0$

Ans. $x_1 = 4,\ x_2 = 6, x_3 = 0$ Max $z = 12$

6. Max $z = 30x_1 + 23x_2 + 29x_3$

s.t. $6x_1 + 5x_2 + 3x_3 \le 26$

$4x_1 + 2x_2 + 5x_3 \le 7$

$x_1, x_2, x_3 \ge 0$

Ans. $x_1 = 0,\ x_2 = \frac{7}{2},\ x_3 = 0,\ \max z = \frac{161}{2}$

7. Max. $z = x_1 + x_2$

s.t. $3x_1 + 2x_2 \le 6$

$x_1 + 4x_2 \le 4$

$x_1, x_2 \ge 0$

Ans. $x_1 = \frac{8}{5},\ x_2 = \frac{3}{5}$, max $z = \frac{11}{5}$

8. Max $z = 5x_1 + 3x_2$

s.t. $3x_1 + 5x_2 \le 15$

$5x_1 + 2x_2 \le 10$

$x_1, x_2 \ge 0$

Ans. $x_1 = \frac{20}{19},\ x_2 = \frac{45}{19},\ Max\ z = \frac{235}{19}$

9. Max $z = 5x_1 + 3x_2$

s.t. $4x_1 + 5x_2 \ge 10$

$5x_1 + 2x_2 \le 10$

$3x_1 + 8x_2 \le 12$

$x_1, x_2 \ge 0$

Ans. $x_1 = \frac{28}{17}, x_2 = \frac{15}{17},\ Max\ z = \frac{185}{17}$

10. Max $z = x_1 + 2x_2 + 3x_3 - x_4$

s.t. $x_1 + 2x_2 + 3x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

$x_1, x_2, x_3, x_4 \geq 0$

Ans. $x_1 = \frac{5}{2},\ x_2 = \frac{5}{2},\ x_3 = \frac{5}{2}, x_4 = 0$, Max $z = 15$.

2.9 Bounded Variable Problems

A bounded variable linear programming problem (BVLPP) is difined as :

Max or Min $z = CX$

s.t. $A\,X \leq, =, \geq b$

$$l_j \leq x_j \leq u_j\ ,\ \forall\ j = 1,2,3,\ldots n \qquad \ldots(16)$$

and $X \geq 0$

Here each variable x_j is bounded from both sides i.e. from upper bound u_j and lower bound l_j. These problems can be solved by simplex method with some modifications.

Bounded Variable Simplex Alogrithm

(i) Convert the objective function into maximization if it is in minimization and introducing slack and surplus variables write the problem in standrad form.

(ii) Find initial basic feasible solution.

(iii) If lower bound of any bounded variable is positive then make it zero by substituting additional variable. For example if $2 \leq x_1 \leq 5$, then put $x_1' = x_1 - 2$

$2 - 2 \leq x_1 - 2 \ \leq 5 - 2$

or $0 \leq x_1' \leq 3$

(iv) Construct the simplex table and test the sign of $z_j - c_j$. In case of $z_j - c_j \geq 0$, the optimal solution is obtained, if $z_j - c_j \ngeq 0$, then entering and departing vectors can be found as follows :

(v) Let $\min\{z_j - c_j\} = z_r - c_r$ then take α_r as entering vector.

(vi) To find departing vector following quatntities are calculated :

$$\theta_1 = \min_i \left\{ \frac{x_{Bi}}{y_{ir}}, y_{ir} > 0 \right\}$$

$$\theta_2 = \min_i \left\{ \frac{u_i - x_{Bi}}{-y_{ir}}, y_{ir} < 0 \right\}$$

$$\theta = \min\{\theta_1, \theta_2, u_r\}$$

where u_r is the upper bound of variable x_r. Clearly when $y_{ir} > 0,\ \theta_2 \to \infty$.

(a) if $\theta = \min\{\theta_1, \theta_2, u_r\} = \theta_1$ and it is corresponding to x_{Bk} then y_k will be departing vector.

(b) If $\theta = \min\{\theta_1, \theta_2, u_r\} = \theta_2$ and it is corresponding to x_{Bk} will be departing vector. If x_{Bk} is non basic on the upper bound, then following substitution is made i.e. all basic variables are updated.

$$(x_{Bk})_r = (x_{Bk})'_r - y_{kr}u_r, \text{ where } 0 \le (x_{Bk})'_r \le u_r$$

and non basic variable x_r on upperbound is made at zero level by substituting $x_r = u_r - x_r^1$, $0 \le x'_r \le u_r$.

(c) If $\theta = \min\{\theta_1, \theta_2, u_r\} = u_r$, then x_r is substituted on the upper bound till then x_r becomes non basic variable and it is being made at zero level using $x_r = u_r - x'_r$.

(vii) Choosing entering and departing vector from steps (v) & (vi) we make simplex table and test the sign of $z_j - c_j$. In case $z_j - c_j \ge 0$, the optimal solution is obtained and if $z_j - c_j \ngeq 0$ repeat steps (iv) to (vii) until we get optimal solution.

2.10 Illustrative Examples

Exampe 6 : Using bounded variable technique, solve the following l.p.p.

Max $\quad z = x_1 + 3x_2$

S.t. $\quad x_1 + x_2 + x_3 \le 10$

$x_1 \quad - 2x_3 \ge 0$

$2x_2 - x_3 \le 10$

and $\quad 0 \le x_1 \le 8,\ 0 \le x_2 \le 4,\ x_3 \ge 0$

Solution : Introducing slack variables x_4, x_5, x_6 the standard form of l.p.p. is as a

Max $\quad z = x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6$

S.t. $\quad x_1 + x_2 + x_3 + x_4 + 0x_5 + 0x_6 = 10$

$x_1 + 0x_2 - 2x_3 + 0x_4 + x_5 + 0x_6 = 0$

$0x_1 + 2x_2 - x_3 + 0x_4 + 0x_5 + x_6 = 10$

and $\quad 0 \le x_1 \le 8, 0 \le x_2 \le 4, x_3, x_4, x_5, x_6 \ge 0$

Initial B.F.S. is $x_4=10, x_5=0, x_6=10$ and Basis $B=I_3$. In the given problem there is no upper bound for basic variables x_4, x_5, x_6 and non basic variable x_3. Thus all the upper bounds are taken at inifinity i.e. $u_4=u_5=u_6=\infty=u_3$.

Simplex Table - 1

			c_j	0	1	3	0	0	0	u_i-x_{Bi}	
c_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_6		
0	α_4	x_4	10	1	1	1	1	0	0	$\infty-10=\infty$	
0	α_5	x_5	0	-1	0	$\boxed{2}$	0	1	0	$\infty-0=\infty$	$\rightarrow$
0	α_6	x_6	10	0	2	-1	0	0	1	$\infty-10=\infty$	
z_j-c_j				0	-1	-3	0	0	0		
u_j				8	4	∞ $\uparrow$	∞	∞	∞		

Since $z_j-c_j \not\geq 0, \forall j$, therefore above BFS is not optimal. Here $Min(z_j-c_j)=z_3-c_3=-3$, hence to improve BFS we introduce x_3 into the basis. For departing vector

$$\theta_1=\min\left\{\frac{x_{Bi}}{y_{i3}}, y_{i3}>0\right\}=\min\{10,0\}=0 \qquad \text{(corresponding to } \alpha_5\text{)}$$

$$\theta_2=\min\left\{\frac{u_i-x_{Bi}}{-y_{i3}}, y_{i3}<0\right\}=\infty \qquad \text{(corresponding to } \alpha_5\text{)}$$

and $\quad u_3=\infty$

$\therefore \quad \min\{\theta_1,\theta_2,u_3\}=\min\{0,\infty,\infty\}=0=\theta_1$

Hence we take α_5 as departing vector.

Simplex Table-2

			c_j	0	1	3	0	0	0	u_i-x_{Bi}
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_6	
0	α_4	x_4	10	$\frac{3}{2}$	1	0	1	-1	0	$\infty-10=\infty$
3	α_3	x_3	0	$-\frac{1}{2}$	0	1	0	1	0	$\infty-0=\infty$
0	α_6	x_6	10	$-\frac{1}{2}$	2	0	0	1	1	$\infty-10=\infty$
z_j-c_j				$-\frac{3}{2}$	-1	0	0	3	0	
u_j				8	4	∞	∞	∞	∞	

$\because\ z_j - c_j \ngeq 0, \forall j$, therefore above BFS is not optimal. $Min(z_j - c_j) = -\frac{3}{2} = z_1 - c_1$, so we take α_1 as entering vector. For departing vector, we have

$$\theta_1 = \underset{i}{Min}\left\{\frac{x_{Bi}}{y_{i1}}, y_{i1} > 0\right\} = \min\left\{\frac{10}{\frac{3}{2}}\right\} = \frac{20}{3} \text{ (for } \alpha_4\text{)}$$

$$\theta_2 = \infty, \text{ and } u_1 = 8$$

$$\therefore \quad \theta = \min\{\theta_1, \theta_2, u_1\} = \min\left\{\frac{20}{3}, \infty, 8\right\} = \frac{20}{3} = \theta_1$$

Hence α_4 is taken as departing vector,

Simplex Table - 3

			c_j	0	1	3	0	0	0	
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_6	
0	α_1	x_1	$\frac{20}{3}$	1	$\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{-2}{3}$	0	
3	α_3	x_3	$\frac{10}{3}$	0	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{2}{3}$	0	
0	α_6	x_6	$\frac{40}{3}$	0	$\frac{7}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1	
		$z_j - c_j$		0	0	0	1	2	0	

$\because\ z_j - c_j \geq 0, \forall j$ so above BFS is optimal. Hence optimal solution is

$$x_1 = \frac{20}{3}, x_2 = 0, x_3 = \frac{10}{3}, \text{ Max } z = 10$$

Example 7 : Using the bounded variable technique, solve the following l.p.p.

Max $\quad z = 3x_1 + 5x_2 + 2x_3$

S.t. $\quad x_1 + 2x_2 + 2x_3 \leq 14$

$\quad 2x_1 + 4x_2 + 3x_3 \leq 23$

and $\quad 0 \leq x_1 \leq 4, 2 \leq x_2 \leq 5, 0 \leq x_3 \leq 3$.

Solution : Since the lower bound of x_2 is positive, therefore let $x_2' = x_2 - 2$ or $x_2 = x_2^1 + 2$, then $0 \leq x_2^1 \leq 3$. Introducing slack variables $x_4, x_5 \geq 0$, the standard form of B.V.L.P.P. is as :

Max $\quad (z - 10) = 3x_1 + 5x_2^1 + 2x_3 + 0x_4 + 0x_5$

s.t. $x_1 + 2x_2^1 + 2x_3 + x_4 + 0x_5 = 10$

$2x_1 + 4x_2^1 + 3x_3 + 0x_4 + x_5 = 15$

$0 \le x_1 \le 4, 0 \le x_2^1 \le 3, 0 \le x_3 \le 3$

$x_4, x_5 \ge 0$

Initial BFS $x_4 = 10, x_5 = 15$, initial basis $B = I_2$

Simplex Table -1

			c_j	3	5	2	0	0	$u_i - x_{Bi}$
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	
0	α_4	x_4	10	1	2	2	1	0	$\infty - 10 = \infty$
0	α_5	x_5	15	2	4	3	0	1	$\infty - 15 = \infty$
		$z_j - c_j$			-3	-5	-2	0	0
			u_j	4	3	3	∞	∞	

$z_j - c_j \not\geq 0$, therefore above $b.f.s.$ is not optimal. $Min(z_j - c_j) - 5(z_2 - c_2)$, so to improve b.f.s. we introduce α_2 into the basis. For departing vector-

$$\theta_1 = \min\left\{\frac{10}{2}, \frac{15}{4}\right\} = \frac{15}{4} \text{ (corresponding to } \alpha_5)$$

$$\theta_2 = \infty,\ u_2 = 3,$$

$\theta = \min\left\{\frac{15}{4}, \infty, 3\right\} = 3 = u_2$, therefore we substitute x_2^1 on the upper bound till then x_2^1 becomes non-basic.

$$x_2^1 = u_2 - x_2'' = 3 - x_2'', \text{ where } 0 \le x_2'' \le 3$$

and update basic variables as

$$x_{B1} = x'_{B1} - y_{12}u_2 = 10 - 2\times 3 = 4$$

$$x_{B2} = x'_{B2} - y_{22}u_2 = 15 - 4\times 3 = 3$$

Simplex Table - 2

			c_j	3	-5	2	0	0	$u_i - x_{Bi}$
C_B	B	X_B	b	y_1	y_2''	y_3	y_4	y_5	
0	α_4	x_4	4	1	-2	2	1	0	$\infty - 4 = \infty$
0	α_5	x_5	3	2	-4	3	0	1	$\infty - 3 = \infty$
$z_j - c_j$				-3	5	-2	0	0	
u_j				4	3	3	0	0	

$\because \quad z_j - c_j \not\geq 0, \forall j$ therefore above *b.f.s.* is not optimal. $Min(z_j - c_j) = -3(z_1 - c_1)$ so we take α_1 as entering vector. For departing vector

$$\theta_1 = Min\left\{\frac{4}{1}, \frac{3}{2}\right\} = \frac{3}{2} \quad \text{(corresponding to } \alpha_5\text{)}$$

$$\theta_2 = \infty, \text{ and } u_1 = 4$$

$$\theta = Min\{\theta_1, \theta_2, u_1\} = Min\left\{\frac{3}{2}, \infty, 4\right\} = \frac{3}{2} = \theta_1$$

Hence α_5 is departing vector.

Simplex Table - 3

			c_j	3	-5	2	0	0	$u_i - x_{Bi}$
C_B	B	X_B	b	y_1	y_2''	y_3	y_4	y_5	
0	α_4	x_4	$\frac{5}{2}$	0	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\infty - \frac{5}{4} = \infty$
3	α_1	x_1	$\frac{3}{2}$	1	$\boxed{-2}$	$\frac{3}{2}$	0	$\frac{1}{2}$	$4-3=\frac{5}{2}$
	$z_j - c_j$		0	0	-1 ↑	$\frac{5}{2}$	0	$\frac{3}{2}$	
	u_j		4	4	3	3	∞	∞	

$\because \quad z_j - c_j \not\geq 0, \ \forall j$ so the above b.f.s is not optimal $\text{Min}(z_j - c_j) = -1\,(z_2 - c_2)$, therefore α_2'' will be introducing vector. Since $y_2'' \leq 0$, so for departing vector

$$\theta_1 = \infty, \theta_2 = Min\left\{\infty, \frac{\frac{5}{2}}{-(-2)}\right\} = \frac{5}{4}, u_2 = 3 \quad \text{(Corresponds to } \alpha_1\text{)}$$

$$\therefore \quad \theta = Min.\left\{\infty, \frac{5}{4}, 3\right\} = \frac{5}{4} = \theta_2$$

$\therefore \alpha_1$ is departing vector. Since upper bound of x_1 is 4 .

Simplex Table - 4

			c_j	3	-5	2	0	0	
C_B	B	X_B	b	y_1	y_2''	y_3	y_4	y_5	
0	α_4	x_4	$\frac{5}{2}$	0	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	
5	α_2''	x''_2	$-\frac{3}{4}$	$-\frac{1}{2}$	1	$-\frac{3}{4}$	0	$-\frac{1}{4}$	
	$z_j - c_j$			$-\frac{1}{2}$	0	$\frac{7}{4}$	0	$\frac{5}{4}$	
	u_j			4	3	3	∞	∞	

so we update the basic variables

$$x_{B1} = x'_{B1} - y_{11}u_1 = \frac{5}{2} - 0 \times 4 = \frac{5}{2}$$

$$x_{B2} = x'_{B2} - y_{21}u_1 = \frac{-3}{4} - \left(-\frac{1}{2}\right) \times 4 = \frac{5}{4}$$

For zero level of non basic variable x_1, substituting $x_1 - 4 = x_1^1$

Simplex Table-5

			c_j	3	-5	2	0	0	
C_B	B	X_B	b	y_1'	y_1''	y_3	y_4	y_5	
0	α_4	x_4	$\frac{5}{2}$	0	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	
5	α_2''	x''_2	$\frac{5}{4}$	$\frac{1}{2}$	1	$-\frac{3}{4}$	0	$-\frac{1}{4}$	
$z_j - c_j$				$\frac{1}{2}$	0	$\frac{7}{4}$	0	$\frac{5}{4}$	
u_j				4	3	3	∞	∞	

Since $z_j - c_j \geq 0, \forall_j$ therefore above b.f.s. in optimal.

The optimal solution from the table

$$x'_1 = 0, x''_2 = \frac{5}{4}, x_3 = 0$$

But $\quad x'_1 = 4 - x_1$ and $x'_2 = 3 - x''_2$

$$\Rightarrow \quad x_1 = 4 - x'_1 = 4 - 0 = 4,\ x_2 = 3 - \frac{5}{4} = \frac{7}{4}$$

$$\therefore \quad x_2 = x'_2 + 2 = \frac{7}{4} + 2 = \frac{15}{4}$$

$$\therefore \quad x_1 = 4, x_2 = \frac{15}{4}, x_3 = 0 \text{ and } Max\ z = 3 \times 4 + 5 \times \frac{15}{4} + 2 \times 0 = \frac{123}{4}$$

Example 8 : Using the bounded variable technique, solve the following linear programing problem :

Max $\quad z = 2x_1 + x_2$

s.t. $x_1 + 2x_2 \le 10$

$x_1 + x_2 \le 6$

$x_1 - x_2 \le 2$

$x_1 - 2x_2 \le 1$

and $0 \le x_1 \le 3$, $0 \le x_2 \le 2$

Solution : Introducing slack variables $x_3, x_4, x_5, x_6 \ge 0$ the standard form of given problem is as :

$Max\ \ z = CX$

s.t. $AX = b,\ 0 \le x_1 \le 3,\ 0 \le x_2 \le 2$

$x_3, x_4, x_5, x_6 \ge 0$

$$\text{where } A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{bmatrix},\ b = \begin{bmatrix} 10 \\ 6 \\ 2 \\ 1 \end{bmatrix} \text{ and } \overline{C} = (2,1,0,0,0,0)$$

Initial BFS $x_3 = 10, x_4 = 6, x_5 = 2, x_6 = 1$ and initial basis $B = I_4$

Simplex Table - 1

			c_j	2	1	0	0	0	0	$u_i - x_{Bi}$
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_6	
0	α_3	x_3	10	1	2	1	0	0	0	$\infty - 10 = \infty$
0	α_4	x_4	6	1	1	0	1	0	0	$\infty - 6 = \infty$
0	α_5	x_5	2	1	-1	0	0	1	0	$\infty - z = \infty$
0	α_6	x_6	1	1	2	0	0	0	1	$\infty - 1 = \infty$ →
$z_j - c_j$				-2	-1	0	0	0	0	
u_j				3	2	∞	∞	∞	∞	

$\because$ $z_j - c_j \ngeq 0, \forall\, j$ therefore BFS is not optimal.

Min $(z_j - c_j) = -2$ (for α_1), so α_1 is taken as entering vector. For departing vector

$$\theta_1 = \min\left\{\frac{10}{1}, \frac{6}{1}, \frac{2}{1}, \frac{1}{1}\right\} = 1 \text{ (corresponding to } \alpha_6\text{)}$$

$\theta_2 = \infty$ and $u_1 = 3$

$\theta = \min\{\theta_1, \theta_2, u_1\} = 1 = \theta_1$

Hence α_6 is taken as departing vector.

Simplex Table-2

			c_j	2	1	0	0	0	0	$u_i - x_{Bi}$	
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_6		
0	α_3	x_3	9	0	4	1	0	0	-1	∞	
0	α_4	x_4	5	0	3	0	1	0	-1	∞	
0	α_5	x_5	1	0	1	0	0	1	-1	∞	$\rightarrow$
2	α_1	x_1	1	1	-2	0	0	0	2	$3-1=2$	
$z_j - c_j$				0	-5	0	0	0	2		
u_j				3	2	∞	∞	∞	∞		

$\because$ $z_j - c_j \not\geq 0, \forall j$ $\therefore$ above BFS is not optimal. Min $(z_j - c_j) = -5$ (for α_1) so α_1 is entering vector. For departing vector

$$\theta_1 = \min\left\{\frac{9}{4}, \frac{5}{3}, \frac{1}{1}\right\} = 1$$

$$\theta_2 = \min\left\{\frac{u_i - x_{Bi}}{-y_{i2}}, y_{i2} < 0\right\} = \frac{2}{-(-2)} = 1$$

$$u_2 = 2$$

$$\theta = \min\{\theta_1, \theta_2, u_2\} = 1 = \theta_1 \text{ or } \theta_2$$

Let $\theta = \theta_1$, then α_5 is taken as departing vector.

Simplex Table - 3

			c_j	2	1	0	0	0	0	$u_i - x_{Bi}$
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_6	
0	α_3	x_3	5	0	0	1	0	-4	3	∞
0	α_4	x_4	2	0	0	0	1	-3	2	∞
1	α_2	x_2	1	0	1	0	0	1	-1	$2-1=1$
2	α_1	x_1	3	1	0	0	0	2	-1	$3-3=0$
$z_j - c_j$			0	0	0	0	5	-3		
u_j			3	2	∞	∞	$\uparrow \infty$	∞		

$\because$ $z_j - c_j \not\geq 0$, $\forall j$ $\therefore$ Above BFS is not optimal

$Min\{z_j - c_j\} = -3$ (for α_6), so α_6 is entering vector.

For departing vector

$$\theta_1 = Min\left\{\frac{5}{3}, \frac{2}{2}\right\} = 1, \ \theta_2 = \min\left\{\frac{1}{-(-1)}, \frac{0}{-(-1)}\right\} = 0$$

(corresponds to α_4) (corresponds to α_1)

and $u_6 = \infty$

$\theta = \min\{\theta_1, \theta_2, u_6\} = 0 = \theta_2$

$\therefore$ α_1 is departing vector.

Simplex Table - 4

			c_j	2	1	0	0	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_6
0	α_3	x_3	14	3	0	1	0	2	0
0	α_4	x_4	8	2	0	0	1	1	0
1	α_2	x_2	-2	-1	1	0	0	-1	0
0	α_6	x_6	-3	-1	0	0	0	-2	1
$z_j - c_j$				-3	-3	0	0	-1	0
u_j				3	2	∞	∞	∞	∞

$\because$ upper bound of x_1 is 3 we update basic variables as :

$x_{B1} = x'_{B1} - y_{11}u_1 = 14 - (3) \times 3 = 5$

$x_{B2} = x'_{B2} - y_{21}u_1 = 8 - (2) \times 3 = 2$

$x_{B3} = x'_{B3} - y_{31}u_1 = -2 - (-1) \times 3 = 1$

$x_{B4} = x'_{B4} - y_{41}u_1 = -3 - (-1) \times 3 = 0$

The non basic variable x_1 can be found by substituting x_1 on upper bound at zero level as $x_1 = 3 - x'_1$ Applying above formula

Simplex Table - 5

			c_j	-2	1	0	0	0	0	$u_i - x_{Bi}$
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_6	
0	α_3	x_3	5	-3	0	1	0	2	0	∞
0	α_4	x_4	2	-2	0	0	1	1	0	∞
1	α_2	x_2	1	1	1	0	0	-1	0	$2-1=1$
0	α_6	x_6	0	1	0	0	0	-2	1	∞
$z_j - c_j$				3	0	0	0	-1	0	
u_j				3	2	∞	∞	∞	∞	

$\because$ $z_j - c_j \ngeq 0$, $\forall\, j$ so above BFS is not optimal

$\min(z_j - c_j) = -1$ (for α_5)

So α_5 is taken as entering vector.

For departing vector $\theta_1 = \min\left\{\frac{5}{1}, \frac{2}{1}\right\} = 2$

(corresponds to α_4)

$$\theta_2 = \frac{1}{-(-1)} = 1, u_5 = \infty$$

$$\theta = \min\{\theta_1, \theta_2, u_5\} = 1 = \theta_2$$

Hence α_2 will be departing vector.

Simplex Table - 6

			c_j	-2	1	0	0	0	0
c_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_6
0	α_3	x_3	7	-1	2	1	0	0	0
0	α_4	x_4	3	-1	1	0	1	0	0
0	α_5	x_5	-1	-1	-1	0	0	1	0
0	α_6	x_6	-2	-1	-2	0	0	0	1
$z_j - c_j$				2	-1	0	0	0	0
u_j				3	2	∞	∞	∞	∞

$\because$ x_2 has upper bound 2, therefore updating the basic variable as :

$$x_{B1} = x'_{B1} - 2 \times 2 = 3$$

$$x_{B2} = x'_{B2} - 1 \times 2 = 1$$

$$x_{B3} = x'_{B3} - (-1) \times 2 = 1$$

$$x_{B4} = x'_{B4} - (-2) \times 2 = 2$$

The non basic variable x_2 can be found by substituting x_2 on upper bound at zero level as $x_2 = 2 - x'_2$. Applying the above formula.

Simplex Table - 7

			c_j	-2	-1	0	0	0	0	
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_6	
0	α_3	x_3	3	-1	-2	1	0	0	0	
0	α_4	x_4	1	-1	-1	0	1	0	0	
0	α_5	x_5	1	-1	1	0	0	1	0	
0	α_6	x_6	2	-1	2	0	0	0	1	
$z_j - c_j$				2	1	0	0	0	0	
u_j				3	2	∞	∞	∞	∞	

$\because z_j - c_j \geq 0, \forall j$ therefore above BFS is optimal.

Optimal solution is $x'_1 = 0, x'_2 = 0$

$\therefore \quad x_1 = 3 - x'_1 = 3 - 0 = 3$

$x_2 = 2 - x'_1 = 2 - 0 = 2$

Max $\quad z = z = 2 * 3 + 2 = 8$

2.11 Self-Learning Exercise - 2

1. What do you mean by bounded variables?
2. How can you find the departing vector in the bounded variable algorithm?
3. If a bounded variable has lower bound positive, then how can it made zero?

2.12 Exercise

1. Using bounded variable technique, solve the following l.p.p.

Max $\quad z = 4x_1 + 4x_2 + 3x_3$

s.t. $\quad -x_1 + 2x_2 + 3x_3 \leq 15$

$-x_2 + x_3 \leq 4$

$2x_1 + x_2 - x_3 \leq 6$

$x_1 - x_2 + 2x_3 \leq 10$

$0 \leq x_1 \leq 8, \; 0 \leq x_2 \leq 4, 0 \leq x_3 \leq 4$

Ans : $\quad x_1 = \dfrac{17}{5}, x_2 = \dfrac{16}{5}, x_3 = 4, \; Max\ z = \dfrac{192}{5}$

2. Solve the following bounded variable problem :

$Max\, z = 4x_1 + 2x_2 + 6x_3$

s.t. $\quad 4x_1 - x_2 \quad \leq 9$

$-x_1 + x_2 + 2x_3 \le 8$

$-3x_1 + x_2 + 4x_3 \le 12$

and $1 \le x_1 \le 3, 0 \le x_2 \le 5,\ 0 \le x_3 \le 2$

Ans. $x_1 = 3, x_2 = 5, x_3 = 2,\ Max\, z = 34$

3. Solve:

Max $z = 3x_1 + 5x_2 + 2x_3$

s.t. $x_1 + x_2 + 2x_3 \le 14$

$2x_1 + 4x_2 + 3x_3 \le 34$

and $0 \le x_4 \le 4, 7 \le x_2 \le 10, 0 \le x_3 \le 3$

Ans. $x_1 = 4, x_2 = \frac{35}{4}, x_3 = 0$, Max $z = \frac{223}{4}$

□□□

Unit - 3
Integer Programming : Gomory's Algorithm

Structure of the Unit

3.0 Objective

The objective of this unit is to introduce the concept of integer programming. After studying this unit one may be able to understand the importance and need of it. A method to solve these problems and suffcient exercise to understand the method is also prosented in this unit.

3.1 Introduction

Integer programming problems are those linear programming problems in which all or some of the variables in the optimal solutions are restricted to take non-negative integer values. Such problems are called **'all integer'** or **'mixed integer programming problems** depending, on whether all or some of the variables are restricted to integer values respectively.

In 1956, R.E. Gomory presented a systematic procedure to find optimum integer solution to an "**all integer programming problem"**. Later he extended the method to deal with the more complicated case of **"mixed integer programming problems"** when some of the variables are required to be integer. These algorithms converge to the optimal integer solution in a finite number of iterations making use of familiar dual simplex method. This is called **"cutting plane algorithm"** because it introduces an idea of constructing "secondary" constraints which, when added to the optimal (non-integer) solution, will effectively cut the solution space towards the required result.

Another important approach, called the **"branch and bound"** technique for solving both the all integer and the mixed integer programming problems, has originated the straight forward idea of finding all feasible integer solutions.

"Branch-and-bound" technique was developed by A.H. Land and A.G. Doig (1960). This technique for solving both the all integer and the mixed integer problems, has orginiated the straight forward idea of finding all feasible integer solutions. Egon Balas (1965) introduced an interasting enumerative algorithm for linear programming problem with the variables having the value zero or one, called the zero one programming problem.

Several algorithms have been developed to solve linear integer programming problems. In this unit we discuss Gomory's cutting plane method, and in the next unit we will discuss branch and bound method.

3.2 Importance of Integer Programming Problems

We know that most industrial applications of large scale programming models are oriented towards planning decisions. There are frequently occuring circumstances in business and industry that lead to planning models involving integer valued variables. For example, in production, manufacturing is frequently scheduled in term of batches, lots or runs. In allocation of goods, a shipment must involve a diserete number of trucks, freight, cars or aircrafts. In such cases, the fractional value of the variables may be meaningless in context of the actual dicision problem. For example it is not possible to use 3.5 boilers in a thermal power station, 9.4 men in a project or 4.6 lathes in a workshop.

3.3 Necessity of Integer Programming

We can think that it is sufficient to obtain an integer solution to a given linear programming problem by first obtaining the non-integer optimal solution using simplex method (or graphical method for two variables problems) and then rounding off the fractional values of decision variables occuring in the optimal solution. But, in some cases, the deviation from the "exact" optimal integer values (obtained as a result of rounding) may become large enough to give an infeasible solution. Hence it was necessary to develop a systametic procedure to determine optimal integer solution to such problems. The following example will give more clarity of the concept.

Example : Consider an I.P.P.

Max $Z = 10x_1 + 4x_2$, subject to the constraints.

$3x_1 + 4x_2 \le 8$, $x_1, x_2 \ge 0$ and x_1, x_2 are integers.

Ignoring the integer restriction we obtain the optimal solution :

$x = 2\frac{2}{3}$, $x_2 = 0$, Max $Z = 26\frac{2}{3}$ by using graphical method. By rounding off the fractional value of $x = 2\frac{2}{3}$, the optimum solution becomes $x_1 = 3$, $x_2 = 0$ with Max $Z = 30$. But this solutions does not satisfy the constraints $3x_1 + 4x_2 \le 8$ and thus this solution is not feasible.

Now again, if we round off the solution to $x_1 = 2$, $x_2 = 0$ obviously this is the feasible solution and also integer valued. But this solution gives $Z = 20$ which is far away from the optimum value of $Z = 26\frac{2}{3}$. So, this is another disadvantage of getting an integer valued solution by rounding off the exact optimum solution. Still there is no guarantee that the "rounding down" solution will be optimal one. Thus a systematic procedure to find an exact optimum integer solution to the integer programming problems is needed.

3.4 Definitions

Integer Programming Problem (I.P.P.) : A linear programming problem :

Max $Z = cx$, subject to $A\overline{X} = b$, $\overline{X} \geq 0$ and some $x_j \in X$ are integers, where C, $X \in R^n$, $b \in R^m$ and A is an $m \times n$ real matrix, is called integer programming problem (I.P.P.).

All Integer Programming Problem (All I.P.P.) : An integer programming problem is said to be an "All Integer Programming Problem" if all $x_j \in X$ are integers.

Mixed Integer Programming Problem (Mixed I.P.P.) : An integer programming problem is said to be "Mixed Integer Programming Problem" if not all $x_j \in X$ are integers.

3.5 Gomory's All I.P.P. Method

Consider a pure linear integer programming problem. First we find optimal solution using regular simplex method ignoring integer valued restriction. Then we observe the following :

(i) If all the variables is the optimum solution thus obtained have integer values, then the current solution will be the desired integer solution.

(ii) If not, the considered l.p.p. requires a modification by introducing secondary constraints (also called Gomory's constraint) that reduces some of the non-integer values of variables to integer one, but does not eliminate any feasible integer.

(iii) Now the optimum solution to this modified l.p.p. is obtained by using any standard algorithm. If all the variables in this solution are integers, then the opotimal integer solution is obtained. Otherwise another secondary constraint is added to the l.p.p. and the whole procedure is repeated.

Thus the optimum integer solution will be obtained definitely after introducing the sufficient number of new constraints. The main work in this method is to construct Gomory's secondary constraints. Now we will discuss the method to construct this secondary construct.

3.6 Construction of Gomory's Constraint

The procedure to construct a secondary constraint is based on the fact that a solution which satsifies the constraint in the I.P.P. (3.4), also satisfies any other derived constraint obtained by employing only row transformation (adding or subtracting two or more constraints or multiply a constraint by non-zero number).

Thus if $\sum_{j=1}^{n} a_j x_j = b$ ...(1)

is any such constraint (obtained by employing row transformations only) then any feasible solution of the problem will also satisfy (1)

Before going further we discuss some rotations as : [p] denotes the integral part and f is fractional part of a number p, where $0 \leq f < 1$,

thus $p = [p] + f$

For example $5\frac{2}{3} = 5 + \frac{2}{3} \Rightarrow \left[5\frac{2}{3}\right] = 5$ and $f = \frac{2}{3}$

and $-5.2 = -6 + 0.8 \Rightarrow [-5.2] = -6$ and $f = 0.8$

using these rotations, let

$$a_j = [a_j] + f_j \quad , \qquad b = [b] + f$$

$$0 \le f_j < 1 \qquad\qquad 0 \le f < 1$$

where f_j and f represent the positive fractional parts of a_j and b respectively. Substituting these values in (1), we get

$$\sum\left([a_j] + f_j\right)x_j = [b] + f$$

$$\Rightarrow \quad \sum f_j x_j - f = [b] - \sum [a_j] x_j \qquad \text{...(2)}$$

Let $h = -\sum f_j x_j + f$ and suppose $h \ge 0$, then since R.H.S. in integer valued so left side must, which shows that $h \ge 1 \Rightarrow f = h + \sum f_j x_j \ge 1$

which constradicts that $0 \le f < 1$

$$\Rightarrow \quad h \ngtr 0 \Rightarrow h \le 0$$

$$\Rightarrow \quad -\sum f_j x_j + f \le 0$$

$$\Rightarrow \quad -\sum f_j x_j \le -f \qquad \text{...(3)}$$

This inequality can be converted into an equation by introducing slack variable x_s, then (3) becomes

$$-\sum f_j x_j + x_s = -f \qquad \text{...(4)}$$

This is the Gomory's secondary constraint and it is introduced in the given problem to form a new l.p.p.

To understand the process more precisely, suppose that in the optional solution of the I.P.P. by simplex method one basic variable, say X_{B_r} (in the r^{th} row) is not an integer. Let $x_{B_r} = x_1(say) = 3\frac{3}{4}$.

Now suppose that in the optimal tableau of the simplex method, the equation corresponding to r^{th} row, in which $x_1 = 3\frac{3}{4}$ occurs, is

$$x_1 + 1\frac{2}{3}x_2 + \frac{5}{3}x_3 - x_4 - 2\frac{1}{3}x_5 = 3\frac{3}{4}$$

This can be written as

$$(1+0)x_1+\left(1+\frac{2}{3}\right)x_2+\left(1+\frac{2}{3}\right)x_3+(-1+0)x_4+\left(-3+\frac{2}{3}\right)x_5=3+\frac{3}{4}$$

$$\Rightarrow \quad \frac{2}{3}x_2+\frac{2}{3}x_3+\frac{2}{3}x_5=\frac{3}{4}+\left[3-x_1-x_2-x_3+x_4+3x_5\right]$$

$$\left[as \;\; +x_1+x_2+x_3-x_4-3x_5\le 3\right]$$

$$\Rightarrow \quad \frac{2}{3}x_2+\frac{2}{3}x_3+\frac{2}{3}x_5\ge\frac{3}{4}$$

$$\Rightarrow \quad -\frac{2}{3}x_2-\frac{2}{3}x_3-\frac{2}{3}x_3\le-\frac{3}{4}$$

$$\Rightarrow \quad -\frac{2}{3}x_2-\frac{2}{3}x_3-\frac{2}{3}x_s+x=-\frac{3}{4}$$

where x_s is a slack variables.

This is the required Gomory's secondary constraint which can be amended to the given I.P.P.

3.7 All I.P.P. Algorithm or Cutting Plane Algorithm

The step by step procedure for the solution of all integer programming problem is as follows :

Step 1 : If the I.P.P. is in minimization form, convert it into maximization form.

Step 2 : Convert all inequality constraints into equalities by introducing slack or surplus variables, if necessary. Now obtain the optimum solution of l.p.p. ignoring integers restrictions by usual simplex method.

Step 3 : Test integrality of the optimum solution thus obtained in step 2.

(i) If an optimum solution contains all the variables have integer values, then an optimum integer basic feasible solution has been achieved.

(ii) If not, go to next step.

Step 4 : If only one variable has the fractional value, then corresponding to the row in which the fractional variables lies in the optimal table of step 2, form a secondary constrant of the form (4).

However if more than one variables are fractional, then select that variable which has largest fractional part.

Step 5 : Modify the l.p.p. by introducing the secondary constraint formed in step 4. Then find the new optimal solution of the modified l.p.p. by the dual simplex algorithm.

Step 6 : If the optimal solution thus obtained is integer valued, then this is the required optimal solution of the original l.p.p. otherwise go to step 4 and modify the l.p.p. by a new contraint. Repeating the process iteratively can definitely obtain the required optimum solution of the l.p.p.

This method is known as cutting plane method as the secondary constraints cut the unuseful area of the feasible region in the graphical solution of the problem i.e. cut that area which has no integer valued feasible solution. Thus these secondary constraints eliminate all the non integer solution without loosing any integer valued solution.

3.8 Illustrative Examples

Example 1. Find the optimum integer solution to the l.p.p.

$$Max\ z = x_1 + 2x_2$$

S.t. $2x_2 \le 7$

$x_1 + x_2 \le 7$

$2x_1 \le 11$

x_1, x_2 are integers and ≥ 0

Solution : First we solve the given l.p.p. using simplex method by ignoring integer restrictions. For this we write it in standard form. Introducing slack variables x_3, x_4, x_5 in the constraints, the problem becomes

$$Max\ z = x_1 + 2x_2 + 0.x_3 + 0.x_4 + 0.x_5$$

s.t. $2x_2 + x_3 = 7$

$x_1 + x_2 + x_4 = 7$

$2x_1 + x_5 = 11$

$x_1, x_2, x_3, x_4, x_5 \ge 0$

Taking initial BFS as $x_1 = x_2 = 0$

$x_3 = 7, x_4 = 7, x_5 = 11$

Simplex Table - 1

			c_j	1	2	0	0	0	$\theta = \frac{x_{Bi}}{y_{ik}},\ y_{ik>0}$
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	
0	α_3	x_3	7	0	[2]	1	0	0	$\frac{7}{2} \rightarrow$
0	α_4	x_4	7	1	1	0	1	0	$\frac{7}{1}$
0	α_5	x_5	11	2	0	0	0	1	--
		$z_j - c_j$		-1	-2 ↑	0	0	0	$\min\theta = \frac{7}{2}(\alpha_3)$

Simplex Table - 2

			c_j	1	2	0	0	0	$\theta = \frac{x_{Bi}}{y_{ik}}, y_{ik>0}$
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	
2	α_2	x_2	$\frac{7}{2}$	0	1	$\frac{1}{2}$	0	0	---
0	α_4	x_4	$\frac{7}{2}$	1	0	$-\frac{1}{2}$	1	0	$\frac{7}{2}/1 \rightarrow$
0	α_5	x_5	11	2	0	0	0	1	$\frac{11}{2}$
		$z_j - c_j$		-1 ↑	0	1	0	0	$\min \theta = \frac{7}{2}$

Simplex Table - 3

			c_j	1	2	0	0	0	$\theta = \frac{x_{Bi}}{y_{ik}}, y_{ik>0}$
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	
2	α_2	x_2	$\frac{7}{2}$	0	1	$\frac{1}{2}$	0	0	
1	α_1	x_1	$\frac{7}{2}$	1	0	$-\frac{1}{2}$	1	0	
0	α_3	x_s	4	2	0	1	-2	1	
		$z_j - c_j$		0 ↑	0	$\frac{1}{2}$	1	0	$\min \theta =$

Since all $z_j - c_j \geq 0$, so this BFS is optimal one, which is $x_1 = 3\frac{1}{2}, x_2 = 3\frac{1}{2}$

This solution does not satisfy the integer restrictions. To obtain this, we use Gomory's cutting plane algorithm. In the above solution, two variables x_1 and x_2 are involving the fractional parts, but both have equal fractional part $\frac{1}{2}$. Let us choose the first row, as source row to form the Gomory's secondary constraint.

The corresponding equation

$$0.x_1 + 1.x_2 + \frac{1}{2}x_3 + 0x_4 + 0.x_5 = \frac{7}{2}$$

or $\quad x_2 + \left(0 + \frac{1}{2}\right)x_3 \qquad = 3 + \frac{1}{2}$

or $\quad \frac{1}{2}x_3 = \frac{1}{2} + (3 - x_2)$

$\Rightarrow \quad \frac{1}{2}x_3 \geq \frac{1}{2}$

$\Rightarrow \quad -\frac{1}{2}x_3 \leq -\frac{1}{2}$

$\Rightarrow \quad -\frac{1}{2}x_3 + x_{s1} = -\frac{1}{2}$

which is Gomory's secondary constraint. Now introducing this constraint in the above optimum table (third table), we get the new table as :

Simplex Table - 4

			C_j	1	2	0	0	0	0	
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_{s1}	
2	α_2	x_2	$\frac{7}{2}$	0	1	$\frac{1}{2}$	0	0	0	
1	α_1	x_1	$\frac{7}{2}$	1	0	$-\frac{1}{2}$	1	0	0	
0	α_3	x_3	4	0	0	1	-2	1	0	
0	y_{s1}	x_{s1}	$-\frac{1}{2}$	0	0	$\boxed{-\frac{1}{2}}$	0	0	1	$\rightarrow$
$z_j - c_j$				0	0	$\frac{1}{2}$	1	0	0	
Max $y_{ij} < 0 \left(\frac{z_j - c_j}{y_{ij}}\right)$				–	–	$\frac{1}{2} \Big/ -\frac{1}{2}$	0	0	0	

Here one variable is negative i.e. the present basic solution is not feasible, so to make it feasible we use dual simplex algorithm.

(i) Since min $x_{B_i} = -\frac{1}{2}$ (for x_{s1}) so we delete x_{s1} from the basis.

(ii) Now $\max\limits_{y_{ij}<0}\left\{\frac{z_j - c_j}{y_{ij}}\right\} = \max\left\{\frac{\frac{1}{2}}{-\frac{1}{2}}\right\} = \frac{z_3 - c_3}{y_{43}}$

$\Rightarrow$ we must enter α_3 vector into the basis.

New simplex Table-5 is as follows :

			c_j	1	2	0	0	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_{s_1}
2	α_2	x_2	3	0	1	0	0	0	1
1	α_1	x_1	4	1	0	1	0	0	-1
0	α_5	x_5	1	0	0	0	0	1	-2
0	α_3	x_3	3	0	0	2	1	0	2
$z_j - c_j$				0	0	1	0	0	1

$\because \quad x_{Bi} \geq 0, \ \forall i$

Thus the above Basic solution is feasible and optimum. i.e,

$x_1 = 4, \ x_2 = 3$

It also satisfies integerality condition, so it is a desired optimal integer solution,

Example 2 : Find the optimum integer solution to the l.p.p.

Max $\quad Z = 3x_1 + 4x_2$

s.t. $\quad 3x_1 + 2x_2 \leq 8$

$x_1 + 4x_2 \leq 10$

$x_1, x_2 \geq 0$, and are integers.

Solution : Introducing slack variables x_3, x_4 the standard form of l.p.p. is

Max $\quad Z = 3x_1 + 4x_2 + 0x_3 + 0x_4$

s.t. $\quad 3x_1 + 2x_2 + x_3 = 8$

$x_1 + 4x_2 + x_4 = 10$

$x_1, x_2, x_3, x_4 \geq 0$

initial B.F.S. is $x_1 = 0 = x_2, \ x_3 = 8, \ x_4 = 10$

Simplex Table - 1

			C_j	3	4	0	0	$\theta = \frac{x_{Bi}}{y_{ik}}$, $y_{ik} > 0$
C_B	B	X_B	b	y_1	y_2	y_3	y_4	
0	α_3	x_3	8	3	2	1	0	$\frac{8}{2}$
0	α_4	x_4	10	1	$\boxed{4}$	0	1	$\frac{10}{4}$ $\rightarrow$
		$Z_j - C_j$		-3	-4 $\uparrow$	0	0	$\min\theta = \frac{10}{4}$

Simplex Table - 2

			C_j	3	4	0	0	$\theta = \frac{x_{Bi}}{y_{ik}}$, $y_{ik} > 0$
C_B	B	X_B	b	y_1	y_2	y_3	y_4	
0	α_3	x_3	3	$\boxed{\frac{5}{2}}$	0	1	$-\frac{1}{2}$	$\frac{6}{5}$ $\rightarrow$
4	α_2	x_2	$\frac{5}{2}$	$\frac{1}{4}$	1	0	$\frac{1}{4}$	10
		$Z_j - C_j$		-2 $\uparrow$	0	0	1	$\min\theta = \frac{6}{5}$

Simplex Table - 3

			C_j	3	4	0	0	
C_B	B	X_B	b	y_1	y_2	y_3	y_4	
3	α_1	x_1	$\frac{6}{5}$	1	0	$\frac{2}{5}$	$-\frac{1}{5}$	
4	α_2	x_2	$\frac{11}{5}$	0	1	$-\frac{1}{10}$	$\frac{3}{10}$	
		$Z_j - C_j$		0	0	$\frac{4}{5}$	$\frac{3}{5}$	

$\because Z_j - C_j \geq 0, \forall j$, therefore optimal non integer solution is $x_1 = \frac{6}{5} = 1\frac{1}{5}$, $x_2 = \frac{11}{5} = 2\frac{1}{5}$

Now, we introduce Gomory's secondary constraint.

The fractional parts of the two variables are same $\left(\frac{1}{5}\right)$, we choose the second row as source row.

$$(0+0)x_1+(1+0)x_2+\left(1+\frac{9}{10}\right)x_3+\left(0+\frac{3}{10}\right)x_4=2+\frac{1}{5}$$

The Gomory's constraint

$$\frac{9}{10}x_3+\frac{3}{10}x_4\geq\frac{1}{5}$$

$$\Rightarrow \qquad -\frac{9}{10}x_3-\frac{3}{10}x_4+x_{S_1}=-\frac{1}{5}$$

The simplex table for modified l.p.p. is as follows :

Simplex Table - 4

			C_j	3	4	0	0	0	
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}	
3	α_1	x_1	$6/5$	1	0	$2/5$	$-1/5$	0	
4	α_2	x_2	$11/5$	0	1	$-1/10$	$3/10$	0	
0	y_{s1}	x_{s1}	$-\frac{1}{5}$	0	0	$\boxed{-9/10}$	$-3/10$	1	→
		Z_j-C_j		0	0	$4/5$	$3/5$		
		$\underset{y_{3j}<0}{Max}\frac{z_j-c_j}{y_{3j}}$		–	–	$\frac{4/5}{-9/10}$ ↑	$\frac{3/5}{-3/10}$		

Here we use dual simplex alogrithm and take x_{s1} as deleting variable and x_3 as entering variable. The next iterative table is as follows :

Simplex Table - 5

			C_j	3	4	0	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}
3	α_1	x_1	$10/9$	1	0	0	$-\frac{1}{3}$	$4/9$
4	α_2	x_2	$20/9$	0	1	0	$\frac{1}{3}$	$-\frac{1}{9}$
0	α_3	x_3	$2/9$	0	0	1	$\frac{1}{3}$	$-\frac{10}{9}$
		Z_j-C_j		0	0	0	$\frac{1}{3}$	$8/9$

Still the optimal solution is not integer, so a new secondary constraint must be added. Choose second row as source row we get the new constraint $-\frac{1}{3}x_4 - \frac{8}{9}x_{s1} + x_{s2} = -\frac{2}{9}$

Introducing this constraint the modified table is

Simplex Table - 6

			C_j	3	4	0	0	0	
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}	y_{s2}
3	α_1	x_1	$\frac{10}{9}$	1	0	0	$-\frac{1}{3}$	$\frac{4}{9}$	0
4	α_2	x_2	$\frac{20}{9}$	0	1	0	$\frac{1}{3}$	$-\frac{1}{9}$	0
0	α_3	x_3	$\frac{2}{9}$	0	0	1	$\frac{1}{3}$	$-\frac{10}{9}$	0
0	y_{s2}	x_{s2}	$-\frac{2}{9}$	0	0	0	$-\frac{1}{3}$	$-8/9$	1 →
$Z_j - C_j$				0	0	0	$\frac{1}{3}$	$8/9$	0
$\underset{y_{4j}<0}{Max} \frac{Z_j - C_j}{y_{4j}}$				–	–	–	$\frac{1}{3} \Big/ \left(-\frac{1}{3}\right)$ ↑	$\frac{8/9}{-8/9}$	–

Now deleting x_{s2} and introducing α_4, by dual simplex algorithm, we get the next iterative table as follows :

Simplex Table - 7

			b	3	4	0	0	0	0
C_B	B	X_B	C_j	y_1	y_2	y_3	y_4	y_{s1}	y_{s2}
3	α_1	x_1	$4/3$	1	0	0	0	$4/3$	-1
4	α_2	x_2	2	0	1	0	0	-1	1
0	α_3	x_3	0	0	0	1	0	-2	1
0	α_4	x_4	$2/3$	0	0	0	1	$8/3$	-3
$Z_j - C_j$				0	0	0	0	0	1

Still this optimal solution does not satisfy integer restriction as $x_1 = \frac{4}{3}$ is fractional. Taking the fourth row as source row the Gomory's constraint is

$$-\frac{2}{3}x_{s1} + x_{s2} = -\frac{2}{3}$$

Introducing this in the above table 7, we get the modified table as

Simplex Table - 8

			C_j	3	4	0	0	0	0	0	
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}	y_{s2}	y_{s3}	
3	α_1	x_1	$4/3$	1	0	0	0	$4/3$	-1	0	
4	α_2	x_2	2	0	1	0	0	-1	1	0	
0	α_3	x_3	0	0	0	1	0	-2	1	0	
0	α_4	x_4	$2/3$	0	0	0	1	$8/3$	-3	0	
0	y_{s1}	x_{s3}	$-2/3$	0	0	0	0	$\boxed{-\frac{2}{3}}$	0	1	→
$Z_j - C_j$				0	0	0	0	0	1	0	
$\underset{y_{sj}<0}{Max}\frac{Z_j - C_j}{y_{sj}}$				–	–	–	–	$\frac{0}{-2/3}$ ↑	–	–	

Now deleting x_{s3} and introducing x_{s1}, we get the next iteration tableau as follows :

Simplex Table - 9

			C_j	3	4	0	0	0	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}	y_{s2}	y_{s3}
3	α_1	x_1	0	1	0	0	0	1	-1	2
4	α_2	x_2	3	0	1	0	0	0	1	$-3/2$
0	α_3	x_3	2	0	0	1	0	0	1	-3
0	α_4	x_4	2	0	0	0	1	0	-3	4
0	y_{s1}	x_{s1}	1	0	0	0	0	1	0	$-3/2$
$Z_j - C_j$				0	0	0	0	0	1	0

Obviously this optimal solution is the required integral solution, which is as follows :

$x_1 = 0$, $x_2 = 3$, Max $Z = 12$

Example 3 : Solve the following integer programming problem :

$$\text{Max} \quad Z = 2x_1 + 10x_2 - 10x_3$$

$$\text{s.t.} \quad 2x_1 + 20x_2 + 4x_3 \le 15$$

$$6x_1 + 20x_2 + 4x_3 = 20$$

$x_1, x_2, x_3 \ge 0$ and integers.

Solve the problem as a (continuous) linear program, then show that it is impossible to obtain feasible integer solution by using simple rounding. Solve the problem using any integer program algorithm.

Solution : Ignoring the integer restrictions, on solving the problem by simplex table, we get the following optimum table :

Simplex Table - 1

			C_j	2	20	-10	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4
20	α_1	x_2	$\frac{5}{8}$	0	1	$\frac{1}{5}$	$\frac{3}{40}$
2	α_2	x_1	$\frac{5}{4}$	1	0	0	$-\frac{1}{4}$
		$Z_j - C_j$		0	0	14	1

Where x_4 is a slack variable and α_4 is the associated vector. The optimum solution is $x_1 = \frac{5}{4}$, $x_2 = \frac{5}{8}$, $x_3 = 0$. The simple rounding reduces to $x_1 = 1$, $x_2 = 0$, $x_3 = 0$ and it does not satisfy the second constraint. Instead, if we take $x_1 = 1$, $x_2 = 1$, $x_3 = 0$ or $x_1 = 2$, $x_2 = 0$, $x_3 = 0$, $x_1 = 2$, $x_2 = 1$, $x_3 = 0$ even then these solutions do not satisfy the constraints. Hence by simple rounding, we cannot obtain an integral solution of the given problem.

Now we use Gomory's cutting plane algorithm to obtain the desired integer solution.

Note that two variables are non-integer and maximum fractional part is $\frac{5}{8}$ (of x_2). So we choose the first row (in which x_2 is available) as a source row for the secondary constraint

$$(0+0)x_1 + (1+0)x_2 + \left(0 + \frac{1}{5}\right)x_3 + \left(0 + \frac{3}{40}\right)x_4 = 0 + \frac{5}{8}$$

$$\Rightarrow \quad \frac{1}{5}x_3 + \frac{3}{40}x_4 = \frac{5}{8} - x_2$$

$$\Rightarrow \quad \frac{1}{5}x_3 + \frac{3}{40}x_4 \geq \frac{5}{8}$$

$$\text{or} \quad -\frac{1}{5}x_3 - \frac{3}{40}x_4 \leq -\frac{5}{8}$$

$$\Rightarrow \quad -\frac{1}{5}x_3 - \frac{3}{40}x_4 + x_{s1} = -\frac{5}{8}$$

Introducing this constraint in the above table we obtain modified table as follows :

Simplex Table - 2

			C_j	2	20	-10	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}
20	α_2	x_2	$\frac{5}{8}$	0	1	$\frac{1}{5}$	$\frac{3}{40}$	0
2	α_1	x_1	$\frac{5}{8}$	1	0	0	$-\frac{1}{4}$	0
0	y_{s1}	x_{s1}	$-\frac{5}{8}$	0	0	$-\frac{1}{5}$	$\boxed{-\frac{3}{40}}$	0 →
		$Z_j - C_j$		0	0	14	1	0
		$\underset{y_{zj} \leq 0}{Max}\left\{\frac{Z_j - C_j}{y_{zj}}\right\}$		–	–	$\frac{14}{-1/5}$	$\frac{1}{-3/40}$ ↑	–

The next iterative table is as : (deleting y_{s1} and entering α_4)

Simplex Table - 3

			C_j	2	20	-10	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}
20	α_2	x_2	0	0	1	0	0	1
2	α_1	x_1	$\frac{10}{3}$	1	0	$2/3$	0	$-\frac{10}{3}$
0	α_4	x_4	$\frac{25}{3}$	0	0	$8/3$	1	$-\frac{40}{3}$
		$Z_j - C_j$		0	0	$\frac{34}{3}$	0	$\frac{40}{3}$

Still this optimal solution does not satisfying integer constraint as $x_1 = \frac{10}{3}$, $x_4 = \frac{25}{3}$, so again one secondary constraint is to be introduced.

Since both the variables have same fractional parts so we can take randomly third row as source row.

$$(0+0)x_1 + (0+0)x_2 - \left(2+\frac{2}{3}\right)x_3 + (1+0)x_3 + (1+0)x_4 + \left(-14+\frac{2}{3}\right)x_{s1} = \left(8+\frac{1}{3}\right)$$

$$\Rightarrow \quad \frac{2}{3}x_3 + \frac{2}{3}x_{s1} = \frac{1}{3} + (8 - 2x_3 - x_4 + 14x_{s1})$$

$$\Rightarrow \quad \frac{2}{3}x_3 + \frac{2}{3}x_{s1} \geq \frac{1}{3}$$

$$\text{or} \quad -\frac{2}{3}x_3 - \frac{2}{3}x_{s1} \leq -\frac{1}{3}$$

$$\Rightarrow \quad -\frac{2}{3}x_3 - \frac{2}{3}x_{s1} + x_{s2} = -\frac{1}{3}$$

Which is the secondary constraint. Adding this constraint in the last table, we get the modified table as follows :

Simplex Table - 4

			C_j	2	20	-10	0	0	0	
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}	y_{s2}	
20	α_2	x_2	0	0	1	0	0	1	0	
2	α_1	x_1	$10/3$	1	0	$2/3$	0	$-10/3$	0	
0	α_4	x_4	$25/3$	0	0	$8/3$	1	$-40/3$	0	
0	y_{s1}	x_{s2}	$-1/3$	0	0	$\boxed{-2/3}$	0	$-2/3$	1	$\rightarrow$
		$Z_j - C_j$		0	0	$34/3$	0	$40/3$	0	
$\underset{y_{4j}<0}{Max}\left(\frac{Z_j - C_j}{y_{4j}}\right)$				–	–	$\frac{34/3}{-2/3}$ $\uparrow$	–	$\frac{40/3}{-2/3}$	–	

Next iterative table is as follows :

Simplex Table - 5

			C_j	2	20	−10	0	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}	y_{s2}
20	α_2	x_2	0	0	1	0	0	1	0
2	α_1	x_1	3	1	0	0	0	−4	1
0	α_4	x_4	7	0	0	0	1	−16	4
−10	α_3	x_3	$\frac{1}{2}$	0	0	1	0	1	$-3/2$
		$Z_j - C_j$		0	0	0	0	2	17

Still the solution does not satisfy the integral restriction and so one more Gomory's constraint will be introduced. We take fourth row as source row which gives

$$x_3 + x_{s1} - \frac{3}{2}x_{s2} = \frac{1}{2}$$

$$\Rightarrow \quad (0+0)x_1 + (0+0)x_{s1} + \left(-2+\frac{1}{2}\right)x_{s2} = \left(0+\frac{1}{2}\right)$$

$$\Rightarrow \quad \frac{1}{2}x_{s2} \geq \frac{1}{2}$$

or $\quad -\frac{1}{2}x_{s2} \leq -\frac{1}{2}$

$$\Rightarrow \quad -\frac{1}{2}x_{s2} + x_{s3} = -\frac{1}{2}$$

Now introducing this secondary constraint in the last table as follows :

Simplex Table - 6

			C_j	2	20	−10	0	0	0	0	
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}	y_{s2}	y_{s3}	
20	α_2	x_2	0	0	1	0	0	1	0	0	
2	α_1	x_1	3	1	0	0	0	4	1	0	
0	α_4	x_4	7	0	0	0	1	−16	4	0	
−10	α_3	x_3	$\frac{1}{2}$	0	0	1	0	1	$-\frac{3}{2}$	0	
0	y_{s2}	x_{s2}	$-\frac{1}{2}$	0	0	0	0	0	$\boxed{-\frac{1}{2}}$	1	→
		$Z_j - C_j$		0	0	0	0	2	17	0	
	$\underset{y_{sj}<0}{Max}\left(\frac{Z_j - C_j}{y_{sj}}\right)$			–	–	–	–	–	$\frac{17}{-1/2}$ ↑	–	

Simplex Table - 7

		C_j		2	20	−10	0	0	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}	y_{s2}	y_{s3}
20	α_2	x_2	0	0	1	0	0	1	0	0
2	α_1	x_1	2	1	0	0	0	−4	0	2
0	α_4	x_4	3	0	0	0	1	−16	0	8
−10	α_3	x_3	2	0	0	1	0	1	0	−3
0	y_{s2}	x_{s2}	1	0	0	0	0	0	1	−2
	$z_j - c_j$			0	0	0	0	2	0	34

Above optimum solution is integer one, so required solution is

$x_1 = 2$, $x_2 = 0$, $x_3 = 2$, Max $Z = -16$

Example 4 : A manufacturer of baby-doll makes two types of dolls, doll x and doll y. Processing of these two dolls is done on two machines, A and B, Doll x requires two hours on machine A and 6 hours on machine B. Doll y requires 5 hours on machine A and also five hours on machine B. There are sixteen hours of time per day available on machine A and thirty hours on machine B. The profit gained on both the dolls is same, i.e., one rupee per doll. What should be the daily production of the two dalls for maximum profit?

(a) Set up and solve the l.p.p.

(b) If the optimum solution is not integer valued, use the Gomory's technique to derive the optimal solution.

Solution : Let x_1, x_2 denote the number of dolls manufactured per day of type x and y respectively, then the corresponding l.p.p. is formulated as follows :

Max $Z = x_1 + x_2$

s.t. $2x_1 + 5x_2 \le 16$

$6x_1 + 5x_2 \le 30$, $x_1, x_2 \ge 0$, are integers.

Introducing slack variables x_3, x_4 and solving the problem by simplex method, the optimal table giving the optimal solution is as follows :

Simplex Table - 1

		C_j		1	1	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4
1	α_2	x_2	$\frac{9}{5}$	0	1	$3/10$	$-1/10$
1	α_1	x_1	$\frac{7}{2}$	1	0	$-\frac{1}{4}$	$\frac{1}{4}$
	$Z_j - C_j$			0	0	$\frac{1}{20}$	$\frac{3}{20}$

Here both the variables are fractional. But their fractional parts are $4/5$ and $\frac{1}{2}$. Out these $\frac{4}{5}$ is largest which is of x_2 lying in the first row of the table. Taking first row as the source row, the corresponding equation is

$$(0+0)x_1+(1+0)x_2+\left(0+\frac{3}{10}\right)x_3+\left(-1+\frac{9}{10}\right)x_4=1+\frac{4}{5}$$

$$\Rightarrow \quad \frac{3}{10}x_3+\frac{9}{10}x_4\ge\frac{4}{5}$$

or $$-\frac{3}{10}x_3-\frac{9}{10}x_4\le-\frac{4}{5}$$

Hence the Gomory's constraint is

$$-\frac{3}{10}x_3-\frac{9}{10}x_4+x_{s1}=-\frac{4}{5}$$

where x_{s1} is a slack variable. The modified table is

Simplex Table - 2

			C_j	1	1	0	0	0	
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}	
1	α_2	x_2	$\frac{9}{5}$	0	1	$\frac{3}{10}$	$-\frac{1}{10}$	0	
1	α_1	x_1	$\frac{7}{2}$	1	0	$-\frac{1}{4}$	$\frac{1}{4}$	0	
0	y_{s1}	x_{s1}	$-\frac{4}{5}$	0	0	$\boxed{-\frac{3}{10}}$	$-\frac{9}{10}$	1	→
		Z_j-C_j		0	0	$\frac{1}{20}$	$3/20$	0	
		$\underset{y_{3j}<0}{Max}\left\{\frac{Z_j-C_j}{y_{3j}}\right\}$		–	–	$\frac{\frac{1}{20}}{-3/10}$ ↑	$\frac{+\frac{3}{20}}{-9/10}$		

Using dual simplex algorithm entering α_3 and removing x_{s1} from the basis, we get new table as follows :

Simplex Table - 3

			C_j	1	1	0	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}
1	α_2	x_2	1	0	1	0	-1	0
1	α_1	x_1	$25/6$	1	0	0	1	$-5/6$
0	α_3	x_3	$8/3$	0	0	1	3	$-10/3$
		$Z_j - C_j$		0	0	0	0	$1/6$

This optimum solution is still not integer. Again, we construct a Gomory's constraint. This time taking second row involving fractional variable $x_1 = \frac{25}{6}$, as a source row, we get the corresponding equation

$$(1+0)x_1 + (1+0)x_4 + \left(-1+\frac{1}{6}\right)x_{s1} = 4 + \frac{1}{6}$$

$$\Rightarrow \quad \frac{1}{6}x_{s1} \geq \frac{1}{6}$$

or $$-\frac{1}{6}x_{s1} \leq -\frac{1}{6}$$

$$\Rightarrow \quad -x_{s1} + x_{s2} = -1$$

Introducing this constraint in the last table, we have the modified table as :

Simplex Table - 4

			C_j	1	1	0	0	0	0	
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}	y_{s2}	
1	α_2	x_2	1	0	1	0	-1	1	0	
1	α_1	x_1	$25/6$	1	0	0	1	$-5/6$	0	
0	α_3	x_3	$8/3$	0	0	1	3	$-10/3$	0	
0	y_{s1}	x_{s1}	-1	0	0	0	0	$\boxed{-1}$	1	$\rightarrow$
		$Z_j - C_j$		0	0	0	0	$1/6$	0	
		$\underset{y_{4j}<0}{Max}\left\{\frac{Z_j - C_j}{y_{4j}}\right\}$		–	–	–	–	$\frac{\frac{1}{6}}{-1}$ $\uparrow$	–	

The next iterative table is

Simplex Table - 5

C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}	y_{s2}
			C_j	1	1	0	0	0	0
1	α_2	x_2	0	0	1	0	1	0	0
1	α_1	x_1	5	1	0	0	1	0	$-5/6$
0	α_3	x_3	6	0	0	1	3	0	$-10/3$
0	y_{s1}	x_{s1}	1	0	0	0	0	1	-1
		$Z_j - C_j$		0	0	0	0	0	$1/6$

This iterative optimal solution having integer value has been reached, which is as :

$x_1 = 5$, $x_2 = 0$ and Max $Z = 5$

3.9 Geometrical Interpretation of Gomory's Cutting Plane Method

We take last example 4 for the geometrical interpretation

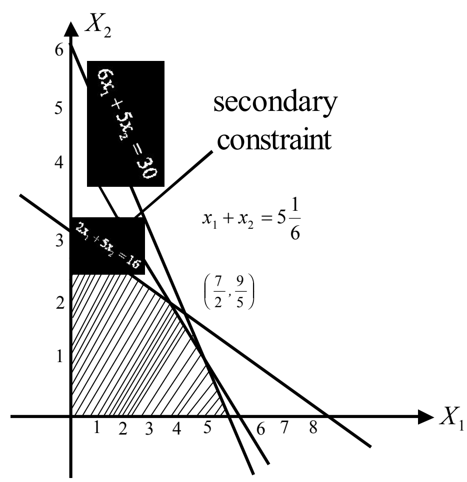


Figure 3.1

The feasible region, is as shown in the above fig. 3.1 Optimum solution $x_1 = \frac{7}{2}$, $x_2 = \frac{9}{5}$. Since the solution is not integer. We introduce first Gomory's constraint

$$\frac{3}{10}x_3 + \frac{9}{10}x_4 \geq \frac{4}{5}$$

To express this in terms of x_1 and x_2, we know that

$$2x_1 + 5x_2 + x_3 = 16$$

$$6x_1 + 5x_2 + x_4 = 30$$

as x_3 and x_4 are slack variables introduced in the begining to convert the inequalities into equations.

These give $\quad x_3 = 16 - 2x_1 - 5x_2$

and $\quad x_4 = 30 - 6x_1 - 5x_2$

substituting in the Gomory's constraint, we get

$$\frac{3}{10}(16 - 2x_1 - 5x_2) + \frac{9}{10}(30 - 6x_1 - 5x_2) \geq \frac{4}{5}$$

$$\Rightarrow \quad x_1 + x_2 \leq 5\frac{1}{6}$$

This constraint cuts-off some part of the feasible region (in this case very minute) and hence now the feasible region is some what less then the previous one (see fig.3.1). Similarly the second Gomory's constraint is $x_{s1} \geq 1$

But $\quad -\frac{3}{10}x_3 - \frac{9}{10}x_4 + x_{s1} = -\frac{4}{5} \quad$ or $\quad x_{s1} = \left(\frac{3}{10}x_3 + \frac{9}{10}x_4\right) - \frac{4}{5}$

$$\Rightarrow \quad x_{s1} = \frac{3}{10}(16 - 2x_1 - 5x_2) + \frac{9}{10}(33 - 6x_1 - 5x_2) - 4/5$$

$$\Rightarrow \quad x_{s1} = 31.8 - 6x_1 - 6x_2$$

$$\therefore \quad x_{s1} \geq 1 \Rightarrow \quad 31.8 - 6x_1 - 6x_2 \geq 1$$

$$\Rightarrow \quad 6x_1 + 6x_2 \leq 30.8$$

$$\Rightarrow \quad x_1 + x_2 \leq 5.103$$

This constraint also cut off some part of feasible region so why this is not plotted here. Due to these cuttings, the method is called cutting plane method.

Example 5 : Solve the integer programming problem :

Max $\quad Z = 7x_1 + 9x_2$

S.t. $\quad -x_1 + 3x_2 \leq 6$

$\quad 7x_1 + x_2 \leq 35$

$\quad x_1 \geq 0,\ x_2 \geq 0$ and x_1, x_2 are integers.

Solution : Introducing slack variables x_3 and x_4 and solving by simplex method, we get the optimal solution as follows :

Simplex Table - 1

			C_j	7	9	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4
9	α_2	x_2	$3\frac{1}{2}$	0	1	$7/22$	$1/22$
7	α_1	x_1	$4\frac{1}{2}$	1	0	$-1/22$	$3/22$
		$Z_j - C_j$		0	0	$28/11$	$15/11$

The non-integer solution thus obtained is :

$$x_1 = 4\frac{1}{2}, \qquad x_2 = 3\frac{1}{2}, \qquad \text{Max } Z = 63$$

Since both the variables have same fractional parts so the first constraint is choosen as the source row to make Gomory's constraint, which is as :

$$(0+0)x_1 + (1+0)x_2 + \left(0 + \frac{7}{22}\right)x_3 + \left(0 + \frac{1}{22}\right)x_4 = 3 + \frac{1}{2}$$

$$\Rightarrow \quad \frac{7}{22}x_3 + \frac{1}{22}x_4 \geq \frac{1}{2}$$

$$\text{or} \quad -\frac{7}{22}x_3 - \frac{1}{22}x_4 \leq -\frac{1}{2}$$

$$\Rightarrow \quad -\frac{7}{22}x_3 - \frac{1}{22}x_4 + x_{s1} = -\frac{1}{2}$$

with Gomory's secondary constraint introducing in the above table we get

Simplex Table - 2

			C_j	7	9	0	0	0	
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}	
9	α_2	x_2	$3\frac{1}{2}$	0	1	$\frac{7}{22}$	$\frac{1}{22}$	0	
7	α_1	x_1	$4\frac{1}{2}$	1	0	$-\frac{1}{22}$	$\frac{3}{22}$	0	
0	y_{s1}	x_{s1}	$-\frac{1}{2}$	0	0	$\boxed{-\frac{7}{22}}$	$-\frac{1}{22}$	1	→
		$Z_j - C_j$		0	0	$28/11$	$3/22$	0	
		$\underset{y_{3j}<0}{Max}\left\{\frac{Z_j - C_j}{y_{3j}}\right\}$		–	–	$\frac{28/11}{-7/22}$ ↑	$\frac{15/11}{-1/22}$		

Using dual simplex algorithm the next iterative table is as follows :

Simplex Table - 3

			C_j	7	9	0	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}
9	α_2	x_2	3	0	1	0	0	1
7	α_1	x_1	$4\frac{4}{7}$	1	0	0	$1/7$	$-\frac{1}{7}$
0	α_3	x_3	$1\frac{4}{7}$	0	0	1	$1/7$	$-22/7$
		$Z_j - C_j$		0	0	0	1	8

The above optimal solution still does not satisfy integer restriction. Choose second row as source row to construct Gomory's secondary constraint.

$$(0+1)x_1 + (0+0)x_2 + (0+0)x_3 + \left(0+\frac{1}{7}\right)x_4 + \left(-1+\frac{6}{7}\right)x_{s1} = 4 + \frac{4}{7}$$

$$\Rightarrow \quad \frac{1}{7}x_4 + \frac{6}{7}x_{s1} \geq \frac{4}{7}$$

$$\text{or} \quad -\frac{1}{7}x_4 - \frac{6}{7}x_{s1} \leq -\frac{4}{7}$$

$$\Rightarrow \quad -\frac{1}{7}x_4 - \frac{6}{7}x_{s1} + x_{s2} = -\frac{4}{7}$$

Introducing this constraint is the above table and applying dual simplex algorithm, we get the transformed table as below :

Simplex Table - 4

			C_j	7	9	0	0	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}	y_{s2}
9	α_2	x_2	3	0	1	0	0	1	0
7	α_1	1	$4\frac{4}{7}$	1	0	0	$1/7$	$-1/7$	0
0	α_3	x_3	$1\frac{4}{7}$	0	0	1	$1/7$	$-22/7$	0
0	y_{s2}	x_{s2}	$-\frac{4}{7}$	0	0	0	$\boxed{-1/7}$	$-6/7$	1 →
		$Z_j - C_j$		0	0	0	1	8	0
$\underset{y_{4j}<0}{Max}\left\{\frac{Z_j - C_j}{y_{4j}}\right\}$				–	–	–	$\frac{1}{-1/7}$ ↑	$\frac{8}{-6/7}$	–

The next iterative table is

Simplex Table - 5

			C_j	7	9	0	0	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}	y_{s2}
9	α_2	x_2	3	0	1	0	0	1	0
7	α_1	x_1	4	1	0	0	0	-1	1
0	α_3	x_3	1	0	0	1	0	-4	1
0	x_4	x_4	4	0	0	0	1	6	-7
		$Z_j - C_j$		0	0	0	0	2	7

In this optimal table all the variables have integer valued, so this is required optimal integer solution, which is as

$$x_1 = 4,\ x_2 = 3, \qquad \text{Max} \quad Z = 55$$

Example 6 : Find the optimum integer solution to the following I.P.P.

Max $\quad Z = x_1 + 4x_2$

S.t. $\quad 2x_1 + 4x_2 \le 7$

$5x_1 + 3x_2 \le 15$

$x_1, x_2 \ge 0$ and are integers.

Solution : Introducing slack variables x_3, x_4 and solving above problem by usual simplex method the optimum non-integer solution is given as follows :

Simplex Table - 1

			C_j	1	4	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4
4	α_2	x_2	$7/4$	$\frac{1}{2}$	1	$\frac{1}{4}$	0
0	α_4	x_4	$39/4$	$\frac{7}{2}$	0	$-\frac{3}{4}$	1
		$Z_j - C_j$		1	0	1	0

In the above solution both the variables have same fractional parts, so consider the first row as source row, which is

$$\left(0+\frac{1}{2}\right)x_1 + (1+0)x_2 + \left(0+\frac{1}{4}\right)x_3 + (0+0)x_4 = 1+\frac{3}{4}$$

$$\Rightarrow \quad \frac{1}{2}x_1 + \frac{1}{4}x_3 \geq \frac{3}{4}$$

$$\text{or} \quad -\frac{1}{2}x_1 - \frac{1}{4}x_3 \leq -\frac{3}{4}$$

$$\Rightarrow \quad -\frac{1}{2}x_1 - \frac{1}{4}x_3 + x_{s1} = -\frac{3}{4}$$

Introducing this secondary constraint in the above table, the modified table is as follows :

Simplex Table - 2

			C_j	1	4	0	0	0	
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}	
4	α_2	x_2	$\frac{7}{9}$	$\frac{1}{2}$	1	$\frac{1}{4}$	0	0	
0	α_4	x_4	$\frac{39}{4}$	$\frac{7}{2}$	0	$-\frac{3}{4}$	1	0	
0	y_{s1}	x_{s1}	$-\frac{3}{4}$	$\boxed{-\frac{1}{2}}$	0	$-\frac{1}{4}$	0	1	→
		$Z_j - C_j$		1	0	1	0	0	
		$\underset{y_{3j}<0}{Max}\left\{\frac{Z_j - C_j}{y_{3j}}\right\}$		$\frac{1}{-\frac{1}{2}}$ ↑	–	$\frac{1}{-1/4}$	–	–	

The next iterative table is as :

Simplex Table - 3

			C_j	1	4	0	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}
4	α_2	x_2	1	0	1	0	0	1
0	α_4	x_4	$\frac{9}{2}$	0	0	$-5/2$	1	7
1	α_1	x_1	$\frac{3}{2}$	1	0	$\frac{1}{2}$	0	-2
		$Z_j - C_j$		0	0	$\frac{1}{2}$	0	2

Since the optimum solution is still not integer valued, we introduce second Gomorian constraint taking second row as source row

$$(0+0)x_1+(0+0)x_2+\left(-3+\frac{1}{2}\right)x_3+(1+0)x_4+(1+0)y_{s1}=4+\frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{2}x_3 \geq \frac{1}{2} \quad \text{or} \quad -\frac{1}{2}x_3 \leq -\frac{1}{2}$$

$$\Rightarrow \quad -\frac{1}{2}x_3+x_{s2}=-\frac{1}{2}$$

Introducing this secondary constraint, the modified table is as :

Simplex Table - 4

			C_j	1	4	0	0	0	0	
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}	y_{s2}	
4	α_2	x_2	1	0	1	0	0	1	0	
0	α_4	x_4	$9/2$	0	0	$-5/2$	1	7	0	
1	α_1	x_1	$3/2$	1	0	$1/2$	0	-2	0	
0	y_{s2}	x_{s2}	$-\frac{1}{2}$	0	0	$-1/2$	0	0	1	→
		Z_j-C_j		0	0	$\frac{1}{2}$	0	2	0	
		$\underset{y_{4j}<0}{Max}\left\{\frac{Z_j-C_j}{y_{4j}}\right\}$		–	–	$\frac{\frac{1}{2}}{-\frac{1}{2}}$ ↑	–	–	–	

The next iterative table is as follows :

Simplex Table - 5

			C_j	1	4	0	0	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_{s1}	y_{s2}
4	α_2	x_2	1	0	1	0	0	1	0
0	α_4	x_4	7	0	0	0	1	7	-5
1	α_1	x_1	1	1	0	0	0	-2	1
0	α_3	x_3	1	0	0	1	0	0	-2
		Z_j-C_j		0	0	0	0	2	1

This table shows that an optimum basis feasible integer solution has been reached. Hence the optimum solution is

$$x_1=1,\ x_2=1, \quad \text{Max} \quad Z=5$$

3.10 Self-Learning Exercise - I

1. How can you construct Gomory's constraint?
2. Gomory's method to solve I.P.P. is called a cutting plane method, Why?
3. Give geometrical interpretation of Gomory's cutting plane algorithm?

3.11 Gomory's Mixed I.P.P. Method (Fractional Cut Method)

In the mixed integer programming problems some of the variables are restructed to take integer values, while other variables may take integer or continuous values. The iterative procedure to solve such programming problems is as follows :

Step 1 : Determine an optimum solution to the given l.p.p. using simplex method ignoring integer restrictions.

Step 2 : Test the integrality of the optimum solution thus obtained in step 1.

(i) If all the variables has integer values, then it the optimum integer solution.

(ii) If integer restricted variables are not integers go to next step.

Step 3 : Choose largest fractional value among the basic variables which are restricted to integers. Consider the row corresponding to above variable and form Gomory's secondary constraint.

Step 4 : Introducing this secondary constraint and modify the table, then apply dual simplex algorithm and follows the procedure as in all IPP method 3.7 until the restricted integer variables becomes integers.

Example 7 : Solve the following mixed integer programming problem :

Maximize $Z = 4x_1 + 6x_2 + 2x_3$

Subject to $4x_1 - 4x_2 \le 5$

$-x_1 + 6x_2 \le 5$

$-x_1 + x_2 + x_3 \le 5$

$x_1, x_2, x_3 \ge 0$ and x_1, x_3 are integers.

Solution : Introducing slack variables x_4, x_5 in first two constraints and solve the l.p.p. by usual simplex method ignoring integer restrictions, we have

Simplex Table - 1

			C_j	4	6	2	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5
4	α_1	x_1	$5/2$	1	0	0	$3/10$	$1/5$
6	α_2	x_2	$5/4$	0	1	0	$1/20$	$1/5$
2	α_3	x_3	$25/4$	0	0	1	$1/4$	0
		$Z_j - C_j$		0	0	0	2	2

$\because$ x_1, x_3 both are not integers and x_1 has maximum fractional part, so we take it (first) as source row which is

$$(1+0)x_1+(0+0)x_2+(0+0)x_3+\left(0+\frac{3}{10}\right)x_4+\left(0+\frac{1}{5}\right)x_5=2+\frac{1}{2}$$

$\Rightarrow$ $\frac{3}{10}x_4+\frac{1}{5}x_5 \geq \frac{1}{2}$

or $-\frac{3}{10}x_4-\frac{1}{5}x_5 \leq -\frac{1}{2}$

$\Rightarrow$ $-\frac{3}{10}x_4-\frac{1}{5}x_5+x_{s1}=-\frac{1}{2}$

where x_{s1} is a slack variable.

Introducing this second constraint, the modified table is :

Simplex Table - 2

			C_j	4	6	2	0	0	0	
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_{s1}	
4	α_1	x_1	$5/2$	1	0	0	$3/10$	$1/5$	0	
6	α_2	x_2	$5/4$	0	1	0	$1/20$	$1/5$	0	
2	α_3	x_3	$25/4$	0	0	1	$1/4$	0	0	
0	y_{s1}	x_4	$-1/2$	0	0	0	$\boxed{-3/10}$	$-1/5$	1	$\rightarrow$
	$Z_j - C_j$			0	0	0	2	2		
	$\underset{y_{4j}<0}{Max}\left\{\frac{Z_j-C_j}{y_{4j}}\right\}$			–	–	–	$\frac{2}{-3/10}$ $\uparrow$	$\frac{2}{-1/5}$		

Applying dual simplex algorithm, we get the transformed table as :

Simplex Table - 3

		C_j		4	6	2	0	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_{s1}
4	α_1	x_1	2	1	0	0	0	0	1
6	α_2	x_2	$7/6$	0	1	0	0	$\frac{1}{6}$	$1/6$
2	α_3	x_3	$5/6$	0	0	1	0	$-\frac{1}{6}$	$5/6$
0	α_4	x_4	$5/3$	0	0	0	1	$2/3$	$-10/3$
		$Z_j - C_j$		0	0	0	0	$2/3$	$20/3$

Since x_3 is still not an integer, we write from the third row of the this iteration

$$(0+0)x_1 + (0+0)x_2 + (1+0)x_3 + (0+0)x_4 + \left(-1+\frac{5}{6}\right)x_5 + \left(0+\frac{5}{6}\right)x_{s1} = 5+\frac{5}{6}$$

$$\Rightarrow \quad \frac{5}{6}x_5 + \frac{5}{6}x_{s1} \geq \frac{5}{6}$$

or $$-\frac{5}{6}x_5 - \frac{5}{6}x_{s1} \leq -\frac{5}{6}$$

$$\Rightarrow \quad -\frac{5}{6}x_5 - \frac{5}{6}x_{s1} + x_{s2} = -\frac{5}{6}$$

Introducing the secondary constraint in the above table the now defined table as :

Simplex Table - 4

		C_j		4	6	2	0	0	0	0	
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_{s1}	y_{s2}	
4	α_1	x_1	2	1	0	0	0	0	1	0	
6	α_2	x_2	$7/6$	0	1	0	0	$1/6$	$1/6$	0	
2	α_3	x_3	$35/6$	0	0	1	0	$-1/6$	$5/6$	0	
0	α_4	x_4	$5/3$	0	0	0	1	$2/3$	$-10/3$	0	
0	y_{s2}	x_{s2}	$-5/6$	0	0	0	0	**$-5/6$**	$-5/6$	1	→
		$Z_j - C_j$		0	0	0	0	$2/3$	$20/3$	0	
		$\underset{y_{5j}<0}{Max}\left(\frac{Z_j - C_j}{y_{5j}}\right)$		–	–	–	–	$\frac{2/3}{-5/6}$ ↑	$\frac{20/3}{-5/6}$		

The next iterative table is as follows :

Simplex Table - 5

			C_j	4	6	2	0	0	0	0
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	y_{s1}	y_{s2}
4	α_1	x_1	2	1	0	0	0	0	1	0
6	α_2	x_2	1	0	1	0	0	0	$1/6$	$1/5$
2	α_3	x_3	6	0	0	1	0	0	$5/6$	$-1/5$
0	α_4	x_4	1	0	0	0	1	0	$-10/3$	$4/5$
0	α_5	x_5	1	0	0	0	0	1	1	$-6/5$
		$Z_j - C_j$		0	0	0	0	0	$20/3$	$4/5$

Since x_1, x_3 are integers so it is required optimal integer solution, which is $x_1 = 2$, $x_2 = 1$, $x_3 = 6$ Max $Z = 26$

3.12 Self-Learning Exercise - II

1. What do you mean by mixed integer programming problem?
2. What is fractional cut?

3.13 Summary

In this unit we have studied the linear programming problems in which some or all variables are restricted to accept integer values, called mixed or pure integer programming problems, respectively. We have presented Gomory's cutting plane method to solve these problems. A procedure to find Gomory's secondary constraint is given. We modify the optimum simplex table by introducing above constraint, then use dual simplex algorithm to find optimum integer solution.

3.14 Answer to Self-Learning Exercise - I

1-3 See corresponding articles

3.15 Answer to Self-Learning Exercise - II

1-2 See corresponding articles

3.16 Exercise

1. Solve the following I.P.P.

Maximize $Z = 2x_1 + 3x_2$

s.t $-3x_1 + 7x_2 \le 14$

$7x_1 - 3x_2 \le 14$

$x_1, x_2 \geq 0$ and integers.

2. Describe any method to solve I.P.P. $u(x_1 - 3, x_2 = 3, \max z = 15)$ use it to solve the problem :

Maximize $\quad Z = 2x_1 + 2x_2$

s.t. $\quad 5x_1 + 3x_2 \leq 8$

$x_1 + 2x_2 \leq 4 \quad (x_1 = x_2 = 1, \max\ z = 4)$

x_1, x_2 are non-negative integers.

3. Solve the following I.P.P.

Minimize $\quad Z = 9x_1 + 10x_2$

s.t. $\quad x_1 \leq 9$

$x_2 \leq 8$

$4x_1 + 3x_2 \geq 40 \qquad (x_1 = 9, x_2 = 2, \min z = 101)$

$x_1, x_2 \geq 0$ and are integers.

4. Find optimum integer solution to the following all I.P.P. :

Maximize $\quad Z = x_1 + 2x_2$

s.t. $\quad x_1 + x_2 \leq 7$

$2x_1 \leq 11$

$2x_2 \leq 7 \qquad (x_1 = 4, x_2 = 3, \max z = 10)$

$x_1, x_2 \geq 0$ and are integers.

5. Solve the following mixed I.P.P. problem :

Maximize $\quad Z = -3x_1 + x_2 + 3x_3$

s.t. $\quad -x_1 + 2x_2 + x_3 \leq 4$

$4x_2 - 3x_3 \leq 2$

$x_1 - 3x_2 + 2x_3 \leq 3 \qquad \left(x_1 = 0, x_2 = \frac{8}{7}, x_3 = 1, \max z = \frac{29}{7}\right)$

x_1 and x_3 are integers and $x_1,\ x_2, x_3 \geq 0$

□□□

Unit - 4
Integer Programming : Branch and Bound Algorithm

Structure of the Unit

4.0 Objective

Integer programming introduced in unit-3 was dealt with an algorithm called Gomory's cutting plane method. The objective of this unit is to discuss another algorithm called **Branch and Bound Technique** to solve integer programming problems.

4.1 Introduction

Branch and Bound algorithm was developed by Land and Doig to solve all-integer and mixed integer programming problems. It is the most general technique to solve integer programming problems in which all or a few variable are constrained by their upper and lower bound or by both.

The concept behind this method is to divide the entire feasible solution space of linear programming problem into smaller parts called sub-problems and then search each of them for an optimalsolution. This approach is useful in those cases where there is a large number of feasible solutions and enumeration of those becomes economically impractical or impossible.

4.2 The Branch and Bound Method

This technique is applicable to both the L.P.P., pure as well as mixed. In this method first we solve the continuous I.P.P. ignoring the integer-valued restrections. If in the optimal solution one of the variables say x_i is not an integer, then we divide or partition the given L.P.P. into two sub problems.

We have $\left[x_r^*\right] < x_r^* < \left[x_r^*\right] + 1$

where x_r^* is the value of x_r in the optimal solution.

Hence any feasible value of x_r must satisfy one of the two conditions

$$x_r \le \left[x_r\right] \quad \text{or} \quad x_r \ge \left[x_r\right] + 1$$

Note that these two constraints are mutually execlusive (i.e. both can not be true simulteneously) and hence both can not be amended in the L.P.P. simulteneously.

By adding these constraints separately to the continuous L.P.P. we form two sub L.P.P. Thus we have branched the original subproblem into two sub problems. According the geometrical interpretation, we observe that the branching process discards that portion of the feasible region which involves no feasible integer solution.

To understand it, we take an example. Suppose we have optimal solution of an L.P.P. as

$$x_1^* = \frac{7}{2} \quad \text{and} \quad x_2^* = \frac{9}{5}$$

clearly $x_1 = \frac{7}{2}$ gives that $3 < x_1^* > 4$

$\Rightarrow$ for an integer valued solution, either

$x_1 \leq 3$ or $x_1 \geq 4$

Thus there will be no integer valued feasible solution in the strip $x_1 = 3$ and $x_1 = 4$ (Actually draw two lines $x_1 = 3$ and $x_1 = 4$ and verify the fact). We should search for optimum value of Z in either the first region $(x_1 \leq 3)$ or second region $(x_1 \geq 4)$.

After branching in this way two subproblems are formed by adding $x_r \leq [x_r^*]$ and $x_r \geq [x_r^*]+1$ one by one to the origional set of constraints. Now these two subproblems are solved. If for any of the subproblems optinum integer solution is obtained then that problem is not further branched. But if ever any subproblem involves non-integer variable then it is again branched and this process of branching contin- ues. Wherever applicable until each subproblem either admits an integer valued optimum solution or there is evidence that it cannto yeild a better one. Then that optimum integer valued solution among all the subproblem is selected which gives the over all optimum value of the objective functions.

4.3 The Branch and Bound Algorithm

The iterative procedure of this method is given as below :

Step 1 : Obtain the optimum solution of the given L.P.P. ignoring the integer restriction.

Step 2 : Test the integrability of the optimum solution obtained in step 1. There are two cases :

(i) If the solution is in integers, the current solution is optimum to the given integer program ming problem.

(ii) If the solution is not in integers, go to next step.

Step 3 : Considering the value of objective function as upper bound, obtain the lower bound by rounding off to integral values of the decision varibales.

Step 4 : Let the optinum value x_j^* of the variable x_j be not an integer. Then subdivide (branch) the given L.P.P. in two subproblems.

SUB-PROBLEM-1 : Given L.P.P. with an additional constraint $x_j \leq [x_j^\infty]$

SUB-PROBLEM-2 : Given L.P.P. with an additional constraint $x_j \geq [x_j^\infty]$

where $[x_j^\infty]$ is the largest integer contained in x_j^∞.

Step 5 : Solve the two problems obtained in step 4. There may arise three cases :

(i) If the optinum solutions of the two subproblems are integral, then the required solution is one that gives larger value of Z.

(ii) If the optimum solution of one subproblem is integer and the other subproblem has no feasible optimal solution, the required solution is same as that of the suproblem having integr valued solution.

(iii) If the optimum solution of one subproblem is integer while that of the other is not integer valued then record the integer valued solutions and repeat step 3 and 4 for the non-integer valued subproblem.

Step 6 : Repeat steps 3 to 5 until all integer valued solutions are recorded.

Step 7 : Choose the solution amongst the recorded integer valued solutions that yields optinum value of $Z \cdot$

4.4 Illustrative Examples

Example 1 : Solve the following I.P.P. by branch and bound technique.

Max. $Z = x_1 + x_2$

Subject to $3x_1 + 2x_2 \le 12$

$x_2 \le 2$

$x_1, x_2 \ge 0$ and integers.

a

Solution :

Step 1 : By Graphical method, the optimum solution of the problem ignoring the integer valued restriction, is

$x_1 = \frac{8}{3}$, $x_2 = 2$ (See Fig. 4.1)

Now x_1 is non integer and $x_1^* = \frac{8}{3}$ gives $Z \le x_1^* \le 3$

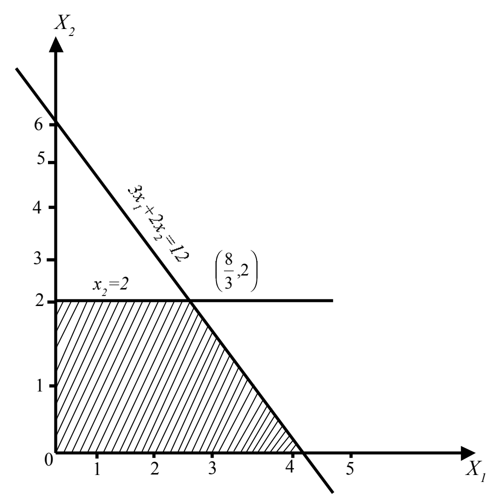


Figure 4.1

Step 2 : Then we form two subproblems given below :

Problem 2	**Problem 3**
Max $Z = x_1 + x_2$	Max $Z = x_1 + x_2$
S.t. $3x_1 + 2x_2 \le 12$	S.t. $3x_1 + 2x_2 \le 12$
$x_2 \le 2$	$x_2 \le 2$

$$x_1 \le 2 \qquad\qquad x_1 \ge 3$$

$$x_1, x_2 \ge 0 \qquad\qquad x_1, x_2 \ge 0$$

For the solution of these problems see fig. 4.2 and 4.3 as given below :

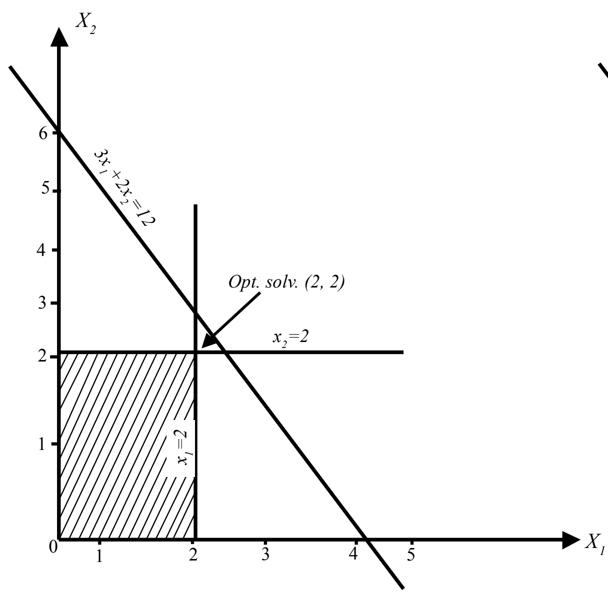


Figure 4.2

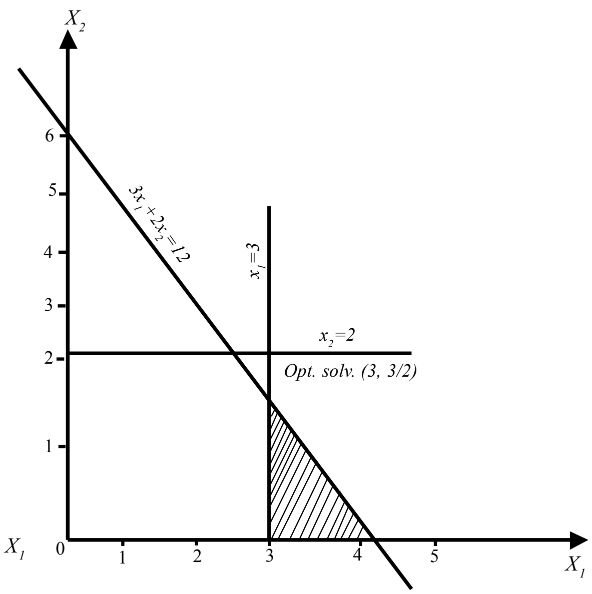


Figure 4.3

Optinal solution of problem 2 is $x_1 = 2$, $x_2 = 2$, Max. $z = 4$

Since in this solution all the variables are integer therefore there is no need to branch this problem further.

The optimal problem of problem **3** is

$$x_1 = 3, \qquad x_2 = \frac{3}{2}, \qquad \text{Max } Z = \frac{9}{2}$$

Step 3 : Since x_2 is non-integer, it needs further subdivision. Here $x_2^* = \frac{3}{2} \Rightarrow 1 \le x_2 \le 2$

Hence, we form two subproblems by introducing the constraints $x_2 \le 1$ and $x_2 \ge 2$ one by one in problem **3**. Now problems are **:**

Problem 4	**Problem 5**
Max $Z = x_1 + x_2$	Max $Z = x_1 + x_2$
S.t. $3x_1 + 2x_2 \le 12$	S.t. $3x_1 + 2x_2 \le 12$
$x_2 \le 2$	$x_2 \le 2$
$x_1 \ge 3$	$x_1 \ge 3$
$x_1, x_2 \ge 0$	$x_1, x_2 \ge 0$

The problem **5** has no feasible solution and in problem **4** the constraint $x_2 \le 2$ is redundant. The optimum solution to this problem is $x_1 = \frac{10}{3}$, $x_2 = 1$ and Max $Z = \frac{13}{3}$. From Figure 4.4 it is clear that any

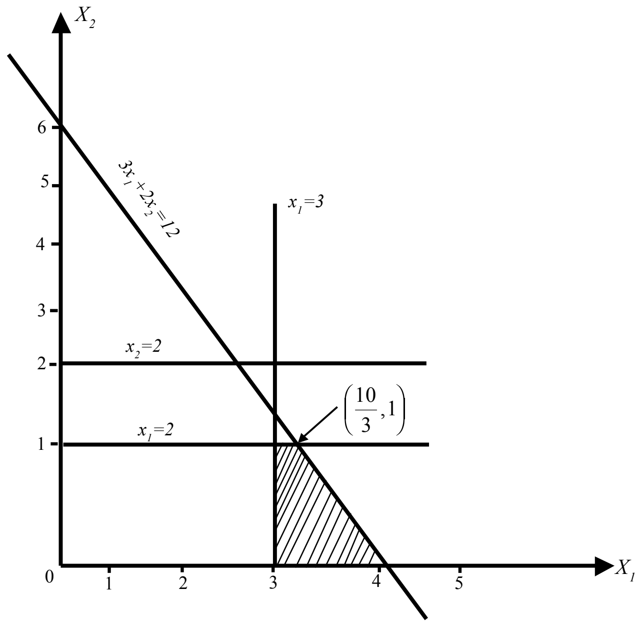


Figure 4.4

further branching of the problem will not improve the value of objective function as next subdivision will improve the value of objective function as next subdivision will impose that restrietions $x_1 \le 3$, $x_1 \ge 4$. Then optimal solution are $x_1 = 3$ and $x_2 = 1$ and $x_1 = 4$ and $x_2 = 0$ respectively. There solutions also gives $Z = 4$.

Step 4 : Hence overall maximum value of the objective function $Z = 4$ and integer valued solutions is any of these

$$x_1 = 2,\ x_2 = 2,\ x_1 = 3,\ x_2 = 1\ ;\ x_1 = 4,\ x_2 = 0$$

Example 2 : Use branch and bound method to solve following L.P.P. :

Maximize $Z = 7x_1 + 9x_2$

Subject to $-x_1 + 3x_2 \le 6$

$7x_1 + x_2 \le 35$

$x_2 \ge 7$

Solution :

Step 1 : Ignoring the integer restriction, the optimal solution to the given L.P.P. can easily be obtained by graphical or simplex method as $x_1 = \frac{9}{2}$, $x_2 = \frac{7}{2}$ and Max. $Z = 63$.

Step 2 : Since the solution is not in integers, let us choose x_1, i.e. $x_1^* = \frac{9}{2}$ being the largest fractional value.

Step 3 : Considering the value of Z as initial upper bound i.e. $Z = 63$. The lower bound is obtained by rounding off the value of x_1, x_2 to the nearest integers, i.e., $x_1 = 4$, $x_2 = 3$ then the lower bound is $Z_1 = 55$.

Step 4 : Since $\left[x_1^*\right] = \left[\frac{9}{2}\right] = 4$; we have

Sub-problem 1 Max $Z = 7x_1 + 9x_2$

s.t. $-x_1 + 3x_2 \le 6$

$7x_1 + x_2 \le 35$

$x_2 \le 7$

$x_1 \le 4$

$x_1, x_2 \ge 0$ and are integers.

Sub-problem 2 Max $Z = 7x_1 + 9x_2$

s.t. $-x_1 + 3x_2 \le 6$

$7x_1 + x_2 \le 35$

$x_2 \le 7$

$x_1 \ge 5$

$x_1, x_2 \ge 0$ and are integers.

Step 5 : On solving the above two subproblems by graphical or simplex method the optimum solutions are

<u>Sub-problem 1</u> $x_1 = 4$, $x_2 = \frac{10}{3}$ Max. $z = 58$

Sub-problem 2 $x_1 = 5$, $x_2 = 0$ and Max. $z = 35$

Since the solution to subproblem 1 is not in integers, we subdivide it into following two subproblems.

Sub-problem 3 Max $Z = 7x_1 + 9x_2$ s.t.

$-x_1 + 3x_2 \le 6$, $7x_1 + x_2 \le 35$

$x_1 \le 4$, $x_2 \le 3$ $x_1, x_2 \ge 0$

Sub-problem 4 Max $Z = 7x_1 + 9x_2$ s.t.

$-x_1 + 3x_2 \le 6$, $7x_1 + x_2 \le 35$

$x_1 \le 4$, $x_2 \ge 4$

$x_1, x_2 \ge 0$

Step 6 : The optimum solutions to the subproblems **3** and **4** are :

Sub-problem 3 $x_1 = 4$, $x_2 = 3$ and Max. $z = 55$

Sub-problem 4 No feasible solution.

Step 7 : Among the recorded integer valued solutions, since the largest value of Z is 55, the required optimum solution is

$$x_1 = 4,\ x_2 = 3 \text{ and Max. } Z = 55$$

The whole branch and bound procedure for the given problem is shown below :

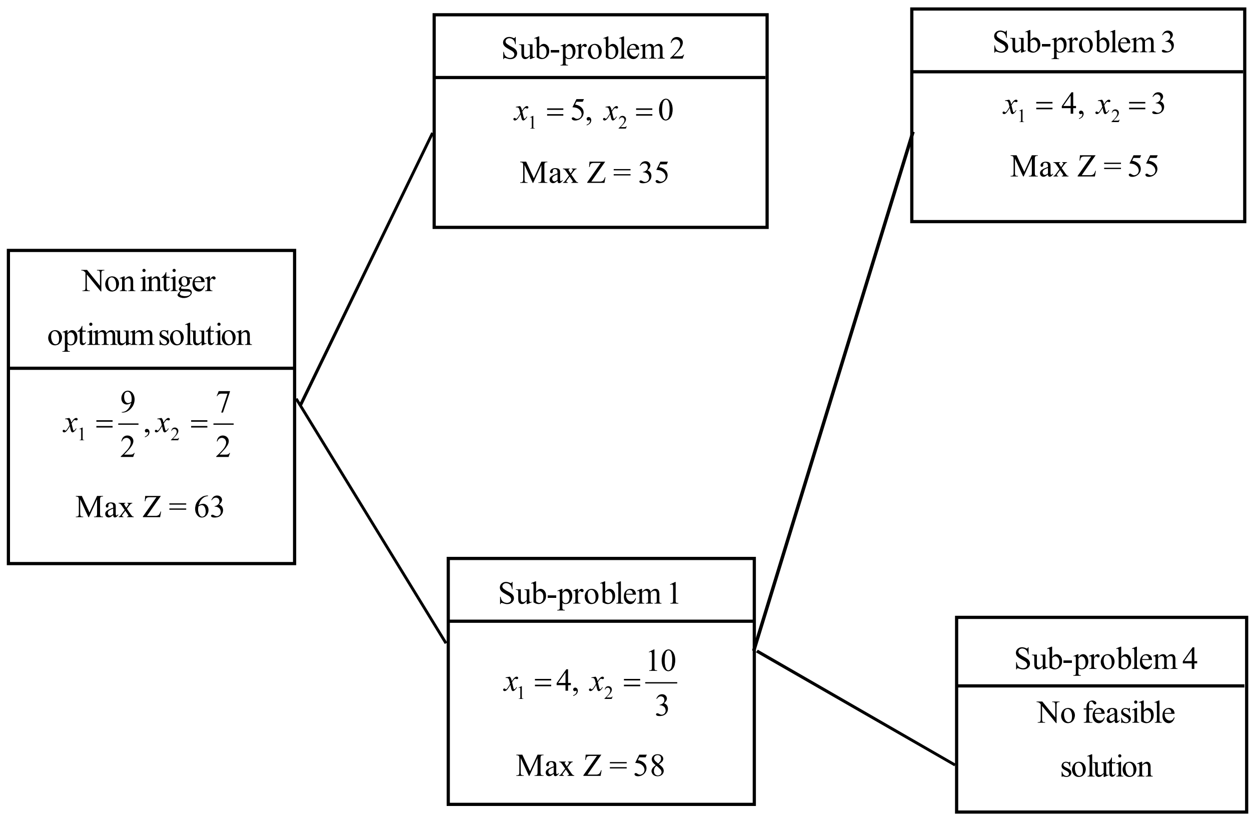


Figure 4.5

Example 3 : Use Branch and Bound Method to solve the following I.P.P. :

Minimize $Z = 4x_1 + 3x_2$

Subject to $5x_1 + 3x_2 \geq 30$

$x_1 \leq 4$

$x_2 \leq 6$

$x_1, x_2 \geq 0$ and are integers.

Solution : Ignoring the integer restrictions, the optinum solution to the L.P.P. can easily be obtained as (Use Graphical or Simplex method)

$$x_1 = 4,\ x_2 = \frac{10}{3} \text{ and Min. } Z = 26$$

Since the value of x_2 is not an integer, we branch on this variable. Since $[x_2] = \left[\frac{10}{3}\right] = 3$, the two branches are $x_2 \leq 3$ and $x_2 \geq 4$. Thus we have.

Sub-problem 1 Minimize $Z = 4x_1 + 3x_2$

subject to $5x_1 + 3x_2 \geq 30$

$x_1 \leq 4$

$x_2 \leq 6$

$x_2 \leq 3$

$x_1, x_2 \geq 0$

Sub-problem 2 Minimize $Z = 4x_1 + 3x_2$

subject to $5x_1 + 3x_2 \geq 30$

$x_1 \leq 4$

$x_2 \leq 6$

$x_2 \geq 4$

$x_1, x_2 \geq 0$

The optimum solutions of above sub-problems are obtained by graphical or simplex method as :

Sub-problem 1 No feasible solution

Sub-problem 2 $x_1 = \frac{18}{5}$, $x_2 = 4$, Min. $Z = \frac{132}{5}$

Since the value of x_1 in sub-problem 2 is not an integer, we branch on this variable. The two branches are $x_1 \leq 3$ and $x_1 \geq 4$, since $\left[\frac{18}{5}\right] = 3$

Thus we have

Sub-problem 3 Minimize $Z = 4x_1 + 3x_2$

subject to $5x_1 + 3x_2 \geq 30$

$x_1 \leq 4$

$x_2 \leq 6$

$x_2 \geq 4$

$x_1 \leq 3$

$x_1, x_2 \geq 0$

Sub-problem 4 Minimize $Z = 4x_1 + 3x_2$

subject to $5x_1 + 3x_2 \geq 30$

$x_1 \leq 4$

$$x_2 \le 6$$
$$x_2 \ge 4$$
$$x_1 \ge 4$$
$$x_1, x_2 \ge 0$$

The optimum solutions to these sub-problems are obtained as :

Sub-problem 3 $x_1 = 3$, $x_2 = 5$ and minimum $Z = 27$

Sub-problem 4 $x_1 = 4$, $x_2 = 4$ and minimum $Z = 28$

Among the feasible solutions to the integer programming problem, since the minimum value of Z is 27; the required optimum solution is

$x_1 = 3$, $x_2 = 5$ and minimum $Z = 27$

The complete Branch and Bound procedure for the I.P.P. is shown below :

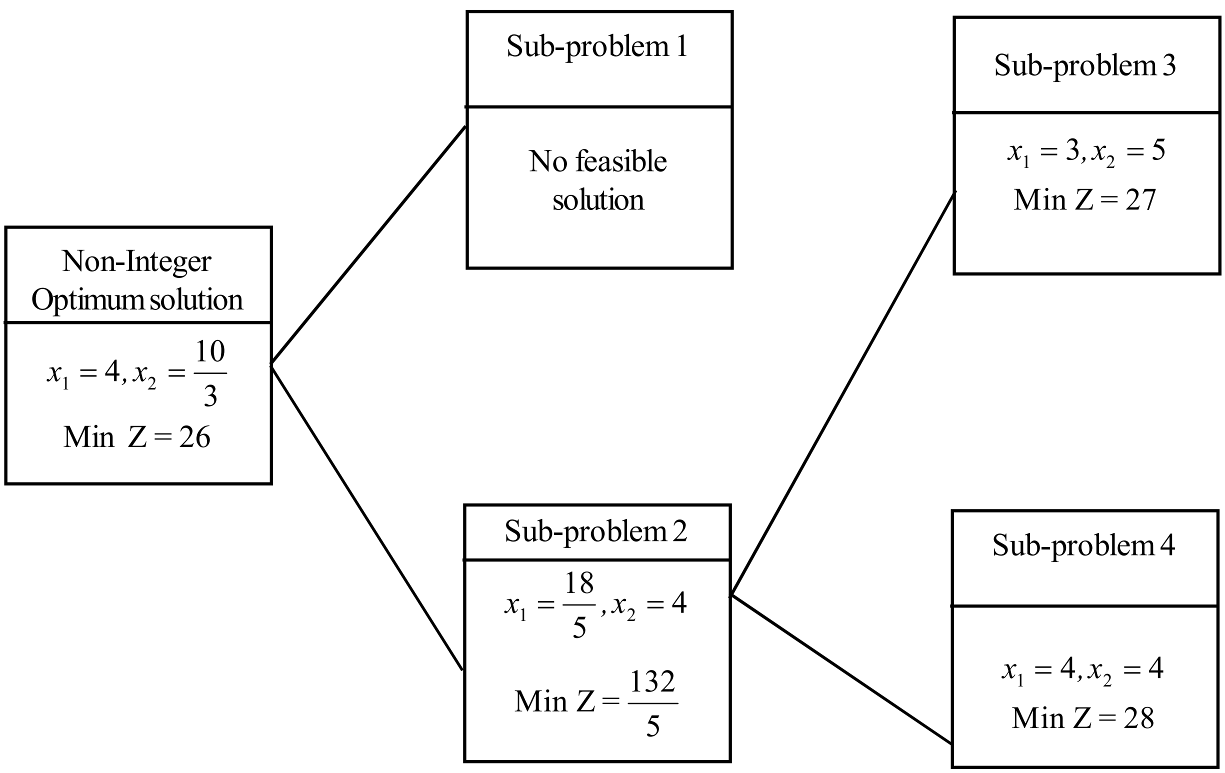


Figure 4.6

Example 4 : Use Branch and Bound technique to solve the following problem :

Max. $Z = 3x_1 + 3x_2 + 13x_3$

s.t. $-3x_1 + 6x_2 + 7x_3 \le 8$

$6x_1 - 3x_2 + 7x_3 \le 8$

$0 \le x_j \le 5$

and x_j are integers for $j = 1,2,3$.

Solution :

Step 1 : Introducing slack variable x_4, x_5 is the first two constraints, the standard form for simplex method (since it is a three variables problem so it cannot be solved by graphical method)

Max. $Z = 3x_1 + 3x_2 + 13x_3 + 0x_4 + 0x_5$

s.t. $-3x_1 + 6x_2 + 7x_3 + x_4 = 8$

$6x_1 - 3x_2 + 7x_3 + x_5 = 8$

$0 \le x_1 \le 5$, $0 \le x_2 \le 5$, $0 \le x_3 \le 5$, $x_4, x_5 \ge 0$

Initital BFS $x_4 = 8$, $x_5 = 8$, $x_1 = x_2 = x_3 = 0$

			C_j	3	3	13	0	0	$\theta = \frac{x_{Bv}}{y_{ik}}, y_{ik} > 0$
C_B	B	X_B	b	y_1	y_2	y_3	y_4	y_5	
0	α_4	x_4	8	-3	6	7	1	0	$\frac{8}{7}$
0	α_5	x_5	8	6	-3	$\boxed{7}$	0	1	$\frac{8}{7} \rightarrow$
		$Z_j - C_j$		-3	-3	-13	0	0	Min $\theta = \frac{8}{7}$
0	α_4	x_4	0	-9	$\boxed{9}$	$\uparrow$ 0	1	-1	$\rightarrow$
13	α_3	x_3	$\frac{8}{7}$	$\frac{6}{7}$	$-\frac{3}{7}$	1	0	$\frac{1}{7}$	
		$Z_j - C_j$		$\frac{57}{7}$	$\frac{-60}{7}$	0	0	$\frac{13}{7}$	Min $\theta = 0$
3	α_2	x_2	0	-1	$\uparrow$ 1	0	$\frac{1}{9}$	$-\frac{1}{9}$	
13	α_3	x_3	$\frac{8}{7}$	$\boxed{\frac{3}{7}}$	0	1	$\frac{1}{21}$	$\frac{2}{21}$	$\rightarrow$
		$Z_j - C_j$		$-\frac{3}{7}$	0	0	$\frac{20}{21}$	$\frac{19}{21}$	Min $\theta = \frac{8}{3}$
3	α_2	x_2	$\frac{8}{3}$	$\uparrow$ 0	1	$\frac{7}{3}$	$\frac{2}{9}$	$\frac{1}{9}$	
3	α_1	x_1	$\frac{8}{3}$	1	0	$\frac{7}{3}$	$\frac{1}{9}$	$\frac{2}{9}$	
		$Z_j - C_j$		0	0	1	1	1	

The optimum non-integer solution to the given L.P.P.

$$x_1 = \frac{8}{3},\ x_2 = \frac{8}{3},\ x_3 = 0,\ \text{Max } Z = 16$$

Step 2 : Since x_1, x_2 are non-integer valued, we choose x_1 for branching

$$\because \qquad [x^*] = \left[\frac{8}{3}\right] = 2$$

The two sub-problmes are as

Sub-problem 1 Max. $Z = 3x_1 + 3x_2 + 13x_3$

S.t. $-3x_1 + 6x_2 + 7x_3 \le 8$

$6x_1 - 3x_2 + 7x_3 \le 8$

$0 \le x_j \le 5, \quad j = 1, 2, 3$

Step 3 : Now we solve sub-problem (1) & (2) using simplex method as before we find that sub-problem (2) has no feasible solution.

The sub-problem (1) has an optimal solution

$$x_1 = x_2 = 2,\ x_3 = \frac{2}{7},\quad \text{Max. } Z = 15\frac{5}{7}$$

Clearly this is not integer valued, so we branch this sub-problem (1) into two on the variable x_3.

Since $[x_3^*] = \left[\frac{2}{7}\right] = 0$

Sub-problem 3 Max. $Z = 3x_1 + 3x_2 + 13x_3$

s.t. $-3x_1 + 6x_2 + 7x_3 \le 8$

$6x_1 - 3x_2 + 7x_3 \le 8$

$0 \le x_1 \le 2$

$0 \le x_2 \le 5$

$1 \le x_3 \le 5$

Sub-problem 4 Max. $Z = 3x_1 + 3x_2 + 13x_3$

s.t. $-3x_1 + 6x_2 + 7x_3 \le 8$

$6x_1 - 3x_2 + 7x_3 \le 8$

$0 \le x_1 \le 2$

$0 \le x_2 \le 5$

$0 \le x_3 \le 0$

Here we observe that sub-problem (3) & (4) differ from sub-problem (1) only in the bounds of x_3.

Step 4 : Now, we solve sub-problem (3), the optimal solution is obtained as $x_1 = x_2 = \frac{1}{3}$, $x_3 = 1$, $Z^* = 15$ select x_2, $\left[x_2^*\right] = \left[\frac{1}{3}\right] = 0$, so we branch this sub-problem into two sub-problem as follows :

Sub-problem 5 Max. $Z = 3x_1 + 3x_2 + 13x_3$

s.t. $-3x_1 + 6x_2 + 7x_3 \le 8$

$6x_1 - 3x_2 + 7x_3 \le 8$

$0 \le x_1 \le 2$, $0 \le x_2 \le 5$, $1 \le x_3 \le 5$, $x_2 \ge 1$

Sub-problem 6 Max. $Z = 3x_1 + 3x_2 + 13x_3$

s.t. $-3x_1 + 6x_2 + 7x_3 \le 8$

$6x_1 - 3x_2 + 7x_3 \le 8$

$0 \le x_1 \le 2$, $0 \le x_2 \le 0$, $1 \le x_3 \le 5$

Step 5 : We can easily see that sub-problem 5 has no feasible solution. The optimal solution to sub-problem (6) is as follows:

$$x_1 = 0,\ x_2 = 0,\ x_3 = 1\frac{1}{7}, \quad \text{Max } Z = 14\frac{6}{7}$$

$\because$ x_3 is fractional, so we again branch this sub-problem on x_3, $\left[x_3^*\right] = \left[1\frac{1}{7}\right] = 1$

Sub-problem 7 First two constraints of sub-problem 6 and

$0 \le x_1 \le 2$, $0 \le x_2 \le 0$, $2 \le x_3 \le 5$

Sub-problem 8 First two constraints of sub-problem 6 and

$0 \le x_1 \le 2$, $0 \le x_2 \le 0$, $1 \le x_3 \le 1$

Step 6 : We see that sub-problem (7) has no feasible solution. The optimal solution of sub-problem (8) is $x_1 = x_2 = 0$, $x_3 = 1$, Max $Z = 13$

Returning to step 3, we observe that only sub-problem 4 is now left to solve, the optimal solution of this problem is

$$x_1 = 2\ x_2 = 2\frac{1}{3},\ x_3 = 0, \text{ Max } Z = 13$$

Since the optimum value of the objective function of sub-problem 8 and sub-problem 5 are same and is equal to $Z = 13$. Hence we stop computations. The optimal solution to given I.P.P. is as follows :

$$x_1 = 0,\ x_2 = 0,\ x_3 = 1 \quad \text{Max } Z = 13$$

Tree-Diagram of Example 7

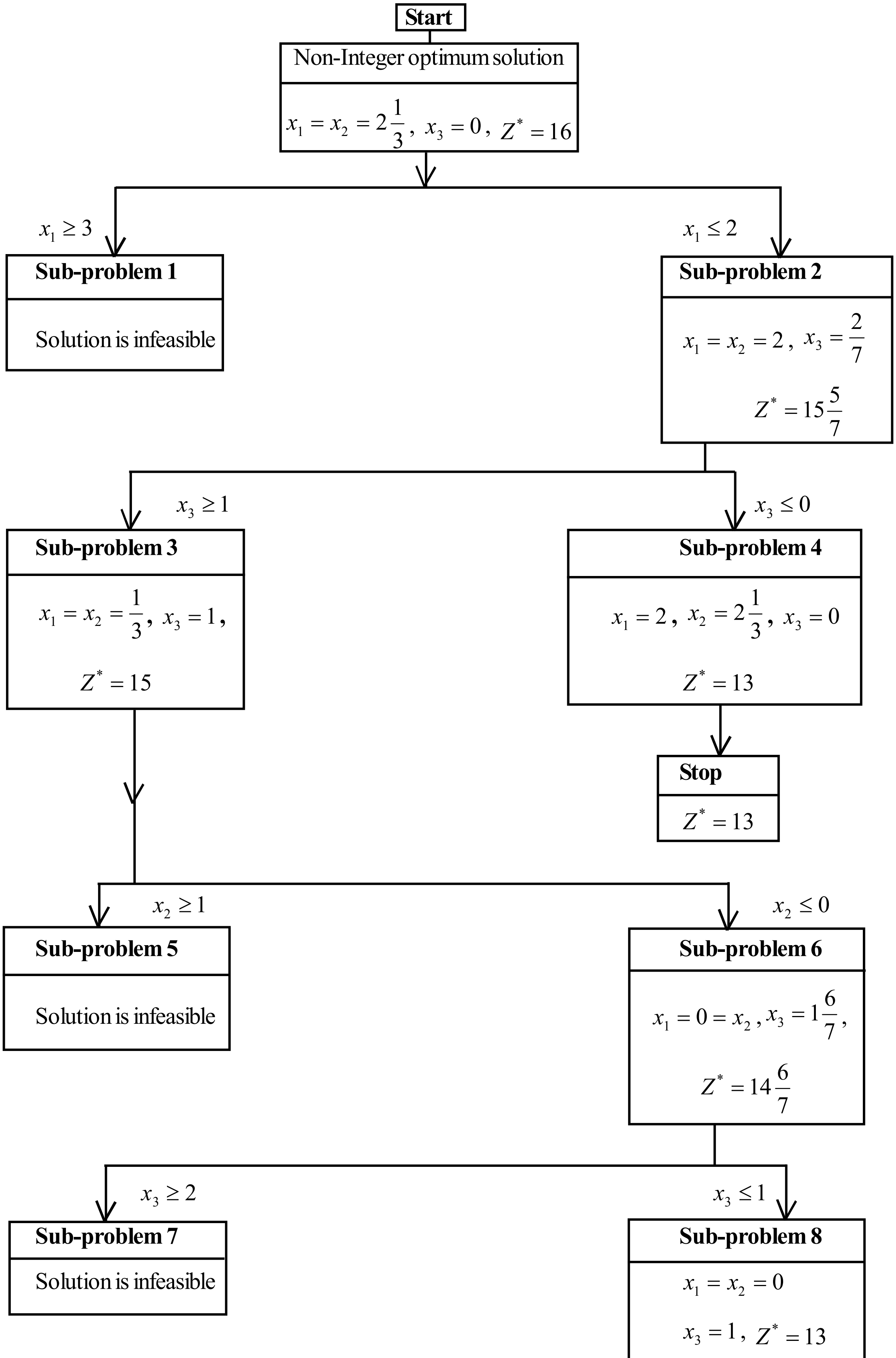


Figure 4.7

4.5 Geometrical Interpretation of Branch and Bound Method

The geometrical interpretation of Branch and Bound Method can easily be understood by a two variable I.P.P. which we solve by graphical method. Example 1 is given for this purpose. To be more clear consider one more example as follows :

Example 5 : Solve the following I.P.P. using branch and bound algorithm.

$$\text{Max} \quad Z = 2x_1 + 6x_2$$

$$\text{s.t.} \quad 3x_1 + x_2 \le 5$$

$$4x_1 + 4x_2 \le 9$$

$$x_1, x_2 \ge 0 \text{ and are integers.}$$

Solution : The graphical solution of given problem gives the optimal solution :

$$x_1 = 0\,,\; x_2 = \frac{9}{4}, \quad \text{Max. } Z^* = \frac{27}{2}$$

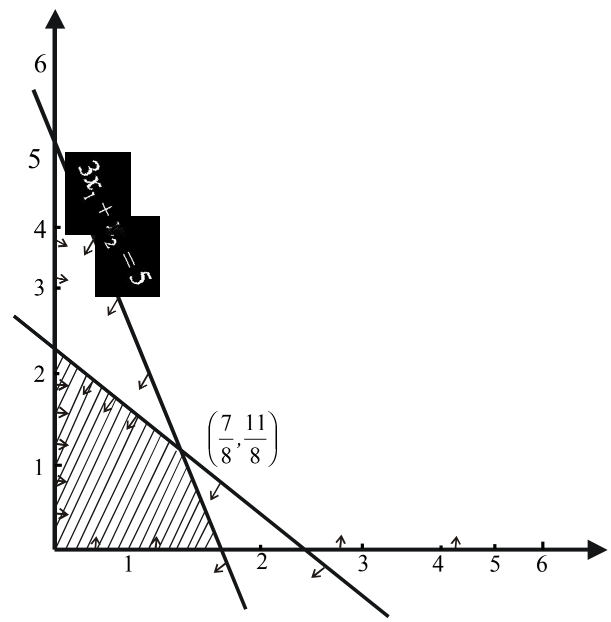


Figure 4.8

Since the variable x_2 has non integer value and x_2 has largest fractional part, so we branch the problem on x_2

$$[x_2^*] = \left[\frac{9}{4}\right] = 2$$

Sub-problem 1 Max. $Z = 2x_1 + 6x_2$

s.t. $3x_1 + x_2 \le 5$

$4x_1 + 4x_2 \le 9$

$x_2 \le 2$

$x_1, x_2 \ge 0$

Sub-problem 2 Max. $Z = 2x_1 + 6x_2$

s.t. $3x_1 + x_2 \le 5$

$4x_1 + 4x_2 \le 9$

$x_2 \ge 3$

$x_1, x_2 \ge 0$

The sub-problem 2 has no feasible solution.

See Figure 4.10

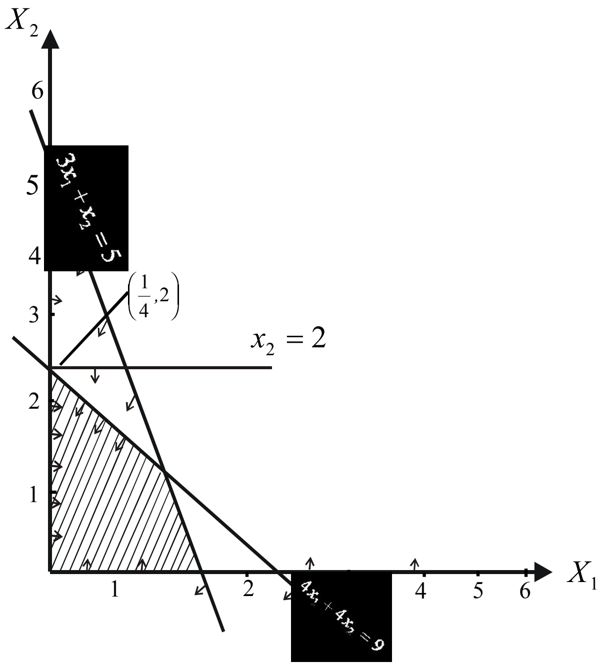


Figure 4.9

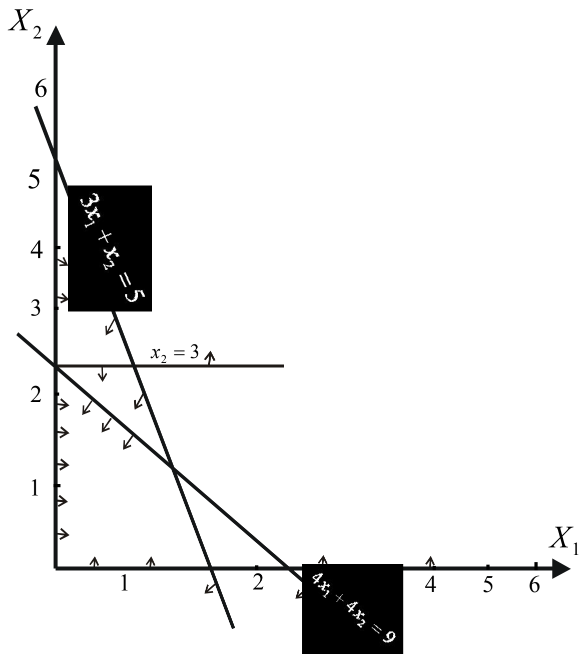


Figure 4.10

The sub-problem 1 has optinum solution as follows :

$$x_1 = \frac{1}{4},\ x_2 = 2, \qquad \text{Max } Z = \frac{25}{2}$$

Since x_1 is not integer, so we branch the above sub-problem 1 on x_1, $\left[x_1^*\right] = \left[\frac{1}{4}\right] = 0$

Sub-problem 3 Max. $Z = 2x_1 + 6x_2$

s.t. $3x_1 + x_2 \le 5$

$4x_1 + 4x_2 \le 9$

$x_2 \le 2$, $x_1 \le 0$

$x_1, x_2 \ge 0$

Sub-problem 4 Max. $Z = 2x_1 + 6x_2$

s.t. $3x_1 + x_2 \le 5$

$4x_1 + 4x_2 \le 9$

$x_2 \le 2$

$x_1 \ge 1$

$x_1, x_2 \ge 0$

Sub-problem **3** has the optimum solution

$x_1 = 0$, $x_2 = 0$, Max. $Z = 12$

See the Figure 4.11 (Feasible region is only the line segment form (0, 0) to (0, 2))

2

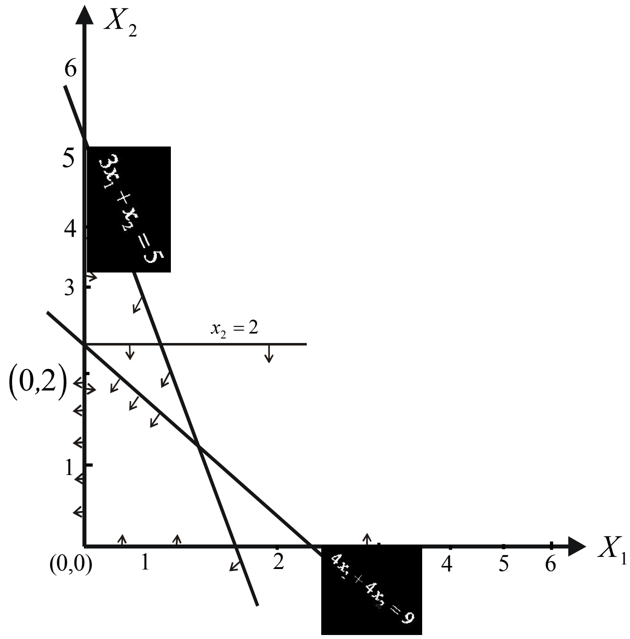


Figure 4.11

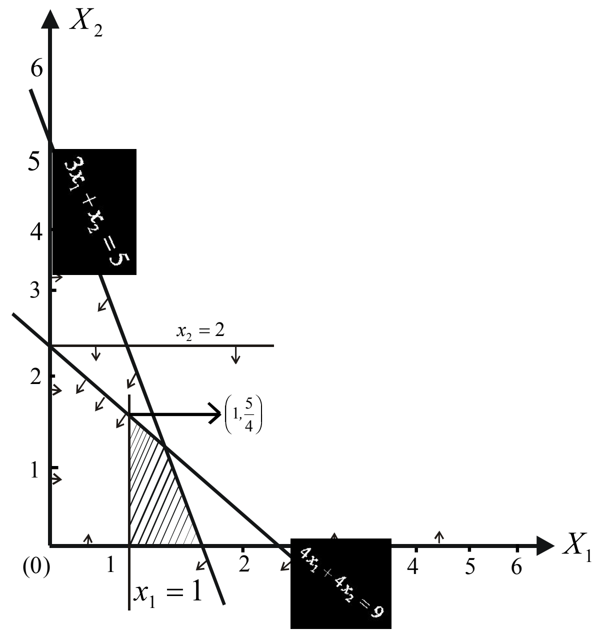


Figure 4.12

Sub-problem **4** has optimal solution (see fig. 4.12)

$$x_1 = 1, \ x_2 = \frac{5}{4}, \qquad \text{Max. } Z = 9\frac{1}{2}$$

The value of objective function in sub-problem (3) has greater value then sub-problem **4**. Hence, the optimum solution of the problem is

$$x_1 = 0, \ x_2 = 2, \qquad \text{Max. } Z = 12$$

4.6 Self-Learning Exercise

Sort the correct answers :

1. Branch and Bound Method divides the feasible region into smaller parts by
   (a) enumerating (b) branching
   (c) bounding (d) all of the above
2. While solving an I.P.P., any non-integer variable in the solution is picked up to
   (a) enter the solution (b) leave the solution
   (c) obtain the cut constant (d) all of the above
3. In a mixed integer programming problems :
   (a) different objective function are mixed together
   (b) all the decision variables require integer solution
   (c) only few of the decision variables required integer solutions
   (d) none of the above
4. Sketch the Branch and Bound Method is integer programming.

5. Distinguish between pure and mixed integer programming.

6. Use Branch and Bound method to solve the following I.P.P.

Max $Z = x_1 + 2x_2$ s.t. $x_1 + x_2 \le 7$

$2x_1 \le 11$, $2x_2 \le 7$ $x_1, x_2 \ge 0$ and are integers.

7. What is the difference between continuous and integer programming?

4.7 Summary

In this unit, Branch and Bound Algorithm has been discussed to solve integer programming problems. In this method, a L.P.P. is branched on a variable by bounding it into two sub-problems. These sub-problems are solved by graphical or Simplex method. The main disadvantages of this method is that it requires the optimum solution of each sub-linear programming problem. In large number of problems, this could be very tediuous job. But in spite of its drawback, this is the most effective method for solving I.P.P. thus when choice is to be made between Cutting Plane and Branch and Bound method; the latter is prefered.

4.8 Answers to Self-Learning Exercise

1. (b) 2. (c) 3. (c)

4.9 Exercises

Use Branch and Bound method to solve the following integer linear programming problems :

1. Maximize $Z = 2x_1 + 3x_2$

Subject to $5x_1 + 7x_2 \le 35$

$4x_1 + 9x_2 \le 36$

$x_1, x_2 \ge 0$ and are integers.

2. Maximize $Z = 2x_1 + 3x_2$

Subject to $x_1 + x_2 \le 7$,

$0 \le x_1 \le 5$, $0 \le x_2 \le 4$; x_1, x_2 are integers

3. Maximize $Z = x_1 + 2x_2$

Subject to $x_1 + 2x_2 \le 12$

$4x_1 + 3x_2 \le 14$

$x_1 \ge 0$, $x_2 \ge 0$ and are integers.

4. Maximize $Z = 2x_1 + 3x_2$

Subject to $6x_1 + 5x_2 \le 25$

$x_1 + 3x_2 \le 10$

$x_1 \ge 0$, $x_2 \ge 0$ and are integers.

5. Maximize $Z = 2x_1 + x_2$

Subject to $x_1 \le \frac{3}{2}$, $x_2 \le \frac{3}{2}$

$x_1, x_2 \ge 0$ and are integers.

6. Maximize $Z = 3x_1 + 2x_2$

Subject to $x_1 \le 2$, $x_2 \le 2$

$x_1 + x_2 \le \frac{7}{2}$

$x_1, x_2 \ge 0$ and are integers.

7. Minimize $Z = 10x_1 + 9x_2$

Subject to $x_1 \le 8$, $x_2 \le 10$

$5x_1 + 3x_2 \ge 45$

$x_1, x_2 \ge 0$ and x_1 is integer.

8. Maximize $Z = x_1 + 5x_2$

Subject to $x_1 + 10x_2 \le 20$

$x_2 \le 2$

$x_1, x_2 \ge 0$ and are integers.

□□□

Unit - 5
Quadratic form and Lagrangian Function

Structure of the Unit

5.0 Objective

The objective of this unit is to present some more about quadratic forms in respect of unit-1. The Lagrangian method to optimize the non-linear functions has also been given in this unit. Using this method we can optimize a non linear function with equality constraint.

5.1 Introduction

The concept of quadratic form has been introduced in the unit-1. The positive and negative definiteness of a quadratic form have also been defined. Several texts for this has also been discussed. In this unit we learn has also been discussed. In this unit we learn more about quadratic form.

The optimization i.e. to find maximum or minimum value of an objective function, is studied in lower classes. In this unit we start our study to optimize a function without any constraint. The main stress will be given on constrained problems of maxima and minima. If there are some constraints under which we optimize a function, we use Lagrange's method.

Now in this unit-I we study quadratic forms.

5.2 Quadratic form

Recall that a quadratic form is a function of n-variables which can be expressed as

$Q(X)=\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j$, where a_{ij} are constants. It can also be written as $Q(X)=X^T AX$ where $X=[x_1,x_2,....x_n]$ and $A=[a_{ij}]$ is a $n\times n$ symmetric matrix.

Example-1 (a) $$(x_1,x_2)\begin{bmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=x_1^2+x_1x_2+x_2^2$$

(b) $$(x_1,x_2,x_3)\begin{bmatrix}1 & \frac{1}{2} & 2\\ \frac{1}{2} & 0 & 0\\ 2 & 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=x_1^2+x_1x_2+4x_1x_3+2x_3^2$$

(c) $$(x_1,x_2,x_3)\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=x_1^2+2x_2^2+3x_3^2$$

Note that matrix representation of A in a quardatic form is not unique. However, A can always be taken to be symmetric without loss of generality.

Example-2 Write the quadratic form $Q(X)=x_1^2+2x_2^2-7x_3^2-4x_1x_2+6x_1x_3-5x_2x_3$ in matrix form.

Solution : $$Q(X)=x_1^2+2x_2^2-7x_3^2+(-2-2)x_1x_2+(3+3)x_1x_3+\left(-\frac{5}{2}-\frac{5}{2}\right)x_2x_3$$

$$=(x_1,x_2,x_3)\begin{bmatrix}1 & -2 & 3\\ -2 & 2 & -\frac{5}{2}\\ 3 & -\frac{5}{2} & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Example-3 Determine which of the following equations are quadratic form :

(i) $z=x_1^2+2x_2$

(ii) $z=x_1^2-x_2^2$

(iii) $z = x_1 x_2$

(iv) $z = 3x_1^2 + 3x_1x_2 + x_2^2$

Solution : For z to be a quadratic form, we must be able to express it in the form

$z = X^T AX$

(i) It is not a quadratic form, because it is linear in x_2

(ii) It is a quadratic form, because

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(iii) It is a quadratic form, because

$$A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

(iv) It is a quadratic form, because

$$A = \begin{bmatrix} 3 & \frac{3}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$$

Example-4 In each of the following cases write the objective function in the form

$z = X^T AX + q^T X$

(i) $z = x_1^2 + 2x_1x_2 + 46x_1x_3 + 3x_2^2 + 2x_2x_3 + 5x_3^2 + 4x_1 - 2x_2 + 3x_3$

(ii) $z = 5x_1^2 + 12x_1x_2 - 16x_1x_3 + 10x_2^2 - 26x_2x_3 + 17x_3^2 - 2x_1 - 4x_2 - 6x_3$

(iii) $z = x_1^2 - 4x_1x_2 + 6x_1x_3 + 5x_2^2 - 10x_2x_3 + 8x_3^2$

Solution : (i) $$z = (x_1, x_2, x_3)\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + (4, -2, 3)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Here $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 5 \end{bmatrix}, q = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$

(ii) $$z = (x_1, x_2, x_3)\begin{bmatrix} 5 & 6 & -8 \\ 6 & 10 & -13 \\ -8 & -13 & 17 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + (-2, -4, -6)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Here $A = \begin{bmatrix} 5 & 6 & -8 \\ 6 & 10 & -13 \\ -8 & -13 & 17 \end{bmatrix}, q = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$

(iii) $$z = (x_1, x_2, x_3)\begin{bmatrix} 1 & -2 & 3 \\ -2 & 5 & -5 \\ 3 & -5 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + (0, 0, 0)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Here $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 5 & -5 \\ 3 & -5 & 8 \end{bmatrix}, q = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

5.3 Positive and Negative Definiteness of Quadratic Forms

You have studied in 1.10, the positive and negative definite, semi-definiteness and indefinite of quadratic forms. There are several tests we may perform on the matrix of the quadratic form to find the character of quadratic form under consideration. Some of these have been discussed in 1.10.

Sylvester's law :

A quadratic form $X^T AX$ is positive definite if and only if all the successive principal minors of the matrix A are positive.

The successive principal minors are determinants of the square submatrices obtained by successively deteting lower rows and right hand columns. For $n \times n$ matrix, there are n-principal minors.

For example, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

then three principal minors of this determinant are

$$a_{11}, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

A quadratic form $X^T AX$ is negative definite if $-X^T AX$ is positive definite, since $-(X^T AX) = X^T(-A)X$, sylvester's theorem can be applied to $-A$ to test the negative definiteness of A.

We cannot test whether or not a matrix is positive definite by simply saying that all the successive principal minors to be non-negative (≥ 0) instead of positive (> 0). Rather all the principal minors must be non-negative. The matrix must be permuted in all possible combinations to determine all the $\left({}^{n}C_{r}\right)^{2}$ principal minors of order r, r = 1, 2, ...n. It is seldom feasible. For a real symmetric matrix, if successive principal minors are positive, then all the prinicipal minors are positive.

A matrix which is not positive definite, negative definite, positive semi definite, or negative semi-definite is indefinite.

Example-5 Determine the sign of definiteness for each of the following matrices.

(a) $\begin{bmatrix} 3 & 1 & 2 \\ 1 & 5 & 0 \\ 2 & 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 2 \\ 1 & -3 & 3 \\ 2 & 0 & -5 \end{bmatrix}$

Solution : (a) $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 5 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

$$a_{11} = 3, \quad \begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} = 15 - 1 = 14$$

$$\begin{vmatrix} 3 & 1 & 2 \\ 1 & -5 & 0 \\ 2 & 0 & -2 \end{vmatrix} = 52$$

A is not possible definite, so form $-A$:

$$-A = \begin{bmatrix} -3 & -1 & -2 \\ -1 & 5 & 0 \\ -2 & 0 & 2 \end{bmatrix}$$

Now

$$a_{11} = -3, \begin{vmatrix} -3 & -1 \\ -1 & 5 \end{vmatrix} = -16, \begin{vmatrix} -3 & -1 & -2 \\ -1 & 5 & 0 \\ -2 & 0 & 2 \end{vmatrix} = -52$$

So A is negative definite.

Example-6 Test the definitiness of the quadratic form :

$$X^{T}AX = (x_1, x_2, x_3)\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution : The characterstic equation for the matrix A is given by

$$|A-\lambda I|=0$$

or $$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & -2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}=0$$

$\Rightarrow$ $(3-\lambda)(-2,-\lambda)(1-\lambda)=0$

$\Rightarrow$ $\lambda=1,3,-2$

Since two eigenvalues are positive and one is negative, therefore the given quadratics form is indefinite.

Example-7 Determine whether or not the quadratic forms $A^T AX$ are positive definite, where

(i) $A=\begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$, (ii) $A=\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, (iii) $A=\begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix}$

Solution : We first check the principal minors to use the Sylvester's theorem.

(i) $|1|>0$, $\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix}=4>0$ and therefore A is positive definite.

(ii) $|1|>0$, $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}=1$, $|A|=0$, and therefore A is not positive definite.

(iii) $|1|>0$, $\begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix}=z>0$ and therefore A is positive definite.

Example-8 Determine the properties of sign definiteness for the following quadratic form :

$$z=x_1^2-4x_1x_2+6x_1x_3+5x_2^2-10x_2x_3+8x_3^2$$

Solution : Here $A=\begin{bmatrix} 1 & -2 & 3 \\ -2 & 5 & -5 \\ 3 & -5 & 8 \end{bmatrix}$

There successive principal minors of A are

$$|1|=1, \begin{vmatrix} 1 & -2 \\ -2 & 5 \end{vmatrix}=5-4=1, |A|=-2$$

Thus using sylvestor's law, α is not positive definite. Similarly, for $-A$, $|-1|=-1, |-A|=2$, so A is not negative definite. Hence A is either positive semidefinite, negative semi-definite, or indefinite. We observe that A is certainly indefinite by showing two points which make Z positive and negative, respectively.

5.4 Self Evaluation Exercise-I

1. Identify the incorrect statement : A quadratic form $Q(X)$ is :

   (a) Positive definite if and only if $Q(X) > 0$,

   (b) Negative definite if and only if $Q(X) < 0$,

   (c) Indefinite if $Q(X) > 0$ for some X and $Q(X) < 0$ for some other X.

   (d) Positive definite as well negative definite irrespective of sign of $Q(X)$.

2. The quadratic form with the associated matrices $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -1 & 14 \end{bmatrix}$ is :

   (a) $x_1^2 + 6x_2^2 + 14x_3^2 + 4x_1x_2 + 8x_1x_3 - 4x_2x_3$

   (b) $x_1^2 + 6x_2^2 + 14x_3^2 + 4x_1x_3 + 8x_2x_3 - 4x_1x_2$

   (c) $x_1^2 + 6x_2^2 + 14x_3^2 + 4x_2x_3 + 8x_1x_3 - 4x_1x_2$

   (d) $x_1^2 + 6x_2^2 + 14x_3^2 + 8x_1x_2 + 4x_1x_3 - 4x_1x_2$

3. Write the quadratic form in matrix vector notation

   $f(X) = x_1^2 - 2x_1x_2 + 4x_2^2$

4. Write down the quadratic form whose associated matrices are :

   (i) $\begin{bmatrix} 2 & -3 & 1 \\ -3 & 4 & 2 \\ 1 & 2 & -6 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & -2 \\ 4 & -2 & 14 \end{bmatrix}$

5. Which of the following are quadratic form?

   (i) $z = x_1^2 + 2x_2^2$ (ii) $z = \frac{x_1}{x_2}$

   (iii) $z = x_1^2 - x_2^2 + 4$ (iv) $z = x_1^2 - 2x_1x_2 + x_2^2 + 4x_1$

6. Determine the sign definiteness of each of the quadratic forms $X^T AX$:

   (i) $A = \begin{bmatrix} 2 & 1 & 4 \\ 6 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ (iii) $A = \begin{bmatrix} 1 & -2 & 1 \\ -4 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

7. Write objective function in the form $z = X^T AX + q^T X$.

(i) $z = 2x_1^2 + x_1x_2 + 9x_1x_2 + 3x_2^2 + x_2x_3 + 2x_2$

(ii) $z = x_1^2 - 6x_1x_2 + x_3^2 + 9x_3$

8. Write the quadratic form in the form $X^T AX$

(i) $x_1^2 + 8x_1x_2 + 16x_3^2 - 3x_3^2$

(ii) $2x_1^2 - 6x_1x_2 + 2x_1x_3 + 2x_3^2 + 6x_2x_3 - 5x_3^2$

9. Determine whether of the following quadratic form :

$x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$

is positive definite.

10. Determine whether each of the following quadratic forms is positive definite or negative definite :

(a) $2x_1^2 + 6x_2^2 - 6x_1x_2$ and (b) $-x_1^2 - x_2^2 - 4x_3^2 + x_1x_2 - 2x_2x_3$

5.5 General Non-Linear Programming Problem

A general non-linear programming problem (GNLPP) is defined as :

Find $(x_1, x_2, x_3, \ldots x_n)$ which

Optimize (Max. or Mini) $Z = f(x_1, x_2, \ldots\ldots x_n)$

Subject to $g^1(x_1, x_2, \ldots. x_n) \leq, =$ or $\geq b_1$

$g^2(x_1, x_2, \ldots. x_n) \leq, =$ or $\geq b_2$

....

....

$g^m(x_1, x_2 \ldots.. x_n) \leq, =$ or $\geq b_m$

and $x_j \geq 0, j = 1, 2 \ldots. n$.

where $Z, g^i{}'s$ real valued functions of n variables $x_1, x_2, \ldots., x_n$. Here either $f(x_1, x_2 \ldots.. x_n)$ or some $gi(x_1, x_2, \ldots\ldots x_n)$; $i = 1, 2 \ldots. m_j$ or both are non-linear.

In matrix notation a GNLPP may be written as

Determine $X^T \in R^n$ so as to maximize or minimize $Z = f(X)$ subject to the constraints :

$g^i(X) \leq, =$ or $\geq b_i$, $X \geq 0$

$i = 1, 2 \ldots.. m.$

Where $f(X)$ or some $g^i(X)$ or both are non-linear in X.

5.6 Constrained Optimization with Equality constraints (Lagrange's Multipler Method)

If the non-linear programming problem is composed of some differential objective function and equality constraints, the optimization can be done by the use of Lagrange multiplier. To understand the method we consider a simple GNLPP with one equality constraint with two variables :

Maximize or Minimize $Z = f(x_1, x_2)$

Subject to the constraint $g(x_1, x_2) = c$

and $x_1, x_2 \geq 0$

Where c is a constant.

Here we assume that $f(x_1, x_2), g(x_1, x_2)$ are differentiable with respect to x_1 and x_2. Now we introduce another differentiable function $h(x_1, x_2)$ defined as

$$h(x_1, x_2) = g(x_1, x_2) - c$$

Then the above problem is restated as

Max. or Min. $Z = f(x_1, x_2)$ subject to the coustraint $h(x_1, x_2) = 0$ and $x_1, x_2 \geq 0$

To find necessary conditions for the maximum or minimum (stationary) value of $z = f(x_1, x_2)$ new function is formed by using some multiplier λ, as

$$L(x_1, x_2, \lambda) = f(x_1 x_2) - Ah(x_1, x_2)$$

Here λ is an unknown constant, called Lagrange's Multiplier and the function $L(x_1, x_2, \lambda)$ is called Lagrange's Function. The necessary conditions for stationary value of $f(x_1, x_2)$ are given by

$$\frac{\partial L(x_1, x_2, \lambda)}{\partial x_1} = 0, \frac{\partial L(x_1, x_2 \lambda)}{\partial x_2} = 0, \frac{\partial L(x_1, x_2, \lambda)}{\partial \lambda} = 0$$

Now these partial derivatives are given by

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1},$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2},$$

$$\frac{\partial L}{\partial \lambda} = -h,$$

where L, f and h stand for the functions $L(x_1, x_2, \lambda)$, $f(x_1, x_2)$ and $h(x_1, x_2)$ respectively or simply by $L_1 = f_1 - \lambda h_1, L_2 = f_2 - \lambda h_2, L_\lambda = -h$

The necessary conditions for maximum or minimum value of $f(x_1, x_2)$ are thus given by

$f_1 = \lambda h_1, f_2 = \lambda h_2$ and $-h(x_1, x_2) = 0$

Example-9 Obtain the necessary conditions for the optimum solution of the following non-linear programming problem :

Min. $Z = f(x_1, x_2) = 3e^{2x_1+1} + 2e^{x_2+5}$

subject to the constraints : $x_1 + x_2 = 7$ and $x_1, x_2 \geq 0$

Solution : Let us define the Lagrange's function as $L(x_1, x_2, \lambda) = \lambda(x_1 + x_2 - 7)$

$$= 3e^{2x_1+1} + 2e^{x_2+5} - \lambda(x_1 + x_2 - 7)$$

Where λ is Lagrange's multiplier.

The necessary conditions for the minimum value of $f(x_1, x_2)$ are given by

$$\frac{\partial L}{\partial x_1} = \Rightarrow 6e^{2x_1+1} - \lambda = 0 \text{ or } \lambda = 6^{2x_1+1} \quad ...(1)$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2e^{x_2+5} - \lambda = 0 \text{ or } \lambda = 2e^{x_2+5} \quad ...(2)$$

$$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 - 7) = 0 \text{ or } x_1 + x_2 = 7 \quad ...(3)$$

(1) & (2) $\Rightarrow 6e^{2x_1+1} = 2e^{x_2+5}$

$$= 2e^{7-x_1+5}$$

$\Rightarrow \quad 3e^{2x_1+1} = e^{|2-x|}$

$\therefore \quad \log 3 + 2x_1 + 1 = 12 - x_1$

$\Rightarrow \quad x_1 = \frac{1}{3}[11 - \log 3]$

From (3) $x_2 = 7 - \frac{1}{3}(11 - \log 3)$

5.7 Necessary Conditions for General NLPP

Consider general non-liner programming problem (GNLPP) as :

Maximize (or Minimize) $Z = f(x_1, x_2 x_3, \ldots\ldots x_n)$

Subject to $g^i(x_1, x_2, \ldots x_n) = c_i; i = 1, 2, \ldots m$

$$x_j \geq 0; j = 1,2,3,\dots.n.(m < n)$$

If we take $h^i(x_1, x_2, \dots x_n) = g^i(x_1, x_2, \dots x_n) - c_i$ for all $i = 1,2,\dots.m$. Then the constraints reduce to $h^i(x_1, x_2, x_3 \dots. x_n) = 0; i = 1,2,3,\dots.m$. The problem in matrix form can be written as

Max (or Mini) $Z = f(x)$

Subject to $h^i(X) = 0 \quad i = 1,2,\dots.m$.

$$X \geq 0, \quad X \in R^n$$

To find maximum and minimum value of $f(X)$ we define Lagrange's function by introducing m Lagrange's multipliers $\lambda = (\lambda_1, \lambda_2, \dots \lambda_m)$ as :

$$L(X, \lambda) = f(X) - \sum_{i=1}^{m} \lambda_i h^i(X)$$

Let us assume that L, f and h^i are all differentiable partially with respect $x_1, x_2, x_3 \dots. x_n$ and $\lambda_1, \lambda_2, \dots. \lambda_m$. The necessary conditions for a maximum (minimum) of $f(x)$ are :

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial h^i(X)}{\partial x_j} = 0; \quad j = 1,2,\dots.n$$

$$\frac{\partial L}{\partial \lambda_i} = -h^i(X) = 0; \quad i = 1,2,\dots..m$$

There $m + n$ necessary conditions can be represented in the following form.

$$L_j = f_j - \sum_{i=1}^{m} \lambda_i h_j^i = 0 \text{ or } f_j = \sum_{i=1}^{m} \lambda_i h_j^i;$$

and $L_i = -h^i = 0$ or $h^i = 0$;

where $f_j = \dfrac{\partial f(X)}{\partial x_j}, h^i = h^i(X)$ and $h_j^i = \dfrac{\partial h^i(X)}{\partial x_j}$

5.8 Sufficient Condition for GNLPP

If in a general non-linear programming problem, the constraints are in equations. The necessary conditions will be sufficient for a maximum value of objective function if the objective function is concave and for minimum value of objective function if the objective function is convex.

When concavity and convexity of objective cannot be determined then we state sufficient conditions as follows :

(a) Sufficient conditions for NLPP with one equality constraint :

The Lagrange's function for a general NLPP involving n variables and one constraint is :

$$L(X,\lambda)=f(X)-\lambda h(X)$$

The necessary conditions for stationary point, are

$$\frac{\partial L}{\partial x_i}=\frac{\partial f}{\partial x_j}-\lambda\frac{\partial h}{\partial x_j}=0,\ j=1,2,3,\ldots.,n$$

and $$\frac{\partial L}{\partial \lambda}=-h(x)=0$$

The value of λ is defined by

$$\lambda=\frac{\frac{\partial f}{\partial x_j}}{\frac{\partial h}{\partial x_j}} \qquad \text{(for } j=1,2,\ldots n\text{)}$$

The sufficient conditions for miximum or minimum value of $f(X)$ require the evaluation at each stationary point of $n-1$ principal minors of the determinant given below:

$$\Delta_{n+1}=\begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \ldots.. & \frac{\partial h}{\partial x_n} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_1}-\lambda\frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2}-\frac{\partial^2 h}{\partial x_1 \partial x_2} & \ldots.. & \frac{\partial y}{\partial x_1 \partial x_n}-\lambda\frac{\partial^2 y}{\partial x_1 \partial x_n} \\ \frac{\partial h}{\partial x_2} & \frac{\partial y}{\partial x_2 \partial x_1}-\lambda\frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2}-\lambda\frac{\partial^2 h}{\partial x_2^2} & \ldots... & \frac{\partial y}{\partial x_2 \partial x_n}-\lambda\frac{\partial^2 h}{\partial x_2 \partial x_n} \\ \\ \frac{\partial h}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1}-\lambda\frac{\partial^2 h}{\partial x_n \partial x_1} & \frac{\partial y}{\partial x_n \partial x_2}-\lambda\frac{\partial^2 h}{\partial x_n \partial x_2} & \ldots.. & \frac{\partial^2 h}{\partial x_n^2}-\lambda\frac{\partial h}{\partial x_n^2} \end{vmatrix}$$

(i) If $\Delta_3>0, \Delta_4<0, \Delta_5>0,$ …… the sign pattern being alternate, the stationary point is local maximum.

(ii) If $\Delta_3<0, \Delta_4<0,$ ……$\Delta_{n+1}<0$, the sign being always negative, the stationary point is local minimum.

Example-10 Obtain the necessary and sufficient conditions for the following NLPP.

Minimize $Z=2x_1^2-24x_1+2x_2^2-8x_2+2x_3^2-12x_3+200$

Subject to $x_1 + x_2 + x_3 = 11$

$x_1, x_2, x_3 \geq 0$

Solution : The Lagrangian function for the given problem is

$$L(x_1, x_2, x_3, \lambda) = 2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200 - \lambda(x_1 + x_2 + x_3 - 11)$$

The necessary conditions for the stationary point are

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 4x_1 - 24 - \lambda = 0$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 4x_2 - 8 - \lambda = 0$$

$$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 4x_3 - 12 - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 0 - (x_1 + x_2 + x_3 - 11) = 0$$

Solving above four simulteneous equations, we get the stationary point $X_0 = (x_1, x_2, x_3) = (6,2,3); \lambda = 0$

For sufficient condition, Here $n = 3$

$$\therefore \Delta_4 = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} & \frac{\partial y}{\partial x_1 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_3} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_3} \\ \frac{\partial h}{\partial x_3} & \frac{\partial^2 f}{\partial x_3 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} - \lambda \frac{\partial^2 h}{\partial x_3^2} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{vmatrix} = -48$$

$$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 4 \end{vmatrix} = -8$$

$\because$ Δ_3, Δ_4 both are negative, therefore the above necessary conditions are sufficient i.e. $X_0 = (6, 2, 3)$ gives minimum value of the objective function.

(b) Sufficient conditions for General NLPP with $(m < n)$ equality constraints :

First we write Lagrange's function fo a GNLPP with more than one constraint by introducing m lagrange multipliers $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_m)$

$$L(X, \lambda) = f(X) - \sum_{i=1}^{m} \lambda_i h^i(X) \qquad (m < n)$$

The necessary conditions for stationary points of $f(x)$ can be obtained from the equations :

$$\frac{\partial L}{\partial x_j} = 0, \quad j = 1, 2, 3, \ldots n$$

$$\frac{\partial L}{\partial \lambda_i} = 0, \quad i = 1, 2, 3, \ldots m$$

Thus the optimization of $f(x)$ subject to $h^i(X) = 0$ is equivalent to the optimization of $L(X, \lambda)$. To write sufficient conditions for stationary point of $f(X)$, we assume that the function $L(X, \lambda), f(X)$ and $h(X)$ all possess partial derivalines of order one and two with respect to the decision variables.

Let $V = \left[\dfrac{\partial^2 L(X, \lambda)}{\partial x_i \, \partial x_j}\right]_{n \times n}$

be the matrix of second order partial derivaties of $L(X, \lambda)$ w.r. to decision variables

$$\cup = \left[h_j^i(X)\right]_{m \times n}$$

Where $h_j^i(X) = \dfrac{\partial h^i(X)}{\partial x_j}, i = 1, 2, \ldots m; \; j = 1, 2, \ldots n$

Define the square matrix $H^B = \left[\begin{array}{c|c} O & U \\ \hline U^T & V \end{array}\right]_{(m+n) \times (m+n)}$

Where O is the null matrix of order $m \times m$. The matrix H^B is called bordered Hessian Matrix.

Now the suficient conditions for maximum and minimum stationary points are given below :

Let (X_0, λ_0) be the stationary point for the function $L(X, \lambda)$ and H_0^B be the corresponding bordered Hassian matrix computed at (X_0, λ_0), then X_0 is a

(i) Maximum point, if starting with principal minors of order $(2m+1)$, the last $(n-m)$ principal minors of H_0^B form an alterating sign pattern starting with $(-1)^{m+n}$; and

(ii) Minimum point, if starting with prinicpal minor of order $(2m+1)$, the last $(n-m)$ principal minors of H_0^B have the sign of $(-1)^m$.

Note : It can be observed that above conditions are only sufficient for identifying an extreme point, but not necessary. That is, a stationary point may be an extreme point without satisfying the above condition.

5.9 Illustrative Examples

Example-11 Obtain the necessary and sufficient conditions for the optimum solution of the following NLPP.

Minimize $Z = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

subject to $x_1 + x_2 + x_3 = 15$

$$2x_1 - x_2 + 2x_3 = 20, \quad x_1, x_2, 2x_3 \geq 0$$

Solution : Here, we have

$$f(X) = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$$

$$h^1(X) = x_1 + x_2 + x_3 - 5 \ ; \qquad h^2(X) = 2x_1 + x_2 + 2x_3 - 30$$

The Lagrangian function is defined as

$$L(X, \lambda) = f(X) - \lambda_1 h^1(X) - \lambda_2 h^2(X)$$

$$4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2 - \lambda_1(x_1 + x_2 + x_3 - 15)$$

$$-\lambda_2(2x_1 + x_2 + 2x_3 - 30)$$

The necessary conditions for the stationary values of $f(x)$ are as :

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 8x_1 - 4x_2 - \lambda_1 - 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 4x_2 - 4x_1 - \lambda_1 - \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_3} = 0 \Rightarrow 3x_3 - \lambda_1 - 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow -(x_1 + x_2 + x_3 - 15) = 0$$

$$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow -[2x_1 - x_2 + 2x_3 - 20] = 0$$

Solving above simultenceous equations we get stationary point (X_0, λ_0) as :

$$X_0 = (x_1, x_2, x_3) = \left(\frac{33}{9}, \frac{10}{3}, 8\right) \text{ and}$$

$$\lambda_0 = (\lambda_1, \lambda_2) = \left(\frac{40}{9}, \frac{52}{9}\right)$$

The Bordered Hessian matrix at (X_0, λ_0) is given by

$$H_0^B = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 2 \\ 1 & 2 & 8 & -4 & 0 \\ 1 & -1 & -4 & 4 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{bmatrix}$$

Since $m=2, n=3$, therefore $n-m=1\ \ 2m+1=5$. It means one needs the check the determinant of H_0^B only and it must have the sign of $(-1)^2$.

Now, det $H_0^B = 90 > 0$, therefore x_0 is a minimum point.

Example-12 Obtain a set of necessary condition for the non-linear programming problem :

$$\textit{Maximize} \quad Z = x_1^2 + 3x_2^2 + 5x_3^2$$

subject to $5x_1 + 2x_2 + x_3 = 5$

$$x_1, x_2 x_3 \geq 0$$

Solution : Here, we have $X = (x_1, x_2, x_3)$ $f(X) = x_1^2 + 3x_2^2 + 5x_3^2$, $g^1(X) = x_1 + x_2 + 3x_3$,

$g^2(X) = 5x_1 + 2x_2 + x_3$ and $c_1 = 2,\ c_2 = 5$

Definging $h^1(X) = g^1(X) - c_1$, $h^2(X) = g^2(X) - c_2$

Thus we have the constraint

$$h^i(X) = 0,\ i = 1, 2$$

The Lagrange's function is defined as :

$$L(X,\lambda)=f(X)-\lambda_1 h^1(X)-\lambda_2 h^2(X)$$

$$\lambda=(\lambda_1,\lambda_2)$$

This finds the following necessary conditions

$$\frac{\partial L}{\partial x_1}=0 \Rightarrow 2x_1-\lambda_1-5\lambda_2=0$$

$$\frac{\partial L}{\partial x_2}=0 \Rightarrow 6x_2-\lambda_1-2\lambda_2=0$$

$$\frac{\partial L}{\partial x_3}=0 \Rightarrow 10x_3-3\lambda_1-\lambda_2=0$$

$$\frac{\partial L}{\partial x_1}=0 \Rightarrow -(x_1+x_2+3x_3-2)=0$$

$$\frac{\partial L}{\partial x_2}=0 \Rightarrow -(5x_1+2x_2+x_3-5)=0$$

Examples-13 Find the dimension of a rectangular parallelopiped with largest volume whose sides are parallel to the coordinate planets, to be inscribed in the ellpsoide $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution : Let the dimensions of a rectangular parallelopiped be x, y and z. Its volume is given by

$$f(x,y,z)=xyz$$

Thus the problem is

$$Max.\ f(x,y,z)=xyz$$

s.t. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

and $x,y,z\geq 0$

The necessary conditions for maximum value of $f(x,y,z)$ are as :

$$\frac{\partial L}{\partial x}=0 \Rightarrow yz-\frac{2\lambda x}{a^2}=0 \qquad ...(1)$$

$$\frac{\partial L}{\partial y}=0 \Rightarrow zx-\frac{2\lambda y}{b^2}=0 \qquad ...(2)$$

$$\frac{\partial L}{\partial z}=0 \Rightarrow xy-\frac{2\lambda z}{c^2}=0 \qquad ...(3)$$

$$\frac{\partial L}{\partial \lambda}=0 \Rightarrow -\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=0 \qquad ...(4)$$

from (1) $\quad yz=\frac{2\lambda x}{b^2}$

dividing we get $\frac{y}{x}=\frac{x}{y}\frac{b^2}{a^2}$

$$\Rightarrow \frac{x}{a}=\frac{y}{b}$$

Similarly $\frac{y}{b}=\frac{z}{c}$

$\therefore \quad \frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\frac{1}{\sqrt{3}}$ $\qquad$ using (4)

$\therefore \quad x=\frac{a}{\sqrt{3}}, y=\frac{b}{\sqrt{3}}, z=\frac{c}{\sqrt{3}}$

which are the required dimennsions

Example-14 A positive quantity b is to be divided into n parts in such a way that the product of the n parts is to be maximum. Use Lagrange multipler technique to obtain the optimal subdivision.

Solution : Let b be divided into n parts $x_1, x_2 ...., x_n$, so that we have to maximize the function

$$z=x_1.x_2.x_3....x_n \qquad ...(1)$$

subject to

$$x_1+x_2+x_3+....+x_n=b \qquad ...(2)$$

$$x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0$$

The Lagrange's Function is defined as :

$$L(x_1,x_2,x_3,...,x_n,\lambda)=x_1x_2,x_3,....x_n+\lambda(x_1+x_2+....+x_n-b)$$

The necessary condition are

$$\frac{\partial L}{\partial x_1}=0 \Rightarrow x_2x_3....x_n-\lambda=0 \qquad ...(3)$$

$$\frac{\partial L}{\partial x_2}=0 \Rightarrow x_1x_3....x_n-\lambda=0 \qquad ...(4)$$

$$\frac{\partial L}{\partial \lambda}=0 \Rightarrow (x_1+x_2+x_3+....+x_n-b)=0$$

Dividing (3) by (4) $\frac{x_2}{x_1} = 1 \quad \Rightarrow x_1 = x_2$

Similarly $x_2 = x_3 = x_4 = ... = x_n$

Thus (6) $\Rightarrow \quad x_1 = x_2 = x_3 = ... x_n = \frac{b}{n}$

$\therefore$ Max. value of $z = \frac{b}{n} . \frac{b}{n} ..... \frac{b}{n} = \left(\frac{b}{n}\right)^n$ (n times)

Example-15 A manufacturing concern produces a product consisting of two raw materials, say A_1 and A_2. The production function is estimated as

$$z = f(x_1, x_2) = 3.6x_1 - 0.4x_1^2 + 1.6x_2 - 0.2x_2^2$$

Where z represents the quantity (in tons) of the product produced and x_1 and x_2 disignate the input amounts of raw materials A_1 and A_2. The company has Rs 50,000 to spend on there two raw materials. The unit price of A_1 is Rs 10000 and of A_2 is Rs 5000. Determine how much input amounts of A_1 and A_2 be decided so as to maximize the production output.

Solution : Since the company must operate within the available funds, the budgetary constraint is $10000x_1 + 5000x_2 \le 50000$ or $2x_1 + x_2 \le 10$ we reduce this inequality constraint to an equality by imposing an additional assumption that the company has to spend every available single paisa on these raw materials. Then, the constraint is $2x_1 + x_2 = 10$. Also, obiviously $x_1 \ge 0, x_2 \ge 0$. The problem of the company can thus be written as :

Maximize $z = f(x_1, x_2) = 3.6x_1 - 0.4x_1^2 + 1.6x_2 - 0.2x_2^2$

s.t. $2x_1 + x_2 = 10$

and $x_1, x_2 \ge 0$

The Lagrange's Function is

$$L(x_1, x_2, \lambda) = 3.6x_1 - 0.4x_1^2 + 1.6x_2 - 0.2x_2^2 - \lambda(2x_1 + x_2 - 10)$$

The necessary conditions are

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 3.6 - 0.8x_1 - 2\lambda = 0$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 1.6 - 0.4x_1 - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow -(2x_1 + x_2 - 0) = 0$$

Solving above simulteneous equations we get $(x_1, x_2, \lambda) = (3,5,3)$

$\because$ z is a concave function so the necessry conditions are sufficient therefore z is maximum at

$x_1 = 3.5, x_2 = 3$

$\therefore$ Max $z = f(3,5,3)$

$= 3.6(3.5) - 0.4(3.5)^2 + 1.6(3) - 0.2(3)^2$

10.7 *tons*.

Thus in order to have a maximum production of 10.7 tons, the company must input 3.5 units or raw material A and 3 units of raw material B.

5.10 Self-Learning Exercise-II

1. Define Lagrange's functions.
2. What are Lagrange's multipliers?
3. State whether true or false :
   - (i) The necessary conditions will be sufficient to maximize a concave function.
   - (ii) The necessary any conditions wil be sufficient to minimize a convesfunction.
   - (iii) The necessary condition will be sufficient minimize a concave function.

5.11 Summary

Quadratic forms have been introduced in the unit-1. A further study have been done in this unit. Tests for the positiveness and negativeness are defined. There are two tests for this, Eigenvalue test and principal minor test. You are able to test prositive/negativeness of quadratic form by doing ample examples given in this unit. In the second part of this unit you have learnt the method to solve non-linear programming problem with equality constraints. The necessary and sufficient conditions are given with the help of Lagrange's multipliers and Lagrange's function. The necessary conditions are sufficient for maximization of an objective function if it is concave and for minimization of an objective function it is convex.

If convcavity and convexity is not known of the objective function, then principal miners of hassian matrix are evaluated.

5.12 Answers to Self-Learning Exercise-I

1. (b) 2. (a) 3. $(x_1, x_2)\begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

4. (i) $2x_1^2 + 4x_2^2 - 6x_3^2 - 6x_1x_2 + 4x_2x_3 + 2x_3x_1$

(ii) $x_1^2 + 6x_2^2 + 14x_3^2 + 4x_1x_2 - 4x_2x_3 + 8x_3x_1$

5. (i), (iii), (iv)

6. (i) Indefinite, (ii) Indefinite, (iii) Indifinite

7. (i) $H = \begin{bmatrix} 2 & \frac{1}{2} & \frac{9}{2} \\ \frac{1}{2} & 3 & \frac{1}{2} \\ \frac{9}{2} & \frac{1}{2} & 0 \end{bmatrix}, \ q = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

(ii) $H = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \ q = \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}$

5.13 Answer to Self-Learning exercise-II

3. (i) True (ii) True (iii) False

5.14 Exercise

Solve the following non-linear programming problems, using lagrange's multiplier method :

1. Minimize $z = 6x_1^2 + 5x_2^2$

   Subject to $x + 5x_2 = 3, x_1, x_2 \geq 0$

2. Minimize $z = 3x_1^2 = x_2^2 - 2x_1x_2 + 6x_1 - 2x_2$

   Subject to $2x_1 + x_2 = 4, x_1, x_2 \geq 0$

3. Minimize $z = 2x_1^2 + x_2^2 = 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$

   Subject to $x_1 + x_2 + x_3 = 20, x_1, x_2, x_3 \geq 0$

4. Maximize $z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$

   Subject to $x_1 + 2x_2 = z, \ x_1, x_2 \geq 0$

5. Maximize $z = 5x_1 + x_2 - (x_1 - x_2)^2$

   Subject to $x_1 + x_2 = 4, x_1, x_2 \geq 0$

6. Minimize $z = x_1^2 + x_2^2 + x_3^2$

   Subject to $4x_1 + x_2^2 + 2x_3 = 14, \quad x_1, x_2, x_3 \geq 0$

7. Minimize $z = x_1^2 + x_2^2 + x_3^2$

   Subject to $x_1 + x_2 + 3x_3 = 5$

   $5x_1 + 2x_2 + x_3 = 5$

   $x_1, x_2 \geq 0$

8. Minimize $z = 6x_1 + 8x_2 - x_1^2 - x_2^2$

Subject to $4x_1 + 3x_2 = 16$,

$3x_1 + 5x_2 = 15$

$x_1, x_2 \geq 0$

9. Solve the following NLPP :

Optimize $z = 4x_1 + 9x_2 - x_1^2 - x_2^2$

Subject to $4x_1 + 3x_2 = 15$

$3x_1 + 5x_2 = 14$

$x_1, x_2 \geq 0$

10. Determine optimum solution for the following NLPP and check whether it maximizes or minimizes the objective function :

$z_1 = x_1^2 - 10x_1 + x_2^2 - 6x_2 + x_3^2 - 4x_3$

Subject to $x_1 + x_2 + x_3 = 7$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.

□□□

Unit - 6
Non Linear Programming Problems

Structure of the Unit

6.0 Objective

The objective of writing this unit is to get students acquainted with the programming problems that are not linear by nature. Such problems are of great importance and are solved by different methods. One such method is the method of Lagrange multipliers which provides a necessary condition for the optimum of the objective function, when the constraints are in the form of equations.

6.1 Introduction

The unit begins with the formal definition of mathematical programming problem followed by the introduction of general nonlinear programming problem. The construction of Lagrangain function and its relation with the minimal point of the objective function is briefly discussed. The necessary and sufficient conditions for the function $F(X,\lambda)$ to have a saddle point are also derived. In the last, graphical method for solving nonlinear programming problem in two variables is also explained through few examples.

6.2 Mathematical Programming Problem

A general mathematical programming problem (MPP) can be stated as given below :

Minimize $f(X)$,

Subject to $\quad g_i(X) \geq 0,$ ...(1)

$h_j(X) = 0,$ ...(2)

$X \in S,$ ...(3)

Where $X = (x_1, x_2, \ldots, x_n)^T$ is a vector of decision variables (that are known) and

$f, g_i\ (i=1,2,...,m)$ and $h_j\ (j=1,2,....,p)$ are the real valued functions of variables $x_1, x_2, ..., x_n$.

The function f in the above formulation is called the objective function. The inequalities (1), equations (2) and the set restrictions (3) are called the constraints. The above mathematical programming problem is a minimization problem, which is considered without any loss of generality, since a maximization problem can always be converted into a minimization problem by using the fact max $f(X)=$ min$(-f(X))$. That intends to say that the maximization of $f(X)$ is equivalent to the minization of $-f(X)$.

Usually, the functions f, g_i and h_j are assumed to be continuous or continuously differentiable functions. Also the set S is considered as a connected subset of R^n. If $S=R^n$ and all the functions appearing in the mathematical programming problem (MPP) are linear in the decision variables X, the mathematical programming problem is called a Linear Programming Problem (LPP). A mathematical programming problem, that is not a linear programming problem is called a **Nonlinear Programming Problem** (NLPP).

The set T of all those points $X \in S$, which satisfy constraints (1) to (3) is known as the feasible region, feasible set or feasible constraint set of the MPP and every point of this set T is called a feasible solution of the MPP. If the constraint set T is empty, then we say that there is no feasible solution to the MPP and the problem is said to be inconsistent.

A feasible solution $X_0 \in T$ of the MPP is said to be an **optimal solution** or a **global optimal solution**, if $f(X) \geq f(X_0)$ for all $X \in T$. This global optimal solution $X_0 \in T$ of the MPP is actually a global minimum point of the MPP. $X_0 \in T$ is referred to as a global maximum point of the function f over the set T if X_0 is a global minimum point of $-f$ over T.

A point $X^* \in T$ is said to be a **local minimum** or **relative minimum** point of the function $f(X)$ over T if there exists a positive number δ such that $f(X) \geq f(X^*)$ for all $X \in T \cap N_\delta(X^*)$, where $N_\delta(X^*)$ is the neighbourhood of X^* with radius δ. The point $X^* \in T$ is a local miximum or a relative maximum point of the function *f* over T if X^* is a local minimum point of $-f$ over T. A point $X^* \in T$ referred to as a local **extremum point** if it is either a local minimum point or a local maximum point. It is noticeable from the above definitions that a global minimum (maximum) point is also a local minimum (maximum) point, but not conversely.

In fact a mathematical programming problem can be classified into two different categories- **unconstrained optimization problem** and **constrained optimization problem**. If the constraint set T is the whole of the space R^n, the problem is said to be an unconstrained optimization problem, for in this case, we are to find a point in R^n that gives an optimum value to the objective function. If T is a proper subset of R^n, then the problem becomes a constrained optimization problem.

Example 1 : Maximize $z=(x_1-1)^2+\left(x_2-\frac{1}{2}\right)^2$

subject to $x_1+x_2 \leq 2$

$x_1, x_2 \geq 0$

The shaded region OAB in the figure 6.1 shows the feasible region. The objective contour $(x_1-1)^2+\left(x_2-\frac{1}{2}\right)^2=z$ is a circle whose centre is $\left(1,\frac{1}{2}\right)$ and radius $\sqrt{z}$. Since we are looking for the maximum value of z, we must find the circle with the largest radius that intersects the feasible region. We see that the point B(0, 2) is the optimal solution with the objective value $\frac{13}{4}$. It can be noticed from the objective contours (dotted circles) that the point A(2, 0) is a point of local maximum but not of global maximum with the objective value $\frac{5}{4}$.

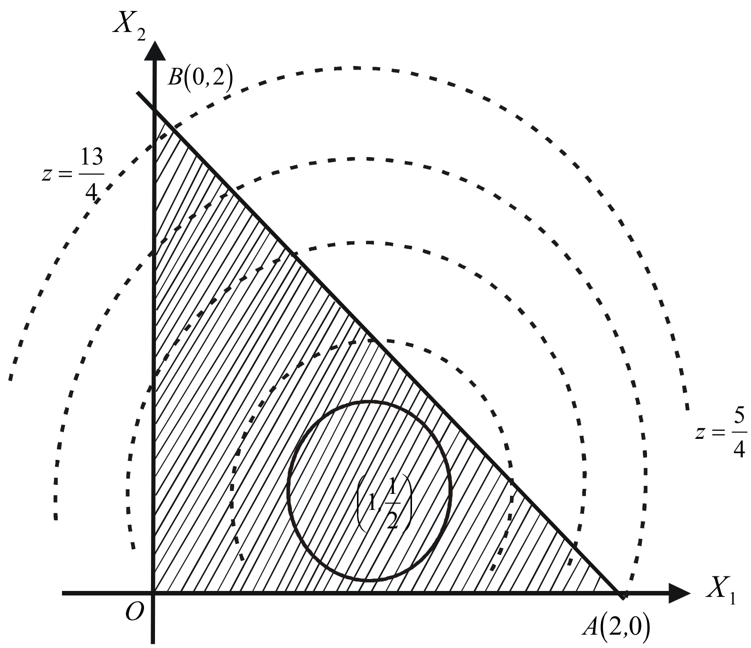


Figure : 6.1

The above example confirms that a local optimum neet not be a global optimum. This is the reason that the derivations of algorithms for non-linear programming problems are difficult to some extent.

Example-2 Minimize $z=(x_1-1)^2+\left(x_2-\frac{1}{2}\right)^2$

subject to $x_2^2-x_1-1\le 0$

$x_1+x_2\le 2$

$x_1,x_2\ge 0$

The feasible region of the given NLPP is shown as the shaded region OABC in the figure 6.2. The objective contour $(x_1-1)^2+\left(x_2-\frac{1}{2}\right)^2=z$ is a circle with centre $\left(1,\frac{1}{2}\right)$ and radius $\sqrt{z}$. Since we are to minimize z, therefore we must look for the circle having the smallest radius that intersects the feasible region. Clearly such a circle with smallest radius is the point circle (i.e. circle that has radius zero), since the point $\left(1,\frac{1}{2}\right)$ lies inside the feasible region. Therefore, the optimal solution to the problem is $x_1=1$ and

$x_2 = \frac{1}{2}$, with the objective value 0.

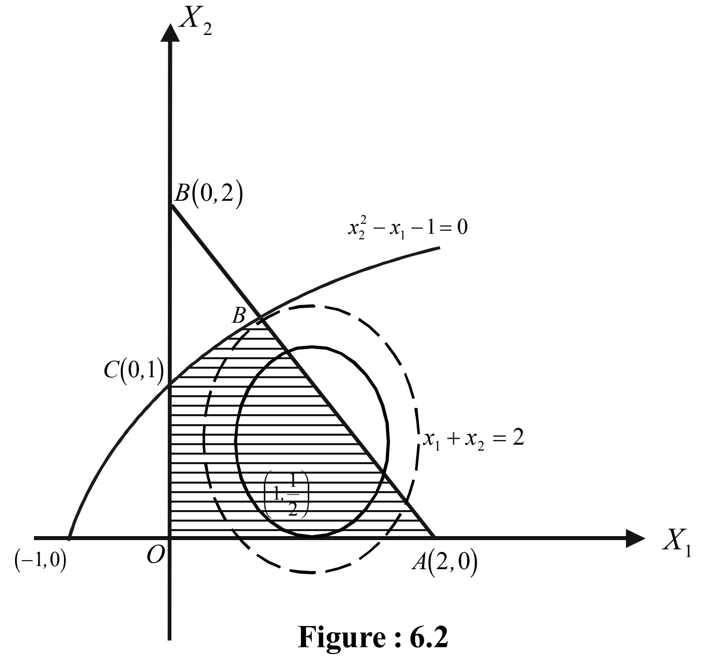


Figure : 6.2

From the above example one can notice that the optimal solution to the NLPP could be any point of the feasible region. This adds to difficulties in solving the NLPP.

6.3 General Nonlinear Programming Problem (GNLPP)

A general nonlinear programming problem (GNLPP) can be formulated as :

Suppose that we are looking for a solution of nonnegative variables $x_j \geq 0$; $j = 1,2,\ldots,n$, which maximize or minimize the real valued function (called the objective function)

$$z = f(x_1, x_2, \ldots x_n),$$

and satisfies the set of m constraints

$$g_1(x_1, x_2, \ldots x_n) \qquad \{\leq, \geq or =\}\ b_1$$

$$g_2(x_1, x_2, \ldots, x_n) \qquad \{\leq, \geq or =\}\ b_2$$

$$g_m(x_1, x_2, \ldots, x_n) \qquad \{\leq, \geq or =\}\ b_m$$

where either $f(x_1, x_2, \ldots, x_n)$ or some $g_i(x_1, x_2, \ldots, x_n)$; $i = 1,2,\ldots,m$ or both are nonlinear real valued functions of n variables $x_1, x_2, \ldots., x_n$.

In matrix form, the GNLPP can be written as :

Determine $X^T \in R^n$ that maximize or minimize the objective function

$$z = f(X)$$

subject to the constraints

$$g_i(\overline{X})\{\leq, \geq or =\}\ b_i; \qquad i=1,2,\ldots,m$$

$$\overline{X} \geq 0$$

where either $f(X)$ or some $g_i(X)$ or both are nonlinear in X.

It is some-times convenient to write the constraints $g_i(X)\{\leq, \geq or =\}b_i$ as $h_i(X)\{\leq, \geq or =\}0$, for $h_i(X)=g_i(X)-b_i$.

6.4 Lagrangian Function and Saddle Point

Let us consider the NLPP a follows :

Minimize $z=f(X); \quad X \in R^n$ ...(1)

subject to $\quad g_i(X) \leq 0;\ i=1,2,\ldots,m$ ...(2)

$X \geq 0$ ...(3)

Where $f(X)$ and $g_i(X)$ are convex functions of $X \in R^n$. ...(4)

In fact, if $f(X)$ is a convex function, then it has a unique relative minimum which is also a global minimum. It can also be learnt that if $f(X)$ is convex, then $-f(X)$ is concave and that $\min f(X) \max(-f(X))$. At present, we relax the condition (3) and (4) (i.e. there is no restriction on X and functions $f(X)$ and $g_i(X)$ are not necessarily convex functions) and consider the problem of minimizing $f(X)$ subject to the constraint set (2) only.

Let us define the function $F(X,\lambda)$ as

$$F(X,\lambda)=f(X)+\sum_{i=1}^{m}\lambda_i g_i(X)$$

$$=f(X)+\lambda^T G(X) \qquad \text{...(5)}$$

where $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_m)^T$ and

$$G(X)=(g_i(X),g_2(X),\ldots,g_m(X))^T \qquad \text{...(6)}$$

Equation (5) shows that $F(X,\lambda)$ is nothing but the Lagrangian function, with the m componnts of λ as the lagrange multipliers.

A point (X_0,λ_0) is said to be a saddle point of the Lagrangian function $F(X,\lambda)$ if

$$F(X_0,\lambda) \leq F(X_0,\lambda_0) \leq F(X,\lambda_0)$$

in some neighbourhood of (X_0, λ_0). The Saddle point of the lagrangian function $F(X, \lambda)$, if at all exists, and the minimal point of the objective function $f(X)$ have a theoretical relationship with each other. This theoretical relationship with each other. This theoretical relationship has led not only to various important theoretical developements but also to algorithms for solving NLPP. This relationship is established through a number of theorems which are various constituents of what we know as **Kuhn-Tucker theory.**

6.4.1 Relation between Saddle Point of $F(X,\lambda)$ and minimal point of $F(X)$

Let $F(X)$ be a real-valued function in R^n and $G(X)$ a vector function consisting of real-valued functions $g_i(X); i=1,2,....,m$.

Consider

Minimize $\quad z=f(X)$ ...(1)

subject to $\quad G(X)\leq 0$ ...(2)

and $\quad F(X,\lambda)=f(X)+\lambda G(X)$ ...(3)

where $\lambda=(\lambda_1,\lambda_2,...,\lambda_m)^T$, and $\lambda\geq 0$. ...(4)

Theorem-1 : If $F(X,\lambda)$ has a saddle point (X_0,λ_0), for each $\lambda\geq 0$, then

$$G(X_0)\leq 0 \text{ and } \lambda_0^T G(X_0)=0.$$

Proof : Let (X_0,λ_0) be the saddle point of the function $F(X,\lambda)$ where $\lambda\geq 0$. Then from the definition

$$F(X_0\lambda)\leq F(X_0,\lambda_0)\leq F(X,\lambda_0)$$

or $f(X_0)+\lambda^T G(X_0)\leq f(X_0)+\lambda_0^T G(X_0)\leq f(X)+\lambda_0^T G(X)$ ...(5)

The left hand side of the inequality (5) shows that

$\lambda^T G(X_0)\leq \lambda_0^T G(X_0)$ ...(6)

If possible, let $g_i(X_0)>0$ for some i. Then whatever may be λ_0, the i^{th} component λ_i of λ can be chosen sufficiently large, so that $\lambda^T G(X_0)$ is large enough to disobey the inequality (6). Hence we must have

$$g_i(X_0)\leq 0 \text{ for all } i=1,2,...,m.$$

Or, $\quad G(X_0)\leq 0$ ...(7)

Now since $\lambda_0\geq 0$ and $G(X)\leq 0$, therefore,

$\lambda_0^T G(X_0)\leq 0.$ ...(8)

Also inequality (6) holds for all $\lambda \geq 0$, therefore, it holds for $\lambda = 0$ also and so

$$\lambda_0^T G(X_0) \geq 0 \qquad ...(9)$$

From equations (9) and (10), we have

$$\lambda_0^T G(X_0) = 0 \qquad ...(10)$$

Theorem-2 If (X_0, λ_0) is a saddle point of the function $F(X, \lambda)$ for every $\lambda \geq 0$, then X_0 is a minimal point of $f(X)$ subject to the constraints $G(X) \leq 0$.

Proof : Using the right hand side inequality of (5) and the reult (1) of theorem-1, we have

$$f(X_0) \leq f(X) + \lambda_0^T G(X)$$

and since $\lambda_0 \geq 0$ and $G(X) \leq 0$, therefore,

$f(X_0) \leq f(X)$ for all those points X which satisfy $G(X) \leq 0$,

The converse of the above theorem need not be true always.

Theorem-3 Let X_0 be a solution of the NLPP

Minimize $z = f(X)$; $X \in R^n$

subject to $G(X) \leq 0$, where

$G(X) = (g_1(X), g_2(X), \ldots, g_m(X))^T$ and

$f(X), g_i(X); i = 1, 2, \ldots, m$ are all convex functions

Let the set of points X such that $G(X) < 0$ be nonempty. The there exists a vector $\lambda_0 \geq 0$ in R^m such that

$$f(X) + \lambda_0^T G(X) \geq f(X_0).$$

Proof : Let $b = (b_0, b_1, \ldots, b_m)^T$ be a vector in R^{m+1} and let

$$C_1 = \{b : b_0 \geq f(X) - f(X_0); g_i(X) \leq b_i, i = 1, 2, \ldots, m\}$$

where for each such b, there is atleast one X for which the above conditions for b hold. It is clear that C_1 is a convex set. Note that $g_i(X)$ are convex functions for $i = 1, 2, \ldots, m$.

Let us consider another set $C_2 \subset R^{m+1}$ defined by

$$C_2 = \{b : b < 0\}.$$

Then it can be seen that C_2 is also a convex set. Further $C_1 \cap C_2 = \phi$, since $b_0 \geq f(X) - f(X_0) \geq 0$ for $b \in C_1$ and $b < 0$ for $b \in C_2$. Now C_1 and C_2 are disjoint convex sets, therefore there can be constructed a hyperplane separating C_1 and C_2. The point $b=0$ is the boundary point of C_1 and C_2 and so the separating hyperplane must pass through this point $b=0$. Let this separating hyperplane be

$$Cb=0,\ C\neq 0$$

Where $Cb\geq 0$, for $b\in C_1$ ...(1)

and $Cb<0$, for $b\in C_2$ ...(2)

The vector C is bound to be nonnegative since if $C\ngeq 0$, then it means that there is some component c_i of C such that $c_i<0$. Now if $b^{(2)}$ is any point in C_2, then $b^{(2)}<0$. Let $b_i^{(2)}$ be the i^{th} component of $b^{(2)}$. Then let $b_i^{(2)}=-M$ for $M>0$. The i^{th} term $c_ib_i^{(2)}$ in C b is clearly positive and by taking M sufficiently large, this term $c_ib_i^{(2)}$ can be made dominating over all other terms in C b, which is against the inequality (2). Thus we conclude that $C\geq 0$.

Now let $b=\left(f(X)-f(X_0),g_1(X),g_2(X),...,g_m(X)\right)^T$ be any point in C_1, Then from (1)

$$c_0f(X)-c_0f(X_0)+c_1g_1(X)+c_2g_2(X)+....+c_mg_m(X)\geq 0 \qquad ...(3)$$

where, $C'=(c_0,c_1,c_2,...,c_m)$

Or, $c_0f(X)+c_1g_1(X)+c_2g_2(X)+...+c_mg_m(X)\geq c_0f(X_0)$

It can be proved that $c_0\neq 0$, since if $c_0=0$, then (3) becomes

$$c_1g_1(X)+c_2g_2(X)+...+c_mg_m(X)\geq 0 \qquad ...(4)$$

Now let X be a point such that $G(X)<0$ and (condition given in th theorem). Also since $C\geq 0$ and $C\neq 0$, therefore (4) is a contradiction for such a point X. But it holds for all X, therefore, $c_0\neq 0$.

Now dividing (3) by c_0 and taking $\frac{c_i}{c_0}=\lambda_{i_0}$; $i=1,2,...,m$, we get

$$f(X)+\lambda_0G(X)\geq f(X_0) \qquad ...(5)$$

$$\lambda_0\geq 0 \qquad ...(6)$$

6.5 Necessary and Sufficient Conditions for the function $f(X,\lambda)$ to have a saddle point at (X_0,λ_0)

Necessary Condition :

Suppose that the function $F(X,\lambda)$ has a saddle point at (X_0,λ_0). Then it means that there exists a possitive number $\in$ such that for all points X in the $\in$-neighbourhood $|X-X_0|<\in$ and for all λ in the $\in-$ neighbourhood $|\lambda-\lambda_0|<\in$ we have

$$F(X_0,\lambda) \le F(X_0,\lambda_0) \le F(X,\lambda_0) \qquad ...(1)$$

where $X = (x_1, x_2, \ldots, x_n)$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ are n-component and m-component vectors, respectively.

Let us partition the components of X and λ satisfying the above condition into three categories, $X = \left[X^{(1)}, X^{(2)}, X^{(3)}\right]$ and $\lambda = \left[\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}\right]$ where,

$X^{(1)} = (x_1 x_2, \ldots, x_p) \le 0$ has p components.

$X^{(2)} = (x_{p+1}, x_{p+2}, \ldots, x_q) \ge 0$ has $q-p$ components

$X^{(3)} = (x_{q+1}, x_{q+2}, \ldots, x_n)$ unrestricted in sign has $n-q$ components.

$\lambda^{(1)} = (\lambda_1, \lambda_2, \ldots, \lambda_r) \ge 0$ has r components.

$\lambda^{(2)} = (\lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_s) \le 0$ has $s-r$ components.

$\lambda^{(3)} = (\lambda_{s+1}, \lambda_{s+2}, \ldots, \lambda_m)$ unrestricted in sign has $m-s$ components

Let us denote by W_1 the set of points X such that the components of X satisfy the above conditions, by W_2, the set of points λ such that the components of λ satisfy the above conditions and by W the set of points $[X,\lambda]$ where $X \in W_1$ and $\lambda \in W_2$, Then the function $F(X,\lambda)$ is said to have a saddle point at (X_0,λ_0) for $(X,\lambda) \in W$ if $(X_0,\lambda_0) \in W$ and there exists an $\in > 0$ such that (1) holds for all $X \in W_1$ in the $\in$-neighbourhood of X_0 and for all $\lambda \in W_2$ in the $\lambda \in W_2$-neighbourhood of λ_0.

Suppose that $F(X,\lambda) \in C^1$ (i.e. all the first derivatives of F are continuous in E^n). If $F(X,\lambda)$ has a saddle point at (X_0,λ_0) for $(X,\lambda) \in W$, then we must have $F(X_0,\lambda_0)$ minimum at X_0 and $F(X_0,\lambda_0)$ maximum at λ_0,

$$\left.\begin{aligned} &\left[\frac{\partial}{\partial \mathrm{x_j}} \mathrm{F}(\mathrm{X},\lambda_0)\right]_{\mathrm{X}=\mathrm{X_0}} = 0, \text{for all j for which } \mathrm{x_j^0} \ne 0 \\ \text{and} \quad &\left[\frac{\partial}{\partial \lambda_{\mathrm{i}}} \mathrm{F}(\mathrm{X_0},\lambda)\right]_{\lambda=\lambda_0} = 0, \text{for all i for which } \lambda_{\mathrm{i}}^0 \ne 0 \end{aligned}\right\} \qquad ...(2)$$

$$\text{i.e. } \frac{\partial}{\partial x_j} F(\overline{X}_0,\lambda_0) = 0 \begin{bmatrix} \text{for all } \mathrm{j = q+1, q+2, \ldots n}, \text{since } \mathrm{x_j^0} \text{ is unrestricted in sign for these j's.} \\ \text{for all } \mathrm{j = 1, 2, \ldots, p}, \text{for which } \mathrm{x_j^0} \ne 0 \\ \text{for all } \mathrm{j = p+1, p+2, \ldots, q} \text{ for which } \mathrm{x_j^0} = 0 \end{bmatrix}$$

and

$$\frac{\partial}{\partial \lambda_i} F(X_0, \lambda_0) = 0 \begin{bmatrix} \text{for all } i = s+1, s+2, \ldots m, \text{since } \lambda_i^0 \text{ is unrestricted in sign for these } i\text{'s.} \\ \text{for all } i = 1, 2, \ldots, r, \text{for which } \lambda_i^0 \neq 0 \\ \text{for all } i = r+1, r+2, \ldots, s \text{ for which } \lambda_i^0 = 0 \end{bmatrix} \quad \ldots(3)$$

Now let us see the nature of $\frac{\partial}{\partial x_j} F(X_0, \lambda_0)$ when $x_j^0 = 0, j = 1,2,\ldots,q$ and $\frac{\partial}{\partial \lambda_i} F(X_0, \lambda_0)$ when $\lambda_i^0 = 0$ and $i = 1,2,\ldots,s$, in order that (1) may hold true.

First, let us assume that $x_j^0 = 0$ for $j = 1,2,\ldots,p$. For this case we shall show that

$$\frac{\partial}{\partial x_j} F(X_0 \lambda_0) \geq 0 \quad \ldots(4)$$

If possible, let $\frac{\partial}{\partial x_j} F(X_0 \lambda_0) < 0$. Since we have assumed that $F(X, \lambda) \in C^1$ i.e. $\frac{\partial}{\partial x_j} F(X, \lambda)$ is continuous, therefore, for a given $\in_0 > 0$, there exists an $\in_0$-neighbourhood of (X_0, λ_0) such that in this $\in$- neighbourhood of $(X_0, \lambda_0), \frac{\partial}{\partial x_j} F(X, \lambda) < 0$. ...(5)

We now select a positive number $\in$ such that $0 < \in < \in_0$ and consider points in the $\in$-neighbourhood of (X_0, λ_0) of the form $(X_0 + h e_j, \lambda_0)$, $0 < h < \in_0$. Then by Taylor's theorem

$$F(X_0 + h e_j, \lambda_0) = F(X_0, \lambda_0) + h \frac{\partial}{\partial x_j} F(X_0 + \theta h e_j, \lambda_0);\ 0 < \theta < 1$$

But $(X_0, \theta h e_j, \lambda_0)$ is in the $\in$-neighbourhood of (X_0, λ_0), therefore, from above

$$F(X_0 + h \hat{e}_j, \lambda_0) < F(X_0, \lambda_0) \qquad [\text{from (5)}] \quad \ldots(6)$$

for all $h, 0 < h < \in_0$

Therefore, every $\in$-neighbourhood of (X_0, λ_0) contains points $(X, \lambda_0) \in W$, such that (6) holds, i.e.,

$$F(X, \lambda_0) < F(X_0 \lambda_0)$$

This contradicts the fact that (X_0, λ_0) is a saddle point of $F(X, \lambda)$ for $(X, \lambda) \in W$. Thus our assumption is not correct. Hence (4) holds true,

i.e., $\frac{\partial}{\partial x_j} F(X_0, \lambda_0) \geq 0$, for $x_j^0 = 0$; $j = 1,2,\ldots,p$. ...(7)

In a similar way we can prove that $\frac{\partial}{\partial x_j} F(X_0, \lambda_0) \leq 0$ for x_j^0; $j = p+1, +2, \ldots, q$ ...(8)

and also if $\lambda_i^0 = 0$, then

$$\frac{\partial}{\partial \lambda_i} F(X_0, \lambda_0) \le 0 \text{ for } i = 1, 2, \ldots, r \qquad \text{...(9)}$$

$$\frac{\partial}{\partial \lambda_i} F(X_0, \lambda_0) \ge 0 \text{ for } i = r+1, r+2, \ldots, s \qquad \text{...(10)}$$

Thus we have shown that either

$$\frac{\partial}{\partial x_j} F(X_0, \lambda_0) = 0 \text{ or, } x_j^0 = 0$$

and either $\frac{\partial}{\partial \lambda_i} F(X_0, \lambda_0) = 0$ or, $\lambda_i^0 = 0$

Hence if $F(X, \lambda)$ has a saddle point at (X_0, λ_0) for $(X, \lambda) \in W$, and if $F(X, \lambda) \in C^1$, then (X_0, λ_0) must satisfy

$$\left.\begin{array}{l} \frac{\partial}{\partial x_j} F(X_0, \lambda_0) \ge 0 \quad , \quad j = 1, 2, \ldots, p \\ \frac{\partial}{\partial x_j} F(X_0, \lambda_0) \le 0 \quad , \quad j = p+1, p+2, \ldots, q \\ \frac{\partial}{\partial x_j} F(X_0, \lambda_0) = 0 \quad , \quad j = q+1, q+2, \ldots, n \end{array}\right\} \qquad \text{...(10)}$$

$$\left.\begin{array}{l} x_j^0 \le 0, j = 1, 2, \ldots, p \\ x_j^0 \ge 0, j = p+1, \ldots, q \\ x_j^0 \; unrestricted \; in \; sign, j = q+1, \ldots, n \end{array}\right\} \qquad \text{...(11)}$$

$$x_j^0 \frac{\partial}{\partial x_j} F(\overline{X}_0, \lambda_0) = 0, \; j = 1, 2, \ldots, n \qquad \text{...(12)}$$

and

$$\left.\begin{array}{l} \frac{\partial}{\partial \lambda_i} F(X_0, \lambda_0) \le 0, \; i = 1, 2, \ldots, r \\ \frac{\partial}{\partial \lambda_i} F(X_0, \lambda_0) \ge 0, \; i = r+1, r+2, \ldots, s \\ \frac{\partial}{\partial \lambda_i} F(X_0, \lambda_0) = 0, \; i = s+1, s+2, \ldots, m \end{array}\right\} \qquad \text{...(13)}$$

$$\left.\begin{array}{l}\lambda_i^0 \geq 0, i=1,2,\ldots,r \\ \lambda_i^0 \leq 0, i=r+1,r+2,\ldots,s \\ \lambda_i^0 unrestricted\ in\ sign,\ i=s+1,s+2,\ldots,m\end{array}\right\} \qquad \ldots(14)$$

$$\lambda_i^0 \frac{\partial}{\partial \lambda_i} F(\overline{X}_0, \lambda_0) = 0,\ i=1,2,\ldots,m \qquad \ldots(15)$$

Equations (10) to (15) are the necessary canditions, which the point (X_0, λ_0) must satisfy if the function $F(X,\lambda)$ has a saddle point at (X_0,λ_0) for $(X,\lambda) \in W$, provided that $F(X,\lambda) \in C^1$

Sufficient condition The conditions (10) to (15) become sufficient if there exists a positive number $\in > 0$ such that $F(X_0,\lambda)$ is a concave function of λ in the $\in$-neighbourhood of λ_0 and $F(X,\lambda_0)$ is a convex function of X in the $\in$-neighbourhood of X_0.

Now if $F(X_0,\lambda)$ is a concave function of λ, then

$$F(X_0,\lambda) \leq F(X_0,\lambda_0) + \nabla_\lambda F(X_0,\lambda_0)(\lambda - \lambda_0) \qquad \ldots(16)$$

where $\nabla_\lambda F(X_0,\lambda_0) = \left(\frac{\partial}{\partial \lambda_1} F(X_0,\lambda_0),\ldots,\frac{\partial}{\partial \lambda_m} F(X_0,\lambda_0)\right)$ is the gradient of $F(X,\lambda)$ with respect to λ at the point (X_0,λ_0).

Similarly if $F(X,\lambda_0)$ is a convex function of X, then

$$F(X,\lambda_0) \geq F(X_0,\lambda_0) + \nabla_X F(X_0,\lambda_0)(X - X_0) \qquad \ldots(17)$$

Where $\nabla_X F(X_0,\lambda_0) = \left(\frac{\partial}{\partial x_1} F(X_0,\lambda_0),\ldots,\frac{\partial}{\partial x_n} F(X_0,\lambda_0)\right)$ is the gradient of $F(X,\lambda)$ with respect to X at (X_0,λ_0).

Inequalities (16) and (17) hold good for all λ in the $\in$-neighbourhood of λ_0 and for all X in the $\in -$ neighbourhood of X_0.

Now $\nabla_\lambda F(X_0 \lambda_0)(\lambda - \lambda_0) = \nabla_\lambda F(X_0 \lambda_0).\lambda - \nabla_\lambda F(X_0,\lambda_0).\lambda_0$

$$= \nabla_\lambda F(X_0,\lambda_0)\lambda \qquad \text{(using (15))} \qquad \ldots(18)$$

and since

$$\lambda_i \geq 0, \frac{\partial}{\partial \lambda_i} F(X_0,\lambda_0) \leq 0,\ \ i=1,2,\ldots,r$$

$$\lambda_i \leq 0, \frac{\partial}{\partial \lambda_i} F(X_0,\lambda_0) \geq 0,\ \ i=r+1,\ldots,s$$

λ_i unrestricted, $\frac{\partial}{\partial \lambda_i} F(X_0, \lambda_0) = 0, \quad i = s+1, \ldots, m$

therefore, $\nabla_\lambda F(X_0, \lambda_0). \lambda = \sum_{i=1}^{m} \frac{\partial}{\partial \lambda_i} F(X_0, \lambda_0). \lambda_i \leq 0$ ...(19)

Thus (18) represents that

$$\nabla_\lambda F(X_0, \lambda_0)(\lambda - \lambda_0) \leq 0$$

Then from (16), we have

$$F(X_0, \lambda) \leq F(X_0, \lambda_0) \quad \text{...(20)}$$

Similarly from (17), we have

$$F(X, \lambda_0) \geq F(X_0, \lambda_0) \quad \text{...(21)}$$

Now from (20) and (21) we conclude that

$$F(X_0, \lambda) \leq F(X_0, \lambda_0) \leq F(X, \lambda_0) \quad \text{...(22)}$$

which holds for all X in the $\in$-neighbourhood of X_0 and for all λ in the $\in -$ neighbourhood of λ_0.

i.e., $F(X, \lambda)$ has a saddle point at (X_0, λ_0).

Note : Consider the following nonlinear programming problem:

Optimize $f(X)$, $\quad X = (x_1, x_2, \ldots, x_n)^T$

subject to $\quad h_i(X) = 0, i = 1, 2, \ldots, m \quad (m < n)$

Introducing Lagrangian multipliers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, the Lagrangian function is

$$L(X, \lambda) = f(X) + \sum_{i=1}^{m} \lambda_i h_i(X), \quad m < n$$

The necessary conditions for stationary points of $f(X)$ at which $f(X)$ may have a maximum or minimum are

$$\frac{\partial L(X, \lambda)}{\partial x_j} = \frac{\partial f(X)}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial h_i(X)}{\partial x_j} = 0; \quad j = 1, 2, \ldots, n$$

and $$\frac{\partial L(\overline{X}, \lambda)}{\partial \lambda_i} = \frac{\partial h_i(\overline{X})}{\partial \lambda_i} = 0, \quad i = 1, 2, \ldots, m \qquad (m < n)$$

Let
$$U = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \cdots\cdots \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \cdots\cdots \frac{\partial h_2}{\partial x_n} \\ ------ & ------ \\ \frac{\partial h_m}{\partial x_1} & \frac{\partial h_m}{\partial x_2} \cdots\cdots \frac{\partial h_m}{\partial x_n} \end{bmatrix}$$

which is an mxn matrix

and
$$V = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \cdots\cdots \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \cdots\cdots \frac{\partial^2 L}{\partial x_2 \partial x_n} \\ ---------- & ---------- \\ \frac{\partial^2 L}{\partial x_n \partial x_1} & \frac{\partial^2 L}{\partial x_n \partial x_2} \cdots\cdots \frac{\partial^2 L}{\partial x_n^2} \end{bmatrix}$$

which is an $n \times n$ matrix.

Also let $O = (O_{ij})$ be an $n \times n$ null matrix.

Then the square matrix H^B of order $(m+n) \times (m+n)$ is called the bordered Hessian matrix and is defined as :

$$H^B = \begin{bmatrix} O & \vdots & U \\ \cdots & \vdots & \cdots \\ U^T & \vdots & V \end{bmatrix}$$

Now if (X_0, λ_0) is a stationary point for the Lagrangian function $L(X_0, \lambda)$ and H_0^B the value of the corresponding bordered Hessian matrix H^B at this stationary point, then

(i) The point X_0 gives the maximum value of the objective function $f(X)$, if, starting with the principal minor of order $(2m+1)$, the last $(n-m)$ principal minors of H_0^B are of alternate signs, starting with $(-1)^{m+n}$ sign.

(ii) The point X_0 gives the minimum value of the objective function, starting with the principal minor of order $(2m+1)$, the last $(n-m)$ principal minors of H_0^B are of the sign of $(-1)^m$.

For example :

(i) If $n=2, m=1$, then the order of H^B is 3×3 (since $m+n=1+2=3$) and since

$2m+1=3,(-1)^{n+m}=(-1)^3=-1$ and $n-m=1,(-1)^m=(-1)^1=-1$. Therefore the extreme point X_0, gives the maximum value of the objective function if $|H^B|<0$ and minimum value of the objective function if $\Delta_3=|H^B|>0$.

(ii) When $n=3, m=1$, then the order of H^B is 4×4 (since $n+m=3+1=4$) and since $2m+1=3,(-1)^{n+m}=(-1)^4=1, n-m=3-1=2,(-1)^m=(-1)^1=-1$. Therefore, the extreme point X_0 gives the maximum value of the objective function $f(X)$ if $\Delta_4=|H^B|<0$ and $\Delta_3>0$ and minimum value of the objective function if $\Delta_4>0$ and $\Delta_3<0$

(iii) When $n=3, m=2$, then the order of H^B is 5×5 (since $n+m=3+2=5$) and since $2m+1=5,(-1)^{n+m}=(-1)^5=-1, n-m=3-2=1,\ (-1)^m=(-1)^2=1$, therefore the extreme point X_0 gives the maximum value of the objective function $f(X)$ if $\Delta_5=|H^B|<0$ and the minimum value of the objective function if $\Delta_5=|H^B|<0$.

Note : If $f(X)$ is a real valued continuous differentiable function of $X=(x_1,x_2,\ldots,x_n)$, then the Hessian matrix of $f(X)$ is

$$H^B(X)=\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\partial x_2} \cdots\cdots\cdots & \frac{\partial^2 f}{\partial x_1\partial x_n} \\ \frac{\partial^2 f}{\partial x_2\partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \cdots\cdots\cdots & \frac{\partial^2 f}{\partial x_2\partial x_n} \\ ------- & ------- & ------- \\ \frac{\partial^2 f}{\partial x_n\partial x_1} & \frac{\partial^2 f}{\partial x_n\partial x_2} \cdots\cdots\cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

The function $f(X)$ is convex if the Hessian matrix $H^B(X)$ of $f(X)$ is positive definite i.e., if all the leading principal minors of $H^B(X)$ are positive in sign.

The function $F(X)$ is concave if the Hessian matrix $H^B(X)$ of $f(X)$ is negative definite, i.e., if the signs of leading principal minors of $H^B(X)$ are alternately negative and positive.

Example-3 : Obtain the necessary conditions for the following nonlinear programming problem :

Minimize $f(X)=3x_1^2+x_2^2+2x_1x_2+6x_1+2x_2$

subject to $2x_1-x_2=4$

$x_1,x_2\geq 0$

Solution : The Lagrangian function for the given problem is

$$L(X,\lambda)=f(X)+\lambda(2x_1-x_2-4)$$

or $$L(X,\lambda)=3x_1^2+x_2^2+2x_1x_2+6x_1+2x_2+\lambda(2x_1-x_2-4)$$

The necessary conditions for the minimum of $f(X)$ are

$$\frac{\partial L}{\partial x_1}=0 \quad \text{or,} \quad 6x_1+2x_2+6+2\lambda=0 \qquad ...(1)$$

$$\frac{\partial L}{\partial x_2}=0 \quad \text{or,} \quad 2x_2+2x_1+2-\lambda=0 \qquad ...(2)$$

$$\frac{\partial L}{\partial \lambda}=0 \quad \text{or,} \quad 2x_1-x_2-4=0 \qquad ...(3)$$

Example-4 : Solve the following non linear programming problem using the method of Lagrangian multipliers :

Minimize $f(X)=x_1^2+x_2^2+x_3^2$

subject to $4x_1+x_2^2+2x_3=14 \qquad (\equiv g(x)-14)$

$x_1,x_2,x_3\geq 0$

Solution : The Lagrangian function is

$$L(X,\lambda)=f(X)+\lambda(4x_1+x_2^2+2x_3-14)$$

or, $$L(X,\lambda)=x_1^2+x_2^2+x_3^2+\lambda(4x_1+x_2^2+2x_3-14)$$

The necessary condition for $f(X)$ to have a maximum or minimum are

$$\frac{\partial L}{\partial x_1}=0\text{, or} \quad 2x_1+4\lambda=0 \qquad ...(1)$$

$$\frac{\partial L}{\partial x_2}=0\text{, or} \quad 2x_2+2\lambda x_2=0 \qquad ...(2)$$

$$\frac{\partial L}{\partial x_3}=0\text{, or} \quad 2x_3+2\lambda=0 \qquad ...(3)$$

$$\frac{\partial L}{\partial \lambda}=0\text{, or} \quad 4x_1+x_2^2+2x_3-14=0 \qquad ...(4)$$

From (2), $x_2(1+\lambda)=0$

or $x_2 = 0$ or $\lambda = -1$.

Also from (1) $x_1 = -2\lambda$ and from (3) $x_3 = -\lambda$. If we put $x_2 = 0$ in (4), then we get

$\lambda = -1.4$ and so $x_1 = 2.8$, $x_3 = 1.4$

If we put $\lambda = -1$ then we get $x_1 = 2, x_3 = 1$ and then from (4), we get $x_2 = 2$.

Therefore, we get the following stationary points :

$$(2.8, 0, 1.4),\ \lambda = -1.4$$

and $(2, 2, 1)$, $\lambda = -1$

We now consider the bordered Hessian matrix

$$H^B = \left[\begin{array}{c|ccc} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} \\ \hline \frac{\partial g}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial g}{\partial x_2} & \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial g}{\partial x_3} & \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{array}\right] = \left[\begin{array}{c|ccc} 0 & 4 & 2x_2 & 2 \\ \hline 4 & 2 & 0 & 0 \\ 2x_2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{array}\right]$$

At the stationary point (2.8, 0, 1.4)

$$H^B = \begin{bmatrix} 0 & 4 & 0 & 2 \\ 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

Here $n = 3, m = 1$, therefore $n - m = 3 - 1 = 2$ and $2m + 1 = 2 \times 1 + 1 = 3$

We check the signs of the principal minors D_3 and D_4

Now $D_3 = \begin{vmatrix} 0 & 4 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -32$

and $D_4 = \begin{vmatrix} 0 & 4 & 0 & 2 \\ 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{vmatrix} = -80$

Since both D_3 and D_4 have the same sign negative, which is the sign of $(-1)^m = (-1)^1$ i.e. negative, therefore $f(X)$ has a minimum at the point $(2.8, 0, 1.4)$ and at this point the minimum of $f(X)$ is 9.8.

And, at the stationary point $(2,2,1)$

$$H^B = \begin{bmatrix} 0 & 4 & 4 & 2 \\ 4 & 2 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

Here $D_3 = \begin{vmatrix} 0 & 4 & 4 \\ 4 & 2 & 0 \\ 4 & 0 & 2 \end{vmatrix} = -64 < 0$

and $D_4 = \begin{vmatrix} 0 & 4 & 4 & 2 \\ 4 & 2 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{vmatrix} = -144 < 0$

Thus $f(X)$ has a minimum value at $(2,2,1)$ which is 9

Since the least among 9 and 9.8 is 9, therefore, $f(X)$ has minimum at the stationary point $(2,2,1)$ and the minimum of $f(X)$ at this stationary point is 9.

Example-5 Use Lagrangian function to find the optimal solution f the following nonlinear programming problem :

Maximize $\quad f(X) = -3x_1^2 - 4x_2^2 - 5x_3^2$

subject to $\quad x_1 + x_2 + x_3 = 10$

$\quad x_1, x_2, x_3 \geq 0$

Solution : Here the Lagrangian function for the given problem is

$$L(X,\lambda) = f(X) + \lambda(10 - x_1 - x_2 - x_3)$$

or $\quad L(X,\lambda) = -3x_1^2 - 4x_2^2 - 5x_3^2 + \lambda(10 - x_1 - x_2 - x_3)$

The necessary condtions for stationary values of $L(X,\lambda)$ are

$$\frac{\partial L}{\partial x_1} = 0 \text{ , or } -6x_1 - \lambda = 0, \text{ or } x_1 = -\frac{1}{6}\lambda$$

$$\frac{\partial L}{\partial x_2}=0, \text{ or } -8x_2-\lambda=0, \text{ or } x_2=-\frac{1}{8}\lambda$$

$$\frac{\partial L}{\partial x_3}=0, \text{ or } -10x_3-\lambda=0 \text{ or } x_3=-\frac{1}{10}\lambda$$

$$\frac{\partial L}{\partial \lambda}=0, \text{ or } 10-x_1-x_2-x_3=0, \text{ or } x_1+x_2+x_3=10$$

Putting the values of x_1, x_2, x_3 in the above equation

$$\frac{1}{6}\lambda+\frac{1}{8}\lambda+\frac{1}{10}\lambda=-10$$

or $\lambda=-\frac{1200}{47}$

Thus $x_1=200/47$; $x_2=150/47$; $x_3=120/47$. Since, $-3x_1^2-4x_2^2-5x_2^2$ is strictly concave function and $x_1+x_2+x_3=10$ is a linear function, therefore, $L(X,\lambda)$ is strictly concave. Thus the Lagrangian necessary conditions are sufficient also for the global maximum.

Hence, the optional solution to the given problem is $x_1=\frac{200}{47}, x_2=\frac{150}{47}, x_3=\frac{120}{47}$

Example-6 Use Lagrangian multiplier method to solve the following nonlinear programming problem:

Minimize $f(X)=2x_1^2+2x_2^2+2x_3^2-24x_1-8x_2-12x_3+10$

subject to $x_1+x_2+x_3=11$

$x_1, x_2, x_3 \geq 0$

Solution : The Lagrangian function for the given problem is

$$L(X,\lambda)=2x_1^2+2x_2^2+2x_3^2-24x_1-8x_2-12x_3+10+\lambda(x_1+x_2+x_3-11)$$

The necessary condition for minimum of $f(X)$ are

$$\frac{\partial L}{\partial x_i}=0, \; i=1,2,3 \text{ and } \frac{\partial L}{\partial \lambda}=0$$

i.e. $\frac{\partial L}{\partial x_1}=0$, or $4x_1-24+\lambda=0$ ...(1)

$\frac{\partial L}{\partial x_2}=0$, or $4x_2-8+\lambda=0$ ...(2)

$$\frac{\partial L}{\partial x_3} = 0, \qquad \text{or} \qquad 4x_3 - 12 + \lambda = 0 \qquad ...(3)$$

$$\frac{\partial L}{\partial \lambda} = 0, \qquad \text{or} \qquad x_1 + x_2 + x_3 - 11 = 0 \qquad ...(4)$$

From (1), (2) and (3)

$$x_1 = \frac{24-\lambda}{4}, x_2 = \frac{8-\lambda}{4}; x_3 = \frac{12-\lambda}{4}$$

Putting these values of x_1, x_2, x_3 in (4)

$$\frac{24-\lambda+8-\lambda+12-\lambda}{4} = 11, \text{ or } \lambda = 0$$

Thus $x_1 = 6, x_2 = 2, x_3 = 3$

Here the minimization function $f(X)$ is the sum of a positive definite quadratic form and a linear function, so is a convex function. Thus $L(X,\lambda)$ is also a convex function as the constraint is a linear equation. Hence $x_1 = 6, x_2 = 2, x_3 = 3$ is the optimal solution of the given nonlinear programming problem.

Example -7 Use method of Lagrangian multipliers to solve the following nonlinear programming problem :

Optimize $f(X) = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$

subject to $x_1 + x_2 + x_3 = 20$

$x_1, x_2, x_3 \geq 0$

Does the solution maximize or minimize the objective function?

Solution : The Lagrangian function is

$$L(X,\lambda) = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100 + \lambda(x_1 + x_2 + x_2 - 20)$$

The necessary condition for the maxima or minima are

$$\frac{\partial L}{\partial x_1} = 0, \qquad \text{or} \qquad 4x_1 + 10 + \lambda = 0 \qquad ....(1)$$

$$\frac{\partial L}{\partial x_2} = 0, \qquad \text{or} \qquad 2x_2 + 8 + \lambda = 0 \qquad ...(2)$$

$$\frac{\partial L}{\partial x_3} = 0 \qquad \text{or} \qquad 6x_3 + 6 + \lambda = 0 \qquad ...(3)$$

$$\frac{\partial L}{\partial \lambda} = 0, \qquad \text{or} \qquad x_1 + x_2 + x_3 - 20 = 0 \qquad ...(4)$$

From (1), (2) and (3) we have

$$x_1 = -\frac{\lambda+10}{4}, x_2 = -\frac{\lambda+8}{2}, x_3 = -\frac{\lambda+6}{6}$$

Therefore, from (4)

$$\frac{\lambda+10}{4} + \frac{\lambda+8}{2} + \frac{\lambda+6}{6} = -20$$

or, $\lambda = -30$

Thus $x_1 = 5, x_2 = 11, x_3 = 4$

Hence the stationary point is (5, 11, 4)

To determine, whether this stationary point results in maximization or minimization of the objective function, (n-1) principal minors of the following determinant are solved :

$$\Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix} = -44$$

and $$\Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -6$$

Since Δ_4 and Δ_3 both are negative, therefore the sationary point is a point of minima.

Thus the optimal solution is

$x_1 = 5, x_2 = 11, x_3 = 4$ and the minimum value of $f(X)$ is

$$f(X) = 2\times 25 + 121 + 3\times 16 + 50 + 88 + 24 - 100$$
$$= 281.$$

Note : Another way to check whether the objective function $f(x_1, x_2, x_3)$ has a minimum value or maximum value at the stationary point $(5,11,4)$ we find the Hessian of the objective function $f(x_1, x_2 x_3)$ at the point $(5,11,4)$, which is

$$H(X) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

The principal minors of $H(x)$ are :

$$|4|=4,\quad \begin{vmatrix}4 & 0\\0 & 2\end{vmatrix}=8 \text{ and } \begin{vmatrix}4 & 0 & 2\\0 & 2 & 0\\0 & 0 & 6\end{vmatrix}=48$$

which are all positive. Therefore, $H(X)$ is positive definite, i.e., $f(x_1, x_2, x_3)$ is convex. Hence $f(x_1, x_2, x_3)$ is minimum at the statonary point $(5,11,4)$.

6.6 Graphical Method for Solving a Nonlinear Programming Problem

We know that in linear programming problem the optimal solution is attained at one of the extreme points of the convex region generated by the constraints. But in case of nonlinear programming problem, it is not necessary that the optimal solution of the problem lies at a corner or edge of the feasible region.

The method of solving a nonlinear programming problem involving only two variables is explained through the following examples :

Examples-8 : Solve the following nonlinear programming problem graphically :

Maximize $f(x_1, x_2) = 8x_1 + 8x_2 - x_1^2 - x_2^2$

subject to $x_1 + x_2 \le 12$

$x_1 - x_2 \ge 4$

$x_1, x_2 \ge 0$

Solution : Considering the given constraints as equalities and drawing the lines an the x_1x_2 – plane, we get the admissible region to be ABDA.

The objective function $f(x_1, x_2)$ is $8x_1 - x_1^2 + 8x_2 - x_2^2$ i.e. $32 - (x_1 - 4)^2 - (x_2 - 4)^2$

which is a circle with centre at $(4,4)$ as, shown in figure 6.3

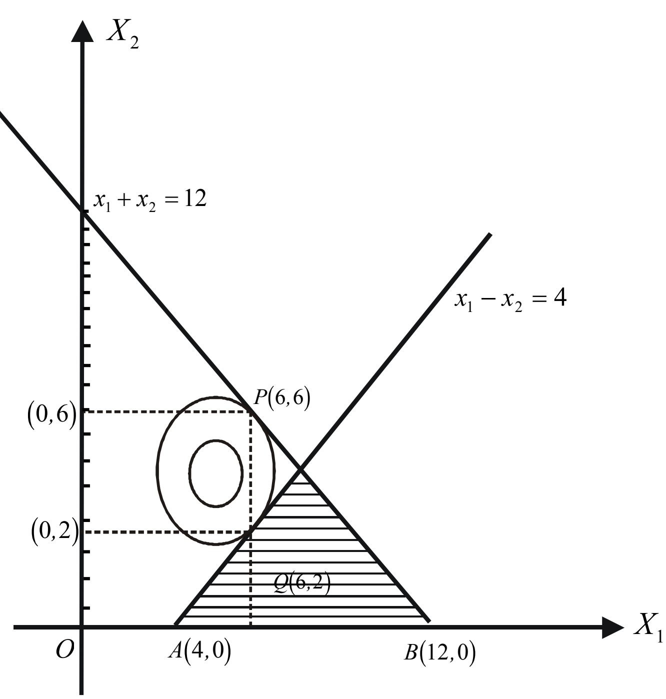


Figure 6.3

The point that gives the maximum value of $f(x_1, x_2)$ is the point at which the feasible region is tangent to the circle given by the objective function $8x_1 - x_1^2 + 8x_2 - x_2^2$

Differentiating $f(x_1, x_2)$ w.r.t. x_1

$$8 - 2x_1 + 8\frac{dx_2}{dx_1} - 2x_2\frac{dx_2}{dx_1} = 0$$

or $\dfrac{dx_2}{dx_1} = \dfrac{2x_1 - 8}{8 - 2x_2} = \dfrac{x_1 - 4}{4 - x_2} = m_1(say)$

for the line $x_1 + x_2 = 12, \dfrac{dx_2}{dx_1} = -1 = m_2(say)$

The circle will touch the line $x_1 + x_2 = 12$, where, $m_1 = m_2$, i.e., $\dfrac{x_1 - 4}{4 - x_2} = -1$, i.e., $x_2 = x_1$. Therefore, putting $x_1 = x_2$ in $x_1 + x_2 = 12$, we get $x_1 = x_2 = 6$.

Thus the circle touches the line at the point $P(6,6)$. But this point $P(6,6)$ is not a point of the feasible region ABDA

Again for the line $x_1 - x_2 = 4$, we have

$$\frac{dx_2}{dx_1} = 1 = m_3(say)$$

The circle touches this line at the point where $m_1 = m_3$, i.e., $\dfrac{x_1 - 4}{4 - x_2} = 1$, i.e., $x_2 = 8 - x_1$

Putting these values in $x_1 - x_2 = 4$, we get $x_1 = 6$ and $x_2 = 2$

i.e., the circle touches the line $x_1 - x_2 = 4$ at the point $Q(6,2)$, which lies in the feasible region. Also for $x_1 = 6, x_2 = 2$, we have $f(x_1, x_2) = 24$

Thus the optimal solution of the given problem is $x_1 = 6, x_2 = 2$ and maximum value of $f(x_1, x_2) = 24$.

Example-9 Solve the following nonlinear programming problem graphically :

Maximize $f(x_1, x_2) = x_1 + 2x_2$

Subject to $x_1^2 + x_2^2 \le 1$

$2x_1 + x_2 \le 2$

$x_1, x_2 \ge 0$

Solution : Considering the given constraints as equations and drawing them in $x_1\,x_2$ – plane the feasible region is OABCO as shown in figure 6.4.

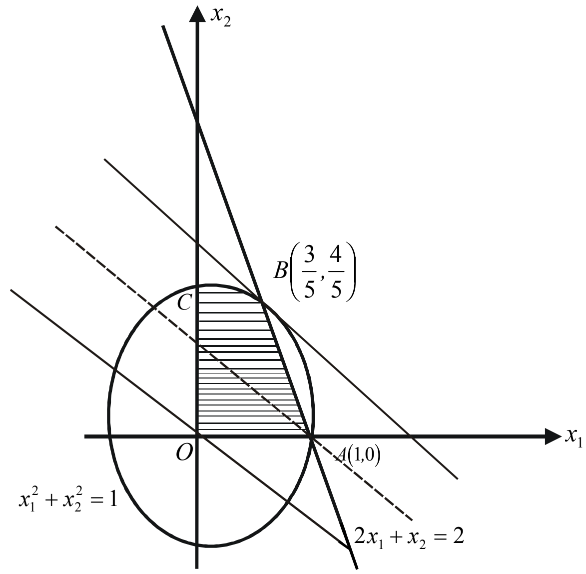


Figure : 6.4

The objective function $f(x_1,x_2)$ is the line $x_1+2x_2=z(say)$. Drawing the objective function through $(0,0)$ and then drawing the lines parallel to this objective functional line, we reach the extremity B of the feasible region OABCO. The point B is the point of intersection of the circles $x_1^2+x_2^2=1$ and the line $2x_1+x_2=2$ and is the most distant point of the feasible region. Thus B is the point of optimal solution of the problem. Solving $x_1^2+x_2^2=1$ and $2x_1+x_2=2$, we get $B\left(\frac{3}{5},\frac{4}{5}\right)$ and $f(x_1,x_2)=\frac{11}{5}$.

Hence the optimal solution of the given nonlinear programming problem is

$$x_1=\frac{3}{5},x_2=\frac{4}{5} \text{ and max. } f(x_1,x_2)=\frac{11}{5}$$

Example-10 Solve the following programming problem graphically :

Minimize $\quad f(x_1,x_2)=x_1^2+x_2^2$

Subject to $\quad x_1+x_2\geq 4$

$2x_1+x_2\geq 5$

$x_1,x_2\geq 0$

Solution : Considering the constraints as equalities and drawing them on the x_1x_2 – plane, feasible region is x_1ABCx_2 which actually is an infinite region. Thus the desired point minimizing the objective function $f(x_1,x_2)$ must be somewhere in this unbounded region. Since our search is for such a point (x_1,x_2) which gives a minimum value of $x_1^2+x_2^2$ and lies in the convex region, the desired point will be that point of the infinite region at which a side of the convex region is tangent to the circle $x_1^2+x_2^2=r^2(say)$ as shown in figure 6.5.

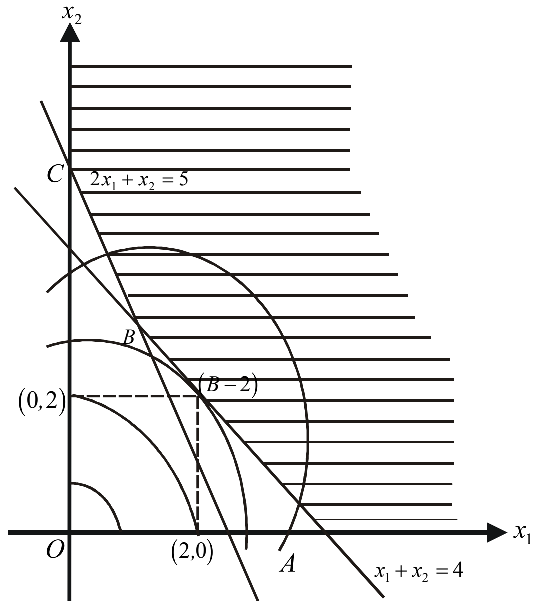


Figure 6.5

Differentiating $x_1^2 + x_2^2 = r^2$, w.r.t. x_1, we have

$$\frac{dx_2}{dx_1} = -\frac{x_1}{x_2} = m_1(say)$$

Differentiating the equation $x_1 + x_2 = 4$ w.r.t. x_1, we have

$$\frac{dx_2}{dx_1} = -1 = m_2(say)$$

The circle touches the line $x_1 + x_2 = 4$ at the point where $m_1 = m_2$

i.e., $\frac{-x_1}{x_2} = -1$, i.e., $x_1 = x_2$

Thus from $x_1 + x_2 = 4$, we get the point $P(2,2)$.

Therefore, the circle touches the line $x_1 + x_2 = 4$ at the point $P(2,2)$, which lies in the convex region bounded by the constraints.

Again differetiating the equation $2x_1 + x_2 = 5$ w.r.t. x_1, we get

$$\frac{dx_2}{dx_1} = -2 = m_3(say).$$

The circle $x_1^2 + x_2^2 = r^2$ will touch the line $2x_1 + x_2 = 5$ at the point where

$m_1 = m_3$, i.e., $\frac{-x_1}{x_2} = -2$, i.e., $x_1 = 2x_2$

Therefore, from $2x_1 + x_2 = 5$, we get the point $Q(2,1)$. Thus the circle touches the line at the point $Q(2,1)$, which does not lie in the convex region bounded by the constraits and so is to be discarded.

Hence the optimal solution to the problem is $x_1 = 2, x_2 = 2$ and minimum value of $f(x_1, x_2) = 2^2 + 2^2 = 8$.

6.7 Self-Learning Exercise

1. A point $X^* \in T$ is a local (relative) minimum of the function $f(X)$ over T if there is a positive number δ such that for all $X \in T \cap N_\delta(X^*)$, we have ....

2. The Lagrangian function for the nonlinear programming problems Min $f(X)$, subject to

   $G(X) \le 0,\ X \ge 0$ is......

3. If the Lagrangian function $F(X, \lambda)$ for the nonlinear programming problems Min $f(X)$, subject to $G(X) \le 0, X \ge 0$ has a saddle point (X_0, λ_0) for each λ, then........

4. If (X_0, λ_0) is a saddle point of the Lagrangian function $F(X, \lambda)$ for the problems Min $f(X)$, subject to $G(X) \le 0, X \ge 0$, then.......

6.8 Summary

In the present unit we discussed about the mathematical programming problem and the general nonlinear programming problem. We studied the Lagraingian function and the saddle point of the Lagrangian function. We derived the necessary and sufficient canditions for the Lagrangian function to have a saddle point. We also saw in brief, how a nonlinear programming problem can be solved graplically.

6.9 Answers to Self-Learning Exerices

1.	$f(X) \ge f(X^*)$	2.	$f(X) + \lambda^T G(X)$
3.	$G(X_0) \le 0,\ \lambda_0^T G(X_0) = 0$	4.	$F(X_0, \lambda) \le F(X_0, \lambda_0) \le F(X, \lambda_0)$

6.10 Exercise

1. Define saddle point and indicate its significance.
2. What is the Lagrange multiplier method?
3. What is a general nonlinear programming proble? Establish the relation between saddle point and the minimal point of the nonlinear programming problem.
4. Solve the following nonlinear programming problems, using the method of Lagrange multipliers :

   (a) Min. $f(x_1, x_2, x_3) = 2x_1^2 - 24x_1 + 2x_2^2 - 8x_2 + 2x_3^2 - 12x_3 + 200$

   subject to $x_1 + x_2 + x_3 = 11$

   $x_1, x_2, x_3 \ge 0$

(Ans. $x_1 = 6,\ x_2 = 2; x_3 = 3$ and minimum $f = 102$)

(b) Min $f(x_1, x_2, x_3) = 4x_1^2 + 2x_2^2 + x_3^2 - 4x_1x_2$

Subject to $x_1 + x_2 + x_3 = 15$

$2x_1 - x_2 + 2x_3 = 20$

$x_1, x_2, x_3 \geq 0$

(Ans. $x_1 = \frac{11}{3}, x_2 = \frac{10}{3}, x_3 = 8$; minimum $f = \frac{820}{9}$)

(c) Min. $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$

subject to $x_1 + x_2 + 3x_3 = 2$

$5x_1 + 2x_2 + x_3 = 5$

$x_1, x_2, x_3 \geq 0$

(Ans. $x_1 = 0.81, x_2 = 0.35, x_3 = 0.928$; minimum $f = 0.857$)

(d) Max. $f(x, y, z) = xyz$

subject to $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$x, y, z \geq 0$

(Ans. $x = a\sqrt{3}, y = b\sqrt{3}, z = c\sqrt{3}$; maximum $f = 3\sqrt{3}abc$)

□□□

Unit 7
Constrained Optimization in Nonlinear Programming Problems; Kuhn-Tucker Conditions

Structure of the Unit

7.0 Objective

The present unit is confined to discuss the theory which has been developed for locating the points of maxima and minima of constrained nonlinear optimization problems. The theory popularly known as Kuhn-Tucker theory, provides a set of necessary and sufficient conditions for check, whether a given point is a point of optimality. The objective of writing this unit is to study the Kuhn-Tucker theory for nonlinear programs.

7.1 Introduction

The unit bigins with the definition of convex programming problem. The theoretical concept of Langrangian function of the general non-linear programming problem and its relation with the saddle point is the next section of the unit that is of fundamental importance. The major part of the unit deals with the Kuhn-Tucker Theory, The Kuhn-Tucker necessary conditions for the optimum of the nonlinear programming problem and their derivation, which is called the Kuhn-Tucker theorem.

7.2 Covex Programming Problems

The general mathematical programming problem consists in finding the minimum value of the function $f(X)$ for all real X, satisfying the conditions $g_i(X) \le, =\ge 0$, $(X) \ge 0$, where $f(X)$ and $g_i(X)$, $i = 1,2,\dots\dots,m$ are all real valued functions of $X = (x_1, x_2, \dots, x_n)$ in E^n. The problem stated above is called a nonlinear programming problem (NLPP) if some or all of the functions $f(X)$, $g_i(X)$ are nonlinear for $i = 1,2,\dots\dots,\mathrm{m}$.

If $f(X)$ and $g_i(X)$ are all convex functions, the problem is said to be a convex programming problem. A convex programming problem can thus be stated as follows:

Minimize $f(X)$, $\quad X = (x_1, x_2, \dots\dots, x_n) \in \mathrm{E}^n$

subject to : $\quad g_i(X) \le 0$; $\mathrm{i} = 1,2,\dots\dots,\mathrm{m}$

$X \ge 0$

where $f(X)$ and $g_i(X)$ are all convex functions.

The convex programming problem has a little advantage over the general nonlinear programming problem as in convex programming problem all the constraint functions $g_i(x)$ are convex functions. Therefore the set S of points, satisfying the constraints. $g_i(X) \le 0$, i =............,m, $X \ge 0$ is a convex set. However this may not be so if $g_i(X)$ are not all convex. Also, if $f(X)$ is a convex function, then the relative minimum of $f(X)$ is also a global minimum, which infact is unique. This may not be possible if the NLPP is a non convex programming problem. It may be noticed that if $f(X)$ is convex, then $-f(X)$ is a concave function and so minimum of f is equal to maximum of $-f$. Thus the statement that a function is convex is equivalent of saying if it is a concave function.

We now begin with some theoretical concepts that are of fundamental importance.

7.2.1 Lagrangin function and saddle point

Let us consider the problem

Minimize $f(X)$, $X = (x_1, x_2, \ldots\ldots, x_n) \in \mathrm{E}^n$

subject to $g_i(X) \le 0; i = 1, 2, \ldots\ldots\ldots m$

Where $f(X)$ and $g_i(X)$ are not necessarily convex functions and also there is no restriction on X.

Let $\lambda = (\lambda_1, \lambda_2, \ldots\ldots, \lambda_m)^T \in \mathrm{E}^m$ be any vector in E^m. We define the function $F(X, \lambda)$ as

$$F(X, \lambda) = f(X) + \sum_{i=1}^{m} \lambda_i g_i(X) = f(X) + \lambda^T G(X)$$

where $G(X) = (g_1(X), g_2(X) \ldots\ldots\ldots, g_m(X))^T$.

The function $F(X, \lambda)$ is then called the lagrangian function, with the components of λ as the Lagrange multiplirs. We recall that (X_0, λ_0) is said to be a saddle point of the Lagrangian function $F(X, \lambda)$ if.

$$F(X_0, \lambda) \le F(X_o, \lambda_o) \le F(X, \lambda_o) \text{ in some neighbourhood of } (X_0, \lambda_0).$$

Infact the saddle point of the Lagrangian function $F(X, \lambda)$, if it exists, and the minimal point of the minimizing function $f(X)$ bear a strang theoretical bond between each other. This has led not only to important theoretical results but also to practical algorithms for solving mathematical programming problems. This relationship is a part of what is commonly known as Kuhn-Tucker theory.

7.3 Kuhn-Tucker Conditions and Kuhn-Tucker Theorem

In this section we shall develop primarily the necessary form of Kuhn-Tucker conditions for getting the stationary points of the constrained nonlinear programming problems. These conditions are also sufficient under certain restrictions.

Consider the NLPP

Minimize $f(X)$, $(X)=(x_1,x_2,\ldots\ldots,x_n)$ ...(1)

Subject to $g_i(X)\le 0$; $i=1,2,\ldots\ldots.,m$ ...(2)

We also assume that $f(X)$ and all $g_i(X)$, $i=1,2,\ldots\ldots.,m$ are differentiable functions in E^n.

Let us form the Lagrangian function

$$F(X,\lambda)=f(X)+\sum_{i=1}^{m}\lambda_i\, g_i(X)=f(X)+\lambda^T G(X)$$

Where $\left[G(X)=\left(g_1(X),\ g_2(X)\ldots\ldots\ldots,g_m(X)\right)^T\right]$

and $\lambda=\left(\lambda_1,\lambda_2,\ldots.,\lambda_m^T\right)\in E^m$

We start with the statement

$$F(X_0,\lambda)\le F(X_0,\lambda_0)\le F(X,\lambda_0) \qquad ...(4)$$

which intends that (X_0,λ_0) is a saddle point of the lagrangian function $F(X,\lambda)$ Let X be any point in the neighbourhood of X_0. Since X is unrestricted, therefore X is an interior point in the neighbourhood of X_0. Thus, the right side inequality of (4) implies that (X,λ_0) is a local minimum of $F(X,\lambda_0)$ and so we must have

$$\left(\frac{\partial F(X,\lambda_o)}{\partial x_j}\right)_{X=X_0}=0 \quad ; \quad j=1,2,\ldots\ldots\ldots..,n \qquad ...(5)$$

Let λ be a point in the neighbourhood of λ_0. since every $\lambda\ge 0$, therefore if we denote by l_{io} the components of λ_0, then let

(i) $\lambda_{i0}>0$ for $i=1,2,\ldots\ldots\ldots\ldots,k$

(ii) $\lambda_{i0}\ge 0$ for $i=k+1,k+2\ldots\ldots\ldots\ldots,m$

Clearly $\lambda_o>0$ is $k=m$ and $\lambda_0=0$ if $k=0$. Let us suppose that the neighbouring point λ differ from λ_0 only in the i^{th} component, the other components in λ and λ_0 being equal. Then by Taylor's series

$$F(X_0,\lambda)=F(X_0,\lambda_0)+(\lambda_i-\lambda_{i0})\left(\frac{\partial F(X_0,\lambda)}{\partial\lambda_i}\right)_{\lambda=\lambda_0}+\ldots\ldots.$$

or
$$F(X_0,\lambda)-F(X_0,\lambda_0)=(\lambda_i-\lambda_{i0})\left(\frac{\partial F(X_o,\lambda)}{\partial\lambda_i}\right)_{\lambda=\lambda_0}+\ldots\ldots. \qquad ...(6)$$

Choosing $(\lambda_i-\lambda_{io})$ sufficiently small so that other higher order terms in the above expansion that are very-very small to be neglected, the sign of the left hand function i.e. of the function $F(X_0,\lambda) - F(X_0,\lambda_0)$ depends upon the sign on the right hand term. Now if $\lambda_{io} > o$ (i.e. of category (i)), then $\lambda_i - \lambda_{io}$ can be made positive or negative by some suitable choice of λ_i which can be greater than or less than λ_{i0}, remembering that the only restriction on λ_i is that $\lambda_i \geq 0$, which can be maintained in either case. Thus $F(X_0,\lambda) - F(X_0,\lambda_0)$ can be made positive or negative by a suitable choice of $\lambda \geq 0$. But by the fact of left side inequality of (4) this is never positive. Therefore if $\lambda_{io} > 0$ for i = 1, 2, ...,k, necessarily we must have

$$\left(\frac{\partial F(X_0,\lambda)}{\partial \lambda_i}\right)_{\lambda=\lambda_0} = 0 \; ; \qquad i = 1,2,\ldots\ldots,k \qquad \text{...(7(a))}$$

We now consider the other possibility. Let $\lambda_{io} = 0$, i.e., λ_{io} belong to the category (ii). In this case $\lambda_i - \lambda_{i0}$ is always positive since $\lambda_{i0} \geq 0$ and $\lambda_i \neq \lambda_{i0}$. Also since $F(X_0,\lambda) - F(X_0,\lambda_0)$ is never positive, therefore in (6) we must have

$$\left(\frac{\partial F(X_0,\lambda)}{\partial \lambda_i}\right)_{\lambda=\lambda_0} = 0 \; ; \qquad i = k+1, k+2\ldots\ldots\ldots\ldots, m \qquad \text{...(7(b))}$$

(7(a)) and (7(b)) together imply that

$$\left(\frac{\partial F(X_0,\lambda)}{\partial \lambda_i}\right)_{\lambda=\lambda_0} \leq 0 \; ; \qquad i = 1,2,\ldots\ldots,m \qquad \text{...(7)}$$

Now, for category (i), $\lambda_{io} > 0$; i =1, 2,.............,k

Therefore, from (7(a))

$$\lambda_{i0}\left(\frac{\partial F(X_0,\lambda)}{\partial \lambda_i}\right) = 0 \qquad \text{...(8)}$$

Similary, for category (ii), $\lambda_{i0} = 0$; i = k+1, k+2............., m. Therefore from (7(b))

$$\lambda_{i0}\left[\frac{\partial F(X_0,\lambda)}{\partial \lambda_i}\right] = 0 \qquad \text{...(9)}$$

Thus we have, from (8) and (9)

$$\lambda_{i0}\left[\frac{\partial F(X_0,\lambda)}{\partial \lambda_i}\right] = 0, \quad \text{for all } i = 1,2,\ldots\ldots\ldots,m \qquad \text{....(10)}$$

Using (3), we can replace $F(X,\lambda)$ by $f(X) + \lambda^T G(X)$ in (5), (7) and (10) and get these conditions in the following from

$$\frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} l_i \frac{\partial g_i}{\partial x_j} = 0 \; ; \qquad j = 1,2,\ldots\ldots,n \qquad \text{...(11)}$$

$$g_i(X) \le 0 \quad ; \quad i = 1,2,......., m \qquad ...(12)$$

$$\lambda_i\, g_i(X) = 0 \quad ; \quad i = 1,2,......., m \qquad ...(13)$$

$$\lambda_i \ge 0 \quad ; \quad i = 1,2,......., m \qquad ...(14)$$

where all the expressions have been evaluated at $(X_0,\ \lambda_0)$.

So far, we have not imposed any restriction on X. Most of the nonlinear programming problems do have the nonnegativity condition on X (i.e. $X \ge 0$). In such a case, when $X \ge 0$, the above discussion remains unchanged except that we define a nonnegative saddle point $(X_0,\ \lambda_0)$ of the Lagrangian function $F(X,\lambda)$ as $F(X\ ,\lambda) \le F(X_0\ ,\lambda_0) \le F(X\ ,\lambda_0), X \ge 0,\ \bar{\lambda} \ge 0$.

Also then the condition (5) is modified to take into account the possibility of X_0 being a boundary point, i.e., some or all of the components being zero. As we argued in deriving (7) and (8), (5) is then replaced by the condition

$$\left[\frac{\partial F\,(X_0,\ \lambda_0)}{\partial x_j}\right]_{X=X_0} \ge 0 \quad ; \quad j = 1,2,......., n \qquad ...(16)$$

$$X_{j0}\left[\frac{\partial F\ (X,\ \lambda_0)}{\partial x_{j0}}\right]_{X=X_0} = 0 \quad ; \quad X_0 \ge 0 \qquad ...(17)$$

Again using (3), we may rewrite the conditions (7), (8), (16) and (17) corresponding to the nonnegative saddle point (X_0,λ_0) [defined by (15)] as :

$$\frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} \ge 0 \ ; \quad j = 1,2,......., n \qquad ...(18)$$

$$x_j\left(\frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j}\right) = 0 \ ; \quad j = 1,2,......., n \qquad ...(19)$$

$$g_i(X) \le 0 \quad ; \quad i = 1,2,......., m \qquad ...(20)$$

$$\lambda_i\, g_i(X) = 0 \quad ; \quad i = 1,2,......., m \qquad ...(21)$$

$$x_j \ge 0 \quad ; \quad j = 1,2,......., n \qquad ...(22)$$

$$\lambda_j \ge 0 \quad ; \quad i = 1,2,......., m \qquad ...(23)$$

The sets of conditions (11) to (14) or (18) to (23) are called the **Necessary form of Kuhn-Tucker (K-T) conditions.** The conditions (11) to (14) are the necessary conditions which (X_0,λ_0) must satisfy if it is a saddle point of the Lagrangian function $F(X,\lambda)$, with the variable X unrestricted in sign,whereas the conditions (18) to (23) are the necessary conditions which (X_0,λ_0), satisfies , if it is a nonnegative saddle point of the function $F(X_0,\lambda_0)$, with $X_0 \ge 0$.

The above conditions are not however sufficient conditions for (X_0, λ_0) to be a saddle point of $F(X,\lambda)$. The reason is quite simple. The condition (11) implies that the gradient of the Lagrangian function $F(X,\lambda)$ with respect to X is zero, which is necessary but not a sufficient condition for the existence of the minimum of $F(X,\lambda)$ with respect to X.

If $f(X)$ and all $g_i(X)$ are convex functions, then the saddle point (X_0, λ_0), $\lambda_0 \geq 0$ of $F(X,\lambda)$ does exist such that X_0 is a point of minima of the function $f(X)$ subject to the constraints $g_i(X) \leq 0$, $i = 1, 2, \ldots\ldots, m$ and $X \geq 0$. With the additional restriction $X \geq 0$, the saddle point. (X_0, λ_0) is nonnegative. Since $f(X_0)$ is convex, it has only one optimum which is the minimum.

Hence, if $f(X_0)$ and all $g_i(X)$ are convex functions, then the solution of the corresponding K-T conditions gives rise the required saddle point and so the minimal point of $f(X)$. If $f(X)$ and $g_i(X)$ are not convex,the K-T conditions can still be obtained and we may look for its solution. The solution so obtained may still give the solution to the corresponding programming problem but not necessarily always.

We have so far assumed that the constranits are $g_i(X) \leq 0$; i = 1, 2, ...,m. However if the constraints are in the form $g_i(X) \leq 0$ then we face no difficulty as we can write them as $-g_i(X) \leq 0$, and while constructing the K-T conditions, we may take the Lagrange multiplier as $-\lambda_i$ instead of λ_i, with $\lambda_i \geq 0$, i = 1, 2, ...,m.

The equality constraint $g_i(X) = 0$ leads to a slightly different case. In this case we shall only observe that the Lagrange multiplier λ_i is unrestricted in sign. In a general way , the constraint $g_i(X) = 0$ is replaced by two inequality constraints $g_i(X) \leq 0$ and $g_i(X) \geq 0$, with the result that the corresponding Lagrange multipliers $\lambda_i^{(1)}$ and $\lambda_i^{(2)}$ both non-negative, would contribute to the term $\left(\lambda_i^{(1)} - \lambda_i^{(2)}\right) g_i(X)$ in the Lagrangian function with the Lagrangian multiplier $\lambda_i = \lambda_i^{(1)} - \lambda_i^{(2)}$ becoming unrestricted in sign.

We now summarize the general form of Kuhn-Tucker conditions which are used to solve the constrained nonlinear programming problems.

If we have the optimization problem:

$$\text{Minimize} \quad f(X) \quad ; \; X = (x_1, x_2, \ldots\ldots, x_n)$$

$$\text{Subject to} \quad g_i(X) \leq 0 \quad ; \; i = 1, 2, \ldots\ldots\ldots\ldots m$$

$$h_j(X) = 0 \quad ; \quad j = 1,2,\ldots\ldots, m \qquad \ldots(24)$$

then the Kuhn-Tucker conditions are :

$$\nabla f(X) + \sum_{i=1}^{m} \lambda_i \nabla g_i(X) - \sum_{j=1}^{p} u_j \nabla h_j(X) = 0$$

$$\lambda_i \, g_i(X) = 0 \quad ; \qquad i = 1, 2, \ldots\ldots\ldots\ldots, m$$

$$g_i(X) \le 0 \quad ; \qquad i = 1, 2, \ldots\ldots\ldots\ldots, m$$

$$h_j(X) = 0 \quad ; \qquad j = 1, 2, \ldots\ldots\ldots\ldots, p$$

$$\lambda_i \ge 0 \quad ; \qquad i = 1, 2, \ldots\ldots\ldots\ldots, m \qquad \ldots(25)$$

Where λ_i and u_j are the Lagrange multipliers associated with the constraints $g_i(X) \le 0$ and $h_j(X) = 0$ respectively. The above form of Kuhn-Tucker conditions represents only the necessary conditions of optimality. In the followin, we specify the precise conditions for the Kuhn-Tucker conditions to be satisfied, which are known as the sufficient conditons.

The Kuhn-Tucker necessary conditions derived above are sufficient for the function $f(X)$ to have a minimum at $X = X_0$, if $f(X)$ is convex, $g_i(X)$ is convex if $\lambda_{i0} \ge 0$ and $g_i(X)$ is concave if $\lambda_{i0} \ge 0$ for i = 1, 2,..........,m.

From the saddle point theorem, $F(X,\lambda)$ has a saddle point at (X_0,λ_0) if

$$F(X_0,\lambda) \le F(X_0,\lambda_0) \le F(x,\lambda_0) \qquad \ldots(26)$$

Now $\quad F(X,\lambda_0) = f(X) + \sum_{i=1}^{m} \lambda_{io} \, g_i(X)$

and since $\lambda_{io} \ge 0, \; g_i(X) \le 0$ imply that

$\sum_{i=1}^{m} \lambda_{io} \, g_i(X) \le 0$, therfore, we have

$$F(X,\lambda_0) \le f(X) \qquad \ldots(27)$$

Also $\quad F(X_0,\lambda_0) = f(X_0) + \sum_{i=1}^{m} \lambda_{i0} \, g_i(X_0)$

$$= f(X_0) + 0$$

Therfore $F(X,\lambda_0) = f(X_0) \qquad \ldots(28)$

(26), (27) and (28) together imply that

$$f(X_0) = F(X_0,\lambda_0) \le F(X,\lambda_0) \le f(X)$$

or, $\quad f(X_0) \le f(X)$ for all $X \ge 0$

i.e. $\quad f(X)$ attains absolute minimum at X_0.

General sufficient form of Kuhn-Tucker conditions can be stated as follows:

Let X_0 be feasible solution to the problem (24). If $\nabla g_i(X_0)\ i \in I$ where I is the set of constraints $g_i(X) \leq 0$ which are satisfied as exact equalities at $X = X_0$ and $\nabla h_j(X_0)$, j = 1, 2,.........,p are linearly independent, then there exist λ_0 and u_0 such that (X_0, λ_0, u_0) satisfy (25).

The condition that $\nabla g_i(X_0)\ i \in I$, where I is the set of constraints $g_i(X) \leq 0$, which are satisfied as exact equalities at X_0 and $\nabla h_j(X_0)$, j = 1, 2, ...,p, be linearly independent, is called constraint qualification. If the constraint qualification fails to hold good at the optimum point, then (25) may or may not have a solution. It is not easy to verify the constraint qualification without knowing X_0 in prior. However the constraint qualification is always satisfied if :

(i) all the inequality and equality constraints are linear.

(ii) all the inequality constraints are convex and all the equality constraints are linear. Also atleast one feasible solution X_0 exists which lies inside the feasible region, so that

$$g_i(X_0) < 0 \ ; \ i = 1, 2, \ldots\ldots\ldots\ldots, m$$

and $$h_j(X_0) < 0 \ ; \ j = 1, 2, \ldots\ldots\ldots\ldots\ldots\ldots, p$$

(iii) The problem is a convex programming problem.

The conditions that ensure that a point satisfying the Kuhn-Tucker conditions is the desired point of optima, can be summarized in the following tables. First table ensures the conditions, which the functions appearing in the given problem must satisfy in order for the solution of Kuhn-Tucker conditions to yield the optimal solution, while the second table ensures the conditions that must be satisfied by the Lagrange multipliers of a point satisfying Kuhn-Tucker conditions to be the point of optimality.

Table -1

Senes of Optimization	Required Conditions	
	Objective Functions	Solution Space
Maximization	Concave	Convex Set
Minimization	Convex	Convex Set

Table- 2

Sense of Optimization	Required Conditions		
	$f(X)$	$g_i(X)$	λ_i
Maximization	Concave	Convex	≥ 0
		Concave	≤ 0
		linear equation	unrestricted
		convex	≤ 0
Minimization	Convex	Concave	≥ 0
		linear equation	unrestricted

Example 1 : Write the Kuhn-Tucker necessary and sufficient conditions for the following nonlinear programming problem to have on optional solution.

Min. $f(x_1, x_2) = x_1^2 - 2x_1 - x_2$

s.t. $2x_1 + 3x_1 \leq 6$

$2x_1 + x_2 \leq 4$

$x_1 + x_2 \geq 0$

Solution : The Lagrangian function for the given minimization problem is :

$$F(X, \lambda) = x_1^2 - 2x_1 - x_2 + \lambda_1 (2x_1 + 3x_2 - 6) + \lambda_2 (2x_1 + x_2 - 4)$$

the necessary conditions are :

(i) $\dfrac{\partial F(X, \lambda)}{\partial x_j} \geq 0$; j=1, 2

i.e. $2x_1 - 2 + 2\lambda_1 + 2\lambda_2 \geq 0$

$-1 + 3\lambda_1 + \lambda_2 \geq 0$

i.e. $2x_1 + 2\lambda_1 + 2\lambda_2 + 2\lambda_2 - u_1 = 2$

$3\lambda_1 + \lambda_2 \quad -u_2 = 1$

(on adding surplus variables) u_1 and u_2)

(ii) $\dfrac{\partial F(X, \lambda)}{\partial \lambda_i} \leq 0$; i = 1 , 2

i.e. $2x_1 + x_2 - 6 \leq 0$

$2x_1 + x_2 - 4 \leq 0$

or $2x_1 + 3x_2 + y_1 = 6$

$2x_1 + x_2 \quad + y_2 = 4$

(on adding slack variables y_1 and y_2)

(iii) $\dfrac{\partial F(X, \lambda)}{\partial x_j} . x_j = 0$; j = 1, 2

i.e. $(2x_1 - 2 + 2\lambda_1 + 2\lambda_2)\, x_1 = 0$

$(-1 + 3\lambda_1 + \lambda_2)\, x_2 = 0$

(iv) $\dfrac{\partial F(X, \lambda)}{\partial \lambda_i} . \lambda_i = 0$; i= 1, 2 i.e. $(2x_1 + 3x_2 - 6)\lambda_1 = 0$

$(2x_1 + x_2 - 4)\,\lambda_2 = 0$

(v) $x_{1,}\, x_2,\, \lambda_1,\, \lambda_2,\, u_1,\, u_{2,}\, y_{1,}\; y_2 \geq 0$

Since the function min $f(x_{1,}\, x_2) = x_1^2 - 2x_1 - x_2$ is convex, therefore the above conditions are suffcient also.

Example 2 : Use Kuhn - Tucker condition to solve the following non-linear programming problem :

Max $f(x) = 8x - x^2$

subject to $x \leq 3$

$x \geq 0$

Solution : We have the Lagrangian function

$$F(x, \lambda) = 8x - x^2 + \lambda = (3 - x)$$

The Kuhn-Tucker conditions are ;

$$\frac{\partial F(x,\lambda)}{\partial x} \leq 0 \quad \text{, or} \quad 8 - 2x - \lambda \leq 0$$

$$\frac{\partial F(x,\lambda)}{\partial \lambda} \geq 0 \quad \text{, or} \quad 3 - x \geq 0$$

$$\frac{\partial F(x,\lambda)}{\partial x}\, x = 0 \quad \text{, or} \quad (8 - 2x - \lambda)\, x = 0$$

$$\frac{\partial F(x,\lambda)}{\partial x} . \lambda = 0 \quad \text{, or} \quad (3 - x)\, \lambda = 0$$

$$x, \lambda \geq 0$$

i.e. $8 - 2x - \lambda \leq 0$, $x \geq 0$, $x(8 - 2x - \lambda)\; 0$ ... (1)

$3 - x \geq 0$, $\lambda \leq 0$, $\lambda(3 - x) = 0$ ...(2)

By combinatorial nature of the equations atleast one of the inequality in (1) must be satisfied in equality form, and similary for (2). Hence we have the following four possible combinations :

(i) $8 - 2x - \lambda = 0$, $3 - x = 0$, i.e. $x = 3$, $\lambda = 2$

This solution satisfies $x \geq 0$ and $\lambda \geq 0$.

(ii) $8 - 2x - \lambda = 0$, $1 = 0$ i.e. $x = 4$, $\lambda = 0$, which violates the condition $3 - x \geq 0$

(iii) $x = 0$, $3 - x = 0$, which is inconsistent

(iv) $x = 0$, $1 = 0$, which violates the condition $8 - 2x - \lambda \leq 0$

Thus only the first combination gives a solution to Kuhn-Tucker conditions. Since both functions $f(x) = 8x - x^2$ and $g(x) = 3 - x$ are concave (note it), the solution $x = 3$, $\lambda = 2$ represents a global maximum of $f(x)$ Hence the optimal solution is $x = 3$, $\lambda = 2$.

Example 3 : Solve the following programming problem graphically and verify the Kuhn-Tucker conditions for the same:

Maximize $f(x_1, x_2) = 2x_1 + 3x_2$

Subject to $x_1^2 + x_2^2 \leq 20$

$x_1 x_2 = 8$

$x_1, x_2 \geq 0$

Solution : In the figure shown below, the constraint curves $x_1^2 + x_2^2 = 0$ and $x_1 x_2 = 8$ are plotted (see fig. 7.1) Since $x_1, x_2 \geq 0$, the feasible region falls in the first quadrant only. The curve $x_1^2 + x_2^2 = 20$ represents a circle with its centre at $(0,0)$ and radius $(20)^{1/2}$ and the curve $x_1, x_2 = 8$ represents a rectangular hyperbola having its asymptotes as the co-ordinate axes. The two curves intersect each other at points A (4,2) and B (2,4). The points (x_1, x_2) lying in the first quadrant shaded by the horizontal lines staisfy the constraints $x_1^2 + x_2^2 \leq 20$, $x_1 \geq 0$, $x_2 \geq 0$; while the points (x_1, x_2) lying in the first quadrant shaded by the vertical lines do satisfy the the constraints $x_1 x_2 = 8$; $x_1 \geq 0$, $x_2 \geq 0$. Thus the required solution must be somewhere in the double shaded region.

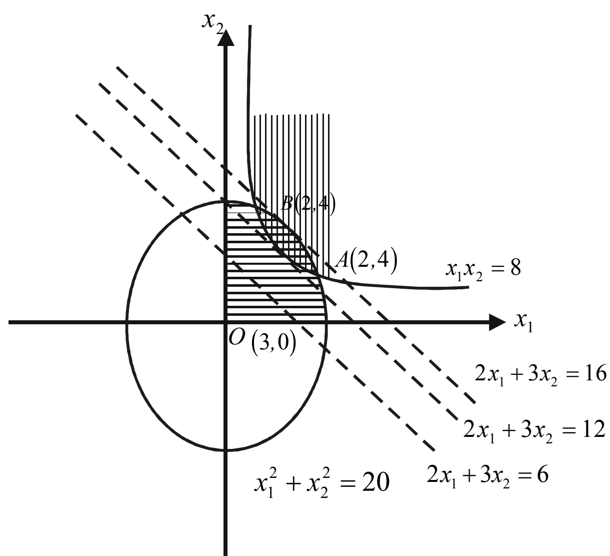


Figure : 7.1

Now in the feasible region for the point (x_1, x_2) that maximizes the function $f(x_1, x_2) = 2x_1 + 3x_2$ adn lies in the feasible region, we draw the lines paralled to the line $2x_1 + 3x_2 = c$ (c is chosen arbitrarily) moving away from origin till the line parallel to $2x_1 + 3x_2 = c$ touches the extreme boundary of the feasible region. It is noticed that the point B (2,4) gives the maximum value of $f(x_1, x_2) = 16$. Thus the graphical solutin of the give problem is :

$x_1 = 2$, $x_2 = 4$; Max ; $f(x_1, x_2) = 16$.

In order to verify that this optimal solution satisfies the Kuhn-Tucker conditions also, we first find the Lagrangian function of the given problem,which is

$$F(X,\lambda) = 2x_1 + 3x_2 + \lambda_1\left(20 - x_1^2 - x_2^2\right) + \lambda_2\left(8 - x_1 x_2\right)$$

Then the Kuhn-Tucker conditions are :

$$\frac{\partial F(X,\lambda)}{\partial x_j} \le 0\ ,\ \ j = 1, 2 \quad \text{or,} \quad \left.\begin{array}{r} 2 - 2\lambda_1 x_1 - \lambda_2 x_2 \le 0 \\ 3 - 2\lambda_1 x_2 - \lambda_2 x_1 \le 0 \end{array}\right\} \qquad \text{...(1)}$$

$$\frac{\partial F(X,\lambda)}{\partial \lambda_i} \ge 0\ ,\ \ i = 1, 2 \quad \text{or,} \quad \left.\begin{array}{r} 20 - x_1^2 - x_2^2 \ge 0 \\ 8 - x_1 x_2 \ge 0 \end{array}\right\} \qquad \text{...(2)}$$

$$\frac{\partial F(X,\lambda)}{\partial x_j} \cdot x_j = 0\ ; \qquad j = 1, 2$$

$$\text{or,} \quad \left.\begin{array}{r} \left(2 - 2\lambda_1 x_1 - \lambda_2 x_2\right) x_1 = 0 \\ \left(3 - 2\lambda_1 x_2 - \lambda_2 x_1\right) x_2 = 0 \end{array}\right\} \qquad \text{...(3)}$$

$$\frac{\partial F(X,\lambda)}{\partial \lambda_i} \cdot \lambda_i = 0\ ; \qquad i = 1, 2$$

$$\text{or,} \quad \begin{array}{r} \left(20 - x_1^2 - x_2^2\right)\lambda_1 = 0 \\ \left(8 - x_1 x_2\right)\lambda_2 = 0 \end{array} \qquad \text{...(4)}$$

$$x_1\ ,\ x_2\ ,\ \lambda_1\ ,\lambda_2 \ge 0 \qquad \text{...(5)}$$

We see that if the point (2,4) satisfies these conditions, then from (1), we have $\lambda_1 = \frac{1}{3}$ and $\lambda_2 = \frac{1}{6}$ which do satisfy (2) , (3) and (4). Thus the optimal solution obtained by graphical method also satisfies the Kuhn-Tucker conditions for optima.

Example 4 : Determine the optimal solution of the following nonlinear programming problem, using the Kuhn-Tucker conditons :

Minimize $\quad f(x_1,\ x_2) = x_1^2 + 2x_2^2 - x_1 x_2$

subject to $\quad x_1 + x_2 \ge 8$

$x_1, x_2 \ge 0$

Solution : The Lagrangian function for the given programming problem is :

$$f(X,\lambda) = x_1^2 + 2x_2^2 - x_1 x_2 + \lambda\left(8 - x_1 - x_2\right)$$

Then the Kuhn-Tucker conditions are

$$\frac{\partial F(X,\lambda)}{\partial x_j} \geq 0 \ ; \ j = 1, 2 \qquad \text{or} \ \ 2x_1 - x_2 - \lambda \geq 0 \qquad \ldots(1)$$

$$-x_1 + 4x_2 - \lambda \geq 0 \qquad \ldots(2)$$

$$\frac{\partial F(X,\lambda)}{\partial \lambda} \leq 0, \qquad \text{or,} \ \ 8 - x_1 - x_2 \leq 0$$

$$\text{i.e.,} \qquad x_1 + x_2 \geq 8 \qquad \ldots(3)$$

$$\frac{\partial F(X,\lambda)}{\partial x_j} \cdot x_j = 0 \ ; \ j = 1, 2 \qquad \text{or} \quad (2x_1 - x_2 - \lambda)x_1 = 0 \qquad \ldots(4)$$

$$(-x_1 + 4x_2 - \lambda)x_2 = 0 \qquad \ldots(5)$$

$$\frac{\partial F(X,\lambda)}{\partial \lambda} \cdot \lambda = 0 \qquad \text{or} \qquad (8 - x_1 - x_2)\lambda = 0 \qquad \ldots(6)$$

$$x_1, x_2, \lambda \geq 0 \qquad \ldots(7)$$

It can easily be seen that if $\lambda = 0$, then $x_1 = 0$, $x_2 = 0$ is the only point satisfying the conditions (1), (2), (4) and (5). But $x_1 = 0$, $x_2 = 0$ does not satisfy the condition (3).

Hence $\lambda = 0$ and therefore

$$x_1 + x_2 = 8 \qquad [\text{from eq. } (6)] \qquad \ldots(8)$$

Now , if $x_1 = 0$ then $x_2 = 8$. But then inequality (1) is not satisfied. Therefore, $x_1 \neq 0$. similarly if $x_2 = 0$, then $x_1 = 8$ and then inequality (2) is not satisfied. Therefore $x_2 \neq 0$.

Thus $x_1 \neq 0$ and $x_2 \neq 0$. In this case (4) and (5) imply that

$$2x_1 - x_2 - \lambda = 0 \qquad \ldots(9)$$

$$-x_1 + 4x_2 - \lambda = 0 \qquad \ldots(10)$$

Solving equations (8), (9) and (10), we get $x_1 = 5$, $x_2 = 3$ and $\lambda = 7$

which satisy all the Kuhn-Tucker conditions from (1) to (7)

Thus the optimal solution to the given problem is $x_1 = 5$, $x_2 = 3$ and the minimum value of $f(x_1, x_2)$ is

$$f(x_1, x_2) = 5^2 + 2(3)^2 - 5 \times 3 = 28$$

Example 5 : Use Kuhn-Tucker conditions to determine x_1 , x_2 , x_3 so as to Minimize

subject to $\quad x_1 + x_2 \leq 2 \qquad f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 4x_1 - 6x_2$

$$2x_1 + 3x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

Solution : The Lagrangian function for the given problem is :

$$F(X,\lambda) = x_1^2 + x_2^2 + x_3^2 - 4x_1 - 6x_2 + \lambda_1(x_1 + x_2 - 2) + \lambda_2(2x_1 + 3x_2 - 12)$$

The Kuhn-Tucker conditions are

$$2x_1 - 4 + \lambda_1 + 2\lambda_2 \geq 0 \qquad\qquad 2x_1 + \lambda_1 + 2\lambda_2 \geq 4 \qquad ...(1)$$

$$2x_2 - 6 + \lambda_1 + 3\lambda_2 \geq 0 \quad \text{or} \quad 2x_2 + \lambda_1 + 3\lambda_2 \geq 6 \qquad ...(2)$$

$$2x_3 \geq 0 \qquad\qquad 2x_3 \geq 0 \qquad ...(3)$$

$$x_1 + x_2 - 2 \leq 0 \quad \text{or} \quad x_1 + x_2 \leq 2 \qquad ...(4)$$

$$2x_1 + 3x_2 - 12 \leq 0 \qquad\qquad 2x_1 + 3x_2 \leq 12 \qquad ...(5)$$

$$(2x_1 - 4 + \lambda_1 + 2\lambda_2)x_1 = 0 \qquad ...(6)$$

$$(2x_1 - 6 + \lambda_1 + 3\lambda_2)x_2 = 0 \qquad ...(7)$$

$$(x_1 + x_2 - 2)\lambda_1 = 0 \qquad ...(8)$$

$$(2x_1 + 3x_2 - 12)\lambda_2 = 0 \qquad ...(9)$$

$$x_1, x_2, x_3, \lambda_1, \lambda_2 \geq 0 \qquad ...(10)$$

The following four different cases arise

(i) If $\lambda_1 = \lambda_2 = 0$, then from (1), (2) and (3), we have $x_1 = 2$, $x_2 = 3$, $x_3 = 0$. But this solution violates the inequalities (4) and (5)

(ii) When $\lambda_1 = 0$, $\lambda_2 \neq 0$. In this case from (1), (2) and (9)

$$2x_1 + 2\lambda_2 = 4,\ 2x_2 + 3\lambda_2 = 6 \text{ and } 2x_1 + 3x_2 - 12 = 0$$

which give $x_1 = 24/13$, $x_2 = 36/13$ and $\lambda_2 = 2/13$ Also from (3), $x_3 = 0$. However this solution violates inequality (4), so this solution is also ruled out.

(iii) When $\lambda_1 \neq 0$, $\lambda_2 = 0$. In this case (8) gives

$x_1 + x_2 = 2$, which along with (1) and (2) i.e., along with $2x_1 + \lambda_1 = 4$ and $2x_2 + \lambda_1 = 6$ give $x_1 = \frac{1}{2}$, $x_2 = \frac{3}{2}$, $\lambda_1 = 3$. Further from (3), $x_3 = 0$. This solution does not violate any of the conditon.

(iv) When $\lambda_1 \neq 0$, $\lambda_2 \neq 0$. In case (8) and (9) give $x_1 + x_2 = 2$ and $2x_1 + 3x_2 = 12$, where from $x_1 = -6$, $x_2 = 8$. Thus from (1), (2) and (3), we get $\lambda_1 = 68$, $\lambda_2 = -26$ and $x_3 = 0$. This violates the condition $x_1 \geq 0$ and $\lambda_2 \geq 0$. Hence $x_1 = -6$, $x_2 = 8$, $x_3 = 0$ is also discarded.

Thus the optimal solution to the given programming problem is given by case (iii) i.e. optimal solution is :

$$x_1 = \frac{1}{2}, \quad x_2 = \frac{3}{2}, \quad x_3 = 0, \text{ with } \lambda_1 = 3 \text{ and } \lambda_2 = 0.$$

The minimum value of $f(x_1, x_2, x_3)$ is

$$\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + 0 - 4\left(\frac{1}{2}\right) - 6\left(\frac{3}{2}\right) = \frac{-17}{2}$$

Example 6 : Solve the following nonlinear programming problem

Minimize $\quad f(x_1, x_2) = (x_1 - 2)^2 + (x_2 - 1)^2$

subject to $\quad x_1^2 - x_2 \le 0$

$x_1 + x_2 \le 2$

$x_1, x_2 \ge 0$

Solution : The Hessian matrix for $f(x_1, x_2)$ is :

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The principal minors are $D_1 = 2$, $D_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4$, which are both positive. So $f(x_1, x_2)$ is a convex function. Also, the given constraint functions are convex functions, therefore, the Kuhn-Tucker conditions for the minimization of $f(x_1, x_2)$ are both necessary and sufficient.

The Lagrangian function is :

$$F(X, \lambda) = (x_1 - 2)^2 + (x_2 - 1)^2 + \lambda_1(x_1^2 - x_2) + \lambda_2(x_1 + x_2 - 2)$$

The Kuhn-Tucker conditions, therefore, are

$$2(x_1 - 2) + 2\lambda_1 x_1 + \lambda_2 \ge 0$$

$$2(x_2 - 1) - \lambda_1 + \lambda_2 \ge 0$$

i.e. $\quad 2x_1 + 2\lambda_1 x_1 + \lambda_2 - 4 \ge 0 \qquad ...(1)$

$2x_2 - \lambda_1 + \lambda_2 - 2 \ge 0 \qquad ...(2)$

$x_1^2 - x_2 \le 0 \qquad ...(3)$

$$x_1 + x_2 - 2 \le 0 \quad ...(4)$$

$$\left(2\left(x_1 - 2\right) + 2\lambda_1 x_1 + \lambda_2\right)x_1 = 0 \quad ...(5)$$

$$\left(2\left(x_2 - 1\right) - \lambda_1 + \lambda_2\right)x_2 = 0 \quad ...(6)$$

$$\left(x_1^2 - x_2\right) \lambda_1 = 0 \quad ...(7)$$

$$\left(x_1 + x_2 - 2\right)\lambda_2 = 0 \quad ...(8)$$

$$x_1, x_2, \lambda_1, \lambda_2 \ge 0 \quad ...(9)$$

The following four cases arise

(i) When $\lambda_1 = 0 = \lambda_2$. In this case from (1) and (2) $2x_1 - 4 = 0; 2x_2 - 2 = 0$ i.e. $x_1 = 2$, $x_2 = 1$, which do not satisfy conditions (3) and (4) Thus this solution is not acceptable.

(ii) When $\lambda_1 = 0,\ \lambda_2 \neq 0$. Then from (8)

$x_1 + x_2 = 2$. Also from (1) and (2)

$$2x_1 + \lambda_2 - 4 = 0,\ 2x_2 + \lambda_2 - 2 = 0$$

Which give $x_1 + x_2 = 2$ and $x_1 - x_2 = 1$

or $\quad x_1 = \dfrac{3}{2},\ x_2 = \dfrac{1}{2}$

This solution violates the conditins (3), so is ruled out.

(iii) When $\lambda \neq 0;\ \lambda_2 = 0$. In this case from (1), (2) and (7)

$$2x_1 + 2\lambda_1 x_1 - 4 = 0$$

$$2x_1 - \lambda_1 - 2 = 0$$

$$x_1^2 - x_2 = 0$$

From the first of these two equations

$$2x_1 + 2x_1\left(2x_2 - 2\right) - 4 = 0$$

or $\quad -x_1 + 2x_1x_2 - 2 = 0$

which using $x_1^2 - x_2 = 0$ gives

$$2x_1^3 - x_1 - 2 = 0 \quad \text{or} \quad x_1 = 1.52$$

and then $x_2 = 2.31$

But these values of x_1 and x_2 do not satisfy conditions (4), so the solution $x_1 = 1.52$, $x_2 = 2.31$ is also discarded.

(iv) When $\lambda_1 \neq 0, \lambda_2 \neq 0$. In this case from (7) and (8), we have $x_1^2 - x_2 = 0$ and $x_1 + x_2 - 2 = 0$

From these two equations

$$x_1^2 + x_1 - 2 = 0$$

or $(x_1 + 2)(x_1 - 1) = 0$

or $x_1 = 1$ (since $x_1 \geq 0$)

Thus $x_2 = 1$

These values of x_1 and x_1 when put in conditions (1) and (2), give

$$2\lambda_1 + \lambda_2 = 2 \text{ and } -\lambda_1 + \lambda_2 = 0$$

or $\lambda_1 = 2/3$, $\lambda_2 = 2/3$

The solution $x_1 = 1$, $x_2 = 1$, $\lambda_1 = 2/3$, $\lambda_2 = 2/3$ does satisfy all the conditions from (1) to (9) and so is the optimal solution of the problem.

Hence the optimal solution of the given problem is $x_1 = 1$, $x_2 = 1$ and minimum value of

$$f(x_1, x_2) = (1-2)^2 + (1-1)^2 = 1$$

Example 7 : Use Kuhn-Tucker conditions to solve the following non linear programming problem :

Maximize $f(x_1, x_2) = 7x_1^2 - 6x_1 + 5x_2^2$

subject to $x_1 + 2x_2 \leq 10$

$x_1 - 3x_2 \leq 9$

$x_1, x_2 \geq 0$

Solution : The Lagrangian function for the given programming problem is

$$F(X, \lambda) = 7x_1^2 - 6x_1 + 5x_2^2 + \lambda_1(10 - x_1 - 2x_2) + \lambda_2(9 - x_1 + 3x_2)$$

The Kuhn-Tucker conditions are :

$14x_1 - 6 - \lambda_1 - \lambda_2 \leq 0$ or $14x_1 - \lambda_1 - \lambda_2 \leq 6$ ...(1)

$10x_2 - 2\lambda_1 + 3\lambda_2 \leq 0$ $10x_2 - 2\lambda_1 + 3\lambda_2 \leq 0$ ...(2)

$x_1 + 2x_2 - 10 \leq 0$ ...(3)

$x_1 - 3x_2 - 9 \leq 0$ ...(4)

$(14x_1 - 6 - \lambda_1 - \lambda_2)x_1 = 0$ ...(5)

$$\left(10x_2 - 2\lambda_1 - 3\lambda_2\right)x_2 = 0 \qquad \text{...(6)}$$

$$\left(x_1 + 2x_2 - 10\right)\lambda_1 = 0 \qquad \text{...(7)}$$

$$\left(x_1 - 3x_2 - 9\right)\lambda_2 = 0 \qquad \text{...(8)}$$

$$x_1, x_2, \lambda_1, \lambda_2 \geq 0 \qquad \text{...(9)}$$

The following four possibilities arise

(i) When $\lambda_1 = 0 = \lambda_2$. In that case from (1) and (2) $x_1 = \frac{3}{7}$ and $x_2 = 0$

This solution satisfies (3), (4) and (9) and so is a feasible solution with $f\left(x_1, x_2\right) = -9/7$.

(ii) When $\lambda_1 \neq 0,\ \lambda_2 = 0$. In this case equations (1), (2) and (7) are

$$14x_1 - \lambda_1 = 6\ ;\ 10x_2 - 2\lambda_1 = 0; \quad x_1 - 2x_2 = 10$$

which give $x_1 = \frac{62}{33},\ x_2 = \frac{134}{33},\ \lambda_1 = \frac{670}{33}$.

This solution also satisfies all the other conditions and so is a feasible solution with

$$f\left(x_1, x_2\right) = 95.78$$

(iii) When $\lambda_1 = 0,\ \lambda_2 \neq 0$. In this case we have from (1), (2) and (8)

$$14x_1 - \lambda_2 = 0\ ,10x_2 + 3\lambda_2 = 0,\ x_1 - 3x_2 = 9$$

which gives $x_1 = \frac{288}{17},\ x_2 = -\frac{45}{17}$

This is an infeasible solution and so is ruled out.

(iv) When $\lambda_1 \neq 0\ ;\ \lambda_2 \neq 0$. In this case from equations (7) and (8) we have,

$x_1 = 48/5$, $x_2 = 1/5$. These values of x_1 and x_2 when put in (1) and (2) give $\lambda_1 = \frac{1936}{25}$, $\lambda_2 = \frac{1274}{25}$. This solution also satisfies all the other conditions and so is acceptable with $f\left(x_1, x_2\right) = 587.72$

Hence the optimal solution is

$x_1 = \frac{48}{5}$, $x_2 = \frac{1}{5}$ and maximum value of $f(x_1, x_2) = 587.72$.

Example 8 : Use Kuhn-Tucker conditions to solve the following nonlinear programming problem :

Optimize $f\left(x_1, x_2, x_3\right) = 2x_1 + 3x_2 - \left(x_1^2 + x_2^2 + x_3^2\right)$

subject to $x_1 + x_2 \leq 1$

$2x_1 + 3x_2 \leq 6$

$x_1, x_2 \geq 0$

Solution : Before applying Kuhn-Tucker conditions we would determine, whether the given problem is of maximization or of minimization type. We construct the bordered Hessian matrix

$$H^B = \begin{bmatrix} O & \vdots & P \\ \cdots & \vdots & \cdots \\ P^T & \vdots & Q \end{bmatrix}_{m+n,\, m+n}$$

$$= \begin{bmatrix} 0 & 0 & \vdots & 1 & 1 & 0 \\ 0 & 0 & \vdots & 2 & 3 & 0 \\ \cdots & \cdots & \vdots & \cdots & \cdots & \cdots \\ 1 & 2 & \vdots & -2 & 0 & 0 \\ 1 & 3 & \vdots & 0 & -2 & 0 \\ 0 & 0 & \vdots & 0 & 0 & -2 \end{bmatrix} = -10$$

where $m = 2$, $n = 3$; $n - m = 1$, $2m + 1 = 5$. For maximization type, the sign of the Hessian matrix must be $(-1)^{m+1}$ i.e. negative, whereas for minimization it must be $(-1)^m$ i.e. positive. Since $H^B = -10 < 0$, therefore we have to maximize. $f(x_1, x_2, x_3)$ The Lagrangian function is :

$$F(X, \lambda) = 2x_1 + 3x_2 - (x_1^2 + x_2^2 + x_3^2) + \lambda_1(1 - x_1 - x_2) + \lambda_2(6 - 2x_1 - 3x_2)$$

The Kuhn-Tucker conditions, therefore, are

$$\frac{\partial F(X, \lambda)}{\partial x_j} \leq 0 \; ; \text{j}=1,2,3$$

or $\quad 2 - 2x_1 - \lambda_1 - 2\lambda_2 \leq 0 \qquad$ ...(1)

$3 - 2x_2 - \lambda_1 - 3\lambda_2 \leq 0 \qquad$ ...(2)

$-2x_3 \leq 0$

$$\frac{\partial F(X, \lambda)}{\partial \lambda_i} \geq 0 \; ; \text{i}=1,2$$

or $\quad 1 - x_1 - x_2 \geq 0 \qquad$ ...(3)

$6 - 2x_2 - 3x_2 \geq 0 \qquad$ ...(4)

$$\frac{\partial F(X, \lambda)}{\partial x_j} . x_j = 0 \quad ; \text{j}=1,2$$

or $\quad (2 - 2x_1 - \lambda_1 - 2\lambda_2).x_1 = 0 \qquad$ ....(5)

$(3 - 2x_2 - \lambda_1 - 3\lambda_2)x_2 = 0 \qquad$ ...(6)

$$\frac{\partial F(X,\lambda)}{\partial \lambda_i}.\lambda_i = 0 \quad ; \ \mathrm{i} = 1,2$$

or $\quad (1 - x_1 - x_2)\lambda_1 = 0$ ...(7)

$(6 - 2x_1 - 3x_2)\lambda_2 = 0$ ...(8)

$x_1, x_2, \lambda_1, \lambda_2 \geq 0$ ...(9)

Now, there arise the following four different possibilities

(i) When $\lambda_1 = 0$, $\lambda_2 = 0$. In this case equations (1), (2) and (3) give $x_1 = 1$, $x_2 = \frac{3}{2}$, $x_3 = 0$.

This solution does not satisfy the condition (3) and so is ruled out.

(ii) When $\lambda_1 = 0$; $\lambda_2 \neq 0$. Then from (8), (1), (2) and(3), we have

$$6 - 2x_1 - 3x_2 = 0$$

$$2 - 2x_1 - 2\lambda_2 = 0$$

$$3 - 2x_2 - 3\lambda_2 = 0$$

Solving these equations , we get $\mathrm{x}_1 = \frac{12}{13}$, $\mathrm{x}_2 = \frac{18}{13}$, $\mathrm{x}_3 = 0$, $\lambda_2 = \frac{1}{13}$ This solution again does not satisfy equation (3) and so is discarded.

(iii) When $\lambda_1 \neq 0$, $\lambda_2 = 0$. In this case from conditions (1), (2), (3) and (7), we get

$$2 - 2x_1 - \lambda_1 = 0$$

$$3 - 2x_2 - \lambda_1 = 0$$

$$x_3 = 0$$

$$1 - x_1 - x_2 = 0$$

Which give solution

$$x_1 = \frac{1}{4},\ x_2 = \frac{3}{4},\ x_3 = 0 \text{ and } \lambda_1 = \frac{3}{2}$$

This solution satisfies all the Kuhn-Tucker conditions and has $f(x_1, x_2, x_3) = 17/8$.

(iv) When $\lambda_1 \neq 0, \lambda_2 \neq 0$. In this case equations (1), (2), (3), (7) and (8) give $x_1 = -3$, $x_2 = 4$, $x_3 = 0$, $\lambda_1 = -34$, $\lambda_2 = 13$. This solution violates the conditions (9) and so is infeasible and thus discarded.

Since there is only one solution that satisfies all the conditions, therefore it is optimal.

Hence the optimal solution the given programming problem is

$x_1 = \frac{1}{4},\ x_2 = \frac{3}{4},\ x_3 = 0$ with maximum value of $f(x_1, x_2, x_3) = \frac{17}{8}$.

7.4 Self-Learning Exercise

1. If the objective function $f(X)$ and all the constraints $g_i(X)$ are convese functions, then the solution of the corresponding Kuhn-Tucker conditions gives rise the ....of $f(X)$.
2. If a concave function $f(X)$ is to be maximized subject to constraints convex in nature then the lagrange multipliers must be ....and when constraints are concave then they must be..........
3. If a concave function is to be maximized subject to linear constraints then λ_i are ....
4. When a convex objective function is to be minimized, then the solution space is a ....
5. When a concave objective function is to be maximized, then the solution space is a........

7.5 Summary

In this unit we discussed the Kuhn-Tucker conditions for the nonlinear programming problems. We also derived these conditions in the form a theorem known as Kuhn-Tucker theorem.

7.6 Answers to Self-Learning Exercise

1. Minimal point.
2. $\geq 0, \leq 0$
3. Unrestricted in sign.
4. Convex set
5. Convex set

7.7 Exercise

1. Define a general non-linear programming problem.
2. What are the Kuhn-Tucker conditions and how are they of fundamental improtance in the theory of nonlinear programming.
3. Formulate the Kuhn-Tucker necessary conditions for the following problem :

Maximize $\quad f(X)$

subject to $\quad g_i(X) \geq 0 \quad ; \mathrm{i} = 1, 2, \ldots\ldots\ldots, \mathrm{m}$

$g_i(X) \leq 0 \quad ; \mathrm{i} = \mathrm{m}+1,\ \mathrm{m}+2, \ldots\ldots\ldots, \mathrm{p}$

$h_i(X) = 0 \quad ; \mathrm{j} = 1, 2, \ldots\ldots., \mathrm{q}$

$X \geq 0$

4. Use Kuhn-Tucker conditions to solve the following nonlinear programming problems:

(i) Maximize $\quad f(X) = 8x_1 + 10x_2 - x_1^2 - x_2^2$

subject to $3x_1 + 2x_2 \le 0$

$x_1, x_2 \ge 0$

(Ans : $x_1 = x = 4/13$, $x_2 = 33/13$, maximum value = 21.3)

(ii) Max. $f(X) = 10x_1 + 10x_2 - x_1^2 - x_2^2$

subject to $x_1 + x_2 \le 14$

$-x_1 + x_2 \le 6$

$x_1, x_2 \ge 0$

(Ans : $x_1 = 5$, $x_2 = 5$, Max. $f(x) = 50$)

(iii) Max. $f(X) = 12x_1 + 21x_2 + 2x_1x_2 - 2x_1^2 - 2x_2^2$

subject to $x_1 + x_2 \le 10$

$x_2 \le 8$

$x_1, x_2 \ge 0$

(Ans : $x_1 = 17/4$, $x_2 = 23/4$ Max. $f(X) = 1734/16$)

(iv) Minimize $f(X) = x_1^2 + x_2^2 + x_3^2$

subject to $2x_1 + x_2 - x_3 \le 0$

$x_1 \ge 1$

$x_2 \ge 2$

$x_3 \ge 0$

□□□

Unit - 8
Quadratic Programming

Structure of the Unit

8.0 Objective

In the previous unit, it was discussed, how the optimal solution of a nonlinear programming problem could be obtained by solving its Kuhn-Tucker conditions. It can be experienced that solving Kuhn-Tucker conditions, which are a set of nonlinear equations and inequalities is not that easy in most of the problems. Alternative methods, therefore, are required to be developed for solving such nonlinear programming problems.

In this unit, special category of nonlinear programming problems, for which specific computational algorithm are developed, is considered. The problems under this special category are **Quadratic Programming Problems.**

8.1 Introduction

The problem of optimizing a quadratic function subject to a set of linear constraints is called a **quadratic programming problem**. The quadratic programming problems are computationally least difficult to handle, when we solve the other nonlinear programming problems. The quadratic programming problems are not only helpful in the application to real life situations but also serve as sub problems in number of algorithms developed for general nonlinear programming problems. In this unit we shall discuss some of the algorithms.

8.2 Quadratic Programming Problems

The quadratic programming problem is the simple most case amongst all nonlinear convex programming problems, which arises when the objective function is quadratic but the constraints in the given programming problem are all linear in nature. In such problems, the Kuhn-Tucker conditions of the problem can be expressed in a form which can be solved using a computational procedure based on the simplex method.

In general the nonlinear programming problem :

Maximize $\quad f(X) = C^T X + \frac{1}{2} X^T G X$

subject to $\quad AX \leq 0 \quad$ ...(1)

$\quad X \geq 0$

where X and $C \in E^n$, $b \in E^m$, G is $n \times n$ symmetric matrix and A is an $m \times n$ matrix, is called a general quadratic programming problem.

We recall that $X^T GX$ which represents a quadratic form is said to be positive definite (negative-definite) if $X^T G\,X > 0(< 0)$ for $X \neq 0$ and positive semidefinite (negative semidefinite) if $X^T G\,X \geq 0(\leq 0)$ for all X such that there is one $X \neq 0$ satisfying $X^T GX = 0$

It can easily be varified that if

(i) $X^T GX$ is positive semi definite (negative semi definite), then it is convex (concave) in X over E^n.

(ii) $X^T GX$ is positive definite (negative definite), then it is strictly convex (stricly concave in X over E^n.

The above two points will help us in determining whether the quadratic objective function $f(X)$ is concave (convex) and then we can simply the same on the sufficiency conditions of Kuhn-Tucker conditions for the maxima (minima) of $f(X)$.

A general constrained optimization problem, like the general linear programming problem, may have

(a) no feasible solution

(b) an unbounded solution or

(c) an optimal solution

The following theorem gives the conditions under which the objective function of the quadratic programming problem (1) may have finite maximum.

Theorem 1 : In the quadratic programming (1) the function $f(X)$ cannot have an unbounded maximum if $X^T GX$ is negative definite or if $C=0$. If $C\neq 0$ and $X^T GX$ is negative semidefinite then $f(X)$ may have an unbounded maximum.

Proof : Consider the quadratic programming (1)

Let $X \neq 0$, then the objective function $f(X)$ can be written as

$$f(X) = X^T GX\left(\frac{1}{2} + \frac{C^T X}{X^T GX}\right) \quad \text{...(2)}$$

Let X be any point on the hypersphere $|X| = r$, where $|X|^2 = X^T X$, Then $X = r\,\hat{X}$, where $\left|\hat{X}\right| = 1$. Therefore,

$$X^T G X = r^2 \hat{X}^T G \hat{X}$$

Let M be the maximum value of $\hat{X}^T G X$. Now since $X^T G X$ is negative definite, therefore,

$$X^T G X \le r^2 M < 0$$

and so $\quad X^T G X \to -\infty \quad$ as $\quad |X| = r \to \infty \quad$ ...(3)

Now let m be the minimum value of $\left|\dfrac{C^T X}{X^T G X}\right|$. Then

$$\left|\frac{C^T X}{X^T G X}\right| = \frac{1}{r}\left|\frac{C^T \hat{X}}{\hat{X}^T G \hat{X}}\right| \ge \frac{m}{r}$$

and therefore,

$$\frac{C^T X}{X^T G X} \to 0 \quad \text{as} \quad r \to \infty \qquad \text{...(4)}$$

Thus from relations (2), (3) and (4) it follows that

$$f(X) \to -\infty \quad \text{as} \quad |X| \to \infty$$

or, $\quad \lim\limits_{|X| \to \infty} f(X) = -\infty$

Thus we see that $\lim\limits_{|X| \to \infty} f(X) \ne \infty$ and so maximum of $f(X)$ is not unbounded.

However if $X^T G X$ is negative semidefinite, i.e., if $X^T G X \le 0$, then there is an X for which $f(X) = C^T X$ and then for $C \ne 0$, it may be possible that $f(X) \to \infty$ as $|X| \to \infty$, in which case $f(X)$ can have an unbounded maximum. Again if C = 0, then clearly $f(X)$ cannot have an unbounded maximum.

8.3 Computational Procedure for Solving Quadratic Programming Problems (Wolfe's Algorithm)

Let us consider the quadratic programming problem (1), i.e.,

Maximize $\quad f(X) = C^T X + \dfrac{1}{2} X^T G X$

subject to $\quad AX \le b$

$\quad X \ge 0$

in the following form

Maximize $\quad f(x_1, x_2, ..., x_n) = \sum_{j=1}^{n} c_j x_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} x_j d_{jk} x_k$

subject to $\quad \sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad , i = 1, 2, ..., m$

$x_j \ge 0 \qquad j = 1, 2, ..., n$

where $\quad d_{jk} = d_{jk}$ $\quad$ for all j and $k = 1, 2, ..., n$ and

$b_i \ge 0$ $\quad$ for all $i = 1, 2, ..., m$.

the Kuhn-Tucker conditions for the above problem are

(i) $\quad f_j - \sum_{i=1}^{m} \lambda_i h_{ij} + \mu_{m+j} = 0 \qquad ; \; j = 1, 2, ..., n$

or $\quad c_j + \frac{1}{2}\left(2\sum_{k=1}^{n} d_{jk} x_k\right) - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_{m+j} = 0 \qquad ; \; j = 1, 2, ..., n$

(ii) $\quad \lambda_i\left(\sum_{i=1}^{n} a_{ij} x_j - b\right) = 0 \qquad ; \; i = 1, 2, ..., m.$

(iii) $\quad \mu_{m+1}(-x_j) = 0$

i.e., $\quad -\mu_{m+j} x_j = 0 \qquad ; \; i = 1, 2, ..., n.$

(iv) $\quad \sum_{j=1}^{n} a_{ij} x_j - b_i \le 0 \qquad ; \; i = 1, 2, ..., m.$

(v) $\quad \lambda_i, \mu_{m+j}, x_j \ge 0 \qquad ; \; i = 1, 2, ..., m.$ and

$j = 1, 2, ..., n$

Thus the Kuhn-Tucker conditions for the optimal solution to the quadratic programming problem (1) are

(a) $\quad c_j + \sum_{k=1}^{n} d_{jk} x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \mu_{m+j} = 0 \qquad ; \; j = 1, 2, ..., n$

(b) $\quad \lambda_i\left(\sum_{i=1}^{n} a_{ij} x_j - b_i\right) = 0 \qquad ; \; i = 1, 2, ..., m.$

(c) $\quad -x_j \mu_{m+1} = 0 \qquad ; \; j = 1, 2, ..., n$

(d) $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$; $i = 1, 2, ..., m.$

and $\lambda_i, \mu_{m+j}, x_j \geq 0$; $i = 1, 2, ..., m.$ and

$j = 1, 2, ..., n$

If we consider $y_i \geq 0$ to be the slack variable introduced in the i^{th} constraint of (d) so that (d) becomes

(e) $\sum_{j=1}^{n} G_{ij} + x_j + y_i = b_i$; $i = 1, 2, ..., m.$

and also assume $u_j = \mu_{m+j}$ for ; $j = 1, 2, ..., n$, then

the conditions (b) and (c) become

(f) $\lambda_i \ y_i = 0$; $i = 1, 2, ..., m.$

(g) $x_j \ u_j = 0$; $j = 1, 2, ..., n$

With the newly defined variable u_j, the condition (a) can be rewritten as

(h) $\sum_{k=1}^{n} d_{jk}x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + u_j = -c_j$; $j = 1, 2, ..., n$

If the quadratic form $\sum_{j=1}^{n} \sum_{k=1}^{n} x_j d_{jk} x_k$

is assumed to be negative semidefinite, then the function $f(X)$ is concave in X and hence the conditions (a) to (e) become necessary and sufficient conditions for the optimal solution to the quadratic programming problem (1). Under this assumption we are to find nonnegative variables λ_i, y_i, x_j, u_j so that conditions (e), (f), (g) and (h) are satisfied and then such x_j determines an optimal solution to the given problem (1).

Iterative Procedure

The iterative procedure for the solution of the quadratic programming problem (1) using Wolfe's method can be summarised as follows :

Step I

Introduce slack variable y_i in the i^{th} constraint, $i = 1, 2, ...$ m and slack variable y_{m+j} in the j^{th} nonnegative constraint, $j = 1, 2, ...$ n.

Step II

Construct the Lagrangian function

$$L(X,\lambda,U,Y) = f(X) - \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij}x_j - b_i + y_i \right) - \sum_{j=1}^{n} u_j\left(-x_j + y_{m+j}\right)$$

where $X = (x_1, x_2, ..., x_n)$; $\lambda = (\lambda_1, \lambda_2, ... , \lambda_m)$

$U = (u_1, u_2, ..., u_n)$; $Y = (y_1, y_2, ..., y_{m+n})$

Differentiate the Lagrangian function partially w.r.t. the components of X, λ, U and Y and equate them to zero. Derive the Kuhn-Tucker conditions from the resulting equations.

Step III

Introduce non negative artificial variables $v_1, v_2, ..., v_n$ in the Kuhn-Tucker condition

$$c_j + \sum_{k=1}^{n} d_{jk}x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + u_j = 0 \qquad \text{for ; } j = 1, 2, ..., n$$

i.e., construct

$$c_j + \sum_{k=1}^{n} d_{jk}x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + u_j + v_j = 0$$

and construct an objective function

$z = -v_1 - v_2 ... -v_n$

Step IV

Obtain an initial basic feasible solution to the linear programming problem

Maximize $z = -v_1 - v_2 ... -v_n$

subject to $\sum_{k=1}^{n} a_{jk}x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + u_j + v_j = -c_j$; $j = 1, 2, ..., n$

$\sum_{j=1}^{n} a_{ij}x_j + y_i = b_i$; $i = 1, 2, ..., m.$

$v_i, \lambda_i, y_i, u_j, x_j \geq 0$; $i = 1, 2, ..., m$ and

$j = 1, 2, ..., n$

where $\lambda_i \; y_i = 0$ and

$x_j u_j = 0$ for $i = 1, 2, ..., m$ and

$j = 1, 2, ..., n$

Step V

Use two - phase method (simplex method) to obtain an optimal solution of the problem in Step IV. The optimal solution so obtained is the optimal solution of the given quadratic programming problem (1).

Note

(1) If the given quadratic programming problem is given in minimization form, then convert it into maximization form by suitable modifications in the objective function $f(X)$ Also convert all the constraints into $\geq$ form.

(2) Alongwith the additional conditions of complementary slackness, (i.e., the conditions $\lambda_i\, y_i = 0$ and $x_j\, u_j = 0$ for $i = 1, 2, \ldots\ldots, m$ and $j = 1, 2, \ldots\ldots, n$) the problem in Step IV becomes a linear programming problem. Thus we need only to modify Simplex algorithm to include the complementary slackness conditions. For example while deciding to introduce y_i into the basis, we must ensure that (i) either λ_i does not exist in the basis or (ii) λ_i is going to be out of the basis when y_i enters. This additional check must be performed at every iteration of the Simplex algorithm.

(3) The solution to the given problem is obtained by using Phase - I of the two - phase method. Since our motto is to obtain a feasible solution, it does not require the use of Phase - II. The only important thing is to maintain the complementary slackness conditions $\lambda_i\, y_i = 0$ and $x_j\, u_j = 0$ every time. This imply that if λ_i remains in the basic solution at positive level, then y_i cannot be a basic solution with positive value. In a similar way both x_j and u_j can not be positive simultaneously.

(4) It must also be observed that the Phase - I of the problem in Step IV will terminate in usual manner with the sum of all artificial variables equal to zero only when the feasible solution to the problem does exist.

Example 1: Solve the following quadratic programming problem by Wolfe's Method :

Min. $f(x_1, x_2) = -10x_1 - 25x_2 + 10x_1^2 + x_2^2 + 4x_1x_2$

subject to $x_1 + 2x_2 \leq 10$

$x_1 + x_2 \leq 9$

$x_1, x_2 \geq 0$

Solution : Changing into maximizatgion form the problem is :

$$Max\ \left[-f(x_1, x_2)\right] = 10x_1 + 25x_2 - 10x_1^2 - x_2^2 - 4x_1x_2$$

subject to $x_1 + 2x_2 \leq 10$

$x_1 + x_2 \leq 9$

$x_1, x_2 \geq 0$

The Lagrangian function for the above problem, therefore, is

$$L(X, \lambda) = 10x_1 + 25x_2 - 10x_1^2 - x_2^2 - 4x_1x_2 + \lambda_1(10 - x_1 - 2x_2) + \lambda_2(9 - x_1 - x_2).$$

The Kuhn-Tucker conditions for the quadratic programming problem are

$$10 - 20x_1 - 4x_2 - \lambda_1 - \lambda_2 \le 0$$

$$25 - 4x_1 - 2x_2 - 2\lambda_1 - \lambda_2 \le 0$$

$$x_1 + 2x_2 \le 0$$

$$x_1 + x_2 \le 9$$

or,

$$10 - 20x_1 - 4x_2 - \lambda_1 - \lambda_2 + u_1 = 0$$

$$25 - 4x_1 - 2x_2 - 2\lambda_1 - \lambda_2 + u_2 = 0$$

$$x_1 + 2x_2 + y_1 = 10$$

$$x_1 + x_2 + y_2 = 9$$

$$x_1, x_2, y_1, y_2, \lambda_1, \lambda_2, u_1, u_2 \ge 0$$

(on adding slack, variables)

where $\lambda_1 y_1 = \lambda_2 y_2 = u_1 x_1 = u_2 x_2 = 0$

(complementary slackness conditions)

The above can be again written as

$$20x_1 + 4x_2 + \lambda_1 + \lambda_2 - u_1 = 10 \quad \text{... (1)}$$

$$4x_1 + 2x_2 + 2\lambda_1 + \lambda_2 - u_2 = 25 \quad \text{... (2)}$$

$$x_1 + 2x_2 + y_1 = 10 \quad \text{... (3)}$$

$$x_1 + x_2 + y_2 = 9 \quad \text{... (4)}$$

where $x_1, x_2, y_1, y_2, \lambda_1, \lambda_2, u_1, u_2 \ge 0$

and $\lambda_1 y_1 = \lambda_2 y_2 = u_1 x_1 = u_2 x_2 = 0$

Introducing the artificial variables v_1, v_2 in (1) and (2) respectively, we have to

Maximize $w = -v_1 - v_2$

subject to

$$20x_1 + 4x_2 + \lambda_1 + \lambda_2 - u_1 + v_1 = 10$$

$$4x_1 + 2x_2 + 2\lambda_1 + \lambda_2 - u_2 + v_2 = 25$$

$$x_1 + 2x_2 + y_1 = 10$$

$$x_1 + x_2 + y_2 = 9$$

$$x_1, x_2, y_1, y_2, \lambda_1, \lambda_2, u_1, u_2, v_1, v_2 \ge 0$$

and $\lambda_1 y_1 = \lambda_2 y_2 = u_1 x_1 = u_2 x_2 = 0$

The Simplex iterations leading to the optimal solution are shown below. The $c_j'^s$ for all the variables except v_1 and v_2 are zero, whereas the $c_j'^s$ for v_1 and v_2 are – 1 each.

Simplex Table -1

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2	
v_1	–1	10	$\boxed{20}$	4	1	1	–1	0	0	0	1	0	
v_2	–1	25	4	2	2	1	0	–1	0	0	0	1	→
y_1	0	10	1	2	0	0	0	0	1	0	0	0	
y_2	0	9	1	1	0	0	0	0	0	1	0	0	
		–35	–24	–6	–3	–2	1	1	0	0	0	0	
			↑										

Simplex Table -2

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2	
x_1	0	$\frac{1}{2}$	1	$\boxed{\frac{1}{5}}$	$\frac{1}{20}$	$\frac{1}{20}$	$-\frac{1}{20}$	0	0	0	$\frac{1}{20}$	0	→
v_2	–1	23	0	$\frac{6}{5}$	$\frac{9}{5}$	$\frac{4}{5}$	$\frac{1}{5}$	–1	0	0	$-\frac{1}{5}$	1	
y_1	0	$\frac{19}{2}$	0	$\frac{9}{5}$	$-\frac{1}{20}$	$-\frac{1}{20}$	$\frac{1}{20}$	0	1	0	$-\frac{1}{20}$	0	
y_2	0	$\frac{17}{2}$	0	$\frac{4}{5}$	$-\frac{1}{20}$	$-\frac{1}{20}$	$\frac{1}{20}$	0	0	1	$-\frac{1}{20}$	0	
		–23	0	$-\frac{6}{5}$	$-\frac{9}{5}$	$-\frac{4}{5}$	$-\frac{1}{5}$	1	0	0	$\frac{6}{5}$	0	
				↑									

Simplex Table -3

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2	
x_2	0	$\frac{5}{2}$	5	1	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	0	0	0	$\frac{1}{4}$	0	
v_2	–1	20	–6	0	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	–1	0	0	$-\frac{1}{2}$	1	
y_1	0	5	–9	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\boxed{\frac{1}{2}}$	0	1	0	$-\frac{1}{2}$	0	→
y_2	0	$\frac{13}{2}$	–4	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	0	0	1	$-\frac{1}{4}$	0	
		–20	6	0	$-\frac{3}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	1	0	0	$\frac{3}{2}$	0	
									↑				

Simplex Table - 4

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2	
x_2	0	5	$\frac{1}{2}$	1	0	0	0	0	$\frac{1}{2}$	0	0	0	
v_2	–1	15	3	0	[2]	1	0	–1	–1	0	0	1	→
u_1	0	10	–18	0	–1	–1	1	0	2	0	–1	0	
y_2	0	4	$\frac{1}{2}$	0	0	0	0	0	$-\frac{1}{2}$	1	0	0	
		–15	–3	0	–2 ↑	–1	0	1	1	0	1	0	

Simplex Table - 5

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2
x_2	0	5	$\frac{1}{2}$	1	0	0	0	0	$\frac{1}{2}$	0	0	0
λ_1	0	$\frac{15}{2}$	$\frac{3}{2}$	0	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$
u_1	0	$\frac{35}{2}$	$\frac{33}{2}$	0	0	$-\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	–1	$\frac{1}{2}$
y_2	0	4	$\frac{1}{2}$	0	0	0	0	0	$-\frac{1}{2}$	1	0	0
		0	0	0	0	0	0	0	0	0	1	1

The optimal solution to the problem, therefore, is $x_1 = 0;\ x_2 = 5$

and Min. $f(x_2, x_2) =$ Max. $(-f(x_1, x_2)) = 100$.

Note

In the Simplex table-2, λ_1 were supposed to enter the basis but as y_1 was already in the basis and was not in a position to leave the basis, so we did select x_2 to enter the basis. Similarly in Simplex Table - 3, λ_1 and λ_2 could not enter the basis, since y_1 and y_2 were present in the basis, so we selected the next variable u_1 to enter.

Example-2 Minimize $f(x_1, x_2) = -8x_1 - 10x_2 + x_1^2 + 2x_2^2$

subject to $x_1 + x_2 \le 5$

$x_1 + 2x_2 \le 8$

$x_1,\ x_2 \ge 0$

Solution : Converting into maximization form the problem can be written as

Max $\left[-f(x_1,\ x_2)\right] = 8x_1 + 10x_2 - x_1^2 - 2x_2^2$

subject to $x_1 + x_2 \le 5$

$x_1 + 2x_2 \le 8$

$x_1,\ x_2 \ge 0$

The Lagragian function, therefore, is

$$L(X, \lambda) = 8x_1 + 10x_2 - x_1^2 - 2x_2^2 + \lambda_1(5 - x_1 - x_2) + \lambda_2(8 - x_1 - 2x_2).$$

The Kuhn-Tucker conditions for the quadratic programming problem are :

$$8 - 2x_1 - \lambda_1 - \lambda_2 \le 0$$

$$10 - 4x_2 - \lambda_1 - 2\lambda_2 \le 0$$

$$x_1 + x_2 \le 5$$

$$x_1 + 2x_2 \le 8$$

or,

$$2x_1 + \lambda_1 + \lambda_2 - u_1 = 8 \quad \text{.... (1)}$$

$$4x_2 + \lambda_1 + 2\lambda_2 - u_2 = 10 \quad \text{.... (2)}$$

$$x_1 + x_2 + y_1 = 5 \quad \text{.... (3)}$$

$$x_1 + 2x_2 + y_2 = 8 \quad \text{.... (4)}$$

$$x_1, x_2, y_1, y_2, \lambda_1, \lambda_2, u_1, u_2 \ge 0$$

where $u_1,\ u_2,\ y_1$ and y_2 are surplus and slack variables. Also

$\lambda_1 y_1 = \lambda_2 y_2 = u_1 x_1 = u_2 x_2 = 0$ are the complementary slackness conditions.

Now introducing the artificial variables $v_1,\ v_2$ in (1) and (2) respectively, we have to

Maximize $w = -v_1 - v_2$

subject to

$$2x_1 + \lambda_1 + \lambda_2 - u_1 + v_1 = 8$$

$$4x_2 + \lambda_1 + 2\lambda_2 - u_2 + v_2 = 10$$

$$x_1 + x_2 + y_1 = 5$$

$$x_1 + 2x_2 + y_2 = 8$$

$$x_1, x_2, y_1, y_2, \lambda_1, \lambda_2, u_1, u_2, v_1, v_2 \ge 0$$

and $\lambda_1 y_1 = \lambda_2 y_2 = u_1 x_1 = u_2 x_2 = 0$.

Simplex Table -1

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2
v_1	–1	8	2	0	1	1	–1	0	0	0	1	0
v_2	–1	10	0	[4]	1	2	0	–1	0	0	0	1 →
y_1	0	5	1	1	0	0	0	0	1	0	0	0
y_2	0	8	1	2	0	0	0	0	0	1	0	0
		–18	–2	–4 ↑	–2	–3	1	1	0	0	0	0

Simplex Table -2

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2
v_1	–1	8	2	0	1	1	–1	0	0	0	1	0
x_2	0	$\frac{5}{2}$	0	1	$\frac{1}{4}$	$\frac{1}{2}$	0	$-\frac{1}{4}$	0	0	0	$\frac{1}{4}$
y_1	0	$\frac{5}{2}$	[1]	0	$-\frac{1}{4}$	$-\frac{1}{2}$	0	$\frac{1}{4}$	1	0	0	$-\frac{1}{4}$ →
y_2	0	3	1	0	$-\frac{1}{2}$	–1	0	$\frac{1}{2}$	0	1	0	$-\frac{1}{2}$
		–8	–2 ↑	0	–1	–1	1	0	0	0	0	1

Simplex Table -3

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2
v_1	–1	3	0	0	[$\frac{3}{2}$]	2	–1	$-\frac{1}{2}$	–2	0	1	$\frac{1}{2}$ →
x_2	0	$\frac{5}{2}$	0	1	$\frac{1}{4}$	$\frac{1}{2}$	0	$-\frac{1}{4}$	0	0	0	$\frac{1}{4}$
x_1	0	$\frac{5}{2}$	1	0	$-\frac{1}{4}$	$-\frac{1}{2}$	0	$\frac{1}{4}$	1	0	0	$-\frac{1}{4}$
y_2	0	$\frac{1}{2}$	0	0	$-\frac{1}{4}$	$-\frac{1}{2}$	0	$\frac{1}{4}$	–1	1	0	$-\frac{1}{4}$
		–3	0	0	$-\frac{3}{2}$ ↑	–2	1	$\frac{1}{2}$	2	0	0	$\frac{1}{2}$

Simplex Table -4

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2
λ_1	0	2	0	0	1	$\frac{4}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{4}{3}$	0	$\frac{2}{3}$	$\frac{1}{3}$
x_2	0	2	0	1	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	0	$-\frac{1}{6}$	$\frac{1}{6}$
x_1	0	3	1	0	0	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$	0	$\frac{1}{6}$	$-\frac{1}{6}$
y_2	0	1	0	0	0	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$	$-\frac{4}{3}$	1	$\frac{1}{6}$	$-\frac{1}{6}$
		0	0	0	0	0	0	0	0	0	1	1

The optimal solution is $x_1 = 3;\ x_2 = 2$ and

Min. $f(x_1, x_2) =$ Max. $\left[-f(x_1, x_2)\right] = -27$

Example-3 Solve the following quadratic programming problem using Wolfe's method.

Min. $f(x_1, x_2) = x_1^2 - x_1x_2 + 2x_2^2 - x_1 - x_2$

subject to $2x_1 + x_2 \le 1$

$x_1, x_2 \ge 0$

Solution : Changing the given problem into the maximization form, we are to

Max. $\left[-f(x_1, x_2)\right] = -x_1^2 + x_1x_2 - 2x_2^2 + x_1 + x_2$

subject to $2x_1 + x_2 \le 1$

$x_1, x_2 \ge 0$

The Lagrangian function for the problem is

$$L(X,\lambda) = -x_1^2 + x_1x_2 - 2x_2^2 + x_1 + x_2 + \lambda(1 - 2x_1 - x_2)$$

The Kuhn-Tucker conditions are

$$1 - 2x_1 + x_2 - 2\lambda \le 0$$

$$1 + x_1 - 4x_2 - \lambda \le 0$$

$$2x_1 + x_2 \le 0$$

which, on introducing slack and surplus variables, can be written as

$$2x_1 - x_2 + 2\lambda - u_1 = 1 \quad \ldots (1)$$

$$-x_1 + 4x_2 + \lambda \qquad -u_2 \qquad = 1 \qquad \text{... (2)}$$

$$2x_1 + x_2 \qquad +y_1 \qquad = 1 \qquad \text{.... (3)}$$

$$x_1, x_2 y, \lambda\, u_1, u_2 \geq 0$$

and $\lambda\, y = u_1 x_1 = u_2 x_2 = 0$

Introducing the artificial variables v_1 and v_2 in (1) and (2) respectively, we are to

maximize $w = -v_1 - v_2$

subject to

$$2x_1 - x_2 + 2\lambda - u_1 \qquad +v_1 \qquad = 1$$

$$-x_1 + 4x_2 + \lambda \qquad -u_2 \qquad +u_2 \quad = 1$$

$$2x_1 + x_2 \qquad +y \qquad = 1$$

$$x_1,\, x_2,\, y,\, \lambda,\, u_1, u_2,\, v_1, v_2 \geq 0$$

and $\lambda y = u_1 x_1 = u_2 x_2 = 0$.

Simplex Table -1

basic variable	C_B	b	x_1	x_2	λ	u_1	u_2	y	v_1	v_2	
v_1	–1	1	2	–1	2	–1	0	0	1	0	
v_2	–1	1	–1	[4]	1	0	–1	0	0	1	→
y	0	1	2	1	0	0	0	1	0	0	
		–2	–1	[–3] ↑	–3	1	1	0	0	0	

Simplex Table -2

basic variable	C_B	b	x_1	x_2	λ	u_1	u_2	y	v_1	v_2	
v_1	–1	$\frac{5}{4}$	$\frac{7}{4}$	0	$\frac{9}{4}$	–1	$-\frac{1}{4}$	0	1	$\frac{1}{4}$	
x_2	0	$\frac{1}{4}$	$-\frac{1}{4}$	1	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	
y	0	$\frac{3}{4}$	[$\frac{9}{4}$]	0	$-\frac{1}{4}$	0	$\frac{1}{4}$	1	0	$-\frac{1}{4}$	→
		$-\frac{5}{4}$	$-\frac{7}{4}$	0	$-\frac{9}{4}$	1	$\frac{1}{4}$	0	–1	$-\frac{1}{4}$	

In the above table, although λ must enter the basis but y does not go out of the basis. Since both λ and y cannot remain simultaneously in the basis, therefore instead of λ we select next variable x_1 to enter the basis (since u_1 is not in the basis).

Simplex Table -3

basic variable	C_B	b	x_1	x_2	λ	u_1	u_2	y	v_1	v_2
v_1	-1	$\frac{2}{3}$	0	0	$\boxed{\frac{22}{9}}$	-1	$-\frac{4}{9}$	$-\frac{7}{9}$	1	$\frac{4}{9}$ →
x_2	0	$\frac{1}{3}$	0	1	$\frac{2}{9}$	0	$-\frac{2}{9}$	$\frac{1}{9}$	0	$\frac{2}{9}$
x_1	0	$\frac{1}{3}$	1	0	$-\frac{1}{9}$	0	$\frac{1}{9}$	$\frac{4}{9}$	0	$-\frac{1}{9}$
		$-\frac{2}{3}$	0	0	$-\frac{22}{9}$ ↑	1	$\frac{4}{9}$	$\frac{7}{9}$	0	$-\frac{4}{9}$

Simplex Table -4

basic variable	C_B	b	x_1	x_2	λ	u_1	u_2	y	v_1	v_2
λ	0	$\frac{3}{11}$	0	0	1	$\frac{-9}{22}$	$\frac{-2}{11}$	$\frac{-7}{22}$	$\frac{9}{22}$	$\frac{2}{11}$
x_2	0	$\frac{3}{11}$	0	1	0	$\frac{1}{11}$	$\frac{-2}{11}$	$\frac{2}{11}$	$\frac{-1}{11}$	$\frac{2}{11}$
x_1	0	$\frac{4}{11}$	1	0	0	$\frac{-1}{22}$	$\frac{1}{11}$	$\frac{9}{22}$	$\frac{1}{22}$	$\frac{-1}{11}$
		0	0	0	0	0	0	0	1	1

The optimal solution to the problem is $x_1 = \frac{4}{11}$, $x_2 = \frac{3}{11}$ and

Min. $f(x_1, x_2) = \text{Max.}\left[-f(x_1, x_2)\right] = \frac{-5}{11}$.

Example 4 : Solve by Wolfe's Method

Max. $f(x_1, x_2) = 2x_1 + x_2 - x_1^2$

subject to $2x_1 + 3x_2 \le 6$

$$2x_1 + x_2 \le 4$$

$$x_1,\ x_2 \ge 0$$

Solution : The given quadratic problem is given in the maximization form. The Lagragian function is :

$$L(X,\ \lambda) = 2x_1 + x_2 - x_1^2 + \lambda_1(6 - 2x_1 - 3x_2) + \lambda_2(4 - 2x_1 - x_2)$$

and so the Kuhn-Tucker conditions are

$$2 - 2x_1 - 2\lambda_1 - 2\lambda_2 \le 0$$

$$1 - 3\lambda_1 - \lambda_2 \le 0$$

$$2x_1 + 3x_2 \le 6$$

$$2x_1 + x_2 \le 4$$

Introducing slack and surplus variables the above conditions can be written as

$$2x_1 + 2\lambda_1 + 2\lambda_2 - u_1 = 2 \quad \text{... (1)}$$

$$3\lambda_1 + \lambda_2 - u_2 = 1 \quad \text{... (2)}$$

$$2x_1 + 3x_2 + y_1 = 6 \quad \text{... (3)}$$

$$2x_1 + x_2 + y_2 = 4 \quad \text{.... (4)}$$

where $x_1, x_2, \lambda_1, \lambda_2, u_1, u_2, y_1, y_2 \ge 0$

and also $\lambda_1 y_1 = \lambda_2 y_2 = u_1 x_1 = u_2 x_2 = 0$

Now *n* add artificial variables v_1 and v_2 in condition (1) and (2) respectively, so that the problem is to

maximize $w = -v_1 - v_2$

subject to

$$2x_1 + 2\lambda_1 + 2\lambda_2 - u_1 + v_1 = 2$$

$$3\lambda_1 + \lambda_2 - u_2 + v_2 = 1$$

$$2x_1 + 3x_2 + y_1 = 6$$

$$2x_1 + x_2 + y_2 = 4$$

satisfying the condition

$x_1, x_2, \lambda_1, \lambda_2, u_1 u_2, y_1, y_2, v_1, v_2 \geq 0$

and $\lambda_1 y_1 = \lambda_2 y_2 = u_1 x_1 = u_2 x_2 = 0$ and where the $c_j's$ for v_1 and v_2 are each equal to – 1 whereas for all other variables $c_j's$ are zero.

Simplex Table -1

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2	
v_1	–1	2	[2]	0	2	2	–1	0	0	0	1	0	→
v_2	–1	1	0	0	3	1	0	–1	0	0	0	1	
y_1	0	6	2	3	0	0	0	0	1	0	0	0	
y_2	0	4	2	1	0	0	0	0	0	1	0	0	
		–3	–2	0	–5	–3	1	1	0	0	0	0	

Simplex Table -2

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2	
x_1	0	1	1	0	1	1	$\frac{-1}{2}$	0	0	0	$\frac{1}{2}$	0	
v_2	–1	1	0	0	3	1	0	–1	0	0	0	1	
y_1	0	4	0	[3]	–2	–2	1	0	1	0	0	0	→
y_2	0	2	0	1	–2	–2	1	0	0	1	–1	0	
		–1	0	0	–3	–1	0	1	0	0	1	0	
				↑									

Simplex Table -3

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2	
x_1	0	1	1	0	1	1	$\frac{-1}{2}$	0	0	0	$\frac{1}{2}$	0	
v_2	–1	1	0	0	[3]	1	0	–1	0	0	0	1	→
x_2	0	$\frac{4}{3}$	0	1	$\frac{-2}{3}$	$\frac{-2}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	0	
y_2	0	$\frac{2}{3}$	0	0	$\frac{-4}{3}$	$\frac{-4}{3}$	$\frac{2}{3}$	0	$\frac{-1}{3}$	1	–1	0	
		–1	0	0	–3	–1	0	1	0	0	1	0	

Simplex Table -4

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2
x_1	0	$\frac{2}{3}$	1	0	0	$\frac{2}{3}$	$\frac{-1}{2}$	$\frac{1}{3}$	0	0	$\frac{1}{2}$	$\frac{-1}{3}$
λ_1	0	$\frac{1}{3}$	0	0	1	$\frac{1}{3}$	0	$\frac{-1}{3}$	0	0	0	$\frac{1}{3}$
x_2	0	$\frac{14}{9}$	0	1	0	$\frac{-4}{9}$	$\frac{1}{3}$	$\frac{-2}{9}$	$\frac{1}{3}$	0	0	$\frac{2}{9}$
y_2	0	$\frac{10}{9}$	0	0	0	$\frac{-8}{9}$	$\frac{2}{3}$	$\frac{-4}{9}$	$\frac{-1}{3}$	1	-1	$\frac{4}{9}$
		0	0	0	0	0	0	0	0	0	1	1

The optimal solution is $x_1 = \frac{2}{3};\ x_2 = \frac{14}{9}$ and max. $f(x_1, x_2) = \frac{22}{9}$

Example 5

Solve the following quadratic programming problem by Wolfe's method :

Minimize $\quad f(x_1, x_2) = -4x_1 + x_1^2 - 2x_1x_2 + 2x_2^2$

subject to $\quad 2x_1 + x_2 \le 6$

$$x_1 - 4x_2 \le 0$$

$$x_1, x_2 \ge 0$$

Solution :

On changing the given programming problem in maximization form, we have to

Max. $\left[-f(x_1, x_2)\right] = 4x_1 - x_1^2 + 2x_1x_2 - 2x_2^2$

subject to $\quad 2x_1 + x_2 \le 6$

$$x_1 - 4x_2 \le 0$$

$$x_1, x_2 \ge 0$$

The Lagrangian function is

$$L(X, \lambda) = 4x_1 - x_1^2 + 2x_1x_2 - 2x_2^2 + \lambda_1(6 - 2x_1 - x_2) + \lambda_2(-x_1 + 4x_2)$$

Thus the Kuhn-Tucker Condition are

$$4 - 2x_1 + 2x_2 - 2\lambda_1 - \lambda_2 \le 0$$

$$2x_1 - 4x_2 - \lambda_1 + 4\lambda_2 \quad \le 0$$

$$2x_1 + x_2 \quad \le 6$$

$$x_1 - 4x_2 \quad \le 0$$

Introducing slack and surplus variables the above conditions can be written as :

$$2x_1 - 2x_2 + 2\lambda_1 + \lambda_1 - u_1 = 4 \quad \text{... (1)}$$

$$-2x_1 + 4x_2 + \lambda_1 - 4\lambda_2 \quad -u_2 = 0 \quad \text{.... (2)}$$

$$2x_1 + x_2 \quad +y_1 = 6 \quad \text{... (3)}$$

$$x_1 - 4x_2 \quad +y_2 = 0 \quad \text{... (4)}$$

where $\quad x_1, x_2, \lambda_1, \lambda_2, u_1, u_2, y_2, y_2 \ge 0$

and also $\quad \lambda_1 y_1 = \lambda_2 y_2 = u_1 x_1 = u_2 x_2 = 0$

Adding artificial variables in condition (1) and (2) we have to

maximize $\quad w = -v_1 - v_2$

subject to $\quad 2x_1 - 2x_2 + 2\lambda_1 + \lambda_2 - u_1 \quad +v_1 \quad = 4$

$$-2x_1 + 4x_2 + \lambda_1 - 4\lambda_2 \quad -u_2 \quad +v_2 \quad = 0$$

$$2x_1 + x_2 \quad +y_1 \quad = 6$$

$$x_1 - 4x_2 \quad +y_2 \quad = 0$$

$x_1, x_2, y_1, y_2, \lambda_1, \lambda_2, u_1, u_2, v_1, v_2 \ge 0$ and

$\lambda_1 y_1 = \lambda_2 y_2 = u_1 x_1 = u_2 x_2 = 0$

where the $c_j'^s$ corresponding to the artificial variables v_1 and v_2 are –1 each and corresponding to all other variables are 0.

Simplex Table -1

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2
v_1	–1	4	2	–2	2	1	–1	0	0	0	1	0
v_2	–1	0	–2	[4]	1	–4	0	–1	0	0	0	1
y_1	0	6	2	1	0	0	0	0	1	0	0	0
y_2	0	0	1	–4	0	0	0	0	0	1	0	0
		–4	0	–2 ↑	–3	3	1	1	0	0	0	0

Simplex Table -2

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2
v_1	–1	4	1	0	$\frac{5}{2}$	–1	–1	$\frac{-1}{2}$	0	0	1	$\frac{1}{2}$
x_2	0	0	$\frac{-1}{2}$	1	$\frac{1}{4}$	–1	0	$\frac{-1}{4}$	0	0	0	$\frac{1}{4}$
y_1	0	6	$\boxed{\frac{5}{2}}$	0	$\frac{-1}{4}$	1	0	$\frac{1}{4}$	1	0	0	$\frac{-1}{4}$ →
y_2	0	0	–1	0	1	–4	0	–1	0	1	0	1
		–4	–1 ↑	0	$\frac{-5}{2}$	1	1	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$

Simplex Table -3

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2
v_1	–1	$\frac{8}{5}$	0	0	$\boxed{\frac{13}{5}}$	$\frac{-7}{5}$	–1	$\frac{-3}{5}$	$\frac{-2}{5}$	0	1	$\frac{3}{5}$ →
x_2	0	$\frac{6}{5}$	0	1	$\frac{1}{5}$	$\frac{-4}{5}$	0	$\frac{-1}{5}$	$\frac{1}{5}$	0	0	$\frac{1}{5}$
x_1	0	$\frac{12}{5}$	1	0	$\frac{-1}{10}$	$\frac{2}{5}$	0	$\frac{1}{10}$	$\frac{2}{5}$	0	0	$\frac{-1}{10}$
y_2	0	$\frac{12}{5}$	0	0	$\frac{9}{10}$	$\frac{-18}{5}$	0	$\frac{-9}{10}$	$\frac{2}{5}$	1	0	$\frac{9}{10}$
		$\frac{-8}{5}$	0	0	$\frac{-13}{5}$ ↑	$\frac{7}{5}$	1	$\frac{3}{5}$	$\frac{2}{5}$	0	0	$\frac{2}{5}$

Simplex Table -4

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2
λ_1	0	$\frac{8}{13}$	0	0	1	$\frac{-7}{13}$	$\frac{-5}{13}$	$\frac{-3}{13}$	$\frac{-2}{13}$	0	$\frac{5}{13}$	$\frac{3}{13}$
x_2	0	$\frac{14}{13}$	0	1	0	$\frac{-9}{13}$	$\frac{1}{13}$	$\frac{-2}{13}$	$\frac{3}{13}$	0	$\frac{-1}{13}$	$\frac{2}{13}$
x_1	0	$\frac{32}{13}$	1	0	0	$\frac{9}{26}$	$\frac{-1}{26}$	$\frac{1}{13}$	$\frac{5}{13}$	0	$\frac{1}{26}$	$\frac{-1}{13}$
y_2	0	$\frac{24}{13}$	0	0	0	$\frac{-81}{26}$	$\frac{9}{26}$	$\frac{-9}{13}$	$\frac{7}{13}$	1	$\frac{-9}{26}$	$\frac{9}{13}$
		0	0	0	0	0	0	0	0	0	1	1

The optimal solution is $x_1 = \frac{32}{13},\ x_2 = \frac{14}{13}$ and

Min. $f(x_1, x_2)$ = Max. $\left[-f(x_1, x_2)\right] = -\frac{88}{13}$.

Example 6

Use Wolfe's method to solve the following quadratic programming problem :

Minimize $f(x_1, x_2) = x_1^2 + x_2^2 - 2x_1 - 4x_2$

subject to $x_1 + 4x_2 \le 5$

$2x_1 + 3x_2 \le 6$

$x_1, x_2 \ge 0$

Solution Converting the given problem to maximization form, we have to

max. $\left[-f(x_1, x_2)\right] = -x_1^2 - x_2^2 + 2x_1 + 4x_2$

subject to $x_1 + 4x_2 \le 5$

$2x_1 + 3x_2 \le 6$

$x_1, x_2 \ge 0$

The Lagrangian for the given problem, therefore, is

$$L(X, \lambda) = -x_1^2 - x_2^2 + 2x_1 + 4x_2 + \lambda_1(5 - x_1 - 4x_2) + \lambda_2(6 - 2x_1 - 3x_2)$$

Thus The Kuhn-Tucker condition for the problem are :

$$2-2x_1-\lambda_1-2\lambda_2 \le 0$$
$$4-2x_2-4\lambda_1-3\lambda_2 \le 0$$
$$x_1+4x_2 \le 5$$
$$2x_1+3x_2 \le 6$$

On adding, slack and surplus variables the above conditions become

$$2x_1+\lambda_1+2\lambda_2-u_1 = 2 \quad ...(1)$$
$$2x_2+4\lambda_1+3\lambda_2 \quad -u_2 = 4 \quad ...(2)$$
$$x_1+4x_2 \quad +y_1 = 5 \quad ...(3)$$
$$2x_1+3x_2 \quad +y_2 = 6 \quad ...(4)$$

$x_1, x_2, \lambda_1, \lambda_2, u_1, u_2, y_1, y_2 \ge 0$ and also

$$\lambda_1 y_1 = \lambda_2 y_2 = u_1 x_1 = u_2 x_2 = 0$$

Introducing artificial variables v_1 and v_2 to the conditions (1) and (2) respectively we have to

maximize $w=-v_1-v_2$

subject to

$$2x_1+\lambda_1+2\lambda_2-u_1 \quad +v_1 = 2$$
$$2x_2+4\lambda_1+3\lambda_2 \quad -u_2 \quad +v_2 = 4$$
$$x_1+4x_2 \quad +y_1 = 5$$
$$2x_1+3x_2 \quad +y_2 = 6$$
$$x_1, x_2, y_1, y_2, \lambda_1, \lambda_2, u_1, u_2, v_1, v_2 \ge 0$$

where $\lambda_1 y_1 = \lambda_2 y_2 = u_1 x_1 = u_2 x_2 = 0$

and the $C_j^{'s}$ corresponding to the artificial variables are –1 where as corresponding to all the others variables are 0.

Simplex Table -1

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2
v_1	–1	2	[2]	0	1	2	–1	0	0	0	1	0
v_2	–1	4	0	2	4	3	0	–1	0	0	0	1 →
y_1	0	5	1	4	0	0	0	0	1	0	0	0
y_2	0	6	2	3	0	0	0	0	0	1	0	0
		–6	–2 ↑	–2	–5	–5	1	1	0	0	0	0

Simplex Table -2

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2
x_1	0	1	1	0	$\frac{1}{2}$	1	$\frac{-1}{2}$	0	0	0	$\frac{1}{2}$	0
v_2	–1	4	0	[2]	4	3	0	–1	0	0	0	1 →
y_1	0	4	0	4	$\frac{-1}{2}$	–1	$\frac{1}{2}$	0	1	0	$\frac{-1}{2}$	0
y_2	0	4	0	3	1	–2	1	0	0	1	–1	0
		–4	0	–2 ↑	–4	–3	0	1	0	0	1	0

Simplex Table -3

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2
x_1	0	1	1	0	$\frac{1}{2}$	1	$\frac{-1}{2}$	0	0	0	$\frac{1}{2}$	0
x_2	0	2	0	1	2	$\frac{3}{2}$	0	$\frac{-1}{2}$	0	0	0	$\frac{1}{2}$
y_1	0	–4	0	0	[$\frac{-17}{2}$]	–7	$\frac{1}{2}$	2	1	0	$\frac{-1}{2}$	–2 →
y_2	0	–2	0	0	–7	$\frac{-13}{2}$	1	$\frac{3}{2}$	0	1	–1	$\frac{-3}{2}$
		0	0	0	0 ↑	0	0	0	0	0	1	1

Since y_1 and y_2 appear in the solution at negative level, they must be eliminated. Hence introduce λ_1 and drop y_1.

Simplex Table -4

basic variable	C_B	b	x_1	x_2	λ_1	λ_2	u_1	u_2	y_1	y_2	v_1	v_2
x_1	0	$\frac{13}{17}$	1	0	0	$\frac{10}{17}$	$\frac{-8}{17}$	$\frac{2}{17}$	$\frac{1}{17}$	0	$\frac{8}{17}$	$\frac{-2}{17}$
x_2	0	$\frac{18}{17}$	0	1	0	$\frac{-5}{34}$	$\frac{-2}{17}$	$\frac{-1}{34}$	$\frac{4}{17}$	0	$\frac{-2}{17}$	$\frac{1}{34}$
λ_1	0	$\frac{8}{17}$	0	0	1	$\frac{14}{17}$	$\frac{-1}{17}$	$\frac{-4}{17}$	$\frac{-2}{17}$	0	$\frac{1}{17}$	$\frac{4}{17}$
y_2	0	$\frac{22}{17}$	0	0	0	$\frac{-25}{34}$	$\frac{10}{17}$	$\frac{-5}{34}$	$\frac{-14}{17}$	1	$\frac{-10}{17}$	$\frac{-6}{17}$
		0	0	0	0	0	0	0	0	0	1	1

The optimal solution is $x_1 = \frac{13}{17}, x_2 = \frac{18}{17}$

and min $f(x_1, x_2) = \left(\frac{13}{17}\right)^2 + \left(\frac{18}{17}\right)^2 - 2\left(\frac{13}{17}\right) - 4\left(\frac{18}{17}\right)$

$$= -\frac{69}{17}$$

8.4 Beales Method for solving Quadratic Programming Problems

Unlike worlfe's method for solving the quadratic programming problem. the Beale's method does not require the use of Kuhn-Tucker conditions. Instead Beale's method involves the partitioning of variables into basic and non basic variables only.

The Beale's algorithm for solving the quadratic programming problem can be summa sized in the following steps :

Suppose that we have the quadratic programming problem.

Maximize $f(X) = C^T X + \frac{1}{2} X^T G X$

subject to $\quad A\,X \leq, = , \geq b$

$X \leq 0$

where X and C $\in E^n$, b $\in E^m$, G is an n×n symmetric matrix and A is an m×n matrix.

Step I

Convert the given objective function of the problem to maximization form (if it is given in the minimization form). Convert all the inequality constraints into equalities by introducing slack and surplus variables $u's$. The given quadratic programming problem has now been put into standard form.

Step-2

Select arbitrarily any m variables as basic variables, provided the matrix corresponding to these m variables is non singular. The remaining n-m variables thus become non basic variables. Denote the basic variables by

$X_B = \left(x_{B_1}, x_{B_2}, \ldots, x_{B_m}\right)$ and the nonbasic variables by

$X_{NB} = \left(x_{NB_1}, x_{NB_2}, \ldots, x_{NB_{n-m}}\right)$.

Step-3

Express each basic variable x_{B_i} entirely in terms of nonbasic variables $x_{NB_k}'s$ (and $u_j's$ if any) using the given constraints. Now express the objective function $f(X)$ also in terms of the nonbasic variables $X_{NB_k}'s$ (and $u_i's$ if any).

step-4

Obtain the partial derivatives of $f(X)$ formulated above w.r.t. the nonbasic variables $x_{NB_k}'s$ and examine its nature at the point $X_{NB} = 0$.

(i) If $\left(\dfrac{\partial f(X)}{\partial x_{NB_k}}\right)_{\substack{X_{NB=0}\\ u=0}} > 0$ for at least one k, then choose the most positive one. The corresponding nonbasic variable will enter the basis.

(ii) If $\left(\dfrac{\partial f(X)}{\partial x_{NB_k}}\right)_{\substack{X_{NB=0}\\ u=0}} < 0$ for each $k = 1, 2, \ldots, n-m$ but $\left(\dfrac{\partial f(X)}{\partial u_i}\right)_{\substack{X_{NB=0}\\ u=0}} \neq 0$ for some $i = r$, then introduce a new nonbasic variable u_j, defined by $u_j = \dfrac{1}{2}\dfrac{\partial f}{\partial u_r}$ and treat u_r as a basic variable (it will be ignored later). Go to step-3.

(iii) If $\left(\dfrac{\partial f(X)}{\partial X_{N_{B_K}}}\right)_{\substack{X_{NB=0}\\ u=0}} = 0$, for each j,

the current basic solution is optimal. Go to step -7.

Step-5

Let $x_{NB_i} = x_k$ be the entering variable identified in step (1). Now compute the minimu *m* of the ratios

$$min.\left\{\frac{a_{ho}}{|a_{hk}|}, \frac{v_{ko}}{v_{kk}}\right\},$$

for all the basic variables x_h, where a_{ho} is a constant term and a_{hk} is the coefficient of x_k in the expression of the basic variable x_h when expressed in terms of nonbasic variables and v_{ko} is the constant term and v_{kk} is the coefficient of x_k in $\frac{\partial f}{\partial x_k}$

Now if

(i) the minimum of the ratio occurs for some $\frac{a_{ho}}{|a_{hk}|}$, the corresponding basic variable x_h leaves the basis.

(ii) the minimum of the ratio occurs for some $\frac{v_{ko}}{|v_{kk}|}$, then an additional nonbasic variable, called a free variable defined by

$$u_i = \frac{1}{2}\frac{\partial f}{\partial x_k} \quad (u_i \text{ is unrestricted in sign})$$

is introduced. This becomes an additional constraint equation.

Step - 6 Go to step-3 and repeat the procedure until an optimal basic solution is attained.

Step-7 Determine the optimal value of X_B and f(X) by setting X_{NB}=0, in the expression obtained in step-3

Example-7 Use Beale's method to solve the quadratic programming problem

Minimize $f(x_1, x_2) = 6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2$

subject to $x_1 + x_2 \le 2$

$x_1, x_2 \ge 0$

Solution : On changing the given problem into maximization form and introducting *g* slack variable x_3 we get the problem in the following form:

$$Max.\ f(X) = Max[-f(x_1, x_2)] = -6 + 6x_1 - 2x_1^2 + 2x_1x_2 - 2x_2^2$$

subject to $x_1 + x_2 + x_3 = 2$

$x_1, x_2, x_3 \ge 0.$

Let us select x_3 arbitrarily as the basic variable (as there is only one constraint, therefore there will be only one basic variable for the current step)

Then we have

$$X_B = (X_3), X_{NB} = (X_1, X_2)$$

Expressing the basic variable X_B and $f(x_1, x_2)$ in terms of X_{NB}, we have

$$x_3 = 2 - x_1 - x_2 \quad ...(1)$$

and $$f = -6 + 6x_1 - 2x_1^2 + 2x_1x_2 - 2x_2^2 \quad ...(2)$$

The partial derivatives *w.r.t.* X_{NB} are

$$\left(\frac{\partial f}{\partial x_1}\right)_{X_{N_B}=0} = (6 - 4x_1 + 2x_2)_{\substack{x_1=0 \\ x_2=0}} = 6 \quad ...(3)$$

$$\left(\frac{\partial f}{\partial x_1}\right)_{X_{N_B}=0} = (2x_1 - 4x_2)_{\substack{x_1=0 \\ x_2=0}} = 0 \quad ...(4)$$

Since $\left(\frac{\partial f}{\partial x_1}\right)_{x_{N_B}} = 6 > 0$ (most positive), there fore variable x_1 enters the basis.

Now min $\left\{\frac{a_{30}}{|\alpha_{31}|}, \frac{v_{10}}{|v_{11}|}\right\} = \min\left\{\frac{2}{|-1|}, \frac{6}{|-4|}\right\} = \frac{6}{4}$

[Note that α_{30} is the constant 2 in (1) and a_{31} is -1, the coefficient of x_1 in the same equation. Similarly v_{10} is the constant 6 in (3) and v_{11} is -4, which is the coefficient of x_1 in this equation]

Since this minimum, i.e., $\frac{6}{4}$ corresponds to $\frac{v_{10}}{|v_{11}|}$, therefore we cannot remove x_3 from the basis. We, therefore, introduce a new non basic variable u_1 defined by

$$u_1 = \frac{1}{2}\frac{\partial f_k}{\partial x_1} = 3 - 2x_1 + x_2 \quad ...(5)$$

Then the current basis is $X_B = (x_3, x_1)$ and $X_{NB} = (x_2, u_1)$.

We again express the current basis X_B and $f(X)$ in terms *ox* X_{NB}.

$$x_1 = \frac{1}{2} + \frac{1}{2}u_1 + \frac{1}{2}x_2 \quad \text{[from (5)]} \quad ...(6)$$

$$x_3 = \frac{1}{2} + \frac{1}{2}u_1 - \frac{3}{2}x_2 \quad \text{[from (1)]} \quad ...(7)$$

and $f = -6 + \left(\frac{3}{2} - \frac{1}{2}u_1 + \frac{1}{2}x_2\right)\left[6 - 2\left(\frac{3}{2} - \frac{1}{2}u_1 + \frac{1}{2}x_2\right) + 2x_2\right] - 2x_2^2$

$$=-6+\left(\frac{3}{2}-\frac{1}{2}u_1+\frac{1}{2}x_2\right)(3+u_1+x_2)-2x_2^2$$

or , $$f=-\frac{3}{2}-\frac{1}{2}u_1^2+\frac{3}{2}x_2^2+3x_2 \qquad ...(8)$$

The partial derivatives of f *w.r.t.* X_{NB} are

$$\left(\frac{\partial f}{\partial x_2}\right)_{\substack{x_{NB}=0\\u_1=0}}=(3-3x_2)_{\substack{x_2=0\\u_1=0}}=3 \qquad ...(9)$$

$$\left(\frac{\partial f}{\partial u_1}\right)_{\substack{x_{NB}=0\\u_1=0}}=(-u_1)_{\substack{x_2=0\\u_1=0}}=0$$

Clearly x_2 enters the basis

Again, we compute the ratio

$$min\left\{\frac{\alpha_{10}}{|\alpha_{12}|},\frac{\alpha_{30}}{|\alpha_{32}|},\frac{v_{20}}{|v_{22}|}\right\}$$

[α_{10}, α_{30}, v_{20} are the constants in (6), (7) and (9) respectivety and α_{12}, α_{32}, v_{22} are the coefficients of x_2 in (6), (7) and (9) respectively.]

$$= \quad \min\left\{\frac{\frac{3}{2}}{\left|\frac{1}{2}\right|},\frac{\frac{1}{2}}{\left|-\frac{3}{2}\right|},\frac{3}{|-3|}\right\}$$

$$= \quad \min\left\{3,\frac{1}{3},1\right\}=\frac{1}{3}=\frac{\alpha_{30}}{|\alpha_{32}|}$$

Thus x_3 will leave the basis. Now the new

$X_B=(x_1,x_2)$ and $X_{NB}=(u_1,x_3)$

Expressing the new basic variables in terms of variables in X_{NB} and also expressing f in terms of X_{NB}, we have

$$x_1=\frac{3}{2}-\frac{1}{2}u_1+\frac{1}{2}.\frac{2}{3}\left(\frac{1}{2}+\frac{1}{2}u_1-x_3\right) \qquad \text{[from (6) and (7)]}$$

or $$x_1=\frac{5}{3}-\frac{1}{3}u_1-\frac{1}{3}x_3 \qquad ...(10)$$

$$x_2 = \frac{2}{3}\left(\frac{1}{2} + \frac{1}{2}u_1 - x_3\right) \qquad \text{[from (7)]}$$

or $$x_2 = \frac{1}{3} + \frac{1}{3}u_1 - \frac{2}{3}x_3 \qquad ...(11)$$

and $$f = \frac{-3}{2} - \frac{1}{2}u_1^2 + 3\left(\frac{1}{3} + \frac{1}{3}u_1 - \frac{2}{3}x_3\right)\left(1 - \frac{1}{6} - \frac{1}{6}u_1 + \frac{1}{3}x_3\right) \qquad \text{[from (8) and (11)]}$$

or $$f = \frac{-2}{3} - \frac{2}{3}u_1 - \frac{4}{2}x_3 + \frac{2}{3}x_3u_1 - \frac{2}{3}u_1^2 - \frac{2}{3}x_3^2 \qquad ...(12)$$

The partial derivatives of *f* *w.r.t.*

X_{N_B} are

$$\left(\frac{\partial f}{\partial x_3}\right)_{\substack{X_{N_B}=0 \\ u_1=0}} = \left(-\frac{4}{3} + \frac{2}{3}u_1 - \frac{4}{3}x_3\right)_{\substack{x_3=0 \\ u_1=0}} = -\frac{4}{3}$$

$$\left(\frac{\partial f}{\partial u_1}\right)_{\substack{X_{N_B}=0 \\ u_1=0}} = \left(\frac{2}{3} + \frac{2}{3}x_3 - \frac{4}{3}u_1\right) = \frac{2}{3}$$

Since $\frac{\partial f}{\partial x_3} < 0$ and $\frac{\partial f}{\partial u_1} \neq 0$, therefore, the current solution can further be improved. Howeve the entry rule does not allow x_3 to enter the basis. So we introduce another nonbasic variable u_2, defined by

$$u_2 = \frac{1}{2}\frac{\partial f}{\partial u_1} = \frac{1}{2}\left(\frac{2}{3} + \frac{2}{3}x_3 - \frac{4}{3}u_1\right)$$

or $$u_2 = \frac{1}{3} + \frac{1}{3}x_3 - \frac{2}{3}u_1 \qquad ...(13)$$

Treating u_1 as the basic variable and expressing the basic variable $X_B = (x_1, x_2, u_1)$ and the function *f* in terms of nonbasic variables (x_3, u_2), we have

$$x_1 = \frac{5}{3} + \frac{1}{2}\left(u_2 - \frac{1}{3} - \frac{1}{3}x_3\right) - \frac{1}{3}x_3 \qquad \text{[from (10) and (13)]}$$

or $$x_1 = \frac{3}{2} + \frac{1}{2}u_2 - \frac{1}{2}x_3 \qquad ...(14)$$

$$x_2 = \frac{1}{3} - \frac{1}{2}\left(u_2 - \frac{1}{3} - \frac{1}{3}x_3\right) - \frac{2}{3}x_3 \qquad \text{[from (11) and (13)]}$$

or $x_2 = \frac{1}{2} - \frac{1}{2}u_2 - \frac{1}{2}x_3$ ...(15)

$u_1 = \frac{1}{2} - \frac{3}{2}u_2 + \frac{1}{2}x_3$ [from (13)] ...(16)

and $f = -\frac{2}{3} - \left(u_2 - \frac{1}{3} - \frac{1}{3}x_3\right)\left(1 + x_3 + \frac{3}{2}\left(u_2 - \frac{1}{3} - \frac{1}{3}x_3\right)\right) - \frac{4}{3}x_3 - \frac{2}{3}x_3^2$

$= -\frac{2}{3} - \left(u_2 - \frac{1}{3} - \frac{1}{3}x_3\right)\left(1 + x_3 + \frac{3}{2}\left(u_2 - \frac{1}{3} - \frac{1}{3}x_3\right)\right) - \frac{4}{3}x_3 - \frac{2}{3}x_3^2$

or $f = -\frac{1}{2} - \frac{3}{2}u_2^2 - \frac{1}{2}x_3^2 - x_3$ ...(17)

Now since $\left(\frac{\partial f}{\partial x_3}\right)_{\substack{X_{N_B}=0 \\ u_2=0}} = \left(-x_3 - 1\right)_{\substack{x_3=0 \\ u_2=0}} = -1$

and $\left(\frac{\partial f}{\partial u_2}\right)_{\substack{X_{N_B}=0 \\ u_2=0}} = \left(-3u_3\right)_{\substack{x_3=0 \\ u_2=0}} = 0$

Therefore, the current basis $X_B = \left(x_1, x_2, u_1\right)$ gives the optimal solution. Ignoring the variables $u_i^{'s}$ (called the free variables) in the basis, the optimal solution is

$x_1 = \frac{3}{2} + 0 - 0 = \frac{3}{2}$ i.e. $x_1 = 3/2$ [from (14)]

$x_2 = \frac{1}{2} - 0 - 0 = \frac{1}{2}$ i.e. $x_2 = 1/2$ [from (15)]

and min f (x_1, x_2) = (-max f)

$= (-1/2)$ [from (17)]

$= \frac{1}{2}$

Example : Solve the following quadralic programming problem by Beale's method.

Min. $f(x_1, x_2) = 10x_1^2 + x_2^2 + 4x_1x_2 - 10x_1 - 25x_2$

subject to $x_1 + 2x_2 \le 10$

$x_1 + x_2 \le 9$

$x_1, x_2 \ge 0$

Solution : on changing the problem into maximization form and adding slack variables to the constraints, we get

Min. $f\left(X\right) = Max\left[-F\left(x_1 x_2\right)\right] = -10x_1^2 - x_2^2 - 4x_1x_2 + 10x_1 + 25x_2$

subject to $x_1 + 2x_2 + x_3 = 10$

$x_1 + x_2 \quad + x_4 = 9$

$x_1, x_2, x_3, x_4 \geq 0$

Let us select x_1 and x_2 arbitrarily as the basic variables (since there are only two constraints. so we can select only two basic variables). Then

$X_B = (x_1, x_2);\ X_{N_B} = (x_3, x_4)$

Expressing the basic variables x_1, x_2 in terms of nonbasic variables

$x_1 = 8 + x_3 - 2x_4$ ...(1)

$x_2 = 1 - x_3 + x_4$ ... (2)

(by solving the constraints for x_1 and x_2)

Now we express the function $f(X)$ in terms of nonbasic variables x_3, x_4. This is

$$f = 10(8 + x_3 - 2x_4) + 25(1 - x_3 + x_4) - 10(8 + x_3 + 2x_4)^2 - (1 - x_3 + x_4)^2 - 4(8 + x_3 - 2x_4)(1 - x_3 + x_4)$$

or $f = -568 - 145x_3 + 299x_4 - 7x_3^2 - 33x_4^2 + 30x_3x_4$

Now the partial derivatives *w.r.t.* X_{N_B} are

$$\left(\frac{\partial f}{\partial x_3}\right)_{X_{N_B}=0} = (-145 - 14x_3 + 30x_4)_{\substack{x_3=0\\x_4=0}} = -145 \quad ...(3)$$

$$\left(\frac{\partial f}{\partial x_4}\right)_{X_{N_B}=0} = (299 - 66x_4 + 30x_3)_{\substack{x_3=0\\x_4=0}} = 299 \quad ....(4)$$

Since $\left(\frac{\partial f}{\partial x_3}\right)_{\substack{x_3=0\\x_4=0}} < 0$, so we cannot consider

x_3 to be the entering variable. On the other hand

$\left(\frac{\partial f}{\partial x_3}\right)_{\substack{x_3=0\\x_4=0}} > 0$ so x_4 enters the basis.

$$\text{Now min}\left\{\frac{\alpha_{10}}{|\alpha_{13}|}, \frac{\alpha_{20}}{|\alpha_{23}|}, \frac{v_{30}}{|v_{33}|}\right\}$$

$$= \text{min}\left\{\frac{8}{|-2|}, \frac{1}{|1|}, \frac{299}{|-66|}\right\}$$

(Here α_{10}, α_{20} and v_{30} are the constants in (1), (2) and (3) respectively which are nothing but 8,1 and 299, respectively whereas α_{13}, α_{23} and v_{33} are the coerfficients of x_4 in these equations)

$$= \text{min}\left\{4,\ 1\ \frac{299}{66}\right\} = 1$$

Thus x_2 leaves the basis. New $X_B = (x_1, x_4)$ and $X_{N_B} = (x_2, x_3)$.

Expressing the basic variables x_1, x_4 interms of nonbasic variables x_2 and x_3, and the maximization function $f(X)$ in terms of x_2, x_3 we have

$$x_1 = 10 - 2x_2 - x_3 \quad ...(5)$$

$$x_4 = 9 - x_1 - x_2 = 9 - (10 - 2x_2 - x_3) - x_2$$

or $$x_4 = -1 + x_2 + x_3 \quad ...(6)$$

and $$f = 10(10 - 2x_2 - x_3) + 25(x_2 - x_2^2 - 10\ (10 - 2x_2 - x_3)^2 - 4x_2(10 - 2x_2 - x_3)$$

or $$f = -900 + 365x_2 + 190x_3 - 33x_2^2 - 10x_3^2 - 36x_2x_3$$

Now $$\left(\frac{\partial f}{\partial x_2}\right)_{X_{N_B}=0} = (365 - 66x_2 - 36x_3)_{\substack{x_2=0\\x_3=0}} = 365$$

$$\left(\frac{\partial f}{\partial x_3}\right)_{X_{N_B}=0} = (190 - 20x_3 - 36x_2)_{\substack{x_2=0\\x_3=0}} = 190$$

Here $\left(\frac{\partial f}{\partial x_2}\right)_{X_{N_B}=0}$ is most positive so x_2 enters the basis.

We now compute the ratio

$$min\left\{\frac{\alpha_{10}}{|\alpha_{12}|}, \frac{\alpha_{40}}{|\alpha_{42}|}, \frac{V_{20}}{|V_{22}|}\right\}$$

$$min\left\{\frac{10}{|-2|}, \frac{-1}{|1|}, \frac{365}{|-66|}\right\}$$

$$min\left\{\frac{10}{2}, \frac{365}{66}\right\}$$ ($\because$ ratio will not be negative in any case)

$$= \frac{10}{2}$$

Thus x_1 leaves the basis

Now new $X_B = (x_2, x_4)$ and $X_{(N)B} = (x_1, x_3)$

Now new $X_B = (x_2, x_4)$ and $X_{(N)B} = (x_1, x_3)$

We shall obtain x_2, x_4 and f in terms of x_1 and x_3

$$x_2 = \frac{1}{2}(10 - x_1 - x_3) = 5 - \frac{1}{2}x_1 - \frac{1}{2}x_3 \quad ...(5)$$

$$x_4 = 9 - x_1 - x_2 = 9 - x_1 - \left(5 - \frac{x_1}{2} = \frac{x_3}{2}\right)$$

or $$x_4 = 4 - \frac{1}{2}x_1 + \frac{1}{2}x_3 \quad ...(6)$$

and $f = 10x_1 + 25\left(5 - \frac{1}{2}x_1 - \frac{1}{2}x_3\right) - 10x_1^2 - \left(5 - \frac{1}{2}x_1 - \frac{1}{2}x_3\right)\left(5 - \frac{1}{2}x_1 - \frac{1}{2}x_3 + 4x_1\right)$

or $f = 100 - \frac{35}{2}x_1 - \frac{15}{2}x_3 - \frac{33}{4}x_1^2 - \frac{1}{4}x_3^2 + \frac{3}{2}x_1 x_3$

The partial derivatives of f *w.e.f.* x_1 and x_3 are

$$\left(\frac{\partial f}{\partial x_1}\right)_{X_{N_B=0}} = \left(\frac{-35}{2} - \frac{33}{2}x_1 + \frac{3}{2}x_3\right)_{\substack{x_1=0\\x_3=0}} = \frac{-35}{2}$$

$$\left(\frac{\partial f}{\partial x_3}\right)_{X_{N_B=0}} = \left(\frac{-15}{2} - \frac{1}{2}x_3 + \frac{3}{2}x_1\right)_{\substack{x_1=0\\x_3=0}} = \frac{-15}{2}$$

Since both the partial derivatives are negative, therefore optimal solution is attained. The optimal solution is

$x_1 = 0;\ x_2 = 5;\ x_4 = 4$ and

$Min\, F(x_1, x_2) = 25 - 125 = -100$

Example-9 Solve the following quadratic programming problem by Beale's method.

$$Max.\, f(x_1, x_2) = x_1 + x_2 - x_1^2 + x_1 x_2 - 2x_2^2$$

subject to $2x_1 + x_2 \le 1$

$x_1, x_2 \ge 0$

Solution : Introducing the slack variable x_3 to the only constraint we get

$$Max.\, f(x_1, x_2) = x_1 + x_2 - x_1^2 + x_1 x_2 - 2x_2^2$$

subject to $2x_1 + x_2 \le 1$

$x_1, x_2 \ge 0$

Let us select x_1 arbitrarily the basic variable, i.e., let $X_B = (x_1)$. Then expressing the basic variable and the function f in terms of non basic variables x_2, x_3

$$x_1 = \frac{1}{2}(1 - x_2 - x_3) \qquad ...(1)$$

$$f = \frac{1}{2}(1 - x_2 - x_3) + x_2 - \frac{1}{4}\left(1 + x_2^2 + x_3^2 - 2x_2 - 2x_3 + 2x_2 x_3\right) + \frac{1}{2}x_2(1 - x_2 - x_3) - 2x_2^2$$

or $f=\frac{1}{4}+\frac{3}{2}x_2-\frac{11}{4}x_2^2-x_2x_3$ ...(2)

Then $\left(\frac{\partial f}{\partial x_2}\right)_{X_{NB=0}}=\left(\frac{3}{2}-\frac{11}{2}x_2-x_3\right)_{\substack{x_2=0\\x_3=0}}=\frac{3}{2}$

$$\left(\frac{\partial f}{\partial x_3}\right)_{X_{NB=0}}=\left(-\frac{1}{2}x_3-x_2\right)_{\substack{x_2=0\\x_3=0}}=0$$

Since $\left(\frac{\partial f}{\partial x_2}\right)_{X_{NB=0}}=\frac{3}{2}>0,$ so x_2 enters the basis

Now min. $\left\{\frac{\alpha_{10}}{|\alpha_{12}|},\frac{v_{20}}{|v_{22}|}\right\}=min\left\{\frac{\frac{1}{2}}{\left|\frac{-1}{2}\right|},\frac{\frac{3}{2}}{\left|\frac{-11}{2}\right|}\right\}$

$$=min\left\{1,\frac{3}{11}\right\}=\frac{3}{11}$$

Since the minimum occurs corresponding to $\frac{\nu_{20}}{|\nu_{22}|}$, therefore x_1 cannot be removed. We, therefore, define a new nonbasic variable

$$u_1=\frac{1}{2}\frac{\partial f}{\partial x_2}=\frac{1}{2}\left(\frac{3}{2}-\frac{11}{2}x_2-x_3\right)$$

or $u_1=\frac{3}{4}-\frac{11}{4}x_2-\frac{1}{2}x_3$ ...(3)

Then current basis is $X_B=(x_1,x_2)$ and $X_{NB}=(x_3,u_1)$.

Expressing the basic variable x_1 and x_2 in terms of non basic variables u_1 and x_3 also the function f in terms of nonbasic variables, we have

$$x_1=\frac{1}{2}-\frac{1}{2}x_2-\frac{1}{2}x_3$$

$$=\frac{1}{2}+\frac{2}{11}\left(u_1-\frac{3}{4}+\frac{1}{2}x_3\right)-\frac{1}{2}x_3 \qquad \text{(using (3))}$$

or $\quad x_1 = \frac{4}{11} + \frac{2}{11}u_1 - \frac{9}{22}x_3$ ...(4)

$$x_2 = 1_3 - 2x_1 = 1 - x_3 - 2\left(\frac{4}{11} + \frac{2}{11}u_1 - \frac{9}{22}x_3\right)$$

or $\quad x_2 = \frac{3}{11} - \frac{4}{11}u_1 - \frac{2}{11}x_3$ ...(5)

and $f = \frac{1}{4} + \frac{3}{2}\left(\frac{3}{11} - \frac{4}{11}u_1 - \frac{2}{11}x_3\right) - \frac{11}{4}\left(\frac{3}{11} - \frac{4}{11}u_1 - \frac{2}{11}x_3\right)^2 - \frac{1}{4}x_3^2 - x_3\left(\frac{3}{11} - \frac{4}{11}u_1 - \frac{2}{11}x_3\right)$

or $f = \frac{5}{11} - \frac{3}{11}x_3 - \frac{4}{11}u_1^2 - \frac{7}{44}x_3^2$ ...(6)

Now $\left(\frac{\partial f}{\partial x_3}\right)_{\substack{X_{NB}=0\\u_1=0}} = \left(\frac{-3}{11} - \frac{7}{22}x_3\right)_{\substack{X_{NB}=0\\u_1=0}} = -\frac{3}{11}$

$$\left(\frac{\partial f}{\partial u_1}\right)_{\substack{X_{NB}=0\\u_1=0}} = \left(\frac{-8}{11}u_1\right)_{\substack{X_{NB}=0\\u_1=0}} = 0$$

Since $\left(\frac{\partial f}{\partial x_3}\right)_{\substack{X_{NB}=0\\u_1=0}} < 0$ and $\left(\frac{\partial f}{\partial u_1}\right)_{\substack{X_{NB}=0\\u_1=0}} = 0$

therefore, optimal solution is attained. The optimal solution is:

$x_1 = \frac{4}{11} + 0 - 0 = \frac{4}{11}$ [from (4)]

$x_2 = \frac{3}{11} - 0 - 0 = \frac{3}{11}$ [from (5)]

and $Max\, f(x_1, x_2) = \frac{5}{11} - 0 - 0 - 0$ [from (6)]

$$= \frac{5}{11}$$

Example-10 Solve the following quadratic programming problem by Beale's method.

Maximize $f(x_1, x_2) = 2x_1 + 3x_2 - 2x_1^2$

subject to $\quad x_1 + 4x_2 \le 4$

$$x_1 + 2x_2 \le 2$$
$$x_1, x_2 \ge 0$$

Solution : Introducing the stack variables to the constraints, we set

Maximize $f(x_1, x_2) = 2x_1 + 3x_2 - 2x_1^2$

subject to $x_1 + 4x_2 + x_3 = 4$ ...(1)

$x_1 + 2x_2 + x_4 = 2$ ...(2)

$x_1, x_2, x_3, x_4 \ge 0$

Now let $X_B = (x_1, x_2)$ and $X_{N_B} = (x_3, x_4)$. Then expressing x_1, x_2 and f in terms of nonbasic variables x_3 and x_4

$$x_1 = x_3 - 2x_4 \quad ...(3)$$

$$x_2 = \frac{1}{2}[2 - x_1 - x_4] = \frac{1}{2}(2 - x_3 + x_4) \quad ...(4)$$

$$f = 2(x_3 - 2x_4) + \frac{3}{2}(2 - x_3 + x_4) - 2(x_3 - 2x_4)^2$$

or, $f = 3 + \frac{1}{2}x_3 - \frac{5}{2}x_4 - 2x_3^2 - 8x_4^2 - 8x_3x_4$ ...(5)

The partial derivatives of f *w.r.t.* x_3 and x_4 are

$$\left(\frac{\partial f}{\partial x_3}\right)_{X_{N_B=0}} = \left(\frac{1}{2} - 4x_3 - 8x_4\right)_{\substack{x_3=0 \\ x_4=0}} = \frac{1}{2}$$

$$\left(\frac{\partial f}{\partial x_4}\right)_{X_{N_B=0}} = \left(-\frac{5}{4} - 16x_4 - 8x_3\right)_{\substack{x_3=0 \\ x_4=0}} = -\frac{5}{4}$$

Clearly x_3 enters the basis

Now, $$min\left\{\frac{\alpha_{10}}{|\alpha_{13}|}, \frac{\alpha_{20}}{|\alpha_{23}|}, \frac{v_{10}}{|v_{33}|}\right\} = min\left\{\frac{0}{|1|}, \frac{1}{\left|1 - \frac{}{2}\right|}, \frac{\frac{1}{2}}{|-4|}\right\}$$

$= \min\{0, 2, \frac{1}{8}\}$

The ratio cannot be 0 or negative, therefore the minimum ratio is 2 that corresponds to x_2. Thus x_2 leaves the basis. Now new $X_B = (x_1, x_3)$ and $X_{N_B} = (x_2, x_4)$

Expressing x_1, x_3 and f in terms of non basic variables x_2, x_4 we have

$$x_1 = 2 - 2x_2 - x_4 \qquad \text{(from (2))} \qquad \ldots(6)$$

$$x_3 = 4 - x_1 - 4x_2 = 4 - (2 - 2x_2 - x_4) - 4x_2$$

or $x_3 = 2 - 2x_2 + x_4$ ...(7)

and $f = 2x_1(1 - x_1) + 3x_2$

$$= 2(2 - 2x_2 - x_4)(-1 + 2x_2 + x_4) + 3x_2$$

or $f = -4 + 15x_2 + 6x_4 - 8x_4^2 - 2x_4^2 - 8x_2x_4$

The partial derivatives of f w.r.t. x_2 and x_4 are

$$\left(\frac{\partial f}{\partial x_2}\right)_{X_{NB=0}} = (15 - 16x_2 - 8x_4)_{\substack{x_2=0 \\ x_4=0}} = 15$$

$$\left(\frac{\partial f}{\partial x_4}\right)_{X_{NB=0}} = (6 - 4x_2 - 8x_2)_{\substack{x_2=0 \\ x_4=0}} = 6$$

Since $\left(\frac{\partial f}{\partial x_2}\right)_{X_{N_B}} = 15$ is most positive so we allow x_2 to enter the basis. Now

$$min\left\{\frac{\alpha_{10}}{|\alpha_2|}, \frac{\alpha_{30}}{|\alpha_{32}|}, \frac{v_{20}}{|v_{22}|}\right\} = min\left\{\frac{2}{|-2|}, \frac{4}{|-4|}, \frac{15}{|-16|}\right\}$$

$$= min\left\{1, 1, \frac{15}{6}\right\} = \frac{15}{16}$$

which corresponds to, $\frac{v_{20}}{|v_{22}|}$. Thus we define a new non-basic variable

$$u_1 = \frac{1}{2}\frac{\partial f}{\partial x_2} = \frac{1}{2}(15 - 16x_2 - 8x_4)$$

or, $u_1 = \frac{15}{2} - 8x_2 - 4x_4$ ...(8)

Now the current basis is $X_B = (x_1, x_2, x_3)$ and $X_{N_B} = (x_4, u_1)$

Expressing X_B and f in terms of x_4 and u_1 we have

$$x_1 = 2 + \frac{1}{4}\left(u_1 - \frac{15}{2} + 4x_4\right) - x_4 \qquad \text{(from (6) and (8))}$$

or $x_1 = \frac{1}{8} + \frac{1}{4}u_1$ ...(9)

$$x_2 = \frac{15}{16} - \frac{1}{8}u_1 - \frac{1}{2}x_4 \qquad \text{(from (8))} \qquad ...(10)$$

$$x_3 = 2 + \frac{1}{4}\left(u_1 - \frac{15}{2} + 4x_4\right) + x_4 \qquad \text{(from (7) and (8))}$$

or $x_3 = \frac{1}{8} + \frac{1}{4}u_1 + 2x_4$ ...(11)

and $f = 2\left(\frac{1}{8} + \frac{1}{4}u_1\right) + 3\left(\frac{15}{16} - \frac{1}{8}u_1 - \frac{1}{2}x_4\right) - 2\left(\frac{1}{8} + \frac{1}{4}u_1\right)^2$

or, $f = \frac{97}{32} - \frac{3}{2}x_4 - \frac{1}{8}u_1 2$ ...(12)

Then $\left(\frac{\partial f}{\partial x_4}\right)_{x_{N_{B=0}} \atop u_1} = \left(\frac{-3}{2}\right)_{x_4=0 \atop u_1=0} = \frac{-3}{2}$

$$\left(\frac{\partial f}{\partial u_1}\right)_{x_{N_{B=0}} \atop u_1=0} = \left(-\frac{1}{4}u_1\right)_{x_4=0 \atop u_1=0} = 0$$

Since $\left(\frac{\partial f}{\partial x_4}\right)_{x_{N_{B=0}} \atop u_1=0} =< 0$ and $\left(\frac{\partial f}{\partial u_1}\right)_{x_{N_{B=0}} \atop u_1=0} = 0$ therefore, optimal solution is attained. The optimal solution is

$$x_1 = \frac{1}{8} + 0 = \frac{1}{8},\ x_2 = \frac{15}{16} - 0 - 0 = \frac{15}{16}$$

$$x_3 = \frac{1}{8} + 0 + 0 = \frac{1}{8}$$

i.e., $x_1 = \frac{1}{8},\ x_2 = \frac{15}{16}, x_3 = \frac{1}{8}$ and maximum value of $f(x_1, x_2, x_3)$ is $\frac{97}{32}$. (from (9), (10), (11) and (12))

8.5 Self-Learning Exercise

1. The quadratic from $X^T GX$ is called positive definite if $X^T GX$ .....

2. If quadratic from $X^T GX$ is negative semi-definite then $X^T GX$ .... for all X such that, there is one $X \neq 0$ satisfying ....

3. If $X^T GX$ is positive semi definite, then it is .... in X over E^n

4. If $X^T GX$ is negative semi definite, then it is .... in X over E^n

5. In Beale's method, the objective function, at each iteration, is expressed in terms of............

6. Answer true or false :

   Quadratic programming problem is a convex programming problem.

8.6 Summary

In this unit, we studied a specified form of the nonlinear programming problem called the quadratic programming problem. We also studied two algorithms namely the wolfe's algorithm and Beale's algorithm to solve the quadratic programming problems.

8.7 Answers to Self Learning Exercise

1. > 0 for all $X \neq 0$

2. $\leq 0, X^T GX = 0$

3. Convex

4. Concave

5. Non basic variables only

6. True

8.8 Exercise

Apply wolfe's method to solve the following programming problems:

(i) Max $f(X) = 8x_1 + 10x_2 - x_1^2 - x_2^2$

subject to $3x_1 + 2x_2 \leq 6$

$x_1, x_2 \geq 0$

(Ans. $x_1 = \frac{4}{13}, x_2 = \frac{33}{13}, Max\, f(X) = \frac{267}{13}$)

(ii) $Min\, f(X) = x_1^2 + x_2^2 + x_3^2$

subject to $x_1 + x_2 + x_3 = 2$

$$5x_1 + 2x_2 + x_3 = 5$$

(Ans. $x_1 = 0.81, x_2 = 0.35, x_3 = 0.35$ $Max\ f(X) = 0.857$)

(iii) $Max\ f(X) = 6x_1 + 3x_2 - 4x_1x_2 - 2x_1^2 - 3x_2^2$

subject to $x_1 + x_2 \le 1$

$2x_1 + 3x_2 \le 4$

$x_1, x_2 \ge 0$

(Ans. $x_1 = 1, x_2 = 0, Max\ f(X) = 4$)

(iv) Minimize $f(X) = 2x_1^2 + x_2^2 - 4x_1 - 6x_2$

subject to $x_1 + 3x_2 \le 3$

$x_1, x_2 \ge 0$

(Ans. $x_1 = \frac{12}{19}, x_2 = \frac{15}{19}, Minimum\ f(X) = \frac{111}{19}$)

Apply Beale's method to solve the following programming problems:

(i) $Min\ f(X) = 6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2$

subject to $x_1 + x_2 \le 2$

$x_1, x_2 \ge 0$

(Ans. $x_1 = \frac{3}{2},\ x_2 = \frac{1}{2}$)

(ii) $Min\ f(X) = 2x_1^2 + x_2^2 - 4x_1 - 6x_2$

subject to $x_1 + 3x_2 \le 3$

$x_1, x_2 \ge 0$

(Ans. $x_1 = \frac{12}{9},\ x_2 = \frac{15}{19}$)

(iii) Max. $f(X) = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$

subject to $x_1 + 2x_2 \le 2$

$x_1, x_2 \ge 0$

(Ans. $x_1 = \frac{1}{3},\ x_2 = \frac{5}{6}, Min\ f(X) = \frac{25}{6}$)

(iv) Min. $f(X) = x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_1 - 5x_2$

subject to $2x_1 + 3x_2 \le 20$

$3x_1 - 5x_2 \le 5$

$x_1 - x_2 \ge 0$

(Ans. $x_1 = \frac{9}{2},\ x_2 = \frac{7}{2}$; Min. $f(X) = -\frac{53}{4}$)

□□□

Unit-9
Quadratic Programming Problem and Duality Theorem in Quadratic Programming

Structure of the Unit

9.0 Objective

Duality plays a crucial role in the theory and compulational algorithms of linear and non-linear programming. Duality is non-linear programming is related to the reciprocal principles of the calculus of variations, which have been known since as far back as 1927. The purpose of writing the present unit is to introduce the non-linear programming problem and its dual and then to dovelop the duality results of non-linear programming. These results are fruitfully applied to quadratic and linear programming problems.

9.1 Introduction

The plan of the unit is to introduce the quadratic programming problem and its dual and then will develop the duality theory for non-linear programming and quadratic programming. There is an extensive literature on the theory of non-linear programming and quadratic programming, but we shall end the unit with the duality theorem for qudratic programming problem.

9.2 Quadratic Programming and Duality

In recent years, there has been much interest in the duality theory of non-linear programming, especially of quadratic programming. As duality plays an important role in the theory of linear programming, it plays equally important role in the theory of quadratic programming also.

If there exists an optimal solution to the quadratic programming problem max $f(X)$ where X is ≥ 0 or unrestricted in sign) subject to the constraints $g_i(X) = b_i$, $\mathrm{i} = 1,2,.....,\mathrm{m}$, then there also exists an optimal solution to the dual of this quadratic programming problem and the two optimal values are equal. If the set of feasible solutions of the given quadratic programming problem is empty but that of its dual problem is non-empty, then the dual problem has an unbounded solution on the set of feasible solutions. If the set of feasible solutions of the given quadratic programming is non-empty and the set of feasible solutions of its dual is empty, then this implies that the quadratic programming problem has no optimal solution.

Unlike in linear programming problem, it can be shown that the dual of the dual of the quadratic programming problem may not be the quadratic programming itself.

9.3 Duality in Non-Linear Programming

Consider the following non-linear programming problem :

$$\text{Maximize} \quad f(X)$$

$$\text{(P1)} \quad \text{subject to} \quad g_i(X) \geq 0 \quad , i = 1,2,\ldots\ldots,m \qquad \ldots(1)$$

$$h_j(X) = 0 \quad , j = 1,2,\ldots\ldots,p$$

where $X^T = (x_1, x_2, \ldots, x_n)$ and the functions f, g_i and h_j are assumed to be continuously differentiable functions over some open set SCE^n.

The Lagrangian function $L(X, \lambda, u)$ associated with the problem (1) is given by

$$L(X,\lambda,\mu) = f(X) - \sum_{i=1}^{m} \lambda_i \, g_i(X) + \sum_{j=1}^{p} \mu_j \, h_j(X) \qquad \ldots(2)$$

where $X \in E^n$, $\mu \in E^p$ and $\lambda^T = (\lambda_1, \lambda_2, \ldots, \lambda_m) \geq 0$

Let $\quad \Lambda = \{(\lambda,\mu) : \lambda \geq 0, \lambda \in E^m, \mu \in E^p\}$ ...(3)

Then treating $L(X,\lambda,\mu)$ as a function of X and (λ,μ), we have the following known definitions. The point (X_0, λ_0, μ_0) is called a Lagrangian saddle point of L (or of problem (1), if $X_0 \in E^n$, $(\lambda_0, \mu_0) \in \Lambda$ and

$$L(X, \lambda_0, \mu_0) \leq L(X_0, \lambda_0, \mu_0) \leq L(X_0, \lambda, \mu) \text{ for all } X \in E^n \text{ and } (\lambda,\mu) \in \Lambda \qquad \ldots(4)$$

The function

$$L_*(X) = \min_{(\lambda,\mu)\in\wedge} L(X,\lambda,\mu), \quad X \in E^n \qquad \ldots(5)$$

is said to be the **primal function** and the function

$$L^*(\lambda,\mu) = \max_{X \in E^n} L(X,\lambda,\mu), \quad (\lambda,\mu) \in \Lambda \qquad \ldots(6)$$

is called the **dual function.**

The functions $L_*(X)$ and $L^*(\lambda,\mu)$ are related to the saddle points of the Lagrangian function L. To relate the primal function $L_*(X)$ to the primal problem (1), we need to evaluate

$$L_*(X) = \min_{(\lambda,\mu)\in\Lambda} \left[f(X) - \sum_{i=1}^{m} \lambda_i \, g_i(X) + \sum_{j=1}^{p} \mu_j \, h_j(X) \right] \qquad \ldots(7)$$

Now if $g_i(X) \geq 0$ for all $\mathrm{i} = 1,2,\ldots\ldots,\mathrm{m}$ and $h_j(X) = 0$ for all $\mathrm{j} = 1,2,\ldots\ldots,\mathrm{p}$, then $\lambda_i = 0$

$(\text{i} = 1,2,\ldots,\text{m})$ will minimize the Lagrangian. But if some $g_i(X) < 0$, then the Lagrangian can be minimized by taking $\lambda_i \to -\infty$. Likewise if some $h_j(X) \neq 0$, then by letting $\mu_j \to \infty$ or $-\infty$ according as $h_j(X) < 0$ or > 0, we can minimize the Lagrangian. Thus

$$L_*(X) = \begin{cases} f(X), & \text{if} \quad g_i(X) \geq 0 \ \ (i = 1,2,\ldots,m) \\ & \text{and} \quad h_j(X) = 0 \ \ (j = 1,2,\ldots,p) \\ -\infty, & \text{otherwise} \end{cases} \qquad \ldots(8)$$

In view of the $-\infty$ in $L_*(X)$, we must use infimum instead of minimum in equation (5). Now suppose that we maximize $L_*(X)$ for $X \in E^n$. Then the unconstrained maximization problem.

$$\text{Max.} \quad L_*(X) \quad ; \quad X \in E^n \qquad \ldots(9)$$

is equivalent to the primal problem (1), namely

$$\text{Max.} \quad f(X) \quad ; \quad X \in E^n$$

$$\text{s.t.} \quad g_i(X) \geq 0 \quad ; \quad i = 1,2,\ldots,m$$

$$h_j(X) = 0 \quad ; \quad j = 1,2,\ldots,p$$

The equivalence of (9) and the primal problem (1), the primal program is to find an optimal X_0, which solves (9).

Now associated with the primal programme (9) is another program, called the dual program which is :

$$\text{Min.} \quad L^*(\lambda,\mu) \quad \text{for} \quad (\lambda,\mu) \in \Lambda \qquad \ldots(10)$$

The above dual programme is equivalent to :

(DP 1) Minimize $\quad L(X,\lambda,\mu) \qquad \ldots(11)$

subject to $\quad L(X,\lambda,\mu) = \max_{X \in E^n} L(X,\lambda,\mu)$

$$\equiv L^*(\lambda,\mu) \qquad \ldots(12)$$

$$\lambda \geq 0 \qquad \ldots(13)$$

A point (X,λ,μ) is said to be **feasible** for the dual (10) if

$$L(X_1,\lambda_1,\mu_1) = L^*(\lambda_1,\mu_1) \ ; \lambda_1 \geq 0$$

Now if X_1 is feasible for the problem (1), then from equation (8)

$$L_*(X_1) = f(X_1) \qquad \ldots(14)$$

from (5), (6), and (12)

$$L_*(X_1) = \min_{(\lambda,\mu)\in\Lambda} L(X_1,\lambda,\mu) \le L(X_1,\lambda_2,\mu_2)$$

$$\le L(X_2,\lambda_2,\mu_2)$$

$$= L^*(\lambda_2,\mu_2)$$

where (X_2,λ_2,μ_2) is **feasible** for the dual (DP 1). Therefore, it easily follows that

$$\max_{X\in E^n} L_* \le \min_{(\lambda_1,\mu)\in\Lambda} L^*(\lambda_1,\mu)$$

We finally conclude that :

If X_* and (X_0,λ_0,μ_0) are feasible solution to the primal (P1) i.e. problem (1) and dual (DP1), i.e., the problem (11), (12), respectively such that

$L_*(X_*) = L^*(X_0,\mu_0)$, then X_* and (X_0,λ_0,μ_0) are optimal solutions for the problem (P1) and (DP1) respectively, i.e., the point (λ_0,μ_0) is optimal for the dual program (10).

We now state the duality theorem for the convex programming (CP). Recall that the general convex programming problem is

(CP) Maximize $f(X)$

subject to $g_i(X) \ge 0$; $i = 1,2,\ldots\ldots,m$

$h_j(X) = 0$; $j = 1,2,\ldots\ldots,p$

where the functions f, $g_1,g_2,\ldots,g_n$ are concave on E^n and $h_1,h_2,\ldots.,h_p$ all linear. If we assume that the functions f and all $g_i(X)$, $i = 1,2,\ldots\ldots..,m$ are differentiable, then clearly the Lagrangian function

$$L(X,\lambda,\mu) = f(X) - \sum_{i=1}^{m} \lambda_i\, g_i(X) + \sum_{j=1}^{p} \mu_j\, h_j(X)$$

is a function X for all $\lambda \ge 0$

Then $\nabla_X L(X,\lambda,\mu) = 0$ if and only if

$$L(X,\lambda,\mu) = \max_{X\in E^n} L(X,\lambda,\mu)$$

therefore the dual programme (DP1) corresponding to the convex programme (CP) becomes :

(DCP) Minimize $L(X,\lambda,\mu)$

s.t. $\nabla_X L(X,\lambda,\mu) = 0$

$\lambda \ge 0$

In the following section we shall discuss the duality in quadratic programming.

9.4 Duality in Quadratic Programming

For each quadratic programming problem there always esixts another quadratic programming problem having the property that if of these two problems, one has finite optimal solution, then so has the other. Interestingly optimal values of the objective functions of both the problems at their respective optimal solutions are the same. This concept in quadratic programming is called the Daulity in Quadratic Programming.

Let we have the quadratic programming problem

$Max\ f(X)$; X is unrestricted in sign

subjct to $g_i(X) = b_i$, $i = 1, 2, \ldots, m$.

Then the dual of the above programming problem is

Min. $L(X, \lambda)$

subject to $\frac{\partial L(X,\lambda)}{\partial x_j} = 0$; $j = 1, 2, \ldots., n$

where $L(X,\lambda) = f(X) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(X))$

As a particular case if the quadratic programming problem is :

Max $f(X) = C^T X + \frac{1}{2} X^T G X$

subject to $AX = b$ ...(1)

X is unrestricted in sign

then its dual problem has the form

Min $L(X,\lambda) = C^T X + \frac{1}{2} X^T G X + \lambda^T (b - AX)$

subject to $C^T + X^T G - \lambda^T A = 0$ ...(2)

Multiplying (2) on right side by X, we see that

$C^T X + X^T G X - \lambda^T A X = 0$

or, $\lambda^T A X = C^T X + X^T G X$

so that for any X, λ satisfying (2), $L(X,\lambda)$ becomes $L(X,\lambda) = -\frac{1}{2} X^T G X + \lambda^T b$

and so the dual of the quadratic progamming problem

Max $f(X) = C^T X + \frac{1}{2} X^T G X$

subject to $AX = b$

X unrestricted

can be written as :

Min $L(X, \lambda) = -\frac{1}{2} C^T GX + \lambda^T b$

subject to $-GX + A^T \lambda = C$

In the above discussion, we didnot take account of the fact that, in general we need $X \geq 0$.

Suppose that we have $X^* \geq 0$ to be the optimal solution of he quadratic programming problem

Max $f(X) = C^T X + \frac{1}{2} X^T GX$

subject to $AX = b$

$$X \geq 0 \quad ...(3)$$

Then by Kuhn-Tucker Theory, there exists a λ^* such that

$$-GX^* + A^T \lambda^* \geq C$$

$$\text{Max } f(X) = L(X^*, \lambda^*) = C^T X^* + \frac{1}{2} (X^*)^T GX^* + (\lambda^*)^T [b - AX^*]$$

$$= C^T X^* + \frac{1}{2} (X^*)^T GX^* \quad ...(4)$$

since $(\lambda^*)^T (b - AX^*) = 0$

Also it can be seen that

$$(X^*)^T GX^* + (\lambda^*)^T AX^* = CX^* \quad ...(5)$$

Now for any $X \geq 0$ and λ satisfying the condition $-GX + A^T \lambda \geq C$, ...(6)

we obtain $-\lambda^T AX \leq -C^T X - X^T GX$, on multiplying (6) on the left by X and then taking the transpose.

Thefore, $L(X, \lambda) \leq -\frac{1}{2} X^T GX + \lambda^T b$ ...(7)

(since $\lambda^T (b - AX) = 0$)

However, by (5)

$$L(X^*, \lambda^*) = -\frac{1}{2} (X^*)^T GX^* + (\lambda^*)^T b$$

$$= Max\ f(X)$$

Therefore, (X^*, λ^*) is an optimal solution to the quadratic programming problem

$$-GX + A^T \geq C$$

$$X \geq 0 \qquad \ldots(8)$$

$$\text{Min } L(X, \lambda) = -\frac{1}{2} X^T GX + \lambda^T b$$

Furthermore, $Max\ f(x) = Min\ L(X, \lambda)$

We call the quadratic programming (8) to be the dual of (3). We have already shown that if (3) has an optimal solution then (8) also has an optimal solution.

9.5 Duality Theorem for Quadratic Programming Problem

Theorem : For each quadratic programming problem

$$Max\ f(X) = C^T X + \frac{1}{2} X^T GX,$$

subject to $AX = b, X \geq 0$,

there exists another quadratic programming problem (called the dual)

$$Min\ L(X, \lambda) = -\frac{1}{2} X^T GX + \lambda^T b$$

subject to $-GX + A^T \lambda \geq C$

$$X \geq 0$$

and λ unrestricted in sign, such that if one has a finite optimal solution, then so has the other. Furthermore, the optimal values of both the problems are the same.

Proof : Suppose that X^* be a finite optimal solution to the quadratic programming problem

$$Max\ f(X) = C^T X + \frac{1}{2} X^T GX$$

subject to $AX = b$

$$X \geq b \qquad \ldots(1)$$

Then by Kuhn-Tucker theory there exists a λ^* such that

(i) $\nabla_X L(X^*, \lambda^*) \leq 0$

i.e., $C^T + (X^*)^T G - (\lambda^*)^T A \leq 0$

or $-GX^* + A^T \lambda^* \geq C \qquad \ldots(2)$

(ii) $\left[\nabla_X L(X^*, \lambda^*)\right] X^* = 0$

i.e. $C^T X^* + (X^*)^T G X^* - (\lambda^*)^T A X^* = 0$ ...(3)

(iii) $\left[\nabla_\lambda L(X^*, \lambda^*)\right] \lambda^* = 0$

i.e., $(\lambda^*)^T (b - AX^*) = 0$ ...(4)

and (iv) λ_i^* is unrestricted in sign for all

$i = 1, 2, \ldots, m$. ...(5)

Now since X^* is an optimal solution to the quadratic programming problem (1), therefore, $AX^* = b$ and

Maximum of $f(X) = C^T X^* + \frac{1}{2}(X^*)^T G X^*$

$$= C^T X^* + \frac{1}{2}(X^*)^T G X^* + (\lambda^*)^T (b - AX^*) \qquad \text{[using (4)]}$$

$$= L(X^*, \lambda^*) \qquad \text{...(6)}$$

Also since $C^T X^* + \frac{1}{2}(X^*)^T G X^* = \left[(\lambda^*)^T A X^* - (X^*)^T G X^*\right] + \frac{1}{2}(X^*)^T G X^*$ [from (3)]

$$= -\frac{1}{2}(X^*)^T G X^* + (\lambda^*)^T A X^*$$

$$= -\frac{1}{2}(X^*)^T G X^* + (\lambda^*)^T b$$

(since $AX = b$)

Thus maximum of $f(X) = C^T X^* + \frac{1}{2}(X^*)^T G X^*$

$$= -\frac{1}{2}(X^*)^T G X^* + (\lambda^*)^T b$$

But from (6),

Maximum of $f(X) = L(X^*, \lambda^*)$

Therefore, maximum of $f(X)$, i.e., $f(X^*)$ is

$$L(X^*, \lambda^*) = -\frac{1}{2}(X^*)^T G X^* + (\lambda^*) b \qquad \text{...(7)}$$

Now for any $X \geq 0$ and λ satisfying $-GX + A^T\lambda \geq C$, on multiplying by X^T on the left and then taking transpose on both sides, we get

$$-X^T GX + \lambda^T AX \geq C^T X$$

or $$-\lambda^T AX \leq -C^T X - X^T GX$$

or, $$\left[C^T X + \frac{1}{2} X^T GX + \lambda^T b\right] - \lambda^T AX \leq \left[C^T X + \frac{1}{2} X^T GX + \lambda^T b\right] - C^T X - X^T GX$$

(on adding $C^T X + \frac{1}{2} X^T GX + \lambda^T b$ on both sides)

or, $L(X,\lambda) \leq -\frac{1}{2} X^T GX + \lambda^T b = Z(X,\lambda)$ (let)

But from (7)

$$L(X^*,\lambda^*) = Z(X^*,\lambda^*) = -\frac{1}{2}(X^*)^T GX^* + (\lambda^*)^T b$$

$$= \text{maximum of } f(X), \text{ i.e., } f(X^*)$$

Therefore $Z(X^*,\lambda^*)$ is a minimum of $Z(X,\lambda)$

Hence (X^*,λ^*) is an opitmal solution to the quadratic programming problem

Min $z(X,h) = -\frac{1}{2} X^T GX + \lambda^T b$

subject to $-GX + A^T\lambda \geq C$

$$X \geq 0 \qquad \text{...(8)}$$

and λ unrestricted in sign. Further more we observed that $max\, f(x) = min\, z(X,\lambda)$

We call the quadratic programming problem (8), the dual of the quadratic programming problem (1). We could prove that if (1) has a finite optimal solution at the point $X = X^*$, then its dual (8) also has a finite optimal solution at (X^*,λ^*).

Conversely, we shall show that if the quadratic programming problem (8) has a finite optimal solution at (X^*,λ^*), then the quadratic programming problem (1) also has a finite optimal solution for this we only require to show that (1) has a feasible solution if we assume that the objective function of (1) is strictly concave function or is negative definite.

Now (X^*,λ^*) is a finite optimal solution of (8) implies that by Kuhn-Tucker theorem there exists a δ^* such that

$$-GX^* + G\delta^* \geq 0 \qquad \left(\because \nabla_X L(X^*,\lambda^*,\delta^*) \geq 0\right)$$

or $\delta^* \geq X^*$

and $A\delta^* = b$ (by $\nabla_\lambda L(X^*, \lambda^*, \delta^*) = 0$ since λ is unrestricted)

i.e. $A\delta^* = b$ and $\delta^* \geq X^* \geq 0$

which shows that δ^* is a feasible solution of the quadratic programming problem (1) and hence has a finite optimal solution.

Example-1 Derive the dual of the quadratic programming problem :

$$\text{Min } f(X) = C^T X + \frac{1}{2} X^T G X \quad ...(1)$$

$$\text{subject to } AX \geq b \quad ...(2)$$

Where A is an $m \times n$ real matrix and G is an $n \times n$ real positive semidefinite a symmetric matrix.

Solution : The Lagrangian of the given quadratic programming problem is :

$$L(X, \lambda) = C^T X + \frac{1}{2} X^T G X - \lambda^T (AX - b)$$

$$= (C - A^T \lambda)^T X + \frac{1}{2} X^T G X + \lambda^T b \quad ...(3)$$

where $\lambda \geq 0$

The dual of the quadratic programming problem, then is

$$Max.\, L(X, \lambda) = \left[(C - A^T \lambda)^T X + \frac{1}{2} X^T G X + \lambda^T b \right.$$

$$\text{subject to} \quad \nabla_X L(X, \lambda) = 0 \quad ...(4)$$

$$\text{i.e.} \quad C - A^T \lambda + GX = 0 \quad ...(5)$$

$$\lambda \geq 0 \quad ...(6)$$

Using the constraint (5) in (4), we see that the dual quadratic programming problem of (1) is

$$\text{Max. } L(X, \lambda) = -(GX)^T X + \frac{1}{2} X^T G X + \lambda^T b$$

$$= -X^T G X + \frac{1}{2} X^T G X + \lambda^T b$$

$$= -\frac{1}{2} X^T G X + \lambda^T b \quad ...(7)$$

$$\text{subject to} \quad A^T \lambda - GX = C \quad ...(8)$$

$$\lambda \geq 0 \quad ...(9)$$

9.6 Self-Learning Exercise

1. If the set of feasible solutions of the quadratic programming problem is nonempty but of its dual is empty then ....

2. If the set of feasible solution of the quadratic programming problem is empty but of its dual is nonempty, then......

3. The dual of the dual of the quadratic programming problem is the quadratic program itself-true or false?

9.7 Summary

In this unit, we studied the duality in non linear programming and quadratic programming. We also proved the duality theorem for quadratic programming problem.

9.8 Answers to Self-Learning Exercise

1. The quadratic programming problem has no optimal solution.

2. The dual problem of has an unabounded solution.

3. False.

9.9 Exercise

1. Set G be a positive semidefinite symmetric matrix. Then write the dual of the following quadratic programming problem

   Minimize $f(X) = C^T X + \frac{1}{2} X^T G X$

   subject to $AX \geq b$

   $X \geq 0$

2. If $f(X)$ is a concave function, then give the dual of the following quadratic programming problem:

   Max $f(X) = C^T X + \frac{1}{2} X^T G X$

   subject to $AX \leq b$

   $X \geq 0$

□□□

Unit - 10
Convex Separable Programming and Algorithm

Structure of the Unit

10.0 Objective

In convex separable programming, convex non linear programming problems are solved by approximating the non linear functions with piecewise linear functions and then solving the optimization problem through the use of a modified simplex algorithm of linear programming, or in special cases, the ordinary simplex algorithm.

10.1 Introduction

Separable programming was first introduced by C.E. Miller in 1963 : E.M.L. Beale in 1965 refered to separable programming as "Probably the most useful non linear programming technique." Mc Millan stated that any continuous, non linear and convex separable function can be approximated by a piecewise linear function and solved using a linear programming solution technique in his book on mathematical programming", Wiley, New York, 1970. In 1974, Hadley also represented a technique that how one can approxmate a nonlinear separable function.

Convex separable programming is an important and richly studied problem of convex non linear programming problems in which the objective function as well as the constratints are separable and the problem of maximizing a concave function or minimizing a convex function over a convex set.

Piecewise linear approximation can be done for convex as well as concave functions. Curves of non linear objective function and constraints can be approximated by a series of piecewise linear segments or polygonal linear approxmations.

Thus a NLPP can be reduced (approximated) to a L.P.P. and used simplex method can be applied to obtain an optimal solution.

10.2 Definitions

10.2.1 Separable Function

A function $f(x_1, x_2, \ldots, x_n)$ is said to be separable if it can be expressed as the sum of n single valued functions $f_1(x_1),\ f_2(x_2),\ \ldots,\ f_n(x_n)$; i.e.

$$f(x_1,\ x_2,\ \ldots,\ x_n) = f_1(x_1) + f_2(x_2) + \ldots + f_n(x_n).$$

For example, the linear function given by :

$f(x_1,\ x_2,\ \ldots,\ x_n) = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ (Where c's are constants) is a separable function. On the other hand, the function defined by :

$$g(x_1, x_2, \ldots, x_n) = x_1^2\ sin(x_1 + x_2) + x_2^3 + x_3.3x_3 + log(x_1 + x_3)$$

is not a separable function.

10.2.2 Convex Programming Problem

The problem of maximizing a concave function or minimizing a convex function over a convex set is called a convex programming problem.

A general convex programming problem (C.P.P.) can be defined as :

Maximize $f(x)$

$X^T\ DX$

Subject to $x \in s$

where $x \in R^n$, $f(x)$ is a concave function on a convex set $S \subset R^n$ ....(10.2.1)

For Example :

(i) The nonlinear programming problem (N.L.P.P.)

Maximize $f(x)$

Subject to $g_i(x) \leq b_i, i = 1, 2, \ldots, m$ and $x \geq 0$

is a convex programming problem if $f(x)$ is concave and

$g_i(x)$ are convex, $\forall\ i = 1,\ 2,\ \ldots,\ m$ .... (10.2.2)

(ii) The quadratic programming problem

Maximize $f(x) = CX + X^T DX$

Subject to $AX = b$

and $X \geq 0$

is a convex programming problem iff $X^T\ DX$ is negative (negative semi) definite. .... (10.2.3)

10.2.3 Separable Programming Problem

A nonlinear programming problem of the form :

Maximize $$Z = \sum_{j=1}^{n} f_j(x_j)$$

Subject to $$\sum_{j=1}^{n} g_{ij}(x_j)\{\le, =, \ge\}\ b_i,\ i = 1, 2, ..., m$$

and $x_j \ge 0,\ j = 1, 2, ..., n$

in which all the functions (objective function and constraints) are separable is called a separable programming problem.

Some times the functions are not directly separable but can be made separable by simple substitution.

e.g. For non separable term $x_i\ x_j$, we can write

$$x_i x_j = y_1^2 - y_j^2, \text{ where } y_i = \frac{1}{2}(x_i + x_j) \text{ and } y_j = \frac{1}{2}(x_i - x_j)$$

convex separable programming problem : A convex programming problem in which all the functions are separable in called a convex separable programming problem.

10.2.4 Convex separable programming Problem

A convex programming problem in which all the functions are separable is called a convex separable programming problem.

10.3 Theorems

Theorem 1 : Every local maximum of the general convex programming problem is its global maximum.

Proof : Consider the general convex programming problem (10.2.1)

If the constraints set s is empty or singleton then the theorem is trivilly hold good.

If s is neither empty nor singleton then we shall prove this theorem by contradiction.

Let us assume that the C.P.P. has a local maximum at $X_0 \in S$ and global maximum at $X^* \in S$ and $f(x_0) \ne f(x^*)$, then $f(x_0) < f(x^*)$

Since $f(x)$ is a concave function on the convex set S, so for $0 < \lambda < 1$

$$\begin{aligned} f\left[\lambda X^* + (1-\lambda)X_0\right] &\ge \lambda f(X^*) + (1-\lambda) f(X_0) \\ &> \lambda f(X_0) + (1-\lambda) f(X_0),\ \because f(X_0) < f(X^*) \\ &> \lambda f(X_0) + f(X_0) - \lambda f(X_0) = f(X_0) \end{aligned}$$

Now for any $\in >\ 0$, however small, if $0\ <\ \lambda < 1$ is so chosen that $0<\lambda<\frac{\in}{|X^*-X_0|}$, then

$$\left|\left\{\lambda X^* + (1-\lambda) X_0\right\} - X_0\right| = \left|\lambda\left(X^* - X_0\right)\right| = \lambda\left|X^* - X_0\right| < \in$$

i.e. $\lambda X^* + (1-\lambda) X_0$ is a point in any $\in -\ nbd$ of X_0 for which

$$f\left[\lambda X^* + (1-\lambda) X_0\right] > f(X_0)$$

which is contraction of the fact that $f(X_0)$ is local minimum of the C.P.P.

So our assumption $f(X_0) \neq f(X^*)$ is wrong.

Hence $f(X_0) = f(X^*)$

Hence a local maximum of the C.P.P. is a global maximum of it.

Theorem 2 : The set of all optimum solutions (global maximum) of the general convex programming problem is a convex set.

Proof : Consider the C.P.P. (10.2.1)

Let A be the set of all optimal solutions of the C.P.P. If A is either empty or singleton then the theorem is trivial. If A is neither empty nor singleton, then suppose $x_1 \in S$ and $x_2 \in S$ are any two different points of A.

Then $f(x_1) = f(x_2) =$ Global maximum of $f(x) = k^*(say)$

Now, $f\left[\lambda x_2 + (1-\lambda)x_1\right] \geq \lambda f(x_2) + (1-\lambda) f(x_1)$, $0 \leq \lambda \leq 1$

$$\geq \lambda k^* + (1-\lambda) k^*$$

$$\geq k^*$$

Since $f\left[\lambda x_2 + (1-\lambda)x_1\right] > k^*$ cannot be true because k^* is global maximum, therefore

$$f\left(\lambda x_2 + (1-\lambda)x_1\right) = k^*$$

$\Rightarrow$ $\lambda x_2 + (1-\lambda)x_1 \in A,\ \forall\, 0 \leq \lambda \geq 1$

$\Rightarrow$ A is a convex set.

Theorem 3 : If in theorem 1, $f(x)$ is strictly concave then the C.P.P. has unique optimal solution (if it exists).

10.4 Aproximate Optimal Solution of a Aonvex Separable Programming Problem.

In the separable programming problem (10.2.3) some or all functions $f_j(x_j)$ and $g_{i_j}(x_j)$, $j = 1, 2, ..., n$ are non linear. We solve this problem by replacing non linear function into linear function by piecewise linear approximations or polygonal approximations. In general, we shall determine a local maximum for the approximating problem but if the separable programming problem is convex programming also, then local maximum also a global maximum. Thus, if (10.2.3) is a convex separable programming problem then we can find a global maximum for the appromating problem and consequently an approximate optimal solution to (10.2.3).

10.5 Piecewise Linear Approximation of a Non-Linear Continous Function

Consider an arbitrary continuous nonlinear function $f(x)$ of a single variable x, which is defined for all $x,\ 0 \le x \le a$ as shown in figure 10.01. We choose some points (refer to them as grid points) $0 = x_0 < x_1 < x_2 < x_3 < ...... < x_r < a$. Now for each x_k we compute $f_k = f(x_k)$ and connect the points $(x_k,\ f_k)$ and $(x_{k+1},\ f_{k+1})$. We have formed approximation function $\overline{f}(x)$, which is a pieswise linear function.

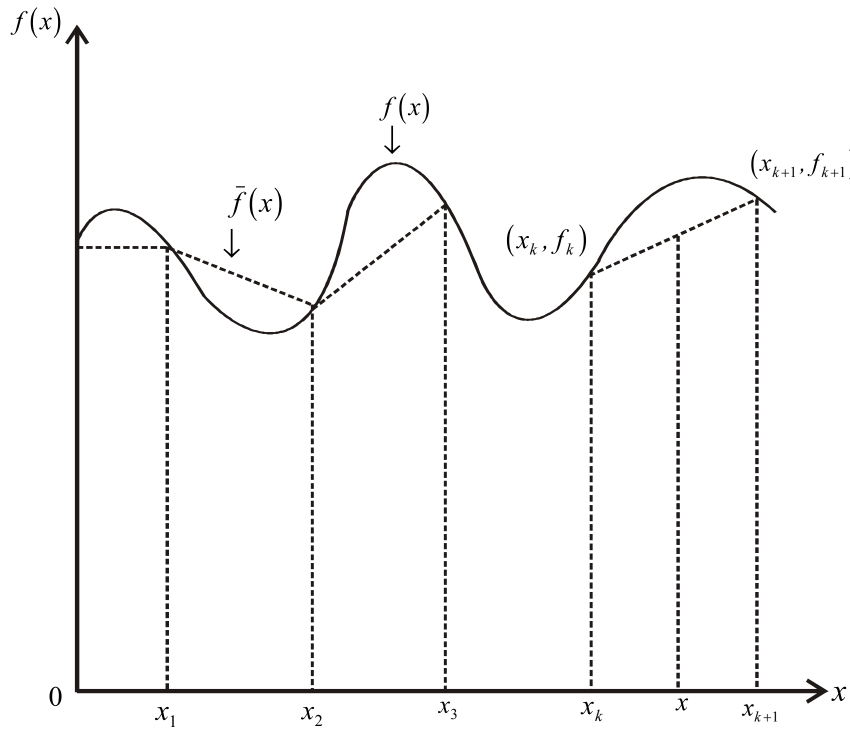


Figure : 10.1

$f(x)$ shown by dark curved.

$\overline{f}(x)$ shown by dashed straight line segment.

From figure, for $x_k < x < x_{k+1}$, we have

$$\overline{f}(x) = f_k + \frac{f_{k+1} - f_k}{x_{k+1} - x_k}(x - x_k),$$

$x \in [x_k,\ x_{k+1}]$ can be written as $x = \lambda_k x_k + \lambda_{k+1} x_{k+1}$,

Where $\lambda_K + \lambda_{k+1} = 1$ and $\lambda_k \ge 0,\ \lambda_{k+1} \ge 0$ (By the definition of

line segment) and then $\overline{f}(x) = \lambda_k f_k + \lambda_{k+1} f_{k+1}$.

Indeed for any $0 = x_0 < x_1 < x_2 < ...... < x_r = a$, we can write

$$x = \sum_{k=0}^{r} \lambda_k x_k,\ \bar{f}(x) = \sum_{k=0}^{r} \lambda_k f_k \text{ , where } \sum_{k=0}^{r} \lambda_k = 1,\ \lambda_k \geq 0,$$

$k = 0, 1, 2, ..., r$ and r is any suitable integer representing the number of segments into which the domain of x is divided. In addition, it is required that no more than two of the λ_k be positive, and if two are positive they must be adjacent. This restriction is called restricted basis entry rule.

By getting polygonal linear approximation (Piecewise linear approximation) of every non linear function in the separable programming problem (10.2.3) and replacing it by its polygonal approximation, we get the approximating problems :

Maximize $\bar{Z} = \sum_{j=1}^{n} \bar{f}_j(x_j)$

Subject to $\sum_{j=1}^{n} \bar{g}_{ij}(x_j)\{\leq,\ =,\ \geq\} b_i,\ i = 1,\ 2,\ .....,\ m$

and $x_j \geq 0,\ \ j = 1,\ 2,\ .....,\ n$

Now, we can solve this linear programming problem by simplex method with restricted basis entry rule.

10.6 Separable Programming Algorithm

The computational procedure to solve this problem is as follows.

Step I

If the objective function is in minimization form, then convert it in to the maximization form and all $b_i,\ \forall\ i = \overline{1,m}$ should be non negative. The separable programming problem should be convex programming problem. If it is not a convex programming problem then the approximate optimum solution (global maximum) may not be found. Since in general, we get a local maximum for the approximating problem.

Step II

Divide the interval $0 \leq x_j \leq a_j,\ j = 1,\ 2,\ ......,\ n$ as subdivided points $0 = x_{j0} < x_{j1} < x_{j2} < ...... < x_{jr_j} = a_j$

compute linear approxmation for each non linear $f_j(x_j)$ and $g_i(x_i)$. Write the approximating problem of the given separable programming problem.

Step III

Solve the approxmated linear programming problem by using simplex method with the use of restricted basis entry rule.

Step IV

Finally, find the optimal solution (approximate) x_j of the original problem by using

$$x_j = \lambda_{j0} x_{j0} + \lambda_{j1} x_{j1} + ... + \lambda_{jr} x_{jr}$$

Note : We may drop the column corresponding $\lambda_{j0},\ j=\overline{1,n}$

which has departing vector in the simplex table because cost of λ_{j0} is 0.

10.7 Illustrative Examples

Example 1 Find an optimal solution of the following convex separable programming problem :

Max. $z = 3x_1 + 2x_2$

Subject to $4x_1^2 + x_2^2 \le 16$

and $x_i,\ x_2 \ge 0$

Solution :

Step I

the objective function in maximization form and $b_i,\ i=1$ is non negative. The objective function is linear so it can be assumed as concave function, the constraint is convex function so the set of feasible solutions is a convex set. Therefore the given problem is a convex separable programming problem, so any local maximum of this problem will be global maximum.

Here, separable functions are

$f_1(x_1) = 3x,$ $\quad f_2(x_2) = 2x_2$ are linear and

$g_{11}(x_1) = 4x_1^2,$ $\quad g_{12}(x_2) = x_2^2$ are non linear

we have to approximate $g_{11}(x_1)$ and $g_{12}(x_2)$

Step II

From the constraint, we observe that $0 \le x_2 \le 4$ and $0 \le x_1 \le 2$ (taking the positive sign)

Subdivide $0 \le x_1 \le 2$ by grid points $x_{10} = 0,\ x_{11} = 1,\ x_{12} = 2$ and $0 \le x_2 \le 4$ by grid points

$x_{20} = 0,\ x_{21} = 1,\ x_{22} = 2,\ x_{23} = 4,\ x_{24} = 4$

Now, the grid points & values of the functions are :

x_1	$g_{11}(x_1) = 4x_1^2$	x_2	$g_{12}(x_2) = x_2^2$
0	0	0	0
1	4	1	1
2	16	2	4
		3	9
		4	16

Linear approximations are :

$x_1 \cong 0\lambda_{10} + 1\lambda_{11} + 2\lambda_{11} + 2\lambda_{12}$

$$x_2 \cong 0\lambda_{20} + 1\lambda_{21} + 2\lambda_{22} + 3\lambda_{23} + 4\lambda_{24} = \lambda_{21} + 2\lambda_{22} + 3\lambda_{23} + 4\lambda_{24}$$

$$g_{11}(x_1) = 4x_1^2 \cong 0\lambda_{10} + 4\lambda_{11} + 16\lambda_{12} = 4\lambda_{11} + 16\lambda_{12}$$

$$g_{12}(x_2) = 4x_2^2 \cong 0\lambda_{20} + 1\lambda_{21} + 4\lambda_{22} + 9\lambda_{23} + 16\lambda_{24} = \lambda_{21} + 4\lambda_{22} + 9\lambda_{23} + 16\lambda_{24}$$

Where $\lambda_{10} + \lambda_{11} + \lambda_{12} = 1$ and $\lambda_{20} + \lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} = 1$

Now, approximating linear programming problem is :

Max. $z = 3\lambda_{11} + 6\lambda_{12} + 2\lambda_{21} + 4\lambda_{22} + 6\lambda_{23} + 8\lambda_{24}$

Such that $\quad 0\lambda_{10} + 4\lambda_{11} + 16\lambda_{12} + 0\lambda_{20} + \lambda_{21} + 4\lambda_{22} + 9\lambda_{23} + 16\lambda_{24}, \le 16$

$$\lambda_{10} + \lambda_{11} + \lambda_{12} = 1$$

$$\lambda_{20} + \lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} = 1$$

and $\lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{20}, \lambda_{21}, \lambda_{22}, \lambda_{23}, \lambda_{24} \ge 0$

With the restriction that not more than two of $\lambda_{10}, \lambda_{11}, \lambda_{12}$

and two of $\lambda_{20}, \lambda_{21}, \lambda_{22}, \lambda_{23}, \lambda_{24}$ are positive if two of them are positive then they correspond to adjacent points.

Now, add the slack varibales in first constraint and solve it by simplex method as given below.

Simplex Table -1

		c_j	0	3	6	0	2	4	6	8	0	Min.
c_B	x_B	b	λ_{10}	λ_{11}	λ_{12}	λ_{20}	λ_{21}	λ_{22}	λ_{23}	λ_{24}	s	Ratio
0	s	16	0	4	16	0	1	4	9	16	1	$\frac{16}{16} = 1$
0	λ_{10}	1	1	1	1	0	0	0	0	0	0	–
0	λ_{20}	1	0	0	0	1	1	1	1	1	0	1
$z_j - c_j$			0	3	–6	0 ↓	–2	–4	–6	–8 ↑	0	

$\because \quad$ For most negative $z_j - c_j = -8$

$\therefore \quad \lambda_{24}$ enters the basis and by minimum ratio rule.

λ_{20} departs from the basis. We can drop this column of λ_{20}

(with zero cost) in the next simplex table.

Simplex Table -2

		c_j	0	3	6	2	4	6	8	0	Min.
c_B	x_B	b	λ_{10}	λ_{11}	λ_{12}	λ_{21}	λ_{22}	λ_{23}	λ_{24}	s	Ratio
0	s	0	0	4	16	–15	–12	–7	0	1	$\frac{16}{16}=1$
0	λ_{10}	1	1	1	1	0	0	0	0	0	$\frac{1}{1}=1$
0	λ_{24}	0	0	0	0	1	1	1	1	0	–
$z_j - c_j$			0	–3	–6	6	4	2	0	0	
				↑						↓	

$z_j - c_j$ most negative for λ_{12} but it can not enter the basis because its entry departs s and then λ_{12}, λ_{10} are not adjacent points so they can not remain in the basis by basis entry rule. Further, take most negative $z_j - c_j$ for λ_{11} which enters the basis as λ_{11}, λ_{10} are adjacent points.

Simplex Table -3

		c_j	0	3	6	2	4	6	8	0	Min.
c_B	x_B	b	λ_{10}	λ_{11}	λ_{12}	λ_{21}	λ_{22}	λ_{23}	λ_{24}	s	Ratio
3	λ_{11}	0	0	1	4	$\frac{-15}{4}$	–3	$\frac{-7}{4}$	0	$\frac{1}{4}$	–
0	λ_{10}	1	1	0	–3	$\frac{15}{4}$	3	$\frac{7}{4}$	0	$-\frac{1}{4}$	$\frac{4}{7}$
8	λ_{24}	1	0	0	0	1	1	1	1	0	$\frac{1}{1}$
$z_j - c_j$			0	0	6	$-\frac{21}{4}$	–5	$\frac{-13}{4}$	0	$\frac{3}{4}$	
			↓					↑			

Since λ_{21}, λ_{22} cannot enter the basis due to restricted basis entry rule, therefore λ_{23} enteres the basis and λ_{10} depasts from the basis, now λ_{23}, λ_{24} adjacent points, λ_{10} column can also be dropped in the next simplex table.

Simplex Table -4

		c_j	3	6	2	4	6	8	0	Min. Ratio
c_B	x_B	b	λ_{11}	λ_{12}	λ_{21}	λ_{22}	λ_{23}	λ_{24}	s	
3	λ_{11}	1	1	1	0	0	0	0	0	0
0	λ_{23}	$\frac{4}{7}$	0	$\frac{-12}{7}$	$\frac{15}{7}$	$\frac{12}{7}$	1	0	0	$\frac{1}{7}$
8	λ_{24}	$\frac{3}{7}$	0	$\frac{12}{7}$	$\frac{8}{7}$	$-\frac{5}{7}$	0	1	$\frac{1}{7}$	
$z_j - c_j$			0	$\frac{3}{7}$	$\frac{12}{7}$	$\frac{4}{7}$	0	0	$\frac{9}{14}$	

Since all $z_j - c_j$ are non negative, therefor it is optimal stage so the approximate optimal solution is given by :

$$\lambda_{11} = 1,\ \lambda_{23} = \frac{4}{7},\ \lambda_{24} = \frac{3}{7}$$

Thus, $x_1 = \lambda_{11} + 2\lambda_{12} = 1 + 0 = 1$

$$x_2 = \lambda_{21} + 2\lambda_{22} + 3\lambda_{23} + 4\lambda_{24}$$

$$= 0 + 2\times 0 + 3\times\frac{4}{7} + 4\times\frac{3}{7} = \frac{24}{7}$$

and optimal value is : Max. $z = 3\times 1 + 2\times\frac{24}{7} = \frac{69}{7}$

Example 2 Solve the following convex separable programming problem :

Min. $z = x_1^2 - 2x_1 - x_2$

Such that $2x_1^2 + 3x_2^2 \le 6$

and $x_1,\ x_2 \ge 0$

Solution : The objective function of the given problem is in minimization form, so convert it into maximization from :

$$\text{Max. } (\hat{z}) = \text{Max. } (-z) = 2x_1 - x_1^2 + x_2$$

It is concave function as $-x_1^2$ is negative definite and in the constraint $2x_1^2 + 3x_2^2$ is convex as it is positive definite. So given problem is convex separable programming problem. Thus every relative maximum will be global maximum and every relative minimum will be global minimum.

Here $f_1(x) = 2x_1 - x_1^2$, $f_2(x_2) = x_2$

$g_{11}(x_1) = 2x_1^2$, $g_{12}(x_2) = 3x_2^2$

are separable functions.

Now, $2x_1^2 + 3x_2^2 \le 6 \Rightarrow 0 \le x_1 \le \sqrt{3}; 0 \le x_2 \le \sqrt{2}$

By taking $0 \le x_1 \le 2$ and $0 \le x_2 \le 2$, the grid points are :

$x_{10} = 0,\ x_{11} = 1,\ x_{12} = 2$ (say) and $x_{20} = 0,\ x_{21} = 1,\ x_{22} = 2$ (say).

Consider the following table :

x_1	$f_1(x_1) = 2x_1 - x_1^2$	$g_{11}(x_1) = 2x_1^2$	x_2	$g_{12}(x_2) = 3x_2^2$
0	0	0	0	0
1	1	2	1	3
2	0	8	2	12

The linear approximation of non linear functions are :

$x_1 \cong 0\lambda_{10} + 1\lambda_{11} + 2\lambda_{12}$; $x_2 \cong 0\lambda_{20} + 1\lambda_{21} + 2\lambda_{22}$

$f_1(x_1) \cong 0\lambda_{10} + 1\lambda_{11} + 0\lambda_{12}$

$g_{11}(x_1) \cong 0\lambda_{10} + 2\lambda_{11} + 8\lambda_{12}$; $g_{12}(x_2) \cong 0\lambda_{20} + 3\lambda_{21} + 12\lambda_{22}$

where $\lambda_{10} + \lambda_{11} + \lambda_{12} = 1$; $\lambda_{20} + \lambda_{21} + \lambda_{22} = 1$

Thus the approximating L.P.P. for the given problem is :

Max $(\hat{z}) = 0\lambda_{10} + 1\lambda_{11} + 0\lambda_{12} + 0\lambda_{20} + 1\lambda_{21} + 2\lambda_{12}$

subject to $0\lambda_{10} + 2\lambda_{11} + 8\lambda_{12} + 0\lambda_{20} + 3\lambda_{21} + 12\lambda_{22} \le 6$

$\lambda_{10} + \lambda_{11} + \lambda_{12} = 1$

$\lambda_{20} + \lambda_{21} + \lambda_{22} = 1$

and $\lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{20}, \lambda_{21}, \lambda_{22} \ge 0$ (with restricted basis entry rule)

After adding slack variables in the first constraint the first simplex table is as follow:

Simplex Table -1

		c_j	0	1	0	0	1	2	0	Min.
c_B	x_B	b	λ_{10}	λ_{11}	λ_{12}	λ_{20}	λ_{21}	λ_{22}	s	Ratio
0	s	6	0	2	8	0	3	12	1	$\frac{6}{2}=3$
0	λ_{10}	1	1	1	1	0	0	0	0	$\frac{1}{1}=1$
0	λ_{20}	1	0	0	0	1	1	1	0	–
$z_j - c_j$			0	–1	0	0	–1	–2	0	
			↓	↑						

Since $z_j - c_j$ is most negative for λ_{22} but it can not enter the basis by restricted basis entry rule. Now there is a tie for most negative $z_j - c_j$ so we consider nearest from the left i.e. $z_2 - c_2$ so λ_{11} enters the basis and λ_{10} departs from the basis (drop it in next table).

Simplex Table -2

		c_j	1	0	0	1	2	0	Min.
c_B	x_B	b	λ_{11}	λ_{12}	λ_{20}	λ_{21}	λ_{22}	s	Ratio
0	s	4	0	6	0	3	12	1	$\frac{4}{3}$
0	λ_{11}	1	1	1	0	0	0	0	–
0	λ_{20}	1	0	0	1	1	1	0	1
$z_j - c_j$			0	1	0	0	–2	0	
					↓	↑			

Since $z_j - c_j$ is most negative for λ_{22} it cannot enter the basis because λ_{22} & λ_{20} are not adjacant points so we consider λ_{21} as entering vector as λ_{11}, λ_{20} already in the basis (consider $z_j - c_j = 0$ from left).

Simplex Table -3

		c_j	1	0	1	2	0	Min.
c_B	x_B	b	λ_{11}	λ_{12}	λ_{21}	λ_{22}	s	Ratio
0	s	1	0	6	0	9	1	$\frac{1}{9}$
1	λ_{11}	1	1	1	0	0	0	–
1	λ_{21}	1	0	0	1	1	0	1
$z_j - c_j$			0	1	0	–1	0	
						↑	↓	

Simplex Table -4

		c_j	1	0	1	2	0	Min. Ratio
c_B	x_B	b	λ_{11}	λ_{12}	λ_{21}	λ_{22}	s	
2	λ_{22}	$\frac{1}{9}$	0	$\frac{2}{3}$	0	1	$\frac{1}{9}$	
1	λ_{11}	1	1	1	0	0	0	
1	λ_{21}	$\frac{8}{9}$	0	$-\frac{2}{3}$	1	0	$-\frac{1}{9}$	
$z_j - c_j$			0	$\frac{5}{3}$	0	0	$\frac{1}{9}$	

$\because$ All $z_j - c_j \geq 0$, therefore at the optimal level the optimal

solution is : $\lambda_{11} = 1,\ \lambda_{21} = \frac{8}{9},\ \lambda_{22} = \frac{1}{9}$

$$\therefore \quad x_1 = \lambda_{11} + 2\lambda_{12} = 1 + 2 \times 0 = 1$$

$$x_2 = \lambda_{21} + 2\lambda_{22} = \frac{8}{9} + \frac{2}{9} = \frac{10}{9}$$

Min. $z = 1 - 2 - \frac{10}{9} = -\frac{19}{9}$

10.8 Summary

In this unit we have studied about the following :

Objective, Introduction, Definitions of separable function, convex programming problem, separable programming problem and convex separable programming problem (CSPP), some important Theorems, Approximate optimal solution of CSPP, Piecewise linear approximation of non-linear continuous function, Separable programming algorithm.

10.9 Exercise

Solve the following convex separable programming problems :

1. Max. $z = x_1 + x_2^4$

   Subject to $3x_1 + 2x_2^2 \leq 9$

   and $x_1, x_2 \geq 0$ $\left(x_1 = 0, x_2 = 2.1, max\, z = 19.45\right)$

2. Max. $z = 2x_1 - x_1^2 + x_2$

   Such that $2x_1 + 3x_2 \leq 6$

   $2x_1 + x_2 \leq 4$

	and	$x_1, x_2 \geq 0$	$\left(x_1 = \frac{2}{3}, x_2 = \frac{14}{9}, max\, z = \frac{22}{9}\right)$
3.	Min.	$z = x_1^2 - 4x_1 + x_2^2 - 2x_3$	
	S.t.	$x_1 + x_2 + x_3 \leq 2$	
		$(x_1 + 1)x_2 \geq 2$	
	and	$x_1, x_2, x_3 \geq 0$	$(x_1 = 1, x_2 = 1, x_3 = 0, min\, z = -2)$
4.	Min.	$z = x_1^2 - 8x_1 + x_2^2 - 10x_2$	
	Subject to	$3x_1 + 2x_2 \leq 6$	
	and	$x_1, x_2 \geq 0$	$\left(x_1 = \frac{4}{13}, x_2 = \frac{33}{13}, min\, z = -\frac{267}{13}\right)$
5.	Max.	$z = (x_1 - 2)^2 + (x_2 - 2)^2$	
	Such that	$x_1 + 2x_2 \leq 4$	
	and	$x_1, x_2 \geq 0$	$(x_1 = 1.6, x_2 = 1.2, max\, z = 0.8)$

□□□

Unit - 11
Dynamic Programming; Bellman's Optimality Principle

Structure of The Unit

11.0 Objective

In most operations research problems the objective is to find the optimal (max. or min.) values of the "decision variables", that is, those variables that can change or be controlled within the problem structure. We come across a number of situations where the decision variables vary with time, and these situations are considered to be dynamic in nature. The teachnique dealing with these types of problem is called **"dynamic programming"**. It will be shown in this unit that time element is not an essential variable rather any multistage situation in which a series of decisions are to be made is considered a dynamic programming problem.

11.1 Introduction

Dynamuic programming is a mathematical technique dealing with the optimization of multistage decision problems. The founding father of dynamic programming, and the man primarily responsible for the development of dynamic programming, is Rechard Bellman. Bellman first developed the concept of dynamic programming in the late 1940s and early 1950s while working as a researcher at the Rand Corporation. By this teachnique decisions regarding a certain problem are typically optimized stages rather than simultaneously. The original problem is broken into subproblems (stages), which can then be solved more efficiently from the computational view point. The optimal solution is attained in an alternative manner starting from first stage to the next and is completed when the final stage is reached. Individually, each decision of the stage may not be optimal but sacrifice at one stage may result in greater gains at some other stage. The technique of dynamic programming aims at optimizing the decision for the situation as a whole, and the decision for the stage may be sub-optimal. So far there is no standard mathematical formulation of a dynamic programming problem but it is often possible to introduce the multi stage nature in the problem so that dynamic programming may be used.

11.2 Basic Features of a Dynamic Programming Problem

1. In dynamic programming problems, decisions regarding a certain problem are typically optimized at subsequent stages rather then simultaneously; i.e. if a programm is to be solved by using dynamic programming, it must be separated into N sub problem.

2. Dynamic programming deals with problems in which choices, or decisions, are to be made at each stage. the set of all possible choices is reflected and/or governed by the state of each stage.

3. There is a return function at every stage that evaluates the choice made at each decision in

terms of the contribution that the decision can make to overall objective (maximization or minimization)

4. Each stage N, the total decision process is related to its adjoining stages by a quantitative relationship called a **transition function**. This transition function can either reflect discrete quantities or continuous quantities depending on the nature of the problem.

5. Given the current state, an optimal policy for the remaining stages in terms of a possible input state is independent of the policy adopted in previous stages.

6. The solution procedure always proceed by finding the optimal policy for each possible input state at the present stage.

7. A recursive relationship is always used to relate the optimal policy at stage n to the (n-1) stages that follow.

8. By using this recursive relation, the solution procedure moves from stage to stage...each time finding an optimal policy for each state at the stage...until the optimal policy for the last stage is found.

11.3 Bellman's Principle of Optimality

The basic concept of the dynamic programming is contained in **Bellman's Principle of Optimality** which says that "An optimal policy (a sequence of decisions) has the property that whatever the initial state and decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision." This principle implies that a wrong decision taken at one stage does not prevent from taking of optimum decisions for the remaining stages.

Mathematically this principle can be written as:

$$f_n(x) = optimum[r(d_n) \otimes f_{n-1}\{T(x \otimes d_n)\}]$$

$$d_n \in \{x\}$$

Where, symbol denotes any mathematical relationship between x and d_n, including addition, subtraction, multiplication, and root operations.

$f_n(x)$: the optimal return from an n-stage process when initial state is x.

$\{x\}$: set of all admissible decisions.

$r(d_n)$: immediate return due to decision d_n.

$T(x \otimes d_n)$: the transfer function which gives the resulting state.

Thus in the light of Bellman's optimality principle we can write a recursive or recurrence relation which enables us to obtain the optimal decision at each state.

11.4 Solution Procedure

We can solve a multistage problem by using dynamic programming as given below:

Step-I

Write the recursive relation connecting the optimum decision function for the n-stage problem with the optimum decision function for $(n-1)$-stage sub problem or to write the Bellman's principle of optimality for the multistage problem.

Step-II

Write the relation giving optimal decision function for one stage and solve it, then further, solve the optimal decisin function for 2, 3, 4, ..., $(n-1)$ stage sub problem successively and finally for n-stage problem.

Note:- (i) "Stage" means point or level at which a decision is made or a device to sequence the decisions.

(ii) "State" means a set of variables at a stage.

(iii) Dynamic programming solves those problems which satisfy Bellman's optimality principle.

(iv) Number of variables in a problem = Number of stages.

(v) Number of constraints in a problem = Number of state parameters in each stage.

11.5 Illustrative Examples

Example-I : Use Bellman's optimality principle to divide a positive quantity 'b' into n parts in such a way that their product is maximum.

Or

Find maximum value of the product of $x_1, x_2, ..., x_n$

When $x_1 + x_2 + ... + x_n = b, x_1, x_2, ..., x_n \geq, 0$, using dynamic programming.

Solution : The problem has n variables and one constraint so we can consider it as n-stage problem with one state parameter at each stage.

Suppose $f_r(b)$ denotes maximum attainable product when the quantity 'b' is divided into r parts; then we have

$$f_r(b) = \underset{x_1, x_2, x_r}{\text{Max.}} \quad (x_1 . x_2 . \ .... \ x_r), \ r = 1, 2, ..., n$$

Subject to $x_1 + x_2 + ... + x_r = b, x_r \geq, 0$

By Bellman's principle of optimality, we have

$$f_r(b) = \underset{x_r}{\text{max.}} \quad [x_r . \underset{x_1, ..., n_{r-1}}{\text{max.}} \quad (x_1 . x_2 . \ .... \ x_{r-1})$$

$$= \underset{x_r}{\text{max.}} \ [x_r . f_{r-1}(b - x_r)]$$

$$= \underset{0 \leq z \leq b}{\text{max.}} \quad [z . f_{r-1}(b - z)] \text{ if } x_r = \text{z to be decision variable.}$$

Now, **Stage-1** For $r = 1$, we get

$f_1(b) = b$ only one part

and optimal policy is: $z = b$

Stage-2 For $r = 2$. we get

$$f_2(b) = \text{max.} \quad [z . f_1(b - z)] = \text{max.} \quad [z(b - z)], \ \therefore f_1(b) = b$$

$$0 \le z \le b \qquad\qquad 0 \le z \le b$$

Now, by using differential calculus, we have

$$\frac{d}{dz}(f_2(b)) = 0 \Rightarrow z = \frac{b}{2} \, and \, \frac{d^2}{dz^2}(f_2(b)) = -2, at z = \frac{b}{2}$$

Therefore $f_2(b)$ in maximum for $z = \frac{b}{2}$.

So optimal policy for $r = 2$ is $\left(\frac{b}{2}, \frac{b}{2}\right)$ and $f_2(b) = \frac{b}{2} . \frac{b}{2} = \left(\frac{b}{2}\right)^2$

For **Stage-3** For $r = 3$, we have

$$f_3(b) = \max. \qquad [z . f_2(b - z)] = \max. \qquad [z\left(\frac{b-z}{2}\right)^2]$$

$$0 \le z \le b \qquad\qquad 0 \le z \le b$$

Now, $\frac{d}{dz}(f_3(b)) = 0 \Rightarrow z = \frac{b}{3} \, and \, \frac{d^2}{dz^2}(f_3(b)) < 0$ at $z = \frac{b}{3}$

i.e. $f_3(b)$ is maximum at $z = \frac{b}{3}$

So optimal policy is : $(\frac{b}{3}, \frac{b}{3}, \frac{b}{3})$ and $f_3(b) = (\frac{b}{3})^3$

By using mathematical induction suppose the result (policy) is true for stage m i.e. $r=m$,

$f_m(b) = (\frac{b}{m})^m$ and optimal policy is $\left(\frac{b}{\mathrm{m}}, \frac{b}{\mathrm{m}}, ... \frac{b}{\mathrm{m}}\right)$

Now, by recurrence relation, we have

$$f_{m+1}(b) = \max. \; [z . f_m(b - z)] = \max. \; z \; (\frac{b-z}{m})^m$$

$$0 \le z \le b \qquad\qquad 0 \le z \le b$$

Now, $\frac{d}{dz} f_{m+1}(b) = 0 \Rightarrow z = \frac{b}{m+1}$ and $\left[\frac{d^2 z}{dz^2} f_{m+1}(b)\right]$ at $z = \frac{b}{m+1}$ is negative

$\therefore f_{m+1}(b)$ is maximum for $z = \dfrac{b}{m+1}$ and optimal policy is $\left(\dfrac{b}{\mathrm{m}+1}, \dfrac{b}{\mathrm{m}+1}, \ldots \dfrac{b}{\mathrm{m}+1}\right)$,

$$f_{m+1}(b) = \left(\frac{b}{m+1}\right)^{m+1}$$

Hence, the result is true for $r = n$ and optimal policy is $\left(\frac{b}{n}, \frac{b}{n}, \ldots, \frac{b}{n}\right)$ and $f_n(b) = \left(\frac{b}{n}\right)^n$

i.e. optimal policy for given problem is:

$x_1 = x_2 = \ldots = x_n = \dfrac{b}{n}$ and optimum value of the objective function = $\left(\dfrac{b}{n}\right)^n$.

Example-2 Make use of dynamic programming to show that

$\sum_{i=1}^{n} \mathrm{p}_i \log \mathrm{p}_i$ subject to $\sum_{i=1}^{n} \mathrm{p}_i = 1, \mathrm{p}_i >, 0$ is minium, when $p_1 = p_2 = \ldots = p_n = \dfrac{1}{n}$ (i in suffix)

Solution : We can consider the problem as an n-stage problem in which 1 can be divided into r parts as r^{th} stage, $r = 1, 2, \ldots, n$.

Suppose $f_r(1) = \min. \sum_{i=1}^{r} \mathrm{p}_i \log \mathrm{p}_i$ when $\sum_{i=1}^{r} \mathrm{p}_i = 1$ and $p_i. \geq, 0, r = 1, n$.

Let z be current decision variable.

Stage-1 For $r = 1$, we get

$f_1(1) = \min. (\mathrm{p}_1 \log \mathrm{p}_1)$, where $\mathrm{p}_1 = 1$

$= 1 \log 1$

i.e. optimal policy for $r = 1$ is 1 and $f_1(1) = 1 \log 1$

State-2 For $r = 2$, we have

$f_1(1) = \min. [\mathrm{p}_1 \log \mathrm{p}_1 + \mathrm{p}_2 \log \mathrm{p}_2]$, where $\mathrm{p}_1 + \mathrm{p}_2 = 1, \mathrm{p}_1, \mathrm{p}_2 \geq, 0$

Here, 1 is divided into two parts. If first part is $\mathrm{p}_1 = z$ (say) then second part is $(1 - z)$.

By Bellman's principle the recurrence relation is:

$$f_r(1) = \min_{0 \leq z \leq 1}. [z \log z + f_{r-1}(1 - z)]$$

For $r = 2$, we get

$$f_2(1) = \min_{0 \leq z \leq 1}. [z \log z + f_1(1 - z)] = \min_{0 \leq z \leq 1}. [z \log z + (1 - z) \log (1 - z)].$$

let S = $z \log z + (1 - z) \log (1 - \mathrm{z})$, then

$$\frac{ds}{dz} = 0 \Rightarrow 1 + \log z - 1 - \log (1 - z) = 0$$

$$\Rightarrow \log\frac{z}{1-z} = 0 \Rightarrow \frac{z}{1-z} = e^0 = 1 \Rightarrow z = \frac{1}{2}$$

and $\frac{d^2s}{dz^2} = \frac{1}{z} + \frac{1}{1-z}$ and $\left[\frac{d^2s}{dz}\right]_{z=\frac{1}{2}} = 4$ (positive)

Thus S is minimum at $z = \frac{1}{2}$

i.e. optimal policy for $r = 2$ is $(\frac{1}{2}, \frac{1}{2})$ and $f_2(1) = 2\,[\frac{1}{2}\log\frac{1}{2}]$

For r = 3, we have

$f_3(1) = \min\limits_{0 \le z \le 1}.\ [z\log z + f_2(1 - z)]$

$= \quad \text{min. } 0 \le z \le 1 \left[z\log z + 2\left(\frac{1-z}{2}\right)\log\left(\frac{1-z}{2}\right)\right] = \min\limits_{0 \le z \le 1}.\ S$ (say).

Then $\frac{ds}{dz} = 0 \Rightarrow z = \frac{1}{3}$ and $\frac{d^2s}{d_z^2} = \frac{9}{2}$ (positive).

Thus S is miximum or $f_3(1)$ is miximum at $z = \frac{1}{3}$.

Optimal policy is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $f_3(1) = \frac{1}{3}\log\frac{1}{3}$

Let us assume that policy for $r = m$ is

$(\frac{1}{m}, \frac{1}{m}, ..., \frac{1}{m})$ and $f_m(1) = [\frac{1}{m}\log\frac{1}{m}]m$

$f_{m+1}(1) = \min\limits_{0 \le z \le 1}.\ [z\log z + f_m(1 - z)]$, by recurrence relation.

$$= \min_{0 \le z \le 1}.\ [z\log z + m(\frac{1-z}{m}\log\frac{1-z}{m})] = \min_{0 \le z \le 1}.\ (S), \text{ say}$$

Then $\frac{ds}{dz} = 0 \Rightarrow 1 + \log z + m\left[-\frac{1}{m} - \frac{1}{m}\log\left(\frac{1-z}{m}\right)\right] = 0 \Rightarrow z = \frac{1}{m+1}$

and $\frac{d^2s}{dz^2} = \frac{1}{z} + \frac{1}{1-z} = \frac{(m+1)^2}{m}$ at $z = \frac{1}{m+1}$ (positive)

$\therefore$ S is miximum at $z = \frac{1}{m+1}$

$\therefore$ Optimal policy for $r = m + 1$ is $\left(\frac{1}{m+1}, \frac{1}{m+1}, \ldots, \frac{1}{m+1}\right)$

and $f_{m+1}(1) = (m+1)\left[(\frac{1}{m+1})\log(\frac{1}{m+1})\right]$

so by mathematical induction the policy for $r = n$ is $\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$ and optimal value is $f_1(1) = n.\frac{1}{n}\log\frac{1}{n}$

Hence, $\sum_{i=1}^{n}$ Pi log Pi is miximum subject to $\sum_{i=1}^{n}$ Pi = 1, Pi$\geq$, 0, $r = 1, n$.

When $P_1 = P_2 = \ldots = P_n = \frac{1}{n}$ and optimal value is log $\frac{1}{n}$ =-log n.

Example-3 Use synamic programming to solve the following problem.

min. $(x_1^2 + x_2^2 + \ldots + x_n^2)$

Subject to $x_1 . x_2 . \ldots . x_n = b$

and $x_1, x_2, \ldots, x_n \geq 0$

Solution : There are n variables and one constraint in the problem so the problem can be considered as an n-stage problem with one state parameter in each stage. The number 'r' parts in which b is factorised may be treated as r^{th} stage.

Suppose $f_r(b) = \min_{x_1,\ldots,x_r} \sum_{i=1}^{r} x_i^2$, $r = 1, 2, \ldots, n$.

subject to $x_1, x_2, \ldots . x_r = b$

and $x_1, x_2, \ldots, x_r \geq 0$.

By using Bellman's principle of optimality the recurrence relation is:

$$f_r(b) = \min_{xr} . [x_r^2 + \min_{x_1,\ldots,x_{r-1}} . \sum_{i=1}^{r} x_i^2] = \min_{0 \leq x_r \leq b} . [x_r^2 + f_{r\text{-}1}(\frac{b}{x_r})] \quad \ldots.(1)$$

Let $x_r = z$ be current decision variable, then

$$f_r(b) = \min_{0 \leq z \leq b} . \left[z^2 + f_{r-1}(\frac{b}{z})\right] \quad \ldots.(2)$$

For $r = 1; f_1(b) = \min. z^2$ where $z = b, z \geq 0$.

$\therefore$ optimal policy is $z = b$ and optimal value is $f_1(b) = b^2$

For $r = 2; f_2(b) = \min_{0 \leq z \leq b} . \left[z^2 + f_1(\frac{b}{z})\right] = \min_{0 \leq z \leq b} . \left[z^2 + (\frac{b}{z})^2\right]$, $\therefore f_1(b) = b^2$

Let S = $z^2 + (\frac{b}{z})^2$, then

$$\frac{ds}{dz} = 0 \Rightarrow 2z + 2(\frac{b}{z})(-\frac{b}{z^2}) = 0 \Rightarrow z - \frac{b^2}{z^3} = 0 \Rightarrow z^4 = b^2 \Rightarrow z = b^{½}$$

and $\frac{d^2s}{dz^2} = 1 + \frac{3b^2}{z^4} = 4$ at $z = b^{½}$ (positive)

$\therefore$ S is minimum, so $f_2(b)$ is minimum at $z = b^{½}$

Hence, optimal policy for $r = 2$ is ($b^{½}$, $b^{½}$) and optimum value is $f_2(b) = 2b$

For $n = 3, f_3(b) = \min_{0 \le z \le b}. \left[z^2 + f_2(\frac{b}{z}) \right] = \min_{0 \le z \le b}. \left[z^2 + 2.\frac{b}{z} \right]$

Let S = $z^2 + \frac{2b}{z}$, then $\frac{ds}{dz} = 0 \Rightarrow 2z - \frac{2b}{z^2} = 0 \Rightarrow z = b^{1/3}$

and $\frac{d^2s}{dz^2} = 1 + \frac{2b}{z^3} = 3$ (positive) at $z = b^{1/3}$

$\therefore$ S is minimum i.e. $f_3(b)$ is minimum at $z = b^{1/3}$

Hence, optimal policy is $(b^{1/3}, b^{1/3}, b^{1/3})$ and $f_3(b) = 3b^{2/3}$

We assume that the optimal policy for $r = m$ is

$(b^{1/m}, b^{1/m}, ..., b^{1/m})$ and $f_m(b) = mb^{1/m}$

Now $f_{m+1}(b) = \min_{0 \le z \le b}. [z^2 + f_m(\frac{b}{z})] = \min_{0 \le z \le b}. [z^2 + m(\frac{b}{z})^{2/m}]$

Let S = $z^2 + m(\frac{b}{z})^{2/m}$, than $\frac{ds}{dz} = 0 \Rightarrow 2z - \frac{2b^{2/m}}{z^{2/m+1}} = 0$ -

$\therefore z = b^{1/m+1}$

and $\frac{d^2s}{dz^2} = 1 + \frac{b^{2/m}(\frac{2}{m}+1)}{z^{2/m+2}} = 1 + \frac{b^{2/m}(\frac{2+m}{m})}{b^{2/m}} = \frac{2(m+1)}{m}$ (positive) at $z = b^{1/m+1}$

$\therefore$ S is minimum i.e. $f_{m+1}(b)$ is minimum at $z = b^{1/m+1}$

Thus, optimal policy is $(b^{1/m+1}, b^{1/m+1}, ..., b^{1/m+1})$ and $f_{m+1}(b) = (m+1)(b)^{2/m+1}$

Hence by law of mathematial induction optimal policy for $r = n$ is $(b^{1/n}, b^{1/n}, ..., b^{1/n})$ and optimal value $f_n(b) = n.b^{2/n}$

11.6 Summary

This unit partains to introduce the

Basic Features of a Dynamic Programming Problem , Belman's Principle of optimality, Solution Procedure.

11.7 Exercises

Solve the following problems by using dynamic programming:

1. Min. $\sum_{i=1}^{n} x_i^2$ subject to $\sum_{i=1}^{n} x_i = b, x_i \geq 0, i = 1, 2, ..., n$

   Hence or otherwise minimize $x_1^2 + x_2^2 + x_3^2$

   subject to $x_1 + x_2 + x_3 \geq 15$

   and $x_1, x_2, x_3 \geq 0$ (Optimal policy $\left(\frac{b}{n}, \frac{b}{n}, ..., \frac{b}{n}\right)$ and $f_n(b) = n\left(\frac{b}{n}\right)^2$ and $f_3(15) = 75$ at $x_1 = x_2 = x_3 = 5$

2. Min. $z = \sum_{i=1}^{n} x_i$

   subject to $\prod_{i=1}^{n} x_i = b$

   and $x_i \geq o, i = 1, 2, ..., n$ (Optimal policy $\left(b^{1/n}, b^{1/n}, ..., b^{1/n}\right)$ and $f_n(\text{b}) = nb^{1/n}$

3. $-\sum_{i=1}^{n} p_i \log p_i$ subject to $\sum_{i=1}^{n} p_i = 1$ is maximum when $p_1 = p_2 = ... = p_n = \frac{1}{n}$

   ($f_n(1) = \log n$ and optimal policy $\left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)$

4. Maximize $z = c_1 x_1 + c_2 x_2 + ... + c_n x_n, c_1 < c_2 < ... < c_n$

   subject to $x_1 + x_2 + ... + x_n = b$

   and $x_1, x_2, ..., x_n \geq 0$ (Optimal plolicy $(0, 0, ... b)$ and $f_n(b) = c_n b$

5. Maximize value of $y_1 y_2 y_3$, subject to $y_1 + y_2 + y_3 \leq 15$ and $y_1, y_2, y_3 \geq 0$.

   (Optimal policy (5, 5, 5) and $f_3(b)_2\ 125 = \max. (y_1, y_2, y_3)$)

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Unit - 12
Solution of Linear Programming Problem Using Dynamic Programming

Structure of the Unit

12.0 Objective

There are several applications of Dynamic programming. Discrete and continuous, deterministic as well as probabilistic Problems can be solved by Dynamic Programming. Thus dynamic programming method is very useful to solving various problems, such as inventory, replacement allocation, linear programming, reliability improvement problem, capital Budgeting problem, cargo loading problem etc.

12.1 Introduction

The dynamic programming can be applied to many real life situations. Many real life problems can be formulated as linear programming problems. We shall study how a linear programming problem can be solved by dynamic programming. Thus we can formulted as a multi-stage decision problem and then can be solved using Bellman's principle of optimality.

12.2 Solution of Linear Programming Problem using Dynamic Programming

Let us consider the following L.P.P.

Max $z = \sum_{j=1}^{n} a_{ij} x_j = b_i$, $i = 1, 2, \ldots\ldots, m$

and $x_j \geq 0$, $j = 1, 2, \ldots., n$

This L.P.P has n varibles with m constraints so it can be expressed as an n-stage problem with m state parameters at each stage.

Suppose $\beta_1^k, \beta_2^k, \ldots.., \beta_m^k$ be state parameters and $f_k\left(\beta_1^k, \beta_2^k, \ldots, \beta_m^k\right)$ be the state function at stage $K, K = 1, 2, \ldots, n$. Now the state function can be defined as :

$$f_k\left(\beta_1^k, \beta_2^k, \ldots, \beta_m^k\right) = \underset{x_1, \ldots, x_k}{Max} \sum_{j=1}^{K} c_j x_j , k = 1, 2, \ldots n$$

Subject to $\sum_{j=1}^{K} a_{ij} x_j = \beta_i^K , i = 1, 2, \ldots., m$

and $x_j \geq 0$, $j = 1,2, \ldots, n$

Then by Bellman's principle of optimality, the recurrence relation is given by

$$f_k\left(\beta_1^k,\beta_2^k,....,\beta_m^k\right)=\underset{x_k}{Max.}\left[c_k x_k+f_{k-1}\left(\beta_1^k-a_{1k}x_k,....,\beta_m^k-a_{mk}x_k\right)\right]$$

We can determine $\overset{*}{x}_k$ (optimal value of x_k) at the stage $k, k=\overline{1,n}$. Which yields $f_k\left(\beta_1^k,\beta_2^k,...\beta_m^k\right)$. Thus at the n^{th} stage optimal value of x_n i.e. $\overset{*}{x}_n$ is determined.

Hnece the L.P.P. can be formulated as n-stage decision problem and then it can be solved by dynamic programming.

12.3 Illustrative Examples

Example-1 Use dynamic programming to solve the following L.P.P. :

Max $z=2x_1+5x_2$

Such that $2x_1+x_2\leq 43$

$2x_2\leq 46$

and $x_1,x_2\geq 0$

Solution : The given L.P.P. has 2 variables with two constraints, so it can be considered as 2-stage problem with two state parameters at each stage.

Let β_1^k and β_2^k be two state paramaters and

$f_k\left(\beta_1^k,\beta_2^k\right)$ be state function at stage $k, k=1,2$. The given L.P.P. can be written as the 2-stage problem as given by

$$f_k\left(\beta_1^k,\beta_2^k\right)=\underset{x_1,x_2}{Max.}\sum_{j=1}^{k}c_j x_j, k=1,2$$

Such that $\sum_{j=1}^{k}c_j x_j\leq\beta_i^k, i=1,2$

and $x_j\geq 0, j=1,2$

The recurrence relation by Bellman's principle is :

$$f_k\left(\beta_1^k,\beta_2^k\right)=\underset{x_k}{Max.}\left[c_k x_k+f_{k-1}\left(\beta_1^k-a_{1k}x_k,\beta_2^k-a_{2k}x_k\right)\right]$$

on replacing β_1^k and β_2^k by u_k,v_k (for simplicity), we get.

For stage k = 1; $f_1(u_1,v_1)=\underset{x_1}{Max.}(2x_1)$

Such that $2x_1\leq u_1$

$0\leq v_1$

i.e. $f_1(u_1, v_1) = \underset{x_1}{Max}.(2x_1)$

Such that $x_1 \leq \frac{u_1}{2}, x_1 \geq 0 \Rightarrow \quad 0 \leq x_1 \leq \frac{u_1}{2}$, where $v_1 \geq 0$

$\therefore \; x_1^* = \frac{u_1}{2}$ and $f_1(u_1, v_1) = 2.\frac{u_1}{2} = u_1$

For stage $k = 2$, we have

$f_2(u_2, v_2) = \underset{x_1, x_2}{Max}.[2x_1 + 5x_2]$; such that $2x_2 \leq v_2, 2x_1 + x_2 \leq u_2, x_1, x_2 \geq 0$

$$\therefore \; f_2(u_2, v_2) = \underset{x_2}{Max}.\left[5x_2 + \underset{x_1}{Max}.(2x_1)\right] = \underset{x_2}{Max}.[5x_2 + f_1(u_1, v_1)]$$

$$= \underset{x_2}{Max}.[5x_2 + f_1(u_2 - x_2, v_2 - 2x_2)]$$

$$= \underset{x_2}{Max}.[5x_2 + (u_2 - x_2)] = \underset{x_2}{Max}.[4x_2 + u_2]$$

Where, $x_1 \geq 0, (v_2 - 2x_2) \geq 0, \; 0 \leq \frac{u_2 - x_2}{2}$

$$\therefore \quad 0 \leq x_2 \leq \min.\left(u_2, \frac{v_2}{2}\right)$$

i.e. $\quad 0 \leq x_2 \leq \min.\left(43, \frac{46}{2}\right)$ at $u_2 = 43$, $v_2 = 46 \backslash$

i.e. $\quad 0 \leq x_2 \leq 23 \; \Rightarrow x_2^* = 23$

Now, $\because 2x_1 + x_2 \leq 43 \Rightarrow u_1 + x_2 \leq u_2 \Rightarrow u_1 = u_2 - x_2 \Rightarrow u_1 = 43 - 23 = 20$

$$\therefore \; x_1^* = \frac{u_1}{2} = 10$$

Thus optimal solution is $x_1 = 10$ and $x_2 = 23$ with optimal value $Max \; z = 135$.

Example-2 Solve the following L.P.P by using dynamic programming :

Max $z = 3x_1 + 5x_2$

subject to $x_1 \leq 4$

$x_2 \leq 6$

$3x_1 + 2x_2 \leq 18$

and $x_1, x_2 \geq 0$

Solution : The given L.P.P. has 2 variables and 3 constraints so it can be expressed as a 2-stage problem with 3-state prarameters at each stage. Suppose ((u_1, v_1, w_1) and (u_2, v_2, w_2) be sate parameters at each stage.

Then the subproblems are :

$$f_1(u_1, v_1, w_1) = Max.(3x_1)$$

Subject to $x_1 \le u_1$

$0x_1 \le v_1$ For stage 1

$3x_1 \le w_1$

and $x_1 \ge 0$

and $f_2(u_2, v_2, w_2) = \underset{x_1, x_2}{Max.}[5x_2 + 3x_1]$

subject to $x_1 + 0x_2 \le u_2$

$0x_1 + x_2 \le v_2$ For stage 2

$3x_1 + 2x_2 \le w_2$

and $x_1, x_2 \ge 0$, where $u_2 = 4, v_2 = 6$ and $w_2 = 18$

Now for stage-1, we get

$$f_1(u_1, v_1, w_1) = Max.(3x_1), \text{ where } v_1 \ge 0,\ 0 \le x_1 \le min.\left(u_1, \frac{w_1}{3}\right)$$

$$3\ Min.\left\{u_1, \frac{w_1}{3}\right\} \text{ at } x_1^* = Min.\left(u_1, \frac{w_1}{3}\right).$$

For stage-2, we have

$$f_2(u_2, v_2, w_2) = \underset{x_1, x_2}{Max.}[5x_2 + 3x_1] = \underset{x_2}{Max.}\left[5x_1 + \underset{x_1}{Max.}(3x_1)\right]$$

$$= \underset{x_2}{Max.}[5x_2 + f_1(u_1, v_1, w_1)]$$

$$= \underset{x_2}{Max.}[5x_2 + f_1(u_1 - 0x_2, v_2 - x_2, w_2 - 2x_2)]$$

$$= \underset{x_2}{Max.}\left[5x_2 + 3\min.\left\{u_2, \frac{w_2 - 2x_2}{3}\right\}\right]$$

$$= \underset{x_2}{Max.}\left[5x_2 + 3\min.\left\{4, \frac{18 - 2x_2}{3}\right\}\right]$$

Thus, $Max.z = f_2(4,6,18) = \underset{x_2}{Max.}\left[5x_2 + 3\,Min.\left\{4, \frac{18-2x_2}{3}\right\}\right]$

Where $x_2 \geq 0,\ v_2 - x_2 \geq 0 \Rightarrow 0 \leq x_2 \leq v_2 = 6$

Now, $Min.\left\{4, \frac{18-2x_2}{3}\right\} = \{4, if\ 0 \leq x_2 \leq 3$

$\frac{18-2x_2}{3}, if\ 3 < x_2 \leq 6$

$= 2$; at $x_2 = 6$

$\therefore\ x_2^* = 6$ and $Max.z = 5x_2 + 6 = 5\times 6 + 6 = 36$

Now, $x_1^* = Min.\left\{u_1, \frac{w_1}{3}\right\} = Min.\left[u_2, \frac{w_2 - 2x_2}{3}\right]$

$= Min.\left[4, \frac{18-12}{3}\right] = 2$

Hence, optimal solution is $x_1 = 2, x_2 = 6$ and optimum value $Max.z = 36$

Example-3 Solve by dynamic programming :

$Max.z = 8x_1 + 7x_2$

Subject to $2x_1 + x_2 \leq 8$

$2x_1 + 2x_2 \leq 15$

and $x_1, x_2 \geq 0$

Solution : Hint : $f_1(u_1, v_1) = \underset{x_1}{Max.}(8x_1), \left[where\ \ x_1 \geq 0, x_1 \leq \frac{u_1}{2}, x_1 \leq \frac{v_1}{2}\right]$

i.e. $0 \leq x_1 \leq Min.\left(\frac{u_1}{2}, \frac{v_1}{2}\right)$

$= 8\ Min.\left[\frac{u_1}{2}, \frac{v_1}{2}\right]$

$\therefore\ x_1^* = 0$

$f_2(u_2, v_2) = \underset{x_2}{Max.}\left[7x_2 + \underset{x_1}{Max}(8x_1)\right] = \underset{x_2}{Max}\left[7x_2 + 8\min.\left(\frac{u_1}{2}, \frac{v_1}{2}\right)\right]$

Where $x_2 \geq 0, x_2 \leq u_2 - u_1, x_2 \leq \frac{v_2 - v_1}{2}$

i.e. $0 \le x_2 \le Min.\left(u_2 - u_1, \frac{v_2 - v_1}{2}\right)$

$$0 \le x_2 \le Min.\left(8 - u_1, \frac{15 - v_1}{2}\right)$$

i.e. $0 \le x_2 \le Min.\left(8, \frac{15}{2}\right) = \frac{15}{2}$

$\therefore \quad x_2^* = \frac{15}{2}$

Hence optimal solution is $x_1 = 0, x_2 = \frac{15}{2}$ and $Max.z = \frac{105}{2}$ Answer

Example-4 Solve by dynamic programming

$$Max.z = x_1 + 9x_2$$

Subject to $2x_1 + x_2 \le 25$

$$x_2 \le 11$$

and $x_1 \ge 0, x_2 \le 0$

Solution : Hint : $f_1(u_1, v_1) = Max.(x_1)$, where $x_1 \ge 0, v_1 \ge 0, x_1 \le \frac{u_1}{2}$

$$= \frac{u_1}{2}, \because 0 \le x_1 \le \frac{u_1}{2}$$

$$f_2(u_2, v_2) = \underset{x_2}{Max.}\left[9x_2 + f_1(u_2 - x_2, v_2 - x_2)\right)$$

$$= \underset{x_2}{Max.}\left[\frac{17}{2}x_2 + \frac{u_2}{2}\right], \text{ where } 0 \le x_2 \le Min(u_2, v_2) = Min.(25, 11)$$

$$= \ = 106 \text{ at } x_2^* = 11$$

Hence optimal solution is $x_1 = 7, x_2 = 1$ and $Max.z = 106$ Answer

12.4 Summary

This unit deal with the following :

Objectives, Introduction, Solution of L.P.P. using dynamic programming, Illustrative examples, Self Learnign Exercises.

12.5 Exercises

Solve the following L.P.P. using dynamic programming :

1. $Max. z = 3x_1 + 7x_2$

subject to $x_1 + 4x_2 \le 8$

$x_2 \le 8$

and $x_1, x_2 \ge 0$ $\quad (x_1 = 8, x_2 = 0, max\, z = 24)$

2. $Max. z = 2x_1 + 3x_2$

subject to $x_1 - x_2 \le 1$

$x_1 + x_2 \le 3$

and $x_1, x_2 \ge 0$ $\quad (x_1 = 0, x_2 = 3, max\, z = 9)$

3. $Max. z = 10x_1 + 30x_2$

subject to $3x_1 + 6x_2 \le 168$

$12x_2 \le 240$

and $x_1, x_2 \ge 0$ $\quad (x_1 = 16, x_2 = 20, max\, z = 760)$

4. $Max. z = 2x_1 + 5x_2$

subject to $3x_1 + x_2 \le 2$

$x_2 \le 3$

and $x_1, x_2 \ge 0$ $\quad (x_1 = 3, x_2 = 3, max\, z = 21)$

5. $Max. z = 3x_1 + x_2$

subject to $2x_1 + x_2 \le 6$

$x_1 \le 2$

$x_2 \le 4$

and $x_1, x_2 \ge 0$ $\quad (x_1 = 2, x_2 = 2, max\, z = 8)$

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