MA/MSc MT-09

Vardhaman Mahaveer Open University, Kota

Integral Transforms and Integral Equation

Vardhaman Mahaveer Open University, Kota

Integral Transforms and Integral Equation

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PREFACE

The present book entitled **"Integral Transforms and Integral Equation"** has been designed so as to cover the unit-wise syllabus of Mathematics-09 course for M.A./M.Sc. (Final) students of Vardhaman Mahaveer Open University, Kota. It can also be used for competitive examinations. The basic principles and theory have been explained in a simple, concise and lucid manner. Adequate number of illustrative examples and exercises have also been included to enable the students to grasp the subject easily. The units have been written by various experts in the field. The unit writers have consulted various standard books on the subject and they are thankful to the authors of these reference books.

Unit - 1
Laplace Transform

Structure of Unit

1.0 Objective

The object of this unit is to define Laplace transform with its existence conditions and simple properties. We shall prove some important theorems regarding its derivatives, integrals, multiplication and division by power of 't'. We shall also discuss the evaluation of integrals by using this transformation.

1.1 Introduction

The English Engineer Heaviside (1850-1925) used the operational methods in solving physical problems which gave birth to operational calculus, knwon as Laplace transformation. The name is due to French Mathematician, Pierre de Laplace (1749-1827) who used such transformation in his research work. The Laplace transform methods are useful as well as effective in solving differential equations in initial value problems and hence gained importance amongst Engineers and Scientists. By using Laplace transformation, certain partial differential equations can be reduced to ordinary differential equations and ordinary differential equations can be reduced to algebraic equations.

1.2 Integral Transforms

The Integral transform of a function $f(t)$ defined in $a \le t \le b$ is denoted by $I[f(t);p] = \bar{I}(p)$ and is defined by

$$I[f(t);p] = \bar{I}(p) = \int_a^b k(p,t) f(t)\, dt \qquad ...(1)$$

where $k(p,t)$, given function of two variables p and t, is called the kernel of the transform. The operator I is usually called an integral transform operator or simply integral transformation. The transform function $\bar{I}(p)$ is often referred to as the image of the given object function $f(t)$ and p is called the transform variable.

A formula which gives $f(t)$ back is called the inversion formula

i.e. $$f(t) = \int_\alpha^\beta \theta(p,t) \bar{f}(p) dp \qquad ...(2)$$

By taking $k(p,t)$ as specific function, several integral transforms such as Fourier Transforms, Mellin transform, Hankel Transform and Laplace Transform have been introduced. Out of this, Laplace transform is most extensively studied and used.

1.2.1 Some Important Integral Transforms

S.No.	$k(p,t)$	$I\{f(t);p\}$	Name of the transform	Notation
(i)	$\begin{cases} 0 & , for\ t<0 \\ e^{-pt}, & for\ t\geq 0 \end{cases}$	$\int_0^\infty e^{-pt} f(t)dt$	Laplace	$L[f(t);p]$
(ii)	$0 \quad , for\ t<0$ $t^{p-1}, for\ t\geq 0$	$\int_0^\infty t^{p-1} f(t)dt$	Mellin	$M[f(t);p]$
(iii)	$0 \quad , for\ t<0$ $\sqrt{\frac{2}{\pi}}\sin pt, for\ t\geq 0$	$\sqrt{\frac{2}{\pi}}\int_0^\infty \sin pt\ f(t)dt$	Fourier Sine	$F_S[f(t);p]$
(iv)	$0 \quad , for\ t<0$ $\sqrt{\frac{2}{\pi}}\cos pt, for\ t\geq 0$	$\sqrt{\frac{2}{\pi}}\int_0^\infty \cos pt\ f(t)dt$	Fourier Cosine	$F_C[f(t);p]$
(v)	$\frac{1}{\sqrt{2\pi}}e^{ipt}, -\infty<t<\infty$	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{ipt} f(t)dt$	Complex Fourier	$F[f(t);p]$
(vi)	$0 \quad , for\ t<0$ $t\ J_\upsilon(pt), for\ t\geq 0$ $J_\upsilon(pt)$ is the Bessel Function of the first kind of order υ.	$\int_0^\infty t\ J_\upsilon(pt)f(t)dt$	Hankel	$H_\upsilon[f(t);p]$

1.3 Laplace Transform

1.3.1 Definition : Let $f:t\to f(t)$ be a real valued function defined over the interval $]-\infty,\infty[$ such that $f(t)=0\ \forall\ t<0$ i.e. $f(t)$ be a function of t defined for all positive values of t. Then the Laplace transform of $f(t)$, denoted by $L[f(t);p]$, is defined by

$$L[f(t);p]=\int_0^\infty e^{-pt}f(t)\ dt$$

provided that the integral exists and finite. It means that the integral converges for some values of p. The parameter p, is a real or complex number but independent of t. In general $Re(p)>0$, $Re(p)$ means real part of p.

$L[f(t);p]$ be clearly be a function of p which is written as $\bar{f}(p)$.

Thus $\quad L[f(t);p]=\bar{f}(p)$

To get the Laplace transform of $f(t)$, we multiply it by e^{-pt} and integrate the result with respect to t for 0 to ∞. This operation is called Laplace transformation. Here e^{-pt} is known as the kernel of the Laplace transform and operator L which transforms $f(t)$ into $\bar{f}(p)$ is called the Laplace transformation operator.

1.3.2 Piecewise Continuity or Sectionally Continous

Definition : A function $f(t)$ is said to be piecewise continuous over the closed interval $a \le t \le b$, if that interval can be divided into a finite number of sub intervals $t_{i-1} \le t \le t_1$ ($i = 1,.....,n$ with $t_0 = a$ and $t_n = b$) such that in each sub interval,

(i) $f(t)$ is continuous in the open interval $t_{i-1} < t < t_i$ $(i = 1,....,n)$

(ii) at the end points of these intervals, the right-hand and left-hand limits exists and finite i.e. $\underset{t \to t_{i-1}+0}{Lim} f(t)$ and $\underset{t \to t_i - 0}{Lim} f(t)$ both exist and are finite.

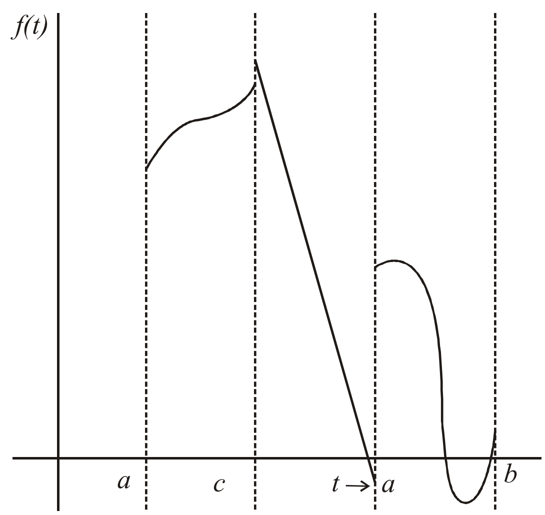


Figure 1.1

(A sectionally continuous function)

Here it is observed that piecewise continuity of f on closed interval simply indicates that a function f has a finite number of discontinuities of the first kind in $[a,b]$.

Example 1 : Consider

$$f(t) = \begin{cases} t^2 + 1; & 1 \le t < \pi/2 \\ \sin t \ ; & \pi/2 \le t < \pi \\ |t| \ ; & \pi \le t < 4 \end{cases}$$

Here $f(t)$ is sectionally continuous function in [1, 4] as the function is continuous in each of the subintervals $(1, \pi/2)$, $(\pi/2, \pi)$ and $(\pi, 4)$ and has finite right-hand and left-hand limits at $t = 1$, $\pi/2$, π

and 4 of these subintervals.

1.3.3 Functions of Exponential Order

Definition : A function $f(t)$ to said to be of exponential order $a(a>0)$ as $t \to \infty$ if

$$\lim_{t\to\infty} e^{-at} f(t) = \text{finite quantity}$$

This means that for a given positive integer t_0, there exists a real number $M > 0$ such that

$$\left|e^{-at} f(t)\right| < M \qquad \forall\ t \ge t_0$$

which implies that

$$\left|f(t)\right| < Me^{at} \qquad \forall\ t \ge t_0$$

We may also write it as $f(t) = 0\left(e^{at}\right)$ as $t \to \infty$.

If a function $f(t)$ is of exponential order a and $b > a$, then $f(t)$ is of exponential order b. (as $e^{bt} > e^{at}; t > 0)$

Example 2 : Bounded functions such as $\sin at, \cos at$ are of exponential order since $|\sin at| \le 1$ and $|\cos at| \le 1$

Example 3 : Show that $f(t) = t^2$ is of exponential order 3.

Solution : Since $\lim_{t\to\infty} e^{-3t} t^2 = \lim_{t\to\infty} \frac{t^2}{e^{3t}} = 0$ [using L' Hospital rule]

$\therefore\ t^2$ is of exponential order

Again since $\left|t^2\right| = t^2 < e^{3t}, \forall\ t > 0$, hence t^2 is order 3.

Example 4 : Show that $f(t) = e^{t^3}$ is not of exponential order.

Solution : Since $\lim_{t\to\infty} e^{-at} e^{t^3} = \lim_{t\to\infty} \frac{e^{t^3}}{e^{at}}$

$$= \lim_{t\to\infty} e^{t\left(t^2 - a\right)} = \infty\ \forall\ a$$

Hence we can not find a number M such that $e^{t^3} < M e^{at}$

Therefore e^{t^3} is not of exponential order.

1.3.4 Existance Conditions of Laplace Transform

Theorem 1 : A function $f(t)$ is sectionally continuous in every finite interval $0 \le t \le t_0$ and is

of exponential order 'a', as $t \to \infty$ then $L[f(t);p]$ exists $\forall\ p > a$.

Proof : Let $t_0 > 0$. Then $L[f(t);p] = \int_o^\infty e^{-pt} f(t)\,dt = \int_0^{t_0} e^{-pt} f(t)\,dt + \int_{t_0}^\infty e^{-pt} f(t)\,dt$ ...(3)

Since $f(t)$ is sectionally continuous in every finite interval $0 \le t \le t_0$, the integral $\int_0^{t_0} e^{-pt} f(t)\,dt$ exists.

Now $$\left|\int_{t_0}^\infty e^{-pt} f(t)\,dt\right| \le \int_{t_0}^\infty \left|e^{-pt} f(t)\right| dt$$

$$\le \int_{t_0}^\infty e^{-pt} |f(t)|\,dt$$

[since $f(t)$ is of exponential order a i.e. $|f(t)| \le M\,e^{at}$]

$$\le M\int_{t_0}^\infty e^{-pt}.e^{at}\,dt = M\int_{t_0}^\infty e^{-(p-a)t}\,dt$$

$$= \frac{Me^{-(p-a)t_0}}{(p-a)} \text{ if } p > a$$

Finally we get $$\left|\int_{t_0}^\infty e^{-pt} f(t)\,dt\right| \le \frac{Me^{-(p-a)t_0}}{(p-a)} \quad \text{if} \quad p > a$$

The term $\frac{Me^{-(p-a)t_0}}{(p-a)}$ can be made as small as we please by choosing t_0 sufficiently large for $p > a$.

Hence $\int_{t_0}^\infty e^{-pt} f(t)\,dt$ exists and accordingly $\int_0^\infty e^{-pt} f(t)\,dt$ also exists for $p > a$.

Thus $L[f(t);p]$ exists for $p > a$.

Remarks : The conditions given in Theorem 1 are sufficient but not necessary for the existence of the Laplace transform. If these conditions are satisfied, the Laplace transform must exist. If the above conditions are not satisfied then the Laplace transform may or may not exist. There are functions whose Laplace transform exists even when the conditions of theorem 1 (one) are not satisfied.

For example $f(t)=\frac{1}{\sqrt{t}}$ is not sectionally continuous in every finite interval in the range $t \geq 0$.

Since the right-hand limit at $t=0,\ \underset{h\to 0}{Lim} f(0+h)=\underset{h\to o}{Lim}\frac{1}{\sqrt{0+h}}=\infty$

$\Rightarrow$ $t^{-1/2}$ has no finite right hand limit at $t=0$

$\therefore$ $f(t)=t^{-1/2}$ is not sectionally continuous in the range $t \geq 0$.

Also $|f(t)|=\left|\frac{1}{\sqrt{t}}\right|=\frac{1}{\sqrt{t}}$ when $t>0$

For exponential order, $|f(t)|<M\,e^{at},\ t\geq t_0$

For $t>0$ with $M=1$ and $a=0,1,2,3.....$ at $t\to\infty$

$$\frac{1}{\sqrt{t}}<M\,e^{at}\Rightarrow\frac{1}{\sqrt{t}}<e^{at}$$

$\Rightarrow$ $\frac{1}{\sqrt{t}}$ is of exponential order as $t\to\infty$

But $$L\left[t^{-1/2};p\right]=\int_0^\infty e^{-pt}\frac{1}{\sqrt{t}}dt$$

$$=\int_0^\infty e^{-u}\sqrt{\frac{p}{u}}\,\frac{du}{p} \qquad \text{(puting } pt=u\text{)}$$

$$=\frac{1}{\sqrt{p}}\int_0^\infty e^{-u}\,u^{-1/2}\,du$$

$$=\frac{1}{\sqrt{p}}\int_0^\infty e^{-u}\,u^{1/2-1}\,du$$

$$=\frac{1}{\sqrt{p}}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{\sqrt{p}}=\sqrt{\frac{\pi}{p}} \quad \text{for } p>0$$

Thus we have shown that Laplace Transform of $\frac{1}{\sqrt{t}}$ exists for $p>0$ even if $\frac{1}{\sqrt{t}}$ is not sectionally continuous in the range $t\geq 0$ and is of exponential order as $t\to\infty$.

1.3.5 Functions of Class A

Definition : A function which is sectionally continuous in every finite interval and is of exponential order 'a' as $t\to\infty$ is known as a function of class A.

1.4 Some Important Properties of Laplace Transforms

We assume, unless otherwise stated, that all functions satisfy the conditions of theorem 1, so that their Laplace transform exist.

1.4.1 Linearity Property

Theorem 2 : If $L[f_1(t);p]$ and $L[f_2(t);p]$ be the Laplace transform of the functions $f_1(t), f_2(t)$ respectively and if C_1, C_2 are any two constants, then

$$L[C_1f_1(t)+C_2f_2(t);p]=C_1L[f_1(t);p]+C_2L[f_2(t);p]$$

Proof : By definition, we have

$$L[C_1f_1(t)+C_2f_2(t);p]=\int_0^\infty e^{-pt}[C_1f_1(t)+C_2f_2(t)]\,dt$$

$$=C_1\int_0^\infty e^{-pt}f_1(t)\,dt+C_2\int_0^\infty e^{-pt}f_2(t)\,dt$$

$$=C_1\,L[f_1(t);p]+C_2L[f_2(t)]$$

1.4.2 Change of Scale Property

Theorem 3 : If $L[f(t);p]=\bar{f}(p)$ then $L[f(at);p]=\frac{1}{a}\bar{f}\left(\frac{p}{a}\right)$

Proof :

$$L[f(at);p]=\int_0^\infty e^{-pt}f(at)\,dt$$

$$=\frac{1}{a}\int_0^\infty e^{-\left(\frac{p}{a}\right)t'}f(t')\,dt' \qquad \text{(putting } at=t'\text{)}$$

$$=\frac{1}{a}\bar{f}\left(\frac{p}{a}\right)$$

1.4.3 First Translation or Shifting Theorem

Theorem 4 : If $L[f(t);p]=\bar{f}(p)$, then $L[e^{at}f(t);p]=\bar{f}(p-a)$ where a is real or complex number.

or

If $\bar{f}(p)$ is the Laplace Transform of $f(t)$, then $\bar{f}(p-a)$ is the Laplace Transform of $e^{at}f(t)$

Proof : We have

$$L[e^{at}f(t);p]=\int_0^\infty e^{-pt}e^{at}f(t)\,dt$$

$$= \int_0^\infty e^{-(p-a)t} f(t)\,dt$$

$$= L[f(t); p-a]$$

$$= \bar{f}(p-a) \qquad \left(\because \bar{f}(p) = \int_0^\infty e^{-pt} f(t)\,dt\right)$$

1.4.4 Second Translation or Shifting Theorem

Theorem 5 : If $L[f(t); p] = \bar{f}(p)$ and a fucntion $g(t)$ is defined as

$$g(t) = f(t-a)\,; \quad t > a$$
$$= 0 \qquad t < a$$

then $L[g(t); p] = e^{-ap}\bar{f}(p)$

Proof : We have

$$L[g(t); p] = \int_0^\infty e^{-pt} g(t)\,dt$$

$$= \int_0^a e^{-pt} g(t)\,dt + \int_a^\infty e^{-pt} g(t)\,dt$$

$$= \int_0^a e^{-pt}.0.\,dt + \int_a^\infty e^{-pt} f(t-a)\,dt$$

$$= \int_0^\infty e^{-p(t'+a)}\, f(t')\,dt' \qquad \text{(putting } t-a = t'\text{)}$$

$$= e^{-ap}\int_0^\infty e^{-pt'} f(t')\,dt'$$

$$= e^{-ap} L[f(t'); p]$$

$$= e^{-ap}\bar{f}(p)$$

1.4.5 Alternate Statement of Second Shifting Theorem

If $\bar{f}(p)$ is the Laplace transform of $f(t)$ and $a > 0$, then $e^{-ap}\bar{f}(p)$ is the Laplace transform of $f(t-a)u(t-a)$ where $u(t-a)$ is the Heaviside unit step function defined as

$$u(t-a) = 1 \quad ; \quad \text{if} \quad t > a$$
$$= 0 \quad ; \quad \text{if} \quad t < a$$

Proof : We have

$$L[f(t-a).u(t-a); p] = \int_0^\infty e^{-pt} f(t-a)\,u(t-a)\,dt$$

$$= \int_0^a e^{-pt} f(t-a) u(t-a)\,dt + \int_a^\infty e^{-pt} f(t-a) u(t-a)\,dt$$

Now using the definition of unit step function, we have

$$L\left[f(t-a)u(t-a); p\right]$$

$$= \int_0^a e^{-pt} f(t-a).0.dt + \int_a^\infty e^{-pt} f(t-a).1.dt$$

$$= \int_a^\infty e^{-pt} f(t-a)\,dt$$

$$= \int_0^\infty e^{-p(a+t')} f(t')\,dt' \qquad \text{(putting } t-a=t' \Rightarrow dt=dt'\text{)}$$

$$= e^{-ap} \int_0^\infty e^{-pt'} f(t')\,dt' = e^{-ap} \bar{f}(p)$$

Hence the result.

1.5 Table of Laplace Transforms

S.No.	$f(t)$	$L[f(t); p] = \bar{f}(p)$	
1.	1	$\dfrac{1}{p}$,	$p > 0$
2.	e^{at}	$\dfrac{1}{p-a}$,	$p > a$
3.	t^n	$\dfrac{n!}{p^{n+1}}$,	$p > 0,\ n \in N$
4.	t^α	$\dfrac{\Gamma(\alpha+1)}{p^{n+1}}$,	if $Re(\alpha) > -1, p > 0$
5.	$\sin at$	$\dfrac{a}{p^2+a^2}$,	$p > 0$
6.	$\cos at$	$\dfrac{p}{p^2+a^2}$,	$p > 0$
7.	$\sinh at$	$\dfrac{a}{p^2-a^2}$,	$p > \lvert a \rvert$
8.	$\cosh at$	$\dfrac{p}{p^2-a^2}$,	$p > \lvert a \rvert$

The following results can be obtained by using first shifting property

(a) $L\left[e^{at}t^{\alpha};p\right]=\dfrac{\Gamma(\alpha+1)}{(p-a)^{\alpha+1}}$

(b) $L\left[e^{at}\sin bt\right]=\dfrac{b}{(p-a)^2+b^2}$

(c) $L\left[e^{at}\cos bt\right]=\dfrac{p-a}{(p-a)^2+b^2}$

(d) $L\left[e^{at}\sinh bt\right]=\dfrac{b}{(p-a)^2-b^2}$

(e) $L\left[e^{at}\cosh bt\right]=\dfrac{p-a}{(p-a)^2-b^2}$

Example 5 : Find the Laplace transform of :

(i) $e^{-2t}-\sin 5t+4\cos 7t+9t^3-5$ (ii) $\cosh^2 4t$

(iii) $\left(1+t\,e^{-t}\right)^3$ (iv) $\dfrac{\cosh at}{\sqrt{t}}$ (v) $t^2e^t\sin 4t$

Solution : We have

(i) $L\left[e^{-2t}-\sin 5t+4\cos 7t+9t^3-5;p\right]$

$$=L\left[e^{-2t};p\right]-L\left[\sin 5t;p\right]+4L\left[\cos 7t;p\right]+9L\left[t^3;p\right]-5L\left[1;p\right]$$

$$=\frac{1}{p+2}-\frac{5}{p^2+25}+\frac{4p}{p^2+49}+9\frac{3!}{p^4}-5\frac{1}{p}$$

$$=\frac{1}{p+2}-\frac{5}{p^2+25}+\frac{4p}{p^2+49}+\frac{54}{p^4}-\frac{5}{p}$$

(ii) $L\left[\cosh^2 4t;p\right]=L\left[\dfrac{1+\cosh 8t}{2};p\right]$

$$=\frac{1}{2}L\left[1;p\right]+\frac{1}{2}L\left[\cosh 8t;p\right]$$

$$=\frac{1}{2p}+\frac{p}{2\left(p^2-64\right)}$$

(iii) $L\left[\left(1+t\,e^{-t}\right)^3;p\right]=L\left[1+3t\,e^{-t}+3t^2e^{-2t}+t^3e^{-3t};p\right]$

$$= L[1;p] + 3L[t\,e^{-t};p] + 3L[t^2 e^{-2t};p] + L[t^3 e^{-3t};p]$$

$$= \frac{1}{p} + 3L[t;p+1] + 3L[t^2;p+2] + L[t^3;p+3]$$

$$= \frac{1}{p} + \frac{3}{(p+1)^2} + \frac{3.2!}{(p+2)^3} + \frac{3!}{(p+3)^4}$$

$$= \frac{1}{p} + \frac{3}{(p+1)^2} + \frac{6}{(p+2)^3} + \frac{6}{(p+3)^4}$$

(iv) $$L\left[\frac{\cosh at}{\sqrt{t}};p\right] = L\left[\frac{1}{\sqrt{t}}\left(\frac{e^{at}+e^{-at}}{2}\right);p\right]$$

$$= \frac{1}{2}L[e^{at}t^{-1/2};p] + \frac{1}{2}L[e^{-at}t^{-1/2};p]$$

$$= \frac{1}{2}L[t^{-1/2};p-a] + \frac{1}{2}L[t^{-1/2};p+a]$$

$$= \frac{1}{2}\left[\frac{\Gamma(1/2)}{(p-a)^{1/2}} + \frac{\Gamma(1/2)}{(p+a)^{1/2}}\right]$$

$$= \frac{\sqrt{\pi}}{2}\left[\frac{1}{(p-a)^{1/2}} + \frac{1}{(p+a)^{1/2}}\right]$$

(v) $L[t^2 e^t \sin 4t;p]$

Since $L[t^2;p] = \dfrac{2}{p^3}$

$\therefore$ $L[e^{4it}.t^2;p] = \dfrac{2}{(p-4i)^3}$ (by first shifting theorem)

$$= \frac{2}{(p-4i)^3} \times \frac{(p+4i)^3}{(p+4i)^3}$$

$$= \frac{2[(p^3-48p)+i(12p^2-64)]}{(p^2+4^2)^3}$$

$$\therefore \qquad L\left[(\cos 4t + i\sin 4t)t^2; p\right] = \frac{2\left(p^3 - 48p\right)}{\left(p^2+4^2\right)^3} + i\frac{2\left(12p^2-64\right)}{\left(p^2+4^2\right)^3}$$

Now equating the imaginary parts on both the sides, we get

$$L\left[t^2 \sin 4t; p\right] = \frac{2\left(12p^2-64\right)}{\left(p^2+4^2\right)^3} = \frac{8\left(3p^2-16\right)}{\left(p^2+4^2\right)^3}$$

Again applying the first shifting theorem

$$L\left[e^t t^2 \sin 4t; p\right] = \frac{8\left[3(p-1)^2-16\right]}{\left[(p-1)^2+4^2\right]^3}$$

$$= \frac{8\left(3p^2-6p-13\right)}{\left(p^2-2p+17\right)^3}$$

Example 6 : Show that $L\left[f(t); p\right]$ is $\left(\frac{1}{p^2}+\frac{1}{p}\right)e^{-p} - \left(\frac{1}{p^2}+\frac{2}{p}\right)e^{-2p}$ where $f(t) = \begin{cases} 0, & 0 < t < 1 \\ t, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$

Solution : We have $L\left[f(t); p\right] = \int_0^\infty e^{-pt} f(t)\,dt$

$$= \int_0^1 e^{-pt} f(t)\,dt + \int_1^2 e^{-pt} f(t)\,dt + \int_2^\infty e^{-pt} f(t)\,dt$$

$$= \int_1^2 e^{-pt}\, t\,dt = \left(\frac{t\,e^{-pt}}{-p}\right)_1^2 + \frac{1}{p}\int_1^2 e^{-pt}\,dt$$

$$= \frac{-2e^{-2p}}{p} + \frac{e^{-p}}{p} - \frac{1}{p^2}\left(e^{-2p} - e^{-p}\right)$$

$$= e^{-p}\left(\frac{1}{p}+\frac{1}{p^2}\right) - e^{-2p}\left(\frac{2}{p}+\frac{1}{p^2}\right)$$

Example 7 : Find the Laplace Transform of $t^2.u(t-3)$ using second shifting theorem, where $u(t-3)$ is a unit step function.

Solution : Since $t^2 = (t-3+3)^2 = (t-3)^2 + 6(t-3) + 9$

$$\therefore \qquad t^2\, u(t-3) = (t-3)^2.u(t-3) + 6(t-3)\,u(t-3) + 9\,u(t-3)$$

$$\therefore \quad L\left[t^2 u(t-3); p\right] = L\left[(t-3)^2 u(t-3); p\right] + 6L\left[(t-3)u(t-3); p\right] + 9L\left[u(t-3); p\right]$$

$$= e^{-3p} L\left[t^2; p\right] + 6e^{-3p} L\left[t; p\right] + 9e^{-3p} L\left[1; p\right]$$

$$= e^{-3p} . \frac{2}{p^3} + 6e^{-3p} . \frac{1}{p^2} + 9\frac{e^{-3p}}{p}$$

$$= e^{-3p}\left(\frac{2+6p+9p^2}{p^3}\right)$$

Self Learning Exercise - I

If $\bar{f}(p) = L\{f(t); p\}$, then fill-in the blanks in the following :

1. $L\{e^{at} f(bt); p\} =$.....

2. $L\{\sin at; p\} =$.....

3. $L\{r^t f(at); p\} =$.....

4. $L\{f(t-a)u(t-a); p\} =$.....

5. The exponential order of t^3 is ....

6. $L\left\{\frac{1}{\sqrt{t}}; p\right\} =$.....

7. $L\{(t+2)^2; p\} =$.....

1.6 Exercise 1 (a)

1. Prove that the following function :

$$f(t) = \begin{cases} at+b; & 0 \le t < 1 \\ e^{-t} \;; & 1 \le t < 2 \\ t^2 \;; & 2 \le t \le 3 \end{cases}$$

is sectionally continuous in $[0,3]$.

2. Show that the function $f(t) = t^n \; (n > 0)$ is of exponential order 'a' if $a > 0$, $n \in N$.

3. Show that the function $f(t) = t^3$ is of exponential order 4.

4. Prove that $L\left[e^{at} f(bt); p\right] = \frac{1}{b}\bar{f}\left(\frac{p-a}{b}\right)$, where $L\left[f(t); p\right] = \bar{f}(p)$ and a and b are constants.

5. Find the Laplace Transform of :

(*i*) $e^{-2t}\sin 4t$ **Ans.** $\dfrac{4}{p^2+4p+20}$

(*ii*) $t^2\, e^{3t}$ **Ans.** $2/(p-3)^2$

(*iii*) $\sin^3 t$ **Ans.** $\dfrac{6}{(p^2+1)(p^2+9)}$

(*iv*) $e^{-t}(3\sinh 2t-5\cosh 2t)$ **Ans.** $\dfrac{1-5p}{p^2+2p-3}$

(*v*) $e^{-2t}(3\cos 6t-5\sin 6t)$ **Ans.** $\dfrac{3p-24}{p^2+4p+40}$

6. Evaluate $L[f(t);p]$ where $f(t)=\begin{cases}\cos t\ ; & 0<t<\pi\\ \sin t\ ; & t>\pi\end{cases}$ **Ans.** $\dfrac{p+(p-1)\,e^{-\pi p}}{(p^2+1)}$

7. State and prove the translation properties of the Laplace transform. Also obtain the Laplace transform of the following functions :

(*i*) $f(t)=\begin{cases}\sin\left(t-\pi/3\right), & t>\pi/3\\ 0 & , t<\pi/3\end{cases}$ **Ans.** $e^{-\left(p\pi/3\right)}.\dfrac{1}{p^2+1};p>0$

(*ii*) $f(t)=\begin{cases}t/T; & 0<t<T\\ 1\ ; & t>T\end{cases}$ **Ans.** $\dfrac{-1}{p^2T}\left(1-e^{-pT}\right);p>0$

(*iii*) $f(t)=\begin{cases}\cos\left(t-2\pi/3\right); & t>2\pi/3\\ 0 & ;\ t<2\pi/3\end{cases}$ **Ans.** $e^{-(2\pi p)/3}.\dfrac{p}{p^2+1}\ ;\ p>0$

8. Find the Laplace Transform of :

(*i*) $\sin 5t\cos 3t$ **Ans.** $\dfrac{1}{2}\left[\dfrac{8}{p^2+4}+\dfrac{1}{p^2+4}\right]$

(*ii*) $\cosh at\cos bt$ **Ans.** $\dfrac{1}{2}\left[\dfrac{p-a}{(p-a)^2+b^2}+\dfrac{p+a}{(p+a)^2+b^2}\right]$

(*iii*) $(t+2)^2 e^t$ **Ans.** $\dfrac{4p^2-4p+2}{(p-1)^3}$

(iv) $e^{-2t}\cos^2 t$ **Ans.** $\dfrac{p^2+4p+6}{(p+2)(p^2+4p+8)}$

(v) $\sinh bt \cos at$ **Ans.** $\dfrac{1}{2}\left[\dfrac{p-b}{(p-b)^2+a^2}-\dfrac{p+b}{(p+b)^2+a^2}\right]$

9. Find the Laplace Transform of :

(i) $\cosh^2 3t$ **Ans.** $\dfrac{p^2-18}{p(p^2-36)}$

(ii) $\cos^3 t$ **Ans.** $\dfrac{1}{4}\left[\dfrac{p}{p^2+9}+\dfrac{3p}{p^2+1}\right]$

10. If $L[f(t);p]=\bar{f}(p)$, show that

(i) $L[(\sinh at)f(t);p]=\dfrac{1}{2}[\bar{f}(p-a)-\bar{f}(p+a)]$

(ii) $L[(\cosh at)f(t);p]=\dfrac{1}{2}[\bar{f}(p-a)+\bar{f}(p+a)]$

11. Using second shifting theorem, find the Laplace transform of :

(i) $e^{t-a}\,u(t-a)$ **Ans.** $\dfrac{1}{p-1}e^{-ap}$

(ii) $(t-1)^2\,u(t-1)$ **Ans.** $\dfrac{2}{p^3}e^{-p}$

1.7 Laplace Transform of Derivatives

Theorem 6 : Let

(i) **$f(t)$ is continuous for $0 \le t \le N$**

(ii) **$f(t)$ is of exponential order 'a' for $t > N$**

(iii) **$f'(t)$ is sectionally continuous for $0 \le t \le N$**

(iv) **$L[f(t);p]=\bar{f}(p)$**

Then $L[f'(t);p]=p\,\bar{f}(p)-f(0)$ for $p>a$ ...(4)

Proof : We have $L[f'(t);p]=\int_0^\infty e^{-pt}f'(t)\,dt$

Using integration by parts, we have

$$= \left[\left[e^{-pt} f(t)\right]_0^\infty + p\int_0^\infty e^{-pt} f(t)\,dt\right]$$

Since $f(t)$ is of exponential order $'a'$, therefore

$$\underset{t\to\infty}{Lim}\; e^{-pt} f(t) = 0 \qquad \text{for} \qquad p > a$$

Thus $L\left[f'(t);p\right] = \left(0 - f(0)\right) + p\bar{f}(p)$

$$= p\bar{f}(p) - f(0) \text{ for } p > a$$

Theorem 7 : Let

(i) $f(t)$, $f'(t)$.... $f^{(n-1)}(t)$ are continuous for $0 \le t \le N$ and are of exponential order $'a'$ for $t > N$.

(ii) $f^{(n)}(t)$ is sectionally continuous for $0 \le t \le N$.

Then $$L\left[f^{(n)}(t);p\right] = p^n\bar{f}(p) - p^{n-1}f(0) - p^{n-2}f'(0) - \ldots - p\,f^{(n-2)}(0) - f^{(n-2)}(0) \quad \ldots(5)$$

$$(for\; p > a)$$

where $\bar{f}(p) = L\left[f(t);p\right]$

Proof : We shall prove the theorem by using mathematical induction.

By the theorem 6 we have

$$L\left[f'(t);p\right] = p\,L\left[f(t);p\right] - f(0) \qquad (p > a) \qquad \ldots(6)$$

Hence the theorem is true for $n = 1$.

Let the theorem be true for $n = m$ (a fixed positive integer)

Then $$L\left[f^{(m)}(t);p\right] = p^m\bar{f}(p) - p^{m-1}f(0) - p^{m-2}f'(0) - \ldots - p\,f^{(m-2)}(0) - f^{(m-1)}(0)$$

Now $$L\left[f^{(m+1)}(t);p\right] = L\left[\frac{d}{dt}f^{(m)}(t);p\right]$$

$$= p\,L\left[f^{(m)}(t);p\right] - f^{(m)}(0)$$

$$= p\left[p^m\bar{f}(p) - p^{m-1}f(0) - p^{m-2}f'(0) - \ldots - p\,f^{(m-2)}(0) - f^{(m-1)}(0)\right] - f^{(m)}(0)$$

$$= p^{m+1}\bar{f}(p) - p^m f(0) - p^{m-1}f'(0) - \ldots - p^2 f^{(m-2)}(0) - p\,f^{(m-1)}(0) - f^{(m)}(0)$$

Therefore the theorem is true for $n = m + 1$

Hence by the principle of mathematical induction, the theorem 7 is true for all $n \in N$.

1.8 Laplace Transform of Integrals

Theorem 8 : If $L[f(t);p] = \bar{f}(p)$, then

$$L\left[\int_0^t f(u)\,du; p\right] = \frac{\bar{f}(p)}{p}$$

Proof : Let $G(t) = \int_0^t f(u)\,du$

then $G'(t) = f(t)$ and $G(0) = 0$

Since $L[G'(t);p] = pL[G(t);p] - G(0)$

$\therefore$ $L[f(t);p] = p\,L\left[\int_0^t f(u)\,du; p\right]$

or $L\left[\int_0^t f(u)\,du; p\right] = \dfrac{\bar{f}(p)}{p}$

Remark : The generalization of above result is

$$L\left[\int_0^t \int_0^{u_n} \cdots \int_0^{u_2} f(u_1)\,du_1 \ldots\ldots d\,u_n; p\right] = p^{-n}\bar{f}(p)$$

where $\bar{f}(p) = L[f(t);p]$

1.9 Multiplication and Division by Powers of 't'

Theorem 9 : If $L[f(t);p] = \bar{f}(p)$, then

$$L[t^n f(t);p] = (-1)^n \frac{d^n}{dp^n}\bar{f}(p)$$

Proof : We have

$$\bar{f}(p) = L[f(t);p] = \int_0^\infty e^{-pt} f(t)\,dt \qquad \ldots(7)$$

Differentiating the above equation (7) on both sides with respect to p, and applying Leibnitz's rule for differentiation under the sign of integration, we have

$$\frac{d}{dp}\bar{f}(p) = \frac{d}{dp}\int_0^\infty e^{-pt} f(t)\,dt$$

$$= \int_0^\infty -t\,e^{-pt} f(t)\,dt$$

Thus $L\left[t\,f(t);p\right]=-\frac{d}{dp}\bar{f}(p)$ ...(8)

which proves that the theorem is true for $n=1$. To establish the theorem 9, we use principle of mathematical induction. Let the theorem be true for $n=m$ (a fixed positive integer), then

$$L\left[t^m f(t);p\right]=(-1)^m\frac{d^m}{dp^m}\bar{f}(p)$$

or $\int_0^\infty e^{-pt}\left\{t^m f(t)\right\}dt=(-1)^m\frac{d^m}{dp^m}\bar{f}(p)$ ...(9)

Differentiating both sides of (9) with respect to $'p'$ and applying Leibnitz's rule, we get

$$-\int_0^\infty e^{-pt}\left\{t^{m+1}f(t)\right\}dt=(-1)^m\frac{d^{m+1}}{dp^{m+1}}\bar{f}(p)$$

or $L\left[t^{m+1}f(t);p\right]=(-1)^{m+1}\frac{d^{m+1}}{dp^{m+1}}\bar{f}(p)$ ...(10)

Therefore the result (10) is true for $n=m+1$.

Hence by principle of mathematical induction, the theorem 9 is true for all positive integers.

Remark : Leibnitz's rule for differentiation under the sign of integration

If $f(x,\alpha)$ and $\frac{\partial f}{\partial\alpha}(x,\alpha)$ are continuous functions of x and α, then $\frac{d}{d\alpha}\left[\int_a^b f(x,\alpha)\,dx\right]=\int_a^b\frac{\partial f}{\partial\alpha}(x,\alpha)\,dx$, where a,b are constants independent of α.

Theorem 10 : If $L\left[f(t);p\right]=\bar{f}(p)$, then $L\left[\frac{f(t)}{t};p\right]=\int_p^\infty\bar{f}(u)\,du$, provided that the integral exists.

Proof : Let $G(t)=\frac{f(t)}{t}$, so that $f(t)=t\,G(t)$

Taking Laplace Transform of both the sides and using Theorem 9. We have

$$L\left[f(t);p\right]=L\left[t\,G(t);p\right]=-\frac{d}{dp}L\left[G(t);p\right]$$

or $\bar{f}(p)=-\frac{d}{dp}L\left[G(t);p\right]$

Now integrating both the sides with respect to p from p to ∞, we have

$$-\left[L\left\{G(t);p\right\}\right]_p^\infty=\int_p^\infty\bar{f}(p)\,dp=\int_p^\infty\bar{f}(u)\,du$$

or $-\lim_{p\to\infty}L\left[G(t);p\right]+L\left[G(t);p\right]=\int_p^\infty\bar{f}(u)\,du$

or $\quad 0 + L[G(t); p] = \int_p^\infty \bar{f}(u)\,du \qquad \left[\because \underset{p\to\infty}{Lim} L[G(t); p] = \underset{p\to\infty}{Lim} \int_0^\infty e^{-pt} G(t)\,dt = 0\right]$

or $\quad L\left[\frac{f(t)}{t}; p\right] = \int_p^\infty \bar{f}(u)\,du$

1.10 Evaluation of Integrals by using Laplace Transforms

Laplace Transform can be used for evaluation of integrals as shown below:

If $\quad L\{f(t); p\} = \bar{f}(p) \quad$ i.e. $\quad \int_0^\infty e^{-pt} f(t)\,dt = \bar{f}(p)$

Taking the limit as $p \to 0$ and assuming that the resulting intergral is convergent. We have

$$\int_0^\infty f(t)\,dt = \bar{f}(0)$$

Example 8 : Using the derivative formula, show that

$(i) \quad L[\cos at; p] = \frac{p}{p^2 + a^2}$

$(ii) \quad L[t\cos at; p] = \frac{p^2 - a^2}{\left(p^2 + a^2\right)^2}$

Solution : (i) Let $f(t) = \cos at$

so that $f'(t) = -a\sin at$, $f''(t) = -a^2\cos at$,

$f(0) = 1$ and $f'(0) = 0$

Since $\quad L[f''(t); p] = p^2 L[f(t); p] - p f(0) - f'(0) \qquad$ ...(11)

therefore $L\left[-a^2\cos at; p\right] = p^2 L[\cos at; p] - p.1 - 0$

Simplifying, we get

$$L[\cos at; p] = \frac{p}{p^2 + a^2}$$

(ii) Here let $f(t) = t\cos at$

$$f'(t) = \cos at - at\sin at$$

and $\quad f''(t) = -2a\sin at - a^2 t\cos at$

Also $\quad f(0) = 0$, $f'(0) = 1$

Substituting these values in (11), we get

$$L\left[-2a\sin at - a^2 t\cos at; p\right] = p^2 L[t\cos at; p] - 1$$

or $\left(p^2+a^2\right)L\left[t\cos at;p\right]=1-2a.\dfrac{a}{p^2+a^2}$

or $L\left[t\cos at;p\right]=\dfrac{p^2-a^2}{\left(p^2+a^2\right)^2}$

Example 9 : Evaluate Laplace Transform of the following functions :

(i) $\sin at - at\cos at+\dfrac{\sin t}{t}$ (ii) $\dfrac{1-\cos t}{t^2}$

Solution : (i) $L\left[\sin at-at\cos at+\dfrac{\sin t}{t};p\right]$

$$=L\left[\sin at;p\right]-a\,L\left[t\cos at;p\right]+L\left[\frac{\sin t}{t};p\right]$$

$$=\frac{a}{p^2+a^2}-a(-1)\frac{d}{dp}L\left[\cos at;p\right]+\int_p^\infty L\left[\sin t;u\right]du$$

$$=\frac{a}{p^2+a^2}+a\frac{\left(a^2-p^2\right)}{\left(p^2+a^2\right)^2}+\int_p^\infty\frac{1}{1+u^2}du$$

$$=\frac{a}{p^2+a^2}+\frac{a\left(a^2-p^2\right)}{\left(p^2+a^2\right)^2}+\left(\tan^{-1}u\right)_p^\infty$$

$$=\frac{2a^3}{\left(p^2+a^2\right)^2}+\frac{\pi}{2}-\tan^{-1}p$$

$$=\frac{2a^3}{\left(p^2+a^2\right)^2}+\tan^{-1}\left(\frac{1}{p}\right)$$

(ii) Let $f(t)=1-\cos t$

$\therefore$ $L\left[f(t);p\right]=L\left[1;p\right]-L\left[\cos t;p\right]=\dfrac{1}{p}-\dfrac{p}{p^2+1}=\bar{f}(p)$

$\therefore$ $L\left[\dfrac{1-\cos t}{t};p\right]=\displaystyle\int_p^\infty\left(\frac{1}{u}-\frac{u}{u^2+1}\right)du$

$$=\left[\log u-\frac{1}{2}\log\left(u^2+1\right)\right]_p^\infty$$

$$=\left[\log\frac{u}{\sqrt{u^2+1}}\right]_p^\infty=\left[\log\frac{1}{\sqrt{\left(1+\frac{1}{u^2}\right)}}\right]_p^\infty$$

$$= 0 - \log\left[\frac{1}{\sqrt{\left(1+ \frac{1}{p^2}\right)}}\right] = \log\left[\frac{\sqrt{p^2+1}}{p}\right]$$

$$\therefore \qquad L\left[\frac{1-\cos t}{t}; p\right] = \frac{1}{2}\log\left(p^2+1\right) - \log p \qquad ...(12)$$

we have

$$L\left[\frac{1-\cos t}{t^2}; p\right] = \int_p^\infty \left[\frac{1}{2}\log\left(u^2+1\right) - \log u\right] du$$

$$= \frac{1}{2}\left[\left\{u\log\left(u^2+1\right)\right\}_p^\infty - \int_p^\infty \frac{2u^2}{u^2+1}du\right] - \left[\left(u\log u\right)_p^\infty - \int_p^\infty du\right]$$

$$= \frac{1}{2}\left[u\log\left(u^2+1\right) - 2\left(u - \tan^{-1}u\right)\right]_p^\infty - \left[u\log u - u\right]_p^\infty$$

$$= \frac{1}{2}\left[u\log\left(u^2+1\right) + 2\tan^{-1}u - 2u\log u\right]_p^\infty$$

$$= \frac{1}{2}\left[u\log\left(1+\frac{1}{u^2}\right) + 2\tan^{-1}u\right]_p^\infty$$

$$= \frac{1}{2}\lim_{u\to\infty} u\log\left(1+\frac{1}{u^2}\right) - \frac{1}{2}p\log\left(1+\frac{1}{p^2}\right) + \left(\frac{\pi}{2} - \tan^{-1}p\right)$$

$$= \frac{1}{2}\lim_{u\to\infty} u\left\{\frac{1}{u^2} - \frac{1}{2u^4} + ...\right\} - p\log\frac{\sqrt{p^2+1}}{p} + \frac{\pi}{2} - \tan^{-1}p$$

$$= \cot^{-1}(p) + p\log\frac{p}{\sqrt{p^2+1}}$$

Example 10 : Find Laplace Transform of the function $\sin\sqrt{t}$ and hence obtain the Laplace Transform of $\frac{\cos\sqrt{t}}{\sqrt{t}}$

Solution : $\sin\sqrt{t} = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + ...$

$$\therefore L\left[\sin\sqrt{t}; p\right] = \frac{\Gamma(3/2)}{p^{3/2}} - \frac{1}{3!}\frac{\Gamma(5/2)}{p^{5/2}} + \frac{1}{5!}\frac{\Gamma(7/2)}{p^{7/2}} - \frac{1}{7!}\frac{\Gamma(9/2)}{p^{9/2}} + ...$$

$$= \frac{\sqrt{\pi}}{2p^{3/2}} \left\{ 1 - \left(\frac{1}{4p}\right) + \frac{1}{2!}\left(\frac{1}{4p}\right)^2 - \frac{1}{3!}\left(\frac{1}{4p}\right)^3 + \ldots \right\}$$

$$= \frac{\sqrt{\pi}}{2p^{3/2}} \, e^{-(1/4p)}$$

Next let $f(t) = \sin\sqrt{t}$, so that $f'(t) = \dfrac{\cos\sqrt{t}}{2\sqrt{t}}$, $f(0) = 0$

Now using the formula $L[f'(t);p] = p\,L[f(t);p] - f(0)$

$\therefore$ we have $L\left[\dfrac{\cos\sqrt{t}}{2\sqrt{t}};p\right] = p\,L\left[\sin\sqrt{t};p\right] - 0$

$$= p\left(\frac{\sqrt{\pi}}{2p^{3/2}} \quad e^{-(1/4p)}\right)$$

or $\quad L\left[\dfrac{\cos\sqrt{t}}{\sqrt{t}};p\right] = \left(\dfrac{\pi}{p}\right)^{1/2} e^{-(1/4p)}$

Example 11 : If $L[f(t);p] = \bar{f}(p)$, then $\int_0^\infty \dfrac{f(t)}{t}dt = \int_0^\infty \bar{f}(u)\,du$,

assuming that the integrals converge and hence prove that

$$\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}.$$

Solution : We have, by Theorem 10

$$L\left[\frac{f(t)}{t};p\right] = \int_0^\infty e^{-pt}\frac{f(t)}{t}dt = \int_p^\infty \bar{f}(u)\,du$$

Taking $p \to 0$ and assuming both the integrals converge, we get

$$\int_0^\infty \frac{f(t)}{t}dt = \int_0^\infty \bar{f}(u)\,du \qquad \ldots(13)$$

Next, let $f(t) = \sin t$, so that

$$L[\sin t;p] = \frac{1}{p^2+1} = \bar{f}(p)$$

$\therefore$ By using (13), we get

$$\int_0^\infty \frac{\sin t}{t} dt = \int_0^\infty \frac{1}{u^2+1} du = \left(\tan^{-1} u\right)_0^\infty = \pi/2$$

Example 12 : Prove that $L\left[\frac{\sin^2 t}{t}; p\right] = \frac{1}{4}\log\left(\frac{p^2+4}{p^2}\right)$ and deduce that

(i) $\quad \int_0^\infty e^{-t} \frac{\sin^2 t}{t} dt = \frac{1}{4}\log 5$

(ii) $\quad \int_0^\infty \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$

Solution : Since $\sin^2 t = \frac{1-\cos 2t}{2}$, therefore proceeding as in Example 9 (ii) we find that

$$\therefore \quad L\left[\frac{\sin^2 t}{t}; p\right] = \frac{1}{4}\log\left(\frac{p^2+4}{p^2}\right) \qquad ...(14)$$

or $\quad \int_0^\infty e^{-pt} \frac{\sin^2 t}{t} dt = \frac{1}{4}\log\left(\frac{p^2+4}{p^2}\right)$

Taking limit $p \to 1$, we have

$$\int_0^\infty e^{-t} \frac{\sin^2 t}{t} dt = \frac{1}{4}\log 5 \qquad ...(15)$$

Again applying the theorem 10 in equation (14) we have

$$L\left[\frac{\sin^2 t}{t^2}; p\right] = \frac{1}{4}\int_p^\infty \log\left(\frac{u^2+4}{u^2}\right) du$$

$$= \frac{1}{4}\left[\int_p^\infty \log\left(u^2+4\right) du - \int_p^\infty \log u^2 \, du\right]$$

$$= \frac{1}{4}\left[\left\{u\log\left(u^2+4\right)\right\}_p^\infty - \int_p^\infty \frac{2u^2}{u^2+4} du - \left(u\log u^2\right)_p^\infty + 2\int_p^\infty du\right]$$

$$= \frac{1}{4}\left[\left\{u\log\left(u^2+4\right) - u\log u^2\right\} - 2\left\{u - \frac{4}{2}\tan^{-1}\left(\frac{u}{2}\right) - u\right\}\right]_p^\infty$$

$$= \frac{1}{4}\left[\lim_{u\to\infty} u\log\left(1+\frac{4}{u^2}\right) - p\log\left(1+\frac{4}{p^2}\right) + 4\frac{\pi}{2} - 4\tan^{-1}\left(\frac{p}{2}\right)\right]$$

$$= \frac{1}{4}\left[0 - p\log\left(1 + \frac{4}{p^2}\right) + 2\pi - 4\tan^{-1}\left(\frac{p}{2}\right)\right]$$

$$\therefore \qquad \int_0^\infty e^{-pt}\frac{\sin^2 t}{t^2}dt = \frac{1}{4}\left[-p\log\left(1 + \frac{4}{p^2}\right) + 2\pi - 4\tan^{-1}\left(\frac{p}{2}\right)\right]$$

Taking limit $p \to 0$, we get

$$\int_0^\infty \frac{\sin^2 t}{t^2}dt = \frac{\pi}{2}$$

Since $\quad \underset{u\to\infty}{Lim}\, u\log\left(1 + \frac{4}{u^2}\right) = \underset{u\to\infty}{Lim}\, u\left\{\frac{4}{u^2} - \frac{1}{2}\frac{6!}{u^4} + ...\right\} = 0$

and $\quad \underset{p\to 0}{Lim}\, p\log\left(1 + \frac{4}{p^2}\right) = \underset{p\to 0}{Lim}\, p\log\left(p^2 + 4\right) - \underset{p\to 0}{Lim}\, p\log p^2$

$$= 0 - 2\underset{p\to 0}{Lim}\frac{\log p}{(1/p)}$$

$$= -2\underset{p\to 0}{Lim}\frac{(1/p)}{\left(-1/p^2\right)}$$

$$= 0$$

Self-Learning Exercise - II

Assuming the conditions of validity, fill in the blanks in the following :

1. $L\left\{f^{(n)}(t); p\right\} = ...$

2. $L\left\{t^n f(t); p\right\} = ...$

3. $L\left\{\int_0^t f(u)\, du; p\right\} = ...$

4. $L\left\{t^n e^{at}; p\right\} = ...$

5. $L\left\{\left(t\frac{d}{dt}\right)f(t); p\right\} = ...$

6. If $m \geq n$, then

   $L\left\{t^m f^{(n)}(t); p\right\} = ...$

7. If $L\left\{\frac{\sin t}{t};p\right\}=\cot^{-1}p$, then $L\left\{\frac{\sin at}{t};p\right\}=....$

1.11 Exercise 1 (b)

1. Given $L\left\{2\sqrt{\frac{t}{\pi}};p\right\}=\frac{1}{p^{3/2}}$, show that $\frac{1}{p^{1/2}}=L\left\{\sqrt{\frac{1}{\sqrt{\pi t}}};p\right\}$

2. Verify directly that $L\left\{\int_0^t\left(u^2-u+e^{-u}\right)du;p\right\}=\frac{1}{p}L\left\{t^2-t+e^{-t};p\right\}$

3. Evaluate Laplace Transform of the following functions :

(i) $t\sin at$ **Ans.** $\frac{2ap}{\left(p^2+a^2\right)^2}$

(ii) $t^3\cos t$ **Ans.** $\frac{6p^4-36p^2+6}{\left(p^2+1\right)^4}$

(iii) $\left(t^2-3t+2\right)\sin 3t$ **Ans.** $\frac{6p^4-18p^3+126p^2-162p+432}{\left(p^2+9\right)^3}$

(iv) $t^2e^t\sin 4t$ **Ans.** $\frac{8\left\{3(p-1)^2-16\right\}}{\left\{(p-1)^2+16\right\}^3}$

(v) $t\,e^{-t}\cosh t$ **Ans.** $\frac{p^2+2p+2}{p^2(p+2)^2}$

4. Find the Laplace Transform of $\frac{\sin at}{t}$. Does the transform of $\frac{\cos at}{t}$ exist? Also prove that $\int_0^\infty\frac{\sin t}{t}dt=\frac{\pi}{2}$.

Ans. $\cot^{-1}\left(\frac{p}{a}\right)$, does not exists

5. Show that $L\left\{\frac{1-\cosh at}{t};p\right\}=\frac{1}{2}\log\left(\frac{p^2-a^2}{p^2}\right)$

6. Evaluate $L\left\{\frac{e^{-at}-e^{-bt}}{t};p\right\}$ and hence deduce that $\int_0^\infty\frac{e^{-3t}-e^{-6t}}{t}dt=\log 2$.

Ans. $\log\left(\frac{p+b}{p+a}\right)$

7. Using Laplace Transform technique, evaluate the following integrals :

(i) $\int_0^\infty t^3 e^{-t} \sin t \, dt$ ***Ans.*** 0

(ii) $\int_0^\infty \frac{e^{-t} \sin t}{t} dt$ ***Ans.*** $\frac{\pi}{4}$

8. If $L[f(t); p] = \bar{f}(p)$, find the Laplace transforms of $f'(t)$ and $f''(t)$, stating conditions of validity of results.

9. If $f(t) = \begin{cases} t\,G(t), & t > 1 \\ 0, & 0 < t < 1 \end{cases}$

Prove that $L[f(t); p] = \frac{d}{dp}\left[e^{-ap} L\{G(t+1); p\}\right]$

10. Prove that $L\left\{\frac{\cos at - \cos bt}{t}; p\right\} = \frac{1}{2}\log\left(\frac{p^2 + b^2}{p^2 + a^2}\right)$

11. Show that $L\left\{\int_0^t \frac{1 - e^{-u}}{u} du; p\right\} = \frac{1}{p}\log\left(1 + \frac{1}{p}\right)$

12. Given that $f(t) = \begin{cases} 2t, & 0 \le t \le 1 \\ t, & t > 1 \end{cases}$

Find (i) $L\{f(t); p\}$ (ii) $L\{f'(t); p\}$

Does the result $L\{f'(t); p\} = L\{f(t); p\} - f(0)$ hold for this case? Explain.

Ans. (i) $-\frac{e^{-p}}{p} + \frac{2}{p^2} - \frac{e^{-p}}{p^2}$ (ii) $\frac{2}{p} - \frac{e^{-p}}{p}$

The result does not hold as $f(t)$ is discoutinuous at $t = 1$

1.12 Initial Value Theorem

Theorem 11 : Let $f(t)$ be continuous for all $t \ge 0$ and be of exponential ordrer as $t \to \infty$. Also suppose that $f'(t)$ is of class A, then

$$\lim_{t \to 0} f(t) = \lim_{p \to \infty} p\,L[f(t); p]$$

Proof : By Theorem 6, we have

$$L[f'(t); p] = p\,L[f(t); p] - f(0)$$

or $$\int_0^\infty e^{-pt} f'(t)\,dt = p\,L[f(t);p] - f(0) \qquad ...(16)$$

Since $f'(t)$ is sectionally continuous and of exponential order, we have

$$\lim_{p\to\infty} \int_0^\infty e^{-pt} f'(t)\,dt = 0$$

Now taking limit as $p \to \infty$ in (16), we find that

$$0 = \lim_{p\to\infty} p\,L[f(t);p] - f(0)$$

or $$f(0) = \lim_{p\to\infty} p\,L[f(t);p] \qquad ...(17)$$

Since $f(t)$ is continuous at $t = 0$, we have

$$f(0) = \lim_{t\to 0} f(t)$$

$\therefore$ From equation (17), we get

$$\lim_{t\to 0} f(t) = \lim_{p\to\infty} p\,L[f(t);p] \qquad ...(18)$$

1.13 Final Value Theorem

Theorem 12 : Let $f(t)$ be continuous for all $t \geq 0$ and be of exponential order as $t \to \infty$ and if $f'(t)$ is of class A, then

$$\lim_{t\to\infty} f(t) = \lim_{p\to 0} p\,L[f(t);p]$$

Proof : By Theorem 6, we have

$$L[f'(t);p] = p\,L[f(t);p] - f(0) \qquad ...(19)$$

Taking limit as $p \to 0$ in (19), we have

$$\lim_{p\to 0} \int_0^\infty e^{-pt} f'(t)\,dt = \lim_{p\to 0} p\,L[f(t);p] - f(0)$$

or $$\int_0^\infty f'(t)\,dt = \lim_{p\to 0} p\,L[f(t);p] - f(0)$$

or $$[f(t)]_0^\infty = \lim_{p\to 0} p\,L[f(t);p] - f(0)$$

or $$\lim_{t\to\infty} f(t) - f(0) = \lim_{p\to 0} p\,L[f(t);p] - f(0)$$

or $$\lim_{t\to\infty} f(t) = \lim_{p\to 0} p\,L[f(t);p]$$

Hence the final value theorem is verified.

1.14 Periodic Functions

Definition : A function $f(t)$ is said to be periodic if there exists a real number T such that

$$f(t+T)=f(t) \qquad \forall t$$

If T is the smallest positive number for which such a relation is satisfied, then T is called the period of the function. For example, the simplest periodic fucntions are $\sin t$ and $\cos t$ having period 2π. Their reciprocals cosec t and $\sec t$ are also periodic with period 2π and $\tan t$ and $\cot t$ are periodic with period π

Theorem 13 : If $f(t)$ is a periodic function with period $T>0$ i.e. $f(u+T)=f(u)$, $f(u+2T)=f(u)$, etc.

then $$L[f(t);p]=\frac{1}{1-e^{-pT}}\int_0^T e^{-pt}f(t)\,dt$$

Proof : We have

$$L[f(t);p]=\int_0^\infty e^{-pt}f(t)\,dt$$

$$=\int_0^T e^{-pt}f(t)\,dt+\int_T^{2T} e^{-pt}f(t)\,dt+\int_{2T}^{3T} e^{-pt}f(t)\,dt+...$$

$$=\int_0^T e^{-pt}f(t)\,dt+\int_0^T e^{-p(u+T)}f(u+T)\,du+\int_0^T e^{-p(u+2T)}f(u+2T)\,du+...$$

(Putting $t=u+T, t=u+2T$, etc. in the 2nd and 3rd .... integrals respectively)

$$=\int_0^T e^{-pu}f(u)\,du+e^{-pT}\int_0^T e^{-pu}f(u)\,du+e^{-2pT}\int_0^T e^{-pu}f(u)\,du+...$$

$$=\left(1+e^{-pT}+e^{-2pT}+....\right)\int_0^T e^{-pu}f(u)\,du$$

$$=\frac{1}{1-e^{-pT}}\int_0^T e^{-pu}f(u)\,du \qquad \left(\because p>0, e^{-pT}<1\right)$$

$$=\frac{1}{1-e^{-pT}}\int_0^T e^{-pt}f(t)\,dt$$

(Using the relation; $1+r+r^2+.....=\dfrac{1}{1-r}$, $|r|<1$)

1.15 Some Special Functions

A. The Gamma Function : The gamma function is defined by

$$\Gamma(n) = \int_0^\infty e^{-u} \; u^{n-1} \, du, \qquad \text{Re}(n) > 0$$

B. **The sme and cosnie Integrals :** The sme and cosine integrals denoted by $S_i(t)$ and $C_i(t)$ respectively are defined by the equations

$$S_i(t) = \int_0^t \frac{\sin u}{u} du$$

and $$C_i(t) = \int_0^t \frac{\cos u}{u} du$$

C. **The Error Function and Complementary Error Function :**

(i) The error fucntion, denoted by $erf(t)$, is defined by

$$erf(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du$$

(ii) The complementary error function, denoted by $erfc(t)$, is defined by

$$erfc(t) = 1 - erf(t) = 1 - \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du$$

$$= \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2} du$$

D. **The Unit Step Function (or Heaviside's Unit Function) :** The unit step function, denoted by $U(t-a)$ is defined by

$$U(t-a) = \begin{cases} 0; \; t < a \\ 1 \; ; \; t > a \end{cases}$$

E. **The Unit Impulse Function or Dirac Delta Function :** Consider the function $\delta_\in(t)$ given by

$$\delta_\in(t) = \begin{cases} 1/\in, & 0 \le t \le \in \\ 0 \;, & t > \in \end{cases}$$

where $\in > 0$. The graph of this function is shown in the diagram below

The Dirac's delta function or unit impulse function is denoted by $\delta(t)$ and is defined as

$$\delta(t) = \lim_{\in \to 0} \delta_\in(t)$$

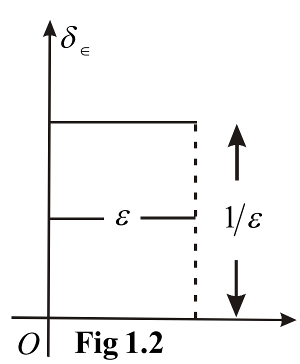


Fig 1.2

F. Bessel Function : Bessel's function of order n is defined by

$$J_n(t) = \frac{t^n}{2^n \Gamma(n+1)}\left[1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2.4(2n+2)(2n+4)} - ....\right]$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \cdot \left(\frac{t}{2}\right)^{n+2r}$$

G. Laguerre Polynomial : Laguerre polynomial is defined by

$$L_n(t) = \frac{e^t}{n!} \cdot \frac{d^n}{dt^n}\left(e^{-t}.t^n\right),\ n = 0,1,2....$$

H. The Exponential Integral : The exponential integral function $E_i(t)$ is defined as

$$E_i(t) = \int_t^{\infty} \frac{e^{-u}}{u}\, du \qquad (t > 0)$$

I. Hypergeometric Function : The function ${}_2F_1(\alpha,\beta;\gamma;t)$ known as Gauss's hypergeometric function or simply hypergeometric function is defined as

$${}_2F_1(\alpha,\beta;\gamma;t) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{t^n}{n!}$$

J. Beta Function : We define the Beta function $B(m,n)$ by the integral

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\, dt$$

where $\operatorname{Re}(m) > 0$, $\operatorname{Re}(n) > 0$

Example 13 : Prove that $L[U(t-a);p] = \dfrac{e^{-ap}}{p}$, where $U(t-a)$ is the Heaviside's unit step function

Solution : By the definition of Heaviside's unit step function,

We have $U(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$

$$L[u(t-a);p] = \int_0^{\infty} e^{-pt} U(t-a)\, dt$$

$$= \int_0^{a} e^{-pt}\, U(t-a)\, dt + \int_a^{\infty} e^{-pt}\, U(t-a)\, dt$$

$$= \int_a^{\infty} e^{-pt}\, dt$$

$$=\left[\frac{e^{-pt}}{-p}\right]_{t=a}^{\infty}=\frac{e^{-ap}}{p}$$

Example 14 : Find $L[\delta_{\in}(t);p]$ where $\delta_{\in}(t)$ is defined in ξ 1.15 and hence show that $\underset{\in\to 0}{Lim}\, L[\delta_{\in}(t);p]=1$.

Solution : We have $\delta_{\in}(t)=\begin{cases}1/\in, & 0\le t\le \in \\ 0\ , & t>\in\end{cases}$

Then $L[\delta_{\in}(t);p]=\int_0^{\infty} e^{-pt}\delta_{\in}(t)\,dt$

$$=\int_0^{\in} e^{-pt}\delta_{\in}(t)\,dt+\int_{\in}^{\infty} e^{-pt}\delta_{\in}(t)\,dt$$

$$=\int_0^{\in} e^{-pt}(1/\in)\,dt+\int_{\in}^{\infty} e^{-pt}.0\,dt$$

$$\therefore\quad L[\delta_{\in}(t);p]=\frac{1}{\in}\int_0^{\in} e^{-pt}\,dt=\frac{1}{\in}\left[\frac{-e^{-pt}}{p}\right]_0^{\in}=\frac{1-e^{-p\in}}{\in p}$$

Thus

$$\underset{\in\to 0}{Lim}\, L[\delta_{\in}(t);p]=\underset{\in\to 0}{Lim}\frac{1-e^{-p\in}}{p\in}$$

$$=\frac{1}{p}\underset{\in\to 0}{Lim}\frac{p\,e^{-p\in}}{1} \qquad \text{(by L' Hospital's Rule)}$$

Example 15 : Evaluate $L[S_i(t);p]=1$, where $S_i(t)=\int_0^t \frac{\sin u}{u}du$.

Solution : We know that

$$S_i(t)=\int_0^t \frac{\sin u}{u}du$$

$$=\int_0^t\left(1-\frac{u^2}{3!}+\frac{u^4}{5!}-\frac{u^6}{7!}+....\right)du$$

$$=t-\frac{t^3}{3(3!)}+\frac{t^5}{5(5!)}-\frac{t^7}{7(7!)}+....$$

Taking Laplace transform of each side, we get

$$L[S_i(t);p]=\frac{1}{p^2}-\frac{1}{3(3!)}.\frac{3!}{p^4}+\frac{1}{5(5!)}.\frac{5!}{p^6}-....$$

$$= \frac{1}{p}\left[\frac{1}{p} - \frac{1}{3}.\frac{1}{p^3} + \frac{1}{5}.\frac{1}{p^5} - \frac{1}{7}.\frac{1}{p^7} + \ldots\right]$$

$$= \frac{1}{p}\tan^{-1}\left(\frac{1}{p}\right) \qquad \left(\because \tan^{-1} t = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \ldots . \ |t| < 1\right)$$

Example 16 : Prove that $L\left(C_i(t); p\right) = L\left[\int_t^\infty \frac{\cos u}{u}\, du; p\right] = \frac{\log\left(p^2+1\right)}{2p}$

Solution : Let $f(t) = C_i(t) = \int_t^\infty \frac{\cos u}{u}\, du = -\int_\infty^t \frac{\cos u}{u}\, du$

so that by Leibnitz's rule

$$f'(t) = -\frac{\cos t}{t}$$

or $\quad t\, f'(t) = -\cos t$

$\therefore \quad L\left[t\, f'(t); p\right] = L\left[-\cos t; p\right] = -\frac{p}{p^2+1}$

or $\quad -\frac{d}{dp} L\left[f'(t); p\right] = \frac{-p}{p^2+1}$

or $\quad \frac{d}{dp}\left[p\bar{f}(p) - f(0)\right] = \frac{p}{p^2+1} \qquad$ where $\bar{f}(p) = L\left[f(t); p\right]$

or $\quad \frac{d}{dp}\left[p\bar{f}(p)\right] = \frac{p}{p^2+1} \qquad \left[\because f(0) \text{ is constant}\right]$

Integrating both sides with respect to $'p'$, we get

$$p\bar{f}(p) = \frac{1}{2}\log\left(p^2+1\right) + A \qquad \ldots(20)$$

But from the final value theorem

$$\lim_{p\to 0} p\bar{f}(p) = \lim_{t\to\infty} f(t) = 0$$

$\therefore$ From (20) as $p \to 0$, we have

$$0 = 0 + A \Rightarrow A = 0$$

or $\quad p\bar{f}(p) = \frac{1}{2}\log\left(p^2+1\right)$

or $$\bar{f}(p) = L[f(t); p] = L[C_i(t); p] = \frac{\log(p^2+1)}{2p}$$

Example 17 : Prove that $L[E_i(t); p] = \dfrac{\log(p+1)}{p}$

Solution : We have

$$E_i(t) = \int_t^\infty \frac{e^{-u}}{u}\,du$$

$$L[E_i; p] = L\left[\int_t^\infty \frac{e^{-u}}{u}\,du\right]$$

$$= L\left[\int_1^\infty \frac{e^{-tv}}{v}\,dv\right] \qquad \text{putting } u = tv$$

$$= \int_0^\infty e^{-pt}\left\{\int_1^\infty \frac{e^{-tv}}{v}\,dv\right\}dt$$

$$= \int_1^\infty \frac{1}{v}\left\{\int_0^\infty e^{-pt}.e^{-tv}\,dt\right\}dv \qquad \text{(changing the order of integration)}$$

$$= \int_1^\infty \frac{1}{v}.\frac{1}{p+v}\,dv$$

$$= \int_1^\infty \frac{1}{p}\left(\frac{1}{v} - \frac{1}{p+v}\right)dv$$

$$= \frac{1}{p}\left[\log v - \log(p+v)\right]_1^\infty$$

$$\therefore \qquad L[E_i(t); p] = \frac{1}{p}\left[-\log\left(\frac{p}{v}+1\right)\right]_1^\infty = \frac{1}{p}\log(p+1)$$

Example 18 : Prove that $L[J_0(t); p] = \dfrac{1}{\sqrt{1+p^2}}$

and hence deduce that

(i) $$L[J_0(at); p] = \frac{1}{\sqrt{p^2+a^2}}$$

(ii) $$L[t\,J_0(at); p] = \frac{p}{(p^2+a^2)^{3/2}}$$

$$(iii) \quad L\left[e^{-at} J_0(bt); p\right] = \frac{1}{\sqrt{\left(p^2 + 2ap + a^2 + b^2\right)}}$$

$$(iv) \quad \int_0^\infty J_0(t)\, dt = 1$$

and $$(v) \quad \int_0^\infty t\, \bar{e}^{3t} J_0(4t)\, dt = 3/125$$

Solution : We know that $J_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{t}{2}\right)^{n+2r}$

$$\therefore \quad J_0(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \cdot \left(\frac{t}{2}\right)^{2r}$$

$$= 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \ldots..$$

$$\therefore \quad L\left[J_0(t); p\right] = L[1; p] - \frac{1}{2^2} L\left[t^2; p\right] + \frac{1}{2^2.4^2} L\left[t^4; p\right] + \frac{2}{2^2.4^2.6^2} L\left[t^6; p\right] + \ldots.$$

$$= \frac{1}{p} - \frac{1}{2^2} \cdot \frac{2!}{p^3} + \frac{1}{2^2.4^2} \frac{4!}{p^5} - \frac{1}{2^2.4^2.6^2} \frac{6!}{p^7} + \ldots..$$

$$= \frac{1}{p}\left[1 - \frac{1}{2}\left(\frac{1}{p^2}\right) + \frac{1.3}{2.4}\left(\frac{1}{p^2}\right)^2 - \frac{1.3.5}{2.4.6}\left(\frac{1}{p^2}\right)^3 + \ldots..\right]$$

$$= \frac{1}{p}\left(1 + \frac{1}{p^2}\right)^{-1/2} = \frac{1}{\sqrt{\left(1+p^2\right)}}$$

Deductions :

(i) Since $L\left[f(at); p\right] = \frac{1}{a} \bar{f}\left(\frac{p}{a}\right)$, where $\bar{f}(p) = \left[f(t); p\right]$

$$\therefore \quad L\left[J_0(at); p\right] = \frac{1}{a} \cdot \frac{1}{\sqrt{\left\{1 + \left(\frac{p}{a}\right)^2\right\}}} = \frac{1}{\sqrt{p^2 + a^2}}$$

$$(ii) \quad L\left[t\, J_0(at); p\right] = -\frac{d}{dp} L\left[J_0(at); p\right]$$

$$= -\frac{d}{dp}\left[\frac{1}{\sqrt{(p^2+a^2)}}\right] = \frac{p}{(p^2+a^2)^{3/2}}$$

(iii) Since $L\left[e^{-at} f(t); p\right] = \bar{f}(p+a)$, where $\bar{f}(p) = L\left[f(t); p\right]$

$$\therefore \quad L\left[e^{-at} J_0(bt); p\right] = \frac{1}{\sqrt{(p+a)^2+b^2}} \qquad \left[\because \{J_0(bt); p\} = \frac{1}{\sqrt{p^2+b^2}}\right]$$

$$= \frac{1}{\sqrt{(p^2+2ap+a^2+b^2)}}$$

(iv) we have $L\left[J_0(t); p\right] = \int_0^\infty e^{-pt} J_0(t)\, dt = \frac{1}{\sqrt{(1+p^2)}}$

putting $p = 0$, we have $\int_0^\infty J_0(t)\, dt = 1$

(v) From deduction (ii), we have

$$\int_0^\infty e^{-pt}\, t\, J_0(at)\, dt = \frac{p}{(p^2+a^2)^{3/2}}$$

putting $p = 3$ and $a = 4$, we get

$$\int_0^\infty e^{-3t}\, t\, J_0(4t)\, dt = \frac{3}{(9+16)^{3/2}} = \frac{3}{125}$$

Self-Learning Exercise - III

Fill in the blanks in the following

1. If $L\{J_0(t); p\} = \frac{1}{\sqrt{p^2+1}}$, then

   (i) $L\{J_1(t); p\} = ...$

   (ii) $L\{t\, J_1(t); p\} = ...$

   (iii) $L\{J_1(at); p\} = ...$

2. If $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$ then $L\{f(t); p\} = ...$

3. If $\bar{f}(p) = L\{f(t); p\}$, then $L\left\{\int_0^\infty \frac{t^u f(u)}{\Gamma(u+1)}\, du; p\right\} = \ldots.$

4. If $L\left\{erf\left(\sqrt{t}\right); p\right\} = \frac{1}{p\sqrt{p+1}}$, then

   (a) $L\left\{erfc\left\{\sqrt{t}\right\}; p\right\} = \ldots..$

   (b) $L\left\{t\, erf\left\{2\sqrt{t}\right\}; p\right\} = \ldots..$

   (c) $L\left\{e^{3t}\, erf\left(\sqrt{t}\right); p\right\} = \ldots..$

5. $L\{\delta(t-a); p\} = \ldots..$

   where $\delta(t)$ is the Dirac delta function.

1.16 Summary

In this unit you studied an important integral transform known as Laplace transform, with existence conditions. You also studied some basic properties and results giving the Laplace transform of derivatives, integrals, multiplication and division by powers of $'t'$. A number of problems on Laplace transform are also solved to felicitate the understanding of this transform.

1.17 Answers to Self-Learning Exercises

Exercise - I

1. $\frac{1}{b}\bar{f}\left(\frac{p-a}{b}\right)$
2. $\frac{a}{p^2+a^2}$
3. $\frac{1}{a}\bar{f}\left(\frac{p-\log r}{a}\right)$
4. $e^{-ap}\bar{f}(p)$
5. 4
6. $\sqrt{\frac{\pi}{p}}$, $p > 0$
7. $\frac{2}{p^3} + \frac{4}{p^2} + \frac{4}{p}$

Exercise - II

1. $p^n\bar{f}(p) - p^{n-1}(0) - p^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$
2. $(-1)^n \frac{d^n}{dp^n}\bar{f}(p)$
3. $\frac{\bar{f}(p)}{p}$
4. $\frac{n!}{(p-a)^{n+1}}$
5. $-\frac{d}{dp}\left(p\bar{f}(p)\right)$

6. $\left(-\frac{d}{dp}\right)^m.\left(p^n\bar{f}(p)\right)$ 7. $\tan^{-1}(a/p)$

Exercise - III

1. (a) $1-\frac{p}{\sqrt{p^2+1}}$ (b) $\frac{1}{\left(p^2+1\right)^{3/2}}$ (c) $\frac{1}{a}\left(1-\frac{p}{\sqrt{p^2+a^2}}\right)$

2. $\frac{1+e^{-p\pi}}{p^2+1}$ 3. $\frac{\bar{f}(\log p)}{p}$

4. (a) $\frac{1}{\sqrt{p+1}\left(\sqrt{p+1}+1\right)}$ (b) $\frac{3p+8}{p^2\left(p^2+4\right)^{3/2}}$ (c) $\frac{1}{(p-3)\sqrt{p-2}}$

5. e^{-ap}

1.18 Exercise 1 (c)

1. If $f(t)=t^2$, $0<t<2$ and $f(t+2)=f(t)$, find $L[f(t)]$.

$$\left[\textbf{Ans.: } \frac{\left[2-\left(4p^2+4p+2\right)e^{-2p}\right]}{p^3\left(1-e^{-2p}\right)}\right]$$

2. If $f(t)$ be a periodic function with period 4, where

$$f(t)=\begin{cases}3t, & 0<t<2\\ 6, & 2<t<4\end{cases}$$

then prove that $L[f(t);p]=\frac{3\left(1-e^{-2p}-2p\,e^{-4p}\right)}{p^2\left(1-e^{-4p}\right)}$

3. Verify the initial value theorem for

(i) $3-2\cos t$ (ii) $(2t+3)^2$

4. Verify the final value theorem for

(i) $t^3\,e^{-2t}$ (ii) $1+e^{-t}(\sin t+\cos t)$

5. Show that

(i) $\int_0^\infty J_0\left(2\sqrt{tu}\right)\cos u\,du=\sin t$ (ii) $\int_0^\infty J_0\left(2\sqrt{tu}\right)\sin u\,du=\cos t$

[Hint. First evaluate $L\left[J_0 2\sqrt{tu}\;e^{iu};p\right]$, and then compare real and imaginary parts of the result thus obtained]

6. Show that $L\left[\frac{d^2}{dt^2}\, e^{-2t} J_0(2t); p\right] = \frac{p^2}{\sqrt{p^2-4p+8}} - p - 2$

7. Find the Laplace Transform of the following functions :

(i) $t\,e^{-2t} J_0\left(t\sqrt{2}\right)$ (ii) $t\,S_i(t)$ (iii) $t^2 C_i(t)$

(iv) $t\,E_i(t)$ (v) $e^{-3t} E_i(t)$ (vi) $t\,e^{-2t} E_i(3t)$

[**Ans.:** (i) $\frac{p+2}{\left(p^2+4p+6\right)^{3/2}}$ (ii) $\frac{\tan^{-1}(1/p)}{p^2} + \frac{1}{p\left(p^2+1\right)}$

(iii) $\frac{\log\left(p^2+1\right)}{p^3} - \frac{3p^2+1}{p\left(p^2+1\right)^2}$ (iv) $\frac{\log(p+1)}{p^2} - \frac{1}{p(p+1)}$

(v) $\frac{\log(p+4)}{p+3}$ (vi) $\frac{1}{(p+2)^2}\log\left(\frac{p+5}{3}\right) - \frac{1}{(p+2)(p+5)}$]

8. Find Laplace transform of :

(i) $\int_0^t \left(\frac{1-e^{-2u}}{u}\right) du$ (ii) $\int_0^t erf\left(\sqrt{u}\right) du$ (iii) $\int_0^\infty \cos\left(xt^2\right) dt$

[**Ans.:** (i) $\frac{1}{p}\log\left(1+\frac{2}{p}\right)$ (ii) $\frac{1}{p^2\sqrt{p+1}}$ (iii) $\frac{\pi}{2\sqrt{2p}}$]

9. If $L\left[f''(t); p\right] = \tan^{-1}\left(\frac{1}{p}\right)$, $f(0)=2$ and $f'(0)=-1$, find $L\left[f(t); p\right]$.

[**Ans.:** $\frac{1}{p^2}\left\{2p-1+\tan^{-1}\left(\frac{1}{p}\right)\right\}$]

10. Find (i) $L\left[t\,u(t-1)+t^2\delta(t-1); p\right]$ and (ii) $L\left[\cos t \log t\, \delta(t-\pi); p\right]$

[**Ans.:** (i) $\frac{e^{-p}}{p^2}\left(p^2+p+1\right)$ (ii) $-e^{-\pi p}\log\pi$]

11. Find the Laplace Transform of the periodic function having period $2k$ and defined by

$f(t) = 1$ $0 < t < k$

$$= -1 \qquad k < t < 2k$$

$\left[\textbf{Ans.: } \frac{1}{p}\tanh\left(\frac{pk}{2}\right)\right]$

12. Prove that $L\left[J_1(t);p\right] = 1 - \frac{p}{\left(p^2+1\right)^{1/2}}$

and deduce that (i) $\quad L\left[t\,J_1(t);p\right] = \frac{a}{\left(p^2+1\right)^{3/2}}$

(ii) $\quad L\left[J_1(at);p\right] = \frac{1}{a}\left\{1 - \frac{p}{\left(p^2+a^2\right)^{1/2}}\right\}$

13. Prove that for $Re(p) > a > 0$

$$L\left[t^{\upsilon} J_{\upsilon}(at);p\right] = \frac{(2a)^{\upsilon}\,\Gamma(\upsilon+1/2)}{\sqrt{\pi}\,\left(p^2+a^2\right)^{\upsilon+1/2}},\quad \upsilon > -\frac{1}{2}$$

14. Prove that if $a > 0, Re(p) > 0$,

$$L\left[t^{\upsilon/2} J_{\upsilon}\left(2\sqrt{at}\right);p\right] = a^{\upsilon/2}\,p^{-\upsilon-1}\,e^{-a/p}$$

Hence deduce that

(i) $\quad L\left\{t^{\upsilon/2}e^{at}\int_0^{\infty} u^{-\upsilon/2}e^{-au}J_{\upsilon}\left(2\sqrt{ut}\right)f(u)\,du;p\right\} = (p-a)^{-\upsilon-1}\;L\left[f(u);a+(p-a)^{-1}\right]$

(ii) $\quad L\left\{t^{\upsilon/2}\int_0^{\infty} u^{-\upsilon/2}J_{\upsilon}\left(2\sqrt{ut}\right)f(u)\,du;p\right\} = p^{-\upsilon-1}\;L\left[f(t);p^{-1}\right]$

15. Prove that $\int_0^{\infty} \frac{J_0(t)-\cos t}{t}\,dt = \log 2$

16. Using Laplace transform, evaluate

$$\int_0^{\infty} t\,e^{-t^2}J_0(at)\,dt$$

$\left[\textbf{Ans.: } \frac{1}{2}\,e^{-a^2/4}\right]$

17. If $L_n[t,p] = \frac{e^t}{n!}\frac{d^n}{dt^n}\left(e^{-t}.t^n\right)$, then prove that $L\left[L_n(t);p\right] = \frac{(p-1)^n}{p^{n+1}},\ p > 1$

18. Obtain $L\left[erf\ t;p\right]$. Hence deduce the value of $L\left[erf\,(bt);p\right]$.

$\left[\textbf{Ans.: } p^{-1}\,e^{p^2/4}\,erfc(p/2),\ p^{-1}\,e^{p^2/4b^2}\,erfc(p/2b)\right]$

19. Evaluate $L\{t^{\upsilon+1}J_\upsilon(at);p\}$ for $Re(p)>a>0$ and $\upsilon>-3/2$

$$\left[\textbf{Ans.:}\ \frac{2^{\upsilon+1}a^\upsilon p\,\Gamma(\upsilon+3/2)}{\sqrt{\pi}\,(p^2+a^2)^{3/2}}\right]$$

20. Prove that $L\left\{\int_0^\infty \frac{x^{u-1}f(u)}{\Gamma(u)}\,du;p\right\}=\bar{f}(\log p)$ where $\bar{f}(p)=L[f(t);p]$

21. Find the Laplace Transform of the function

$$t^{-3/2}\,e^{-k^2/4t}\ (k>0)$$

$$\left[\textbf{Ans.:}\ \frac{2\sqrt{\pi}}{k}\,e^{-k\sqrt{p}}\right]]$$

22. Show that for $Re(p)>1$,

$$L[J_\upsilon(t);p]=2^{-\upsilon}p^{-\upsilon-1}{}_2F_1\left(\frac{\upsilon+1}{2},\frac{\upsilon+2}{2};\upsilon+1;-\frac{1}{p^2}\right)$$

using the result

$${}_2F_1\left(a,a+\frac{1}{2};2a;x\right)=2^{2a-1}(1-x)^{-1/2}\left[1+\sqrt{(1-x)}\right]^{1-2a}$$

Show that

$$L\{J_\upsilon(t);p\}=(p^2+1)^{-1/2}\left\{p+\sqrt{p^2+1}\right\}^{-\upsilon}$$

and deduce that

(i) $$L\{J_\upsilon(at);p\}=\frac{\left\{\sqrt{p^2+a^2}-p\right\}^\upsilon}{a^\upsilon\sqrt{p^2+a^2}}$$

(ii) $$\int_0^\infty J_\upsilon(t)\,dt=1$$

(iii) $$L\{t\,J_\upsilon(t);p\}=\frac{p+\upsilon\sqrt{p^2+1}}{(p^2+1)^{3/2}\left(p+\sqrt{p^2+1}\right)^\upsilon}$$

(iv) $$\int_0^\infty t\,J_\upsilon(t)\,dt=\upsilon$$

(v) $$L\left\{\frac{J_\upsilon(t)}{t};p\right\}=\frac{\left(p+\sqrt{p^2+1}\right)^{-\upsilon}}{\upsilon}$$

Unit - 2
The Inverse Laplace Transform

Structure of the Unit

2.0 Objective

The object of this unit is to define inverse Laplace transform with its simple properties. We shall prove some imporatant theorems regarding its derivatives, integrals, multiplication and division by powers of p. We shall also discuss the convolution theorem and complex inversion formula for Laplace transform.

2.1 Introduction

In the last unit we studied the Laplace transform and its properties. In this unit we define the inverse Laplace transform and establish various properties and results associated with inverse Laplace Transform.

2.2 The Inverse Laplace Transform

2.2.1 Definition : If $\bar{f}(p)$ is the Laplace transform of a function $f(t)$,

i.e. $L[f(t); p] = \bar{f}(p)$

then $f(t)$ is called the inverse Laplace transform of the function $\bar{f}(p)$ and is written as

$$f(t) = L^{-1}[\bar{f}(p)]$$

L^{-1} is called the inverse Laplace transformation operator.

Example 1 : $L[t^n; p] = \frac{n!}{p^{n+1}}$

$$\therefore \qquad t^n = L^{-1}\left[\frac{n!}{p^{n+1}}; t\right]$$

2.2.2 Null Function

If $N(t)$ be a function of t such that $\int_0^t N(t)\,dt = 0, \quad \forall\ t > 0$

Then $N(t)$ is called a Null function.

Example 2 : The function $f(t) = \begin{cases} 1, & t = 1 \\ -1, & t = 2 \\ 0, & otherwise \end{cases}$

is a Null function.

2.2.3 Uniqueness

We know that $L[N(t); p] = 0$

$$\therefore \qquad N(t) = L^{-1}(0)$$

Further, if $L\left[f(t);p\right]=\bar{f}(p)$, then

$$L\left[f(t)+N(t);p\right]=\bar{f}(p)$$

Consequently if $L^{-1}\left[\bar{f}(p)\right]=f(t)$

then $\quad L^{-1}\left[\bar{f}(p)\right]=f(t)+N(t)$

which implies that, we can have two different functions, with the same Laplace transform.

Example 3 : The two different functions

$$f_1(t)=e^{-\alpha t} \text{ and } f_2(t)=\begin{cases} 0\,, & for\ \ t=1 \\ e^{-\alpha t}, & otherwise \end{cases}$$

have the same Laplace Transform i.e. $\dfrac{1}{(p+\alpha)}$, $\left(Re(\alpha)>0\right)$.

Hence the inverse Laplace transform of a function is unique if we do not allow Null functions. This is indicated in Lerch's theorem given below :

Lerch's Theorem : Let $L\left[f(t)\right]=\bar{f}(p)$ and $f(t)$ be piecewise continous in every finite interval $0\le t\le N$ and of exponential ordre for $t>N$, then the inverse Laplace transform of $\bar{f}(p)$ is unique.

Remark : Throught this unit, we shall assume such uniqueness unless otherwise stated.

2.3 Inverse Laplace Transform of Certain Elementary Functions

From the definition 2.2.1 of inverse Laplace transform and Laplace transform of some elementary functions mentioned in the last unit, we obtain

(i) $\quad L\left[t^n;p\right]=\dfrac{n!}{p^{n+1}}\Rightarrow L^{-1}\left[\dfrac{1}{p^{n+1}};t\right]=\dfrac{t^n}{n!}$ where $n\in N$.

(ii) $\quad L\left[t^\alpha;p\right]=\dfrac{\Gamma(\alpha+1)}{p^{\alpha+1}}\Rightarrow L^{-1}\left[\dfrac{1}{p^{\alpha+1}};t\right]=\dfrac{t^\alpha}{\Gamma(\alpha+1)}$ if $Re(\alpha)>-1$

(where α may be a real or complex number)

(iii) $\quad L\left[e^{at};p\right]=\dfrac{1}{p-a}\Rightarrow L^{-1}\left[\dfrac{1}{p-a};t\right]=e^{at}$ if $p>a$.

(iv) $\quad L\left[\sin at;p\right]=\dfrac{a}{p^2+a^2}\Rightarrow L^{-1}\left[\dfrac{1}{p^2+a^2};t\right]=\dfrac{\sin at}{a}$

(v) $L\left[\cos a\,t;p\right]=\dfrac{p}{p^2+a^2}\Rightarrow L^{-1}\left[\dfrac{p}{p^2+a^2};t\right]=\cos a\,t$

(vi) $L\left[\sinh a\,t;p\right]=\dfrac{a}{p^2-a^2}\Rightarrow L^{-1}\left[\dfrac{1}{p^2-a^2}\right]=\dfrac{\sinh a\,t}{a}$

(vii) $L\left[\cos h\,a\,t;p\right]=\dfrac{p}{p^2-a^2}\Rightarrow L^{-1}\left[\dfrac{p}{p^2-a^2};t\right]=\cos h\,a\,t$

(viii) $L\left[J_0(at);p\right]=\dfrac{1}{\sqrt{p^2+a^2}}\Rightarrow L^{-1}\left[\dfrac{1}{\sqrt{p^2+a^2}};t\right]=J_0(at)$

2.4 Some Important Properties

2.4.1 Linearity Property :

Theorem 2 : Let for all $i=1,2,3,\ldots,n$ if $\bar{f}_i(p)$ are Laplace transforms of the functions $f_i(t)$ and c_i are constants, then

$$L^{-1}\left[c_1\bar{f}_1(p)\pm c_2\bar{f}_2(p)\pm\ldots\ldots\pm c_n\bar{f}_n(p)\right]=c_1L^{-1}\left[\bar{f}_1(p)\right]\pm c_2L^{-1}\left[\bar{f}_2(p)\right]\pm\ldots\pm c_nL^{-1}\left[\bar{f}_n(p)\right]$$

Proof : By Linearity property of Laplace transform, we have

$$L\left[c_1f_1(t)\pm c_2f_2(t)\pm\ldots\ldots\pm c_nf_n(t)\right]=c_1L\left[f_1(t)\right]\pm c_2L\left[f_2(t)\right]\pm\ldots\pm c_nL\left[f_n(t)\right]$$

$$=c_1\bar{f}_1(p)\pm c_2\bar{f}_2(p)\pm\ldots\ldots\pm c_n\bar{f}_n(p)$$

or $$c_1f_1(t)\pm c_2f_2(t)\pm\ldots\ldots\pm c_nf_n(t)=L^{-1}\left[c_1\bar{f}_1(p)\pm c_2\bar{f}_2(p)\pm\ldots\ldots\pm c_n\bar{f}_n(p)\right]$$

or $$c_1L^{-1}\left[\bar{f}(p)\right]\pm c_2L^{-1}\left[\bar{f}(p)\right]\pm\ldots\ldots\pm c_nL^{-1}\left[\bar{f}_n(p)\right]$$

$$=L^{-1}\left[c_1\bar{f}(p)\pm c_2\bar{f}_2(p)\pm\ldots\ldots\pm c_n\bar{f}_n(p)\right]$$

(by definition of inverse Laplace transform)

2.4.2 Change of Scale of Property :

Theorem 3 : If $L^{-1}\left[\bar{f}(p);t\right]=f(t)$, then $L^{-1}\left[\bar{f}(ap);t\right]=\dfrac{1}{a}f\left(\dfrac{t}{a}\right)$, $a>0$

Proof : Since $\bar{f}(p)=\int_0^\infty e^{-pt}f(t)\,dt$

$\therefore$ $\bar{f}(ap)=\int_0^\infty e^{-a\,p\,t}f(t)\,dt;$

$$= \int_0^\infty e^{-pu} \left\{ \frac{1}{a} f\left(\frac{u}{a}\right) \right\} du \qquad \text{(putting } at = u\text{)}$$

Hence $\bar{f}(ap) = L\left[\frac{1}{a} f\left(\frac{t}{a}\right); p\right]$

Inverting, we have

$$L^{-1}\left[\bar{f}(ap); t\right] = \frac{1}{a} f\left(\frac{t}{a}\right)$$

2.4.3 First Shifting or Translation Property :

Theorem 4 : If $L^{-1}\left[\bar{f}(p); t\right] = f(t)$, then

$$L^{-1}\left[\bar{f}(p-a); t\right] = e^{at} f(t) = e^{at} L^{-1}\left[\bar{f}(p); t\right]$$

Proof : By definition, we have

$$\bar{f}(p) = \int_0^\infty e^{-pt} f(t)\, dt$$

$$\Rightarrow \qquad \bar{f}(p-a) = \int_0^\infty e^{-(p-a)t} f(t)\, dt = L\left[e^{at} f(t); p\right]$$

Hence, $L^{-1}\left[\bar{f}(p-a); t\right] = e^{at} f(t)$

Remark : The result of this theorem is also expressible as

$$L^{-1}\left[\bar{f}(p); t\right] = e^{at} L^{-1}\left[\bar{f}(p+a); t\right]$$

2.4.4 Second Shifting Property :

Theorem 5 : If $L^{-1}\left[\bar{f}(p); t\right] = f(t)$ and

$$g(t) = \begin{cases} 0 \quad , & t < a \\ f(t-a), & t > a \end{cases}$$

then $L^{-1}\left[e^{-ap} \bar{f}(p); t\right] = g(t)$

or $L^{-1}\left[e^{-ap} \bar{f}(p); t\right] = f(t-a) U(t-a)$

where $U(t-a)$ is the well known unit step function.

Proof : By definition, we have

$$L\left[g(t); p\right] = \int_0^\infty e^{-pt} g(t)\, dt$$

$$= \int_0^a e^{-pt} g(t)\,dt + \int_a^\infty e^{-pt} g(t)\,dt$$

$$= \int_0^a e^{-pt} .0\,dt + \int_a^\infty e^{-pt} f(t-a)\,dt$$

$$= e^{-ap} \int_0^\infty e^{-pu} f(u)\,du \qquad \text{(putting } t-a=u\text{)}$$

Hence $L[g(t); p] = e^{-ap} \bar{f}(p)$

or $\quad \bar{L}^{-}[e^{-ap} \bar{f}(p); t] = g(t)$

2.5 Use of Partial Fractions

If $\bar{f}(p)$ is of the form $\dfrac{g(p)}{h(p)}$, where g and h are polynomials in p, then break $\bar{f}(p)$ into partial fractions and manipulate term by term.

Example 4 : Evaluate $L^{-1}\left\{\dfrac{1}{(p-4)^5} + \dfrac{5}{(p-2)^2+5^2} + \dfrac{p+3}{(p+3)^2+6^2}\right\}$

Solution : By the linearity property, we have

$$L^{-1}\left\{\frac{1}{(p-4)^5}\right\} + L^{-1}\left\{\frac{5}{(p-2)^2+5^2}\right\} + L^{-1}\left\{\frac{p+3}{(p+3)^2+6^2}\right\}$$

$$= e^{4t} L^{-1}\left\{\frac{1}{p^5}\right\} + e^{2t} L^{-1}\left\{\frac{5}{p^2+5^2}\right\} + e^{-3t} L^{-1}\left\{\frac{p}{p^2+6^2}\right\}$$

[Using $L^{-1}\{\bar{f}(p-a); t\} = e^{at} f(t)$]

$$= e^{4t}\frac{t^4}{4!} + e^{2t}\sin 5t + e^{-3t}\cos 6t .$$

Example 5 : Find $L^{-1}\left[\dfrac{p e^{-ap}}{p^2-w^2}\right]$; $a > 0$

Solution : Let $\dfrac{p}{p^2-w^2} = \bar{f}(p) \Rightarrow L^{-1}[\bar{f}(p)] = L^{-1}\left[\dfrac{p}{p^2-w^2}\right] = \cosh wt = f(t)$ (say)

Then using second shifting theorem

$$L^{-1}\left[\frac{p e^{-ap}}{p^2-w^2}\right] = \begin{cases} \cosh w(t-a) & if \quad t > a \\ 0 & if \quad t < a \end{cases}$$

$$= \cosh w(t-a)U(t-a)$$

Example 6 : If $L^{-1}\left\{\frac{p^2-1}{(p^2+1)^2};t\right\} = t\cos t$, then find $L^{-1}\left\{\frac{9p^2-1}{(9p^2+1)^2};t\right\}$

Solution : Since $L^{-1}\left\{\frac{p^2-1}{(p^2+1)^2};t\right\} = t\cos t$

Replacing p by ap, we have by Theorem 3,

$$L^{-1}\left\{\frac{a^2p^2-1}{(a^2p^2+1)^2};t\right\} = \frac{1}{a}\frac{t}{a}\cos\left(\frac{t}{a}\right)$$

$$\therefore \quad L^{-1}\left\{\frac{9p^2-1}{(9p^2+1)^2};t\right\} = \frac{t}{9}\cos\left(\frac{t}{3}\right)$$

Example 7 : Prove that $L^{-1}\left[\frac{e^{-1/p}}{\sqrt{p}};t\right] = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$

Hence deduce the value of

$$L^{-1}\left[\frac{e^{-a/p}}{\sqrt{p}};t\right], \text{ where } a > 0.$$

Solution : Since $\frac{e^{-1/p}}{\sqrt{p}} = \frac{1}{\sqrt{p}}\left(1-\frac{1}{p}+\frac{1}{2!\,p^2}-\frac{1}{3!\,p^3}+.....\right)$

$$= \frac{1}{p^{1/2}} - \frac{1}{p^{3/2}} + \frac{1}{2!}\frac{1}{p^{5/2}} - \frac{1}{3!}\frac{1}{p^{7/2}} + ...$$

$$\therefore \quad L^{-1}\left[\frac{e^{-1/p}}{\sqrt{p}};t\right] = L^{-1}\left\{\frac{1}{p^{1/2}};t\right\} - L^{-1}\left\{\frac{1}{p^{3/2}};t\right\} + \frac{1}{2!}L^{-1}\left\{\frac{1}{p^{5/2}};t\right\} - \frac{1}{31}L^{-1}\left\{\frac{1}{p^{7/2}};t\right\} + ...$$

$$= \frac{1}{\sqrt{\pi t}} - \frac{2t^{1/2}}{\sqrt{\pi}} + \frac{2t^{3/2}}{3\sqrt{\pi}} - \frac{4t^{5/2}}{45\sqrt{\pi}} + ...$$

$$\therefore \qquad L^{-1}\left\{\frac{e^{-1/p}}{\sqrt{p}};t\right\}=\frac{1}{\sqrt{\pi t}}\left\{1-\frac{\left(2\sqrt{t}\right)^2}{2!}+\frac{\left(2\sqrt{t}\right)^4}{4!}-\frac{\left(2\sqrt{t}\right)^6}{!6}+....\right\}=\frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$$

Now let $\bar{f}(p)=\dfrac{e^{-1/p}}{\sqrt{p}}$. Then by theorem 3, we have

$$L^{-1}\left\{\frac{e^{-1/kp}}{\sqrt{kp}};t\right\}=\frac{1}{k}\,\frac{\cos 2\sqrt{t/k}}{\sqrt{\pi(t/k)}}=\frac{1}{\sqrt{k}}\,\frac{\cos 2\sqrt{t/k}}{\sqrt{\pi t}}$$

So that $L^{-1}\left\{\dfrac{e^{-1/kp}}{\sqrt{p}};t\right\}=\dfrac{\cos 2\sqrt{t/k}}{\sqrt{\pi t}}$

Putting $k=1/a$, we find that

$$L^{-1}\left\{\frac{e^{-a/p}}{\sqrt{p}};t\right\}=\frac{\cos 2\sqrt{at}}{\sqrt{\pi t}}$$

Example 8 : Evaluate the inverse Laplace transform of

(i) $\dfrac{p}{(p+3)^{7/2}}$ (ii) $\dfrac{(p+1)e^{-\pi p}}{p^2+p+1}$

Solution : (i) $L^{-1}\left\{\dfrac{p}{(p+3)^{7/2}};t\right\}=L^{-1}\left\{\dfrac{p+3-3}{(p+3)^{7/2}};t\right\}=e^{-3t}L^{-1}\left\{\dfrac{p-3}{p^{7/2}};t\right\}$

$$=e^{-3t}\left[L^{-1}\left\{\frac{1}{p^{5/2}};t\right\}-3L^{-1}\left\{\frac{1}{p^{7/2}};t\right\}\right]$$

$$=e^{-3t}\left[\left\{\frac{t^{3/2}}{\Gamma(5/2)}-\frac{3t^{5/2}}{\Gamma(7/2)}\right\}\right]$$

$$=\frac{e^{-3t}}{\sqrt{\pi}}\left[\frac{4}{3}t^{3/2}-\frac{8}{5}t^{5/2}\right]=\frac{4t^{3/2}\pi^{-1/2}}{15}(5-6t)$$

(ii) Since $L^{-1}\left\{\dfrac{p+1}{p^2+p+1}\right\}=L^{-1}\left\{\dfrac{\left(p+1/2\right)+1/2}{\left(p+1/2\right)^2+3/4};t\right\}=e^{-t/2}L^{-1}\left\{\dfrac{p+1/2}{p^2+3/4};t\right\}$

$$= e^{-t/2} L^{-1}\left\{\frac{p}{p^2+\left(\sqrt{3/2}\right)^2};t\right\} + \frac{1}{2} e^{-t/2} L^{-1}\left\{\frac{1}{p^2+\left(\sqrt{3/2}\right)^2};t\right\}$$

$$= e^{-t/2}\cos\frac{\sqrt{3}\,t}{2} + \frac{1}{2}e^{-t/2}\frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}\right)t$$

$$= \frac{e^{-t/2}}{\sqrt{3}}\left\{\sqrt{3}\cos\frac{\sqrt{3}}{2}t + \sin\frac{\sqrt{3}}{2}t\right\}$$

$$\therefore \qquad L^{-1}\left\{\frac{(p+1)e^{-\pi/p}}{p^2+p+1};t\right\} = \begin{cases} \dfrac{e^{-(t-\pi)/2}}{\sqrt{3}}\left\{\sqrt{3}\cos\dfrac{\sqrt{3}}{2}(t-\pi)+\sin\dfrac{\sqrt{3}}{2}(t-\pi)\right\} & t>\pi \\ 0, & t<\pi \end{cases}$$

$$= \frac{e^{-(t-\pi)/2}}{\sqrt{3}}\left\{\sqrt{3}\cos\frac{\sqrt{3}}{2}(t-\pi)+\sin\frac{\sqrt{3}}{2}(t-\pi)\right\}U(t-\pi)$$

Example 9 : Find $L^{-1}\left\{\dfrac{1}{\left(p^2+a^2\right)^{3/2}};t\right\}$

Hence obtain the inverse Laplace transform of $\dfrac{1}{\left(p^2+2p+5\right)^{3/2}}$.

Solution : We know that $L\left[J_0(at);p\right] = \dfrac{1}{\left(p^2+a^2\right)^{1/2}}$

Now differentiating with respect to $'a'$, we get

$$\frac{d}{da}L\left[J_0(at);p\right] = \frac{d}{da}\left\{\frac{1}{\left(p^2+a^2\right)^{1/2}}\right\} = -\frac{a}{\left(p^2+a^2\right)^{3/2}}$$

or $$L\left[\frac{d}{da}J_0(at);p\right] = L\left[t\,J_0'(at);p\right] = -\frac{a}{\left(p^2+a^2\right)^{3/2}}$$

or $$L^{-1}\left[\frac{1}{\left(p^2+a^2\right)^{3/2}};t\right] = -\frac{t}{a}J_0'(at) = \frac{t}{a}J_1(at) \qquad (\because J_0' = -J_1)$$

Deduction : $$L^{-1}\left[\frac{1}{\left(p^2+2p+5\right)^{3/2}};t\right]=L^{-1}\left[\frac{1}{\left\{(p+1)^2+4\right\}^{3/2}};t\right]$$

$$=e^{-t}\,L^{-1}\left[\frac{1}{\left(p^2+4\right)^{3/2}};t\right]$$

$$=e^{-t}.\frac{t}{2}J_1(2t)=\frac{t\,e^{-t}J_1(2t)}{2}$$

Example 10 : Use partial fractions to find the inverse Laplace Transform of $\dfrac{p^2}{p^4+4a^4}$.

Solution : $$\frac{p^2}{p^4+4a^4}=\frac{p^2}{\left(p^2+2a^2\right)^2-(2ap)^2}=\frac{p^2}{\left(p^2-2ap+2a^2\right)\left(p^2+2ap+2a^2\right)}$$

$$=\frac{1}{4a}\left[\frac{p}{p^2-2ap+2a^2}-\frac{p}{p^2+2ap+2a^2}\right]$$

$$=\frac{1}{4a}\left[\frac{(p-a)+a}{(p-a)^2+a^2}-\frac{(p+a)-a}{(p+a)^2+a^2}\right]$$

$$\therefore\quad L^{-1}\left[\frac{p^2}{p^4+4a^4};t\right]=\frac{1}{4a}\left[e^{at}\,L^{-1}\left\{\frac{p+a}{p^2+a^2};t\right\}-e^{-at}\,L^{-1}\left\{\frac{p-a}{p^2+a^2};t\right\}\right]$$

$$=\frac{1}{4a}\left[e^{at}\,L^{-1}\left\{\frac{p}{p^2+a^2};t\right\}+e^{at}\,L^{-1}\left\{\frac{a}{p^2+a^2};t\right\}\right.$$

$$\left.-e^{-at}\,L^{-1}\left\{\frac{p}{p^2+a^2};t\right\}+e^{-at}\,L^{-1}\left\{\frac{a}{p^2+a^2};t\right\}\right]$$

$$=\frac{1}{4a}\left[e^{at}\cos at+e^{at}\sin at-e^{-at}\cos at+e^{-at}\sin at\right]$$

$$=\frac{1}{2a}\left[\cos at\left(\frac{e^{at}-e^{-at}}{2}\right)+\sin at\left(\frac{e^{at}+e^{-at}}{2}\right)\right]$$

$$=\frac{1}{4a}\left[\cos a\,t\sinh a\,t+\sin a\,t\cosh a\,t\right]$$

Self-Learning Exericse - I

Fill in the blanks :

1. $L^{-1}\left\{\frac{1}{p^{n+1}};t\right\}=.....$

2. $L^{-1}\left\{\frac{1}{p^4}-\frac{3p}{p^2+16}+\frac{5}{p^2+4};t\right\}=.....$

3. $L^{-1}\left\{\frac{1}{p}e^{-1/p};t\right\}=.....$

4. $L^{-1}\left\{\frac{1}{\sqrt{p^2+a^2}};t\right\}=.....$

5. $L^{-1}\left\{\left(1-2p\cos\theta+p^2\right)^{-1/2};t\right\}=.....$

2.6 Exercise 2 (a)

1. Find Inverse Laplace transform of :

(i) $\frac{3p-8}{p^2+4}-\frac{4p-24}{p^2-16}$ (ii) $\frac{5p+4}{p^3+9}-\frac{2p-18}{p^2+9}+\frac{24-30\sqrt{p}}{p^4}$

(iii) $\frac{7}{p^2+9}+\frac{5}{p-7}+\frac{1}{2p^{3/2}}$ (iv) $\frac{5}{p^2}+\left(\frac{\sqrt{p}-1}{p}\right)^2-\frac{7}{3p+2}$

[**Ans.** (i) $3\cos 2t-4\sin 2t-4\cosh 4t+6\sinh 4t$

(ii) $5t+2t^2-2\cos 3t+6\sin 3t+4t^3-\frac{16t^{5/2}}{\sqrt{\pi}}$

(iii) $\frac{7}{3}\sin 3t+5e^{7t}+\sqrt{t/\pi}$ (iv) $6t+1-\frac{4\sqrt{t}}{\pi}-\frac{7}{3}e^{-2t/3}$]

2. Show that

(i) $L^{-1}\left\{\frac{1}{p}\sin\frac{1}{p};t\right\}=t-\frac{t^3}{(3!)^2}+\frac{t^5}{(5!)^2}-\frac{t^7}{(7!)^2}+.....$

(ii) $L^{-1}\left\{\frac{1}{p}e^{-1/p};t\right\}=J_0\left(2\sqrt{t}\right)$

3. Evaluate the inverse Laplace transform of:

(i) $\dfrac{6p-4}{p^2-4p+20}$ (ii) $\dfrac{p}{(p+1)^5}$ (iii) $\dfrac{1}{\sqrt{p^2-4p+20}}$

(iv) $\dfrac{e^{-4p}}{(p+2)^3}$ (v) $\dfrac{3(1+e^{-p\pi})}{p^2+9}$ (vi) $\dfrac{e^{-2p}}{\sqrt{p^2+9}}$

(vii) $\dfrac{1}{\sqrt{2p+3}}$ (viii) $\dfrac{p}{p^2+25}e^{-4\pi p/5}$

[**Ans.:** (i) $6e^{2t}\cos 4t + 2e^{2t}\sin 4t$. (ii) $e^{-t}\dfrac{t^3}{3!} - e^{-t}\dfrac{t^4}{4!}$

(iii) $e^{2t}J_0(4t)$ (iv) $\dfrac{1}{2}(t-4)^2 e^{-2(t-4)}U(t-4)$

(v) $\begin{cases} 0 \;, & t>\pi \\ \sin 3t \;, & t<\pi \end{cases}$

(vi) $\begin{cases} J_0(3t-6), & t>2 \\ 0 \;, & t<2 \end{cases} = J_0(3t-6)U(t-2)$

(vii) $\dfrac{1}{\sqrt{2\pi}}e^{-3t/2}\,t^{-1/2}$ (viii) $\cos 5t\, U\left(t-\dfrac{4\pi}{5}\right)$]

4. Show that $L^{-1}\left\{\dfrac{1}{p}\cos\dfrac{1}{p};t\right\} = 1 - \dfrac{t^2}{(2!)^2} + \dfrac{t^4}{(4!)^2} - \dfrac{t^6}{(6!)^2} + \ldots..$

5. If $L^{-1}\left[\dfrac{1}{p\sqrt{p+1}};t\right] = erf\sqrt{t}$, Find $L^{-1}\left[\dfrac{1}{p\sqrt{p+a}};t\right]$; $a>0$.

[**Ans.:** $\dfrac{erf\sqrt{at}}{\sqrt{a}}$]

6. Show that $L^{-1}\left\{\dfrac{1}{p}J_0\left(\dfrac{2}{\sqrt{p}}\right);t\right\} = 1 - \dfrac{t}{(1!)^3} + \dfrac{t^2}{(2!)^3} - \dfrac{t^3}{(3!)^3} + \ldots..$

7. Find functions whose Laplace transforms are :

(i) $\dfrac{pe^{-ap}}{p^2-\omega^2}$, $a>0$ (ii) $\dfrac{e^{-5p}}{(p-2)^4}$ (iii) $\dfrac{pe^{-2p}}{p^2+3p+2}$

$$\left[\quad \textbf{Ans.:} \quad \text{(i)} \quad \begin{cases} \cosh\omega(t-a), & t>a \\ 0 & , t<a \end{cases} \text{ or } \cosh\omega(t-a)U(t-a)\right.$$

(ii) $\begin{cases} \frac{1}{6}(t-5)^3 e^{2(t-5)}, & t>5 \\ 0 & , t<5 \end{cases}$ or $\frac{1}{6}(t-5)^3 e^{2(t-5)} U(t-5)$

(iii) $\begin{cases} 2e^{-2(t-2)} - e^{-(t-2)}, & t>2 \\ 0 & , t<2 \end{cases}$ or $\left\{2e^{-2(t-2)} - e^{-(t-2)}\right\} U(t-2) \quad]$

8. Use Partial fractions to find inverse Laplace transform of the following functions :

(i) $\dfrac{3p+16}{p^2-p-6}$ (ii) $\dfrac{5p^2-15p-11}{(p+1)(p-2)^3}$ (iii) $\dfrac{3p+1}{(p-1)(p^2+1)}$

(iv) $\dfrac{p^2-4}{(p^2+1)(p^2+4)^2}$ (v) $\dfrac{2p^2+5p-4}{p^3+p^2-2p}$

(vi) $\dfrac{p^2+2p+3}{(p^2+2p+2)(p^2+2p+5)}$

[**Ans.:** (i) $5e^{3t}-2e^{-2t}$ (ii) $-\frac{1}{3}e^{-t} - \frac{7}{2}t^2e^{2t} + 4te^{2t} + \frac{1}{3}e^{2t}$

(iii) $2e^t - 2\cos t + \sin t$ (iv) $-\frac{5}{9}\sin t - \frac{1}{8}t\cos 2t + \frac{49}{144}\sin 2t$

(v) $2+e^{-2t}+e^t$ (vi) $\frac{1}{3}e^{-t}(\sin t + \sin 2t)$]

2.7 Inverse Laplace Trasform of Derivatives

Theorem 6 : If $L^{-1}\left[\bar{f}(p);t\right] = f(t)$, then

$$L^{-1}\left[\bar{f}^{(n)}(p);t\right] = L^{-1}\left[\frac{d^n}{dp^n}\bar{f}(p);t\right] = (-1)^n t^n f(t), \quad n = 1,2,3,\ldots\ldots$$

Proof : Since, we have

$$L\left[t^n f(t);p\right] = (-1)^n \frac{d^n}{dp^n}\bar{f}(p) = (-1)^n \bar{f}^{(n)}(p)$$

$$\therefore \qquad L^{-1}\left[\bar{f}^{(n)}(p);t\right] = L^{-1}\left[\frac{d^n}{d\,p^n}\bar{f}(p);t\right] = (-1)^n t^n f(t)$$

2.8 Inverse Laplace Transform of Integrals

Theorem 7 : If $L^{-1}\left[\bar{f}(p);t\right] = f(t)$, then

$$L^{-1}\left[\int_p^\infty \bar{f}(u)\,du\,;t\right] = \frac{f(t)}{t}$$

Proof : From Theorem 10 of unit 1, we have

$$L\left[\frac{f(t)}{t};p\right] = \int_p^\infty \bar{f}(u)\,du \qquad \text{(provided that } \lim_{t\to 0}\left\{\frac{f(t)}{t}\right\} \text{ exists)}$$

$$\therefore \qquad L^{-1}\left\{\int_p^\infty \bar{f}(u)\,du\right\} = \frac{f(t)}{t}$$

2.9 Multiplication and Division by Powers of p

Theorem 8 : If $L^{-1}\left[\bar{f}(p);t\right] = f(t)$ and $f(0) = 0$, then

$$L^{-1}\left[p\,\bar{f}(p);t\right] = f'(t)$$

Proof : From Theorem 6 of unit 1, we have

$$L\left[f'(t);p\right] = p\bar{f}(p) - f(0) = p\bar{f}(p) \qquad (\because f(0) = 0)$$

Hence $L^{-1}\left[p\bar{f}(p);t\right] = f'(t)$

Remark 1. If $f(0) \neq 0$, then $L^{-1}\left[p\bar{f}(p) - f(0);t\right] = f^1(t)$

or $\quad L^{-1}\left[p\bar{f}(p);t\right] = f'(t) + f(0)\delta(t)$

where $\delta(t)$ is the dirac delta function or unit impulse function.

Remark 2. Generalization to $L^{-1}\left[p^n\bar{f}(p);t\right]$ is possible,

i.e. $\quad L^{-1}\left[p^n\bar{f}(p);t\right] = f^n(t) = \frac{d^n}{dt^n}f(t)$

provided that $f(0) = f'(0) = f''(0) = ... = f^{(n-1)}(0) = 0$

Theorem 9 : Let $L^{-1}\left[\bar{f}(p);t\right] = f(t)$. If $f(t)$ is sectionally continuous and of exponential order 'a' and such that $\lim_{t\to 0}\frac{f(t)}{t}$ exists, then for p > a, we have

$$\boldsymbol{L^{-1}\left\{\frac{\bar{f}(p)}{p};t\right\}=\int_0^t f(u)du}$$

Proof : Let $G(t)=\int_0^t f(u)du$

Then $G'(t)=f(t)$ and $G(0)=0$

$\therefore \qquad L[G'(t);p]=pL[G(t);p]-G(0)=p\,L[G(t);p]$

or $\bar{f}(p)=p\,L[G(t);p]$

$\therefore \qquad L[G(t);p]=\dfrac{\bar{f}(p)}{p}$

Hence $L^{-1}\left[\dfrac{\bar{f}(p)}{p};t\right]=G(t)=\int_0^t f(u)du$

Theorem 10. Let $\boldsymbol{L^{-1}[\bar{f}(p);t]=f(t)}$, then

$$\boldsymbol{L^{-1}\left[\frac{\bar{f}(p)}{p^2};t\right]=\int_0^t\int_0^v f(u)du\,dv}$$

Proof : Let $\quad G(t)=\int_0^t\int_0^v f(u)du\,dv$

Then $\quad G'(t)=\int_0^t f(u)du$ and $G''(t)=f(t)$

Since $\quad G(0)=G'(0)=0$

Now $\quad L[G''(t);p]=p^2L[G(t);p]-p\,G(0)-G'(0)=p^2\,L[G(t);p]$

or $\quad L\{f(t);p\}=p^2L[G(t);p]$

or $\quad \dfrac{\bar{f}(p)}{p^2}=L[G(t);p]$

or $\quad L^{-1}\left[\dfrac{\bar{f}(p)}{p^2};t\right]=G(t)=\int_0^t\int_0^v f(u)\,du\,dv$

The above result may also be written as

$$L^{-1}\left[\frac{\bar{f}(p)}{p^2};t\right]=\int_0^t\int_0^t f(t)\,dt^2$$

In general, we have

$$L^{-1}\left[\frac{\bar{f}(p)}{p^n};t\right]=\int_0^t\int_0^t\ldots\ldots\int_0^t f(t)\,dt^n$$

2.10 Convolution Theorem

2.10.1 Convolution of Two Functions :

Definition : Let $f(t)$ and $g(t)$ be two functions of class A, then the convolution of the two functions $f(t)$ and $g(t)$ denoted by $f*g$ is defined by the relation :

$$f*g=\int_0^t f(u)\,g(t-u)\,du \qquad ...(1)$$

The equation (1) can be written as

$$f*g=\int_0^t f(t-u)\,g(u)\,du \qquad ...(2)$$

The convolution $f*g$ is also known as **Faltung** or **resultant** of f and g.

2.10.2 The Convolution Theorem :

Theorem 11 : Let $f(t)$ and $g(t)$ be two functions of class A and let $L^{-1}\left[\bar{f}(p);t\right]=f(t)$ and $L^{-1}\left[\bar{g}(p);t\right]=g(t)$. Then

$$L^{-1}\left[\bar{f}(p)\,.\,\bar{g}(p);t\right]=\int_0^t f(u)\,g(t-u)\,du=f*g.$$

Proof : Here we shall prove that

$$L\left[\int_0^t f(u)\,g(t-u)\,du;p\right]=\bar{f}(p)\,.\,\bar{g}(p) \qquad ...(3)$$

Let $H(t)=\int_0^t f(u)\,g(t-u)\,du$

$$\therefore \quad L\left[H(t);p\right]=\int_{t=0}^{\infty} e^{-pt}\left\{\int_{u=0}^{t} f(u)\,g(t-u)\,du\right\}dt \qquad ...(4)$$

The region of integration Δ is bounded by the curves $t=0$, $t=\infty$, $u=0$ and $u=t$. Thus Δ is the half of the first quadrant.

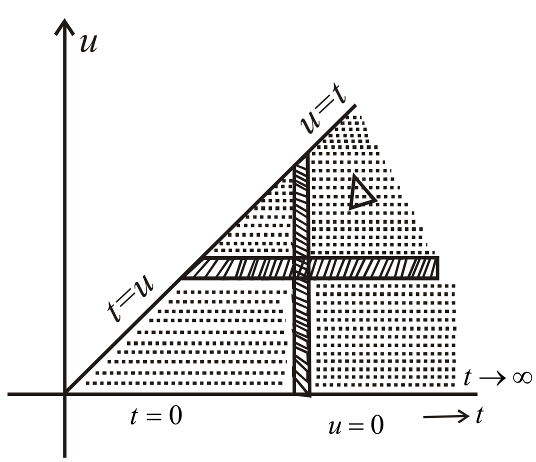


Figure 2.1

We can cover it by horizontal strip which starts from $t=u$ to $t=\infty$. For this strip u varies from 0 to ∞.

$$L[H(t);p] = \int_{u=0}^{\infty} f(u)\,du \int_{t=u}^{\infty} e^{-pt}\, g(t-u)\,dt$$

$$= \int_{u=0}^{\infty} e^{-pu} f(u)\,du \int_{t=u}^{\infty} e^{-p(t-u)}\, g(t-u)\,dt$$

$$= \int_{u=0}^{\infty} e^{-pu} f(u)\,du \int_{v=0}^{\infty} e^{-pv}\, g(v)\,dv \qquad \text{(putting } t-u=v\text{)}$$

$\therefore \quad L[H(t);p] = \bar{f}(p) * \bar{g}(p)$

or $\quad L^{-1}[\bar{f}(t) * \bar{g}(p);t] = H(t)$

or $\quad L^{-1}[\bar{f}(p)\,.\,\bar{g}(p);t] = \int_0^t f(u)\,g(t-u)\,du = f*g$

Remark : The convolution theorem can be rewritten as :

$$L\left[\int_0^t f(u)\,g(t-u)\,du;p\right] = L[f(t);p]\; L[g(t);p]$$

Example 11 : Find the inverse Laplace transform of

(i) $\dfrac{p}{(p^2+a^2)^2}$ (ii) $\dfrac{p+1}{(p^2+2p+2)^2}$

(iii) $\log\left(1+\dfrac{1}{p^2}\right)$ or $\log\left(\dfrac{p^2+1}{p^2}\right)$ (iv) $\cot^{-1}(p+1)$

Solution : (i) Since $\dfrac{d}{dp}\left(\dfrac{1}{p^2+a^2}\right) = \dfrac{-2p}{(p^2+a^2)^2}$

and $\quad L^{-1}\left[\dfrac{1}{p^2+a^2};t\right] = \dfrac{1}{a}\sin at$

$$\therefore \quad L^{-1}\left[\frac{p}{(p^2+a^2)^2};t\right] = L^{-1}\left[-\frac{1}{2}\frac{d}{dp}\left(\frac{1}{p^2+a^2}\right);t\right]$$

$$= -\frac{1}{2}L^{-1}\left[\frac{d}{dp}\left(\frac{1}{p^2+a^2}\right);t\right] \qquad \text{(Using Theorem 6)}$$

$$= -\frac{1}{2}t(-1)L^{-1}\left[\frac{1}{p^2+a^2}\right] = \frac{t}{2a}\sin at$$

(ii) $$L^{-1}\left[\frac{p+1}{(p^2+2p+2)^2};t\right]=L^{-1}\left[\frac{p+1}{\{(p+1)^2+1\}^2};t\right]$$

$$=e^{-t}\,L^{-1}\left[\frac{p}{(p^2+1)^2};t\right]$$ (by first shifting theorem)

$$=\frac{1}{2}\,t\,e^{-t}\sin t$$ (proceeding as above)

(iii) Let $\bar{f}(p)=\log\left(1+\frac{1}{p^2}\right)=L[f(t);p]$

$$=-\log\frac{p^2}{p^2+1}=-2\log p+\log(p^2+1)$$

$\therefore$ $$\bar{f}'(p)=-2\left(\frac{1}{p}-\frac{p}{p^2+1}\right)$$

or $$L^{-1}[\bar{f}'(p);t]=-2\,L^{-1}\left[\frac{1}{p}-\frac{p}{p^2+1};t\right]$$

$$=-2\,L^{-1}\left[\frac{1}{p};t\right]+2\,L^{-1}\left[\frac{p}{p^2+1};t\right]$$

$\therefore$ $$L^{-1}[\bar{f}'(p);t]=-2.1+2\cos t=-2(1-\cos t)$$

But $$L^{-1}[\bar{f}'(p);t]=(-1)\,t\,f(t)$$

which implies that $-t\,f(t)=-2(1-\cos t)$

or $$f(t)=L^{-1}\left[\log\left(1+\frac{1}{p^2}\right);t\right]=\frac{2(1-\cos t)}{t}$$

(iv) Let $\bar{f}(p)=\cot^{-1}(1+p)$

$$\bar{f}'(p)=\frac{(-1)}{1+(1+p)^2}$$

or $$L^{-1}[\bar{f}'(p);t]=-L^{-1}\left[\frac{1}{(p+1)^2+1};t\right]=-e^{-t}\,L^{-1}\left[\frac{1}{p^2+1};t\right]$$

or $\quad -t\,f(t) = -e^{-t}\sin t$

or $\quad f(t) = L^{-1}\left[\cot^{-1}(1+p);t\right] = \dfrac{e^{-t}\sin t}{t}$

Example 12 : Find the inverse Laplace Transform of

(i) $\dfrac{a^2}{p(p+a)^2}$ (ii) $\dfrac{1}{p^3(p^2+1)}$

(iii) $\dfrac{1}{p}\log\dfrac{p+2}{p+1}$ (iv) $\dfrac{1}{p\sqrt{p+4}}$

Solution : (i) Since $L^{-1}\left[\dfrac{a^2}{(p+a)^2};t\right] = -a^2\,L^{-1}\left[\dfrac{d}{dp}\left(\dfrac{1}{p+a}\right);t\right]$

$$= (-a^2)(-1)\,t\,e^{-at} = a^2 t\,e^{-at}$$

$$\therefore \quad L^{-1}\left[\frac{a^2}{p(p+a)^2};t\right] = a^2\int_0^t u\,e^{-au}\,du = \left[-u\,e^{-au} - e^{-au}\right]_{u=0}^{t}$$

$$= -\left[a\,t\,e^{-at} + \left(e^{-at}-1\right)\right]$$

$$= 1 - e^{-at}(at+1)$$

(ii) $\quad L^{-1}\left[\dfrac{1}{p^2+1};t\right] = \sin t$

$$\therefore \quad L^{-1}\left[\frac{1}{p(p^2+1)};t\right] = \int_0^t \sin u\,du = 1-\cos t$$

$$\therefore \quad L^{-1}\left[\frac{1}{p^2(p^2+1)};t\right] = \int_0^t (1-\cos u)\,du = t - \sin t$$

$$\therefore \quad L^{-1}\left[\frac{1}{p^3(p^2+1)};t\right] = \int_0^t (u-\sin u)\,du = \frac{1}{2}t^2 + \cos t - 1$$

(iii) $\quad$ Let $\bar{f}(p) = \log\dfrac{(p+2)}{(p+1)} = \log(p+2) - \log(p+1)$

or $\quad \bar{f}'(p) = \dfrac{1}{p+2} - \dfrac{1}{p+1}$

$\therefore \quad L^{-1}\left[\bar{f}'(p);t\right] = e^{-2t} - e^{-t}$

or $\quad -t\,L^{-1}\left[\bar{f}(p);t\right] = e^{-2t} - e^{-t}$

$\therefore \quad L^{-1}\left[\bar{f}(p);t\right] = \dfrac{e^{-t} - e^{-2t}}{t}$

or $\quad L^{-1}\left[\log\dfrac{p+2}{p+1};t\right] = \dfrac{e^{-t} - e^{-2t}}{t}$

$\therefore \quad L^{-1}\left[\dfrac{1}{p}\log\dfrac{p+2}{p+1};t\right] = \int_0^t \dfrac{e^{-u} - e^{-2u}}{u}\,du$

(iv) We have

$$L^{-1}\left[\frac{1}{\sqrt{p+4}};t\right] = e^{-4t}\,L^{-1}\left[\frac{1}{\sqrt{p}};t\right]$$

$$= e^{-4t}\,\frac{t^{-1/2}}{\Gamma(1/2)} = \frac{e^{-4t}\,t^{-1/2}}{\sqrt{\pi}}$$

$\therefore \quad L^{-1}\left[\dfrac{1}{p\sqrt{p+4}};t\right] = \displaystyle\int_0^t \dfrac{e^{-4u}\,u^{-1/2}}{\sqrt{\pi}}\,du \qquad$ (using Theorem 9)

$$= \frac{1}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-t^2}\,dt \qquad \text{(putting } 4u = t^2\text{)}$$

$\therefore \quad L^{-1}\left[\dfrac{1}{p\sqrt{p+4}};t\right] = \dfrac{1}{2}\,erf\left(2\sqrt{t}\right)$

Example 13 : Calculate

(i) $\quad L^{-1}\left[\dfrac{1}{1+\sqrt{p}};t\right] \qquad$ (ii) $\quad L^{-1}\left[\dfrac{\sqrt{p}}{p-1};t\right]$

(iii) $\quad L^{-1}\left[\log\left(\dfrac{p+\sqrt{p^2+1}}{2p}\right);t\right] \qquad$ (iv) $\quad L^{-1}\left[\dfrac{e^{-p}\left(1-e^{-p}\right)}{p\left(p^2+1\right)};t\right]$

(v) $L^{-1}\left[\dfrac{1}{\left(p^2+a^2\right)^{3/2}};t\right]$

Solution : (i) We have $\dfrac{1}{\sqrt{p}+1}=\dfrac{\left(\sqrt{p}+1\right)-1}{\sqrt{p}\left(\sqrt{p}+1\right)}=\dfrac{1}{\sqrt{p}}-\dfrac{1}{\sqrt{p}\left(\sqrt{p}+1\right)}$

$$\therefore \quad L^{-1}\left[\frac{1}{\sqrt{p}+1};t\right]=L^{-1}\left[\frac{1}{\sqrt{p}};t\right]-L^{-1}\left[\frac{1}{\sqrt{p}\left(\sqrt{p}+1\right)};t\right]$$

$$=\frac{t^{-1/2}}{\sqrt{\pi}}-e^t.erfc\left(\sqrt{t}\right)$$

$$\left\{\because\ L\left[erfc\sqrt{t};p\right]=\frac{1}{\left(\sqrt{p+1}\right)\left[\left(\sqrt{p+1}\right)+1\right]}\right\}$$

(ii) $$L^{-1}\left[\frac{\sqrt{p}}{p-1}\right]=L^{-1}\left[\frac{(p-1)+1}{\sqrt{p}\,(p-1)};t\right]=L^{-1}\left[\frac{1}{\sqrt{p}};t\right]+L^{-1}\left[\frac{1}{\sqrt{p}\,(p-1)};t\right]$$

$$=\frac{t^{-1/2}}{\sqrt{\pi}}+e^t\ erf\left(\sqrt{t}\right)\ \left\{\because L\left[erf\sqrt{t};p\right]=\frac{1}{p\sqrt{p+1}}\right\}$$

(iii) $$\bar{f}(p)=\log\left(\frac{p+\sqrt{p^2+1}}{2p}\right)=\log\left(p+\sqrt{p^2+1}\right)-\log 2p$$

$$\bar{f}'(p)=\frac{1}{\sqrt{p^2+1}}-\frac{1}{p}$$

$$\therefore \quad L^{-1}\left[\bar{f}'(p);t\right]=L^{-1}\left[\frac{1}{\sqrt{p^2+1}}\right]-L^{-1}\left[\frac{1}{p}\right]$$

$$\Rightarrow \quad -t\,L^{-1}\left[f(p);t\right]=J_0(t)-1$$

$$\therefore \quad L^{-1}\left[f(p);t\right]=\frac{1-J_0(t)}{t}$$

or $$L^{-1}\left[\log\left(\frac{p+\sqrt{p^2+1}}{2p};t\right)\right]=\frac{1-J_0(t)}{t}$$

(iv) Since $$\frac{1}{p\left(p^2+1\right)}=\frac{1}{p}-\frac{p}{p^2+1}$$

$$\therefore \quad L^{-1}\left[\frac{1}{p\left(p^2+1\right)};t\right]=L^{-1}\left[\frac{1}{p}\right]-L^{-1}\left[\frac{p}{p^2+1}\right]=1-\cos t$$

Hence $$L^{-1}\left[\frac{e^{-p}\left(1-e^{-p}\right)}{p\left(p^2+1\right)};t\right]=L^{-1}\left[\frac{e^{-p}}{p\left(p^2+1\right)};t\right]-L^{-1}\left[\frac{e^{-2p}}{p\left(p^2+1\right)};t\right]$$

$$=\{1-\cos(t-1)\}U(t-1)-\{1-\cos(t-2)\}U(t-2)$$

$$=\begin{cases} 0 & , \quad t<1 \\ 1-\cos(t-1) & , \quad 1<t<2 \\ \cos(t-2)-\cos(t-1), & \quad t>2 \end{cases}$$

(v) Since $$L\left[J_0(at);p\right]=\frac{1}{\left(\sqrt{p^2+a^2}\right)}$$

Differentiating both sides with respect to 'a', we get

$$\frac{d}{da}L\left[J_0(at);p\right]=\frac{d}{da}\left\{\frac{1}{\sqrt{p^2+a^2}}\right\}$$

$$L\left[\frac{d}{da}J_0(at);p\right]=\frac{-a}{\left(p^2+a^2\right)^{3/2}}$$

or $$L\left[t\ J_0'(at);p\right]=\frac{-a}{\left(p^2+a^2\right)^{3/2}}$$

$$\therefore \quad L^{-1}\left[\frac{1}{\left(p^2+a^2\right)^{3/2}}\right]=-\frac{t}{a}J_0'(at)=\frac{t\,J_1(at)}{a}$$

Example 14 : Apply convolution theorem to prove that

$$L^{-1}\left[\frac{1}{p\sqrt{p+4}};t\right]=\frac{1}{2}erf\left(2\sqrt{t}\right)$$

Solution : Since $L^{-1}\left\{\frac{1}{\sqrt{p+4}};t\right\}=e^{-4t}\,L^{-1}\left\{\frac{1}{\sqrt{p}};t\right\}$

$$=e^{-4t}\cdot\frac{t^{-1/2}}{\Gamma(1/2)}=\frac{e^{-4t}}{\sqrt{\pi t}}$$

and $\quad L^{-1}\left\{\frac{1}{p};t\right\}=1$

$\therefore$ By convolution theorem, we get

$$L^{-1}\left[\frac{1}{\sqrt{p+4}}\cdot\frac{1}{p};t\right]=\int_0^t\frac{e^{-4u}}{\sqrt{\pi u}}.1.du$$

$$=\frac{1}{\sqrt{\pi}}\int_0^{2\sqrt{t}}e^{-t^2}dt \qquad \left(\text{put } 4u=t^2\ \therefore du=\frac{1}{2}t\,dt\right)$$

$$=\frac{1}{2}\cdot\frac{2}{\sqrt{\pi}}\int_0^{2\sqrt{t}}e^{-t^2}dt=\frac{1}{2}erf\left(2\sqrt{t}\right)$$

$$\therefore \qquad L^{-1}\left[\frac{1}{p\sqrt{p+4}};t\right]=\frac{1}{2}erf\left(2\sqrt{t}\right)$$

Example 15 : Evaluate $\int_0^t \sin u\cos(t-u)\,du$.

Solution : Let $f(t)=\int_0^t \sin u\cos(t-u)\,du$

$$L[f(t);p]=L\left[\int_0^t \sin u\cos(t-u)\,du;p\right]$$

$$=L[\sin t;p].L[\cos t;p] \qquad \text{(by convolution theorem)}$$

$$=\frac{1}{p^2+1}\cdot\frac{p}{p^2+1}=\frac{p}{\left(p^2+1\right)^2}$$

$$\therefore \qquad f(t)=L^{-1}\left[\frac{p}{\left(p^2+1\right)^2};t\right]$$

$$= L^{-1}\left[-\frac{1}{2}\frac{d}{dp}\left(\frac{1}{p^2+1}\right);t\right]$$

$$= \left(\frac{-1}{2}\right)(-1)t\, L^{-1}\left(\frac{1}{p^2+1};t\right)$$

$$\therefore \qquad f(t) = \frac{1}{2}t\sin t$$

Example 16 : Apply convolution theorem to prove that

$$B(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}du = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)},\ (m>0, n>0).$$

Hence duduce that

$$\int_0^{\pi/2} \sin^{2m-1}\theta\, \cos^{2n-1}\theta = \frac{1}{2}B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{2\Gamma(m+n)}$$

where $B(m,n)$ is called Beta function.

Solution : Let $f(t) = \int_0^t u^{m-1}(t-u)^{n-1}du$

$\therefore$ By convolution theorem, we have

$$L[f(t);p] = L\left[\int_0^t u^{m-1}(t-u)^{n-1}du;p\right]$$

$$= L\left[t^{m-1};p\right].\, L\left[t^{n-1};p\right]$$

$$= \frac{\Gamma(m)}{p^m}\cdot\frac{\Gamma(n)}{p^n} = \frac{\Gamma(m)\Gamma(n)}{p^{m+n}}$$

$$\therefore \qquad f(t) = L^{-1}\left[\frac{\Gamma(m)\Gamma(n)}{p^{m+n}};t\right] = \Gamma(m)\,\Gamma(n)\, L^{-1}\left[\frac{1}{p^{m+n}};t\right]$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\, t^{m+n-1}$$

Now taking $t=1$, we have

$$\int_0^1 u^{m-1}(1-u)^{n-1}du = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m,n)$$

Deduction : Putting $u = \sin^2\theta$, $du = 2\sin\theta\cos\theta\, d\theta$

$$B(m,n)=\int_0^{\pi/2}(\sin\theta)^{2m-2}(\cos\theta)^{2n-2}.2\sin\theta\cos\theta\,d\theta$$

$$\frac{1}{2}B(m,n)=\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta=\frac{1}{2}\frac{\Gamma(m)\Gamma(n)}{p(m+n)}$$

Thus, we get the required result.

Self-Learning Exercise - II

Fill in the blanks :-

1. $L^{-1}\{\bar{f}^{n}(p);t\}=......$

2. $L^{-1}\{\int_p^\infty \bar{f}(u)\,du;t\}=......$

3. $L^{-1}\left\{\frac{1}{(p-a)^2};t\right\}=......$

4. $L^{-1}\left\{\frac{1}{(p+1)^2};t\right\}=......$

5. State the convolution theorem for Laplace Transform.

2.11 Exercise 2 (b)

1. Find the inverse Laplace Transform of

(i) $\log\frac{p^2+a^2}{p^2+b^2}$ (ii) $\frac{1}{p^2(p+1)^2}$ (iii) $\frac{1}{p(p+1)^3}$

(iv) $\frac{1}{p\sqrt{p^2+a^2}}$ (v) $\frac{p+2}{p^2(p+3)}$ (vi) $\log\left(1+\frac{\omega^2}{p^2}\right)$

[**Ans.** (i) $\frac{2(\cos bt-\cos at)}{t}$ (ii) $t\,e^{-t}+2e^{-t}+t-2$ (iii) $1-e^{-t}\left(1+t+\frac{t^2}{2}\right)$

(iv) $\int_0^t J_0(au)\,du$ (v) $\frac{2}{3}t+\frac{1}{9}-\frac{1}{9}e^{-3t}$ (vi) $\frac{2}{t}(1-\cos\omega t)$]

2. By making use of convolution theorem, find

(i) $L^{-1}\left[\frac{1}{p^2(p+1)^2};t\right]$ (ii) $L^{-1}\left[\frac{1}{p^2(p^2-a^2)};t\right]$ (iii) $L^{-1}\left[\frac{1}{(p^2+4)(p+2)};t\right]$

[**Ans.** (i) $t\,e^{-t}+2e^{-t}+t-2$ (ii) $\frac{1}{a^3}(-at+\sin h\,at)$

(iii) $\frac{1}{8}(\sin 2t-\cos 2t+e^{-2t})$]

3. Use convolution theorem to find

(i) $L^{-1}\left[\dfrac{p}{\left(p^2+a^2\right)^2};t\right]$ (ii) $L^{-1}\left[\dfrac{1}{p\left(p^2+4\right)^2};t\right]$

(iii) $L^{-1}\left[\dfrac{p^2}{\left(p^2+a^2\right)^2};t\right]$ (iv) $L^{-1}\left[\dfrac{p}{\left(p^2+a^2\right)^3};t\right]$

[**Ans.** (i) $\dfrac{t\sin at}{2a}$ (ii) $\dfrac{1}{16}\left[1-\cos 2t-t\sin 2t\right]$

(iii) $\dfrac{1}{2}\left[t\cos at+\dfrac{1}{a}\sin at\right]$ (iv) $\dfrac{t}{8a^3}\left[\sin at-at\cos at\right]$]

4. Prove that $\int_0^t J_0(u)\sin(t-u)\,du=t\,J_1(t)$

5. Evaluate :

(i) $\int_0^\infty \cos t^2 dt$ (ii) $\int_0^\infty e^{-t^2}dt$ (iii) $\int_0^\infty \dfrac{\sin t}{t}dt$

[**Ans.** (i) $\dfrac{1}{2}\sqrt{\dfrac{\pi}{2}}$ (ii) $\dfrac{\sqrt{\pi}}{2}$ (iii) $\dfrac{\pi}{2}$]

2.12 Dirichlet's Conditions

If $f(t)$ satisfies the following conditions

(i) $f(t)$ is defined in the interval $c<t<c+2\alpha$

(ii) $f(t)$ and $f'(t)$ both are piecewise continuous in $c<t<c+2\alpha$

(iii) $f(t+2\alpha)=f(t)$ i.e. $f(t)$ is periodic with period 2α .

The above conditions are sufficient (but not necessary) conditions for the convergence of a Fourier Series.

2.13 Fourier Integral Theorem

Theorem 12 : Let $f(t)$ satisfy the Dirichlet's conditions in every finite interval $-\alpha\le t\le\alpha$ and if $\int_{-\infty}^{\infty}\left|f(t)dt\right|$ converges (or $f(t)$ is absolutely integrable in $(-\infty,\infty)$) then at each point of continuity t of $f(t)$,

$$f(t)=\frac{1}{2\pi}\int_{v=-\infty}^{\infty}dv\int_{u=-\infty}^{\infty}f(u)\cos\{v(t-u)\}\,du \qquad ...(6)$$

If 't' is a point of discontinuity, then the L.H.S. of (6) is replaced by $\frac{1}{2}\left[f(t+0)+f(t-0)\right]$ i.e. the mean value of $f(t)$ at the point of discontinuity.

The above conditions are sufficient but not necessary.

Since we know that $\sin\{v(t-u)\}$ is always an odd function of v, therefore, we have

$$0=\frac{1}{2\pi}\int_{v=-\infty}^{\infty}dv\int_{u=-\infty}^{\infty}f(u)\sin\{v(t-u)\}\,du \qquad ...(7)$$

From (6) and (7), we get

$$f(t)=\frac{1}{2\pi}\int_{v=-\infty}^{\infty}dv\int_{u=-\infty}^{\infty}f(u)\,e^{iv(t-u)}\,du$$

or $$f(t)=\frac{1}{2\pi}\int_{v=-\infty}^{\infty}e^{ivt}dv\int_{u=-\infty}^{\infty}f(u)\,e^{-ivu}\,du \qquad ...(8)$$

This result is known as the complex form of Fourier integral.

2.14 The Complex Inversion Formula

Theorem 13 : If $f(t)$ has a continuous derivative and is of exponential order γ for large positive values of t, where $\gamma>0$ and if $L[f(t)]=\bar{f}(p)$ then $L^{-1}[\bar{f}(p)]=f(t)$ is given by

$$f(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{pt}\,\bar{f}(p)\,dp,\quad t>0 \qquad ...(9)$$

and $\quad f(t)=0\ ;\ t<0$

Proof : If $g(t)$ has a continuous derivative and if $\int_{-\infty}^{\infty}g(t)\,dt$ is absolutely convergent, then $g(t)$ may be represented by the Fourier's integral such that

$$g(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ivt}\,dv\int_{-\infty}^{\infty}g(u)\,e^{ivu}\,du \qquad ...(10)$$

Now let us take $g(t)=\begin{cases}e^{-\gamma t}f(t), & t>0\\ 0\ , & t<0\end{cases}$

then $\int_{-\infty}^{\infty}g(t)\,dt$ is absolutely convergent for $\gamma>0$

Hence from (10), we have for $t>0$

$$e^{-rt}f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ivt}\left[\int_{0}^{\infty}e^{-\gamma u}f(u)\,e^{-ivu}\,du\right]dv,\ r>t$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ivt}\left[\int_0^{\infty} e^{-(\gamma+iv)u} f(u)\, du\right] dv$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ivt}\, \bar{f}(\gamma+iv)\, dv \qquad [\because L[f(t)] = \bar{f}(p)]$$

$$\therefore \quad e^{-\gamma t} f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{t(p-\gamma)}\, \bar{f}(p)\, dp \qquad \text{(putting } \gamma+iv = p\text{, so that } dv = \frac{dp}{i}\text{)}$$

$$\Rightarrow \quad f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\, \bar{f}(p)\, dp, \; t>0$$

Remark 1 : In the above proof, we also assume that $e^{-\gamma u} f(u)$ is absolutely integrable in $(0,\infty)$, i.e. $\int_0^{\infty} e^{-\gamma u}|f(u)|\, du$ converges, so that Fourier's is integral theorem can be applied.

Remark 2 : The integration in (9) is to be performed along a line $Re(p) = \gamma$ in the complex plane where $p = u + iv$. The real number γ is chosen so that the line $p=\gamma$ lies to the right of all the singularities (poles, branch points or essential singularities).

2.15 The Bromwich Contour

The integral (9) in Theorem 13 can also be evaluated by considering the contour integral $\frac{1}{2\pi}\oint_C e^{pt}\, \bar{f}(p)dp$

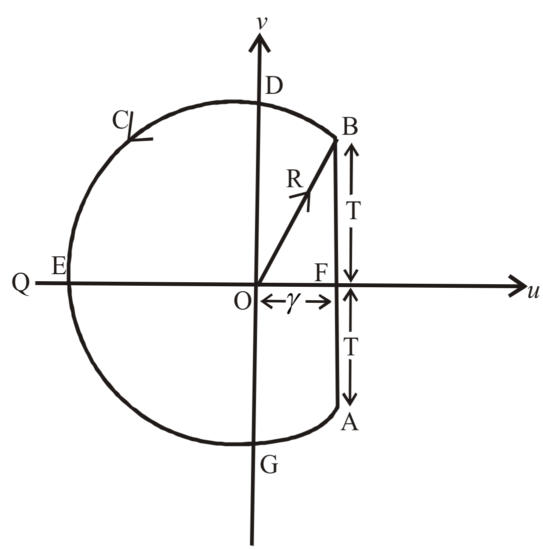


Figure 2.2

where C is the contour, as shown in the Fig. 2.2. The contour C is known as Bromwich contour and is denoted by (i) a line AB (ii) arc BDEGA of a circle of radius R with centre at the origin O. Also let the arc BDEGA be denoted by Γ, then we have

$$\oint_C e^{pt}\, \bar{f}(p)dp = \int_{\gamma-iT}^{\gamma+iT} e^{pt}\, \bar{f}(p)dp + \int_{\Gamma} e^{pt}\, \bar{f}(p)dp$$

or $$\frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} e^{pt}\bar{f}(p)\,dp = \frac{1}{2\pi i}\oint_C e^{pt}\bar{f}(p)\,dp - \frac{1}{2\pi i}\int_\Gamma e^{pt}\bar{f}(p)\,dp$$

Letting $T \to \infty$ or $R \to \infty$ as $R^2 = \gamma^2 + T^2$ and using the integral of the equation (9), we have

$$f(t) = \lim_{R\to\infty}\left[\frac{1}{2\pi i}\oint_C e^{pt}\bar{f}(p)\,dp - \frac{1}{2\pi i}\int_\Gamma e^{pt}\bar{f}(p)\,dp\right] \quad ...(11)$$

2.16 Use of Residue Theorem in obtaining Inverse Laplace Transform

Theorem 14 : Suppose that only singularities of $\bar{f}(p)$ are poles which all lie to the left of the line $Re(p) = \gamma$ for some real constant γ. Also suppose that $|\bar{f}(p)| < \frac{m}{R^k}$, where $k > 0$ and M are constants, such that $\lim_{R\to\infty}\int_\Gamma e^{pt}\bar{f}(p)\,dp = 0$, then the inverse Laplace transform of $\bar{f}(p)$ is given by

$f(t) =$ sum of residues of $e^{pt}\bar{f}(p)$ at all the poles of $\bar{f}(p)$ ...(12)

Proof : $$\oint_C e^{pt}\bar{f}(p)\,dp = \frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} e^{pt}\bar{f}(p)\,dp + \frac{1}{2\pi i}\int_\Gamma e^{pt}\bar{f}(p)\,dp \quad ...(13)$$

where C is the Bromwich contour and Γ is the circular arc BDEGA

Now by Cauchy's residue theorem, we have

$$\frac{1}{2\pi i}\oint_C e^{pt}\bar{f}(p)\,dp = \text{sum of residues of } e^{pt}\bar{f}(p) \text{ at all poles of } \bar{f}(p) \text{ inside } C \quad ...(14)$$

Using (14) in the equation (13), we have

$$\frac{1}{2\pi i}\int_{\gamma-iT}^{\gamma+iT} e^{pt}\bar{f}(p)\,dp = \Sigma \text{ Residues inside } C - \frac{1}{2\pi i}\int_\Gamma e^{pt}\bar{f}(p)\,dp.$$

Taking the limit as $T \to \infty$ (or $R \to \infty$ as $R^2 = \gamma^2 + T^2$), we find that

$$f(t) = \Sigma \text{ Residues inside } C - 0$$

$$\left\{\begin{array}{l} \because \lim_{R\to\infty}\int_\Gamma e^{pt}\bar{f}(p)\,dp = 0 \text{ and} \\ f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\bar{f}(p)\,dp,\ t > 0 \end{array}\right\}$$

$\therefore$ $f(t) =$ sum of residues of $e^{pt}\bar{f}(p)$ at all the poles of $F(p)$.

Example 17 : Use complex inversion formula to obtain the inverse Laplace transform of

$$\frac{p}{(p+1)\,(p-1)^2}.$$

Solution : Let $p = R\,e^{i\theta}$, we have

$$\left|\frac{p}{(p+1)(p-1)^2}\right| = \frac{|p|}{|p+1|\;|p-1|^2} = \frac{|R\,e^{i\theta}|}{|R\,e^{i\theta}+1|\;|R\,e^{i\theta}-1|^2}$$

$$\leq \frac{R}{\left(|R\,e^{i\theta}|1-1\right)\left(|R\,e^{i\theta}|-1\right)^2}$$

$$\leq \frac{R}{(R-1)^3} = \frac{1}{R^2\left[1-\left(1/R\right)\right]^3} \leq \frac{8}{R^2}, \text{ for } R>2$$

$\therefore$ $$\left|\frac{1}{(p+1)(p-1)^2}\right| < \frac{8}{R^2}, \text{ for } R>2$$

$\therefore$ $$L^{-1}\left[\frac{1}{(p+1)(p-1)^2}\right] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\,\frac{p}{(p+1)\,(p-1)^2}\,dp$$

$$= \frac{1}{2\pi i}\oint_C \frac{e^{pt}\,p\,dp}{(p+1)(p-1)^2} \qquad (C \text{ being Bromwich contour})$$

$=$ sum of residues of $e^{pt}\,\dfrac{p}{(p+1)\,(p-1)^2}$ at poles $p=-1$ (simple pole) and $p=1$ (a double pole).

Now the residue at simple pole at $p=-1$ is given by

$$\lim_{p\to -1}(p+1)\left\{\frac{p\,e^{pt}}{(p+1)(p-1)^2}\right\} = -\frac{1}{4}e^{-t}$$

and the residue at the double pole $p=1$ is given by

$$\lim_{p\to 1}\frac{1}{1!}\frac{d}{dp}\left[(p-1)^2\left\{\frac{p\,e^{pt}}{(p+1)(p-1)^2}\right\}\right] = \lim_{p\to 1}\frac{d}{dp}\left\{\frac{p\,e^{pt}}{p+1}\right\}$$

$$= \lim_{p\to 1}\frac{(p+1)\left(e^{pt}+p\,t\,e^{pt}\right)-p\,e^{pt}}{(p+1)^2} = \frac{e^t(1+2t)}{4}$$

$$\therefore \qquad L^{-1}\left[\frac{p}{(p+1)(p-1)^2}\right]=\frac{e^t(1+2t)}{4}-\frac{1}{4}e^t$$

Example 18 : Find $L^{-1}\left[\frac{1}{\left(p^2+1\right)^2}\right]$, using complex inversion formula.

Solution : Here $\bar{f}(p)=\frac{1}{\left(p^2+1\right)^2}=\frac{1}{\left[(p+i)(p-i)\right]^2}=\frac{1}{(p+i)^2(p-i)^2}$

Since $\left|\bar{f}(p)\right|<\frac{16}{R^4}\ (R>2)$ for $p=R\,e^{i\theta}$, therefore

$$L^{-1}\left[\frac{1}{\left(p^2+1\right)^2}\right]=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{pt}}{(p+i)^2(p-i)^2}\,dp$$

$$=\frac{1}{2\pi i}\oint_C\frac{e^{pt}}{(p+i)^2(p-i)^2}\,dp$$

Sum of residues of $\frac{e^{pt}}{(p+i)^2(p-i)^2}$ at poles $p=i$ and $p=-i$ which are double poles.

Now, residue at pole of order 2 at $p=i$

$$=\lim_{p\to i}\frac{d}{dp}\left[(p-i)^2\cdot\frac{e^{pt}}{(p+i)^2(p-i)^2}\right]$$

$$=\lim_{p\to i}\frac{d}{dp}\left[\frac{e^{pt}}{(p+i)^2}\right]$$

$$=\lim_{p\to i}\frac{(p+i)^2\,t\,e^{pt}-2(p+i)e^{pt}}{(p+i)^4}$$

$$=\lim_{p\to i}\frac{(p+i)\,t\,e^{pt}-2\,e^{pt}}{(p+i)^3}=-\frac{1}{4}t\,e^{it}-\frac{1}{4}i\,e^{it}$$

And the residue at pole of order 2 at $p=-i$ is

$$=-\frac{1}{4}t\,e^{-it}+\frac{1}{4}i\,e^{-it}$$

$$\therefore \qquad L^{-1}\left[\frac{1}{\left(p^2+1\right)^2}\right] = -\frac{1}{4}t\,e^{it} - \frac{1}{4}i\,e^{it} - \frac{1}{4}t\,e^{-it} + \frac{1}{4}i\,e^{-it}$$

$$= -\frac{1}{4}t\left(e^{it}+e^{-it}\right) - \frac{1}{4}i\left(e^{it}-e^{-it}\right)$$

$$= -\frac{1}{2}t\cos t + \frac{1}{2}\sin t$$

$$= \frac{1}{2}\left(\sin t - t\cos t\right)$$

Example 19 : Use complex inversion formula to obtain inverse Laplace Transform of

$$\frac{1}{(p+1)\left(p^2+1\right)}.$$

Solution : For $p = R\,e^{i\theta}$, we have $\left|\bar{f}(p)\right| < \frac{8}{R^3}\ (R > 2)$, therefore

$$L^{-1}\left[\frac{1}{(p+1)\left(p^2+1\right)}\right] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{pt}}{(p+1)(p+i)(p-i)}\,dp$$

$$= \frac{1}{2\pi i}\oint_C \frac{e^{pt}}{(p+1)(p+i)(p-i)}\,dp$$

$$= \text{sum of residues of } \frac{e^{pt}}{(p+1)(p+i)(p-i)}$$

at simple poles $p=-1$, $p=-i$ and $p=i$.

Now, Residue at simple pole at $p=-1$

$$= \lim_{p\to -1}(p+1)\left\{\frac{e^{pt}}{(p+1)(p+i)(p-i)}\right\} = \frac{e^{-t}}{2}$$

Residue at simple pole at $p=-i$

$$= \lim_{p\to -i}(p+i)\left\{\frac{e^{pt}}{(p+1)(p+i)(p-i)}\right\} = \frac{e^{-it}}{-2i(1-i)}$$

$$= -\frac{(1+i)\,e^{-it}}{4i}$$

Similarly residue at simple pole at $p=i$ is $\frac{e^{it}(1-i)}{4i}$

$$\therefore \qquad L^{-1}\left[\frac{1}{(p+1)(p^2+1)}\right]=\frac{e^{-t}}{2}-\frac{1}{4i}e^{-it}(1+i)+\frac{1}{4i}e^{it}(1-i)$$

$$=\frac{e^{-t}}{2}+\frac{1}{2}\left(\frac{e^{it}-e^{-it}}{2i}\right)-\frac{1}{2}\left(\frac{e^{it}+e^{-it}}{2}\right)$$

$$=\frac{1}{2}\left(e^{-t}+\sin t-\cos t\right)$$

2.17 Inverse Laplace Transform of Functions with Branch Points

If $\bar{f}(p)$ has branch points then Bromwich contour C is suitably modified e.g. if $\bar{f}(p)$ has only one branch point $p=0$, then the contour given in the fig. 2.3 can be used. Here BDE and LNA are the arcs of the circle of radius 'R' with centre O while FHK is the arc of a small circle centred at O of radius $\in$. This procedure can be understood by the following example.

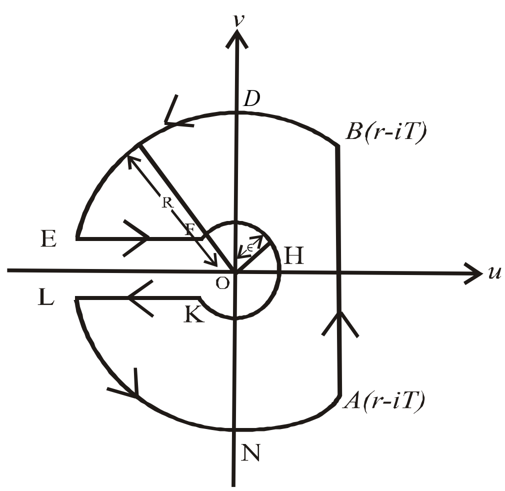


Figure 2.3

Exmaple 20 : Evaluate $L^{-1}\left[\frac{e^{-a\sqrt{p}}}{p}\right]$ by the use of complex inversion formula.

Solution : Using complex inversion formula, we have (by Theorem 13 and equation (9))

$$f(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{pt}.\frac{e^{-a\sqrt{p}}}{p}dp=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{pt-a\sqrt{p}}}{p}dp \qquad ...(15)$$

But the point $p=0$ is a branch point of $\left[\frac{e^{pt-a\sqrt{p}}}{p}\right]$. Therefore we consider the contour C as shown in the above Fig. 2.3 i.e. Bromwich contour which is indented at the point $p=0$ by means of a circle of small radius $\in$ with centre at $p=0$.

$$\therefore \quad \frac{1}{2\pi i}.\oint_C \frac{e^{pt-a\sqrt{p}}}{p}dp = \frac{1}{2\pi i}\int_{AB}\frac{e^{pt-a\sqrt{p}}}{p}dp + \frac{1}{2\pi i}\int_{BDE}\frac{e^{pt-a\sqrt{p}}}{p}dp$$

$$+\frac{1}{2\pi i}\int_{EF}\frac{e^{pt-a\sqrt{p}}}{p}dp + \frac{1}{2\pi i}\int_{FHK}\frac{e^{pt-a\sqrt{p}}}{p}dp$$

$$+\frac{1}{2\pi i}\int_{KL}\frac{e^{pt-a\sqrt{p}}}{p}dp + \frac{1}{2\pi i}\int_{LNA}\frac{e^{pt-a\sqrt{p}}}{p}dp \qquad ...(16)$$

Since the singularity $p = 0$ of the integrand is not inside C, the integral on the left of (16) vanishes by Cauchy's theorem. Also the integrand satisfies the condition of Theorem 14 $\left(i.e.\ \lim_{R\to\infty} e^{pt}\,\bar{f}(p)dp = 0\right)$ so that on taking the limit as $R \to \infty$, the integrals along BDE and LNA tends to zero. It implies that

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{pt-a\sqrt{p}}}{p}dp = \lim_{\substack{R\to\infty\\ \in\to 0}}\frac{1}{2\pi i}\int_{AB}\frac{e^{pt-a\sqrt{p}}}{p}dp$$

$$= -\lim_{\substack{R\to\infty\\ \in\to 0}}\frac{1}{2\pi i}\left[\int_{EF}\frac{e^{pt-a\sqrt{p}}}{p}dp + \int_{FHK}\frac{e^{pt-a\sqrt{p}}}{p}dp + \int_{KL}\frac{e^{pt-a\sqrt{p}}}{p}dp\right] \qquad ...(17)$$

Along EF, $p = u\,e^{\pi i}$ so that $\sqrt{p} = \sqrt{u}\,e^{i\pi/2} = i\sqrt{u}$ and as p goes from $-R$ to $-\in$, u goes from R to $\in$. Hence we have

$$\int_{EF}\frac{e^{pt-a\sqrt{p}}}{p}dp = \int_{-R}^{-\in}\frac{e^{pt-a\sqrt{p}}}{p}dp = \int_R^{\in}\frac{e^{-tu-ai\sqrt{u}}}{u}du$$

Similarly along KL, $p = u\,e^{-\pi i}$, $\sqrt{p} = \sqrt{u}\,e^{-\pi i/2} = -i\sqrt{u}$ and as p goes from $-\in$ to $-R$, u goes from $\in$ to R. Therefore

$$\int_{KL}\frac{e^{pt-a\sqrt{p}}}{p}dp = \int_{-\in}^{-R}\frac{e^{pt-a\sqrt{p}}}{p}dp = \int_{\in}^{R}\frac{e^{-tu+ai\sqrt{u}}}{u}du$$

Also along FHK, $p = \in e^{i\theta}$ and, we have

$$\int_{FHK}\frac{e^{pt-a\sqrt{p}}}{p}dp = \int_{\pi}^{-\pi}\frac{e^{\in e^{i\theta}t-a\sqrt{\in}e^{i\theta/2}}}{\in e^{i\theta}}\,i\in e^{i\theta}\,d\theta$$

$$= i\int_{\pi}^{-\pi} e^{\in e^{i\theta}t-a\sqrt{\in}e^{i\theta/2}}\,d\theta$$

Now substituting these values in the equation (17), we get

$$f(t) = -\underset{\substack{-R\to\infty \\ \in\to 0}}{Lim} \frac{1}{2\pi i}\left[\int_R^{\in} \frac{e^{-tu-ai\sqrt{u}}}{u}\,du + \int_{\in}^{R} \frac{e^{-tu+ai\sqrt{u}}}{u}\,du + i\int_{\pi}^{-\pi} e^{\in e^{i\theta}t - a\left(\sqrt{\in}\right)e^{i\theta/2}}\,d\theta\right]$$

$$= -\underset{\substack{R\to\infty \\ \in\to 0}}{Lim} \frac{1}{2\pi i}\left[\int_{\in}^{R} \frac{e^{-tu}\left(e^{ai\sqrt{u}} - e^{-ai\sqrt{u}}\right)}{u}\,du + i\int_{\pi}^{-\pi} e^{\in e^{i\theta}t - a\left(\sqrt{\in}\right)e^{i\theta/2}}\,d\theta\right]$$

$$= -\underset{\substack{R\to\infty \\ \in\to 0}}{Lim} \frac{1}{2\pi i}\left[2i\int_{\in}^{R} \frac{e^{-tu}\sin a\sqrt{u}}{u}\,du + i\int_{\pi}^{-\pi} e^{\in e^{i\theta}t - a\left(\sqrt{\in}\right)e^{i\theta/2}}\,d\theta\right]$$

But $\underset{\in\to 0}{Lim}\int_{\pi}^{-\pi} e^{\in e^{i\theta}t - a\left(\sqrt{\in}\right)e^{i\theta/2}}\,d\theta = \int_{\pi}^{-\pi} 1.d\theta = -2\pi$

$\therefore \quad f(t) = 1 - \frac{1}{\pi}\int_0^{\infty} \frac{e^{-ut}\sin a\sqrt{u}}{u}\,du$

or $\quad f(t) = 1 - erf\left(\frac{a}{2\sqrt{t}}\right) = erfc\left(\frac{a}{2\sqrt{t}}\right)$

Since $\quad I = \frac{1}{\pi}\int_0^{\infty} \frac{e^{-tu}\sin a\sqrt{u}}{u}\,du = \frac{2}{\pi}\int_0^{\infty} \frac{e^{-\omega^2 t}\sin a\,\omega}{\omega}\,d\omega$

$\Rightarrow \quad \frac{dI}{da} = \frac{2}{\pi}\int_0^{\infty} e^{-\omega^2 t}\cos a\omega\,d\omega$

$$= \frac{2}{\pi}\cdot\frac{\sqrt{\pi}}{2\sqrt{t}}e^{-a^2/4t} \qquad \left(\because \int_0^{\infty} e^{-at^2}\cos pt\,dt = \frac{\sqrt{\pi}}{2\sqrt{a}}e^{-p^2/4a}\right)$$

$\therefore \quad I = \frac{1}{\sqrt{\pi}}\int_0^{a} \frac{e^{-a^2/4t}\,da}{\sqrt{t}} = \frac{2}{\sqrt{\pi}}\int_0^{a/2\sqrt{t}} e^{-\omega^2}\,d\omega$

$\therefore \quad I = erf\left(\frac{a}{2\sqrt{t}}\right)$

Example 21 : Find $L^{-1}\left[e^{-a\sqrt{p}}\right]$.

Solution : Since $f(t) = L^{-1}\left[\frac{e^{-a\sqrt{p}}}{p}\right] = 1 - \frac{2}{\sqrt{\pi}}\int_0^{a/2\sqrt{t}} e^{-\omega^2}\,d\omega$

Also Since $f(0) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-\omega^2}\,d\omega = 1 - \frac{2}{\sqrt{\pi}}.\frac{\sqrt{\pi}}{2} = 0$

Therefore $L^{-1}\left[e^{-a\sqrt{p}}\right]=f'(t)=\dfrac{d}{dt}\left\{1-\dfrac{2}{\sqrt{\pi}}\int_0^{a/2\sqrt{t}} e^{-\omega^2}\,d\omega\right\}$

$$=-\frac{2}{\sqrt{\pi}}\,e^{-a^2/4\sqrt{t}}\,\frac{d}{dt}\left(\frac{a}{2\sqrt{t}}\right)$$

$$=\frac{a}{2\sqrt{t}}\,t^{-3/2}\,e^{-a^2/4t}$$

2.18 Inverse Laplace Transform of Functions with Infinitely Many Singularities

In this case, we have to choose the radius R_m of Bromwich contour of the curved portion such that there exist only a finite number of the singularities inside it and the curved portion does not pass through any singularity. Therefore the required inverse Laplace transform can be obtained by taking an appropriate limit as $m \to \infty$ and this will be clear by the following example.

Example 22 : Find $L^{-1}\left[\dfrac{\cosh u\sqrt{p}}{p\cosh\sqrt{p}}\right]$, where $0<u<1$

Solution : First of all, we have to find out the singularities of

$$\bar{f}(p)=\frac{\cosh u\sqrt{p}}{p\cosh\sqrt{p}},\quad 0<u<1 \qquad ...(18)$$

$$=\frac{1+\left(u\sqrt{p}\right)^2/2!+\left(u\sqrt{p}\right)^4/4!+....}{p\left[1+\left(\sqrt{p}\right)^2/2!+\left(\sqrt{p}\right)^4/4!+....\right]}$$

$$=\frac{1+u^2p/2!+u^4p^2/4!+......}{p\left(1+p/2!+p^2/4!+......\right)} \qquad ...(19)$$

But by inspection, it appears that $p=0$ is branch point due to the presence of $\left(\sqrt{p}\right)$ in the equation (18). But it is not so, therefore it is evident from (19) that there is a simple pole at $p=0$. So the function $\bar{f}(p)$ has infinitely many poles which can be obtained by the root of the equation

$$\cosh\sqrt{p}=\frac{1}{2}\left(e^{\sqrt{p}}+e^{-\sqrt{p}}\right)=0,$$

or $\qquad e^{2\sqrt{p}}=-1=e^{\pi i+2k\pi i},\ k=0,\pm1,\pm2.....$

or $\qquad \sqrt{p}=\left(k+\dfrac{1}{2}\right)\pi i$ or $p=-\left(k+\dfrac{1}{2}\right)^2\pi^2$

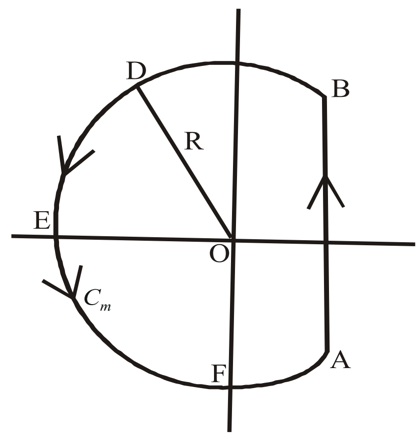


Figure 2.4

which are the simple poles. Hence $\bar{f}(p)$ has simple poles at $p = 0$ and $p = p_n$ where

$$p_n = -\left(n - \frac{1}{2}\right)^2 \pi^2, \qquad n = 1,2,3,\ldots$$

Therefore, the required inverse Laplace transform can be obtained by using the Bromwich contour. The line AB is such that all the poles lie to the left of it. Again we have to choose the Bromwich contour so that the curved portion BDEFA is an arc of circle Γ_m with centre at the origin 'O' and radius $R_m = m^2\pi^2$ where m is a positive integer. This implies that the contour does not pass through any of the poles. Now to find the residues of $e^{pt}\,\bar{f}(p) = e^{pt}\,\dfrac{\cosh u\sqrt{p}}{p\cosh\sqrt{p}}$ at the poles. We have

Residue at $p = 0$ is $\lim\limits_{p\to 0}(p-0)\left\{\dfrac{e^{pt}\cosh u\sqrt{p}}{p\cosh\sqrt{p}}\right\} = 1$

Again residue at $p_n = -\left(n-\dfrac{1}{2}\right)^2\pi^2$, $n = 1,2,3,\ldots\ldots$ is given by

$$\lim_{p\to p_n}(p-p_n)\left\{\frac{e^{pt}\cosh u\sqrt{p}}{p\cosh\sqrt{p}}\right\} = \lim_{p\to p_n}\left\{\frac{(p-p_n)}{\cosh\sqrt{p}}\right\} = \lim_{p\to p_n}\left\{\frac{e^{pt}\cosh u\sqrt{p}}{p}\right\}$$

$$= \lim_{p\to p_n}\left\{\frac{1}{\left(\sinh\sqrt{p}\right)\left(1/2\sqrt{p}\right)}\right\}\lim_{p\to p_n}\left\{\frac{e^{pt}\cosh u\sqrt{p}}{p}\right\}$$

$$= \frac{4(-1)^n}{\pi(2n-1)}\left[e^{-\left(n-\frac{1}{2}\right)^2\pi^2 t}\cos\left(n-\frac{1}{2}\right)\pi u\right]$$

If C_m is the contour of fig. 2.4, then

$$\frac{1}{2\pi i}\oint_{C_m}\frac{e^{pt}\cosh u\sqrt{p}}{\cosh\sqrt{p}}\,dp = 1+\frac{4}{\pi}\sum_{n=1}^{m}\frac{(-1)^n}{(2n-1)}\,e^{-\left(n-\frac{1}{2}\right)^2\pi^2 t}\cosh\left(n-\frac{1}{2}\right)\pi u$$

Now, taking the limit as $m \to \infty$ and the integral around Γ_m tends to zero, we find that

$$L^{-1}\left[\frac{\cosh u\sqrt{p}}{p\cosh\sqrt{p}}\right] = 1+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}\,e^{-\left(n-\frac{1}{2}\right)^2\pi^2 t}\cosh\left(n-\frac{1}{2}\right)\pi u\,.$$

2.19 Summary

In this unit you studied important results for inverse Laplace transform. Various methods for the evaluation of inverse Laplace transform were explained and illustrated with the help of solved and unsolvd problems. We also discussed complex inversion formula and inverse Laplace transform of certain functions were obtained by using this formula.

2.20 Answers to Self-Learning Exercises

Exercise - I

1. $\frac{t^n}{n!}$ 2. $\frac{t^3}{6}-3\cos 4t+\frac{5}{2}\sin 2t$

3. $J_0\left(2\sqrt{t}\right)$ 4. $J_0(at)$ 5. $e^{t\cos\theta}J_0\left(t\sin\theta\right)$

Exercise - II

1. $(-t)^n f^n(t)$ 2. $\frac{f(t)}{t}$ 3. $t\,e^{at}$ 4. $t\,e^{-t}$

2.21 Exercise 2 (c)

1. Find the inverse Laplace Transform of each of the following using complex inversion formula :

(i) $\frac{p}{p^2+a^2}$ (ii). $\frac{1}{(p+1)\,(p-2)^2}$

(iii) $\frac{1}{(p-2)\,(p+3)(p-4)}$ (iv) $\frac{1}{p^3\left(p^2+1\right)}$

(v) $\frac{p^2}{\left(p^2+4\right)^2}$ (vi) $\frac{p^2+3}{(p+1)\left(p^2-2p+5\right)}$

(vii) $\frac{1}{(p+a)(p-b)^2}$ where a and b are any positive constants.

[**Ans.** (i) $\cos at$ (ii) $\frac{1}{9}e^{-t}+\frac{1}{3}t\,e^{2t}-\frac{1}{9}e^{2t}$

(iii) $\frac{1}{35}e^{-3t}+\frac{1}{14}e^{4t}-\frac{1}{10}e^{2t}$ (iv) $\frac{1}{2}t^2+\cos t-1$

(v) $\frac{1}{4}\sin 2t+\frac{1}{2}t\cos 2t$ (vi) $\frac{1}{2}e^{-t}+\frac{1}{2}e^{t}(\cos 2t+\sin 2t)$

(vii) $\frac{e^{-at}}{(a+b)^2}+\frac{t\,e^{bt}}{(a+b)}-\frac{e^{bt}}{(a+b)^2}$]

2. Use the complex inversion formula to evaluate :

(i) $L^{-1}\left[\frac{\sinh pu}{p^2\cosh pa}\right],\ (0<u<a)$ (ii) $L^{-1}\left[\frac{\cosh u\sqrt{p}}{p\cosh a\sqrt{p}}\right],\ (0<u<a)$

(iii) $L^{-1}\left[\frac{p}{(p+1)^3(p-1)^2}\right]$

[**Ans.** (i) $u+\frac{8a}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\frac{(2n-1)\pi u}{2a}\cos\frac{(2n-1)\pi t}{2a}$

(ii) $1+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}e^{-(2n-1)^2\pi^2 t/4a^2}\cos\frac{(2n-1)\pi u}{2a}$

(iii) $\frac{1}{16}e^{-t}(1-2t^2)+\frac{1}{16}e^{t}(2t-1)$]

3. Find $L^{-1}\left[\frac{3p-1}{p(p-1)^2(p+1)}\right]$ by the complex inversion formula.

[**Ans.** $t\,e^{t}+e^{-t}+1$]

4. Evaluate $L^{-1}\left[\frac{\cosh pu}{p^3\cosh pa}\right],\ 0<u<a$

[**Ans.** $\frac{1}{2}(t^2+u^2-a^2)-\frac{16a^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}\cos\frac{(2n-1)\pi t}{2a}\cos\frac{(2n-1)\pi u}{2a}$]

5. Find $L^{-1}\left[\frac{e^{-\sqrt{p}}}{p}\right]$. Hence deduce that $L^{-1}\left[\frac{e^{-\sqrt{p}}}{k^2 p}\right] = erfc\left(\frac{k}{2\sqrt{t}}\right)$

[**Ans.** $erfc\left(\frac{1}{2\sqrt{t}}\right)$]

6. Find $L^{-1}\left[\frac{2}{(p-1)^2(p^2+1)}\right]$ by the complex inversion formula.

[**Ans.** $e^t(t-1)+\cos t$]

7. Evaluate $L^{-1}\left[\frac{\sinh t\sqrt{p}}{p\sinh\sqrt{p}}\right]$, $0 < t < 1$, $p > 0$

[**Ans.** $\sum_{n=0}^{\infty}\left[erfc\left(\frac{1-t+2n}{2\sqrt{t}}\right) - erfc\left(\frac{1+t+2n}{2\sqrt{t}}\right)\right]$]

Unit - 3
Solution of Ordinary Differential Equations with Constant and Variable Coefficients and the Solution of Boundary Value Problems by Laplace Transform

Structure of the Unit

3.0 Objective

The main object of this unit is to give application of the Laplace transform for finding solution of ordinary differential equations with constant and variable coefficients and boundary value problems such as heat conduction equation and wave equation.

3.1 Introduction

The Laplace transform is a Mathematical tool for finding the solution of ordinary and partial differential equations. By the application of Theorems 7 and 9 of Unit-1, the Laplace transform reduces a differential equation into an **algebraic equation** (which is known as subsidiary equation in the transformed function). The required solution is thus obtained by finding the inverse Laplace transform of the transformed function.

This method is very useful specially when the initial conditions i.e. the value of the function and its derivatives at $t = 0$ (say) are given in the problem.

The advantage of this method is that it yields the particular solution directly without the necessity of first finding complementary function and particular integral and then evaluating the arbitrary constants.

3.2 Solution of Ordinary Linear Differential Equations with Constant Coefficients

Let us consider a linear differential equation with constant coefficients

$$\frac{d^n x}{dt^n}+a_1\frac{d^{n-1}x}{dt^{n-1}}+a_2\frac{d^{n-2}x}{dt^{n-2}}+\ldots+a_{n-1}\frac{dx}{dt}+a_n\ x=f(t) \qquad \ldots(1)$$

where $t>0$ and $f(t)$ is a given function of the independent variable t. Suppose we want a solution $x=x(t)$ of this equation satisfying the initial conditions,

$$\left.\begin{aligned} x(0)=x_0,\ x'(0)=x_1,\ x''(0)=x_2,\ldots\ldots,x^{(n-1)}(0)=x_{n-1} \\ and\quad x^{(n)}(0)=x_n\quad when\ t=0 \end{aligned}\right\} \qquad \ldots(2)$$

We also suppose that there exists a transform of the solution of (1) and of its derivatives

$\frac{dx}{dt},\frac{d^2x}{dt^2},\ldots\ldots\frac{d^n x}{dt^n}$. Also let $L[f(t);p]=\bar{f}(p)$ and $L(x)=\bar{x}$

Now multiplying all terms of (1) by e^{-pt} and then integrating w.r. to 't' between limits 0 to ∞ and using the formulae for Laplace transform of derivatives, we have

$$\int_0^\infty e^{-pt}\frac{d^n x}{dt^n}dt+a_1\int_0^\infty e^{-pt}\frac{d^{n-1}x}{dt^{n-1}}dt+\ldots\ldots+a_{n-1}\int_0^\infty e^{-pt}\frac{dx}{dt}dt$$

$$+a_n\int_0^\infty e^{-pt}x\,dt=\int_0^\infty e^{-pt}f(t)dt$$

$$\Rightarrow\quad L\left[\frac{d^n x}{dt^n}\right]+a_1L\left[\frac{d^{n-1}x}{dt^{n-1}}\right]+\ldots\ldots+a_{n-1}\ L\left[\frac{dx}{dt}\right]+a_nL[x]=L[f(t)]$$

$$\Rightarrow\quad \left(p^n\bar{x}-p^{n-1}x_0-p^{n-2}x_1-\ldots-x_{n-1}\right)+a_1\left(p^{n-1}\bar{x}-p^{n-2}x_0-\ldots-x_{n-2}\right)$$

$$+\ldots\ldots+a_{n-1}\left(p\bar{x}-x_0\right)+a_n\ \bar{x}=\bar{f}(p)$$

Now collecting the coefficients of $\bar{x}$, we have

$$\bar{x}\left(p^n+a_1p^{n-1}+a_2p^{n-2}+\ldots+a_{n-1}p+a_n\right)$$

$$=\bar{f}(p)+x_0\left(p^{n-1}+a_1p^{n-2}+a_2p^{n-3}+\ldots+a_{n-1}\right)+x_1\left(p^{n-2}+a_1p^{n-3}+\ldots\ldots+a_{n-2}\right)$$

$$+x_2\left(p^{n-3}+a_1p^{n-4}+\ldots\ldots+a_{n-3}\right)+\ldots+x_{n-2}\left(p+a_1\right)+x_{n-1} \qquad \ldots(3)$$

The equation (3) is called the **subsidiary equation.** Dividing by $\left(p^n+a_1p^{n-1}+\ldots+a_n\right)$, we get $\bar{x}$ is a function of p. Now resolving this into partial fractions and taking inverse Laplace transform we get x is a function of t. This will be the requried solution of (1) under the given conditons (2).

Remark : How to obtain susidiary equation?

If the given diferential equation is written as

$$\left(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n\right)x = f(t),$$

the L.H.S. of the subsidiary equation is obtained by replacing D by p and x by $\bar{x}$. The first term of the R.H.S. is the Laplace transform of $f(t)$ whereas remaining terms are terms in $x_0, x_1, \ldots\ldots, x_{n-1}$ multiplied by some polynomial in 'p'. These polynomials are obtained by dividing $\left(p^n + a_1 p^{n-1} + a_2 p^{n-2} + \ldots . + a_n\right)$ successively by $p, p^2, p^3, \ldots, p^n$ and dropping off any negative power of p.

3.3 Solution of Ordinary Differential Equations with Variable Coefficients

The Laplace transform technique is also useful in solving the differential equations in which the coefficients are variable. For this purpose, we always use the result of theorem 9 of unit -1, because an expression of the form $t^n \dfrac{d^n y}{dt^n}$ is involved in the given differential equation.

Example 1 : Solve $\dfrac{d^4 y}{dx^4} - y = 1$, subject to conditons;

$$y(0) = y'(0) = y''(0) = y'''(0) = 0$$

Solution : We have $\left(D^4 - 1\right)y = 1$

Let $L(y) = \bar{y}$. Taking Laplace transform of both sides, we have

$$\left(p^4 - 1\right)\bar{y} = L(1) + y_0\left(p^3\right) + y_1\left(p^2\right) + y_2(p) + y_3(1)$$

or $$\left(p^4 - 1\right)\bar{y} = \frac{1}{p}$$

$$\left[\because\ y(0) = y_0 = 0,\ y'(0) = y_1 = 0,\ y''(0) = y_2 = 0,\ and\ \ y'''(0) = y_3 = 0\right]$$

$$\therefore \qquad \bar{y} = \frac{1}{p\left(p^4 - 1\right)} = \frac{1}{p\left(p^2 - 1\right)\left(p^2 + 1\right)}$$

$$\bar{y} = -\frac{1}{p} + \frac{p}{2\left(p^2 - 1\right)} + \frac{p}{2\left(p^2 + 1\right)}$$ (resolving into partical fractions)

Taking inverse Laplace transform on both the sides, we have

$$L^{-1}(\bar{y}) = -L^{-1}\left(\frac{1}{p}\right) + \frac{1}{2}L^{-1}\left(\frac{p}{p^2 - 1}\right) + \frac{1}{2}L^{-1}\left(\frac{p}{p^2 + 1}\right)$$

$\therefore \qquad y = -1 + \frac{1}{2}\cosh t + \cos t$

Example 2 : Solve $(D^2 + 9)y = \cos 2t$, if $y(0) = 1$, $y\left(\frac{\pi}{2}\right) = -1$.

Solution : Let $L(y) = \bar{y}$. Then taking Laplace transform of the given differential equation, we have

$$(p^2 + 9)\bar{y} = L(\cos 2t) + p\,y_0 + y_1$$

But $y'(0) = y_1$ is not given, so let us assume

$$y'(0) = y_1 = c \qquad (\because y_0 = 1)$$

$$\therefore \qquad (p^2 + 9)\bar{y} = \frac{p}{p^2 + 4} + p.1 + c$$

or $$\bar{y} = \frac{p}{(p^2 + 4)(p^2 + 9)} + \frac{(p + c)}{(p^2 + 9)}$$

$$= \frac{p}{5(p^2 + 4)} - \frac{1}{5}\frac{p}{(p^2 + 9)} + \frac{p}{(p^2 + 9)} + \frac{c}{(p^2 + 9)}$$

or $$\bar{y} = \frac{4}{5}\frac{p}{(p^2 + 9)} + \frac{c}{(p^2 + 9)} + \frac{p}{5(p^2 + 4)}$$

Hence $y(t) = L^{-1}(\bar{y}) = \frac{4}{5}\cos 3t + \frac{c}{3}\sin 3t + \frac{1}{5}\cos 2t$

But we are given that $y\left(\pi/2\right) = -1$ $\qquad \therefore c = 12/5$

Putting the value of 'c' in the above equation, we get the required solution i.e.

$$y = \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t + \frac{1}{5}\cos 2t.$$

Example 3 : Solve $(D^2 + 1)y = t\cos 2t$, $y = 0$, $\frac{dy}{dt} = 0$ when $t = 0$.

Solution : Taking Laplace transform of both the sides of the given differential equation, we get

$$(p^2 + 1)\bar{y} = L[t\cos 2t] + y_0(p) + y_1(1), \qquad \text{where } L[y] = \bar{y}$$

$$= -\frac{d}{dp}\left(\frac{p}{p^2 + 4}\right) = \frac{p^2 - 4}{(p^2 + 4)^2} \qquad (\because y_0 = 0 = y_1)$$

$$\therefore \quad \bar{y} = \frac{p^2-4}{(p^2+1)(p^2+4)^2} = -\frac{5}{9(p^2+1)} + \frac{5}{9(p^2+4)} + \frac{8}{3(p^2+4)^2}$$

$$\therefore \quad y = L^{-1}(\bar{y}) = -\frac{5}{9}L^{-1}\left[\frac{1}{(p^2+1)}\right] + \frac{5}{9}L^{-1}\left[\frac{1}{(p^2+4)}\right] + \frac{8}{3}L^{-1}\left[\frac{1}{(p^2+4)^2}\right]$$

$$= -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{8}{3}\int_0^t \frac{1}{2}\sin 2u.\frac{1}{2}\sin 2(t-u)\,du$$

(by convolution Theorem and $L^{-1}\left[\frac{1}{p^2+4}\right] = \frac{1}{2}\sin 2t$]

$$= \frac{-5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{1}{3}\int_0^t \left[\cos 2(t-2u) - \cos 2t\right]du$$

$$= \frac{-5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{1}{3}\left[\frac{-1}{4}\sin 2(t-2u) - u\cos 2t\right]_0^t$$

$$= \frac{-5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{1}{12}\sin 2t - \frac{t}{3}\cos 2t + \frac{1}{12}\sin 2t$$

$$\therefore \quad y = \frac{-5}{9}\sin t + \frac{4}{9}\sin 2t - \frac{t}{3}\cos 2t$$

Example 4 : $(2D^2+3D-2)y = 0,\ y(0)=1,\ y(t)\to 0$ as $t\to\infty$.

Solution : Taking Laplace transform on both the sides, we get

$$(2p^2+3p-2)\bar{y} = L(0) + y_0(2p+3) + y_1(2)$$

But $y'(0) = y_1$ is not given, so let us assume $y_1 = A$

$$\therefore \quad (2p^2+3p-2)\bar{y} = (2p+3) + 2A$$

or $$\bar{y} = \frac{2p+2A+3}{(2p-1)(p+2)} = \frac{2p+3}{(2p-1)(p+2)} + \frac{2A}{(2p-1)(p+2)}$$

$$= \frac{p+\frac{3}{2}}{\left(p-\frac{1}{2}\right)(p+2)} + \frac{A}{\left(p-\frac{1}{2}\right)(p+2)}$$

$$= \frac{p+2-\frac{1}{2}}{\left(p-\frac{1}{2}\right)(p+2)} + \frac{A}{\left(p-\frac{1}{2}\right)(p+2)}$$

$$= \frac{1}{\left(p-\frac{1}{2}\right)} + \frac{\left(A-\frac{1}{2}\right)}{\left(p-\frac{1}{2}\right)(p+2)} = \frac{1}{\left(p-\frac{1}{2}\right)} + \frac{\left(A-\frac{1}{2}\right)}{\left(-\frac{5}{2}\right)}\left[\frac{1}{p+2} - \frac{1}{p-\frac{1}{2}}\right]$$

$$= \frac{1}{p-\frac{1}{2}} - \frac{(2A-1)}{5} \cdot \frac{1}{(p+2)} + \frac{2A-1}{5\left(p-\frac{1}{2}\right)}$$

$$= \frac{(2A+4)}{5\left(p-\frac{1}{2}\right)} - \frac{(2A-1)}{5} \cdot \frac{1}{(p+2)}$$

$\therefore \quad y(t) = \left(\frac{2A+4}{5}\right) e^{t/2} - \frac{(2A-1)}{5}\, e^{-2t}$

But $\quad y(t) \to 0$ as $t \to \infty \Rightarrow \frac{2A+4}{5}$ must be zero

$\therefore \quad \frac{2A+4}{5} = 0 \Rightarrow A = -2$

Hence $y(t) = e^{-2t}$ is the required solution.

Example 5 : Solve $2\frac{d^2x}{dt^2} + 8x = CU(t-a)$ if $x(0) = 10$, $x'(0) = 0$ and $U(t-a)$ is a unit step function.

Solution : The given equation may be written as

$$(2D^2 + 8)x = CU(t-a)$$

Taking Laplace transform on both the sides, we get

$$(2p^2 + 8)\bar{x} = CL[U(t-a)] + x_0(2p) + x_1(2)$$

or $\quad (2p^2 + 8)\bar{x} = C\frac{e^{-ap}}{p} + 20p + 0 \qquad (\because x_0 = 10 \text{ and } x'(0) = x_1 = 0)$

or $\quad \bar{x} = C\frac{e^{-ap}}{2p(p^2+4)} + \frac{10p}{(p^2+4)}$

or $\quad \bar{x} = \frac{Ce^{-ap}}{8}\left\{\frac{1}{p} - \frac{p}{p^2+4}\right\} + \frac{10p}{(p^2+4)}$

Taking inverse Laplace transform on both the sides, we get

$$x = L^{-1}(\bar{x}) = \begin{cases} \frac{C}{8}\{1-\cos 2(t-a)\} + 10\cos 2t, & if\ t > a \\ 10\cos 2t, & if\ t < a \end{cases}$$

Example 6 : Solve $(D^2+4)y = f(t)$, $y(0)=0$, $y'(0)=1$

where $f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & t > 1 \end{cases}$

Solution : $L[f(t);p] = \int_0^\infty e^{-pt} f(t)\, dt = \int_0^1 e^{-pt}.1.dt + \int_1^\infty e^{-pt}.0.dt$

$$\therefore \quad L[f(t)] = \int_0^1 e^{-pt} dt = -\left(\frac{e^{-pt}}{p}\right)_0^1 = \frac{1}{p} - \frac{e^{-p}}{p}$$

Taking Laplace Transform of the given differential equation, we get

$$(p^2+4)\bar{y} = L[f(t)] + y_0(p) + y_1(1)$$

$$= \frac{1}{p} - \frac{e^{-p}}{p} + 0 + 1 = 1 + \frac{1}{p} - \frac{e^{-p}}{p}$$

$$\therefore \quad \bar{y} = \frac{1}{p^2+4} + \frac{1}{p(p^2+4)} - \frac{e^{-p}}{p(p^2+4)}$$

since $L^{-1}\left[\frac{1}{p^2+4}\right] = \frac{\sin 2t}{2}$

$$L^{-1}\left[\frac{1}{p(p^2+4)}\right] = \int_0^t \frac{\sin 2u}{2} du = \frac{1-\cos 2t}{4}$$

and $L^{-1}\left[\frac{e^{-p}}{p(p^2+4)}\right] = \begin{cases} \frac{1-\cos 2(t-1)}{4}, & t > 1 \\ 0 & , t < 1 \end{cases}$

$$\therefore \quad y = L^{-1}(\bar{y}) = \begin{cases} \frac{1}{2}\sin 2t + \frac{1-\cos 2t}{4}, & t < 1 \\ \frac{1}{2}\sin 2t - \frac{1}{4}\cos 2t + \frac{1}{4}\cos 2(t-1), & t > 1 \end{cases}$$

Example 7 : $t y'' + y' + 4ty = 0$ if $y(0) = 3$, $y'(0) = 0$.

Solution : On taking Laplace Transform of the given differential equation, we have

$$L[t\,y'']+L[y']+4L[ty]=L(0)$$

or $$-\frac{d}{dp}\left[p^2L(y)-p\,y(0)-y'(0)\right]+\left[p\,L(y)-y(0)\right]-4\frac{d}{dp}L(y)=0$$

or $$-\frac{d}{dp}\left[p^2z-p.3-0\right]+(pz-3)-4\frac{dz}{dp}=0 \quad \text{where} \quad L(y)=z$$

or $$(p^2+4)\frac{dz}{dp}+pz=0$$

or $$\frac{dz}{z}+\frac{p\,dp}{(p^2+4)}=0$$

On integration, we get

$$\log z+\frac{1}{2}\log(p^2+4)=\log c$$

or $$L(y)=z=\frac{c}{\sqrt{p^2+4}}$$

$$\Rightarrow y=c\,L^{-1}=\left[\frac{1}{\sqrt{p^2+4}}\right]$$

$$\therefore\ y=c\,J_0(2t)$$

But $y(0)=3$ $\quad\therefore c=3$

Hence $y=3\,J_0(2t)$

Example 8 : Solve $t\,y''+(t-1)\,y'-y=0$, $y(0)=5$, $y(\infty)=0$.

Solution : Taking Laplace transform of the given equation, we get

$$L[t\,y'']+L[t\,y']-L[y']-L[y]=0$$

or $$-\frac{d}{dp}L[y'']-\frac{d}{dp}L[y']-L[y']-L[y]=0$$

or $$-\frac{d}{dp}\left[p^2\,\bar{y}-p\,y(0)-y'(0)\right]-\frac{d}{dp}\left[p\,\bar{y}-y(0)\right]-\left[p\,\bar{y}-y(0)\right]-\bar{y}=0$$

or $$-\frac{d}{dp}(p^2\bar{y}-5p-c)-\frac{d}{dp}(p\,\bar{y}-5)-(p\,\bar{y}-5)-\bar{y}=0$$

(since $y'(0)$ is not given, let $y'(0)=c$)

or $$\left(p^2+p\right)\frac{d\,\bar{y}}{dp}+\left(3p+2\right)\bar{y}=10$$

or $$\frac{d\,\bar{y}}{dp}+\frac{3p+2}{p\left(p+1\right)}\bar{y}=\frac{10}{p\left(p+1\right)} \qquad ...(4)$$

which is a linear differential equation.

Now $\quad$ I.F. $= e^{\int\left(\frac{3p+2}{p(p+1)}\right)dp} = e^{\int\left(\frac{2}{p}+\frac{1}{p+1}\right)dp} = p^2\left(p+1\right)$

Therefore the solution of the above equation (4) is

$$y\,p^2\left(p+1\right)=\int\frac{10}{p\left(p+1\right)}.p^2\left(p+1\right)dp+c_2$$

$$=5\,p^2+c_2, \qquad \text{where } c_2 \text{ is the constant of integration}$$

or $$\bar{y}=\frac{5}{p+1}+\frac{c_2}{p^2\left(p+1\right)}$$

Thus $$y=L^{-1}\left(\bar{y}\right)=L^{-1}\left[\frac{5}{p+1}\right]+c_2\,L^{-1}\left[\frac{1}{p^2\left(p+1\right)}\right]$$

$$y=5e^{-t}+c_2\left(t+e^{-t}-1\right)$$

Buy $\quad y\left(\infty\right)=0\Rightarrow c_2=0$

Hence $y=5e^{-t}$ is the required solution.

3.4 Exercise 3 (a)

Solve the following differential equations by means of the Laplace transform :

1. $\frac{d^2y}{dt^2}+y=0;\; y\left(0\right)=1,\; y'\left(0\right)=1$ **[Ans.** $y=smt+\cos t$]

2. $\left(D^2+4\right)y=9t,\; y=0,\; \frac{dy}{dt}=7$, when $t=0$. **[Ans.** $y=\frac{9}{4}t+\frac{19}{8}\sin t$]

3. $\left(D^2-3D+2\right)y=1-e^{2t},\; y=1,\; \frac{dy}{dt}=0$ when $t=0$. [**Ans.** $y=\frac{1}{2}+\frac{1}{2}e^{2t}-t\,e^{2t}$]

4. $\left(D^3-D^2-D+1\right)y=8\,t\,e^{-t}$ given that $y\left(0\right)=0,\; y'\left(0\right)=1$ and $y''\left(0\right)=0$.

   [Ans. $y=\left(1+2t+t^2\right)e^{-t}-\left(1-t\right)e^{t}$]

5. $\frac{d^3y}{dt^3}-2\frac{d^2y}{dt^2}-\frac{dy}{dt}+2y=0$, satisfying the conditions, $y(0)=0,\ y'(0)=1,\ y(1)=0$.

$\left[\ \textbf{Ans.}\ \ y=\frac{1}{6}Ce^{-t}-\left(1+\frac{C}{2}\right)e^{t}+\left(1+\frac{C}{3}\right)e^{2t}\ \text{where}\ C=\frac{-6e^2}{(e-1)(2e+1)}\ \right]$

6. $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=t$, given that $y(0)=-3,\ y(1)=-1$

[**Ans.** $y=t-2-e^{-t}+te^{-t}$]

7. $\frac{d^2y}{dt^2}+a^2y=f(t)$, where $y(0)=1,\ y'(0)=-2$

[**Ans.** $y=\cos at-\frac{2}{a}\sin at+\frac{1}{2}\int_0^t f(u)\sin a(t-u)\,du$]

8. $(D^3-2D^2+5D)y=0,\ y(0)=0,\ y'(0)=1,\ y(\pi/8)=1$

[**Ans.** $y=1-e^{t}(\cos 2t-\sin 2t)$]

9. $(D^2+n^2)\,y=a\sin(nt+\alpha),\ y=Dy=0$ when $t=0$.

$\left[\ \textbf{Ans.}\ y=\frac{a}{2n^2}\left[\sin nt\cos\alpha-nt\cos(nt+\alpha)\right]\ \right]$

10. $\left[D^3-3D^2+3D-1\right]y=t^2e^t$

[**Ans.** $y=A_1t^2e^t+A_2te^t+A_3e^t+\frac{t^5e^t}{60}$ where A_1, A_2 and A_3 are arbitrary constants.]

11. $(D^2+1)y=t$ with $y'(0)=1,\ y(\pi)=0$. [**Ans.** $y=t+\pi\cos t$]

12. $(D^3+1)y=1,\ t>0$ if $y=Dy=D^2y=0$ when $t=0$.

$\left[\ \textbf{Ans.}\ y=1-\left(\frac{1}{3}\right)e^{-t}-\left(\frac{2}{3}\right)e^{t/2}\cos\left(\frac{\sqrt{3}t}{2}\right)\ \right]$

13. $2\frac{d^2y}{dt^2}+3\frac{dy}{dt}-2y=0,\ y(0)=1,\ y(t)\to 0$ as $t\to\infty$. **Ans.** $y=e^{-2t}$

14. $(D^2+D+1)\,y=3e^t;\ y(0)=0=y'(0)$ $\left[\ \textbf{Ans.}\ y=e^t-e^{-t/2}\left(\cos\frac{\sqrt{3}t}{2}+\sqrt{3}\sin\frac{\sqrt{3}t}{2}\right)\ \right]$

15. Find the bounded solution of the equation $t^2y''+t\,y'+(t^2-1)\,y=0,\ y(1)=2$.

$\left[\ \textbf{Ans.}\ \ y(t)=\dfrac{2J_1(t)}{J_1(1)}\ \right]$

16. Solve $y''-t\,y'+y=1,\ y(0)=1$ and $y'(0)=2$ **[Ans.** $y(t)=1+2t$]

17. Solve $y''+t\,y'-2y=4,\ y(0)=-1$ and $y'(0)=0$. **[Ans.** $y=t^2-1$]

18. Solve $t\,y''+y'+2y=0;\ y(0)=1$ $\left[\ \textbf{Ans.}\ \ y(t)=J_0\left(2\sqrt{2t}\right)\ \right]$

19. $y''+t\,y'-y=0$ given that $y(0)=0,\ y'(0)=1$. **[Ans.** $y=t$]

20. $t\,y''+(1-2t)\,y'-2y=0,\ y(0)=1,\ y'(0)=2$. **[Ans.** $y=e^{2t}$]

21. $y''+ax\,y'-2ay=1,\ y(0)=y'(0)=0,\ a>0$. **[Ans.** $y=\frac{1}{2}t^2$]

22. $\left[t\,D^2+(t-1)D-1\right]y=0$ if $y(0)=5,\ y(\infty)=0$ **[Ans.** $y=5e^{-t}$]

23. $y''+t\,y'-2y=2,\ y(0)=0=y'(0)$ **[Ans.** $y=t^2$]

24. $t\,y''+y'+t\,y=0,\ y(0)=1$ and $y'(0)=0$. **[Ans.** $y=J_0(t)$]

3.5 Partial Differential Equations and Boundary Value Problems

Many problems in Physics and Engineering are governed by partial differential equations together with certain prescribed conditions (known as boundary conditions) of the function which arise from the physical situation. Such problems are known as **boundary value problems.**

In solving such problems, Laplace transform provides an effective method of attack. In § 3.2, the Laplace transformation was used to reduce ordinary differential equations to algebraic equations. In the same way, a partial differential equation in two variables x and t may be reduced to an ordinary differential equation in x by means of Laplace transformation with respect to t. For this, the equation must be linear and the coefficients of the unknonwn function and its derivatives must be independent of 't'

i.e. the terms of the form $t^2\dfrac{\partial^2 u}{\partial t^2},\ t\dfrac{\partial u}{\partial t}$ etc. are absent.

Theorem 1 : If $u(x,t)$ be a function of two independent variables for $a\le x\le b$, $t>0$, then under suitable restrictions on $u=u(x,t)$, we have

$(i)\qquad L\left[\dfrac{\partial u}{\partial t}\right]=p\,\bar{u}(x,p)-u(x,0)$

(ii) $L\left[\frac{\partial u}{\partial x}\right] = \frac{d\,\bar{u}}{d\,x}$

(iii) $L\left[\frac{\partial^2 u}{\partial t^2}\right] = p^2 u(x,p) - p\,u(x,0) - u_t(x,0)$

(iv) $L\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{d^2 u}{dx^2}$ where $u(x,p) = L[u(x,t)]$ and $u_t(x,0) = \left(\frac{\partial u}{\partial t}\right)_{t=0}$

Proof : (i) $L\left[\frac{\partial u}{\partial t}\right] = \int_0^{\infty} e^{-pt}\frac{\partial u}{\partial t}dt$

$$= \left\{e^{-pt}u(x,t)\right\}_0^{\infty} + p\int_0^{\infty} e^{-pt}u(x,t)\,dt \qquad \text{(by integration by parts)}$$

$$= \underset{T\to\infty}{Lim}\left\{e^{-pt}u(x,t)\right\}_0^{T} + p\,\bar{u}(x,p) \qquad [\because L\,u(x,t) = \bar{u}(x,p)$$

$$= 0 - u(x,0) + p\,\bar{u}(x,p)$$

Assuming that $u(x,t)$ is of exponential order $'a'$ as $T \to \infty$

so that $\underset{T\to\infty}{Lim}\, e^{-pT}\,u(x,T) = 0$

$\therefore$ $L\left[\frac{\partial u}{\partial t}\right] = p\,\bar{u}(x,p) - u(x,0)$.

(ii) By the Leibnitz's rule for differentiation under the integral sign, we have

$$L\left[\frac{\partial u}{\partial x}\right] = \int_0^{\infty} e^{-pt}\frac{\partial u}{\partial x}dt = \frac{d}{dx}\int_0^{\infty} e^{-pt}u(x,t)\,dt$$

$$= \frac{d}{dx}L[u(x,t)] = \frac{d}{dx}\bar{u}(x,p) = \frac{d\,\bar{u}}{dx}$$

$\therefore$ $L\left[\frac{\partial u}{\partial x}\right] = \frac{d\,\bar{u}}{dx}$

(iii) $L\left[\frac{\partial^2 u}{\partial t^2}\right] = L\left[\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)\right] = L\left[\frac{\partial V}{\partial t}\right]$, where $V = \frac{\partial u}{\partial t}$

$$= p\,L\left[V(x,t) - V(x,0)\right]$$

$$= p\left[\left\{p\,\bar{u}(x,p) - u(x,0)\right\} - u_t(x,0)\right]$$

$$\therefore \qquad L\left[\frac{\partial^2 u}{\partial t^2}\right] = p^2\,\bar{u}(x,p) - p\,u(x,0) - u_t(x,0) \qquad \text{where } u_t(x,0) = \left(\frac{\partial u}{\partial t}\right)_{t=0}$$

(iv) Again, by the Leibnitz's rule for differentiation under the integral sign, we have

$$L\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_0^\infty e^{-pt}\frac{\partial^2 u}{\partial x^2}\,dt = \frac{d^2}{dx^2}\int_0^\infty e^{-pt}u(x,t)\,dt$$

$$= \frac{d^2}{dx^2}L\left[u(x,t)\right] = \frac{d^2}{dx^2}\bar{u}(x,p) = \frac{d^2 u}{dx^2}$$

$$\therefore \qquad L\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{d^2\bar{u}}{dx^2}$$

3.6 Solution of Boundary Value Problems

Laplace Transform (with respect to t or x) in one dimensional boundary value problem converts the partial differential equation (or equations) into an ordinary differential equation. The required solution can then be obtained by solving this equation by the methods discussed earlier. In two dimensional problems, we usually apply Laplace transformation twice (for example, with respect to t and with respect to x) and then arrive at ordinary differential equation. In such a case, the required solution is obtained by a double inversion. This process is usually referred to as iterated Laplace transformation. A similar technique can be applied to three (or higher) dimensional problems.

3.7 Heat Conduction Equation

The heat flow in a body of homogeneous material is governed by the heat equation

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \qquad c^2 = \frac{k}{\sigma\rho} \qquad \text{...(5)}$$

or

$$\frac{\partial u}{\partial t} = c^2\nabla^2 u,$$

where $u(x,y,z,t)$ is the temperature in the body, k is the thermal conductivity, σ the specific heat, ρ the density of the material of the body and the constant c^2, is called the diffusivity of the body. Also $\nabla^2 u$ is known as Laplacian operator.

If there is no flow of heat in the $z-$direction, i.e. the temperature in the body is independent of z, then the heat equation (5) becomes

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \text{...(6)}$$

which is called the heat equation for two dimensional flow parallel to $xy-$plane.

If we consider the heat flow in a long thin bar or wire of constant cross-section and homogeneous material which is along x-axis and is perfectly insulated laterally, so that the heat flows in the x-direction only, u depends only x and t and therefore the heat equation becomes

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(7)$$

which is known as one-dimensional heat equation.

3.8 Wave Equation

The transverse displacement u of an elastic string is governed by the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c^2 = \frac{T}{\rho} \qquad ...(8)$$

where the variable $u(x,t)$ is the displacement of any point x of the string at time t. The constant $c^2 = T/\rho$, where T is the (constant) tension in the string and ρ is the (constant) mass per unit length of the string.

This equation is applicable to the small transverse vibration of a taut, flexible string, beam initially located on the x-axis and set into motion. (see Fig.)

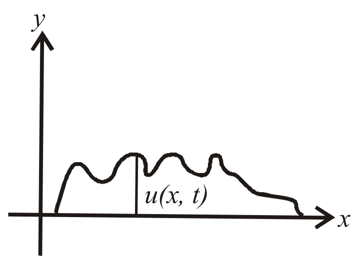


Figure 3.1

If $u(x,y,t)$ is the transverse diplacement of any point (x,y) of a membrane in the x, y plane at any time t, then the vibrations of this membrane, assumed small, are governed by the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \qquad ...(9)$$

which is called the two dimensional wave equation.

Similarly $$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = c^2 \nabla^2 u$$

Where $\nabla^2 u$ is called the Laplacian of $u(x,y,z,t)$ is the equation for the transverse vibrations in three dimensions.

Example 9 : Find the solution of $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, where $0 < x < 1$, $t > 0$ together with the conditions

$u(x,0) = 3\sin 2\pi x$, $u(0,t) = 0$, $u(1,t) = 0$.

Solution : Taking Laplace transform of both the sides of the given partial differential equation and using the initial conditions, we obtain

$$p\,\bar{u}(x,p) - u(x,0) = \frac{d^2\bar{u}}{dx^2}, \qquad \text{where } L[u(x,t)] = \bar{u}(x,p)$$

or $$\frac{d^2\bar{u}}{dx^2} - p\,\bar{u}(x,p) = -3\sin 2\pi x$$

or $$(D^2 - p)\bar{u} = -3\sin 2\pi x \qquad \text{...(10)}$$

which is a second order linear differential equation whose

$$C.F. = C_1 e^{x\sqrt{p}} + C_2 e^{-x\sqrt{p}}$$

and $$P.I. = \frac{-3\sin 2\pi x}{(D^2 - p)} = \frac{3\sin 2\pi x}{(p + 4\pi^2)}$$

Therefore the general solution of the above equation (10) is

$$\bar{u}(x,p) = C_1 e^{x\sqrt{p}} + C_2 e^{-x\sqrt{p}} + \frac{3\sin 2\pi x}{p + 4\pi^2} \qquad \text{...(11)}$$

To evaluate C_1 and C_2, we take the Laplace transform of those boundary conditions which involve t, we have

$$L[u(0,t)] = \bar{u}(o,p) = 0 \text{ and } L[u(1,t)] = \bar{u}(1,p) = 0$$

Using these conditions in equation (11), we get

$$C_1 + C_2 = 0 \quad \text{and} \quad C_1 e^{\sqrt{p}} + C_2 e^{-\sqrt{p}} = 0$$

Solving the above two equations, we find that

$$C_1 = 0,\ C_2 = 0 \text{ and}$$

Therefore, the equation (11) beomes

$$\bar{u}(x,p) = \frac{3}{p + 4\pi^2}\sin 2\pi x \qquad \text{...(12)}$$

Now taking inverse Laplace transform of both sides of (12), we get

$$u(x,t) = 3\,e^{-4\pi^2 t}\sin 2\pi x$$

which is the required solution.

Example 10 : Find the solution of

$$\frac{\partial u}{\partial t}=3\frac{\partial^2 u}{\partial x^2}, \text{ given that } u_x(0,t)=0,\ u\left(\frac{\pi}{2},t\right)=0 \text{ and } u(x,0)=30\cos 5x.$$

Solution : Let $L[u(x,t)]=\bar{u}(x,p)$. Then taking Laplace transform on both sides of the given differential equation, we find that

$$p\bar{u}(x,p)-u(x,0)=3\frac{d^2\bar{u}}{dx^2}$$

or $$\frac{d^2\bar{u}}{dx^2}-\frac{p}{3}\bar{u}=-10\cos 5x$$

The general solution of above linear differential equation is

$$\bar{u}(x,p)=C_1e^{\sqrt{(p/3)}x}+C_2e^{-\sqrt{(p/3)}x}+\frac{30}{75+p}\cos 5x \qquad ...(13)$$

To evaluate C_1, C_2, we take the Laplace transform of boundary condition unvolving t, we obtain

$$L\left[u\left(\pi/2,t\right)\right]=\bar{u}\left(\pi/2,p\right)=0 \quad \text{when } x=\frac{\pi}{2}$$

and $$L\left[\frac{\partial u}{\partial x}\right]=0, \text{ when } x=0 \Rightarrow \frac{d\bar{u}}{dx}=0, \text{ when } x=0$$

$\therefore$ From equation (13) we get

$$0=C_1e^{(\pi/2)\sqrt{(p/3)}}+C_2e^{-(\pi/2)\sqrt{(p/3)}}$$

and $$0=C_1\sqrt{(p/3)}-C_2\sqrt{(p/3)}$$

Solving these equations, we find $C_1=0=C_2$ and equation (13) becomes

$$\bar{u}(x,p)=\frac{30}{75+p}\cos 5x \qquad ...(14)$$

Hence by taking inverse Laplace transform of the above equation (14), we obtain

$$u(x,t)=30e^{-75t}\cos 5x$$

which is the required solution.

Example 11 : Find the solution of the equation

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}$$

which tends to zero as $x\to\infty$ and which satisfies the conditions

$u = f(t)$ when $x = 0,\ t > 0$ and $u = 0$ when $x > 0,\ t = 0$

Solution : Taking Laplace transform of both the sides of the given equation, we have

$$p\bar{u}(x,p) - u(x,0) = k\frac{d^2\bar{u}}{dx^2}$$

or $$\frac{d^2\bar{u}}{dx^2} - \frac{p}{k}\bar{u} = 0 \qquad [\because u(x,0) = 0] \qquad ...(14)$$

$\therefore$ The solution of the equation (14) is given by

$$\bar{u}(x,p) = C_1 e^{\sqrt{(p/k)}x} + C_2 e^{-\sqrt{(p/k)}x} \qquad ...(15)$$

where C_1 and C_2 are arbitrary constants.

Since $u \to 0$ as $x \to \infty \Rightarrow \bar{u}(x,p) \to 0$ as $x \to \infty$

which implies that C_1 must be zero.

Therefore equation (15) gives

$$\bar{u}(x,p) = C_2\ e^{-\sqrt{(p/k)}x} \qquad ...(16)$$

But we are also given that $u = f(t)$ when $x = 0$,

$\Rightarrow$ $\bar{u} = \bar{f}(p)$ when $x = 0$

$\therefore$ From (16), we have $C_2 = \bar{f}(p)$

Hence from (16), we have $\bar{u} = \bar{f}(p)\ e^{-\sqrt{(p/k)}x}$

Applying complex inversion formula, we get

$$u(x,t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt}\bar{f}(p)\ e^{-\sqrt{(p/k)}x}dp.$$

Example 12 : A semi-infinite rod $x > 0$ is initially at temperature zero. At time $t > 0$, a constant temperature $V_0 > 0$ is applied and maintained at the face $x = 0$. Find the temperature at any point of the solid at any time $t > 0$.

Solution : The temperature $u(x,t)$ at any point of the rod at any time $t > 0$ is governed by the one dimensional heat equation

$$\frac{\partial u}{\partial t} = k^2\frac{\partial^2 u}{\partial x^2},\ (x > 0, t > 0) \qquad ...(17)$$

with the boundary conditions

$$u(0,t) = V_0,\ \ u(x,0) = 0$$

Taking Laplace transform of both the sides of the equation (17), we have

$$p\bar{u}(x,p)-u(x,0)=k^2\frac{d^2\bar{u}}{dx^2}$$

or $$\frac{d^2\bar{u}}{dx^2}-\frac{p}{k}\bar{u}=0$$

whose solution is

$$\bar{u}(x,p)=C_1\ e^{x\sqrt{(p/k)}}+C_2\ e^{-x\sqrt{(p/k)}} \qquad ...(18)$$

Since u is finite when $x\to\infty$

$\therefore$ $\bar{u}$ is also finite when $x\to\infty$

$\therefore$ $C_1=0$, otherwise $\bar{u}\to\infty$ as $x\to\infty$

Taking the Laplace transform of the boundary condition $u(0,t)=V_0$, we have

$$\bar{u}(0,p)=\int_{t=0}^{\infty}e^{-pt}\ V_0\ dt=\frac{V_0}{p}$$

$\therefore$ From (18), we have

$$\bar{u}(0,p)=C_2=\frac{V_0}{p}$$

Hence $\bar{u}(x,p)=\frac{V_0}{p}\ e^{-x\sqrt{(p/k)}}$ $\qquad (\text{since } C_1=0)$

$$\therefore\quad u(x,t)=L^{-1}\left[\frac{V_0}{p}\ e^{-x\sqrt{(p/k)}}\right]=V_0\ erfc\left(\frac{x}{2\,k\,t}\right)\qquad\left\{\because\ L^{-1}\left[\frac{e^{-\lambda\sqrt{p}}}{p}\right]=erfc\left(\frac{\lambda}{2\sqrt{t}}\right)\right\}$$

Example 13 : An infinite long string having one end $x=0$ is initially at rest on the x-axis. The end $x=0$ undergoes a periodic transverse displacement given by $A_0\sin\omega t$, $t>0$. Find the displacement of any point on the string at any time.

Solution : Let us suppose that $u(x,t)$ is the transverse displacement of the string at any point x at any time t, then the boundary value problem is governed by the equation

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2},\ x>0,\ t>0 \qquad ...(19)$$

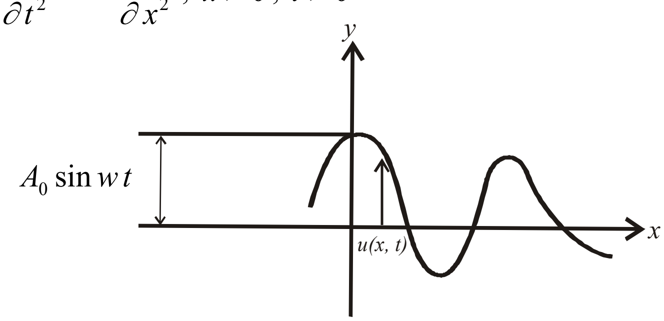


Figure 3.2

with the initial and boundary conditions

$$u(x,0) = 0 \qquad x \geq 0 \qquad ...(20)$$

$$ut(x,0) = 0 \qquad x \geq 0 \qquad ...(21)$$

$$u(0,t) = A_0 \sin(\omega t), \quad t > 0 \qquad ...(22)$$

and the displacement is finite i.e. $u(x,t) < M$

Taking the Laplace Transform of equation (19), we have

$$L\left[\frac{\partial^2 u}{\partial t^2}\right] = c^2 L\left[\frac{\partial^2 u}{\partial x^2}\right]$$

or $\quad p^2\bar{u}(x,p) - u(x,0) - u_t(x,0) = c^2 \dfrac{d^2\bar{u}}{dx^2}$

or $\quad \dfrac{d^2\bar{u}}{dx^2} - \dfrac{p^2\bar{u}}{c^2} = 0 \qquad$ or $\quad \left(D^2 - \dfrac{p^2}{c^2}\right)\bar{u} = 0$

A.E. is $m^2 - \dfrac{p^2}{c^2} = 0$ which gives $m = \pm \dfrac{p}{c}$

$\therefore \quad \bar{u}(x,p) = Ae^{\frac{px}{c}} + Be^{-px/c}$

Since $u(x,t)$ is finite, so $\bar{u}(x,p)$ is also finite $\forall x$

$A = 0$, otherwise $\bar{u}(x,p)$ becomes infinite when $x \to \infty$

$\therefore \quad \bar{u}(x,p) = Be^{-px/c} \qquad ...(23)$

From (22), we have $u(0,t) = A_0 \sin\omega t \qquad \therefore \bar{u}(0,p) = A_0 = \dfrac{\omega}{p^2+\omega^2}$

i.e. $\quad \bar{u} = \dfrac{A_0\omega}{p^2+\omega^2}$ when $x = 0 \Rightarrow B = \dfrac{A_0\omega}{p^2+\omega^2}$

Hence (23) gives

$$\bar{u}(x,p) = \frac{A_0\omega}{p^2+\omega^2}\, e^{-px/c}$$

Taking inverse Laplace transform, we get

$$u(x,t) = A_0\omega\, L^{-1}\left[\frac{e^{-px/c}}{p^2+\omega^2}\right]$$

$$= \begin{bmatrix} A_0 \sin w\left(t - \frac{x}{c}\right), & t > \frac{x}{c} \\ 0 & , \ t < \frac{x}{c} \end{bmatrix} \qquad \begin{pmatrix} L^{-1}\left[e^{-ap}.\bar{f}(p)\right] = f(t-a), t > a \\ = 0, t < a \\ \text{using second shifting theorem} \end{pmatrix}$$

Example 14 : A flexible string has its end points on the x-axis at $x = 0$ and $x = c$. At time $t = 0$, the string is given a shape defined by $b\sin\left(\frac{\pi x}{c}\right)$, $0 < x < c$ and released. Find the displacement of any point x of the string at any time $t > 0$.

or

A string is stretched between two fixed points $(0,0)$ and $(c,0)$. If it is displaced in to curve $u = b\sin\left(\frac{\pi x}{c}\right)$ and released from rest in that position at time $t = 0$. Find the displacement of any time t of any point $0 < x < c$.

Solution : The displacement $u(x,t)$ of any point of the string is governed by the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(24)$$

with the boundary conditions

(i) $u(0,t) = 0$ (ii) $u(c,t) = 0$

(iii) $u_t(x,0) = 0$ (iv) $u(x,0) = b\sin\left(\frac{\pi x}{c}\right)$

Taking Laplace transform of the above equation (24), we get

$$p^2\,\bar{u}(x,p) - p\,u(x,0) - u_t(x,0) = a^2 \frac{d^2\bar{u}}{dx^2}$$

Applying the boundary conditions,

$$\left(D^2 - \frac{p^2}{a^2}\right)u^2 = \frac{-pb}{a^2}\sin\left(\frac{z\,x}{c}\right)$$

whose general solution is

$$\bar{u}(x,p) = C_1\,e^{-px/a} + C_2\,e^{px/a} - \frac{pb}{a^2}\left[\frac{1}{-\left(\frac{\pi^2}{c^2}\right) - \left(\frac{p^2}{a^2}\right)}\right]\sin\left(\frac{\pi x}{c}\right) \qquad ...(25)$$

$$\bar{u}(x,p) = C_1\, e^{-px/a} + C_2\, e^{px/a} + pb\sin\left(\frac{\pi x}{c}\right).\frac{1}{p^2 + \left(\frac{\pi a}{c}\right)^2}$$

But from the boundary conditions (i) and (ii), we have

$$\bar{u}(0,p) = 0 \quad \text{as} \quad \bar{u}(c,p) = 0$$

Applying the above two boundary conditions, we get

$$C_1 + C_2 = 0 \qquad \text{...(26)}$$

and $$C_1\, e^{-pc/a} + C_2\, e^{pc/a} = 0 \qquad \text{...(27)}$$

Solving (26) and (27),

$$C_1 = C_2 = 0$$

Then the equation (25) gives

$$\bar{u}(x,p) = b\sin\left(\frac{\pi x}{c}\right).\frac{p}{p^2 + \left(\frac{\pi a}{c}\right)^2}$$

Taking inverse Laplace Transform of both the sides, we get

$$u(x,t) = b\sin\left(\frac{\pi x}{c}\right).\cos\left(\frac{\pi at}{c}\right)$$

Example 15 : Solve the boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = a^2\,\frac{\partial^2 u}{\partial x^2},\ (x > 0,\ t > 0)$$

with the boundary conditions

$$u(x,0) = 0 \qquad u_t(x,0) = 0; \qquad x > 0$$

$$u(0,t) = f(t) \qquad \lim_{x\to\infty} u(x,t) = 0, \qquad t \geq 0$$

Solution : Taking Laplace Transform of the given equation

$$p^2\,\bar{u}(x,p) - p\,u(x,0) - u_t(x,0) = a^2\,\frac{d^2\bar{u}}{d\,x^2}$$

Applying the boundary conditions, we have

$$p^2\,\bar{u}(x,p) = a^2\,\frac{d^2\bar{u}}{d\,x^2}$$

or $\quad \dfrac{d^2\bar{u}}{d x^2} - \dfrac{p^2}{a^2}\bar{u} = 0$

whose general solution is

$$\bar{u}(x,p) = C_1 e^{-px/a} + C_2 e^{px/a} \qquad ...(28)$$

But we are given that $u(0,t) = f(t)$

$$\Rightarrow \bar{u}(0,p) = \int_0^\infty e^{-pt} f(t)\,dt = \bar{f}(p) \qquad ...(29)$$

and $\quad \lim_{x\to\infty} u(x,t) = 0 \Rightarrow \lim_{x\to\infty} \bar{u}(x,p) = 0 \qquad ...(30)$

In view of (33), we get $C_2 = 0$

Hence $\bar{u}(x,p) = C_1 e^{-px/a}$

Applying the condition (29), we get

$$\bar{f}(p) = C_1$$

$\therefore \quad \bar{u}(x,p) = \bar{f}(p) e^{-px/a}$

Taking inverse Laplace transform, we get

$$u(x,t) = f\left(t - \frac{x}{a}\right) u\left(t - \frac{x}{a}\right)$$

where $U\left(t - \dfrac{x}{a}\right)$ is Heaviside unit step function.

Self-Learning Exercise

Fill in the Blanks :

1. On taking Laplace transform of $2\dfrac{d^2x}{dt^2} + 8x = CU(t-a)$ with $x(0) = 10$ and $x'(0) = 0$ w.r.t. variable t, it converts to

   $\bar{x} =$......... and solution is $x(t) =$............

2. $L\{xy'' + (x-1)y' - y,\ p\} =$...... (with $y(0) = 5,\ y(\infty) = 0$)

3. $L\left\{\dfrac{\partial^2 u}{\partial x^2}; p\right\} =$.....

4. $L\left\{\dfrac{\partial^2 u}{\partial t^2}; p\right\} =$..... where $u(x,0) = 0$, $u_t(x,0) = 5$

5. Write two-dimensional heat conduction equation?

6. What is a boundary value problem?

3.9 Summary

In this unit you studied the solution of ordinary differential equations with constant and variable coefficients and boundary value problems by the method of Laplace transform. This method is illustrated with the help of solved examples.

3.10 Answers to Self-Learning Exercise

1. $\bar{x} = \dfrac{10p}{p^2+4} + \dfrac{c\,e^{-ap}}{2p(p^2+4)}$ and $x(t) = \begin{cases} 10\cos 2t = \dfrac{c}{8}\left[1-\cos 2(t-a)\right], & if \;\; t > a \\ 10\cos 2t & , if \;\; t < a \end{cases}$

2. $3J_0(2x)$

3. $\dfrac{d^2\bar{u}}{dx^2}$, where $\bar{u}(x,p) = L\{u(x,t);p\}$

4. $p^2\,\bar{u}(x,p) - 5$

5. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = k\,\dfrac{\partial u}{\partial t}$

3.11 Exercise 3 (b)

1. Solve $\dfrac{\partial^2 u}{\partial t^2} = 9\,\dfrac{\partial^2 u}{\partial x^2}$ subject to the conditions :

$u(0,t) = 0$, $u(2,t) = 0$, $u(x,0) = 20\sin 2\pi x - 10\sin 5\pi x$ and $u_t(x,0) = 0$

[Ans. $u(x,t) = 20\sin 2\pi x\cos 6\pi t - 10\sin 5\pi x\cos 15\pi t$]

2. Solve $\dfrac{\partial^2 u}{\partial t^2} = 16\,\dfrac{\partial^2 u}{\partial x^2}$, subject to the conditions

$u(x,0) = 0,\ u(3,t) = 0,\ u_x(0,t) = 0$ and $u_t(x,0) = 12\cos\pi x + 16\cos 3\pi x - 8\cos 5\pi x$

[Ans. $u(x,t) = \dfrac{3}{\pi}\cos\pi x\sin 4\pi t + \dfrac{4}{3\pi}\cos 3\pi x\sin 12\pi t - \dfrac{4}{5\pi}\cos 5\pi x\sin 20\pi t$]

3. Solve $\dfrac{\partial u}{\partial t} = 2\,\dfrac{\partial^2 u}{\partial x^2}$, given that

$u(0,t) = 0,\ u(5,t) = 0,\ u(x,0) = 10\sin 4\pi x$

[Ans. $u(x,t) = 10e^{-32\pi^2 t}\sin 4\pi x$]

4. Find the bounded solution $u(x,t),\ 0<x<1,\ t>0$ of the boundary value problem

$$\frac{\partial u}{\partial x}-\frac{\partial u}{\partial t}=1-e^{-t} \qquad \text{provided that } u(x,0)=x.$$

[Ans. $u(x,t)=x+1-e^{-t}$]

5. Find the bounded solution of $\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}$, $x>0$, $t>0$ such that $u(0,t)=1$, $u(x,0)=0$.

[**Ans.** $u(x,t)=erfc\left(\dfrac{x}{2\sqrt{t}}\right)$]

6. Find the solution of the diffusion equation

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2},\ x>0,\ t>0$$

subject to the initial and boundary conditions $u(x,0)=0,\ x>0;-K\left(\dfrac{\partial u}{\partial x}\right)=f(t)$, at $x=0$, $t>0$ and $u(x,t)\to 0$ as $x\to\infty$ and $t>0$ (where k and K are respectively the thermal diffusitivity and conductivity of the material of the given solid).

[**Ans.** $u(x,t)=\dfrac{x}{K\sqrt{\pi}}\displaystyle\int_{x/2\sqrt{kt}}^{\infty} f\left(t-\frac{x^2}{4kv^2}\right)v^{-2}\,e^{-v^2}\,dv$]

7. A semi-infinite solid $x>0$ has its initial temperature equal to zero. A constant heat flux 'A' is applied at the face $x=0$ so that $-K\,u_x(0,t)=A$. Find the temperature at any point $x>0$ of the solid.

[**Ans.** $u(x,t)=\dfrac{2A}{K}\sqrt{\dfrac{kt}{\pi}}\;e^{-x^2/4kt}-\dfrac{Ax}{K}\;erfc\left(\dfrac{x}{2\sqrt{kt}}\right)$]

8. A bar of length 'l' is at constant temperature u_0. At $t=0$, the end $x=l$ is suddenly given the constant temperature u_1 and the end $x=0$ is insulated. Assuming that the surface of the bar is isulated, find the temperature at any point x of the bar at any time $t>0$.

[**Ans.** $u(x,t)=u_0+(u_1-u_0)\left[\displaystyle\sum_{n=0}^{\infty}(-1)^n\,erfc\left\{\frac{(zn+1)\,l-x}{2\sqrt{kt}}\right\}+erfc\left\{\frac{(2n+1)\,l+x}{2\sqrt{kt}}\right\}\right]$]

9. Solve the one-dimensional diffusion equation in a finite medium

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2},\ 0<x<a,\ t>0$$

under the conditions

$$u(x,0)=0 \text{ for } 0<x<a;\ u(a,t)=u_0 \text{ for } t>0 \text{ and } \frac{\partial u}{\partial x}=0 \text{ for } x=0,\ t>0.$$

$$\left[\ \textbf{Ans.}\ \ u(x,t)=u_0\left[\sum_{n=0}^{\infty}(-1)^n\left\{erfc\left(\frac{(2n+1)\ l-x}{2\sqrt{kt}}\right)+erfc\left(\frac{(2n+1)\ l+x}{2\sqrt{kt}}\right)\right\}\right]\ \right]$$

10. The temperature $u(x,t)$ at any point x at any time t, of the semi-infinite rod $x>0$ is given by the differential equation $\dfrac{\partial u}{\partial t}=k\dfrac{\partial^2 u}{\partial x^2}$, subject to the conditions,

(i) $u=0$ when $t=0$ (ii) $\dfrac{\partial u}{\partial x}=-A$ when $x=0$, $t>0$

and (iii) u is finite when $x\to\infty$

Using Laplace transform show that the temperature at the face $x=0$ after a time t is $A\sqrt{(kt/\pi)}$.

11. A beam of length l which has its end $x=0$ fixed, is initially at rest. A constant force F_0 per unit area is applied longitudinally at the free end. Find the longitudinal displacement at any point x of the beam at any time $t>0$.

$$\left[\ \textbf{Ans.}\ \ u(x,t)=\frac{F_0}{E}\left[x+\frac{8l}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\frac{(2n-1)\ \pi x}{2l}\cos\frac{(2n-1)\pi at}{2l}\right]\ \right]$$

Unit - 4
Fourier Transform

Structure of the Unit

4.0 Objective

The object of this unit is to define complex Fourier, Fourier sine and cosine transforms and establish inversion theorems, convolution theorem and derivative formulas for these transforms.

4.1 Introduction

Many linear boundary value and initial value problems in applied mathematics, mathematical physics and engineering science can be effectively solved by the use of the Fourier transform, the Fourier cosine and sine transforms.

We begin with the definition of complex Fourier and (infinite) Fourier sine and cosine transforms. This is followed by inversion theorems and elementary properties for these transforms. Several examples are included to illustrate different methods for finding out the images of functions under these transforms. Next convolution theorem and Parseval's identity are proved for Fourier transforms. At the end of the unit derivative formulas are given for Fourier transforms.

4.2 Definitions

(a) Fourier Transform or Complex Fourier Transform or Exponential Fourier Transform :

Let $f(t)$ be a function of t defined on $(-\infty,\infty)$ and be piecewise continuously differentiable and absolutely integrable in $(-\infty,\infty)$, then the Fourier transform of $f(t)$, denoted by $\mathsf{F}\ \{f(t);p\}$ or $F(p)(p\in R)$ is defined by the integral

$$\mathsf{F}\ \{f(t);p\}=F(p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipt}f(t)\,dt$$

where $\frac{1}{\sqrt{2\pi}}e^{ipt}$ is known as the kernel of the Fourier transformation and F stands for Fourier transformation operator. This is oftenly called the complex Fourier transform. A sufficient condition for $f(t)$ to have a Fourier Transform is that $f(t)$ is absolutely integrable or $(-\infty,\infty)$. The convergence of the integral follows at once from the fact that $f(t)$ is absolutely integrable. Infact, the integral converges uniformly with respect to p.

(b) Fourier Sine Transform :

If $f(t)$ be a function defined for $t>0$ and be piecewise continuously differentiable and is absolutely integrable in $(0,\infty)$ then the (infinite) Fourier sine transform of $f(t)$, denoted by $\mathsf{F}_s\{f(t);p\}$ or $F_s(p)$ is defined as

$$f_s\{f(t);p\}=F_s(p)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(t)\sin(pt)\,dt\ \ ;\ (p>0)$$

Here $\sqrt{\left(\frac{2}{\pi}\right)}\sin(pt)$ is knwon as the kernel of the Fourier sine transform and F_s stands up Fourier sine transformation operator.

(c) Fourier Cosine Transform :

If $f(t)$ be a function defined for $t>0$ and be piecewise continuously differentiable and absolutely integrable in $(0,\infty)$, then the (infinite) Fourier cosine transform of $f(t)$, denoted by $\mathsf{F}_c\{f(t);p\}$ or $F_c(p)$ is defined as

$$\mathsf{F}_c\{f(t);p\}=F_c(p)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(t)\cos(pt)\,dt\ ,\ (p>0)$$

Here $\sqrt{\left(\frac{2}{\pi}\right)}\cos(pt)$ is known as the kernel of the Fourier cosine transform and F_c stands for Fourier cosine transformation operator.

Remarks : The literature on complex Fourier transform contains minor variations in the choice of kernel and in notation. The kernels e^{-ipt} and e^{ipt} are sometimes used. Some authors use the kernels $\sin pt$ and $\cos pt$ respectively for the Fourier sine transform and Fourier cosine transforms.

4.3 Inversion Theorems

4.3.1 Complex Fourier Transform :

Theorem 1 : Let $F(p)$ be the Fourier transform of $f(t)$, that is, if

$$F(p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipt} f(t)\, dt \qquad \text{...(1)}$$

and if $f(t)$ is piecewise continuously differentiable and absolutely integrable in $(-\infty,\infty)$, then at each point of continuity 't' of $f(t)$,

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipt} F(p)\, dp \qquad \text{...(2)}$$

The function $f(t)$ is called the inverse Fourier transform of $F(p)$.

Proof : From Fourier integral theorem (in unit - 2), we have

$$f(t)=\frac{1}{2\pi}\int_{v=-\infty}^{\infty} e^{ivt}\, dv \int_{u=-\infty}^{\infty} f(u)\, e^{-ivu}\, du$$

$$=\frac{1}{\sqrt{2\pi}}\int_{p=-\infty}^{\infty} e^{-ipt}\, dp . \frac{1}{\sqrt{2\pi}}\int_{u=-\infty}^{\infty} f(u)\, e^{+ipu}\, du \quad \text{(putting } v=-p \text{ so that } dv=-dp\text{)}$$

or $$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipt}\, F(p)\, dp \qquad \text{(from (1))}$$

Remarks : If the kernel of the Fourier transform be taken as e^{ipt}, then the equation (1) and (2) become

$$F(p)=\int_{-\infty}^{\infty} e^{-ipt} f(t)\, dt \quad \text{and} \quad f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipt}\, F(p)\, dp$$

4.3.2 Fourier Sine Transform :

Theorem 2 : Let $F_5(p)$ be the Fourier sine transform of $f(t)$, that is,

$$F_s(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin(pt)\, dt \qquad ...(3)$$

and if $f(t)$ is piecewise continuously differentiable and absolutely by integrable in $(0, \infty)$ then at each point of continuity t of $f(t)$

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(p) \sin(pt)\, dp \qquad ...(4)$$

The function $f(t)$ is called the inverse Fourier sine transform of $F_5(p)$.

Proof : From Fourier integral theorem (in unit - 2), we have

$$f(t) = \frac{1}{\pi} \int_{p=0}^\infty dp \int_{u=-\infty}^\infty f(u) \cos\{p(t-u)\}\, du$$

$$= \frac{1}{\pi} \int_{p=0}^\infty dp \int_{u=-\infty}^\infty f(u) \{\cos(pt)\cos pu + \sin(pt)\sin(pu)\}\, du$$

$$= \frac{1}{\pi} \int_{p=0}^\infty \cos(pt)\, dp \int_{u=-\infty}^\infty f(u)\cos(pu)\, du + \frac{1}{\pi} \int_{p=0}^\infty \sin(pt)\, dp \int_{u=-\infty}^\infty f(u)\sin(pu)\, du$$

or $$f(t) = \frac{1}{\pi} \int_{p=0}^\infty \cos(pt)\, dp \int_{t=-\infty}^\infty f(t)\cos(pt)\, dt + \frac{1}{\pi} \int_{p=0}^\infty \sin(pt)\, dp \int_{t=-\infty}^\infty f(t)\sin(pt)\, dt \qquad ...(5)$$

Now, define $f(t)$ in $(-\infty, 0)$ such that $f(-t) = -f(t)$, then $f(t)$ will be an odd function in $(-\infty, \infty)$. Thus $f(t)\cos(pt)$ is an odd function while $f(t)\sin(pt)$ is an even function. Therefore (5) reduces to

$$f(t) = \frac{2}{\pi} \int_{p=0}^\infty \sin(pt)\, dp \int_{t=0}^\infty f(t)\sin(pt)\, dt$$ (because the first integral vanishes)

or $$f(t) = \sqrt{\frac{2}{\pi}} \int_{p=0}^\infty \sin(pt)\, dp \sqrt{\frac{2}{\pi}} \int_{t=0}^\infty f(t)\sin(pt)\, dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(p)\sin(pt)\, dp$$ (From 3)

Remark : If the kernel of the Fourier sine transform be taken a $\sin pt$ then (3) and (4) become

$$F_s(p)=\int_0^\infty f(t)\sin(pt)\,dt \;\; ; \;\; f(t)=\frac{2}{\pi}\int_0^\infty F_s(p)\sin pt\,dp$$

4.3.3 Fourier Cosine Transform :

Theorem 3 : Let $F_c(p)$ be the Fourier cosine transform of $f(t)$, that is, if

$$F_c(p)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos(pt)\,dt \qquad \text{...(6)}$$

and if $f(t)$ is piecewise continuously differentiable and absolutely integrable in $(0,\infty)$ then at each point of continuity t of $f(t)$

$$f(t)=\sqrt{\frac{2}{\pi}}\int_0^\infty F_c(p)\cos(pt)\,dp \qquad \text{...(7)}$$

The function $f(t)$ is called the inverse Fourier cosine transform of $F_c(p)$.

Proof : Proceeding as in Theroem 2, we have

$$f(t)=\frac{1}{\pi}\int_{p=0}^\infty \cos(pt)\,dp\int_{t=-\infty}^\infty f(t)\cos(pt)\,dt$$

$$+\frac{1}{\pi}\int_{p=0}^\infty \sin(pt)\,dp\int_{t=-\infty}^\infty f(t)\sin(pt)\,dt \qquad \text{...(8)}$$

Now define $f(t)$ is $(-\infty,0)$ such that $f(t)=f(-t)$, then $f(t)$ is an even function of t in $(-\infty,\infty)$ and we have

$$\int_{-\infty}^\infty f(t)\,\sin(pt)\,dt=0 \text{ and } \int_{-\infty}^\infty f(t)\,\cos(pt)\,dt=2\int_0^\infty f(t)\,\cos(pt)\,dt$$

Therefore from (8), we have

$$f(t)=\frac{2}{\pi}\int_{p=0}^\infty \cos(pt)\,dp\int_{t=0}^\infty f(t)\,\cos(pt)\,dt$$

or $$f(t)=\frac{2}{\pi}\int_0^\infty F_c(p)\,\cos(pt)\,dp$$

Remarks : (i) If the kernel of the Fourier cosine transform be taken as $\cos(pt)$, then (6) and (7) become

$$F_c(p)=\int_0^\infty f(t)\cos(pt)\,dt \text{ and } f(t)=\frac{2}{\pi}\int_0^\infty F_c(p)\cos(pt)\,dp$$

(ii) Note that Fourier cosine and sine transforms are special cases of complex Fourier transform.

4.4 Relationship between Fourier Transform and Laplace Transform

Let us define a function $f(t)$ as under:

$$f(t)=\begin{cases} e^{-\lambda t}\,\phi(t), & t>0 \\ 0, & t<0 \end{cases}$$

Then $\mathsf{F}\,[f(t);p]=\int_{-\infty}^{\infty} e^{ipt} f(t)\,dt$ (taking non-symmetrical form)

$$=\int_{-\infty}^{0} e^{ipt} f(t)\,dt+\int_{0}^{\infty} e^{ipt} f(t)\,dt$$

$$=\int_{-\infty}^{0} e^{ipt}.0.dt+\int_{0}^{\infty} e^{ipt}\; e^{-\lambda t}\,\phi(t)\,dt$$

$$=\int_{0}^{\infty} e^{-(\lambda-ip)t}\,\phi(t)\,dt$$

$$=\int_{0}^{\infty} e^{-st}\,\phi(t)\,dt\text{, where } \lambda-ip=s$$

$\therefore$ $\mathsf{F}\,[f(t);p]=L[\phi(t);s]$

which is the required relationship between Fourier transform and Laplace transform.

4.5 Some Useful Results for Direct Applications

(i) $\int e^{at}\sin bt\,dt=\dfrac{e^{at}}{a^2+b^2}(a\sin bt-b\cos bt)$

$\int_0^{\infty} e^{-at}\sin bt\,dt=\dfrac{b}{a^2+b^2}$

(ii) $\int e^{at}\cos bt\,dt=\dfrac{e^{at}}{a^2+b^2}(a\cos bt+b\sin bt)$

$\int_0^{\infty} e^{-at}\cos bt\,dt=\dfrac{a}{a^2+b^2}$

(iii) $\int_0^{\infty}\dfrac{\sin pt}{t}\,dt=\begin{cases} \pi/2, & if\;\; p>0 \\ -\pi/2, & if\;\; p<0 \\ \pi/2, & if\;\; p=1 \end{cases}$

(iv) $\int_{-\infty}^{\infty} e^{-t^2}dt=\sqrt{\pi}$, $\int_0^{\infty} e^{-t^2}dt=\dfrac{\sqrt{\pi}}{2}$

(v) $\displaystyle\int_0^\infty \frac{e^{at}+e^{-at}}{e^{\pi t}-e^{-\pi t}}\,dt = \frac{1}{2}\sec\left(\frac{a}{2}\right)$

(vi) $\displaystyle\int_0^\infty \frac{e^{at}-e^{-at}}{e^{\pi t}-e^{-\pi t}}\,dt = \frac{1}{2}\tan\left(\frac{a}{2}\right)$

4.6 Elementary Properties for Fourier Transforms

4.6.1 Linearity Property

Theorem 4 : Let for all $i = 1,2,\ldots,n,$ $F_i(p)$ be the Fourier transforms of $f_i(t)$ and c_i are any constants, then

$$\mathcal{F}\,\{c_1 f_1(t) \pm c_2 f_2(t) \pm \ldots\ldots \pm c_n f_n(t); p\}$$

$$= c_1\,\mathcal{F}\,\{f_1(t); p\} + c_2\,\mathcal{F}\,\{f_2(t); p\} \pm \ldots\ldots \pm c_n\,\mathcal{F}\,\{f_n(t); p\}$$

$$= c_1 F_1(p) + c_2 F_2(p) \pm \ldots\ldots \pm c_n F_n(p)$$

Proof : We have

$$\mathcal{F}\,\{c_1 f_1(t) \pm c_2 f_2(t) \pm \ldots\ldots \pm c_n f_n(t); p\}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipt}\{c_1 f_1(t) \pm c_2 f_2(t) \pm \ldots\ldots \pm c_n f_n(t); p\}\,dt$$

$$= \frac{c_1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipt} f_1(t)\,dt \pm \frac{c_2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipt} f_2(t)\,dt \pm \ldots\ldots \pm \frac{c_n}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipt} f_n(t)\,dt$$

$$= c_1 F_1(p) \pm c_2 F_2(p) \pm \ldots\ldots \pm c_n F_n(p)$$

Remark : The above property also holds good for sine and cosine transforms

4.6.2 Change of Scale Property

Theorem 5 : If $F(p)$ is the Fourier transform of $f(t)$, then $\dfrac{1}{|a|}F\left(\dfrac{p}{a}\right)$ is the Fourier transform of $f(at)$.

Proof : We have $\mathcal{F}\,\{f(t); p\} = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} e^{ipt} f(t)\,dt = F(p)$

(i) If $a > 0$, then

$$\mathcal{F}\,\{f(at); p\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipt} f(at)\,dt = \frac{1}{a}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{(ip/a)u} f(u)\,du$$

$$= \frac{1}{a} F\left(\frac{p}{a}\right) \qquad \text{(putting } at = u\text{)}$$

(ii) If $a < 0$, let $a = -b \ (b > 0)$.

Now $\mathcal{F}\left[f(at); p\right] = \mathcal{F}\left[f(-bt); p\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipt} f(-bt)\, dt$

$$= \frac{1}{b\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(-p/b)u} f(u)\, du \qquad \text{(putting } -bt = u\text{)}$$

$$= \frac{1}{b} \mathcal{F}\left[f(t); -\frac{p}{b}\right] = -\frac{1}{a} \mathcal{F}\left[f(t); \frac{p}{a}\right]$$

$$= \frac{1}{|a|} \mathcal{F}\left[f(t); \frac{p}{a}\right]$$

Hence in genral

$$\mathcal{F}\left[f(at); p\right] = \frac{1}{|a|} F\left(\frac{p}{a}\right)$$

Theorem 6 : If $F_s(p)$ is the Fourier sine transform of $f(t)$ then Fourier sine transform of $f(at)$ is $\frac{1}{a} - F_s\left(\frac{p}{a}\right), (a > 0)$.

Proof : We have $F_s(p) = \mathcal{F}_s\left[f(t); p\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin pt \ dt$

$$\therefore \qquad \mathcal{F}_s\left[f(at); p\right] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(at) \sin pt \ dt$$

$$= \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \sin\left\{\left(\frac{p}{a}\right)u\right\} du$$

$$\therefore \qquad \mathcal{F}_s\left[f(at); p\right] = \frac{1}{a} F_s\left(\frac{p}{a}\right)$$

Similarly, we can also prove that

Theorem : If $F_c(p)$ is the Fourier cosine transform of $f(t)$, then Fourier cosine transform of $f(at)$ is $\frac{1}{a} F_c\left(\frac{p}{a}\right), (a > 0)$

$$\therefore \qquad F_c\left[f(at); p\right] = \frac{1}{a} F_c\left(\frac{p}{a}\right)$$

4.6.3 Shifting Property

Theorem 8 : If $F(p)$ is the complex Fourier transform of $f(t)$, then complex Fourier transform of $f(t-a)$ is $e^{ipa}F(p)$ i.e. if $\mathcal{F}\left[f(t);p\right]=F(p)$, then $\mathcal{F}\left[f(t-a);p\right]=e^{ipa}F(p)$

Proof : We have

$$F(p)=\mathcal{F}\left\{f(t);p\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipt}f(t)\,dt$$

Now $$\mathcal{F}\left\{f(t-a);p\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipt}f(t-a)\,dt=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ip(a+u)}f(u)\,du$$

$$=e^{ipa}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipu}f(u)\,du=e^{ipa}F(p)$$

4.6.4 Modulation Theorem

Theorem 9 : If $f(t)$ has the Fourier transform $F(p)$, then $f(t)\cos at$ has the Fourier transform $\frac{1}{2}\left[F(p-a)+F(p+a)\right]$.

Proof : We have

$$F(p)=\mathcal{F}\left\{f(t);p\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipt}f(t)\,dt$$

Now $$\mathcal{F}\left\{f(t)\cos at;p\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipt}f(t)\cos at\,dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipt}f(t)\frac{e^{iat}+e^{-iat}}{2}\,dt$$

$$=\frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i(p+a)t}f(t)\,dt+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i(p-a)t}f(t)\,dt\right]$$

$$=\frac{1}{2}\left[F(p+a)+F(p-a)\right]$$

Theorem 10 : If $F_s(p)$ and $F_c(p)$ are Fourier sine and cosine transforms of $f(t)$ respectively, then

(i) $\mathcal{F}_s\left\{f(t)\cos at;p\right\}=\frac{1}{2}\left[F_s(p+a)+F_s(p-a)\right]$

(ii) $\mathcal{F}_c\left\{f(t)\sin at;p\right\}=\frac{1}{2}\left[F_s(p+a)-F_s(p-a)\right]$

(iii) $\mathcal{F}_s\left\{f(t)\sin at;p\right\}=\frac{1}{2}\left[F_c(p-a)-F_c(p+a)\right]$

Proof : (i) We have

$$F_s\left\{f(t)\cos at; p\right\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\cos at \sin pt\, dt$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\left[\sin(p+a)t + \sin(p-a)t\right] dt$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\sin(p+a)t\, dt + \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\sin(p-a)t\, dt\right]$$

$$= \frac{1}{2}\left[F_s(p+a) + F_s(p-a)\right]$$

On similar lines (ii) and (iii) can be easily proved.

Theorem 11 : If $\phi(p)$ is the Fourier transform of $f(t)$ for $p>0$, then for $p<0$, $\mathrm{F}_s\left\{f(t); p\right\} = -\phi(-p)$.

Proof : By definition,

$$\mathrm{F}_s\left\{f(t); p\right\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\sin pt\, dt = \phi(p) \qquad ...(9)$$

for $p<0$, let $p=-u$ with $u>0$, then the right hand integral (9) becomes

$$\mathrm{F}_s\left\{f(t); p\right\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\sin(-ut)\, dt$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\sin ut\, dt = -\phi(u) = -\phi(-p),\ p<0$$

therefore $\mathrm{F}_s\left\{f(t); p\right\} = -\phi(-p),\ p>0$

Hence in genral, we have

$$\mathrm{F}_s\left\{f(t); p\right\} = \begin{cases} \phi(p), & p>0 \\ -\phi(-p), & p<0 \end{cases}$$

Example 1 : Find the Fourier transform of $f(t) = \begin{cases} 1, & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$

Solution : We know that $\mathrm{F}\left[f(t); p\right] = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^\infty e^{ipt} f(t)\, dt$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-1} e^{ipt} f(t)\, dt + \int_{-1}^{1} e^{ipt} f(t)\, dt + \int_1^\infty e^{ipt} f(t)\, dt\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{ipt}\ .\ 1\ .\ dt = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{ipt}}{ip}\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{ip} - e^{-ip}}{ip} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{p} \sin p = \sqrt{\frac{2}{\pi}} \frac{\sin p}{p} \quad (if \;\; p \neq 0)$$

$$\therefore \qquad \mathsf{F} \left[f(t); p \right] = \sqrt{\frac{2}{\pi}} \frac{\sin p}{p}$$

Example 2 : Find the Fourier sine and cosine transform of $f(t)$, if

$$f(t) = \begin{cases} t \;\;, & 0 < t < 1 \\ 2 - t, & 1 < t < 2 \\ 0 \;\;, & t > 2 \end{cases}$$

Solution : By definition, we have

$$\mathsf{F}_s \{ f(t); p \} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin pt \, dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 f(t) \sin pt \, dt + \int_1^2 f(t) \sin pt \, dt + \int_2^\infty f(t) \sin pt \, dt \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 t \sin pt \, dt + \int_1^2 (2-t) \sin pt \, dt + \int_2^\infty 0 \, \sin pt \, dt \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left\{ \frac{-t \cos pt}{p} + \frac{\sin pt}{p^2} \right\}_{t=0}^{1} + \left\{ \frac{-2 \cos pt}{p} \right\}_{t=1}^{2} \left\{ \frac{-t \cos pt}{p} + \frac{\sin pt}{p^2} \right\}_{t=1}^{2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(\frac{-\cos p}{p} + \frac{\sin p}{p^2} \right) - \frac{2}{p} \cos 2p + \frac{2}{p} \cos p + \frac{2 \cos 2p - \cos p}{p} - \left(\frac{\sin 2p - \sin p}{p^2} \right) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2 \sin p - 2 \sin p \cos p}{p^2} \right] = \sqrt{\frac{2}{\pi}} \left[\frac{2 \sin p (1 - \cos p)}{p^2} \right]$$

$$\mathsf{F}_c \{ f(t); p \} = \sqrt{\frac{2}{\pi}} \left[\int_0^1 t \cos pt \, dt + \int_1^2 (2-t) \cos pt \, dt \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(\frac{t \sin pt}{p} + \frac{\cos pt}{p^2} \right)_0^1 + \left(\frac{(2-t) \sin pt}{p} - \frac{\cos pt}{p^2} \right)_1^2 \right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{2\cos p-\cos 2p-1}{p^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{2\cos p-2\cos^2 p}{p^2}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{2\cos p\left(1-\cos p\right)}{p^2}\right]$$

Example 3 : Find the complex Fourier Transform of $f(t)=e^{-a|t|}$, where $a>0$ and t belongs to $(-\infty,\infty)$.

Solution : We know that $\mathsf{F}\left\{f(t);p\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipt}f(t)\,dt=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipt}e^{-a|t|}\,dt$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0}e^{ipt}\,e^{at}\,dt+\int_{0}^{\infty}e^{-ipt}\,e^{-at}\,dt\right]\qquad\left\{\begin{array}{l}\because |t|=\ \ t\ if\ t>0\\ \qquad\ =-t\ if\ t<0\end{array}\right\}$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0}e^{(a+ip)t}\,dt+\int_{0}^{\infty}e^{-(a-ip)t}\,dt\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\left\{\frac{e^{(a+ip)t}}{a+ip}\right\}_{-\infty}^{0}+\left\{\frac{e^{-a(a-ip)t}}{-(a-ip)}\right\}_{0}^{\infty}\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+ip}+\frac{1}{a-ip}\right]=\frac{1}{\sqrt{2\pi}}\,\frac{2a}{\left(a^2+p^2\right)}=\sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+p^2}$$

Example 4 : Find the Fourier cosine transform of e^{-t^2}.

Solution : We know that

$$\mathsf{F}_c\left\{e^{-t^2};p\right\}=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}e^{-t^2}\cos pt\,dt=I\qquad\text{(say)}\qquad ...(10)$$

Differentiating (10) with respect to p, we have

$$\frac{dI}{dp}=-\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}t\,e^{-t^2}\sin pt\,dt$$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\left(-2t\,e^{-t^2}\right)\sin pt\,dt$$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\left(e^{-t^2}\sin pt\right)_{0}^{\infty}-\int_{0}^{\infty}p\,e^{-t^2}\cos pt\,dt\right]$$

$$= 0 - \frac{p}{2} I \ \text{ or } \ \frac{dI}{I} = -\frac{p}{2} dp$$

Integrating, $\log I = -\frac{p^2}{4} + \log A$ where A is the constant of integration

or $\qquad I = A e^{-p^2/4}$ ...(11)

When $p = 0$, then we have

$$I = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t^2} (\cos 0) \, dt = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-t^2} \, dt = \sqrt{\frac{2}{\pi}} \frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}} \qquad ...(12)$$

Also for $p = 0$, equation (11) gives $I = A$

$$\therefore \qquad \frac{1}{\sqrt{2}} = I = A \Rightarrow A = \frac{1}{\sqrt{2}}$$

Hence $I = \mathrm{F}_c \left\{ e^{-t^2}; p \right\} = \frac{1}{\sqrt{2}} e^{-p^2/4}$

Example 5 : Find the Fourier transform of $f(t)$ defined by $f(t) = \begin{cases} 1, & |t| < a \\ 0, & |t| > a \end{cases}$

and hence evaluate

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin pa \cos pt}{p} \, dp$ $\qquad$ (ii) $\displaystyle\int_0^\infty \frac{\sin p}{p} \, dp$

Solution : Here $f(t) = 1$ when $-a < t < a$ and 0 otherwise (given)

Therefore $\mathrm{F} \left\{ f(t); p \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipt} f(t) \, dt = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{ipt} .1. \, dt$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ipt}}{ip} \right]_{-a}^{a} = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ipa} - e^{-ipa}}{ip} \right] = \sqrt{\frac{2}{\pi}} \frac{\sin ap}{p}$$

Thus $\qquad \mathrm{F} \left\{ f(t); p \right\} = F(p) = \sqrt{\frac{2}{\pi}} \frac{\sin ap}{p}$ ...(13)

But from the Fourier inversion theorem, we have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipt} F(p) \, dp = f(t)$$

or $$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipt}\sqrt{\frac{2}{\pi}}\,\frac{\sin pa}{p}\,dp = f(t) \qquad \text{(using 13)}$$

or $$\frac{1}{\pi}\int_{-\infty}^{\infty} e^{-ipt}\,\frac{\sin pa}{p}\,dp = f(t) = \begin{cases}1, & |t| < a\\ 0, & |t| > a\end{cases}$$

or $$\int_{-\infty}^{\infty}\frac{\cos pt\sin pa - i\sin pt\sin pa}{p}\,dp = \begin{cases}\pi, & |t| < a\\ 0, & |t| > a\end{cases}$$

Equating real parts on both sides, we get

$$\int_{-\infty}^{\infty}\frac{\cos pt\sin pa}{p}\,dp = \begin{cases}\pi, & |t| < a\\ 0, & |t| > a\end{cases} \qquad ...(14)$$

Putting $t = 0$ and $a = 1$ in (14), we obtain

$$\int_{-\infty}^{\infty}\frac{\sin p}{p}\,dp = \pi \text{ or } 2\int_{0}^{\infty}\frac{\sin p}{p}\,dp = \pi$$

or $$\int_{0}^{\infty}\frac{\sin p}{p}\,dp = \frac{\pi}{2}$$

Example 6 : Find the Fourier transform of $f(t)$, where $f(t) = \begin{cases}1-t^2, & |t| < 1\\ 0, & |t| > 1\end{cases}$

and hence evaluate $\int_{0}^{\infty}\left(\frac{t\cos t - \sin t}{t^3}\right)\cos\frac{t}{2}\,dt$

Solution : Given that $f(t) = 1 - t^2$ for $-1 < t < 1$ and 0 otherwise

Therefore $$\mathcal{F}\{f(t);p\} = F(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipt} f(t)\,dt = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-t^2)e^{ipt}\,dt$$

$$= \frac{1}{\sqrt{2\pi}}\left[(1-t^2)\,\frac{e^{ipt}}{ip}\right]_{-1}^{1} + \frac{2}{\sqrt{2\pi}}\int_{-1}^{1} t\,\frac{e^{ipt}}{ip}\,dt$$

$$= \frac{1}{ip}\sqrt{\frac{2}{\pi}}\left[\left(\frac{t\,e^{ipt}}{ip}\right)_{-1}^{1} - \int_{-1}^{1}\frac{e^{ipt}}{ip}\,dt\right] = \frac{1}{ip}\sqrt{\frac{2}{\pi}}\left[\frac{e^{ip}+e^{-ip}}{ip} - \frac{\left(e^{ipt}\right)_{-1}^{1}}{(ip)^2}\right]$$

$$= \frac{1}{ip}\sqrt{\frac{2}{\pi}}\left[\frac{2\cos p}{ip} + \frac{e^{ip}-e^{ip}}{p^2}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{-2\cos p}{p^2} + \frac{2\sin p}{p^3}\right]$$

$$= -2\sqrt{\frac{2}{\pi}}\left[\frac{p\cos p - \sin p}{p^3}\right]$$

$\therefore$ By using inversion formula, we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipt} F(p)\, dp = f(t)$$

or $$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{-2\sqrt{\frac{2}{\pi}}\left(\frac{p\cos p - \sin p}{p^3}\right)\right\} e^{-ipt}\, dp = f(t)$$

or $$-\int_{-\infty}^{\infty}\left(\frac{p\cos p - \sin p}{p^3}\right)(\cos pt - i\sin pt)\, dp = \frac{\pi}{2} f(t)$$

Equating real parts, we have

$$-\int_{-\infty}^{\infty}\left(\frac{p\cos p - \sin p}{p^3}\right)\cos pt\, dp = \frac{\pi}{2} f(t)$$

or $$-2\int_{0}^{\infty}\left(\frac{p\cos p - \sin p}{p^3}\right)\cos pt\, dp = \begin{cases} \frac{\pi}{2}(1-t^2), & |t| < 1 \\ 0, & |t| > 1 \end{cases}$$ using given values

Taking $t = \frac{1}{2}$, we obtain

$$-2\int_{0}^{\infty}\left(\frac{p\cos p - \sin p}{p^3}\right)\cos\frac{p}{2}\, dp = \frac{\pi}{2}\left(1 - \frac{1}{4}\right) = \frac{3\pi}{8}$$

or $$\int_{0}^{\infty}\left(\frac{t\cos t - \sin t}{t^3}\right)\cos\frac{t}{2}\, dt = \frac{-3\pi}{16}$$

Example 7 : Find $f(t)$, if its Fourier sine transform is $\frac{p}{(1+p^2)}$.

Solution : Let $F_s(p) = \frac{p}{1+p^2}$

By Fourier sine inversion formula,

$$f(t) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_s(p)\sin pt\, dt = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{p}{1+p^2}\sin pt\, dt \quad ...(15)$$

$$= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{p^2 \sin pt}{p(1+p^2)}\, dp = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}\frac{(p^2+1)-1}{p(1+p^2)}\sin pt\, dp$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin pt}{p}\, dp - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin pt}{p\left(1+p^2\right)}\, dp$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin pt}{p\left(1+p^2\right)}\, dp \qquad \left(\because \int_0^\infty \frac{\sin pt}{p}\, dp = \frac{\pi}{2}\right)$$

so that $$f(t) = \sqrt{\frac{\pi}{2}} - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin pt}{p\left(1+p^2\right)}\, dp \qquad ...(16)$$

Differentiating this twice with respect to t, we get

$$\frac{df}{dt} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos pt}{1+p^2}\, dp \qquad ...(17)$$

and $$\frac{d^2 f}{d\,t^2} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{p \sin pt}{1+p^2}\, dp = f(t) \qquad \text{(using (15)}$$

or $$\frac{d^2 f}{d\,t^2} - f = 0$$

Therefore the solution of above linear differential equation is

$$f(t) = c_1 e^t + c_2 e^{-t} \qquad ...(18)$$

This gives $$\frac{df}{dt} = c_1 e^t - c_2 e^{-t} \qquad ...(19)$$

Putting $t = 0$ in (16) and (18), we get

$$f(0) = \sqrt{\frac{\pi}{2}} \text{ and } f(0) = c_1 + c_2$$

Therefore $$c_1 + c_2 = \sqrt{\frac{\pi}{2}} \qquad ...(20)$$

Putting $t = 0$ in (17) and (19)

$$\left(\frac{df}{dt}\right)_{t=0} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{dp}{1+p^2} = -\sqrt{\frac{2}{\pi}} \left(\tan^{-1} p\right)_0^\infty = -\sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} = -\sqrt{\frac{\pi}{2}}$$

and $$\left(\frac{df}{dt}\right)_{t=0} = c_1 - c_2$$

$$\therefore \quad c_1 - c_2 = -\sqrt{\frac{\pi}{2}} \qquad ...(21)$$

Solving (20) and (21), we get $c_1 = 0$ and $c_2 = \sqrt{\frac{\pi}{2}}$

Hence $f(t) = \sqrt{\frac{\pi}{2}}\, e^{-t}$

Example 8 : Find $f(t)$ if its given transform is $\frac{e^{-ap}}{p}$. Hence deduce $F_s^{-1}\left(\frac{1}{p}\right)$.

Solution : By inversion theorem for Fourier sine transform, we have

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(p) \sin pt\, dp$$

or $$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ap}}{p} \sin pt\, dp \qquad ...(22)$$

Differentiating this with respect to t by Leibnitz's rule,

$$\frac{df}{dt} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ap} \cos pt\, dp = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2 + t^2},\ a > 0$$

On integrating it, we have $f(t) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{t}{a}\right) + A$ ...(23)

where A is constant of integration.

Now, when $t = 0$, $f(0) = 0$ by (22) and $A = f(0)$ by (23),

Hence $A = 0$

Thus $f(t) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{t}{a}\right)$

Deduction : Putting $a = 0$ in the above result, we get

$$F_s^{-1}\left\{\frac{1}{p}\right\} = \sqrt{\frac{2}{\pi}} \tan^{-1} \infty = \sqrt{\frac{2}{\pi}} \times \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$$

Example 9 : Find Fourier sine transform of

$$f(x) = t\left(a^2 - t^2\right)^{v - 3/2} U(a - t), \qquad v > 1/2,\ a > 0$$

where $U(a - t)$ is the Heaviside unit step function

Solution : We have

$$\mathrm{F}_s\left\{t\left(a^2-t^2\right)^{v-3/2}U(a-t);p\right\}=\sqrt{\frac{2}{\pi}}\int_0^\infty t\left(a^2-t^2\right)^{v-3/2}U(a-t)\sin pt\,dt$$

$$=\sqrt{\frac{2}{\pi}}\int_0^a t\left(a^2-t^2\right)^{v-3/2}\sin pt\,dt$$

$$=\sqrt{2}\sum_{r=0}^{\infty}\frac{(-1)^r\left(p/2\right)^{2r+1}}{r!\,\Gamma\left\{r+\left(3/2\right)\right\}}\int_0^a t^{2r+2}\left(a^2-t^2\right)^{v-3/2}dt$$

(Expanding $\sin pt$ and changing order of integration and summation)

$$=\sqrt{2}\sum_{r=0}^{\infty}\frac{(-1)^r\left(p/2\right)^{2r+1}}{r!\,\Gamma\left\{r+\left(3/2\right)\right\}}a^{2v+2r}\int_0^{\pi/2}\sin^{2r+2}\theta\cos^{2v-2}\theta\,d\theta$$

(putting $x=a\sin\theta$)

$$=2^{v-3/2}\,\Gamma\left(v-\frac{1}{2}\right)p^{1-v}a^{v}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(v+r+1)}\left(pa/2\right)^{v+2r}$$

$$=2^{v-3/2}\,\Gamma\left(v-\frac{1}{2}\right)p^{1-v}a^{v}\,J_v(pa)$$

Example 10 : Obtain Fourier transform of $e^{-a^2t^2}$. Hence obtain Fourier cosine transform of $\cos\left(\frac{t^2}{2}\right)$ and $\sin\left(\frac{t^2}{2}\right)$.

Solution : Using $t^2-ipt=\left(t-\frac{1}{2}ip\right)^2+\frac{p^2}{4}$, we get

$$\mathrm{F}\left\{e^{-t^2};p\right\}=\frac{e^{-p^2/4}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\left(t-\frac{ip}{2}\right)^2}dt$$

$$=\frac{e^{-p^2/4}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-y^2}dy \qquad \left(\text{taking } t-\frac{ip}{2}=y\right)$$

$$=\frac{1}{\sqrt{2}}e^{-p^2/4} \qquad \left(\because \int_{-\infty}^{\infty}e^{-n^2}dn=\sqrt{\pi}\right)$$

Using change of scale property for Fourier transform, we find that

$$\text{F}\left\{e^{-a^2t^2};p\right\}=\frac{1}{\left(\sqrt{2}\right)a}e^{-p^2/4a^2}\left(a>0\right) \qquad ...(24)$$

Putting $a=\frac{1}{2}(1-i)$ in (24) and using the result

$\frac{1}{\left(\sqrt{2}\right)a}=\frac{1+i}{\sqrt{2}}$ and $a^2=-\frac{i}{2}$, we find that

$$\text{F}\left\{e^{-it^2/p};p\right\}=\frac{1}{\sqrt{2}}(1+i)e^{-ip^2/2}$$

$$\Rightarrow \quad \text{F}_c\left\{e^{-it^2/2};p\right\}=\frac{1}{\sqrt{2}}(1+i)e^{-ip^2/2}$$

$$\Rightarrow \quad \text{F}_c\left\{\cos\frac{t^2}{2}+i\sin\frac{t^2}{2};p\right\}=\frac{1}{\sqrt{2}}(1+i)\left(\cos\frac{p^2}{2}-i\sin\frac{p^2}{2}\right)$$

$$=\frac{1}{\sqrt{2}}\left[\left(\cos\frac{p^2}{2}+\sin\frac{p^2}{2}\right)+i\left(\cos\frac{p^2}{2}-\sin\frac{p^2}{2}\right)\right]$$

Comparing real and imaginary parts, we get

$$\text{F}_c\left\{\cos\frac{t^2}{2}\right\} \text{ and } =\frac{1}{\sqrt{2}}\left\{\cos\left(\frac{p^2}{2}\right)+\sin\left(\frac{p^2}{2}\right)\right\}$$

and $$\text{F}_c\left(\sin\frac{t^2}{2}\right)=\frac{1}{\sqrt{2}}\left\{\cos\left(\frac{p^2}{2}\right)-\sin\left(\frac{p^2}{2}\right)\right\}$$

Self-Learning Exercise - I

1. Write the inversion formula for Fourier sine transform.
2. State the shifting property for Fourier transform.

Fill in the blanks in the following question :

3. $\text{F}_s\left\{f(x)\cos ax;p\right\}=$..........
4. $\text{F}_c\left\{f(x)\sin ax;p\right\}=$..........
5. If $F(p)=\text{F}\left\{f(x);p\right\}$, then $\text{F}\left\{f(ax);p\right\}=$.....
6. If $\text{F}\left\{e^{-x^2};p\right\}=\frac{1}{\sqrt{2}}e^{-p^2/4}$, then $\text{F}_c\left\{e^{-x^2};p\right\}=$..........

4.7 Exercise 4 (a)

1. Find the Fourier transform of the following functions;

(i) $f(t)=\begin{cases} t, & |t|\le a \\ 0, & |t|>a \end{cases}$ (ii) $f(t)=\begin{cases} e^{i\omega t}, & a<t<b \\ 0, & t>a,\ t>b \end{cases}$

(iii) $f(t)=\begin{cases} t^2, & |t|<a \\ 0, & |t|>a \end{cases}$

[**Ans.** (i) $\frac{i}{p^2}\sqrt{\frac{2}{\pi}}\left(\sin pa - ap\cos pa\right)$ (ii) $\frac{1}{\sqrt{2\pi}}\left[\frac{e^{ib(\omega+p)}-e^{ia(\omega+p)}}{i(\omega+p)}\right]$

(iii) $\sqrt{\frac{2}{\pi}}\cdot\frac{2}{p^3}\left[\left(a^2p^2-2\right)\sin ap+2ap\cos ap\right]$]

2. Find the Fourier sine and cosine transform of the function t^{m-1}.

[**Ans.** (i) $\sqrt{\frac{2}{\pi}}\,\frac{\Gamma(m)}{p^m}\sin\left(\frac{m\pi}{2}\right)$ (ii) $\sqrt{\frac{2}{\pi}}\,\frac{\Gamma(m)}{p^m}\cos\left(\frac{m\pi}{2}\right)$]

3. Find the Fourier sine and cosine transform of the function $f(t)=t^n e^{-at}$, $a>0$, $n>-1$

[**Ans.** (i) $\sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n+1)\sin(n+1)\theta}{\left(a^2+p^2\right)^{(n+1)/2}}$ (ii) $\sqrt{\frac{2}{\pi}}\,\frac{\Gamma(n+1)\cos(n+1)\theta}{\left(a^2+p^2\right)^{(n+1)/2}}$]

where $\theta=\tan^{-1}(x/a)$

4. Find the Fourier cosine transform of $\frac{e^{at}+e^{-at}}{e^{\pi t}-e^{-\pi t}}$.

[**Ans.** $\sqrt{\frac{1}{2\pi}}\,\frac{\left(e^{p/2}+e^{-p/2}\right)\cos\left(a/2\right)}{\cos a+\left\{\left(e^{p}+e^{-p}\right)/2\right\}}$]

5. Find Fourier cosine transform of $\frac{1}{1+t^2}$ and hence find Fourier sine transform of $\left(\frac{t}{1+t^2}\right)$.

[**Ans.** $\sqrt{\frac{\pi}{2}}\,e^{-p}$, $\sqrt{\frac{\pi}{2}}\,e^{-p}$]

6. Find the Fourier cosine transform of

$$f(t)=\left(a^2-t^2\right)^{v-1/2}U(a-t),\ v>1/2,\ a>0$$

Where $U(a-t)$ is a Heaviside's unit step function.

[**Ans.** $2^{\upsilon-1/2}\Gamma\left(\upsilon+\frac{1}{2}\right)p^{1-\upsilon}a^{\upsilon}J_{\upsilon}(pa)$]

7. Find the inverse Fourier transform of $F(p)=e^{-|p|y}$, where $y\in(-\infty,\infty)$,

[**Ans.** $f(t)=\sqrt{\frac{2}{\pi}}\left(\frac{y}{x^2+y^2}\right)$]

8. Find the Fourier sine and cosine transform of e^{-at}, $a>0$. Hence deduce that

(i) $\mathsf{F}_s\left[\frac{e^{-at}}{t};p\right]=\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{p}{a}\right),\ (a>0)$

(ii) $\mathsf{F}_s\left[\frac{1}{t};p\right]=\sqrt{\frac{\pi}{2}}$

(iii) $\mathsf{F}_c\left[t\,e^{-at};p\right]=\sqrt{\frac{2}{\pi}}\frac{(a^2-p^2)}{(a^2+p^2)^2},\ (a>0)$

(iv) $\mathsf{F}_s\left[t\,e^{-at};p\right]=\sqrt{\frac{2}{\pi}}\frac{2\,a\,p}{(a^2+p^2)^2},\ (a>0)$

[**Ans.** $F_c(p)=\sqrt{\frac{2}{\pi}}\frac{a}{a^2+p^2},\ a>0;\ F_s(p)=\sqrt{\frac{2}{\pi}}\cdot\frac{p}{a^2+p^2},\ a>0$]

9. Show that the Fourier transform of $e^{-t^2/2}$ is $e^{-p^2/2}$.

10. Prove that $e^{-t^2/2}$ is a self-reciprocal function under the Fourier cosine transform. Hence obtain the Fourier sine transform $t\,e^{-t^2/2}$.

[**Ans.** $p\,e^{-p^2/2}$]

[Hint : A function is called a self reciprocal function if it coincides with its transform.]

11. Find the Fourier sine transform of $\frac{1}{e^{\pi t}-e^{-\pi t}}$ and deduce that

$$F_s\left[\operatorname{cosec}h\,\pi\,t;p\right]=\frac{1}{\sqrt{2\pi}}\tan h\left(\frac{p}{2}\right).$$

[**Ans.** $F_s(p) = \frac{1}{\left(2\sqrt{2\pi}\right)} \tan h\left(\frac{p}{2}\right)$]

12. Find the Fourier cosine transform of $\frac{1}{a^2+t^2}$ and hence find the Fourier sine transform of

$$\frac{1}{t\left(a^2+t^2\right)} \text{ and } \frac{t}{\left(a^2+t^2\right)}.$$

[**Ans.** $F_c(p) = \frac{1}{a}\sqrt{\frac{\pi}{2}}\, e^{-ap}$, $F_s(p) = \frac{1}{a^2}\sqrt{\frac{\pi}{2}}\left(1-e^{-ap}\right)$ and $\sqrt{\frac{\pi}{2}}\, e^{-ap}$]

13. Show that if $a > 0$, then $\mathsf{F}\left\{\left(a^2-t^2\right)^{-1/2} U\left(a-|t|;p\right)\right\} = \sqrt{\frac{\pi}{2}}\, J_0(ap)$.

14. Find $f(t)$, if its cosine transform is :

$$F_c(p) = \begin{cases} \frac{1}{\sqrt{2\pi}}\left(a - p/2\right), & p < 2a \\ 0, & p \geq 2a \end{cases}$$

[Ans. $\sin^2 at/\pi t^2$]

15. Show that $\mathsf{F}_s^{-1}\left[e^{-\pi p}\right] = \frac{2t}{\pi\left(\pi^2+t^2\right)}$, where

$$F_s(p) = \int_0^\infty \sin pt\, f(t)\, dt.$$

16. Taking $f(t) = \begin{cases} 1, & 0 \leq t < a \\ 0, & t \geq a \end{cases}$ prove that

$$\int_0^\infty \frac{\sin pa \cos pt}{p}\, dp = \begin{cases} \pi/2, & 0 \leq t \leq a \\ 0, & t > a \end{cases}$$

Hence evaluate $\int_0^\infty \frac{\sin p}{p}\, dp$.

4.8 The Convolution or Faltung of Two Functions

(a) Definition : The convolution of two integrable functions $f(t)$ and $g(t)$, where $-\infty < t < \infty$ is denoted and defined as

$$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(u)\, g(t-u)\, du$$

4.7.1 Convolution Theorem for Fourier Transforms

Theorem 12 :

Statement : Let

(i) $f(t)$ **and** $g(t)$ **and their first order derivatives are continuous in** $(-\infty, \infty)$,

(ii) $f(t)$ **and** $g(t)$ **are absolutely integrable in** $(-\infty, \infty)$,

(iii) $F(p)$ **and** $G(p)$ **are Fourier transform of** $f(t)$ **and** $g(t)$ **respectively,**

Then the Fourier transform of the convolution of $f(t)$ **and** $g(t)$ **exists and is the product of the Fourier transform of** $f(t)$ **and** $g(t)$ **i.e.**

$$\mathcal{F}\left[f * g\right] = \mathcal{F}\left[f(t)\right] . \mathcal{F}\left[g(t)\right] = F(p) . G(p) \qquad ...(24)$$

Proof : By definition, we have

$$\mathcal{F}\left[f(t); p\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipt} f(t)\, dt = F(p) \qquad ...(25)$$

and $$\mathcal{F}\left[g(t); p\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipt} g(t)\, dt = G(p) \qquad ...(26)$$

By definition of convolution, we have

$$\mathcal{F}\left[f * g\right] = \mathcal{F}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(t-u)\, du\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipt} \left\{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(t-u)\, du\right\} dt$$

(by definition of Fourier transform)

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left\{\int_{-\infty}^{\infty} e^{ipt}\, g(t-u)\, dt\right\} du$$

(by changing the order of integration)

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left\{\int_{-\infty}^{\infty} e^{ip(u+y)}\, g(y)\, dy\right\} du \qquad \text{(putting } t-u=y\text{)}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipu} f(u) \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipy} g(y) dy \right\} du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipu} f(u)\, G(p) du$$

$$= G(p) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipu} f(u)\, du = G(p) F(p)$$

$$\therefore \quad f[f * g] = f[f(t)] f[g(t)] = G(p)\, F(p) \qquad ...(27)$$

Taking inverse Fourier transform of both the sides of (27), we obtain

$$f * g = \mathsf{F}^{-1}[G(p)\, F(p)]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipt} G(p) F(p) dp$$

or $$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) g(t-u) du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipt} G(p) F(p) dp$$

or $$\int_{-\infty}^{\infty} f(u) g(t-u) du = \int_{-\infty}^{\infty} e^{-ipt} G(p) F(p) dp \qquad ...(28)$$

If we put $t = 0$ in the above equation (28), we get an interesting result

$$\int_{-\infty}^{\infty} f(u) g(-u) du = \int_{-\infty}^{\infty} G(p) F(p) dp$$

4.9 Parseval's Identity for Fourier Transform

Statement : If $F(p)$ and $G(p)$ are the Fourier transform of $f(t)$ and $g(t)$ respectively, then

(i) $$\int_{-\infty}^{\infty} F(p) G^*(p) dp = \int_{-\infty}^{\infty} f(t) g^*(t) dt$$

(ii) $$\int_{-\infty}^{\infty} |F(p)|^2 dp = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

where $*$ signifies the complex conjugate.

Proof : (i) Using the inversion formula for Fourier transform, we get

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(p)\, e^{-ipt}\, dp \qquad ...(29)$$

Now, taking complex conjugate of both the sides of (29), we get

$$g^*(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G^*(p)\, e^{ipt}\, dp \qquad ...(30)$$

Taking $\int_{-\infty}^{\infty} f(t)\, g^*(t)\, dt = \int_{-\infty}^{\infty} f(t)\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G^*(p)\, e^{ipt} dp\right] dt$

[putting the value of $g^*(t)$ from (30)]

$$= \int_{-\infty}^{\infty} G^*(p) \left\{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ipt} dt\right\} dp$$

(by changing the order of integration)

$$= \int_{-\infty}^{\infty} G^*(p)\, F(p)\, dp$$

$\therefore \quad \int_{-\infty}^{\infty} f(t)\, g^*(t)\, dt = \int_{-\infty}^{\infty} F(p)\, G^*(p)\, dp$

(ii) Taking $f(t) = g(t)$ in part (i), we get

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipt} f(t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipt} g(t) \Rightarrow F(p) = G(p) \text{ or } F^*(p) = G^*(p)$$

$\therefore \quad \int_{-\infty}^{\infty} F(p)\, F^*(p)\, dp = \int_{-\infty}^{\infty} f(t)\, f^*(t)\, dt$

$$\int_{-\infty}^{\infty} |F(p)|^2 dp = \int_{-\infty}^{\infty} |f(t)|^2\, dt$$

This relation can also be written as

or $\quad ||F|| = ||f||$

4.10 Fourier Transforms of Derivatives

Theorem 13 : Let

(i) $f(t)$ **is continous and absolutely integrable in** $(-\infty, \infty)$

(ii) $f'(t)$ **is piecewise continously differentiable and absolutely integrable in** $(-\infty, \infty)$

Then $\mathsf{F}\left[f'(t); p\right] = (-ip)\, \mathsf{F}\left[f(t); p\right]$

Proof : We have

$$\mathsf{F}\left[f'(t);\, p\right] = \frac{1}{\sqrt{2p}} \int_{-\infty}^{\infty} f'(t)\, e^{ipt}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[f(t)\, e^{ipt}\right]_{-\infty}^{\infty} - \frac{ip}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ipt}\, dt$$

Since $f(t)$ is absolutely integrable on $(-\infty,\infty)$

$\Rightarrow \quad \lim_{t\to\infty} f(t)=0$, so that the first integral vanishes.

$\therefore \quad f\left[f'(t);p\right]=-i\,p \quad f\left[f(t);p\right].$

Remark : If $f(t)$ has a finite discontinuity at the point $t=a$, then

$$\mathrm{F}\left[f'(t);p\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a-} f'(t)e^{ipt}dt+\frac{1}{\sqrt{2\pi}}\int_{a+}^{\infty} f'(t)e^{ipt}dt$$

Integrating by parts, we get

$$\mathrm{F}\left[f'(t);p\right]=\frac{1}{\sqrt{2\pi}}\left[f(t)\,e^{ipt}\right]_{-\infty}^{a-}+\frac{1}{\sqrt{2\pi}}\left[f(t)e^{ipt}\right]_{a+}^{\infty}$$

$$-\frac{ip}{\sqrt{2\pi}}\int_{-\infty}^{a-} f(t)e^{ipt}dt-\frac{ip}{\sqrt{2\pi}}\int_{a+}^{\infty} f(t)e^{ipt}dt$$

$$=-i\,p\,f\left[f(t);p\right]-\frac{1}{\sqrt{2\pi}}e^{ipa}\left[f\right]_a \qquad \text{...(31)}$$

where $\left[f\right]_a=f\left(a^+\right)-f\left(a^-\right)$

If $f(t)$ has n points of finite discontinuties denoted by $a_1,a_2,....a_n$, then (31) may be written as

$$\mathrm{F}\left[f'(t);p\right]=-i\,p\,\mathrm{F}\left[f(t);p\right]-\frac{1}{\sqrt{2\pi}}\sum_{r=1}^{n}e^{ipa_r}\left[f\right]_{a_r}$$

Theorem 14 : Let

(i) $f(t),\ f'(t),\ f''(t)\ ......\ f^{n-1}(t)$ **are continuous in** $(-\infty,\infty)$,

(ii) $f^{(n)}(t)$ **is piecewise continuously differentiable on** $(-\infty,\infty)$,

(iii) **each of the derivatives** $f^{(r)}(t)\ \ (r=0,2,3,\ldots n)$ **is absolutely integrable on** $(-\infty,\infty)$,

Then the Fourier transform of the function $\frac{d^n f}{dt^n}$ **is** $(-i\,p)^n$ **times the Fourier transform of the function** $f(t)$ **i.e.**

$$\mathrm{F}\left[\frac{d^n f}{dt^n};p\right]=\mathrm{F}\left[f^{(n)}(t);p\right]=(-i\,p)^n\,\mathrm{F}\left[f(t);p\right]$$

Proof : By definition of the Fourier transform, we have

$$\mathsf{F}\left[\frac{d^n f}{dt^n};p\right]=\mathsf{F}\left[f^{(n)}(t);p\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipt}f^{(n)}(t)\,dt$$

$$=\frac{1}{\sqrt{2\pi}}\left[f^{(n-1)}(t)\,e^{ipt}\right]_{-\infty}^{\infty}-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f^{(n-1)}(t)\,i\,p\,e^{ipt}dt$$

$$=-\frac{1}{\sqrt{2\pi}}\,ip\int_{-\infty}^{\infty}f^{(n-1)}(t)\,e^{ipt}dt \qquad \left[\because \underset{|t|\to\infty}{Lim}f^{(n-1)}(t)=0\right]$$

Now, repeating this process of integration by parts $(n-1)$ times more and $\underset{|t|\to\infty}{Lim}f^{(r)}(t)=0$ for $r=1,2,....(n-1)$, we get

$$\mathsf{F}\left[f^{(n)}(t);p\right]=\frac{1}{\sqrt{2\pi}}(-ip)^n\int_{-\infty}^{\infty}f(t)e^{ipt}dt$$

$$\therefore \qquad \mathsf{F}\left[f^{(n)}(t);p\right]=(-ip)^n\,\mathsf{F}\left[f(t);p\right]$$

Theorem 15 : If

(i) $f(t)$ **is continuous and absolutely integrable in** $(-\infty,\infty)$**,**

(ii) $f(t)$ **is absolutely integrable in** $(-\infty,\infty)$**,**

(iii) $F(p)$ **is the Fourier transform of** $f(t)$**, then**

$$F'(p)=\frac{d}{dp}\left[F(p)\right]=i\,F\left[t\,f(t);p\right] \qquad \textbf{....(32)}$$

or $$\mathsf{F}\left[t\,f(t)\right]=(-i)F'(p)=(-i)\frac{d}{dp}\left[F(p)\right]$$

Proof : Here $F(p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipt}f(t)\,dt$ ...(33)

Differentiating the above equation (33) with respect to initial, we get

$$F'(p)=\frac{d}{dp}\left[F(p)\right]=\frac{1}{\sqrt{2\pi}}\frac{d}{dp}\left[\int_{-\infty}^{\infty}e^{ipt}\,f(t)\,dt\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial p}e^{ipt}\right\}f(t)\,dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipt} (it) f(t)\, dt$$

$$= \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipt} (tf(t))\, dt = i\, F[tf(t); p]$$

$\therefore \qquad F'(p) = i \;\; \mathsf{F}\; [tf(t); p]$

If we continue this process r times, then we easily arrive at the following theorem :

Theorem 16 : (i) Let $f(t)$ is continuous in $(-\infty, \infty)$ (ii) each of $t^r f(t)\,(r = 0,1,2,\ldots,n)$ is absolutely integrable on $(-\infty, \infty)$.

Then $F^r(p) = (i)\mathsf{F}\;[t^r f(t); p],\ (r = 0,1,2,\ldots,n)$.

Example 11 : Use parseval's identify to prove that

(i) $$\int_{-\infty}^{\infty} \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)},\ (a > 0,\ b > 0)$$

(ii) $$\int_{-\infty}^{\infty} \frac{\sin at}{(a^2+t^2)}\, dt = \frac{\pi}{2} \left\{ \frac{1-e^{-a^2}}{a^2} \right\}$$

Solution : (i) Let $f(t) = e^{-at}$, $g(t) = e^{-bt}$, $(a > 0,\ b > 0)$

Then $F_c = \sqrt{\frac{2}{\pi}} . \frac{a}{a^2+p^2}$, $G_c = \sqrt{\frac{2}{\pi}} . \frac{b}{b^2+p^2}$

Putting these values in Parseval's identify for Fourier cosine transform, we have

$$\int_0^{\infty} F_c(p) G_c(p)\, dp = \int_0^{\infty} f(t) g(t)\, dt$$

or $$\frac{2}{\pi} \int_0^{\infty} \frac{ab\, dp}{(a^2+p^2)(b^2+p^2)} = \int_0^{\infty} e^{-(a+b)} dt$$

or $$\int_0^{\infty} \frac{dp}{(a^2+p^2)(b^2+p^2)} = \frac{\pi}{2ab} \left[\frac{e^{-(a+b)t}}{-(a+b)} \right]_0^{\infty} = \frac{\pi}{2ab(a+b)}$$

or $$\int_0^{\infty} \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)} \qquad (a > 0,\ b > 0)$$

(ii) Let $f(t) = e^{-at}$ and $g(t) = \begin{cases} 1, & 0 < t < a \\ 0, & r > a \end{cases}$

then $F_c(p) = \sqrt{\frac{2}{\pi}}\ \frac{a}{a^2 + b^2}$

and $G_c(p) = \sqrt{\frac{2}{\pi}}\ \int_0^\infty g(t) \cos pt\, dt$

$$= \sqrt{\frac{2}{\pi}}\ \int_0^a \cos pt\, dt$$

$$= \sqrt{\frac{2}{\pi}}\ \frac{\sin ap}{p}$$

Using Parseval's identify, we have

$$\int_0^\infty F_c(p)\ G_c(p)\ dp = \int_0^\infty f(t) g(t)\ dt$$

Putting all these values, we get

$$\frac{2}{\pi} \int_0^\infty \frac{a}{a^2 + p^2} \cdot \frac{\sin ap}{p}\ dp = \int_0^\infty e^{-at}\ g(t)\ dt$$

or $$\frac{2a}{\pi} \int_0^\infty \frac{\sin ap}{p\left(a^2 + p^2\right)}\ dp = \int_0^\infty e^{-at}.1.\ dt = \frac{\left[e^{-at}\right]_0^a}{-a}$$

$$= \frac{1}{a}\left(1 - e^{-a^2}\right)$$

$\therefore$ $$\int_0^\infty \frac{\sin at}{t\left(a^2 + t^2\right)}\ dt = \frac{\pi}{2a^2}\left(1 - e^{-a^2}\right)$$

Example 12 : Find fourier transform of $f(t)$ defined by $f(t) = \begin{cases} 1, & |t| < a \\ 0, & |t| > a \end{cases}$

and hence prove that $\int_0^\infty \frac{\sin^2 at}{t^2} dt = \frac{\pi a}{2}$

$$\text{F}\ \left[f(t); p\right] = F(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{ipt} f(t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-a}^a e^{ipt}.1.dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ipt}}{ip}\right]_{-a}^a = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ipa} - e^{-ippa}}{ip}\right]$$

$$\therefore \qquad F(p)=\frac{1}{\sqrt{2\pi}}\left[\frac{2\sin pa}{p}\right],\ p\neq 0$$

Using parseval's identify for Fourier transform, we have

$$\int_{-\infty}^{\infty}|f(t)|^2\,dt=\int_{-\infty}^{\infty}|F(p)|^2\,dp$$

or $$\int_{-a}^{a}1^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4\sin^2 pa}{p^2}\,dp$$

or $$\int_{-\infty}^{\infty}\frac{\sin^2 pa}{p^2}\,dp=\frac{\pi}{2}.2a=\pi a$$

or $$\int_{0}^{\infty}\frac{\sin^2 at}{t^2}\,dt=\frac{\pi a}{2}$$

Example 13 : Solve the integral equation for $f(t)$, $\int_0^\infty f(t)\cos pt\,dt=\begin{cases}1-p, & 0\le p\le 1\\ 0, & p>1\end{cases}$

Hence deduce that $\int_0^\infty \frac{\sin^2 t}{t^2}\,dt=\frac{\pi}{2}$

Solution : Let $\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos pt\,dt=\mathrm{F}_c[f(t)]=F_c(p)$

then $$F_c(p)=\begin{cases}\sqrt{\frac{2}{\pi}}\,(1-p), & 0\le p\le 1\\ 0 & ,\ p>1\end{cases}$$

Applying Fourier cosine inversion theorem, we get

$$f(t)=\sqrt{\frac{2}{\pi}}\,F_c(p)\cos pt\,dp$$

$$=\sqrt{\frac{2}{\pi}}\int_0^1\sqrt{\frac{2}{\pi}}\,(1-p)\cos pt\,dp=\frac{2(1-\cos t)}{\pi t^2}$$

$\therefore \qquad f(t)=\frac{2(1-\cos t)}{\pi t^2}$, which is the required solution.

Deduction : Putting the value of $f(t)$ in the given integral equation, we have

$$\int_0^\infty\frac{2(1-\cos t)}{\pi t^2}\cos pt\,dt=\begin{cases}1-p, & 0\le p\le 1\\ 0, & p>1\end{cases}$$

Letting $p \to 0$, this equation becomes

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos t}{t^2}\,dt \ \text{ or } \int_0^\infty \frac{2\sin^2(t/2)}{t^2}\,dt = \frac{\pi}{2}$$

Now, putting $t = 2u$ so that $dt = 2du$, we get

$$\int_0^\infty \frac{2\sin^2 u}{4u^2}.2du = \frac{\pi}{2}$$

or $$\int_0^\infty \frac{\sin^2 u}{u^2}du = \frac{\pi}{2}$$

Example 14 : Show that $\int_0^\infty \frac{t^2 dt}{(t^2+a^2)} = \frac{\pi}{(2a)^5}$, $(a>0)$

Solution : Let $f(t) = \frac{1}{2(t^2+a^2)}$, then $f'(t) = \frac{1}{(t^2+a^2)^2}$

But we know by Example 3,

$$\text{F } \left|e^{-a|t|};p\right| = \sqrt{\frac{2}{\pi}}\ \frac{a}{a^2+p^2}$$

$\therefore$ $$\text{F }^{-1}\left\{\frac{1}{a^2+p^2};t\right\} = \frac{1}{a}\ \sqrt{\frac{2}{\pi}}.\ e^{-a|t|}$$

or $$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-ipt}.\frac{1}{a^2+p^2}\ dp = \frac{1}{a}\ \sqrt{\frac{2}{\pi}}.e^{-a|t|}$$

Now replacing p by $-p$, we find

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \frac{e^{-ipt}}{a^2+p^2}dp = \frac{1}{a}\ \sqrt{\frac{2}{\pi}}.e^{-a|t|}$$

Thus $$\text{F } [f(t);p] = F(p) = \frac{1}{2a}\sqrt{\frac{\pi}{2}}\ e^{-a|t|}$$

Example 15 : Prove that

$$p^n F^{(m)}(p) = i^{m+n}\sum_{r=0}^{q}\frac{m!\ n!}{r!\ (n-r)!\ (m-r)!}\ \text{F } \left\{t^{m-r} f^{(n-r)}(t);p\right\}$$

Solution : Since $F^{(m)}(p) = i^{m+n}\ \text{F } \left\{t^m f(t);p\right\}$

Therefore $p^n F^{(m)}(p) = i^{m+n}\ \text{F } \left\{\frac{d^n}{dt^n}\ t^m f(t);p\right\}$ ...(34)

Using Leibnitz's theorem for n^{th} derivative of a product, we find that

$$\frac{d^n}{dt^n}\{t^m f(t)\} = \sum_{r=0}^{q} \frac{m!n!}{r!(n-r)!(m-r)!} x^{m-r} f^{(n-r)}(t) \qquad ...(35)$$

where $q = \min\{m, n\}$

Using (35) in (34) we get the required result.

Self-Learning Exercise-2

1. State the convolution theorem for Fourier transform.
2. State the Parseval's for Fourier cosine transform.

Fill in the blanks-

3. F $\{f^{(n)}(t); p\} = ...$
4. If $f(t) = \cos t$ and $g(t) = \exp(-a|t|)$, $a > 0$ then $(f * g)(t) = ...$
5. If $\int_0^\infty f(t) \sin pt\, dt = \begin{cases} 1, & 0 \le p < 1 \\ 2, & 1 \le p < 2 \\ 0, & p > 2 \end{cases}$,

then $f(t) = ...$

4.11 Summary

In this unit you studied the complex Fourier transform and Fourier sine and cosine transform and some important thoerems and properties concerned with these transforms.

4.12 Answers to Self-Learning Exercise

Exercise - I

3. $\frac{1}{2}\left[F_c(p+a) + F_s(p-a)\right]$
4. $\frac{1}{2}\left[F_s(p+a) + F_s(p-a)\right]$
5. $\frac{1}{|a|} F\left(\frac{p}{a}\right)$

Exercise - II

3. $(-i p^n)$ F $\{f(t); p\}$
4. $\sqrt{\frac{2}{\pi}} \frac{a \cos t}{1+a^2}$
5. $\frac{2}{\pi t}(1 + \cos t - 2\cos 2t)$

4.13 Exercise 4 (b)

1. Use Parseval's identify to prove that;

(a) $$\int_0^\infty \frac{t^2\,dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2(a+b)} \; ; a>0,\ b>0$$

(b) $$\int_0^\infty \frac{dt}{(1+t^2)^2} = \frac{\pi}{4}$$ (c) $$\int_0^\infty \frac{t^2\,dt}{(1+t^2)^2} = \frac{\pi}{4}$$

2. Show that if $p>0$, $\frac{2}{4}\int_0^\infty \frac{\sin t \cos pt}{t}\,dt = U(1-p)$

3. Making use of the Fourier cosine transform and the Parseval relation, prove that

$$\int_0^\infty \frac{t^{-a}dt}{1+t^2} = \frac{\pi}{2}\sec\frac{1}{2}(\pi a),\ 0<a<1$$

4. Making use of the inversion theorem, show that

(i) $$\int_0^\infty \frac{\cos pt}{a^2+p^2}\,dp = \frac{\pi}{2a}\,e^{-at}$$ (ii) $$\int_0^\infty \frac{p\sin pt}{a^2+p^2}\,dp = \frac{\pi}{2}\,e^{-at}$$

5. Solve the integral equation

$$\int_0^\infty f(t)\cos\lambda t\,dt = e^{-\lambda}$$ [**Ans.** $f(t)=\dfrac{2}{\pi(1+t^2)}$]

6. Evaluate $\int_{-\infty}^\infty \frac{dt}{(t^2+a^2)(t^2+b^2)}$, $a>0$, $b>0$ [**Ans.** $\dfrac{\pi}{ab(a+b)}$]

[**Hint :** Taking $f(t)=e^{-a|t|}$ and $g(t)=e^{-b|t|}$ and apply convolution theorem for Fourier transform]

7. Show that $$\int_{-\infty}^\infty \frac{1-\cos p\pi}{p}\sin pt\,dp = \begin{cases} \frac{\pi}{2}, & 0<t<\pi \\ 0, & t>\pi \end{cases}$$

8. Prove that $$\int_0^\infty \frac{\sin(\lambda t)\sin(\mu t)}{t^2}\,dt = \frac{\pi}{2}\min(\lambda,\mu),$$

where $\min(\lambda,\mu)$ means the lesser of the two positive members λ and μ.

9. Find the Fourier cosine transform of

$$f(t)=t^{-a},\ (0<a<1),\ g(t)=(1-t^2)^{\upsilon-1/2}U(1-t),\ (\upsilon>-1/2)$$

and hence prove that

$$\int_0^\infty t^{a-\upsilon-1}J_\upsilon(t)\,dt = \frac{2^{a-\upsilon-1}\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\upsilon-\frac{a}{2}+1\right)},\ \left(0<a<1,\ \upsilon-\frac{1}{2}\right)$$

[**Ans.** $\sqrt{\frac{2}{\pi}}\,p^{a-1}\Gamma(1-a)\sin\left(\frac{a\pi}{2}\right),\ 2^{\upsilon-\frac{1}{2}}\Gamma\left(\upsilon+\frac{1}{2}\right)p^{-\upsilon}J_\upsilon(p)$]

Unit - 5
Mellin Transform

Structure of the Unit

5.0 Objective

The main object of this unit is to define Mellin transform and to give elementary properties of the Mellin transform, Mellin transform of derivatives and integrals. Important theorems such as Mellin inversion theorem and Convolution (for Faltung) theorem etc are also proved.

5.1 Introduction

The Mellin transform arises in a natural way in the solution of boundary value problems concerning an infinite wedge. In this unit we shall consider the properties of this transform and its inverse.

5.2 Definition

The Mellin transform of the function $f(x)$, where $0 < x < \infty$ is denoted by $M\{f(x); p\}$ or $F(p)$ and is defined as

$$M\{f(x); p\} = F(p) = \int_0^\infty x^{p-1} f(x)\,dx$$

Here x^{p-1} is known as **Kernel** of the Mellin transform and M stands for **Mellin Transformation Operation.**

5.3 Elementry Properties of the Mellin Transform

I. If $M\{f(x); p\} = F(p)$, then

$$M\{f(ax); p\} = a^{-p}F(p)$$

Proof : By definition, we have

$$M\left\{f(ax);p\right\}=\int_0^\infty x^{p-1}f(ax)\,dx$$

$$=\int_0^\infty\left(\frac{t}{a}\right)^{p-1}f(t)\,\frac{dt}{a} \qquad \text{(let } ax=t\text{)}$$

$$=\frac{1}{a^p}\int_0^\infty x^{p-1}f(x)\,dx=a^{-p}F(p)$$

II. **If $M\left\{f(x);p\right\}=F(p)$, then**

$$M\left\{x^a f(x);p\right\}=F(p+a)$$

Proof : By definition, we have

$$M\left\{x^a f(x);p\right\}=\int_0^\infty x^{(p+a)-1}f(x)\,dx$$

$$=F(p+a)$$

III. **If $M\left\{f(x);p\right\}=F(p)$, then**

$$M\left\{f(x^a);p\right\}=\frac{1}{a}F\left(\frac{p}{a}\right), \quad a>0$$

Proof : By definition, we have

$$M\left\{f(x^a);p\right\}=\int_0^\infty x^{p-1}f(x^a)\,dx$$

Substituting $x^a=t$ or $x=t^{t/a}$, $dx=\frac{1}{a}t^{(1/a)-1}\,dt$

in R.H.S., we get

$$M\left\{f(x^a);p\right\}=\int_0^\infty\left(t^{1/a}\right)^{p-1}f(t)\frac{1}{a}t^{(1/a)-1}\,dt$$

$$=\frac{1}{a}\int_0^\infty t^{(1/a)-1}f(t)\,dt \qquad =\frac{1}{a}F\left(\frac{p}{a}\right)$$

IV. **If $M\left\{f(x);p\right\}=F(p)$, then**

$$M\left\{\frac{1}{x}f\left(\frac{1}{x}\right);p\right\}=F(1-p)$$

Proof : By definition, we have

$$M\left\{\frac{1}{x}f\left(\frac{1}{x}\right);p\right\}=\int_0^{\infty}x^{p-2}f\left(\frac{1}{x}\right)dx$$

Substituting $\frac{1}{x}=t$ or $x=\frac{1}{t}$, $dx=\frac{1}{t^2}dt$

in R.H.S., we get

$$M\left\{\frac{1}{x}f\left(\frac{1}{x}\right);p\right\}=\int_0^{\infty}t^{(1-p)-1}f(t)\,dt=F(1-p)$$

V. If $M\{f(x);p\}=F(p)$, then

$$M\{\log x\, f(x);p\}=\frac{d}{dp}F(p)$$

Proof : We know that $\frac{d}{dp}x^{p-1}=(\log x)x^{p-1}$

Multiplying both sides by $f(x)$ and integrating with respect to x between the limits 0 to ∞, we get

$$\int_0^{\infty}\left\{\frac{d}{dp}x^{p-1}\right\}f(x)\,dx=\int_0^{\infty}x^{p-1}\log x\; f(x)\,dx$$

or $$\frac{d}{dp}\left[\int_0^{\infty}x^{p-1}f(x)\,dx\right]=M\{\log x\; f(x);p\}$$ (by definition)

or $$\frac{d}{dp}F(p)=M\{\log x\; f(x);p\}$$

VI If $M\{f(x);p\}=F(p)$, then

$$M\{x^m f(ax^n);p\}=\frac{1}{n}a^{-(p+m)/n}\;F\left(\frac{p+m}{n}\right),\ (a>0)$$

Proof : By definition, we have

$$M\{x^m f(ax^n);p\}=\int_0^{\infty}x^{p+m-1}f(ax^n)\,dx$$

Substituting $ax^n=t$ or $x=\left(\frac{t}{a}\right)^{1/n}$, $dx=\frac{1}{n}\left(\frac{t}{a}\right)^{(1/n)-1}\frac{1}{a}\,dt$

in R.H.S., we get

$$M\left\{x^m f\left(ax^n\right);p\right\} = \frac{1}{n} a^{-(p+m)/n} \int_0^\infty x^{\{(p+m)/n\}-1} f(t)\,dt$$

$$= \frac{1}{n} a^{-(p+m)/n} F\left(\frac{p+m}{n}\right)$$

Example 1 : Prove that

$$M\left\{(1+x)^{-a};p\right\} = \frac{\Gamma(a)\,\Gamma(a-p)}{\Gamma(a)}, \qquad 0 < Re(p) < Re(a)$$

Solution : By definition of Beta fucntion, we have

$$\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}, \qquad Re(m) > 0,\ Re(n) > 0$$

Replacing $m+n=a$ or $m=a-n$, we get

$$\int_0^\infty \frac{x^{n-1}}{(1+x)^{a}}\,dx = \frac{\Gamma(a-n)\,\Gamma(n)}{\Gamma(a)}, \qquad 0 < Re(n) < Re(a)$$

Again replacing n by p, we get

$$\int_0^\infty x^{p-1}\,(1+x)^{-a}\,dx = \frac{\Gamma(a-p)\,\Gamma(p)}{\Gamma(a)}$$

or $$M\left\{(1+x)^{-a};p\right\} = \frac{\Gamma(p)\,\Gamma(a-p)}{\Gamma(a)}, \qquad 0 < Re(n) < Re(a)$$

Particular case : When $a=1$, then

$$M\left\{(1+x)^{-1};p\right\} = \Gamma(p)\,\Gamma(1-p) \qquad \left[\because \Gamma(1)=1\right]$$

Using the formula $\Gamma(p)\,\Gamma(1-p) = \dfrac{\pi}{\sin p\pi}$, $\qquad 0 < Re(p) < Re(1)$

in R.H.S., we have

$$M\left\{(1+x)^{-1};p\right\} = \frac{\pi}{\sin p\pi}$$

or $$M\left\{(1+x)^{-1};p\right\} = \pi \operatorname{cosec} p\pi, \qquad 0 < Re(p) < Re(1)$$

Example 2 : Prove that

$$M\left\{\left(1+x^a\right)^{-b};p\right\} = \frac{\Gamma\left(p/a\right)\,\Gamma\left(b-p/a\right)}{a\,\Gamma(b)}, \qquad 0 < Re(p) < Re(ab)$$

Solution : Freom Ex. 1, we have just seen that

$$M\left\{(1+x)^{-b};p\right\}=\frac{\Gamma(p)\,\Gamma(b-p)}{\Gamma(b)},\qquad 0<Re(p)<Re(b) \qquad ...(1)$$

From the property (III) of 5.3, we have

$$M\left\{f(x^{a});p\right\}=\frac{1}{a}\,F\left(p/a\right),\quad a>0$$

Using this property in (1), we get

$$M\left\{(1+x^{a})^{-b};p\right\}=\frac{1}{a}\,\frac{\Gamma\left(p/a\right)\,\Gamma\left(b-p/a\right)}{\Gamma(b)} \qquad ...(2)$$

provided that $0<Re(p)<Re(ab)$

Particular Cases : (i) Taking $b=1$ in (2), we get

$$M\left\{(1+x^{a})^{-1};p\right\}=\frac{1}{a}\,\frac{\Gamma\left(p/a\right)\,\Gamma\left(1-p/a\right)}{\Gamma(1)}$$

or $$M\left\{(1+x^{a})^{-1};p\right\}=\frac{\pi}{a}\,\text{cosec}\,\frac{\text{p}\pi}{\text{a}},\qquad 0<Re(p)<Re(a) \qquad ...(3)$$

(ii) Taking $a=2$ in (3), we get

$$M\left\{(1+x^{2})^{-1};p\right\}=\frac{\pi}{2}\,\text{cosec}\,\frac{p\pi}{2},\qquad 0<Re(p)<2$$

Example 3 : Prove that

$$M\left\{e^{-ax}J_{v}(bx);p\right\}=\frac{b^{v}\,2^{p-1}}{\sqrt{\pi}\;\Gamma(v+1)}\,\left(a^{2}+b^{2}\right)^{-(v+p)/2}$$

$$\times\,\Gamma\left(\frac{v+p}{2}\right)\Gamma\left(\frac{v+p+1}{2}\right)2F_{1}\left(\frac{v+p}{2},\frac{v-p+1}{2};v+1;\frac{b^{2}}{a^{2}+b^{2}}\right),$$

$$\left(Re(a)>0,\ v>-\frac{1}{2}\right)$$

Hence deduce that

(i) $$M\left\{J_{v}(bx);p\right\}=\frac{b^{-p}\,2^{p-1}\,\Gamma\left(\frac{v+p}{2}\right)}{\Gamma\left(\frac{v-p+2}{2}\right)},\qquad -v<p<v+2$$

(ii) $$M\left\{x^{-\nu}J_\nu(x);p\right\}=\frac{2^{p-\nu-1}\,\Gamma\left(p/2\right)}{\Gamma\left(\nu-\frac{1}{2}p+1\right)},\quad 0<Re(p)<1,\ \nu>-\frac{1}{2}$$

Solution : By definition, we have

$$M\left\{e^{-ax}J_\nu(bx);p\right\}=\int_0^\infty x^{p-1}e^{-ax}J_\nu(bx)\,dx$$

$$=\int_0^\infty x^{p-1}e^{-ax}\sum_{r=0}^\infty\frac{(-1)^r}{r!\,\Gamma\,(\nu+r+1)}\left(\frac{bx}{2}\right)^{\nu+2r}dx$$

$$=\sum_{r=0}^\infty\frac{(-1)^r}{r!\,\Gamma\,(\nu+r+1)}\left(\frac{b}{2}\right)^{\nu+2r}\int_0^\infty e^{-ax}x^{p+\nu+2r-1}\,dx$$

$$=\sum_{r=0}^\infty\frac{(-1)^r}{r!\,\Gamma(\nu+r+1)}\left(\frac{b}{2}\right)^{\nu+2r}\frac{\Gamma(\nu+p+2r)}{a^{p+\nu+2r}}$$

(by definition of Gamma function)

$$=\sum_{r=0}^\infty\frac{(-1)^r}{r!\,\Gamma(\nu+r+1)}\left(\frac{b}{2}\right)^{\nu+2r}\frac{\Gamma\left(\frac{p+\nu}{2}+r\right)\Gamma\left(\frac{p+\nu+1}{2}+r\right)2^{p+\nu+2r-1}}{\sqrt{\pi}\,a^{p+\nu+2r}}$$

(by duplication formula for gamma function)

$$=\frac{b^\nu\,2^{p-1}}{\sqrt{\pi}\,a^{p+\nu}}\sum_{r=0}^\infty\frac{(-1)^r}{r!\,(\nu+1)_r\,\Gamma(\nu+1)}\left(\frac{b^2}{a^2}\right)^r\left(\frac{p+\nu}{2}\right)_r$$

$$\Gamma\left(\frac{\nu+p}{2}\right)\left(\frac{p+\nu+1}{2}\right)_r\ \Gamma\left(\frac{p+\nu+1}{2}\right)$$

$$=\frac{b^\nu\,2^{p-1}\,\Gamma\left(\frac{p+\nu}{2}\right)\Gamma\left(\frac{p+\nu+1}{2}\right)}{\sqrt{\pi}\,a^{p+\nu}\,\Gamma(\nu+1)}\,{}_2F_1\left(\frac{p+\nu}{2},\frac{p+\nu+1}{2};\ \nu+1;-\frac{b^2}{a^2}\right)\quad\text{...(5)}$$

Now, using the result

$${}_2F_1(\alpha,\beta;\gamma;x)=(1-x)^{-\alpha}\,{}_2F_1\left(\alpha,\gamma-\beta,\gamma;\frac{x}{x-1}\right)\quad\text{...(6)}$$

in (5), we get

$$M\left\{e^{-ax}J_\nu(bx);p\right\}=\frac{b^\nu\,2^{p-1}\,\Gamma\left(\frac{\nu+p}{2}\right)\Gamma\left(\frac{\nu+p+1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu+1)}$$

$$\times \left(a^2+b^2\right)^{-(p+\nu)/2} {}_2F_1\left(\frac{p+\nu}{2}, \frac{\nu-p+1}{2}; \nu+1; \frac{b^2}{b^2+a^2}\right) \qquad ...(7)$$

Deduction :

(i) Taking $a=0$ in (7), we get

$$M\{J_\nu(bx); p\} = \frac{b^{-p}\, 2^{p-1}\, \Gamma\left(\frac{\nu+p}{2}\right) \Gamma\left(\frac{\nu+p+1}{2}\right)}{\sqrt{\pi}\, \Gamma(\nu+1)} \times {}_2F_1\left(\frac{p+\nu}{2}, \frac{\nu-p+1}{2}; \nu+1; 1\right) \quad ...(8)$$

Using the Gauss summation theorem

$$ {}_2F_1(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\, \Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\, \Gamma(\gamma-\beta)}$$

in (8), we get

$$M\{J_\nu(bx); p\} = \frac{b^{-p}\, 2^{p-1}\, \Gamma\left(\frac{\nu+p}{2}\right)}{\Gamma\left(\frac{\nu-p+2}{2}\right)}, \qquad -\nu < p < \nu+2 \qquad ...(9)$$

(ii) From property (II) of 5.3, we have

$$M\{x^\alpha f(x); p\} = F(p+\alpha)$$

Using this property in (9) with $b=1$, we get

$$M\{x^{-\upsilon} J_\nu(x); p\} = \frac{2^{p-\nu-1}\, \Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\nu - \frac{1}{2}p + 1\right)}, \qquad 0 < Re(p) < 1,\ \nu > -\frac{1}{2} \qquad ...(10)$$

Example 4 : Prove that

$$M\left\{x^\rho (1-x)^{c-1}\, {}_2F_1(a,b;c;\, 1-x)\, H(1-x); p\right\} = \frac{\Gamma(c)\, \Gamma(p+\rho)\, \Gamma(p-a-b+c+\rho)}{\Gamma(p-a+c+\rho)\, \Gamma(p-b+c+\rho)}$$

Solution : By definition, we have

$$M\left\{x^\rho (1-x)^{c-1}\, {}_2F_1(a,b;c;1-x)\, U(1-x); p\right\}$$

$$= \int_0^\infty x^{p-1} x^\rho (1-x)^{c-1}\, {}_2F_1(a,b;c;1-x)\, U(1-x)\, dx$$

$$= \int_0^1 x^{p+\rho-1}\, (1-x)^{c-1}\, {}_2F_1(a,b;c;1-x)\, .1\, .\, dx$$

$$= \int_0^1 x^{p+\rho-1} (1-x)^{c-1} \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r! (c)_r} (1-x)^r dx$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r! (c)_r} \int_0^1 x^{p+\rho-1} (1-x)^{c+r-1} dx$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r! (c)_r} \frac{\Gamma(p+\rho)\Gamma(c+r)}{\Gamma(p+\rho+c+l)} \quad \text{(by definition of Beta function)}$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r!} \frac{\Gamma(p+\rho)\Gamma(c)}{\Gamma(p+\rho+c)_r \Gamma(p+\rho+c)}$$

$$= \frac{\Gamma(p+\rho)\Gamma(c)}{\Gamma(p+\rho+c)} \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{r! (p+\rho+c)_r}$$

$$= \frac{\Gamma(p+\rho)\Gamma(c)}{\Gamma(p+\rho+c)} {}_2F_1(a,b;p+\rho+c;1)$$

$$= \frac{\Gamma(p+\rho)\Gamma(c)\Gamma(p+\rho+c)\Gamma(p+\rho+c-a-b)}{\Gamma(p+\rho+c)\Gamma(p+\rho+c-a)\Gamma(p+\rho+c-b)}$$

(by Gauss summation theorem)

$$= \frac{\Gamma(c)\Gamma(p+\rho)\Gamma(p+\rho+c-a-b)}{\Gamma(p+\rho+c-a)\Gamma(p+\rho+c-b)}$$

5.4 Mellin Transform of Derivatives

By the definition of Mellin Transform, we have

$$M\{f'(x);p\} = \int_0^{\infty} x^{p-1} f'(x)\,dx \qquad \text{where } f'(x) = \frac{df}{dx}$$

$$= \left[x^{p-1} f(x)\right]_0^{\infty} - (p-1)\int_0^{\infty} x^{p-2} f(x)\,dx \qquad \text{[integration by parts]} \quad ...(11)$$

Here if there exist α_1, α_2 such that

$$\underset{x\to 0}{Lim}\, x^{p-1} f(x) = 0, \qquad \underset{x\to\infty}{Lim}\, x^{p-1} f(x) = 0$$

when $\alpha_1 < Re(p) < \alpha_2$ and if $F(p-1)$ exists in the band, then (11) reduces to

$$M\{f'(x);p\} = -(p-1)\int_0^{\infty} x^{(p-1)-1} f(x)\,dx$$

or $\quad M\{f'(x);p\} = -(p-1)\,M\{f(x);p-1\}$

$$= -(p-1)\,F(p-1) \qquad ...(12)$$

Applying this formula twice, we have

$$M\{f''(x);p\} = -(p-1)\,M\{f'(x);p-1\}$$

$$= -(p-1)\{-(p-2)f(p-2)\}$$

or $\quad M\{f''(x);p\} = (-1)^2\,(p-1)(p-2)\;\; M\{f(x);p-2\} \qquad ...(13)$

Therefore by the Principle of mathematical induction (PMI), we find that

$$M\{f^{(n)}(x);p\} = (-1)^n\,(p-2).....(p-n)\,F(p-n)$$

or $\quad M\{f^{(n)}(x);p\} = \dfrac{(-1)^n\,\Gamma(p)}{\Gamma(p-n)}\,M\{f(x);p-n\}$

or $\quad M\{f^{(n)}(x);p\} = \dfrac{(-1)^n\,\Gamma(p)}{\Gamma(p-n)}\,F(p-n) \qquad ...(14)$

provided that $\underset{x\to 0}{Lim}\,x^{p-r-1}f^{(r)}(x) = 0$, $r = 0,1,....,n-1$ and $F(p-n)$ exits.

Now, by property (II) of 5.3, we have

$$M\{x^n f(x);p\} = F(p+n) = M\{f(x);p+n\}$$

$\therefore \quad M\{x^n f^{(n)}(x);P\} = M\{f^{(n)}(x);p+n\}$

or $\quad M\{x^n f^{(n)}(x);p\} = (-1)^n\,\dfrac{\Gamma(p+n)}{\Gamma(p)}\,M\{f(x);p\}$

or $\quad M\{x^n f^{(n)}(x);p\} = \dfrac{(-1)^n\,\Gamma(p+n)}{\Gamma(p)}\,F(p) \qquad$ [Replacing p by $p+n$ in (14)] ...(15)

Example 5 : Prove that if n is a positive integer,

$$M\left[\left(x\frac{d}{dx}\right)^n f(x);p\right] = (-1)^n\,p^n F(p) \qquad ...(16)$$

where $\quad M\{f(x);p\} = F(p)$

Solution : We shall prove the result by using Mathematical induction.

By the property (II) of 5.3 with n =1, we have

$$M\{xf'(x);p\}=-p\,M\{f(x);p\}=-p\,F(p)$$

$$=-(p+1-1)\,M\{f(x);p+1-1\}$$

or $\quad M\{xf'(x);p\}=-p\,M\{f(x);p\}=-pF(p) \qquad$ [using (12)]

or $$M\left\{\left(x\frac{d}{dx}\right)f(x);p\right\}=-\,pF(p) \qquad ...(17)$$

Let the result (17) is true for n = m (a fixed positive integer), them

$$M\left\{\left(x\frac{d}{dx}\right)^m f(x);p\right\}=(-1)^m p^m F(p) \qquad ...(18)$$

Let us assume that $\left(x\frac{d}{dx}\right)^m f(x)=g(x)$

Now, $\quad M\left\{\left(x\frac{d}{dx}\right)g(x);p\right\}=-p\,M\{g(x);p\} \qquad$ [using (17) by setting $f(x)=g(x)$]

or $$M\left\{\left(x\frac{d}{dx}\right)^{m+1} f(x);p\right\}=-\,p\,M\left\{\left(x\frac{d}{dx}\right)^m f(x);p\right\}$$

$$=-p(-1)^m p^m F(p)$$

$$=(-1)^{m+1} p^{m+1} F(p)$$

Hence the result is true for n = m+1

Hence by Principle of mathematical Induction, the result (16) is true all $n \in N$.

Example 6 : Prove that if $\boldsymbol{m}$ is a positive integer $\boldsymbol{\alpha \neq 0}$

$$M\left\{\left(x^{1-\alpha}\frac{d}{dx}\right)^m f(x);p\right\}=(-1)^m \alpha^m \frac{\Gamma\left(p/\alpha\right)}{\Gamma\left(\frac{p}{\alpha}-m\right)} F(p-m\alpha) \qquad ...(19)$$

where $\quad M\{f(x);p\}=F(p)$

Solution : We shall prove the result by using mathematical induction.

By the definition of Mellin Transform, we have

$$M\left[\left(x^{1-\alpha}\frac{d}{dx}\right)f(x);p\right]=\int_0^\infty x^{p-1}\left(x^{1-\alpha}\frac{d}{dx}\right)f(x)\,dx$$

$$= \int_0^\infty x^{p-\alpha} \frac{d}{dx} f(x)\, dx$$

$$= \left[x^{p-\alpha} f(x)\right]_0^\infty - (p-\alpha) \int_0^\infty x^{p-\alpha-1} f(x)\, dx$$

[Integration by parts]

or $$M\left[\left(x^{p-\alpha} \frac{d}{dx}\right) f(x); p\right] = -(p-\alpha)\, M\left[f(x); p-\alpha\right] \qquad ...(20)$$

provided that $x^{p-\alpha} f(x) \to 0$ as $x \to 0$ and $x \to \infty$.

Let the result (19) is true for $m = k$ (a fixed positive integer) then

$$M\left[\left(x^{1-\alpha} \frac{d}{dx}\right)^k f(x); p\right] = (-1)^k \alpha^k \frac{\Gamma\left(p/\alpha\right)}{\Gamma\left(\frac{p}{\alpha} - k\right)} F(p - k\alpha) \qquad ...(21)$$

Taking $\left(x^{1-\alpha} \frac{d}{dx}\right)^k f(x) = g(x)$, then we have

$$M\left[\left(x^{1-\alpha} \frac{d}{dx}\right) g(x); p\right] = -(p-\alpha)\, M\left[g(x); p-\alpha\right]$$

[By using (19(after putting $f(x) = g(x)$]

or $$M\left[\left(x^{1-\alpha} \frac{d}{dx}\right)^{k+1} f(x); p\right] = -(p-\alpha)\, M\left[\left(x^{1-\alpha} \frac{d}{dx}\right)^k f(x); p-\alpha\right]$$

$$= -(p-\alpha)\,(-1)^k\, \alpha^k \frac{\Gamma\left(\frac{p-\alpha}{\alpha}\right)}{\Gamma\left(\frac{p-\alpha}{\alpha} - k\right)} F(p-\alpha-k\alpha)$$

[Using 21]

$$= (-1)^{k+1}\, \alpha^{k+1} \frac{\Gamma\left(p/\alpha\right)}{\Gamma\left(p/\alpha - k - 1\right)} F\left[p - (k+1)\alpha\right]$$

Hence the result is true for $m = k+1$.

Hence by principle of mathematical induction (19) is true for all $m \in N$.

5.5 Mellin Transform of Integrals

Case I : **If $M\{f(x);p\}=F(p)$, then show that**

(i) $$M\left\{\int_0^x f(u)\,du\,;p\right\}=-\frac{1}{p}F(p+1)$$

(ii) $$M\left\{\int_0^x dy\int_0^y f(u)\,du\,;p\right\}=\frac{1}{p(p+1)}F(p+2)$$

(iii) $$M\{I_n\ f(x);p\}=(-1)^n\frac{\Gamma(p)}{\Gamma(p+n)}F(p+n)$$

Where the n^{th} repeated integral of $f(x)$ is denoted by $I_n\ f(x)$ in the sense that

$$I_n\,f(x)=\int_0^x I_{n-1}f(u)\,du$$

Proof : Let $\int_0^x f(u)du=g(x)$ ...(22)

so that $g'(x)=f(x)$ ...(23)

Now, $M[g'(x);p]=-(p-1)\,M[g(x);p-1]$ [using (12)] ...(24)

or $M[f(x);p]=-(p-1)\,M\left\{\int_0^x f(u)du;p-1\right\}$ [using (22) and (23)]

Now replacing p by $p+1$ in the above result, we get

$$M[f(x);p+1]=-p\,M\left\{\int_0^x f(u)du;p\right\}$$

$\therefore$ $M\left\{\int_0^x f(u)du;p\right\}=-\frac{1}{p}M\{f(x);p+1\}=-\frac{1}{p}\,F(p+1)$ ...(25)

which prove the required result (i).

To prove the result (ii), we shall repeat the process explained in above part (i). Thus, we have

$$M\left\{\int_0^x dy\int_0^y f(u)du;p\right\}=-\frac{1}{p}M\left\{\int_0^y f(u)du;p+1\right\}\qquad\text{[by part (i)]}$$

$$=-\frac{1}{p}\left(-\frac{1}{p+1}F(p+2)\right)\qquad\text{[by part (i) again]}$$

$$=\frac{1}{p(p+1)}F(p+2).$$

This is the required result (ii).

To prove the result (iii), we can be use mathematical induction over n.

Case II : If $M\{f(x);p\}=F(p)$, then show that

(i) $$M\left\{\int_0^x f(u)\,du;p\right\}=\frac{1}{p}F(p+1)$$

(ii) $$M\left\{\int_x^\infty dy\int_y^\infty f(u)\,du;p\right\}=\frac{1}{p(p+1)}F(p+2)$$

(iii) $$M\left\{\int_n^\infty f(x);p\right\}=\frac{\Gamma(p)}{\Gamma(p+n)}F(p+n)$$

where the n^{th} repeated integral of $I_n^\infty f(x)$ is denoted in the sense that

$$I_n^\infty f(x)=\int_x^\infty I_{n-1}^\infty f(u)\,du$$

Proof : Let $\int_x^\infty f(u)\,du=g(x)$ ...(26)

so that $g'(x)=-f(x)$ ...(27)

Now $M\{g'(x);p\}=-(p-1)\,M\{g(x);p-1\}$, [using (12)]

or $M\{-f(x);p\}=-(p-1)\,M\left[\int_x^\infty f(u)\,du;p-1\right]$ [using (26) and (27)]

or $M\{f(x),p\}=(p-1)\,M\left[\int_x^\infty f(u)\,du;p-1\right]$

Now replacing p by $p+1$, we get

$$M\{f(x);p+1\}=p\,M\left[\int_x^\infty f(u)\,du;p\right]$$

$$\therefore\quad M\left\{\int_x^\infty f(u)\,du;p\right\}=\frac{1}{p}M[f(x);p+1]=\frac{1}{p}F(p+1)$$

which is the required result (i).

By repeated use of result (i), part (ii) can be proved as explained in part (ii) of case I.

To prove the part (c), we can use the mathematical induction as usual.

Example 7 : Prove that

$$M\left\{x^\alpha\int_0^\infty u^\beta f(xu)\,g(u)\,du;p\right\}=M\{f(x);p+\alpha\}\;M\{g(x);1+\beta-\alpha-p\}$$

Solution : By the definition of Mellin transform, we have

$$M\left\{\int_0^\infty u^\beta f(xu)\,g(u)du; p\right\} = \int_0^\infty x^{p-1}\left[\int_0^\infty u^\beta f(xu)\,g(u)du\right]dx$$

$$= \int_0^\infty u^\beta g(u)\left[\int_0^\infty x^{p-1} f(xu)\,dx\right]du$$

[changing the order of integration]

$$= \int_0^\infty u^\beta g(u)\left[\int_0^\infty \left(\frac{t}{u}\right)^{p-1} f(t)\frac{dt}{u}\right]du$$

[by putting $xu = t$ so that $u\,dx = dt$]

$$= \int_0^\infty u^{\beta-p} g(u)\left[\int_0^\infty t^{p-1} f(t)\,dt\right]du$$

$$= \int_0^\infty u^{\beta+1-p-1} g(u)du \,.\int_0^\infty t^{p-1} f(t)dt$$

$$= M\{g(x); 1+\beta-p\}\; M\{f(x); p\}$$

or $M\left\{\int_0^\infty u^\beta f(xu)g(u)du; p\right\} = M\{f(x); p\}\, M\{g(x); 1+\beta-p\}$ ...(28)

Now by the Property (II) of 5.3, we have

$$M\{x^\alpha h(x); p\} = M\{h(x); p+\alpha\}$$

Using this property in (28), we get

$$M\left\{x^\alpha \int_0^\infty u^\beta f(xu)g(u)du; p\right\} = M\{f(x); p+\alpha\}\;\; M\{g(x); 1+\beta-\alpha-p\}$$

$$= F(p+\alpha)G(1+\beta-\alpha-p) \qquad ...(29)$$

where F(p) and G(p) are mellin transforms of $f(x)$ and and $g(x)$.

Special case : Putting $\alpha = 0 = \beta$ in (29), we obtain

$$M\left\{\int_0^\infty f(xu)\,g(u)du; p\right\} F(p)\,G(1-p)$$

Example 8 : If $F(p)$ and $G(p)$ are the Mellin transform of $f(x)$ and $g(x)$ respectively. Find the mellin transform of

$$x^\lambda \int_0^\infty u^\mu f\left(\frac{x}{u}\right)g(u)du$$

where λ and μ are constants.

Solution : By the definition of Mellin transform, we have

$$M\left\{x^{\lambda}\int_0^{\infty} u^{\mu} f\left(\frac{x}{u}\right) g(u)du; p\right\} = \int_0^{\infty} x^{p-1}\left[x^{\lambda-1}\int_0^{\infty} u^{\mu} f\left(\frac{x}{u}\right) g(u)du\right]dx$$

$$= \int_0^{\infty} u^{\mu} g(u)\left[\int_0^{\infty} x^{p+\lambda-1} f\left(\frac{x}{u}\right)dx\right]du$$

[changing the order of integration]

$$= \int_0^{\infty} u^{\mu} g(u)\left[\int_0^{\infty} (ut)^{p+\lambda-1} f(t)\, u\, dt\right]du$$

[putting $x = ut$ so that $dx = udt$]

$$= \int_0^{\infty} u^{\mu+\lambda+p} g(u)du \int_0^{\infty} t^{p+\lambda-1} f(t)dt$$

$$= M\{g(x); 1+\lambda+\mu+p\}\, M\{f(x); p+\lambda\}$$

Hence

$$M\left\{x^{\lambda}\int_0^{\infty} u^{\mu} f\left(\frac{x}{u}\right) g(u)du\, ; p\right\} = F(p+\lambda)\, G(1+\lambda+\upsilon+p) \qquad ...(30)$$

Special cases : (i) Putting $\lambda = 0$ in (30), we get

$$M\left\{\int_0^{\infty} u^{\mu} f\left(\frac{x}{u}\right) g(u)du; p\right\} = F(p)\, G(1+\mu+p) \qquad ...(31)$$

(ii) Putting $\lambda = 0 = \mu$ in (30), we got

$$M\left\{\int_0^{\infty} f\left(\frac{x}{u}\right) g(u)du; p\right\} = F(p)\, G(1+p) \qquad ...(32)$$

(iii) Putting $\lambda = 0, \mu = -1$ in (31), we get

$$M\left\{\int_0^{\infty} f\left(\frac{x}{u}\right) g(u)\frac{du}{u}; p\right\} = F(p)\, G(p) \qquad ...(33)$$

The relation (33) is often used in its inverse form

$$M^{-1}\{F(p)\, G(p); x\} = \int_0^{\infty} f\left(\frac{x}{u}\right) g(u)\frac{du}{u}$$

Self-Learning Exercise - I :

Fill in the blanks :

1. $M\{x^2 f(3x); p\} = \ldots.$

2. $M\{e^{-ax};p\}=....$

3. $M\{\sin x;p\}=....$

4. $M\{\cos x;p\}=....$

5. $M\{f'''(x);p\}=....$

6. $M\{x^2 f''(x);p\}=....$

5.6 Excercise 5 (a)

1. If $M[f(x);p]=F(p)$, then prove that

$$M\left\{\frac{1}{x^m} f\left(\frac{1}{x^n}\right);p\right\}=\frac{1}{n}F\left(\frac{m-p}{n}\right)$$

2. Find Mellin transform of $x^{\alpha}\left(1+x^{a}\right)^{-b}$. Mention the conditions of validity.

$$\left[\ \textbf{Ans.}\ \frac{M\left(\frac{p+\alpha}{a}\right)M\left(b-\frac{p+\alpha}{a}\right)}{a\Gamma(b)};\ 0<\operatorname{Re}(p+\alpha)<\operatorname{Re}(ab)\ \right]$$

3. If $M\{f(x);p\}=F(p)$, then prove that

$$M\left\{x^2\frac{d^2 f}{dx^2}+x\frac{df}{dx};p\right\}=p^2F(p)$$

4. If m is a positive integer and $\alpha\neq 0$, then prove that

$$M\left\{\left(\frac{d}{dx}x\right)^m f(x);p\right\}=(-p)^m F(p)$$

where $M\{f(x);p\}=F(p)$

5. Prove that

$$M\left\{x^m f\left(ax^{-n}\right);p\right\}=\frac{1}{n}a^{(p+m)/n}F\left(-\frac{p+m}{n}\right)$$

where $F(p)=M\{f(x);p\}$

6. Prove that

$$M\left\{\left(x^{1-\alpha}\frac{d}{dx}\right)^m x^{\beta}g(x);p\right\}=(-1)^m\frac{\alpha^m\Gamma(p/\alpha)}{\Gamma\left(\frac{p}{\alpha}-m\right)}$$

$$.\,M\{g(x);p-m\alpha+\beta\}$$

7. Find mellin transforms of $x^m = H(x-a)$ and $E_i(x)$

$$\left[\ \textbf{Ans.}\ -\frac{a^{p+m}}{p+m},\mathrm{Re}(p)<-\mathrm{Re}(m)\ \text{and}\ p^{-1}\Gamma(p)\ \right]$$

8. Prove that

$$M\{W_\mu[f(u);p]\}=\frac{\Gamma(p)}{\Gamma(p+u)}F(p+u)$$

where $M\{f(x);p\}=F(p)$ and

$$W_\mu[f(u);x]=\frac{1}{\Gamma(\mu)}\int_x^\infty(u-x)^{\mu-1}f(u)\,du$$

9. Starting from the definition

$$P_n(x)=\frac{(2x)^n(1/2)_n}{n!(p-n)}\,{}_2F_1\left(\frac{1}{2}-\frac{n}{2},-\frac{n}{2};\frac{1}{2}-n;\frac{1}{x_2}\right)$$

of the Legendre polynomial $P_n(x)$, show heat

$$M\left[P_n\left(\frac{1}{x}\right)H(1-x);p\right]=2^{p-1}\pi^{-1/2}\frac{\Gamma[(p+n+1)/2]\Gamma[(p-n)/2]}{\Gamma(p+1)}$$

where $H(x)$ is the Heaviside unit funcitoin.

10. Find $M\{\Delta_2 f(\rho,\theta);p\}$ where

$$\Delta_2 f(\rho,\theta)=\frac{\partial f}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial f}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\rho^2}$$

represents the two dimensional. Laplace operator in the plane of polar co-ordinates ρ and θ.

$$\left[\ \textbf{Ans.}\ \left[(p-2)^2+\frac{\partial^2}{\partial\theta^2}\right]f^*(p-2,\theta)\ \right]\quad \text{where } f^*(p,\theta)=m\{f(\rho,\theta);\rho\to p\}$$

11. If n is a positive integer, then show that

$$M\{x^n P_n(x)H(1-x);p\}=(p+n)^{-1}\,{}_2F_1\left(-n,\,n+1,\,n+p+1;\frac{1}{2}\right)$$

5.7 Inverse Mellin Transform

(i) **Definition :** Let $M\{f(x);p\}=F(p)$

then $f(x)$ is called the inverse Mellin Transform of $F(p)$ and we write

$$M^{-1}\{F(p);x\}=f(x)$$

(ii) **The Mellin Inversion Theorem :**

Statement : If the integral $\int_0^{\infty} x^{k-1}|f(x)|\,dx$ is bounded for some $k>0$ and if

$$M\{f(x);p\}=\int_0^{\infty} x^{p-1} f(x)\,dx=F(p)$$

then the Mellin inverse transform $f(x)$ of $F(p)$ is given by

$$f(x)=M^{-1}\{F(p);x\}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-p}F(p)\,dp \qquad \text{where } c>k$$

Proof : By the definition, the complex fourier transform $G(\xi)$ of $g(t)$, $-\infty<t<\infty$, is given by

$$G(\xi)=\int_{-\infty}^{\infty} e^{i\xi t} g(t)\,dt \qquad ...(34)$$

and the inverse complex fourier transform $g(t)$ of $G(\xi)$ is given by

$$g(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi t} G(\xi)\,d\xi \qquad ...(35)$$

If we put $p=c+i\,\xi$ and $t=\log x$ in (34) and (35), we get

$$G(ic-ip)=\int_0^{\infty} x^{p-c} g(\log x)\,\frac{dx}{x}$$

or $$G(ic-ip)=\int_0^{\infty} x^{p-1}x^{-c} g(\log x)\,dx \qquad ...(36)$$

and $$g(\log x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(ic-ip)\,x^{c-p}\,dp \qquad ...(37)$$

Now writing $x^{-c}g(\log x)=f(x)$ and $G(ic-ip)=F(p)$

Then (36) and (37) become

$$F(p)=\int_0^{\infty} x^{p-1} f(x)\,dx \qquad ...(38)$$

and $$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-p}F(p)\,dp \qquad ...(39)$$

5.8 Convolution or Faltung Theorem for the Mellin Transform

Statement : If $F(p)$ and $G(p)$ are the Mellin Transforms of the functions $f(x)$ and $g(x)$. Then

$$M\{f(x)g(x);p\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)\,G(p-z)\,dz$$

Proof : By the definition of the Mellin Transform, we have

$$M\{f(x)g(x);p\} = \int_0^{\infty} x^{p-1} f(x)\,g(x)\,dx$$

$$= \int_0^{\infty} x^{p-1} g(x)\left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-z} F(z)\,dz\right] dx \qquad \text{[using (39)]}$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)\left[\int_{\infty}^{\infty} x^{p-z-1}\, g(x)\,dx\right] dz$$

[changing the order of integrations]

or $$M\{f(x)\;g(x);p\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)\,G(p-z)\;dz \qquad \text{[Using (38)]}$$

Special Case : Taking $p=1$ in the main Theorem, we get

$$M\{f(x)\;g(x);1\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)\,G(1-z)\;dz$$

or $$\int_0^{\infty} f(x)\,g(x)\;dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)\,G(1-z)\;dz \qquad ...(40)$$

(40) is also knwon as **Parseval's Theorem.**

Example 9 : Obtain the Mellin Transform of

$$f(x) = \frac{2\left(1-x^2\right)^{\lambda-1} H(1-x)}{\Gamma(\lambda)},\;\; g(x) = \frac{2\left(1-a^2x^2\right)^{\mu-1} H(1-ax)}{\Gamma(\mu)}$$

with $\lambda > 0$, $\mu > 0$, $0 < a < 1$. Hence or otherwise establish that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{z}{2}\right)\Gamma\left(\alpha-\frac{z}{2}\right)a^{z}}{\Gamma\left(\beta+\frac{z}{2}\right)\Gamma\left(\gamma-\frac{z}{2}\right)}\,dz = \frac{2a^{2\alpha}}{\Gamma(\alpha+\beta)\,\Gamma(\gamma-\alpha)}$$

$$\times {}_2F_1\left(\alpha, 1+\alpha-\gamma; \alpha+\beta; a^2\right)$$

with $0 < \alpha \le 1$, $0 < \alpha < \gamma$, $\beta > 0$

Solution : By the definition of Mellin transform, we have

$$F(z) = M\{f(x); z\} = \int_0^\infty x^{z-1} f(x)\, dx$$

$$= \int_0^\infty x^{z-1} \frac{2\left(1-x^2\right)^{\lambda-1} H(1-x)}{\Gamma(\lambda)}\, dx$$

$$= \frac{2}{\Gamma(\lambda)} \int_0^1 x^{z-1}\left(1-x^2\right).1\, dx$$

$$= \frac{2}{\Gamma(\lambda)} \int_0^{\pi/2} (\sin t)^{z-1}\left(\cos^2 t\right)^{\lambda-1} \cos t\, dt \quad \text{[putting } x = \sin t \text{ so that } dx = \cos t\, dt \text{]}$$

$$= \frac{2}{\Gamma(\lambda)} \int_0^{\pi/2} (\sin t)^{z-1} (\cos t)^{2\lambda-1}\, dt$$

$$= \frac{2}{\Gamma(\lambda)} \frac{\Gamma\left(z/2\right)\Gamma(\lambda)}{\Gamma\left(\lambda+\frac{z}{2}\right)}$$

or $$F(z) = \frac{\Gamma\left(z/2\right)}{\Gamma\left(\lambda+\frac{z}{2}\right)}, \quad (\lambda > 0) \qquad \text{...(41)}$$

Similarly $G(z) = M\{g(x); z\} = \int_0^\infty x^{z-1} g(x)\, dx$

$$= \int_0^\infty x^{z-1} \frac{2\left(1-a^2x^2\right)^{\mu-1}}{\Gamma(\mu)} H(1-x)\, dx$$

$$= \frac{2}{\Gamma(\mu)} \int_0^{1/a} x^{z-1}\left(1-a^2x^2\right)^{\mu-1} .1.dx$$

$$= \frac{2}{\Gamma(\mu)} \int_0^{\pi/2} \left(\frac{\sin\theta}{a}\right)^{z-1} \left(\cos^2\theta\right)^{\mu-1} \frac{\cos\theta}{a}\, d\theta$$

[putting $ax = \sin\theta$, so that $dx = \dfrac{\cos\theta}{a}\, d\theta$]

$$= \frac{2}{\Gamma(\mu)\, a^{z}} \int_0^{\pi/2} (\sin\theta)^{z-1} (\cos\theta)^{2\mu-1}\, d\theta$$

$$= \frac{2}{a^{z}\, \Gamma(\mu)}\, \frac{\Gamma(z/2)\, \Gamma(\mu)}{\Gamma(\mu + z/2)}$$

or $$G(z) = \frac{a^{-z}\, \Gamma(z/2)}{\Gamma(\mu + z/2)}, \qquad \mu > 0,\ 0 < a < 1 \qquad ...(41)$$

Using there pairs in convolution theorem, we have

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(z/2)}{\Gamma(\lambda + z/2)}\, a^{z-p}\, \frac{\Gamma\left(\frac{p-z}{2}\right)}{\Gamma\left(\mu + \frac{p-z}{2}\right)}\, dz$$

$$= \frac{4}{\Gamma(\lambda)\,\Gamma(\mu)} \int_0^1 x^{p-1} \left(1-x^2\right)^{\lambda-1} \left(1-a^2x^2\right)^{\mu-1} dx \qquad [\because a \le 1]$$

$$= \frac{4}{\Gamma(\lambda)\,\Gamma(\mu)} \int_0^1 x^{p-1} \left(1-x^2\right)^{\lambda-1} \sum_{r=0}^{\infty} \frac{(1-\mu)_r (ax)^{2r}}{r!}\, dx$$

$$\left[\because \left(1-a^2x^2\right) = \sum_{r=0}^{\infty} \frac{(1-\mu)_r}{r!}\, (ax)^{2r}\right]$$

$$= \frac{4}{\Gamma(\lambda)\,\Gamma(\mu)} \sum_{r=0}^{\infty} \frac{(1-\mu)_r}{r!}\, a^{2r} \int_0^1 x^{p+2r-1} \left(1-x^2\right)^{\lambda-1} dx$$

$$= \frac{4}{\Gamma(\lambda)\,\Gamma(\mu)} \sum_{r=0}^{\infty} \frac{(1-\mu)_r}{r!}\, a^{2r} \int_0^{\pi/2} (\sin\phi)^{p+2r-1} (\cos\phi)^{2\lambda-2} \cos\phi\, d\phi$$

[putting $x = \sin\phi$ so that $dx = \cos\phi\, d\phi$]

$$= \frac{4}{\Gamma(\lambda)\,\Gamma(\mu)} \sum_{r=0}^{\infty} \frac{(1-\mu)_r}{r!}\, a^{2r}\, \frac{\Gamma\left(\frac{p}{2}+r\right)\Gamma(\lambda)}{2\,\Gamma\left(\frac{p}{2}+\lambda+r\right)}$$

$$= \frac{4}{\Gamma(\lambda)\,\Gamma(\mu)} \sum_{r=0}^{\infty} \frac{(1-\mu)_r}{r!}\, a^{2r} \, \frac{\left(\frac{p}{2}\right)_r \Gamma\left(\frac{p}{2}\right) \Gamma(\lambda)}{\left(\frac{p}{2}+\lambda\right)_r \Gamma\left(\frac{p}{2}+\lambda\right)}$$

$$= \frac{2\,\Gamma\left(p/2\right)}{\Gamma\left(\frac{p}{2}+\lambda\right)} \sum_{r=0}^{\infty} \frac{(1-\mu)_r \left(\frac{p}{2}\right)_r}{r!\left(\frac{p}{2}+\lambda\right)_r} \left(a^2\right)^r$$

or $$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(z/2\right)}{\Gamma\left(\lambda+\frac{z}{2}\right)}\, a^{z-p}\, \frac{\Gamma\left(\frac{p-z}{2}\right)}{\Gamma\left(\mu+\frac{p-z}{2}\right)}\, dz$$

$$= 2\,\frac{\Gamma\left(p/2\right)}{\Gamma\left(\frac{p}{2}+\lambda\right)}\; {}_2F_1\left(1-\mu, \frac{p}{2}; \frac{p}{2}+\lambda; a^2\right) \qquad \text{...(42)}$$

Now taking $\frac{p}{2}=\alpha$, $\lambda=\beta$, $\mu+\alpha=\gamma$, (42) reduces to

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(p/2\right)\Gamma\left(\alpha-z/2\right)}{\Gamma\left(\beta+\frac{z}{2}\right)\Gamma\left(\gamma-\frac{z}{2}\right)}\, dz$$

$$= \frac{2a^{2\alpha}\,\Gamma(\alpha)}{\Gamma(\alpha+\beta)\,\Gamma(\gamma-\alpha)}\; {}_2F_1\left(\alpha, 1+\alpha-\gamma; \alpha+\beta; a^2\right)$$

with $0<\alpha\le 1$, $0<\alpha<\gamma$, $\beta>0$.

Example 10 : Find the Mellin Transform of $\sin x$ and show that

$$M^{-1}\left\{\Gamma(p)\sin\left(\frac{p\pi}{2}\right) f^*(1-p); x\right\} = \sqrt{\frac{\pi}{2}}\; F_s\{f(t); x\}$$

where $f^*(p) = M\{f(t); p\}$

Solution : We know that

$$F_s\{x^{p-1}; p\} = \sqrt{\frac{2}{\pi}}\; p^{-\rho}\; \Gamma(\rho)\sin\left(\frac{\pi\rho}{2}\right), \qquad 0<\rho<1$$

$$\Rightarrow \quad \sqrt{\frac{2}{\pi}} \int_0^\infty x^{p-1} \sin p x \; dx = \sqrt{\frac{2}{\pi}} \; p^{-\rho} \; \Gamma(\rho) \sin\left(\frac{\pi \rho}{2}\right)$$

$$\therefore \quad \int_0^\infty x^{p-1} \sin x \; dx = \Gamma(p) \sin\left(\frac{\pi p}{2}\right), \qquad 0 < Re(p) < 1 \qquad ...(43)$$

From (43) and property (I) for Mellin transform, we have

$$\int_0^\infty x^{p-1} \sin t x \; dx = t^{-p} \, \Gamma(p) \sin\left(\frac{\pi p}{2}\right) \qquad ...(44)$$

Now $\quad M\left[F_s f(t); p\right] = M\left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin t x \; dt; p\right]$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x^{p-1} \left\{\int_0^\infty f(t) \sin t x \; dt\right\} dx$$

Changing the order of integration, we get

$$M\left[F_s\{f(t);x\};p\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \left\{\int_0^\infty x^{p-1} \sin t x \; dx\right\} dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty t^{-p} \, \Gamma(p) \sin\left(\frac{\pi p}{2}\right) f(t) dt$$

$$= \sqrt{\frac{2}{\pi}} \, \Gamma(p) \sin\left(\frac{\pi p}{2}\right) f^*(1-p)$$

On taking inverse Mellin transform, we get

$$M^{-1}\left\{\Gamma(p) \sin\left(\frac{p\pi}{2}\right) f^*(1-p); x\right\} = \sqrt{\frac{\pi}{2}} F_s\{f(t); x\}$$

Self-Learning Exercise - II

1. State the Mellin inversion theorem.

2. State the convolution theorem for Mellin transform.

5.9 Summary

In this unit you have studied the Mellin transform and its inversion theorem. You have also studied the elementry properties and important results concerning this transform. These reulst were illustrated with the help of examples.

5.10 Answers to Self-Learning Exercises

Exercise - I

1. $3^{-(p+2)} M\{f(x); p+2\}$ 2. $a^{-p}\ \Gamma(p) \qquad (p>0)$

3. $\Gamma(p)\sin(p\pi/2) \qquad (0<Re(p)<1)$ 4. $\Gamma(p)\cos(p\pi/2) \qquad (0<Re(p)<1)$

5. $-\dfrac{\Gamma(p)}{\Gamma(p-3)}\ M\{f(x); p-3\}$ 6. $\dfrac{\Gamma(p+2)}{\Gamma(p)}\ M\{f(x); p\}$

5.11 Exercise 5 (b)

1. Show that

$$M^{-1}\left\{\cos\left(\frac{\pi p}{2}\right)\Gamma(p)\ f^*(1-p); x\right\} = \sqrt{\frac{\pi}{2}}\ F_c\{f(t); x\}$$

2. Show that

(i) $$M^{-1}\left\{\frac{\Gamma\left(p/2\right)}{\Gamma\left(\frac{p+1}{2}\right)}\ f^*(p); x\right\} = \frac{2}{\sqrt{\pi}}\int_x^{\infty}\frac{f(t)\ dt}{\sqrt{(t^2-x^2)}}$$

(ii) $$M^{-1}\left\{\frac{\Gamma\left(\frac{1-p}{2}\right)}{\Gamma\left(1-\frac{p}{2}\right)}\ f^*(p); x\right\} = \frac{2}{\sqrt{\pi}}\int_0^{x}\frac{f(t)\ dt}{\sqrt{(x^2-t^2)}}$$

where $f^*(p) = M\{f(x); p\}$

3. Prove that

$$M^{-1}\{\Gamma(p)\ F(1-p); x\} = L\{(f(t); x)\}$$

where $F(p) = M\{f(x); p\}$

Unit - 6
The Infinite Hakel Transform

Structure of the Unit

6.0 Objective

This unit deals with the definition and basic elementary properties of Hankel Transform. Inversion formula for Hankel transform and Hankel transform of derivatives of functions are also proved. In the end Parseval's theorem for Hankel transform is given.

6.1 Introduction

The Hankel Transform :

Definition :

The Hankel transform of the function $f(x)$, $0 < x < \infty$, denoted by $H_v\{f(x)\}$ or $F_v(p)$ or $F(p)$ and is defined as

$$H_v\{f(x); p\} = F_v(p) = \int_0^\infty x\, J_v(px)\, f(x)\, dx$$

Where $J_v(px)$ is the **Bessel function of the first kind of oreder** v**.**

Here $x\, J_v(px)$ is known as **Kernel** of the Hankel transform and H_v stands for **Hankel transformation operation** of order v.

6.2 Some Important Results for Bessel Functions

Since the kernel involves Bessel function, therefore the definition of Bessel function; its some important properties, recurrence relations and integrals involving Bessel functions are given for ready reference.

I. **Bessel Function** of first kind is defined as

$$J_v(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(x/2\right)^{v+2r}}{r!\,\Gamma(v+r+1)}$$

II. **Recurrence Relations for $J_v(x)$:**

(i) $x J_v'(x) = v J_v(x) - x J_{v+1}(x)$

(ii) $x J_v'(x) = -v J_v(x) + x J_{v-1}(x)$

(iii) $2 J_v'(x) = J_{v-1}(x) - J_{v+1}(x)$

(iv) $2 v J_v(x) = x\left[J_{v-1}(x) + J_{v+1}(x)\right]$

(v) $\dfrac{d}{dx}\left[x^{-v} J_v(x)\right] = -x^{-v} J_{v+1}(x)$

(vi) $\dfrac{d}{dx}\left[x^{v} J_v(x)\right] = x^{v} J_{v-1}(x)$

III. **Some Important Infinite Integrals Involving Bessel Functions :**

(i) $\int_0^\infty e^{-ax} J_0(px)\,dx = \left(a^2+p^2\right)^{-1/2}, \qquad a>0$

(ii) $\int_0^\infty e^{-ax} J_1(px)\dfrac{dx}{x} = \dfrac{\left(a^2+p^2\right)^{-1/2} - a}{p}$

(iii) $\int_0^\infty x\, e^{-ax} J_0(px)\,dx = a\left(a^2+p^2\right)^{-3/2}$

(iv) $\int_0^\infty x\, e^{-ax} J_1(px)\,dx = p\left(a^2+p^2\right)^{-3/2}$

(v) $\int_0^\infty e^{-ax} J_1(px)\,dx = \dfrac{1}{p} - \dfrac{a}{p\left(a^2+p^2\right)^{1/2}}$

(vi) $\int_0^\infty \sin a x\, J_0(px)\,dx = \begin{cases} 1/\sqrt{a^2-p^2}, & 0<p<a \\ 0, & p>a \end{cases}$

(vii) $\int_0^\infty \cos a x\, J_0(px)\,dx = \begin{cases} 1/\sqrt{a^2-p^2}, & p>a \\ 0, & 0<p<a \end{cases}$

6.3 Elementary Properties of Hankel Transform

I. Linearity Property :

Let $F_i(p)$ are Hankel transform of $f_i(x)$ for all $i = 1, 2, \ldots\ldots, n$ and C_i are any constants, then

$$H_v\{C_1 f_1(x) + C_2 f_2(x) + \ldots\ldots\ldots + C_n f_n(x); p\}$$

$$= C_1 H_v\{f_1(x); p\} + C_2 H_v\{f_2(x); p\} + \ldots\ldots\ldots + C_n H_v\{f_n(x); p\}$$

$$= C_1 F_1(p) + C_2 F_2(p) + \ldots\ldots\ldots + C_n F_n(p)$$

or $$H_v\left\{\sum_{r=1}^{n} C_r\, f_r(x); p\right\} = \sum_{r=1}^{n} C_r H_v\{f_r(x); p\}$$

Proof : We have

$$H_v\{C_1 f_1(x) + C_2 f_2(x) + \ldots\ldots + C_n f_n(x); p\}$$

$$= \int_0^\infty x\, J_v(px)\, \{C_1 f_1(x) + C_2 f_2(x) + \ldots\ldots + C_n f_n(x)\}\, dx$$

$$= C_1 \int_0^\infty x\, J_v(px)\, f_1(x)\, dx + C_2 \int_0^\infty x\, J_v(px)\, f_2(x)\, dx + \ldots\ldots + C_n \int_0^\infty x\, J_v(px)\, f_n(x)\, dx$$

$$= C_1 F_1(p) + C_2 F_2(p) + \ldots\ldots\ldots + C_n F_n(p)$$

II. Change of Scale Property :

If $H_v\{f(x); p\} = F_v(p)$, then

$$H_v\{f(ax); p\} = \frac{1}{a^2} F_v\left(p/a\right), \qquad a \neq 0$$

Proof : We have

$$H_v\{f(ax); p\} = \int_0^\infty x\, J_v(px)\, f(ax)\, dx \qquad \text{(by definition)}$$

$$= \int_0^\infty \frac{t}{a} J_v\left(\frac{pt}{a}\right) f(t) \frac{dt}{a} \qquad \left[\text{on putting } ax = t \text{ so that } dx = \frac{dt}{a}\right]$$

$$= \frac{1}{a^2} \int_0^\infty t\, J_v\left(\frac{p}{a} t\right) f(t)\, dt = \frac{1}{a^2} F_v\left(p/a\right)$$

Example 1 : Find the Hankel transform of

$$f(x)=\begin{cases}1, & 0<x<a, \quad v>0\\ 0, & x>a, \quad v=0\end{cases}$$

Solution : By definition, we have

$$H_0\{f(x);p\}=\int_0^\infty x\,J_0(px)\,f(x)\,dx$$

$$=\int_0^a x\,J_0(px)\,f(x)\,dx+\int_a^\infty x\,J_0(px)\,f(x)\,dx$$

$$=\int_0^a x\,J_0(px).1.dx+\int_a^\infty x\,J_0(px).0.dx$$

$$=\int_0^a x\,J_0(px)\,dx \qquad ...(1)$$

Now putting $v=1$ and replacing x by px in the recurrence relation II (vi) for Bessel function of § 6.2, we get

$$\frac{1}{p}\frac{d}{dx}\{px\,J_1(px)\}=px\,J_0(px)$$

or $$x\,J_0(px)=\frac{1}{p}\frac{d}{dx}\{x\,J_1(px)\} \text{ §} \qquad ...(2)$$

Using (2) in (1), we get

$$F(p)=\int_0^a\frac{1}{p}\frac{d}{dx}\{x\,J_1(px)\}dx=\frac{1}{p}\left[x\,J_1(px)\right]_0^a$$

or $F(p)=\dfrac{a}{p}J_1(pa)$, since $x\,J_1(px)\to 0$ as $x\to 0$

Example 2 : Find the Hankel transform of the function

$$f(x)=\begin{cases}a^2-x^2, & , 0<x<a\\ 0, & , \quad x>a\end{cases}$$

taking $x\,J_0(px)$ as the kernel.

Solution : By definition, we have

$$H_0\{f(x);p\}=\int_0^\infty x\,J_0(px)\,f(x)\,dx$$

$$=\int_0^a x\,J_0(px)\,f(x)\,dx+\int_a^\infty x\,J_0(px)\,f(x)\,dx$$

$$= \int_0^a x\, J_0(px)\, (a^2 - x^2)\, dx + \int_a^\infty x\, J_0(px).0.\, dx$$

$$= a^2 \int_0^a x\, J_0(px)\, dx - \int_0^a x^j\, J_0(px)\, dx + 0$$

$$= a^2 I_1 - I_2 \quad \text{(say)} \qquad ...(3)$$

Now using (2), we get

$$\therefore \qquad I_1 = \int_0^a x\, J_0(px)\, dx = \frac{1}{p}\int_0^a \frac{d}{dx}[x\, J_1(px)]\, dx = \left[\frac{x}{p} J_1(px)\right]_0^a$$

or $\qquad I_1 = \frac{a}{p} J_1(pa)$, $\qquad$ (since $x\, J_1(px) \to 0$ as $x \to 0$) $\qquad ...(4)$

and $\qquad I_2 = \int_0^a x^3 J_0(px)\, dx = \int_0^a x^2 . \frac{1}{p}\frac{d}{dx}[x\, J_1(px)]\, dx$, $\qquad$ [using (2)]

$$= \left[x^2 . \frac{x}{p} J_1(px)\right]_0^a - \int_0^a 2x . \frac{x}{p} J_1(px)\, dx \qquad \text{[Integration by parts]}$$

$$\therefore \qquad I_2 = \frac{a^3}{p}\, J_1(pa) - \frac{2}{p}\int_0^a x^2 J_1(px)\, dx \qquad ...(5)$$

In recurrence relation (vi) of Bessel function, replacing υ and x by p and px respectively, we get

$$\frac{1}{p}\frac{d}{dx}\left[p^2 x^2 J_2(px)\right] = p^2 x^2 J_1(px)$$

$$\Rightarrow \qquad x^2 J_1(px) = \frac{1}{p}\frac{d}{dx}\left[x^2 J_2(px)\right]$$

$$\therefore \qquad \int_0^a x^2 J_1(x)\, dx = \left[\frac{x^2}{p} J_2(px)\right]_0^a = \frac{a^2}{p} J_2(pa)$$

$\therefore$ From (5), we have

$$I_2 = \frac{a^3}{p} J_1(pa) - \frac{2a^2}{p^2} J_2(pa) \qquad ...(6)$$

Hence from (3), (4) and (6), we finally obtain

$$H_0\{f(x); p\} = \frac{a^2 . a}{p} J_1(pa) - \frac{a^3}{p} J_1(pa) + \frac{2a^2}{p^2} J_2(pa)$$

$$= \frac{2a^2}{p} J_2(pa)$$

Example 3 : Find the Hankel transform of

(i) $\dfrac{\cos a x}{x}$ (ii) $\dfrac{\sin a x}{x}$,

taking $x J_0(px)$ as the kernel.

Solution : (i) By definition, we have

$$H_0\left\{\frac{\cos a x}{x}; p\right\} = \int_0^\infty \frac{\cos a x}{x} x J_0(px)\,dx$$

$$= \int_0^\infty \cos a x \, J_0(px)\,dx$$

$$= \text{Real part of } \int_0^\infty e^{-iax} J_0(px)\,dx$$

$$= \text{Real part of } \left(\mathrm{i}^2 a^2 + p^2\right)^{-1/2} \qquad \text{[by result (1) of 6.2 (III)]}$$

$$= \text{Real part of } \left(p^2 - a^2\right)^{-1/2}$$

$$= \begin{cases} \left(p^2 - a^2\right)^{-1/2}, & , \quad p > a \\ 0 & , \quad , \; 0 < p < a \end{cases}$$

$$H_0\left\{\frac{\sin a x}{x}; p\right\} = \int_0^\infty \frac{\sin a x}{x} . x J_0(px)\,dx$$

$$= \int_0^\infty \sin x . J_0(px)\,dx$$

$$= -\left[\text{Imaginary part of } \int_0^\infty e^{-iax} J_0(px)\,dx\right]$$

$$= -\left[\text{Imaginary part of } \left(\mathrm{i}^2 a^2 + p^2\right)^{-1/2}\right] \qquad \text{[by result (1) of 6.2 (III)]}$$

$$= -\left[\text{Imaginary part of } \left(p^2 - a^2\right)^{-1/2}\right]$$

$$= \begin{cases} 0, & , \quad p > a \\ \left(a^2 - p^2\right)^{-1/2}, & , \; 0 < p < a \end{cases}$$

Example 4 : Find the Hankel transform of the function

$$f(x) = \begin{cases} x^n, & 0 < x < a \\ 0\,, & x > a \end{cases} \qquad (n > -1)$$

taking $x J_n(px)$ as the Kernel.

Solution : By definition, we have

$$H_n\{f(x);p\} = \int_0^\infty x\,J_n(px)\,f(x)\,dx$$

$$= \int_0^a x\,J_n(px).x^n\,dx + \int_0^\infty x\,J_n(px).0.dx$$

$$= \int_0^a x^{n+1} J_n(px)\,dx \qquad ...(7)$$

Now, by recurrence relation of Bessel function of 6.2 II (vi) for $J_n(x)$, we have

$$\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$$

Replacing n by $(n+1)$ and writing px for x, we get

$$\frac{1}{p}\frac{d}{dx}\{x^{n+1} J_{n+1}(px)\} = x^{n+1} J_n(px)$$

Using this result in (7), we get

$$H_n\{f(x);p\} = \int_0^a \frac{1}{p}\frac{d}{dx}\{x^{n+1} J_{n+1}(px)\}\,dx$$

$$= \frac{1}{p}\left[x^{n+1} J_{n+1}(px)\right]_0^a = \frac{a^{n+1}}{p} J_{n+1}(pa)$$

6.4 Relation Between Hankel and Laplace Transform

By the definition of Hankel transform, we have

$$H_v\{e^{-ax} f(x);p\} = \int_0^\infty x\,J_v(px)\,e^{-ax} f(x)\,dx$$

$$= \int_0^\infty e^{-ax}\{x\,J_v(px)\,f(x)\}\,dx = L\{x\,J_v(px)\,f(x);a\}$$

Example 5 : Find the Hankel transform of $x^v e^{-ax}$, taking $x\,J_v(px)$ as the kernel.

Solution : By definition, we have

$$H_v\{x^v e^{-ax};p\} = L\{x^{v+1} J_v(px);a\}$$

$$= \int_0^\infty e^{-ax} x^{v+1} J_v(px)\,dx$$

$$= \int_0^\infty e^{-ax} x^{v+1} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{xp}{2}\right)^{v+2r} dx$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{p}{2}\right)^{v+2r} \int_0^\infty e^{-ax} x^{2v+2r+1} dx$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{p}{2}\right)^{v+2r} \frac{\Gamma(2v+2r+2)}{a^{2v+2r+2}}$$

$$= \left(\frac{p}{2}\right)^{v} \frac{1}{a^{2v+2}} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{p}{2a}\right)^{2r} \frac{\Gamma(v+r+1)\,\Gamma\left(v+r+\frac{3}{2}\right)}{\sqrt{\pi}}, 2^{2v+2r+1}$$

[using duplication formula for gamma function]

$$= \frac{p^v 2^{v+1}}{\sqrt{\pi}\; a^{2v+2}} \Gamma\left(v+\frac{3}{2}\right) \sum_{r=0}^{\infty} \frac{\left(v+\frac{3}{2}\right)_r}{r!} \left(-\frac{p^2}{a^2}\right)^r$$

$$= \frac{p^v 2^{v+1}}{\sqrt{\pi}\; a^{2v+2}} \Gamma\left(v+\frac{3}{2}\right) \left(1+\frac{p^2}{a^2}\right)^{-v-\frac{3}{2}}$$

$$= \frac{p^v 2^{v+1}}{\sqrt{\pi}\; a^{2v+2}} \Gamma\left(v+\frac{3}{2}\right) \left(1+\frac{p^2}{a^2}\right)^{-v-\frac{3}{2}}$$

Example 6 : If $H_v\{f(x);p\} = \int_0^{\infty} f(x) J_v(px)\,(xp)^{1/2}\,dx$, $p>0$, then show that

$$H_v\left\{x^{v-\frac{1}{2}} e^{-ax}; p\right\} = \frac{2^v \Gamma\left(v+\frac{1}{2}\right) p^{v+1/2}}{\sqrt{\pi}\,\left(a^2+p^2\right)^{v+\frac{1}{2}}}$$

where $\operatorname{Re}(a)>0$ and $\operatorname{Re}(v)>\frac{1}{2}$.

Solution : By the given definition of Hankel transform, we have

$$H_v\left\{x^{v-\frac{1}{2}}\; e^{-ax}; p\right\} = \int_0^{\infty} x^{v-\frac{1}{2}}\; e^{-ax} J_v(px)\,(xp)^{1/2}\,dx$$

$$= \int_0^{\infty} p^{1/2} x^v e^{-ax} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{px}{2}\right)^{v+2r} dx$$

$$= p^{1/2} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{p}{2}\right)^{v+2r} \int_0^{\infty} x^{2v+2r}\, e^{-ax} dx$$

$$= p^{1/2} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{p}{2}\right)^{2r} \frac{\Gamma(2v+2r+1)}{a^{2v+2r+1}}$$

$$= \frac{p^{v+1/2}}{2^v a^{2v+1}} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{p}{2a}\right)^{2r} \frac{\Gamma\left(v+r+\frac{1}{2}\right)\Gamma(v+r+1)\,2^{2v+2r}}{\sqrt{\pi}}$$

$$= \frac{2^v\,\Gamma\left(v+\frac{1}{2}\right)p^{v+1/2}}{\sqrt{\pi}\;a^{2v+1}} \sum_{r=0}^{\infty} \frac{\left(v+\frac{1}{2}\right)_r}{r!} \left(-\frac{p^2}{a^2}\right)^r$$

$$= \frac{2^v\,\Gamma\left(v+\frac{1}{2}\right)p^{v+1/2}}{\sqrt{\pi}\;a^{2v+1}} \left(1+\frac{p^2}{a^2}\right)^{-v-\frac{1}{2}}$$

$$= \frac{2^v\;\Gamma\left(v+\frac{1}{2}\right)\,p^{v+\frac{1}{2}}}{\sqrt{\pi}\;\left(a^2+p^2\right)^{v+\frac{1}{2}}}$$

Example 7 : Prove that if $v > -\frac{1}{2}$, then

$$H_v\left\{x^{v-1}e^{-ax};p\right\} = L\left\{x^v J_v(px);a\right\} = \frac{2^v\,p^v\,\Gamma\left(v+\frac{1}{2}\right)}{\sqrt{\pi}\;\left(a^2+p^2\right)^{v+\frac{1}{2}}}$$

Solution : By definition of Hankel transform, we have

$$H_v\left\{x^{v-1}e^{-ax};p\right\} = \int_0^{\infty} x\,J_v(px)\,x^{v-1}e^{-ax}\,dx$$

$$= \int_0^{\infty} x^v\,e^{-ax} J_v(px)\,dx$$

$$= \int_0^{\infty} e^{-ax} x^v \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{px}{2}\right)^{v+2r} dx$$

[by definition of Bessel function]

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{p}{2}\right)^{v+2r} \int_0^{\infty} e^{-ax} x^{2v+2r} dx$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{p}{2}\right)^{v+2r} \frac{\Gamma(2v+2r+1)}{a^{2v+2r+1}}$$

$$= \left(\frac{p}{2}\right)^v \frac{1}{a^{2v+1}} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{p}{2a}\right)^{2r} \frac{\Gamma\left(v+r+\frac{1}{2}\right)\Gamma(v+r+1)\,2^{2v+2r}}{\sqrt{\pi}}$$

[using duplication formula for gamma function]

$$= \frac{p^v 2^v\ \Gamma\left(v+\frac{1}{2}\right)}{a^{2v+1}\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{\left(v+\frac{1}{2}\right)_r}{r!} \left(-\frac{p^2}{a^2}\right)^{-r}$$

$$= \frac{p^v 2^v\ \Gamma\left(v+\frac{1}{2}\right)}{a^{2v+1}\sqrt{\pi}} \left(1+\frac{p^2}{a^2}\right)^{-v-\frac{1}{2}}$$

$$= \frac{p^v 2^v\,\Gamma\left(v+\frac{1}{2}\right)}{\sqrt{\pi}\,\left(a^2+p^2\right)^{v+\frac{1}{2}}}$$

Example 8 : Prove that

$$H_v\left\{e^{-px^2/4} f(x); s\right\} = 2L\left\{f\left(2\sqrt{x}\right) J_v\left(2s\sqrt{x}\right); p\right\}.$$

Deduce that

$$H_v\left\{x^v e^{-px^2/4}; s\right\} = \frac{2^{v+1} s^v}{p^{v+1}} e^{-s^2/p}$$

and hence that

(i) $$H_v\left\{x^v e^{-x^2/a^2}; s\right\} = \left(\frac{a^2}{2}\right)^{v+1} e^{-a^2 s^2/4}$$

(ii) $$H_v\left\{x^v\ e^{-x^2/2}; s\right\} = s^v\ e^{-s^2/2}$$

Solution : By definition of Hankel transform, we have

$$H_v\left\{e^{-px^2/4} f(x); s\right\} = \int_0^{\infty} x\,J_v(sx)\,e^{-px^2/4} f(x)\,dx$$

$$= 2\int_0^{\infty} f\left(2\sqrt{u}\right) J_v\left(2s\sqrt{u}\right)\,e^{-pu}\,du \quad \left[putting\ \frac{x^2}{4} = u, xdx = 2du\right]$$

$$= 2\int_0^{\infty} e^{-pu}\left\{f\left(2\sqrt{x}\right) J_v\left(2s\sqrt{x}\right)\,dx\right.$$

$$= 2\, L\left\{ f\left(2\sqrt{x}\right) J_v\left(2s\sqrt{x}\right); p\right\} \qquad ...(8)$$

Deduction : Taking $f(x) = x^v$ in (8), we get

$$H_v\left\{x^v e^{-px^2/4}; s\right\} = 2L\left\{2^v x^{v/2} J_v\left(2s\sqrt{x}\right); p\right\}$$

$$= 2\int_0^{\infty} e^{-px}\, 2^v x^{v/2}\, J_v\left(2s\sqrt{x}\right) dx$$

$$= 2^{v+1} s^v \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} = s^{2r} \int_0^{\infty} e^{-px} x^{v+r} dx$$

$$= 2^{v+1} s^v \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)}\; s^{2r}\; \frac{\Gamma(v+r+1)}{p^{v+r+1}}$$

$$= \frac{2^{v+1}}{p^{v+1}}\; s^v \sum_{r=0}^{\infty} \frac{1}{r!} \left(-\frac{s^2}{p}\right)^r$$

$$= \frac{2^{v+1} s^v}{p^{v+1}}\; e^{-s^2/p}$$

6.5 Inversion Formula for the Hankel Transform

Statement : If $F_v(p)$ is the Hankel transform of the function $f(x)$ i.e.

$$F_v(p) = \int_0^{\infty} x\, J_v(px)\, f(x)\, dx, \text{ then}$$

$$f(x) = \int_0^{\infty} p\, J_v(px)\, F_v(p)\, dp$$

is known as the inversion formula for the Hankel transform of $F_v(p)$ and written as

$$f(x) = H^{-1}\left\{F_v(p); x\right\}$$

Proof : If $F(p)$ is the Fourier complex transform of the function $f(x)$; that is, if

$$F(p) = \int_{-\infty}^{\infty} f(x)\, e^{ipx} dx$$

then $f(x)$ is given by the inversion formula

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(p) e^{-ipx} dp$$

These results for the complex Fourier transform can be extended to cover functions of two variables. Thus if

$$F(s,t)=\int_{-\infty}^{\infty} f(x,y)\, e^{i(sx+ty)}dx\,dy\,, \qquad ...(9)$$

Then $$f(x,y)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty} F(s,t)\, e^{-i(sx+ty)}ds\,dt \qquad ...(10)$$

Putting $x=r\cos\theta$, $y=r\sin\theta$, $s=p\cos\alpha$, $t=p\sin\alpha$, the equations (9) and (10) become

$$F(p,\alpha)=\int_0^{\infty} r\,dr\int_0^{2\pi} f(r,\theta)e^{i\,pr\cos(\theta-\alpha)}\,d\theta \qquad ...(11)$$

and $$f(r,\theta)=\frac{1}{4\pi^2}\int_0^{\infty} p\,dp\int_0^{2\pi} F(p,\alpha)e^{-i\,pr\cos(\theta-\alpha)}\,d\alpha \qquad ...(12)$$

Let we choose $f(r,\theta)=f(r)e^{-iv\theta}$ ...(13)

Then equation (11) becomes

$$F(p,\alpha)=\int_0^{\infty} f(r)r\,dr\int_0^{2\pi} e^{i\{-v\theta+pr\cos(\theta-\alpha)\}}\,d\theta \qquad ...(14)$$

Now, putting $\phi=\alpha-\theta-\pi/2$, we get

$$\int_0^{2\pi} e^{i\{-v\theta+pr\cos(\theta-\alpha)\}}d\theta=\int_0^{2\pi} e^{i\{v(\phi-\alpha+\pi/2)+pr\cos(-\phi-\pi/2)\}}\,d\phi$$

$$=\int_0^{2\pi} e^{iv(\pi/2-\alpha)}\; e^{i\{v\phi+pr\cos(\phi+\pi/2)\}}\,d\phi$$

$$=e^{iv(\pi/2-\alpha)}\int_0^{2\pi} e^{i\{v\phi-pr\sin\phi\}}\,d\phi$$

$$=e^{iv(\pi/2-\alpha)}2\pi\,J_v(pr) \qquad \left[\because\; J_v(pr)=\frac{1}{2\pi}\int_0^{2\pi} e^{i\{v\phi-pr\sin\phi\}}\,d\phi\right]$$

Putting this value in (14), we get

$$F(p,\alpha)=\int_0^{\infty}\left[f(r)r\,e^{iv(\pi/2-\alpha)}.2\pi\,J_v(pr)\right]dr$$

$$=2\pi\,e^{iv(\pi/2-\alpha)}\int_0^{\infty} f(r)r\,J_v(pr)\,dr \qquad ...(15)$$

If we denote the Hankel transform $f(r)$ by $F_v(p)$, then by definition, we have

$$F_v(p)=\int_0^{\infty} f(r)r\,J_v(pr)\,dr \qquad ...(16)$$

Hence from (15), we see that

$$F(p,\alpha) = 2\pi e^{iv\left(\pi/2-\alpha\right)} F_v(p) \qquad ...(17)$$

Using (13) and (17) in (12), we get

$$f(r)e^{-iv\theta} = \frac{1}{4\pi^2} \int_0^\infty p\ dp \int_0^{2\pi} 2\pi e^{iv\left(\pi/2-\alpha\right)} F_v(p) e^{-i\,pr\cos(\theta-\alpha)}\, d\alpha$$

or $$f(r)e^{-iv\theta} = \frac{1}{2\pi} \int_0^\infty p\,F_v(p)\,dp \int_0^{2\pi} e^{i\left\{v\left(\pi/2-\alpha\right)-pr\cos(\theta-\alpha)\right\}}\, d\alpha \qquad ...(18)$$

Now, putting $\psi = \theta - \alpha + \dfrac{\pi}{2}$, we get

$$\int_0^{2\pi} e^{i\left\{v\left(\pi/2-\alpha\right)-pr\cos(\theta-\alpha)\right\}} d\alpha = \int_0^{2\pi} e^{iv(\psi-\theta)-i\,pr\cos\left(\psi-\pi/2\right)} d\psi$$

$$= e^{-iv\theta} \int_0^{2\pi} e^{i\{v\psi - pr\sin\psi\}} d\psi = e^{-iv\theta}\, 2\pi\, J_v(pr)$$

Therefore (18) reduces to

$$f(r)e^{-iv\theta} = \frac{1}{2\pi}\int_0^\infty p\,F_v(p)\,e^{-iv\theta}\,2\pi\,J_v(pr)\,dp$$

or $$f(r)e^{-iv\theta} = e^{-iv\theta}\int_0^\infty p\,F_v(p)\,J_v(pr)\,dp$$

or $$f(r) = \int_0^\infty p\,F_v(p)\,J_v(pr)\,dp$$

which is the required inversion formula for the Hankel transform.

Example 9 : Prove that

$$H_v\left\{x^v\left(a^2-x^2\right)^{\mu-v-1} U(a-x); p\right\} = 2^{\mu-v-1}\Gamma(\mu-v)\,p^{v-\mu}a^{\mu}\,J_\mu(pa) \quad a>0,\ \mu>v>0$$

Hence deduce

(i) $$H_v\left\{x^v\,U(a-x); p\right\} = \frac{a^{v+1}}{p}\,J_{v+1}(pa),\ a>0 \text{ and}$$

(ii) $$H_v\left\{\frac{x^v\,U(a-x)}{\sqrt{a^2-x^2}}; p\right\} = \sqrt{\frac{\pi}{2p}}\; a^{\frac{a^{v+1}}{2}}\; J_{v+\frac{1}{2}}(pa)$$

Proof : By definition, we have

$$H_v\left\{x^v\left(a^2-x^2\right)^{\mu-v-1}U(a-x);p\right\} = \int_0^\infty x\,J_v(px)\,x^v\left(a^2-x^2\right)^{\mu-v-1}H(a-x)\,dx$$

$$= \int_0^a x^{v+1}\left(a^2-x^2\right)^{-\mu-v-1} J_v(px).1.dx$$

$$= \int_0^a x^{v+1}\left(a^2 - x^2\right)^{\mu-v-1} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{px}{2}\right)^{v+2r} dx$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{p}{2}\right)^{v+2r} \int_0^a x^{2v+2r+1}\left(a^2 - x^2\right)^{\mu-v-1} dx$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{p}{2}\right)^{v+2r} \int_0^{\pi/2} (a\sin t)^{2v+2r+1}\left(a^2 \cos^2 t\right)^{\mu-v-1} a\cos t\, dt$$

[putting $x = a\sin t$, so that $dx = a\cos dt$]

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{p}{2}\right)^{v+2r} a^{2\mu+vr} \int_0^{\pi/2} (\sin t)^{2v+2r+1}(\cos t)^{2\mu-2v-1} dt$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{p}{2}\right)^{v+2r} a^{2\mu+2r} \frac{\Gamma(v+r+1)\,\Gamma(\mu-v)}{2\Gamma(\mu+r+1)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{pa}{2}\right)^{\mu+2r} 2^{\mu-v-1}\,\Gamma(\mu-v)\; p^{\mu-v}\, a^{\mu}$$

$$= 2^{\mu-v-1}\,\Gamma(\mu-v)\, p^{\mu-v}\, a^{u} J_u(pa) \qquad \text{...(19)}$$

Deduction : (i) Taking $\mu = v+1$ in (19), we get

$$H_v\left\{x^v H(a-x); p\right\} = \frac{a^{v+1}}{p}\; J_{v+1}(pa)$$

(ii) Taking $\mu = v + \frac{1}{2}$ in (19), we get

$$H_v\left\{\frac{x^v U(a-x)}{\sqrt{a^2-x^2}}; p\right\} = \sqrt{\frac{\pi}{2p}}\; a^{v+1/2}\, J_{v+\frac{1}{2}}(pa)$$

Example 10 : Prove that

$$H_v\left\{x^{v-\mu} J_\mu(ax); p\right\} = \frac{p^v\left(a^2-p^2\right)^{\mu-v-1}}{2^{\mu-v-1}\Gamma(\mu-v)a^{\mu}}\; U(a-p) \qquad \left(a>0,\ \mu > v \geq 0\right)$$

deduce that

(i) $$H_v\left\{x^{-1} J_{v+1}(ax); p\right\} = \frac{p^v}{a^{v+1}}\; U(a-p),\; a>0$$

(ii) $$H_v\left\{x^{-1/2}J_{v+\frac{1}{2}}(ax);p\right\}=\sqrt{\frac{2}{\pi}}\,\frac{p^v\,U(a-p)}{a^{v+\frac{1}{2}}\left(a^2-p^2\right)^{1/2}},\qquad a>0,\quad v\geq 0$$

and hance that

(iii) $$H_0\left[x^{-2}\{1-J_0(ax);p\}\right]=H(a-p)\log\left(\frac{a}{p}\right)$$

Solution : By the definition of Hankel inversion theorem in Ex. 19, we find that if $\mu > v \geq 0$, then

$$x^v\left(a^2-x^2\right)^{\mu-v-1}U(a-x)=\int_0^\infty p\,J_v(px)2^{\mu-v-1}\,\Gamma(\mu-v)\;p^{v-\mu}\,a^\mu\,J_\mu(pa)\,dp$$

$$=2^{\mu-v-1}\,\Gamma(\mu-v)\,a^\mu\int_0^\infty p\,J_v(px)\,p^{v-\mu}\,J_\mu(pa)\,dp$$

or $$\int_0^\infty p\,J_v(px)\,p^{v-\mu}\,J_\mu(pa)\,dp=\frac{x^v\left(a^2-x^2\right)^{\mu-v-1}U(a-x)}{2^{\mu-v-1}\,\Gamma(\mu-v)\,a^\mu}$$

Interchanging p and x, we get

$$\int_0^\infty x\,J_v(px)\,x^{v-\mu}\,J_\mu(ax)\,dx=\frac{p^v\left(a^2-p^2\right)^{\mu-v-1}U(a-p)}{2^{\mu-v-1}\,\Gamma(\mu-v)\,a^\mu}$$

or $$H_v\left[x^{v-\mu}\,J_\mu(ax);p\right]=\frac{p^v\left(a^2-p^2\right)^{\mu-v-1}U(a-p)}{2^{\mu-v-1}\,\Gamma(\mu-v)\,a^\mu}\qquad ...(20)$$

Deduction :

(i) Taking $\mu = v+1$ in (20), we get

$$H_v\left[x^{-1}\,J_{v+1}(ax);p\right]=\frac{p^v}{a^{v+1}}\,H(a-p),\; a>0\qquad ...(21)$$

(ii) Taking $\mu = v+\frac{1}{2}$ in (20), we get

$$H_v\left[x^{-1/2}\,J_{v+\frac{1}{2}}(ax);p\right]=\sqrt{\frac{2}{\pi}}\,\frac{p^v\,U(a-p)}{a^{v+1/2}\left(a^2-p^2\right)^{1/2}},\qquad a>0,\; v\geq 0\qquad ...(22)$$

(iii) Taking $v=0$ in (21), we obtain

$$H_0\left[x^{-1}\,J_1(ax);p\right]=\frac{U(a-p)}{a},\; a>0\qquad ...(23)$$

If we write the left hand side of equation (23), in terms of integral, we find that

$$\int_0^{\infty} J_1(ax)\, J_0(px)\, dx = \frac{U(a-p)}{a} \qquad \text{...(24)}$$

Now $\displaystyle \int_0^{\infty} J_1(ax)\, dx = \sum_{r=0}^{\infty} \frac{(-1)^r \left(x/2\right)^{2r+1}}{r!\, \Gamma(r+2)} \int_0^a a^{2r+1}\, dr$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r a^{2r+2} x^{2r+1}}{\Gamma(r+2)\, 2^{2r+2} (r+1)\, \Gamma(r+1)}$$

$$= \sum_{r=1}^{\infty} \frac{(-1)^{r-1} a^{2r} x^{2r-1}}{\Gamma(r+1)\, 2^{2r}\, \Gamma(r+1)} + \frac{1}{x} \qquad \text{(replacing } r \text{ by } r-1\text{)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^{r-1} a^{2r} x}{\Gamma(r+1) 2^{vr}\, \Gamma(r+1)} - \frac{1}{x}$$

$$= \frac{1}{x} - \frac{1}{x} \sum_{r=0}^{\infty} \frac{(-1)^r \left(ax/2\right)^{2r}}{r!\, \Gamma(r+1)}$$

Hence $\displaystyle \int_0^a J_1(ax)\, da = \frac{1}{x} - \frac{1}{x} J_0(ax)$ ...(25)

Again

$$\int_0^a \frac{H(a-p)}{a} da = \int_0^p 0.\frac{1}{a} da + \int_p^a 1.\frac{1}{a} da \qquad \left[\because H(a-p) = \begin{cases} 0, & p > a \\ 1, & p < a \end{cases}\right]$$

$$= \log a - \log p \qquad (p < a)$$

$$\therefore \quad \int_0^a \frac{H(a-p)}{a}\, da = H(a-p)\, \log\left(a/p\right) \qquad \text{...(26)}$$

Now integrating (24) with respect to 'a' both sides and using the integrals (25) and (26) there in, we find that

$$\int_0^{\infty} \frac{1 - J_0(ax)}{x}\, J_0(px)\, dx = U(a-p)\, \log\left(a/p\right)$$

or $\quad H_0\left[x^{-2}\{1 - J_0(ax); p\}\right] = U(a-p)\, \log\left(a/p\right)$

6.6 Hankel Transform of Derivatives of Functions

Theorem 1 : **Let $F_v(p)$ and $F_v'(p)$ be the Hankel transforms of order v of $f(x)$ and $f'(x)=\frac{df}{dx}$ respectively. Then**

$$H_v\{f'(x);p\}=F_v'(p)=-\frac{p}{2v}\{(v+1)F_{v-1}(p)-(v-1)F_{v+1}(p)\}$$

Proof : By the definition of Hankel transform, we have

$$H_v\{f'(x);p\}=F_v'(p)=\int_0^\infty x\,J_v(px)\frac{df}{dx}dx$$

Integration by parts of the integral on the R.H.S. (taking $\frac{df}{dx}$ as the second function), we get

$$F_v'(p)=\{x\,J_v(px)f(x)\}_0^\infty-\int_0^\infty f(x)\frac{d}{dx}\{x\,J_v(px)\}dx$$

or $$F_v'(p)=-\int_0^\infty f(x)\{J_v(px)+xp\,J_v'(px)\}dx \quad ...(27)$$

assuming that $x\,f(x)\to 0$ as $x\to 0$ and $x\to\infty$.

By the recurrence relation II (ii) of 6.2, we have

$$x\,J_v'(x)=x\,J_{v-1}(x)-v\,J_v(x)$$

Replacing x by px, we get

$$px\,J_v'(x)=px\,J_{v-1}(px)-v\,J_v(px)$$

or $$J_v(px)+px\,J_v'(px)=J_v(px)+px\,J_{v-1}(px)-v\,J_v(px)$$

or $$J_v(px)+px\,J_v'(px)=-(v-1)\;J_v(px)+px\,J_{v-1}(px)$$

Putting this value in (27), we get

$$F_v'(p)=(v-1)\int_0^\infty f(x)J_v(px)dx-p\int_0^\infty f(x)\,x\,J_{v-1}(px)dx \quad ...(28)$$

In recurrence relation $2v\,J_v(x)=x\{J_{v-1}(x)+J_{v+1}(x)\}$ replacing x by px, we have

$$J_v(px)=\frac{px}{2v}\{J_{v-1}(px)+J_{v+1}(px)\}$$

Hence (28) reduces to

$$F_v'(p)=\frac{p(v-1)}{2v}\int_0^\infty f(x)\,x\left[J_{v-1}(px)+J_{v+1}(px)\right]dx-\int_0^\infty f(x)\,x\,J_{v-1}(px)dx$$

$$= \frac{p(v-1)}{2v} \int_0^\infty f(x) x \, J_{v+1}(px)\, dx - \frac{p(v+1)}{2v} \int_0^\infty f(x) x \, J_{v-1}(px)\, dx$$

$$= \frac{p(v-1)}{2v} \, F_{v+1}(p) - \frac{p(v+1)}{2v} \, F_{v-1}(p)$$

Thus $$F_v'(p) = -\frac{p}{2v}\{(v+1)F_{v-1}(p) - (v-1)F_{v+1}(p)\} \quad ...(29)$$

This is the Hankel transform of order v of the derivative $\frac{df}{dx}$.

The formula for the Hankel transform of higher derivatives of the function $f(x)$ may be obtained of respected applications of equation (29).

Theorem 2 : Prove that

$$\boldsymbol{H_v\{f''(x);p\} = F_v''(p) = \frac{p^2}{4}\left[\frac{v+1}{v-1}\, F_{v-2}(p) - \frac{2(v^2-3)}{v^2-1}\, F_v(p) + \frac{v+1}{v-1}\, F_{v+2}(p)\right]}$$

Proof : Replacing $f(x)$ by $f'(x)$ in (29), we get

$$F_v''(p) = -p\left[\frac{v+1}{2v}\, F_{v-1}'(p) - \frac{v-1}{2v}\, F_{v+1}'(p)\right] \quad ...(30)$$

Replacing v by $(v-1)$ and $(v+1)$ successively in (29), we get

$$F_{v-1}'(p) = -p\left\{\frac{v}{2(v-1)}\, F_{v-2}(p) - \frac{v-2}{2(v-1)}\, F_v(p)\right\} \quad ...(31)$$

and $$F_{v+1}'(p) = -p\left\{\frac{v+2}{2(v+1)}\, F_v(p) - \frac{v}{2(v+1)}\, F_{v+2}(p)\right\} \quad ...(32)$$

Using (31) and (32) in (30), we get

$$F_v''(p) = p^2\left[\frac{v+1}{2v}\left\{\frac{v}{2(v-1)}\, F_{v-2}(p) - \frac{v-2}{2(v-1)}\, F_v(p)\right\}\right.$$

$$\left. - \frac{v-1}{2v}\left\{\frac{v+2}{2(v+1)}\, F_v(p) - \frac{v}{2(v+1)}\, F_{v+2}(p)\right\}\right]$$

or $$F_v''(p) = \frac{p^2}{4}\left[\frac{v+1}{v-1}\, F_{v-2}(p) - \frac{2(v^2-3)}{v-1}\, F_v(p) + \frac{v-1}{v+1}\, F_{v+2}(p)\right]$$

Theorem 3 : Show that

$$\boldsymbol{H_v\left\{\frac{d^2 f}{d\,x^2}+\frac{1}{x}\frac{df}{dx}-\frac{v^2}{x^2}f;p\right\}=-p^2F_v(p)}$$

where $F_v(p)$ is the Hankel transform of order v of the function $f(x)$.

Proof : By definition of Hankel transform, we have

$$H_v\left\{\frac{d^2 f}{d\,x^2}+\frac{1}{x}\frac{df}{dx}-\frac{v^2}{x^2}f;p\right\}=\int_0^\infty x\,J_v(px)\left\{\frac{d^2 f}{d\,x^2}+\frac{1}{x}\frac{df}{dx}-\frac{v^2}{x^2}f\right\}dx$$

$$=\int_0^\infty J_v(px)\left\{\left(x\,\frac{d^2 f}{d\,x^2}+x\,\frac{df}{dx}\right)-\frac{v^2}{x^2}f\right\}dx$$

$$=\int_0^\infty J_v(px)\frac{d}{dx}\left(x\,\frac{df}{dx}\right)dx-v^2\int_0^\infty \frac{1}{x}\,J_v(px)f(x)dx$$

$$=\left[J_v(px)\,x\,\frac{df}{dx}\right]_0^\infty-\int_0^\infty\left\{\frac{d}{dx}\,J_v(px)\right\}x\,\frac{df}{dx}\,dx-v^2\int_0^\infty \frac{1}{x}\,J_v(px)f(x)dx$$

[integrating by parts the first integral]

$$=-\int_0^\infty\left\{x\,\frac{d}{dx}\,J_v(px)\right\}\frac{df}{dx}\,dx-v^2\int_0^\infty \frac{1}{x}\,J_v(px)f(x)dx$$

(provided that $x\,f'(x)\to 0$ as $x\to 0$ and $x\to\infty$)

$$=\int_0^\infty\left[x\frac{d^2}{dx^2}\,J_v(px)+\frac{d}{dx}\,J_v(px)-\frac{v^2}{x}\,J_v(px)\right]f(x)\,dx$$

[integrating by parts the first integral]

$$=\int_0^\infty\frac{1}{x}\left[x^2\,\frac{d^2}{dx^2}\,J_v(px)+x\,\frac{d}{dx}\,J_v(px)-v^2\,J_v(px)\right]f(x)\,dx \quad ...(33)$$

Since $J_v(x)$ satisfies the Bessel's differential equation

$$x\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-v^2\right)y=0.$$

therefore

$$x\frac{d^2}{dx^2}J_v(x)+x\frac{d}{dx}J_v(x)+\left(x^2-v^2\right)J_v(x)=0$$

Replacing x by px, we get

$$p^2x^2\frac{d^2}{p^2dx^2}J_v(px)+px\frac{d}{p\,dx}J_v(px)+(p^2x^2-v^2)J_v(px)=0$$

or $$x^2\frac{d^2}{dx^2}J_v(px)+x\frac{d}{dx}J_v(px)-v^2J_v(px)=-p^2x^2\,J_v(px) \qquad ...(34)$$

Using (34) in (33), we obtain

$$H_v\left\{\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}-\frac{v^2}{x^2}f\,;p\right\}=\int_0^\infty\frac{1}{x}\left\{-p^2x^2\,J_v(px)\right\}fx\,dx$$

$$=-p^2\int_0^\infty x\,J_v(px)\,f(x)\,dx$$

$$=-p^2F_v(p) \qquad ...(35)$$

Example 11 : If $f(x)=\dfrac{e^{-ax}}{x}$, then find (i) the Hankel transform of order zero of the function $\dfrac{d^2f}{dx^2}+\dfrac{1}{x}\dfrac{df}{dx}$ and (ii) the Hankel transform of order one of $\dfrac{df}{dx}$.

Solution : (i) Taking $v=0$ in (35), we get

$$H_0\left[\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx};p\right]=\int_0^\infty\left(\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right)x\,J_0(px)\,dx$$

$$=-p^2F_0(p)$$

$$=-p^2\int_0^\infty f(x)\,x\,J_0(px)\,dx$$

$$=-p^2\int_0^\infty e^{-ax}\;J_0(px)\,dx$$

$$=-p^2\left(a^2+p^2\right)^{-1/2} \qquad \text{[by result (i) of 6.2 (III)]}$$

(ii) Again taking $v=1$ in (29), we get

$$H_1\left\{\frac{df}{dx};p\right\}=\int_0^\infty x\frac{df}{dx}J_1(px)\,dx$$

$$=-p\,F_0(p)=-p\int_0^\infty x\,f(x)\,J_0(px)\,dx$$

$$=-p\int_0^\infty e^{-ax}\,J_0(px)\,dx$$

$$=-p\left(a^2+p^2\right)^{-1/2} \qquad \text{[by result (i) of 6.2 (III)]}$$

6.7 Parseval's Theorem for Hankel Transform

Theorem 4 : **If $F_v(p)$ and $G_v(p)$ are the Hankel transforms of the functions $f(x)$ and $g(x)$ respectively, then**

$$\int_0^\infty x\, f(x) g(x)\, dx = \int_0^\infty p\, F_v(p) G_v(p)\, dp$$

Proof : By definition of Hankel transform, we have

$$F_v(p) = \int_0^\infty x\, J_v(px) f(x)\, dx \qquad ...(36)$$

and $$G_v(p) = \int_0^\infty x\, J_v(px) g(x)\, dx \qquad ...(37)$$

Now $$\int_0^\infty p\, F_v(p)\, G_v(p)\, dp = \int_0^\infty p\, F_v(p) \left[\int_0^\infty x\, J_v(px)\, g(x)\, dx \right] dp \qquad \text{[by (37)]}$$

$$= \int_0^\infty x\, g(x) \left[\int_0^\infty F_v(p)\, p\, J_v(px)\, dp \right] dx$$

[interchanging the order of integration]

$$= \int_0^\infty x\, g(x) f(x)\, dx \qquad \text{[on using inversion formula]}$$

$$= \int_0^\infty x\, f(x) g(x)\, dx$$

Example 12 : Find the Hankel transform of $x^v H(a-x)$ and $x^v H(b-x)$, $v > -\frac{1}{2}$. Hence or otherwise establish that

$$H_v\{x^{-2} J_v(ax); p\} = \begin{cases} \frac{1}{2v}\left(\frac{p}{a}\right)^v, & 0 < p < a \\ \frac{1}{2v}\left(\frac{a}{p}\right)^v, & p > a \end{cases}$$

Solution : Here $f(x) = x^v H(a-x)$

and $g(x) = x^v H(b-x)$, $v > -\frac{1}{2}$, $a > 0$, $b > 0$

By the definition of Hankel transform, we have

$$F_v(p) = H_v\{f(x); p\} = \int_0^\infty x\, J_v(px) f(x)\, dx$$

$$= \int_0^\infty x\, J_v(px) x^v H(a-x)\, dx$$

$$= \int_0^a x^{v+1} J_v(px).1.dx$$

$$= \int_0^a x^{v+1} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{px}{2}\right)^{v+2r} dx$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{p}{2}\right)^{v+2r} \int_0^a x^{2v+2r+1}\, dx$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{p}{2}\right)^{v+2r} \left[\frac{x^{2v+2r+2}}{(2v+2r+2)}\right]_0^a$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+1)} \left(\frac{p}{2}\right)^{v+2r} \frac{a^{2v+2r+2}}{2(v+r+2)}$$

$$= \frac{a^{v+1}}{p} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(v+r+2)} \left(\frac{p}{2}\right)^{v+2r+1}$$

or $$F_v(p) = \frac{a^{v+1}}{p} J_{v+1}(pa)$$

Similarly we can get

$$H_v\{g(x);p\} = G_v(p) = \frac{b^{v+1}}{p} J_{v+1}(pb)$$

Now by using Parseval's theorem,

$$\int_0^\infty p\, F_v(p)\, G_v(p)\, dp = \int_0^\infty x\, f(x)\, g(x)\, dx$$

we have

$$(ab)^{v+1} \int_0^\infty p^{-1} J_{v+1}(pa)\, J_{v+1}(pb)\, dp = \int_0^{\min(a,b)} x^{2v+1}\, dx$$

Suppose $0 < a < b$, then we have

$$(ab)^{v+1} \int_0^\infty p^{-1} J_{v+1}(pa)\, J_{v+1}(pb)\, dp = \frac{a^{2v+2}}{(2v+2)}$$

or $$\int_0^\infty p^{-1} J_{v+1}(pa)\, J_{v+1}(pb)\, dp = \frac{1}{2(v+1)} \left(\frac{a}{b}\right)^{v+1}$$

Replacing $(v+1)$ by v, p by x and b by p, we have

$$\int_0^\infty x^{-1} J_v(ax)\, J_v(px)\, dx = \begin{cases} \dfrac{1}{2v}\left(\dfrac{p}{a}\right)^v, & 0 < p < a \\ \dfrac{1}{2v}\left(\dfrac{a}{p}\right)^v, & p > a, v > -\dfrac{1}{2} \end{cases}$$

or $$H_v\left\{\frac{J_v(ax)}{x^2}; p\right\} = \begin{cases} \dfrac{1}{2v}\left(\dfrac{p}{a}\right)^v, & 0 < p < a \\ \dfrac{1}{2v}\left(\dfrac{a}{p}\right)^v, & p > a, v > -\dfrac{1}{2} \end{cases}$$

Self-Learning Exercise

Fill in the balnks :

1. $H_0\{e^{-x}; p\} =$.............

2. $H_0\left\{\dfrac{e^{-ax}}{x}; p\right\} =$.............

3. $H_1\{e^{-ax}; p\} =$.............

4. $H_1\left\{\dfrac{e^{-ax}}{x}; p\right\} =$.............

5. $H_1\left\{\dfrac{e^{-x}}{x^2}; p\right\} =$.............

6. $H_1^{-1}\left\{\dfrac{e^{-ap}}{p^2}; x\right\} =$.............

7. $H_1^{-1}\{e^{-ap}; x\} =$.............

8. $H_0\{x^{-1} J_1(ax); p\} =$.............

9. $H_0\{x^{-1} \sin a\, x; p\} =$.............

6.8 Summary

In this unit you studied the infinite Hankel transform and various results connected with this transform. The properties and properties of Hankel transform have been illustrated by solving a number of problems.

6.9 Answers to Self-Learning Exercise

1. $\left(1+p^2\right)^{-3/2}$

2. $\left(a^2+p^2\right)^{-1/2}$

3. $p\left(a^2+p^2\right)^{-3/2}$

4. $\dfrac{1}{p}-\dfrac{a}{p\left(a^2+p^2\right)^{1/2}}$

5. $\dfrac{\sqrt{1+p^2}-1}{p}$

6. $\dfrac{\sqrt{\left(a^2+x^2\right)}-a}{x}$

7. $x\left(a^2+x^2\right)^{-3/2}$

8. $\dfrac{H(a-p)}{a}$, $a>0$

9. $\dfrac{H(a-p)}{\sqrt{a^2-p^2}}\,(a>0)$

6.10 Exercise 6

1. Find the Hankel transform of $\dfrac{\sin a\,x}{x}$, taking $x\,J_1(px)$ as the kernel.

$$\left[\ \textbf{Ans.}\ \begin{cases}\dfrac{a}{p\sqrt{p^2+a^2}}, & p>\alpha\\ 0\quad, & 0<p<a\end{cases}\ \right]$$

2. Find the Hankel transform of e^{-ax}, taking $x\,J_0(px)$ as the kernel.

$$\left[\ \textbf{Ans.}\ \ a\left(p^2+a^2\right)^{-3/2}\ \right]$$

3. If $v>-1$, prove that $H_v\left\{e^{-ax^2};p\right\}=\dfrac{p^v}{(2a)^{v+1}}\,e^{-p^2/4a}$

4. Prove that

$$H_v\left\{e^{-ax};p\right\}=\frac{a+v\sqrt{a^2+p^2}}{\sqrt{a^2+p^2}}\left\{\frac{p}{a+\sqrt{a^2+p^2}}\right\}^v$$

5. Prove that

$$H_v\left\{x^{-1}e^{-ax};p\right\}=\frac{1}{\sqrt{a^2+p^2}}\left\{\frac{p}{a+\sqrt{a^2+p^2}}\right\}^v$$

6. Find the Hankel Transform of order zero of $x^2 H(a-x)$.

$\left[\quad \textbf{Ans.} \quad \dfrac{a^2}{p^2}\left\{2\,J_0(pa)+\left(ap-\dfrac{4}{ap}\right)J_1(pa)\right\} \quad \right]$

7. Evaluate Hankel transform of order v of x^{s-1}.

$\left[\quad \textbf{Ans.} \quad \dfrac{2^s\Gamma\left[(s+v+1)/2\right]}{p^{s+1}\;\Gamma\left[(-s+v+1)/2\right]}, \quad -1-v<s<1+v \quad \right]$

8. If $H_v\{f(x);p\}=F_v(p)$, then prove that

$$H_v\{x^{-1}f(x);p\}=\frac{p}{2v}\left[F_{v-1}(p)+F_{v+1}(p)\right]$$

9. Evaluate $H_v^{-1}\left\{\dfrac{e^{-ap}}{p}\right\}$

$\left[\quad \textbf{Ans.} \quad \left(a^2+x^2\right)^{-1/2} \quad \right]$

10. If $H^{-1}\{F_v(p)\}=f(x)$, then prove that

$$H^{-1}\{F_v(ap)\}=\frac{1}{a^2}f\left(\frac{x}{a}\right), \qquad a>0$$

11. Find the Hankel transform of order zero of $\dfrac{1}{x}$ and then apply the inversion formula to get the original function.

$\left[\quad \textbf{Ans.} \quad \dfrac{1}{p} \quad \right]$

12. Find the Hankel transform of $f(x)=e^{-ax}$, $a>0$, taking $x\,J_0(px)$ as the kernel and hence show that

$$H_0\left\{\frac{1}{\left(a^2+x^2\right)^{3/2}}\right\}=\frac{e^{-ap}}{a}, \qquad a>0$$

$\left[\quad \textbf{Ans.} \quad \dfrac{a}{\left(a^2+p^2\right)^{3/2}} \quad \right]$

13. Prove that $H_v\left\{\rho^{v-1}\dfrac{\partial}{\partial\rho}\left\{\rho^{v-1}f(\rho)\right\};\xi\right\}=-\xi\, H_{v-1}\left\{f(\rho);\xi\right\}$ provided that $\lim_{\rho\to 0}\rho^{v+1}f(\rho)=0$ and $\lim_{\rho\to\infty}\rho^{1/2}f(\rho)=0$

14. If $\lim_{\rho\to 0}\rho^{v+1}f(\rho)=0$ and $\lim_{\rho\to\infty}\rho^{1/2}f(\rho)=0$, then prove that

$$H_v\left[\rho^{-v-1}\frac{\partial}{\partial\rho}\left\{\rho^{v+1}f(\rho);\xi\right\}\right]=\xi\, H_{v+1}\left[f(\rho);\xi\right]$$

15. Find the Hankel transform of $\dfrac{\partial^2 f}{\partial t^2}$ of order v with respect to variable x, where f is a function of x and t.

$\left[\ \textbf{Ans.}\ \dfrac{d^2}{dt^2}\, H_v\left\{f(x,t);p\right\}\ \right]$

16. Prove that

$$H_1\left\{x^{-1}\sin a\,x;p\right\}=\frac{aU(p-a)}{p\sqrt{p^2-a^2}}$$

Unit - 7
Application of Fourier and Infinite Hankel Transform to the Solution of Simple Boundary Problems

Structure of the Unit

7.0 Objective

The main object of this unit is to give the application of (sine, cosine and complex) Fourier and Infinite Hankel transforms to the solution of simple boundary problems.

7.1 Introduction

The partial differential equations subject to appropriate boundary conditions are known as boundary value problems. The important tools for solving boundary value problems are Laplace, Fourier and Hankel transforms. The application of Laplace transform to solve boundary value problems has already been studied in unit 3. Now we discuss Fourier transform method and Hankel transform method to solve boundary value problems

7.2 Applications of Fourier Transform

When any one of the variables in the differential equation ranges in the inteval

I. $(-\infty, \infty)$: Then take the complex Fourier transform of both sides of the partial differential equation, thus reducing to the ordinary differential equation, solve it and finally take the corresponding inverse Fourier complex transform of the solution thus obtained.

II $(0, \infty)$: Then take the infinite Fourier sine or cosine transform of both sides of the partial differential equation, thus reducing to the ordinary differential equation, solve it and finally take the corresponding inverse infinite Fourier sine or cosine transform of the solution so obtained.

The choice of sine or cosine transform is decided by the form of the boundary conditions at the lower limit of the variable selected for exclusion. In this connection we require the following results

(i) $F_s\left\{\frac{\partial^2 U}{\partial x^2}\right\} = \int_0^\infty \frac{\partial^2 U}{\partial x^2} \sin px \, dx$ [taking non-symmetrical definition of Fourier sine transform]

$$= \left[\frac{\partial U}{\partial x}\sin px\right]_0^\infty - p\int_0^\infty \frac{\partial U}{\partial x}\cos p\,x\;dx \qquad \text{[integrating by parts]}$$

$$= p\int_0^\infty \frac{\partial U}{\partial x}(\cos p\,x)\;dx \text{, provided } \frac{\partial U}{\partial x}\to 0 \text{ as } x\to\infty$$

$$= -p\left\{[U\cos p\,x]_0^\infty + p\int_0^\infty U\sin p\,x\,dx\right\}$$

$$= -p\left[U(x,t)\right]_{x=0} - p^2 u_s(p,t)$$

[assuming that $U(x,t)\to 0$ as $x\to\infty$]

Thus $$F_s\left\{\frac{\partial^2 U}{\partial x^2}\right\} = pU(0,t) - p^2 u_s(p,t) \qquad ...(1)$$

where $u_s(p,t)$ is the Fourier sine transform of $U(x,t)$ with respect to x.

$$F_c\left\{\frac{\partial^2 U}{\partial x^2}\right\} = \int_0^\infty \frac{\partial^2 U}{\partial x^2}\cos p\,x\,dx \text{ [by non-symmetric definition of Fourier cosine transform]}$$

$$= \left[\frac{\partial U}{\partial x}\cos p\,x\right]_0^\infty + p\int_0^\infty \frac{\partial U}{\partial x}\sin p\,x\,dx$$

$$= -\left(\frac{\partial U}{\partial x}\right)_{x=0} + p\int_0^\infty \frac{\partial U}{\partial x}\sin p\,x\,dx$$

[assuming that $\frac{\partial U}{\partial x}\to 0$ as $x\to\infty$]

$$= -\left(\frac{\partial U}{\partial x}\right)_{x=0} + p\left\{[U\sin p\,x]_0^\infty - p\int_0^\infty U\cos p\,x\,dx\right\}$$

$$= -\left(\frac{\partial U}{\partial x}\right)_{x=0} - p^2\int_0^\infty U(x,t)\cos p\,x\,dx$$

[assuming that $U(x,t)\to 0$ as $x\to\infty$]

Hence $$F_c\left\{\frac{\partial^2 U}{\partial x^2}\right\} = -\left(\frac{\partial U}{\partial x}\right)_{x=0} - p^2 u_c(p,t) \qquad ...(2)$$

where $u_c(p,t)$ is the Fourier cosine transform of $U(x,t)$ with respect to x, where $U(x,t)$ is a function of the variables x and t.

From (1) and (2), it follows that if we want to remove the term $\frac{\partial^2 U}{\partial x^2}$ from a partial differential equation, then we require

(i) $U_x(0,t)$ i.e. $\frac{\partial U}{\partial x}$ when $x=0$ in Fourier cosine transform

(ii) $U(0,t)$ i.e. $U(x,t)$ when $x=0$ in Fourier sine transform

Similarly we can attempt to remove the term $\frac{\partial^4 U}{\partial x^4}$ or any other derivative of even order but these transforms will fail for removing any derivative of odd order. If we can use these transforms for solution of a partial differential equation, there is definetely a considerable advantage over the Laplace transform.

When one of the variables in a partial differential equation ranges from $-\infty$ to ∞, then that variable can be excluded with the help of complex Fourier transforms.

Example 1 : Solve $\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial x^2}$, $x>0$, $t>0$ subject to conditions :

(i) $U(0,t)=0$

(ii) $U=\begin{cases}1, & , 0<x<1 \\ 0, & , \quad x\geq 1\end{cases}$ when $t=0$

(iii) $U(x,t)$ is bounded

Solution : Since $(U)_{x=0}$ is given, therefore taking Fourier sine transforms of both sides of the given equation,

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial U}{\partial t}\sin p\,x\,dx=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 U}{\partial x^2}\sin p\,x\,dx$$

or $$\frac{d}{dt}\sqrt{\frac{2}{\pi}}\int_0^\infty U(x,t)\sin p\,x\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[\frac{\partial U}{\partial x}\sin p\,x\right]_0^\infty - p\int_0^\infty \frac{\partial U}{\partial x}\cos p\,x\,dx\right\}$$

or $$\frac{d\,u_s}{dt}=-p\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial U}{\partial x}\cos p\,x\,dx\,,\quad \frac{\partial U}{\partial x}\to 0 \text{ as } x\to\infty$$

[Assuming that $u_s(p,t)=\sqrt{\frac{2}{\pi}}\int_0^\infty U(x,t)\sin p\,x\;dx$]

or $$\frac{d u_s}{dt} = -p\sqrt{\frac{2}{\pi}}\left\{\left[U(x,t)\cos p x\right]_0^\infty + p\int_0^\infty U(x,t)\sin p x\, dx\right\}$$

or $$\frac{d u_s}{dt} = -p\left\{0 - \sqrt{\frac{2}{\pi}}\, U(0,t) + p u_s\right\},$$ [assuming that $U(x,t) \to 0$ as $x \to \infty$]

or $$\frac{d u_s}{dt} = -p^2 u_s,$$ by boundary condition (i)

or $$\frac{d u_s}{dt} + p^2 u_s = 0 \qquad ...(3)$$

The solution of (3) is $u_s(p,t) = c_1 e^{-p^2 t}$ ...(4)

Now, taking Fourier sine transform of boundary condition (ii), we have

$$u_s(p,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty U(x,0)\sin p x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left\{\int_0^1 U(x,0)\sin p x\, dx + \int_1^\infty U(x,0)\sin p x\, dx\right\}$$

$$= \sqrt{\frac{2}{\pi}}\left\{\int_0^1 1.\sin p x\, dx + \int_1^\infty 0.\sin p x\, dx\right\}$$

or $$u_s(p,0) = \sqrt{\frac{2}{\pi}}\left[-\frac{\cos p x}{p}\right]_0^1 = \sqrt{\frac{2}{\pi}}\,\frac{1-\cos p}{p} \qquad ...(5)$$

Taking $t = 0$ in (4) and using (5), we get

$$u_s(p,0) = c_1 = \sqrt{\frac{2}{\pi}}\,\frac{1-\cos p}{p}$$

Substituting the value of c_1 in (4), we have

$$u_s(p,t) = \sqrt{\frac{2}{\pi}}\,\frac{1-\cos p}{p}e^{-p^2 t}$$

Now, taking inverse Fourier sine transform, we get

$$U(x,t) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos p}{p}e^{-p^2 t}\sin p x\, dp$$

which is the required solution.

Example 2 : The temperature $U(x,t)$ in the semi-infinite rod $0 \le x \le \infty$ is determined by the differential equation :

$$\frac{\partial U}{\partial t} = k \frac{\partial^2 U}{\partial x^2}$$

Subject to the conditions :

(i) $U = 0$ when $t = 0$, $x \geq 0$

(ii) $\frac{\partial U}{\partial x} = -\mu$ (a constant), when $x = 0$, $t \geq 0$

Making use of cosine transform, show that

$$U(x,t) = \frac{2\mu}{\pi} \int_0^\infty \frac{\cos p u}{p^2}\left(1 - e^{-k p^2 t}\right) dp$$

Solution : Since $\left(\frac{\partial U}{\partial x}\right)_{x=0}$ is given, therefore taking Fourier cosine transform of both sides of the given differential equation, we have

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial U}{\partial t} \cos p x \, dx = k \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 U}{\partial x^2} \cos p x \, dx$$

or $$\sqrt{\frac{2}{\pi}} \frac{d}{dt} \int_0^\infty U(x,t) \cos p x \, dx$$

$$= \sqrt{\frac{2}{\pi}}\, k \left\{ \left[\cos p x \frac{\partial U}{\partial x}\right]_0^\infty + p \int_0^\infty \sin p x \frac{\partial U}{\partial x} dx \right\}$$

or $$\frac{d u_c}{dt} = -\sqrt{\frac{2}{\pi}}\, k \left(\frac{\partial U}{\partial x}\right)_{x=0} + \sqrt{\frac{2}{\pi}}\, k\, p \int_0^\infty \frac{\partial U}{\partial x} \sin p x \, dx,$$

if $\frac{\partial U}{\partial x} \to 0$ as $x \to \infty$ Here $u_c = \sqrt{\frac{2}{\pi}} \int_0^\infty U(x,t) \cos p x \; dx$]

we have

$$\frac{d u_c}{dt} = \sqrt{\frac{2}{\pi}}\, k\, \mu + \sqrt{\frac{2}{\pi}}\, k\, p \left\{ \left[U(x,t) \sin p x\right]_0^\infty - \int_0^\infty U(x,t)\, p \cos p x \; dx \right\}$$

or $$\frac{d u_c}{dt} = \sqrt{\frac{2}{\pi}}\, k\, \mu - k\, p^2 u_c, \qquad \text{if } U(x,t) \to 0 \text{ as } x \to \infty.$$

or $$\frac{d u_c}{dt} + k\, p^2 u_c = \sqrt{\frac{2}{\pi}}\, k\, \mu \qquad ...(6)$$

which is a linear differential equation of first order. It's

I.F. is $= e^{\int k p^2 dt} = e^{k p^2 t}$

Thus the general solution of (6) is

$$e^{k p^2 t}\, u_c(p,t) = \sqrt{\frac{2}{\pi}}\, k\,\mu \int e^{k p^2 t}\, dt + c_1$$

$$\therefore \qquad u_c(p,t) = \sqrt{\frac{2}{\pi}}\, \frac{\mu}{p^2} + c_1\, e^{-k p^2 t} \qquad \text{...(7)}$$

Now taking the Fourier cosine transform of boundary condition (i) we have

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} U(x,0) \cos p x\, dx = 0 \quad \text{or} \quad u_c(p,0) = 0 \qquad \text{...(8)}$$

Putting $t = 0$ in (7) and using (8), we get

$$0 = \sqrt{\frac{2}{\pi}}\, \frac{\mu}{p^2} + c_1 \qquad \text{or} \qquad c_1 = -\sqrt{\frac{2}{\pi}}\, \frac{\mu}{p^2}$$

Thus From (7), we get

$$u_c(p,t) = \sqrt{\frac{2}{\pi}}\, \frac{\mu}{p^2} \left(1 - e^{-k p^2 t}\right)$$

Now, taking the inverse Fourier cosine transform, we get

$$U(x,t) = \frac{2}{\pi} \int_0^{\infty} \frac{\mu}{p^2} \left(1 - e^{-k p^2 t}\right) \cos p x\, dp$$

or $$U(x,t) = \frac{2\mu}{\pi} \int_0^{\infty} \frac{\cos p x}{p^2} \left(1 - e^{-k p^2 t}\right) dp$$

Example 3 : Solve $\dfrac{\partial^4 U}{\partial x^4} + \dfrac{\partial^2 U}{\partial x^2} = 0$, $-\infty < x < \infty$, $y \geq 0$ satisfying the conditions :

(i) U and its partial derivatives tend to zero as $x \to \pm\infty$ and

(ii) $U = f(x)$, $\dfrac{\partial U}{\partial y} = 0$ for $y = 0$

Solution : Given

$$\frac{\partial^4 U}{\partial x^4} + \frac{\partial^4 U}{\partial y^2} = 0 \qquad \text{...(9)}$$

with the boundary conditions :

(i) $U \to 0\,,\ \dfrac{\partial U}{\partial x} \to 0\,,\ \dfrac{\partial^2 U}{\partial x^2} \to 0\,,\ \dfrac{\partial^3 U}{\partial x^3} \to 0$ as $x \to \pm\infty$

(ii) $U(x,0) = f(x)$

(iii) $\left(\dfrac{\partial U}{\partial y}\right)_{y=0} = 0$

Taking the complex Fourier transform of (9), we get

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\partial^4 U}{\partial x^4}\, e^{ipx}\, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{\partial^4 U}{\partial y^4}\, e^{ipx}\, dx = 0$$

or $$\frac{1}{\sqrt{2\pi}} \left\{ \left[\frac{\partial^3 U}{\partial x^3}\, e^{ipu} \right]_{-\infty}^{\infty} - i\,p \int_{-\infty}^\infty \frac{\partial^3 U}{\partial x^3}\, e^{ipx}\, du \right\} + \frac{d^2}{dy^2}\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty U\, e^{ipx}\, dx = 0$$

or $$-\frac{i\,p}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{\partial^3 U}{\partial x^3}\, e^{ipx}\, dx + \frac{d^2 \overline{u}}{dy^2} = 0$$

[using boundary condition (i), and assuming

$$\overline{u}(p,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty U(x,y)\, e^{ipx}\, dx\,]$$

or $$-\frac{i\,p}{\sqrt{2\pi}} \left\{ \left[\frac{\partial^2 U}{\partial x^2}\, e^{ipx} \right]_{-\infty}^{\infty} - i\,p \int_{-\infty}^\infty \frac{\partial^2 U}{\partial x^2}\, e^{ipx}\, dx \right\} + \frac{d^2 \overline{u}}{dy^2} = 0$$

or $$\frac{(i\,p)^2}{\sqrt{2\pi}} \left\{ \left(\frac{\partial U}{\partial x}\, e^{ipx} \right)_{-\infty}^{\infty} - i\,p \int_{-\infty}^\infty \frac{\partial U}{\partial x}\, e^{ipx}\, dx \right\} + \frac{d^2 \overline{u}}{dy^2} = 0$$

[using boundary condition (i)]

or $$\frac{(i\,p)^3}{\sqrt{2\pi}} \left\{ \left(U\, e^{ipx} \right)_{-\infty}^{\infty} - i\,p \int_{-\infty}^\infty U(x,y)\, e^{ipx}\, dx \right\} + \frac{d^2 \overline{u}}{dy^2} = 0$$

or $$\frac{(i\,p)^4}{\sqrt{2\pi}} \int_{-\infty}^\infty U(x,y)\, e^{ipx}\, dx + \frac{d^2 \overline{u}}{dy^2} = 0$$

or $$p^2 \overline{u} + \frac{d^2 \overline{u}}{dy^2} = 0$$

where solution is

$$\overline{u}(p,y) = c_1 \cos p^2 y + c_2 \sin p^2 y \qquad ...(10)$$

Taking complex Fourier transform of boundary conditions (iii), we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial U}{\partial y}\, e^{ipx}\, dx = 0 \text{ at } y = 0 \text{ or } \frac{d\bar{u}}{dy} = 0 \text{ at } y = 0 \qquad ...(11)$$

Taking $y = 0$ in (10) and using (11), we get

$$o = c_2 p^2 \Rightarrow c_2 = 0$$

Therefore from (10), we have $\bar{u}(p, y) = c_1 \cos p^2 y$ ...(12)

Again, taking complex Fourier transform of boundary condition (ii), we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(x,0)e^{ipx}\, dx = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} f(x)e^{ipx}\, dx$$

or $\quad \bar{u}(p,0) = \bar{f}(p)$ ...(13)

where $\quad \bar{f}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ipx}\, dx$ ...(14)

Putting $y = 0$ in (12) and using (13), we obtain

$$\bar{f}(p) = c_1$$

Thus from (12), we have

$$\bar{u}(p, y) = \bar{f}(p)\cos p^2 y$$

Further taking inverse Fourier transform, we get

$$U(x, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{f}(p)\cos p^2 y\; e^{-ipx}\, dp$$

where $\bar{f}(p)$ is given by (14).

Example 4 : If the flow of heat is linear so that the variation of θ (temperature) with z and y-axes may be neglected and if it is assumed that no heat is generated in the medium, then solve the differential equation

$$\frac{\partial \theta}{\partial t} = k\,\frac{\partial^2 \theta}{\partial x^2},$$

where $-\infty < x < \infty$ and $\theta = f(x)$ when $t = 0$, $f(x)$ being a given function of x.

Solution : The equation is

$$\frac{\partial \theta}{\partial t} = k\,\frac{\partial^2 \theta}{\partial x^2}, \; -\infty < x < \infty \qquad ...(15)$$

with the initial conditions $\theta(x,0) = f(x)$ ...(16)

Taking the complex Fourier transform of both sides of (15), we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial\theta}{\partial t}\,e^{ipx}\,dx=\frac{k}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial^2\theta}{\partial x^2}\,e^{ipx}\,dx$$

or $$\frac{d}{dt}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\theta(x,t)\,e^{ipx}\,dx=k\,(-ip)^2\,\mathrm{F}\,\{\theta(x,t)\}$$

or $$\frac{d\overline{\theta}}{dt}=-k\,p^2\overline{\theta},\text{ where }\overline{\theta}=\mathrm{F}\,\{\theta(x,t)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\theta(x,t)\,e^{ipx}\,dx$$

whose general solution is

$$\overline{\theta}(p,t)=c_1e^{-kp^2t} \qquad ...(17)$$

Taking the complex Fourier transform of both sides of (16), we get

$$\overline{\theta}(p,0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,e^{ipx}\,dx=\bar{f}(p) \qquad ...(18)$$

Putting $t=0$ in (17) and using (18), we have $\bar{f}(p)=c_1$

Hence from (17), we have

$$\overline{\theta}(p,t)=\bar{f}(p)\,e^{-kp^2t}$$

Now, taking the inverse Fourier transform, we get

$$\theta(x,t)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)\,g(x-u)\,dx, \qquad ...(19)$$

where

$$g(x)=f^{-1}\left\{e^{-kp^2t}\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-kp^2t-ipx}\,dp$$

$$=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{4kt}\right)\int_{-\infty}^{\infty}e^{-kt(p+ix/2kt)^2}\,dp$$

$$=\frac{1}{\sqrt{2\pi\,k\,t}}\exp\left(-\frac{x^2}{4kt}\right)\int_{-\infty}^{\infty}e^{-u^2}\,du$$

$$=\frac{1}{\sqrt{2\,k\,t}}\exp\left(-\frac{x^2}{4kt}\right) \qquad ...(20)$$

Thus, the solution of (19) becomes

$$\theta(x,t)=\frac{1}{\sqrt{4\pi k t}}\int_{-\infty}^{\infty} f(u)\,\exp\left\{-\frac{(x-u)^2}{4kt}\right\}du \qquad ...(21)$$

Example 5 : Find the solution of the linear diffusion equation

$$\frac{\partial^2 U}{\partial x^2}=\frac{1}{k}\frac{\partial U}{\partial t}$$

in a semi-infinite rod $x \geq 0$, satisfying the boundary conditons :

(i) $U(0,t)=f(t),\ t\geq 0$

(ii) $U(x,t)\to 0$ as $x\to\infty$

and the initial condition $U(x,0)=0$

Solution : Taking Fourier sine transform of the given equation and using boundary condition (i), we get

$$\frac{du_s}{dt}+k\,p^2u_s=\sqrt{\frac{2}{\pi}}\,k\,p\,f(t) \qquad ...(22)$$

Further initial condition is equivalent to

$$u_s(p,0)=0 \qquad ...(23)$$

Since (22) is a linear ordinary differential equation whose solution is

$$u_s(p,t)\,e^{k p^2 t}=\sqrt{\frac{2}{\pi}}\,kp\int_0^t f(\tau)\,e^{k p^2 t}d\tau+c$$

Taking $t=0$ and applying the condition (23), we get $c=0$. Thus

$$u_s(p,t)=\sqrt{\frac{2}{\pi}}\,k\,p\int_0^t f(\tau)\,e^{-k(t-\tau)p^2}\,d\tau \qquad ...(24)$$

Taking Fourier inverse sine transform and using the fact that $F_s^{-1}=F_s$, we get

$$u(x,t)=\sqrt{\frac{2}{\pi}}\,k\int_0^t f(\tau)\,F_s\left\{p\,e^{-k(t-\tau)p^2};p\right\}d\tau \qquad ...(25)$$

Now $\quad F_s\left\{p\,e^{-k(t-\tau)p^2};x\right\}=\sqrt{\frac{2}{\pi}}\int_0^{\infty}p\,e^{-k(t-\tau)p^2}\sin px\,dp$

$$=\sqrt{\frac{2}{\pi}}\left\{\left(\frac{-\sin px}{2k(t-\tau)}\,e^{-k(t-\tau)p^2}\right)_0^{\infty}+\frac{x}{2k(t-\tau)}\int_0^{\infty}e^{-k(t-\tau)p^2}\cos px\,dp\right\}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{x}{k(t-\tau)} \int_0^\infty e^{-k(t-\tau)p^2} \cos p\,x\; dp$$

$$= \frac{x}{2\sqrt{2}\left[k(t-\tau)\right]^{3/2}} e^{-x^2/4k(t-\tau)}$$

$$\left[\because \mathrm{F}_c\left\{e^{-a^2x^2}; p\right\} = \frac{1}{\sqrt{2}\,a} e^{-p^2/4a^2}\right]$$

$$= \frac{x}{2\sqrt{2}} \left\{k(t-\tau)\right\}^{-3/2} \exp\left\{-\frac{x^2}{4k(t-\tau)}\right\} \qquad \text{...(26)}$$

Substituting the aforementioned value in (25), we finally obtain

$$u(x,t) = \frac{x}{\sqrt{4\pi k}} \int_0^t f(\tau) \exp\left\{-\frac{x^2}{4k(t-\tau)}\right\} \frac{d\tau}{(t-\tau)^{3/2}}$$

If instead of boundary condition (i), we have the condition

$$U_x(0,t) = f(t), \qquad (t \geq 0)$$

then we must use Fourier cosine transforms and in this case the solution will be

$$u(x,t) = -\sqrt{\frac{k}{\pi}} \int_0^t f(\tau) \exp\left\{-\frac{x^2}{4k(t-\tau)}\right\} \frac{d\tau}{\sqrt{t-\tau}}$$

Example 6 : Solve $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$, $x > 0$, $t > 0$

If (i) $V_x(0,t) = 0$ (ii) $V(x,0) = \begin{cases} x, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$

and (iii) $V(x,t)$ is bounded.

Solution : Since $\left(\frac{\partial V}{\partial x}\right)_{x=0}$ is given. Therefore taking Fourier cosine transform of both the sides of the given equation,

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \left(\frac{\partial V}{\partial t}\right) \cos p\,x\,dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 V}{\partial x^2} \cos p\,x\,dx$$

or
$$\sqrt{\frac{2}{\pi}} \frac{d}{dt} \int_0^\infty V \cos p\,x\,dx = \sqrt{\frac{2}{\pi}} \left\{\left(\frac{\partial V}{\partial x} \cos p\,x\right)_0^\infty + p \int_0^\infty \frac{\partial V}{\partial x} \sin p\,x\,dx\right\}$$

or $$\frac{d\bar{v}_c}{dt} = -\sqrt{\frac{2}{\pi}}\left\{\left(\frac{\partial V}{\partial x}\right)_{x=0} - p\left(V\sin px\right)_0^\infty + p^2\int_0^\infty V\cos px\,dx\right\}$$

$$\left[\frac{\partial V}{\partial x} \to 0 \text{ when } x \to \infty \text{ and } v \to 0 \text{ when } x \to \infty\right]$$

or $$\frac{d\bar{v}_c}{dt} = -p^2\bar{v}_c \quad \text{or} \quad \frac{d\bar{v}_c}{dt} + p^2\bar{v}_c = 0$$

where $\bar{v}_c(p,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty V(x,t)\cos px\,dx$

The solution of this equation is

$$\bar{v}_c = Ae^{-p^2t} \qquad ...(27)$$

Again $V(x,0) = \begin{cases} x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$

Taking Fourier cosine transform, we get

$$\bar{v}_c(p,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty V(x,0)\cos px\,dx = \sqrt{\frac{2}{\pi}}\left[\int_0^1 x\cos px\,dx + \int_1^\infty 0\cos px\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(\frac{x\sin px}{p}\right)_0^1 + \frac{1}{p^2}\left(\cos px\right)_0^1\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin p}{p} + \frac{\cos p - 1}{p^2}\right] \qquad ...(28)$$

Therefore when $t = 0$, then by (27), we have

$$A = \sqrt{\frac{2}{\pi}}\left[\frac{\sin p}{p} + \frac{\cos p - 1}{p^2}\right]$$

$$\therefore \quad \bar{v}_c = \sqrt{\frac{2}{\pi}}\left[\frac{\sin p}{p} + \frac{\cos p - 1}{p^2}\right]e^{-p^2t} \qquad ...(29)$$

Now, taking inverse Fourier cosine transform of (29), we get

$$V(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{v}_c\cos px\,dp$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{\sin p}{p} + \frac{\cos p - 1}{p^2}\right)e^{-p^2t}\cos px\,dp$$

$$= \frac{2}{\pi} \int_0^\infty \left(\frac{\sin p}{p} + \frac{\cos p - 1}{p^2} \right) e^{-p^2 t} \cos p x \, dp$$

Example 7 : Solve the initial value problem for the wave equation

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2}, \; (-\infty < x < \infty, \, t > 0) \qquad ...(30)$$

subject to conditions

$$U(x,0) = f(x) \qquad ...(31)$$

$$U_t(x,0) = g(x), \qquad (-\infty < x < \infty) \qquad ...(32)$$

Solution : Applying complex Fourier transform with respect to variable x in (30), (31) and (32), we get

$$\frac{d^2 \bar{u}}{dt^2} + c^2 p^2 \bar{u} = 0 \qquad ...(33)$$

$$\bar{u}(p,0) = \bar{f}(p); \; \left(\frac{d\bar{u}}{dt} \right)_{t=0} = \bar{g}(p) \qquad ...(34)$$

where $\bar{u}(p,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} U(x,t) e^{ipx} dx$

$$\bar{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ipx} dx$$

Similarly $\bar{g}(p)$ can be defined.

The solution of (33) is

$$U(p,t) = c_1 e^{ipct} + c_2 e^{-ipct} \qquad ...(35)$$

where c_1 and c_2 are constants

Putting $t = 0$ in (35) and its differential coefficient and using (34), we get

$$c_1 + c_2 = \bar{f}(p) \text{ and } c_1 - c_2 = \frac{1}{ipc} \bar{g}(p)$$

Solving for c_1 and c_2 we get

$$\bar{u}(p,t) = \frac{1}{2} \bar{f}(p) \left(e^{ipct} + e^{-ipct} \right) + \frac{\bar{g}(p)}{2icp} \left(e^{ipct} - e^{-ipct} \right)$$

Taking the inverse Fourier transform, we get

$$U(x,t) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(p) \left(e^{ipct} + e^{-ipct} \right) e^{-ipx} dp$$

$$+\frac{1}{2c\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\bar{g}(p)}{ip}\left(e^{ipct}-e^{-ipct}\right)e^{-ipx}dp$$

$$=\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(e^{-ip(x+ct)}+e^{-ip(x-ct)}\right)\bar{f}(p)\,dp$$

$$+\frac{1}{2c\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\bar{g}(p)}{ip}\left(e^{-ip(x-ct)}-e^{-ip(x+ct)}\right)dp$$

$$=\frac{1}{2}\left[f(x+ct)+f(x-ct)\right]+\frac{1}{2c\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{g}(p)\,dp\int_{x-ct}^{x+ct}e^{-ipu}\,du$$

$$=\frac{1}{2}\left[f(x+ct)+f(x-ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}du\,.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ipu}\,\bar{g}(p)\,dp$$

Thus

$$U(x,t)=\frac{1}{2}\left[f(x+ct)+f(x-ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(u)\,du \qquad ...(36)$$

Particular Case : If $g(x)=e^{-x^2}$, $g(x)=0$ and $c=1$, then solution (36) becomes,

$$U(x,t)=\frac{1}{2}\left[e^{-(x-t)^2}+e^{-(x+t)^2}\right]$$

7.3 Application of Hankel Transform to Boundary Value Problems

Now, some special type of differential equations will be solved with the applicaton of Hankel transform. While dealing with boundary value problem having symmetry about an axis, it is convenient to use polar coordinates. If the range of the radial variable is 0 to ∞, it can be removed conveniently by the application of Hankel transform. The solutions of the resulting equation will be a function of p and the remaining variables. Thus solution here to be “inverted” to recover the lost variable. The method will be more clear by the following illustrative examples.

Example 8 : Apply Hankel transform (of zero order) to solve the differential equation

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{\partial^2 U}{\partial z^2}=0, \qquad 0\le r\le\infty,\ z\ge 0$$

Satisfying the following conditions :

(i) $U\to 0$ as $z\to\infty$ and $r\to\infty$

(ii) $U=f(r)$ on $z=0$, $r\ge 0$. It is given that $U(r,z)$ is bounded.

Solution : Let $u(p,z)$ denote the Hankel transform of $U(r,z)$ with respect to r, for which $\nu=0$.

Therefore

$$u(p,z)=\int_0^\infty U(r,z)r\,J_0(pr)\,dr$$

Multiplying the given differential equation by the kernel $r\,J_0(pr)$ and integrating with respect to r from 0 to ∞, we get

$$\int_0^\infty\left(\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}\right)r\,J_0(pr)\,dr+\int_0^\infty\frac{\partial^2 U}{\partial z^2}r\,J_0(pr)\,dr=0$$

Using the result $\int_0^\infty\left(\frac{d^2 f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right)x\,J_0(px)\,dx=-p^2 f$, this yields

$$-p^2u+\frac{d^2u}{dy^2}=0 \qquad \text{or} \qquad \frac{d^2u}{dy^2}-p^2u=0$$

whose solution is $u(p,z)=c_1e^{pz}+c_2e^{-pz}$.

Since U is finite and so u is finite as $z\to\infty$, consequently $c_1=0$ otherwise u becomes infinite as $z\to\infty$.

Thus we have $u(p,z)=c_2\,e^{-pz}$ ...(37)

Also on taking the Hankel transform of order zero of the given boundary condition (ii), we get

$$u(p,0)=\int_0^\infty f(r)r\,J_0(pr)\,dr=\bar{f}(p)\quad\text{(say)} \qquad ...(38)$$

Putting $z=0$ in (37) and using (38), we get $c_2=\bar{f}(p)$.

Hence (37) reduces to

$$u(p,z)=\bar{f}(p)\,e^{-pz}$$

Applying the inversion formula for Hankel transform, we have

$$U(r,z)=\int_0^\infty p\,J_0(pr)\,u(p,z)\,dp=\int_0^\infty p\,J_0(pr)\,\bar{f}(p)\,e^{-pz}\,dp$$

Which is the required solution.

Example 9 : Find the potential $V(r,z)$ of a field due to a flat circular disc of unit radius with its centre at the origin and axis along the $z-$axis satisfying the differential equation

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{\partial^2 V}{\partial z^2}=0,\ 0\le r\le\infty,\ z\ge 0$$

and satisfying the boundary conditions :

$V=V_0$ when $z=0$, $0\le r<1$,

and $\frac{\partial V}{\partial z} = 0$, when $z = 0\ r > 1$,

Solution : Let $v\{p,z\}$ denote the Hankel transform of $V\{r,t\}$ with respect to r, for which $v = 0$. Therefore

$$v(p,z) = \int_0^\infty V(r,z) r\, J_0(pr)\, dr$$

Multiplying the given differential equation by the kernel $r\, J_0(pr)$ and integrating with respect to r from 0 to ∞, we have

$$\int_0^\infty \left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} \right) r\, J_0(pr)\, dr + \int_0^\infty \frac{\partial^2 V}{\partial z^2}\, r\, J_0(pr)\, dr = 0$$

Using the result $\int_0^\infty \left(\frac{d^2 f}{dx^2} + \frac{1}{x}\frac{df}{dx} \right) x\, J_0(px)\, dx = -p^2 f$

This yields

$$-p^2 v(p,z) + \frac{d^2}{dz^2} \int_0^\infty V(r,z)\, r\, J_0(pr)\, dr = 0$$

Therefore

$$\frac{d^2 v}{d z^2} - p^2 v = 0$$

whose general solution is

$$v(p,z) = c_1\, e^{-pz} + c_2\, e^{-pz}$$

For a bounded solution, we must have $c_1 = 0$, for otherwise $v \to \infty$ (since $p > 0$). Therefore

$$v(p,z) = c_2\, e^{-pz}, \qquad ...(39)$$

where c_2 is independent of z i.e. c_2 is a function of p only. Therefore we may write $c_2(p)$ in place of c_2 also. Applying the inversion formula, we have

$$V(r,z) = \int_0^\infty c_2(p)\, e^{-pz}\, p\, J_0(pr)\, dp \qquad ...(40)$$

and $$\frac{\partial V}{\partial z} = \int_0^\infty c_2(p)\, e^{-pz} \left(-p^2\right) J_0(pr)\, dp \qquad ...(41)$$

$c_2(p)$ is determined from the 'dual' integral equations obtained by substituting (40) and (41) in the given boundary conditions. Thus inserting (40) and (41) in the given boundary conditions and setting

$z = 0$, we have

$$\int_0^\infty p\,c_2(p)\; J_0(pr)\,dp = V_0,\ \text{for}\ 0 \le r < 1 \qquad ...(42)$$

and $$\int_0^\infty p^2\,c_2(p)\; J_0(pr)\,dp = 0,\ \text{for}\ r > 1 \qquad ...(43)$$

But we know that

$$\int_0^\infty J_0(pr)\frac{\sin p}{p}\,dp = \frac{\pi}{2}, \qquad 0 \le r < 1 \qquad ...(44)$$

and $$\int_0^\infty J_0(pr)\sin p\; dp = 0, \qquad r > 1 \qquad ...(45)$$

Comparing (42) and (43) with (44) and (45), we have

$$c_2(p) = \frac{2}{\pi}\,V_0\,\frac{\sin p}{p^2}$$

Substituting this in (40), we get

$$V(r,z) = \frac{2V_0}{\pi}\int_0^\infty \frac{e^{-pz}}{p}\;J_0(pr)\;\sin p\;dp$$

which is the required result.

Example 10 : The free symmetric vibrations of a very large membrane are governed by the equation

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2}, \qquad r > 0,\ t > 0$$

with $U = f(t),\ \dfrac{\partial U}{\partial r} = g(r),\ t = 0$

Show that for $t > 0$

$$U(r,t) = \int_0^\infty p\,F(p)\cos(pct)\,J_0(pr)\;dp + \frac{1}{c}\int_0^\infty G(p)\sin(pct)\,J_0(pr)\;dp$$

where $F(p)$ and $G(p)$ are the zero order Hankel transform of $f(r)$, $g(r)$ respectively.

Solution : Let $u(p,t)$ denote the Hankel transform of $U(r,t)$ with respect to r for which $\nu = 0$. Therefore

$$u(p,t) = \int_0^\infty U(r,t)\;r\,J_0(pr)\;dr$$

Multiplying the given differential equation by the kernel $r\,J_0(pr)$ and integrating with respect to r from 0 to ∞, we have

$$\int_0^\infty \left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} \right) r\, J_0(pr)\, dr = \frac{1}{c^2} \int_0^\infty \frac{\partial^2 U}{\partial t^2} r\, J_0(pr)\, dr$$

Using the rsult $\int_0^\infty \left(\frac{d^2 f}{d x^2} + \frac{1}{r} \frac{d f}{d x} \right) x\, J_0(px)\, dx = -p^2 f$, we have ...(46)

$$-p^2 u(p,t) = \frac{1}{c^2} \frac{d^2}{dt^2} \int_0^\infty U(r,t) r\, J_0(pr)\, dr$$

Therefore $\frac{d^2 u}{dt^2} + c^2 p^2 u = 0$

whose general solution is

$$u(p,t) = A\cos(cpt) + B\sin(cpt) \quad ...(47)$$

and $\frac{du}{dt} = -Ac\, p\sin(cpt) + B\, c\, p\cos(cpt)$ ...(48)

Taking the Hankel transform of few order of the given boundary conditions with respect to r , we have

$$u(p,0) = \int_0^\infty f(r) r\, J_0(pr)\, dr = F(p) \text{ (given)} \quad ...(49)$$

and $\frac{du}{dt} = \int_0^\infty g(r) r\, J_0(pr)\, dr = G(p)$ (given) when $t = 0$. ...(50)

Putting $t = 0$ in (47) and (48) and using the conditions (49) and (50), we get

$$A = F(p)\ ;\ Bcp = G(p) \text{ i.e. } B = \frac{G(p)}{cp}$$

Substituting these in equation (47), we get

$$u(p,t) = F(p)\cos(cpt) + \frac{G(p)}{cp}\sin(cpt)$$

Applying the inversion formula, we get

$$U(r,t) = \int_0^\infty p\, F(p)\cos(cpt)\, J_0(pr)\, dp + \frac{1}{c} \int_0^\infty G(p)\sin(cpt)\, J_0(pr)\, dp$$

which is the required solution.

Example 11 : Heat is supplied at a constant rate Q per in the plane $z = 0$ to an infinite solid of conductivity K. Show that the steady temperature at a point distant r from the axis of the circular area and distance z from the plate $r = 0$ is given by

$$\frac{Qa}{2K} \int_0^\infty e^{-pz} J_0(pr) J_1(pa) p^{-1} dp$$

Solution : Let $U(r,z)$ be the temperature at the point (r,z) then it is governed by differential equation

$$\frac{\partial U}{\partial t} = K\left[\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2}\right]$$

Since the temperature is steady, therefore $\frac{\partial U}{\partial t} = 0$. Hence we get

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} = 0 \qquad ...(51)$$

with boundary conditions

$$2\left(-K\frac{\partial U}{\partial z}\right) = Q, \qquad 0 \le r < a$$

and $$2\left(-K\frac{\partial U}{\partial z}\right) = 0, \qquad r \ge a$$

when $z = 0$

Since there is a symmetry about the plane $z = 0$, hence we shall find out temperature only for the case $z > 0$. Let $u(p,z)$ denote the Hankel transform of $U(r,z)$ with respect to r, for which $v = 0$. Therefore

$$u(p,z) = \int_0^\infty U(r,z)\, r\, J_0(pr)\, dr \qquad ...(52)$$

Taking the Hankel transform of order zero of (51), we have

$$\int_0^\infty \left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right) r\, J_0(pr)\, dr + \int_0^\infty \frac{\partial^2 U}{\partial z^2}\, r\, J_0(pr)\, dr = 0$$

Using the result (46) this yields.

$$-p^2 u(p,z) + \frac{d^2}{dz^2}\int_0^\infty U(r,z)\, r\, J_0(pr)\, dr = 0$$

Therefore $$\frac{d^2 u}{dz^2} - p^2 u = 0$$

whose general solution is

$$u(p,z) = A\, e^{pz} + B\, e^{-pz} \qquad ...(53)$$

Also on taking the Hankel transform of order zero of the given boundary conditions with respect to r, we have

$$\frac{du}{dz} = \int_0^a \left(-\frac{Q}{2K}\right) r\, J_0(pr)\, dr + \int_a^\infty 0 . r\, J_0(pr)\, dr = 0, \text{ when } z = 0$$

or $$\frac{du}{dz} = -\frac{Q}{2K} \int_0^a r\, J_0(pr)\, dr, \text{ when } z = 0 \qquad ...(54)$$

Now, writing pr for r and $v = 1$ in the recurrence relation (vi) of 6.2 (II), we have

$$\frac{d}{d(pr)} \{pr\, J_1(pr)\} = pr\, J_0(pr)$$

This gives $\frac{1}{p}\frac{d}{dr}\{r\, J_1(pr)\} = r\, J_0(pr)$

Substituting this in (54) we get

$$\frac{du}{dz} = -\frac{Q}{2K}\int_0^a \frac{1}{p}\frac{d}{dr}\{r\, J_1(pr)\} dr, \quad \text{when } z = 0$$

$$= -\frac{Q}{2K}\frac{1}{p}\left[r\, J_1(pr)\right]_0^a, \quad \text{when } z = 0$$

or $$\frac{du}{dz} = -\frac{Q}{2K} . \frac{a}{p} J_1(ap), \text{ when } z = 0 \qquad ...(55)$$

For a bounded solution, we must have $A = 0$ for otherwise $u \to \infty$ as $z \to \infty$ (since p>0). Therefore

$$u(p,z) = B\, e^{-pz} \qquad ...(56)$$

and $$\frac{du}{dz} = -B\, p\, e^{-pz} \qquad ...(57)$$

Putting $z = 0$ in (57) and using (55), we get

$$-Bp = -\frac{Q}{2K}\frac{a}{p} J_1(pa) \qquad \text{or} \qquad B = \frac{Q}{2K} . \frac{a}{p^2} J_1(ap)$$

$\therefore$ From (56), we get

$$u(p,z) = \frac{Q\, a}{2K\, p^2} e^{-pz} J_1(ap)$$

Applying the inversion formula in above equation, we get

$$U(r,z) = \int_0^\infty u(p,z)\, p\, J_0(pr)\, dp$$

$$= \int_0^\infty \frac{Qa}{2Kp^2} J_1(ap) e^{-pz} p J_0(pr) dp$$

or $$U(r,z) = \frac{Qa}{2K} \int_0^\infty e^{-pz} J_0(pr) J_1(pa) p^{-1} dp$$

Self-Learning Exercise

1. If we want to remove the term $\dfrac{\partial^2 U}{\partial x^2}$ from a p.d.e. then at $x = 0$, what type of condition is we required in the case of (i) Fourier cosine transform and (ii) Fourier sine transform

Fill in the Blanks

2. If the differential equation ranges from $-\infty$ to ∞ then ....Fourier transform can be used to solve a boundary value problem.

3. If we apply Hankel transform of order zero w.r.t. variable r to p.d.e.

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial w}{\partial z^2} = 0,$$

then we get

$$u(p,z) = ...$$

where $u(p,z) = \int_0^\infty U(r,z) r J_0(pr) dr$

7.4 Summary

In this unit you studied the applications of Fourier and Hankel transforms to solve boundary value problems. The different methods were explained with the help of practical problems

7.5 Answers to Self-Learning Exercise

1. (i) $\dfrac{\partial U}{\partial x}$ at $x = 0$ (ii) $U(x,t)$ at $x = 0$

2. Complex 3. $c_1 e^{pz} + c_2 e^{-pz}$ (c_1 and c_2 are constants)

7.6 Exercise-7

1. Solve $\dfrac{\partial V}{\partial t} = \dfrac{\partial^2 V}{\partial x^2}$, if $U(0,t) = 0$, $U(x,0) = e^{-x}$, $x > 0$, $U(x,t)$ is bounded where $x > 0$, $t > 0$.

$$\left[\textbf{Ans. } U(x,t) = \frac{2}{\pi} \int_0^\infty \frac{p}{1+p^2} e^{-2p^2 t} \sin px \, dp \right]$$

2. If θ is the temperature at time t and k the diffusivity of the material, find θ from the partial differential eqaution $\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2}$, $x > 0$, $t > 0$, subject to the boundary condition $\theta = \theta_0$, when $x = 0$, $t > 0$ and the initial condition $\theta = 0$ when $t = 0$, $x > 0$.

$$\left[\textbf{Ans. } \theta(x,t) = \theta_0 \left[1 - \frac{2}{\pi} \int_0^\infty \frac{e^{-k p^2 t}}{p} \sin p x \, dp \right] \right]$$

3. Use a cosine transform to show that the steady temperature in the semi-infinite solid $y > 0$ when the temperature on the surface $y = 0$ is kept at unity over the strip $|x| < a$ and at zero outside the strip, is

$$\frac{1}{\pi} \left[\tan^{-1} \frac{a + x}{y} + \tan^{-1} \frac{a - x}{y} \right]$$

The result $\int_0^\infty e^{-px} x^{-1} \sin r x \, dx = \tan^{-1}\left(\frac{r}{p}\right)$, $r > 0$, $p > 0$ may be assumed.

4. If the function $U(x, y)$ is determined by the differential equation

$\frac{\partial U}{\partial x} = \frac{\partial^2 U}{\partial y^2}$ for $x > 0$, $-\infty < y < \infty$ and $U = f(y)$ when $x = 0$ then show that

$$U(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(p) \, e^{-p^2 x - ipy} dp$$

where $\bar{f}(p)$ is the Fourier transform of $f(x)$.

5. Show that the solution of Laplace equation for U inside the semi-inifinite strip $x > 0$, $0 < y < b$, such that

$U = f(x)$, when $y = 0$, $0 < x < \infty$

$U = 0$, when $y = b$, $0 < x < \infty$

$U = 0$, when $x = 0$, $0 < y < b$

is given by $U = \frac{2}{\pi} \int_0^\infty f(u) du \int_0^\infty \frac{\sin h(b - y) p}{\sin h \, pb} \sin x p \, \sin u p \, dp$

6. Use the method of Fourier transform to determine the displacement $y(x, t)$ of an infinite string, given that the string is initially at rest and that the initial displacement is $f(x)$, $-\infty < x < \infty$. Show that the solution can also be put in the form :

$$y(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

7. Solve the Laplace equation in the half plane

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial z^2} = 0, \qquad (-\infty < x < \infty,\ y \geq 0)$$

with the boundary conditions

$$U(x,0) = f(x),\ -\infty < x < \infty$$

and $U(x,y) \to 0$ as $|x| \to \infty,\ y \to \infty$

$$\left[\textbf{Ans. } U(x,y) = \frac{y}{\pi(x^2+y^2)} \right]$$

8. The magnetic potential U for a circular disc of radius a and strength w, magnetized parallel to its axis, satisfying Laplace's equation, is equal to $2\pi w$ on the disc itself and vanishes at exterior point in the plane of the disc. Show that at the point (r,z), $z>0$

$$U = 2\pi a w \int_0^\infty e^{-pz} J_0(pr) J_1(ap)\, dp$$

Unit - 8
Linear Integral Equation

Structure of the Unit

8.0 Objective

The aim of this unit is to define Linear integral equation, conversion to an initial and boundary value problem to an integral equation. Eigenvalues, eigenfunctions and solution of homogenous Fredholm integral equation of second kind with separable kernels are also dicussed.

8.1 Introduction

In the recent years, the theory of integral equations has become an essential part of mathematical analysis. Foremost among, there are differential equations and operator theory. Many physical problems which are usually solved by ordinary and partial differential equation methods can be solved more effectively by integral equation methods. Many existence and uniqueness results can then be derived from the corresponding results from integral equations.

Integral equation arise in several problems of applied mathematics, mathematical physics and theoretical mechanics. Its importance for physical problems lies in the fact that most differential equation together with their boundary conditions may be reformulated to give a single integral equation. The theory of integral equation also furnishes a uniform method for the study of the eigenvalue problems of mathematical physics.

8.2 Linear Integral Equation : Definition and Classification

Definition I : Integral Equation : An integral equation is an equation in which an unknown function to be determined appears under one are or more integral signs.

For Example

$$g(x)=\int_a^x K(x,t)g(t)dt, \quad ...(1)$$

$$g(x)=f(x)+\int_a^b K(x,t)g(t)dt, \quad ...(2)$$

$$g(x)=\int_a^b K(x,t)[g(t)]^2 dt, \quad ...(3)$$

where $a \le x \le b$ and $a \le t \le b$.

Here the function $g(x)$ is the unknown function while all other functions are known. These functions may be complex valued functions of the real variables x and t.

Definition II : Linear and Non Linear Integral Equations : An integral equation is called linear if only linear operations are performed in it upon the unknown functions. If integral equation is not linear then it is known as non-linear integral equation.

For example, the integral equation (1) and (2) are linear while (3) is non linear.

The most general type of linear integral equation is of the form

$$\alpha(x)g(x)=f(x)+\lambda\int_{\Omega} K(x,t)g(t)dt \quad ...(4)$$

when the upper limit may be variable x or fixed. The functions f, α and K are known functions, while g is to be determined; λ is a non-zero real or complex parameter. The function $K(x,t)$ is known as the kernel of the integral equation. The integration extends over the domain Ω of the auxillary variable t.

Integral equations, which are linear involve the linear operator

$$L[\ ]=\int_{\Omega} K(x,t)[\ ]\,dt$$

having the kernel $K(x,t)$. It satisfies the linearity condition

$$L\{c_1 g_1(t)+c_2 g_2(t)\}=c_1 L\{g_1(t)\}+c_2 L\{g_2(t)\}$$

where $L\{g(t)\} = \int_{\Omega} K(x,t) g(t)\, dt$ and c_1, c_2 are constants.

Linear integral equations are classified into the basic types.

(i) Volterra Integral Equation :

An integral equation is said to be a **Volterra ntegral quation** if the upper limit of integration is a variable, e.g.

$$\alpha(x) g(x) = f(x) + \lambda \int_a^x K(x,t) g(t) dt \qquad ...(5)$$

where a is a constant, $f(x)$, $\alpha(x)$ and $K(x,t)$ are known functions while $g(x)$ is unknown function, λ is a non-zero real or complex parameter. Equation (5) is called **Volterra integral equation of third kind.**

(a) When $\alpha = 0$, the unknown function g appears only under the integral sign and nowhere else in the equation (5) then

$$f(x) + \lambda \int_a^x K(x,t) g(t)\, dt = 0 \qquad ...(6)$$

is called the **Volterra integral equations of first kind.**

(b) When $\alpha = 1$, the equation (5) involves the unknown function g, both sides as well as outside the integral sign, then

$$g(x) = f(x) + \lambda \int_a^x K(x,t) g(t)\, dt \qquad ...(7)$$

is called the **Volterra's integral equation of second kind.**

(c) When $\alpha = 1$, $f(x) = 0$, the equation (5) reduces to

$$g(x) = \lambda \int_a^x K(x,t) g(t)\, dt \qquad ...(8)$$

is called the homogenous **Volterra's integral equation of second kind.**

(ii) Fredholm Integral Equation :

An integral equation is said to be **Fredholm integral equation** if the upper limit of integration is fixed, say b e.g.

$$\alpha(x) g(x) = f(x) + \lambda \int_a^b K(x,t) g(t)\, dt \qquad ...(9)$$

where a and b are both constants, $f(x)$, $\alpha(x)$ and $K(x,t)$ are known functions while $g(x)$ is unknown function and λ is a non-zero real or complex parameter. Equation (9) is called **Fredholm integral equation of third kind.**

(a) When $\alpha = 0$, equation (9) involves unknown function g only under the integral sign, then

$$f(x) + \lambda \int_a^b K(x,t) g(t) dt = 0 \qquad ...(10)$$

is called the **Fredholm integral equation of first kind.**

(b) When $\alpha = 1$, equation (9) involves the unknown function g inside as well out side the integral sign, then

$$g(x) = f(x) + \lambda \int_a^b K(x,t)\, g(t)\, dt \qquad ...(11)$$

is known as **Fredholm integral equation of second kind.**

(c) When $\alpha = 1$, $f(x) = 0$, equation (9) reduces to

$$g(x) = \lambda \int_a^b K(x,t)\, g(t)\, dt \qquad ...(12)$$

is known as the **homogenous Fredholm integral equation of second kind.**

(iii) Singular Integral Equation :

An integral equation is said to be **singular** when one or both limits of integration became infinite, or if the kernel becomes infinite at one or more points of the integral under consideration.

For Example $f(x) = \int_0^{\infty} \sin(x,t)\, g(t)\, dt$,

$$g(x) = f(x) + \lambda \int_{-\infty}^{\infty} K(x,t)\, g(t)\, dt,$$

$$f(x) = \int_a^x \frac{K(x,t)}{(x-t)^{\alpha}}\, g(t)\, dt, \qquad 0 < r < 1$$

$$f(x) = \int_a^x \frac{g(t)}{(x-t)^{\alpha}}\, dt, \qquad 0 < \alpha < 1$$

are singular integral equations.

(iv) Integral Equation of Convolution Type :

If the kernel $K(x,t)$ of the integral equation is defined as a function of the difference $(x-t)$, i.e. $K(x,t) = K(x-t)$, where K is a certain function of one variable, then the integral equation

$$g(x) = f(x) + \lambda \int_a^x K(x-t)\, g(t)\, dt,$$

and the corresponding Fredholm inegral equation

$$g(x) = f(x) + \lambda \int_a^b K(x-t)\, g(t)\, dt,$$

are called integral equation of convolution type.

8.3 Special Kinds of Kernels

(i) Separable or Degenerate Kernel :

A kernel $K(x,t)$ is said to be separable or degenerate if it can be expressed as the sum of a finite number of terms, each of which is the product of a function of x only and a function of t only i.e.

$$K(x,t) = \sum_{i=1}^{n} a_i(x)\; b_i(t) \qquad ...(13)$$

The function $a_i(x)$ can be assumed to be linearly independent, otherwise the number of terms in the relation (13) can be reduced by linear independence of the functions $a_i(x)$. It is meant that, if $c_1 a_1(x) + c_2 a_2(x), ...c_n a_n(x) = 0$, where c_i are arbitrary constants, then $c_1 = c_2 = ... = c_n = 0$.

(ii) Symmetric Kernel :

A complex valued function $K(x,t)$ is called **symmetric** (or Hermition) if $K(x,t) = \overline{K(t,x)}$, where the bar denotes the complex conjugate. For a real kernel this concides with definition $K(x,t) = K(t,x)$.

8.4 Useful Results

(a) Leibnitz's rule of differentiation under the integral sign :

Let $F(x,t)$ and $\frac{\partial F}{\partial x}$ be continuous functions of both x and t and let the first derivatives of $g(x)$ and $h(x)$ be continuous. Then

$$\frac{d}{dx}\int_{g(x)}^{h(x)} F(x,t)\,dt = \int_{g(x)}^{h(x)} \frac{\partial F}{\partial x}\,dt + F[x,h(x)]\frac{dh}{dx} - F[x,g(x)]\frac{dg}{dx} \qquad ...(14)$$

Particular Case :

If g and h are absolutely constants, then (14) reduces to

$$\frac{d}{dx}\int_{g}^{h} F(x,t)\;dt = \int_{g}^{h} \frac{\partial F}{\partial x}\;dt \qquad ...(15)$$

(b) A formula for converting a multiple integral into a single ordinary integral :

$$\int_a^x g(t)\;dt^n = \int_a^x \frac{(x-t)^{n-1}}{(n-1)!}\,g(t)\,dt \qquad ...(16)$$

Note that the integral on the L.H.S. of (16) is a mutliple integral of order n while the integral on the R.H.S. of (16) is ordinary integral of order one.

8.5 Solution of an Integral Equation

A solution of an integral equation is a function $g(x)$, which when substituted into the equation reduces to an identity (wtih respect to x).

Example 1 : Show that the function $g(x) = e^x\left(2x - \frac{2}{3}\right)$ is a solution of the Fredholm equation

$$g(x) + 2\int_0^1 e^{x-t} g(t)\, dt = 2x e^x$$

Solution : Substituting the value of $g(x)$ in L.H.S. of the given equation, we have

$$\text{L.H.S.} = e^x\left(2x - \frac{2}{3}\right) + 2\int_0^1 e^x\left(2t - \frac{2}{3}\right) dt$$

$$= e^x\left(2x - \frac{2}{3}\right) + 2e^x\left[t^2 - \frac{2}{3}t\right]_0^1 = 2x e^x = \text{R.H.S.}$$

Hence $g(x)$ is a solution of given integral equation.

Example 2 : Show that the function $g(x) = \left(1 + x^2\right)^{-3/2}$ is a solution of the Volterra integral equation

$$g(x) = \frac{1}{1+x^2} - \int_0^x \frac{t}{1+x^2} g(t)\, dt$$

Solution : Substituting $g(x)$ in the R.H.S. of the given equation, we have

$$\text{R.H.S.} = \frac{1}{1+x^2} - \int_0^x \frac{t}{\left(1+x^2\right)}\left(1+t^2\right)^{-3/2} dt$$

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2}\int_0^{x^2} \frac{1}{2}(1+z)^{-3/2}\, dz \qquad \left[\text{Putting } t^2 = z \text{ and } 2t\,dt = dz\right]$$

$$= \frac{1}{1+x^2} + \frac{1}{1+x^2}\left[\frac{1}{(1+z)^{1/2}}\right]_0^{x^2}$$

$$= \frac{1}{1+x^2} + \frac{1}{\left(1+x^2\right)}\left[\frac{1}{\left(1+x^2\right)^{1/2}} - 1\right]$$

$$= \frac{1}{\left(1+x^2\right)^{3/2}} = \text{L.H.S.}$$

Hence $g(x)$ is a solution of the given integral equation.

Example 3 : Show that the function $g(x) = x e^x$ is a solution of the Volterra integral equation

$$g(x) = \sin x + 2 \int_0^x \cos(x-t)\, g(t)\, dt$$

Solution : Substituting $g(x)$ in the R.H.S. of the given equation, we have

$$\text{R.H.S.} = \sin x + 2 \int_0^x t\, e^t \cos(t-x)\, dt$$

$$= \sin x + 2 \left[t \cdot \frac{e^t}{2} \{\cos(t-x) + \sin(t-x)\} \right]_0^x$$

$$-2 \int_0^x 1 \cdot \frac{e^t}{2} \{\cos(t-x) + \sin(t-x)\}\, dt$$

{Integrating by parts and using the following standard results :

$$\int e^{ax} \sin(bx+c)\, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin(bx+c) - b\cos(bx+c)\right] \quad \}$$

$$\int e^{ax} \cos(bx+c)\, dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos(bx+c) + b\sin(bx+c)\right]$$

$$\text{Thus R.H.S.} = \sin x + x e^x - \int_0^x e^t \cos(t-x)\, dt - \int_0^x e^t \sin(t-x)\, dt$$

$$= \sin x + x e^x - \left[\frac{e^t}{2} \{\cos(t-x) + \sin(t-x)\} \right]_0^x$$

$$- \left[\frac{e^t}{2} \{\sin(t-x) - \cos(t-x)\} \right]_0^x$$

$$= \sin x + x e^x - \left[\frac{e^x}{2} - \frac{1}{2}(\cos x - \sin x) \right] - \left[-\frac{e^x}{2} - \frac{1}{2}(-\sin x - \cos x) \right]$$

$$= x e^x = g(x) = \text{ L.H.S.}$$

Hence $g(x)$ is a solution of the given integral equation.

Example 4 : Show that the function $g(x) = \sin\left(\frac{\pi x}{2}\right)$ is a solution of the Fredholm integral equation

$$g(x) - \frac{\pi^2}{4} \int_0^1 K(x,t)\, g(t)\, dt = \frac{x}{2}$$

where $K(x,t)=\begin{cases}\dfrac{x\,(2-t)}{2}, & 0\le x\le t\\ \dfrac{t\,(2-x)}{2}, & t\le x\le 1\end{cases}$

Solution : Writing the L.H.S. of the given equation as follows :

$$g(x)-\frac{\pi^2}{4}\int_0^1 K(x,t)\,g(t)\,dt$$

$$\text{L.H.S.} = g(x)-\frac{\pi^2}{4}\left[\int_0^x K(x,t)\,g(t)\,dt+\int_x^1 K(x,t)\,g(t)\,dt\right]$$

$$= g(x)-\frac{\pi^2}{4}\left[\frac{(2-x)}{2}\int_0^x t\,g(t)\,dt+\frac{x}{2}\int_x^1(2-t)\,g(t)\,dt\right]$$

Now substituting $g(x)=\sin\dfrac{\pi x}{2}$ in the above expression, we get

$$\text{L.H.S.} = \sin\frac{\pi x}{2}-\frac{\pi^2}{8}(2-x)\int_0^x t\sin\frac{\pi t}{2}\,dt-\frac{\pi^2 x}{8}\int_x^1(2-t)\sin\frac{\pi t}{2}\,dt$$

$$= \sin\frac{\pi x}{2}-\frac{\pi^2(2-x)}{8}\left[\left\{t\left(\frac{-\cos(\pi t/2)}{\pi/2}\right)\right\}_0^x\right.$$

$$\left.-\int_0^x 1\left(\frac{-\cos(\pi t/2)}{\pi t/2}\right)dt\right.-\frac{\pi^2 x}{8}\left[\left\{(2-t)\left(\frac{-\cos(\pi t/2)}{\pi/2}\right)\right\}_x^1\right.$$

$$\left.-\int_x^1(-1)\left(\frac{-\cos(\pi t/2)}{\pi/2}\right)dt\right]$$

$$= \sin\frac{\pi x}{2}-\frac{\pi^2(2-x)}{8}\left[-\frac{2x}{\pi}\cos\frac{\pi x}{2}+\left(\frac{\sin(\pi t/2)}{(\pi/2)^2}\right)_0^x\right]$$

$$-\frac{\pi^2 x}{8}\left[\frac{2(2-x)}{\pi}\cos\frac{\pi x}{2}-\left(\frac{\sin(\pi t/2)}{(\pi/2)^2}\right)_x^1\right]$$

$$= \sin\frac{\pi x}{2}-\frac{\pi^2(2-x)}{8}\left[-\frac{2x}{\pi}\cos\frac{\pi x}{2}+\frac{4}{\pi^2}\sin\frac{\pi x}{2}\right]$$

$$-\frac{\pi^2 x}{8}\left[\frac{2(2-x)}{\pi}\cos\frac{\pi x}{2}-\frac{4}{\pi^2}+\frac{4}{\pi^2}\sin\frac{\pi x}{2}\right]$$

$$=\sin\frac{\pi x}{2}\left\{1-\frac{1}{2}(2-x)-\frac{x}{2}\right\}+\frac{x}{2}=\frac{x}{2}=\text{ R.H.S.}$$

Hence $g(x)=\sin\frac{\pi x}{2}$ is a solution of the given integral equation.

Self-Learning Exercise - 1

Define the following terms :

1. Integral equation.
2. Linear and non-linear integral equations.
3. Singular integral equation.
4. Convolution integral equation.
5. Fredholim and Volterra integral equation of first and second kinds.

Fill in the blanks :

6. $-x=\int_0^x e^{x-t}g(t)dt$ is ..... integral equation of........ kind.

7. $g(x)-\frac{\pi^2}{4}\int_0^1 K(x,t)g(t)dt=\frac{x}{2}$, where $K(x,t)=\begin{cases}\frac{x}{2}(2-t), & 0\le x\le t\\ \frac{t}{2}(2-x), & t\le x\le 1\end{cases}$ is .... integral equation of...... kind.

8.6 Exercise 8 (a)

1. Verify whether the given functions, $g(x)$ are solution of the corresponding Volterra's integral equations :

(a) $g(x)=1-x;\ \int_0^x e^{x-t}g(t)\,dt=x$

(b) $g(x)=3\ ;\ x^3=\int_0^x (x-t)^2 g(t)\,dt$

(c) $g(x)=x-\frac{x^3}{6}\ ;\ g(x)=x-\int_0^x \sin h(x-t)\,g(t)\,dt$

(d) $g(x)=1-\frac{2\sin x}{[1-(\pi/2)]}\ ;\ g(x)-\int_0^x \cos(x+t)\,g(t)\,dt=1$

2. Examine whether the given functions $g(x)$ solutions of the following Fredholm integral equations

(a) $g(x) = x e^{-x}$; $g(x) - 4 \int_0^\infty e^{-(x+t)} g(t)\,dt = (x-1)e^x$

(b) $g(x) = e^x$; $g(x) + \lambda \int_0^\infty \sin x t \, g(t)\,dt = 1$

(c) $g(x) = \cos x$; $g(x) - \int_0^\pi (x^2 + t)\cos g(t)\,dt = \sin x$

(d) $g(x) = \sqrt{x}$; $g(x) - \int_0^1 K(x,t) g(t)\,dt = \sqrt{x} + \frac{x}{15}(4x^{3/2} - 7)$

where $K(x,t) = \begin{cases} \frac{1}{2} x(2-t), & 0 \le x \le t \\ \frac{1}{2} t(2-x), & t \le x \le 1 \end{cases}$

[Ans. (b) and (c) : given functions are not the solution of the corresponding integral equations]

3. Show that the function $g(x) = 1$ is a solution of the Fredholm integral equation

$$g(x) + \int_0^1 x\left(e^{xt} - 1\right) g(t)\,dt = e^x - x$$

4. Show that the function $g(x) = \frac{1}{2}$ is a solution of the integral equation

$$\int_0^x \frac{g(t)\,dt}{\sqrt{x-t}} = \sqrt{x}$$

5. Show that the function $g(x) = \frac{x}{\left(1+x^2\right)^{5/2}}$ is a solution of integral equation

$$g(x) = \frac{3x + 2x^3}{3\left(1+x^2\right)^2} - \int_0^x \frac{3x + 2x^3 - t}{\left(1+x^2\right)^2} - g(t)\,dt$$

6. Show that $g(x) = \cos 2x$ is a solution of the integral equation

$$g(x) = \cos x + 3\int_0^\pi K(x,t)\, g(t)\, dt \text{, where}$$

$$K(x,t) = \begin{cases} \sin x \cos t, & 0 \le x \le t \\ \cos x \sin t, & t \le x \le \pi \end{cases}$$

7. Show that the function $g(x) = \frac{1}{\pi\sqrt{x}}$ is a solution of the integral equation

$$\int_0^x \frac{g(t)}{\sqrt{x-t}}\, dt = 1$$

8. Give the definition and complete classification of linear integral equation.

8.6 Method of Converting an Intial Values Problem to a Volteira Integral Equation

When an ordinary differential equation is to be solved under conditions involving dependent variable and its derivatives at the same value of the independent variable, then the problem under consideration is said to be an initial value problem.

Consider the ordinary linear differential equation of order n :

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + a_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots.. + a_n(x)y = f(x) \qquad ....(17)$$

With the initial conditions

$$y(a) = C_0, y'(a) = C_1, \ldots., \; y^{(n-1)}(a) = C_{n-1} \qquad ...(18)$$

where the functions $a_1(x), a_2(x), \ldots.., a_n(x)$ and $f(x)$ are defined and continuous in $a \le x \le b$.

Now in order to reduce above initial value problem to the Volterra integral equation, we introduce an unknown function $g(x)$ as

$$\frac{d^n y}{dx^n} = g(x) \qquad ...(19)$$

Integrating (19) both sides with respect to *'x'* from a to x and using the initial conditions (18), we get

$$\frac{d^{n-1}y}{dx^{n-1}} = \int_a^x g(t)dt + C_{n-1} \qquad ...(20)$$

Again integrating (20) and using (18), we get

$$\frac{d^{n-2}y}{dx^{n-2}} = \int_a^x (x-t)\, g(t)dt + (x-a)C_{n-1} + C_{n-2} \qquad ..(21)$$

and so on.

Finally, we get

$$\frac{dy}{dx} = \int_a^x \frac{(x-t)^{n-2}}{(n-2)!} g(t)dt + \frac{(x-a)^{n-2}}{(n-3)!} C_{n-1} + \ldots. + \; (x-a)C_2 + C_1 \qquad ...(22)$$

and $$y=\int_a^x \frac{(x-t)^{n-1}}{(n-1)!}g(t)dt+\frac{(x-a)^{n-1}}{(n-1)!}C_{n-1}+\frac{(x-a)^{n-2}}{(n-2)!}C_{n-2}+\ldots+(x-a)\,C_1+C_0 \quad \ldots(23)$$

Multiplying (20), (21) ...., (22) and (23) by 1, $a_1(x),\ldots\ldots,a_{n-1}(x)$ and $a_n(x)$ respectively and adding, we get

or $$f(x)=g(x)+h(x)-\int_a^x K(x,t)g(t)dt$$

where we have used (17) and assumed the following :

$$h(x)=C_{n-1}\left\{a_1(x)+\ldots.+\frac{(x-a)^{n-1}}{(n-1)!}a_n(x)\right\}$$

$$+C_{n-2}\left\{a_2(x)+\ldots+\frac{(x-a)^{n-2}}{(n-2)!}a_n(x)\right\}+\ldots.+\;C_1\left\{a_{n-1}(x)+(x-a)a_n(x)\right\}+C_0a_n(x) \quad \ldots(24)$$

and $$K(x,t)=-\left[a_1(x)+(x-t)a_2(x)+\ldots.+\frac{(x-t)^{n-1}}{(n-1)!}a_n(x)\right]$$

$$=-\sum_{k=1}^{n}a_k(x)\frac{(x-t)^{k-1}}{(k-1)!} \quad \ldots(25)$$

Again, let $f(x)-h(x)=\phi(x)$, then ...(26)

$$g(x)=\phi(x)+\int_a^x K(x,t)\,g(t)dt \quad \ldots(27)$$

which is the required Volterra integral equation of second kind.

Example 5 : Form an integral equation corresponding to the differential equation

$$\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

with initial conditions $y(0)=1,\; y'(0)=0$

Solution : Let $\frac{d^2y}{dx^2}=g(x)$ ...(28)

Integrating both sides of (28) from 0 to x, we get

$$\frac{dy}{dx}=\int_0^x g(t)\,dt \qquad [\text{using } y'(0)=0] \quad \ldots(29)$$

Again integrating (29) from 0 to x and using $y(0)=1$, we have or $\quad y(x)=1+\int_0^x (x-t)\,g(t)\,dt \quad$ [using (16)] ...(30)

Now putting $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and y from (28), (29) and (30) in the given differential equation, we get

$$g(x)+x\int_0^x g(t)dt+1+\int_0^x (x-t)\,g(t)dt=0$$

or $\quad g(x)=-1-\int_0^x (2x-t)\,g(t)dt$

which is the required Volterra integral equation.

Example 6 : Form an integral equation coresponding to the differential equation :

$$\frac{d^3y}{dx^3}+x\frac{d^2y}{dx^2}+\left(x^2-x\right)y=xe^x+1$$

with initial conditions : $y(0)=1=y'(0)$ and $y''(0)=0$

Solution : Consider $\frac{d^2y}{dx^3}=g(x)$ ...(31)

Integrating (31) w.r.t. 'x' from o to x and using $y''(0)=0$ we have

$$\frac{d^2y}{dx^2}=\int_0^x g(t)dt \quad ...(32)$$

Integrating (32) w.r.t. 'x' from o to x and using $y'(0)=1$, we get

$$\frac{dy}{dx}=1+\int_0^x g(t)dt^2$$

or $\quad \frac{dy}{dx}=1+\int_0^x (x-t)g(t)dt \quad$ [using (16)] ...(33)

Integrating again (33) w.r. to '*x*' from *o* to *x* and using $y(0)=1$, we get

$$y=1+x+\int_0^x \frac{(x-t)^2}{2!}g(t)dt \quad ...(34)$$

Substituting the values of $\frac{d^3y}{dx^3}, \frac{d^2y}{dx^2}, \frac{dy}{dx}$ and *y* in the given differential equation, we have

$$g(x)+x\int_0^x g(t)dt+\left(x^2-x\right)\left[1+x+\int_0^x \frac{(x-t)^2}{2!}g(t)dt\right]=xe^x+1$$

or $$g(x)=xe^x+1-x(x^2-1)-\int_0^x\left[x+\frac{1}{2}(x^2-x)(x-t)^2\right]g(t)dt$$

which is the required integral equation.

8.8 Alternative method of converting an Initial Value Problem into Voltera Integral Equation

This method is very useful in problems we are required to derive the original differential equations with the initial conditions from the integral equation obtained. The method is illustrated with the help of following solved examples :

Example 7 : Convert the following differential equation into an integral equation :

$$\frac{d^2y}{dx^2}+\lambda\,x\,y=f(x)\ ;y(0)=1\,,\ y'(0)=0$$

Solution : Integrating both sides of given differential equation w.r.t. $'x'$ from 0 to x, we have

$$\frac{dy}{dx}-y'(0)=\int_0^x\left[f(x)-\lambda\,x\,y\right]dx$$

or $$\frac{dy}{dx}=\int_0^x\left[f(x)-\lambda\,x\,y\right]dx \qquad [\because y'(0)=0]$$

Integrating again the both sides of the above equations w.r.t. $'x'$ from 0 to x, we have

$$y(x)-y(0)=\int_a^x\left[f(x)-\lambda\,x\,y\right]dx^2$$

or $$y(x)=1+\int_a^x\left[f(t)-\lambda\,t\,y(t)\right]dt^2 \qquad [\because y(0)=1]$$

or $$y(x)=1+\int_a^x(x-t)\left[f(t)-\lambda\,t\,y(t)\right]dt$$

which is the required integral equation.

Example 8 : Reduce the differential equation

$$\frac{d^2y}{dx^2}-3\,\frac{dy}{dx}+2y=4\sin x$$

with the conditions $y(0)=1\,,\ y'(0)=-2$

into a non-homogenous Volterra's integral equation of second kind. Conversely, derive the original differential equation with the initial conditions from the integral equation obtained.

Solution : The given differential equation may be written as

$$y''(x) = 4\sin x - 2y(x) + 3y'(x) \qquad ...(35)$$

Integrating both sides w.r.t. $'x'$ from 0 to x and using $y(0)=1$, $y'(0)=-2$, we get

$$y'(x) - y'(0) = -4(\cos x - 1) - 2\int_0^x y(x)\,dx + 3[y(x) - y(0)]$$

or $$y'(x) = -1 - 4\cos x + 3y(x) - 2\int_0^x y(x)\,dx \qquad ...(36)$$

Integrating (36) w.r.t. $'x'$ from 0 to x, we have

or $$y(x) - y(0) = -x - 4\sin x + 3\int_0^x y(t)\,dt - 2\int_0^x (x-t)\,y(t)\,dt$$

or $$y(x) = (1 - x - 4\sin x) + \int_0^x [3 - 2(x-t)]\,y(t)\,dt \qquad [\because y(0)=1] \qquad ...(37)$$

which represents the non-homogenous Volterra's integral equation of second kind.

Converse Part :

Differentiating (37) w.r.t. x and using Lebnitz'r rule, we have

$$y'(x) = -1 - 4\cos x + \int_0^x \frac{\partial}{\partial x}[3 - 2(x-t)]\,y(t)\,dt$$

$$+ [3 - 2(x-t)]\,y(x)\frac{dx}{dx} - [3 - 2(x-0)]\,y(0)\frac{do}{dx}$$

or $$y'(x) = -1 - 4\cos x + 3y(x) - 2\int_0^x y(t)\,dt \qquad ...(38)$$

Differentiating (38) again w.r.t. x, we get

or $$y''(x) = 4\sin x + 3y'(x) - 2y(x)$$

or $$y''(x) - 3y'(x) + 2y(x) = 4\sin x$$

which is the required given differential equation

Putting $x = 0$ in (37) and (38), we get

$$y(0) = 1,\ y'(0) = -2$$

8.9 Method of Converting a Boundary Value Problem to a Fredholm Integral Equation

When an ordinary differential equation is to be solved under conditions involving dependent variable and its derivatives at two different values of independent variables then the problem under consideration is called boundary value problem.

We explain the method with the help of the following solved examples.

Example 9 : Convert the differential equation

$$\frac{d^2y}{dx^2}+\lambda\, y=0$$

with the conditions $y(0)=0$, $y(l)=0$, into Fredholm integral equation of second kind. Also, recover the original differential equation from the integral equation you obtain.

Solution : Integrating both sides of given differential equation w.r.t. $'x'$ from 0 to x, we have

$$y'(x)-y'(0)=-\lambda\int_0^x y(x)\,dx$$

Let $y'(0)=c$, a constant, then

$$y'(x)=c-\lambda\int_0^x y(x)\,dx \qquad ...(39)$$

Integrating (39) both sides w.r.t. $'x'$, we have

$$y(x)-y(0)=c\int_0^x dx-\lambda\int_0^x y(x)\,dx^2$$

or $$y(x)=cx-\lambda\int_0^x (x-t)\,y(t)\,dt \qquad [\because y(0)=0] \qquad ...(40)$$

Putting $x=l$ in (40), we get

$$y(l)=0=cl-\lambda\int_0^l (l-t)\,y(t)\,dt \qquad [\because y(l)=0]$$

$\Rightarrow$ $$c=\frac{\lambda}{l}\int_0^l (l-t)\,y(t)\,dt \qquad ...(41)$$

Using (41) in (40), we get

$$y(x)=\int_0^l \frac{\lambda\, x(l-t)}{l}\,y(t)\,dt-\int_0^x \lambda\,(x-t)\,y(t)\,dt \qquad ...(42)$$

or $$y(x)=\int_0^x \frac{\lambda\, x(l-t)}{l}\,y(t)\,dt+\int_x^l \frac{\lambda\, x(l-t)}{l}\,y(t)\,dt-\int_0^x (x-t)\,y(t)\,dt$$

or $$y(x)=\lambda\left[\int_0^x \frac{t(l-x)}{l}\,y(t)\,dt+\int_x^l \frac{x(l-t)}{l}\,y(t)\,dt\right]$$

Converse Part :

Differentiating both sides of (42) w.r.t. $'x'$, and using Lebnitz's rule therein, we have

$$y'(x) = \int_0^l \frac{\lambda(l-t)}{l} y(t)\,dt - \int_0^x \lambda\, y(t)\,dt$$

Differentiating it again w.r.t. $'x'$ and applying Lebnitz's rule therein, we get

$$y''(x) + \lambda\, y(x) = 0$$

which is the required differential equation

Putting $x = 0$ and $x = l$, we get

$$y(0) = 0 \text{ and } y(l) = 0$$

Example 10 : Transform $\frac{d^2y}{dx^2} + xy = 1$; $y(0) = 0$, $y(1) = 1$ into an intergral equation :

Solution : The given differential equation may be written as

$$y''(x) = 1 - x\,y(x)$$

Integrating both sides w.r.t. $'x'$ from 0 to x and using $y'(0) = c$, a constant, we get

$$y'(x) = c + x - \int_0^x x\,y(x)\,dx \qquad ...(44)$$

Integrating (44) both sides w.r.t. $'x'$ from 0 to x, we get

$$y(x) - y(0) = cx + \frac{x^2}{2} - \int_0^x t\,y(t)\,dt^2$$

or $$y(x) = cx + \frac{x^2}{2} - \int_0^x (x-t)\,y(t)\,dt \qquad [\because y(0) = 0] \qquad ...(45)$$

Putting $x = 1$ in (45), we get

$$y(1) = 1 = c + \frac{1}{2} - \int_0^1 (1-t)\,t\,y(t)\,dt \qquad [\because y(1) = 1]$$

or $$c = \frac{1}{2} + \int_0^1 (1-t)\,y(t)\,dt \qquad ...(46)$$

Using (46) in (45), we have

$$y(x) = \frac{1}{2}x(1+x) + \int_0^1 x\,t\,(1-t)\,y(t)\,dt - \int_0^x t\,(x-t)\,y(t)\,dt$$

or $$y(x) = \frac{1}{2}x(1+x) + \int_0^x x\,t\,(1-t)\,y(t)\,dt + \int_x^1 x\,t\,(1-t)\,y(t)\,dt - \int_0^x t\,(x-t)\,y(t)\,dt$$

or $$y(x)=\frac{1}{2}x(1+x)+\int_0^x t^2(1-x)\,y(t)\,dt+\int_x^1 x\,t(1-t)\,y(t)\,dt$$

or $$y(x)=\frac{1}{2}x(1+x)+\int_0^1 K(x,t)\,y(t)\,dt$$

where $$K(x,t)=\begin{cases} t^2(1-x), & when\;\; t<x \\ x\,t(1-t), & when\;\; t>x \end{cases}$$

Example 11 : If $y(x)$ is continuous and satisfies the integral equation $y(x)=\lambda\int_0^1 K(x,t)\,y(t)\,dt$

where $$K(x,t)=\begin{cases} (1-t)\,x, & 0\le x\le t \\ (1-x)\,t, & t\le x\le 1 \end{cases}$$

Then prove that $\boldsymbol{y(x)}$ is also the solution of the boundary value problem

$$\frac{d^2y}{dx^2}+\lambda\,y=0\,,\;\; y(0)=0\,,\;\; y(1)=0\,.$$

Solution : Given integral equation may be written as

$$y(x)=\lambda\left[\int_0^x K(x,t)\,y(t)\,dt+\int_x^1 K(x,t)\,y(t)\,dt\right]$$

or $$y(x)=\lambda(1-x)\int_0^x t\,y(t)\,dt+\lambda\,x\int_x^1(1-t)\,y(t)\,dt \qquad ...(47)$$

Putting $x=0$ and $x=1$ by turn in (47), we get

$$y(0)=0 \text{ and } y(1)=0 \qquad ...(48)$$

Differentiating both sides of (47) w.r.t. $'x'$ and using Lebnitz's rule, we get

$$\frac{dy}{dx}=-\int_0^x \lambda\,t\,y(t)\,dt+\int_x^1 \lambda(1-t)\,y(t)\,dt \qquad ...(49)$$

Differentiating again both sides of (49) w.r.t. $'x'$, we get

or $$\frac{d^2y}{dx^2}+\lambda\,y=0$$

Thus $y(x)$ is the solution of the boundary value problem

$$\frac{d^2y}{dx^2}+\lambda\,y=0\;\;;\;\; y(0)=0=y(1)$$

Self-Learning Exercise - II

Fill in the blanks :

1. The linear differential equation of second order $y'' + a_1(x)y' + a_2(x)y'' = f(x)$ with $y(0) = c_0$ and $y'(0) = c_1$ can be converted into .... integral equation .... kind.

2. The equation $\frac{dy}{dx} - y = 0$; $y(0) = 1$ can be transformed to ....

3. The equation $\frac{d^3y}{dx^3} - 2x\,y = 0$; $y(0) = \frac{1}{2}$, $y'(0) = y''(0) = 1$ can be transformed to .... integral equation of .... kind.

4. The differential equation $\frac{d^2y}{dx^2} + xy = 1$, $y(0) = 0$, $y(1) = 1$ can be converted into .... integral equation of .... kind.

5. The problem $\frac{d^2y}{dx^2} + y = x$, $y(0) = 0$, $y'(1) = 0$ can be converted into .... integral equation of .... kind.

8.10 Exercise 8 (b)

Reduce the following intital value problems to Volterra integral equation of second kind :

1. $\frac{d^2y}{dx^2} + xy = 1$, $y(0) = y'(0) = 0$

   [Ans. $g(x) = 1 - \int_0^x x(x-t)\,g(t)\,dt$ **]**

2. $\frac{d^2y}{dx^2} + (1+x^2)\,y = \cos x$; $y(0) = 0$, $y'(0) = 2$

   [Ans. $g(x) = \cos x - 2x(1+x^2) - \int_0^x (1+x^2)(x-t)\,g(t)\,dt$ **]**

3. $\frac{d^3y}{dx^3} - 2\,x\,y = 0$; $y(0) = \frac{1}{2}$, $y'(0) = y''(0) = 1$

   [Ans. $g(x) = x(x+1)^2 + \int_0^x x\,(x-t)^2\,g(t)\,dt$ **]**

4. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4\sin x$; $y(0) = 1$, $y'(0) = -2$

   [Ans. $g(x) = 4\sin x + 4x - 8 + \int_0^x (3 - 2x - 2t)\,g(t)\,dt$ **]**

5. $\dfrac{d^2y}{dx^2} + y = 0$ when $y(0) = 0$, $y'(0) = 0$

[Ans. $g(x) = -\int_0^x (x-t)\,g(t)\,dt$ **]**

6. Prove that the linear differential equation of second order

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = f(x)$$

with initial conditions $y(0) = c_0$ and $y'(0) = c_1$ can be transformed into non-homogeneous Volterra's integral equation of second kind.

7. Reduce the following initial value problem into an integral equation

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \ ; \ y(0) = 1, \ y'(0) = 1$$

[Ans. $y(x) = 1 + x - \int_0^x t\,y(t)\,dt$ **]**

8. Convert the following differential equation into an integral equation :

$$\frac{d^2y}{dx^2} + \lambda\,y = f(x) \text{ when } y(0) = 1, \ y'(0) = 0$$

[Ans. $y(x) = 1 + \int_0^x (x-t)\left[f(t) - \lambda\,y(t)\,dt\right]$]

9. Reduce the initial value problem

$$\frac{d^2y}{dx^2} - \sin x\frac{dy}{dx} + e^x y = x$$

with the initial condition $y(0) = 1$, $y'(0) = -1$ to a Volterra's integral equation of second kind.

Conversly, derive the original differential equation with the initial conditions from the intergral equation defined.

[Ans. $g(x) = x - \sin x - e^x(1-x) + \int_0^x \sin x - e^x$ **]**

10. Reduce the boundary value problem

$$\frac{d^2y}{dx^2} + \lambda\,y = x \ ; \ y(0) = y(\pi) = 0$$

into an integral equation

$$y(x) = \frac{1}{6}x^3(1-\pi^2) + \lambda\int_0^\pi K(x,t)\,y(t)\,dt$$

where $K(x,t)=\begin{cases}\dfrac{x}{\pi}(x-t), & when\ \ t>x\\ \left(\dfrac{x}{\pi}-1\right)(x-t), & when\ \ t<x\end{cases}$

11. Converte the problem $y''+\lambda y=0$; $y(0)=y'(0)$, $y(\pi)=y'(\pi)$ into an integral equation

[Ans. $y(x)=\lambda\int_0^{\pi}K(x,t)\,y(t)\,dt$

Where $K(x,t)=\begin{cases}\dfrac{(t+1)(\pi-x-1)}{\pi}, & 0\le t\le x\\ \dfrac{(x+1)(\pi-t-1)}{\pi}, & x\le t\le \pi\end{cases}$

12. Transform the boundary value problem

$$\frac{d^2y}{dx^2}+y=x\ ;\ y(0)=0,\ y'(1)=0$$

to a Fredholm integral equation

[Ans. $y(x)=\dfrac{x^3}{3}-\dfrac{x}{2}+\int_0^{x}K(x,t)\,y(t)\,dt$

where $K(x,t)=\begin{cases}t, & t<x\\ x, & t>x\end{cases}$

8.11 Eigenvalues and Eigenfunctions

Consider the homogeneous Fredholm integral equation of second kind

$$g(x)=\lambda\int_a^{b}K(x,t)\,g(t)\,dt \qquad ...(50)$$

Obiviously (50) has always, the solution $g(x)=0$, which is known as 'zero' or 'trivial' solution of (50). The values of the parameter λ for which (50) has 'non-zero' solution $g(x)\neq 0$ are known as eigen values (or characteristics number) of (50) of the kernel $K(x,t)$. Further every 'non-zero' solution of (50) is called eignfunction (or characteristic function) corresponding to the eigenvalue λ.

If the kernel $K(x,t)$ is continuous in rectangle $a\le x\le b$, $a\le t\le b$ and the numbers a and b are finite, then to every value λ there exist a finite number of linearly independent eigenfucntions; the number of such functions is called the index of the eigenvalues. Different eigenvalues have indices.

Remark :

1. The number $\lambda = 0$ is not an eigenvalue since for $\lambda = 0$ (50) gives $g(x) = 0$, which is a zero solution.

2. If $\phi(x)$ is an eigenfunction of (50) corresponding to eigenvalue λ_0, then $c\phi(x)$ is also eigenfunction of (50) corresponding to the same eigenvalue where c is an arbitrary constant.

3. A homogeneous Fredholm integral equation may, generally, have no eigenvalue and eigenfunction or it may not have any real eigenvalues and eigenfunctions.

8.12 Solution of Homogeneous Fredholm Integral Equation of the Second Kind with Separable (or Degenerate) Kernel

Consider a homogeneous Fredholm integral equation of second kind

$$\phi(x) = \lambda \int_a^b K(x,t)\phi(t)\,dt \qquad ...(51)$$

By the definition of the degenerate or separable kernel (given in § 8.3), we have

$$K(x,t) = \sum_{i=1}^{n} g_i(x) h_i(t) \qquad ...(52)$$

Using (52) in (51) and interchanging the order of integration and summaration, we get

$$\phi(x) = \lambda \sum_{i=1}^{n} g_i(x) \int_a^b h_i(t)\phi(t)\,dt \qquad ...(53)$$

To solve (53), let

$$\int_a^b h_i(t)\phi(t)\,dt = c_i\,(i = 1,2,3,......) \qquad ...(54)$$

Using (54) in (53), we get

$$\phi(x) = \lambda \sum_{i=1}^{n} c_i\, g_i(x) \qquad ...(55)$$

where $c_i\,(i = 1,2,3,....,n)$ are unknown constants, as the function $\phi(x)$ is unknown. Thus (55) is the required solution of the integral equation (51).

We now proceed to evaluate c_i's as follows :

Multiplying both sides of (55) succesively by $h_1(x), h_2(x),.....h_n(x)$ and integrating over the interval (a,b), we have

$$\int_a^b h_1(x)\phi(x)\,dx = \lambda \sum_{i=1}^{n} c_i \int_a^b h_1(x) g_i(x)dx \qquad ...(A_1)$$

$$\int_a^b h_2(x)\phi(x)\,dx = \lambda\sum_{i=1}^{n} c_i \int_a^b h_2(x)\,g_i(x)dx \qquad ...(A_2)$$

....

and $$\int_a^b h_n(x)\phi(x)\,dx = \lambda\sum_{i=1}^{n} c_i \int_a^b h_n(x)\,g_i(x)dx \qquad ...(A_n)$$

Let $$\alpha_{ij} = \int_a^b h_j(x)\,g_i(x)dx \qquad (i, j = 1,2,3,......) \qquad ...(56)$$

Using (54) and (56) in (A_1), we get

$$(1-\lambda\alpha_{11})c_1 - \lambda\alpha_{12}c_2 - .... - \lambda\alpha_{1n}c_n = 0$$

Similarly, we may simplify $(A_2),...(A_n)$. Thus we obtain the following system of homogeneous linear equations to determine $c_1, c_2, ...c_n$:

$$(1-\lambda\alpha_{11})c_1 - \lambda\alpha_{12}c_2 .... \lambda\alpha_{1n}c_n = 0 \qquad ...(B_1)$$

$$-\lambda\alpha_{21}c_1 + (1-\lambda\alpha_{22})c_2 - .... \lambda\alpha_{2n}c_n = 0 \qquad ...(B_2)$$

....

$$-\lambda\alpha_{n1}c_1 - \lambda\alpha_{n2}c_2 - .... + (1-\lambda\alpha_{nn})c_n = 0 \qquad ...(B_n)$$

The determinant $D(\lambda)$ of this system is

$$D(\lambda) = \begin{vmatrix} 1-\lambda\alpha_{11} & -\lambda\alpha_{12} & -\lambda\alpha_{1n} \\ -\lambda\alpha_{21} & 1-\lambda\alpha_{22} & -\lambda\alpha_{2n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ -\lambda\alpha_{n1} & -\lambda\alpha_{n2} & -\lambda\alpha_{nn} \end{vmatrix} \qquad ...(57)$$

which is a polynomical in λ with degree atmost n. Now following case arise :

I. **If $D(\lambda) \neq 0$,** the system of equation $(B_1),(B_2),......,(B_n)$ has only one trival solution i.e. $c_1 = c_2 = ... = c_n = 0$ and hence from (55), we find that (51) has only zero or trivial solution i.e. $\phi(x) = 0$.

II. **If $D(\lambda) = 0$,** then at least one of the c_j's can be assigned arbitrary value and the remaining c_j's can be determined accordingly. Hence when $D(\lambda) = 0$, infinitely many solution of the integral equation (51) exist.

The characterstic number of (51) are given by $D(\lambda) = 0$ *i.e.*,

$$\begin{vmatrix} 1-\lambda\alpha_{11} & -\lambda\alpha_{12} & -\lambda\alpha_{2n} \\ -\lambda\alpha_{21} & 1-\lambda\alpha_{22} & -\lambda\alpha_{2n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ -\lambda\alpha_{n1} & -\lambda\alpha_{n2} & 1-\lambda\alpha_{nn} \end{vmatrix} = 0 \qquad ....(58)$$

So the degree of equation (58) in λ is $p \geq n$. It follows that if integral equation (51) has separable kernal given by (52), then (51) has atmost n eigenvalues.

8.13 Reality of Characteristic Numbers

Theorem 1 : The Characteristic numbers of a symmentric kernal are real.

Proof : Consider a homogenours Fredholim integral equation of the first and kind

$$\phi(x) = \lambda \int_a^b K(x,t)\,\phi(t)\,dt \qquad ...(59)$$

If possible, suppose that (59) has an eigenvalue λ_0, which is not real. Thus, let

$$\lambda_0 = \alpha + i\beta$$

Let $g_0(x) = u + iv$ be the complex eigenfunction corresponding to the eigenvalue λ_0. Then we know that the complex conjugate number $\overline{\lambda}_0$ would necessary be an eigenvalue corresponding to the eigenfunction $\overline{g}_0(x)$ which is the complex conjugate of $g_0(x)$. Thus, we have

$$\overline{\lambda}_0 = \alpha - i\beta \text{ and } \overline{g}_0(x) = u - iv$$

Since $g_0(x)$ and $\overline{g}_0(x)$ are eigenfunctions corresponding to eiganvalues λ_0 and $\overline{\lambda}_0$, therefore they must satisfy (59). Thus, we have

$$g_0(x) = \lambda_0 \int_a^b K(x,t)\,g_0(t)\,dt \qquad ...(60)$$

and $$\overline{g}_0(x) = \overline{\lambda}_0 \int_a^b \overline{K}(x,t)\,\overline{g}_0(t)\,dt \qquad ...(61)$$

Interchanging x and t and simplifying we have

or $$\int_a^b \overline{K}(t,x)\overline{g}_0(x)\,dx = \frac{1}{\overline{\lambda}_0}\overline{g}_0(t) \qquad ...(62)$$

As $K(x,t)$ is symmetic, we have

$$K(x,t) = \overline{K}(t,x) \qquad ...(63)$$

Multilying both sides of (60) by $\overline{g}_0(x)$ and integrating both sides w.r.t. 'x' from a to b, we get

$$\int_a^b \bar{g}_0(x)\, g_0(x)\, dx = \lambda_0 \int_a^b \bar{g}_0(x) \left\{ \int_a^b K(x,t) g_0(t) dt \right\} dx$$

$$= \lambda_0 \int_a^b g_0(t) \left\{ \int_a^b K(x,t) \bar{g}_0(x) dx \right\} dt \qquad \text{[on changing the order of integrations)}$$

or $$\int_a^b \bar{g}_0(x)\, g_0(x)\, dx = \lambda_0 \int_0^a g_0(t) \left\{ \int_a^b \bar{K}(t,x)\, \bar{g}_0(x)\, dx \right\} dt \qquad \text{...(64)}$$

Using (62) in (64), we get

$$\int_a^b \bar{g}_0(x)\, g_0(x)\, dx = \lambda_0 \int_0^a g_0(t) \left\{ \frac{1}{\bar{\lambda}_0}\, \bar{g}_0(t) \right\} dt$$

or $$\bar{\lambda}_0 \int_a^b \bar{g}_0(x)\, g_0(x)\, dx = \lambda_0 \int_0^a g_0(x)\, \bar{g}_0(x)\, dx$$

or $$\left(\lambda_0 - \bar{\lambda}_0\right) \int_a^b \bar{g}_0(x)\, g_0(x)\, dx = 0 \qquad \text{...(65)}$$

Substituting the values of λ_0 and $\bar{\lambda}_0$, in (65), we find that

$$2i\beta \int_a^b g_0(x)\, \bar{g}_0(x)\, dx = 0 \qquad \text{...(66)}$$

Since $g_0(x)$ is characteristic function, hence $g_0(x) \neq 0$.

Therefore $$\int_a^b \left| g_0(x) \right|^2 dx \neq 0$$

and so (66) implies that $\beta = 0$ i.e. the imaginery part of the eigenvalues $\lambda_0 = (\alpha + i\beta)$ is zero. Hence $\lambda_0 = \alpha$, which is real. Since λ_0 is any eigenvalue of (59), it follows that all eigenvalues of symmetric kernel are real.

8.14 Orthogonality of Eigenfunctions

Theorem 2 : If $K(x,t)$ is a symmetric kernel of homogeneous integral equation of second kind

$$\boldsymbol{g(x) = \lambda \int_a^b K(x,t)\, g(t)\, dt} \qquad \textbf{...(67)}$$

and $g_0(x)$ and $g_1(x)$ are eigenfunctions of $K(x,t)$ corresponding to eigenvalues λ_0 and λ_1 respectively $(\lambda_0 \neq \lambda_1)$, then $g_0(x)$ and $g_1(x)$ are orthogonal on the interval $[a,b]$ i.e.,

$$\boldsymbol{\int_a^b g_0(x)\, \bar{g}_1(x)\, dx = 0}$$

or

The eigenfunctions of a symmetric kernel, corresponding to different eigenvalues are orthogonal.

Proof : By the definition of eigenfunction, $g_0(x)$ and $g_1(x)$ must satisfy (67), therefore

$$g_0(x) = \lambda_0 \int_a^b K(x,t)\, g_0(t)\, dt \qquad ...(68)$$

and $$g_1(x) = \lambda_1 \int_a^b K(x,t)\, g_1(t)\, dt \qquad ...(69)$$

Also $$K(x,t) = \overline{K}(t,x) \qquad ...(70)$$

because $K(x,t)$ is symmetric.

Multiplying both sides of (68) by $\overline{g}_1(x)$ and integrating with respect to 'x' over the interval (a,b), we get

$$\int_a^b \overline{g}_1(x)\, g_0(x)\, dx = \lambda_0 \int_a^b g_1(x) \left\{ \int_a^b K(x,t)\, g_0(t)\, dt \right\} dx$$

On changing the order of integration, we get

or $$\int_a^b \overline{g}_1(x)\, g_0(x)\, dx = \lambda_0 \int_a^b g_0(t) \left\{ \int_a^b K(x,t)\, \overline{g}_1(x)\, dx \right\} dt \qquad ...(71)$$

Now equation (69) can be written as

$$g_1(t) = \lambda_1 \int_a^b K(t,x)\, g_1(x)\, dx$$

or $$\overline{g}_1(t) = \overline{\lambda}_1 \int_a^b K(t,x)\, \overline{g}_1(x)\, dx$$

or $$\int_a^b \overline{K}(t,x)\, \overline{g}_1(x)\, dx = \frac{1}{\lambda_1} \overline{g}_1(t) \qquad \left(\because \overline{\lambda}_1 = \lambda_1\right) \qquad ...(72)$$

So equations (71) and (72) yeild

$$\int_a^b \overline{g}_1(x)\, g_0(x)\, dx = \lambda_0 \int_a^b g_0(t) \left\{ \frac{1}{\lambda_1} \overline{g}_1(t) \right\} dt$$

or $$\lambda_1 \int_a^b \overline{g}_1(x)\, g_0(x)\, dx = \lambda_0 \int_a^b g_0(x)\, \overline{g}_1(x)\, dx$$

or $$(\lambda_1 - \lambda_0) \int_a^b \overline{g}_1(x)\, g_0(x)\, dx = 0$$

or $$\int_a^b \overline{g}_1(x)\, g_0(x)\, dx = 0 \qquad [\because \lambda_1 \neq \lambda_0]$$

or g_0 and g_1 are orthogonal

This completes the proof of the theorem.

Example 12 : Solve the homogeneous Fredholm integral equation

$$\phi(x) = \lambda \int_0^1 e^{x+t}\, \phi(t)\, dt$$

Solution : The given equation may be written as

$$\phi(x) = \lambda\, e^x \int_0^1 e^t\, \phi(t)\, dt \qquad ...(73)$$

or $$\phi(x) = \lambda\, e^x c \qquad ...(74)$$

where $$c = \int_0^1 e^t\, \phi(t)\, dt \qquad ...(75)$$

From (74) and (75), we have

$$c = \int_0^1 e^t\, \lambda\, e^t c = \lambda_c \left(\frac{e^{2t}}{2}\right)_0^1 = \frac{\lambda_c}{2}\left(e^2 - 1\right)$$

or $$c\left[1 - \frac{\lambda}{2}\left(e^2 - 1\right)\right] = 0 \qquad ...(76)$$

If $c = 0$, then (74) gives $\phi(x) = 0$. We therefore, assume that for non-zero solution of (73), $c \neq 0$, then (76) gives

$$1 - \frac{\lambda}{2}\left(e^2 - 1\right) = 0 \qquad \text{or} \qquad \lambda = \frac{2}{e^2 - 1}$$

which is an eigenvalue.

Putting this value of λ in (74), the corresponding eigenfunction is given by

$$\phi(x) = \frac{2c}{e^2 - 1}\, e^x$$

Hence corresponding to eigenvalue $\dfrac{2}{\left(e^2 - 1\right)}$, there corresponds the eigenfunction e^x.

Remark : While writing eigenfunction the constant c is taken as unity.

Example 13 : Find the eigenvalues and eigenfunction of the homogeneous integral equation

$$g(x) = \lambda \int_0^{\pi} \left[\cos^2 x\, \cos 2t + \cos 3x \cos^3 t\right] g(t)\, dt$$

Solution : The given integral equation may be written as

$$g(x) = \lambda\left[c_1 \cos^2 x + c_2 \cos 3x\right] \qquad ...(77)$$

where $c_1 = \int_0^{\pi} \cos 2t\, g(t)\, dt$ ...(78)

and $c_2 = \int_0^{\pi} \cos^3 t\, g(t)\, dt$ ...(79)

Using (77) in (78), we get

$$c_1 \left[1 - \lambda \int_0^{\pi} \cos 2t \cos^2 t\, dt\right] - \lambda c_2 \int_0^{\pi} \cos 2t \cos 3t\, dt = 0 \qquad ...(80)$$

Now, $\int_0^{\pi} \cos 2t \cos^2 t\, dt = \int_0^{\pi} \cos 2t \frac{(1+\cos 2t)}{2} dt$

$$= \frac{1}{2}\int_0^{\pi} \cos 2t\, dt + \frac{1}{2}\int_0^{\pi} \cos^2 2t\, dt$$

$$= \frac{1}{2}\left[\frac{\sin 2t}{2}\right]_0^{\pi} + \frac{1}{2}\int_0^{\pi} \frac{1+\cos 4t}{2} dt$$

$$= 0 + \frac{1}{4}\left[t + \frac{\sin 4t}{4}\right]_0^{\pi} = \frac{\pi}{4}$$

and $\int_0^{\pi} \cos 2t \cos 3t\, dt = \frac{1}{2}\int_0^{\pi} [\cos 5t + \cos t]\, dt$

$$= \frac{1}{2}\left[\frac{\sin 5t}{5} + \sin t\right]_0^{\pi} = 0$$

Thus (80) reduces to

$$c_1 \left(1 - \frac{\lambda \pi}{4}\right) = 0 \qquad ...(81)$$

Similarly using (77), (79) becomes

$$\lambda c_1 \int_0^{\pi} \cos^5 t\, dt + c_2 \left[\lambda \int_0^{\pi} \cos^3 t \cos 3t\, dt - 1\right] = 0 \qquad ...(82)$$

Now $\int_0^{\pi} \cos^5 t\, dt = 0$, $\left[\text{since } \cos^5(\pi - t) = -\cos^5 t\right]$ ...(83)

and $\int_0^{\pi} \cos^3 t \cos 3t\, dt = \frac{1}{4}\int_0^{\pi} \cos 3t (\cos 3t + 3\cos t)\, dt$

$$= \frac{1}{4}\int_0^{\pi} \cos^2 3t\, dt + \frac{3}{4}\int_0^{\pi} \cos 3t \cos t\, dt$$

$$= \frac{1}{8}\int_0^{\pi} (1 + \cos 6t)\, dt + \frac{3}{8}\int_0^{\pi} (\cos 4t + \cos 2t)\, dt$$

$$= \frac{1}{8}\left[t + \frac{\sin 6t}{6}\right]_0^{\pi} + \frac{3}{8}\left[\frac{\sin 4t}{4} + \frac{\sin 2t}{2}\right]_0^{\pi}$$

$$\therefore \qquad \int_0^{\pi} \cos^3 t \cos 3t \, dt = \frac{\pi}{8} \qquad ...(84)$$

Using (83) and (84) in (82), we get

$$c_2\left(1 - \frac{\lambda\pi}{8}\right) = 0 \qquad ...(85)$$

For non-zero solution of the system of equations (81) and (85), we must have

$$\begin{vmatrix} 1 - \frac{\lambda\pi}{4} & 0 \\ 0 & 1 - \frac{\lambda\pi}{8} \end{vmatrix} = 0 \Rightarrow \lambda_1 = \frac{4}{\pi} \text{ and } \lambda_2 = \frac{8}{\pi} \qquad ...(86)$$

Putting $\lambda = \lambda_1 = \frac{4}{\pi}$ in (81) and (85), we have

$$0.c_1 = 0 \text{ and } \frac{1}{2}.c_2 = 0$$

If follows that $c_2 = 0$ and c_1 is arbitrary. Putting these values is (77), we get

$$g_1(x) = c_1\lambda\cos^2 x = \cos^2 x, \text{ if } \lambda c_1 = c_1(4/\pi) = 1$$

Putting $\lambda = \lambda_1 = \frac{8}{\pi}$ in (81) and (85), we get

$$-c_1 = 0 \text{ and } 0.c_2 = 0$$

If follows that $c_1 = 0$ and c_2 is arbitrary. Putting these values in (77), we get

$$g_1(x) = c_2\lambda\cos 3x = \cos 3x, \quad \text{if } c_2\lambda = c_2(8/\pi) = 1$$

Thus the eigenfunctions are

$$g_1(x) = \cos^2 x \text{ and } g_2(x) = \cos 3x$$

corresponding to the eigenvalues $\lambda = 4/\pi$ and $\lambda_2 = 8/\pi$ respectively.

Example 14 : Solve the homogeneous Fredholm integral equation of the second kind

$$g(x) = \lambda\int_0^{2\pi} \sin(x+t)\, g(t)\, dt$$

Solution : The given integral equation can be written as

$$g(x)=\lambda\left[\sin x\int_0^{2\pi}\cos t\,g(t)\,dt+\cos 3x\int_0^{2\pi}\sin t\,g(t)\,dt\right]$$

or $$g(x)=\lambda\left[c_1\sin x+c_2\cos x\right] \qquad ...(87)$$

where $$c_1=\int_0^{2\pi}\cos t\,g(t)\,dt \qquad ...(88)$$

and $$c_2=\int_0^{2\pi}\sin t\,g(t)\,dt \qquad ...(89)$$

Using (87) in (88) and evaluating the integrals, we get

or $$c_1=\frac{\lambda c_1}{2}\int_0^{2\pi}\sin 2t\,dt+\frac{\lambda c_2}{2}\int_0^{2\pi}(1+\cos 2t)\,dt$$

or $$c_1=\frac{\lambda c_1}{2}\left[-\frac{\cos 2t}{2}\right]_0^{2\pi}+\frac{\lambda c_2}{2}\left[t+\frac{\sin 2t}{2}\right]_0^{2\pi}$$

or $$c_1-\lambda\pi c_2=0 \qquad ...(90)$$

Similarly using (87) in (89) and evaluating the integrals, we get

$$c_2=\int_0^{2\pi}\sin t\left[\lambda c_1\sin t+\lambda c_2\cos t\right]dt$$

or $$c_2=\frac{\lambda c_1}{2}\int_0^{2\pi}(1-2\cos t)\,dt+\frac{\lambda c_2}{2}\int_0^{2\pi}\sin 2t\,dt$$

$$c_2=\frac{\lambda c_1}{2}\left[t-\frac{\sin 2t}{2}\right]_0^{2\pi}+\frac{\lambda c_2}{2}\left[\frac{\cos 2t}{2}\right]_0^{2\pi}$$

or $$\pi\lambda c_1-c_2=0 \qquad ...(91)$$

For non-zero solution of the system of equations (90) and (91), we have

$$\begin{vmatrix}1 & -\lambda\pi\\ \lambda\pi & -1\end{vmatrix}=0\Rightarrow -1+\lambda^2\pi^2=0\Rightarrow\lambda=\pm\frac{1}{\pi}$$

Hence the eigenvalues are

$$\lambda_1=\frac{1}{\pi}\text{ and }\lambda_2=-\frac{1}{\pi}$$

Now for $\lambda=\lambda_1=\frac{1}{\pi}$, equations (90) and (91) reduce to

$$c_1-c_2=0\Rightarrow c_1=c_2$$

Hence from (87), we have

$$g_1(x) = \frac{1}{\pi} c_1 (\sin x + \cos x) = \sin x + \cos x \qquad \left[\text{taking } \frac{c_1}{\pi} = 1\right]$$

Similarly for $\lambda = \lambda_2 = -\frac{1}{\pi}$, equations (90) and (91) reduce to

$$c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

Hence from (87), we get

$$g_2(x) = -\frac{1}{\pi}[c_1 \sin x - c_1 \cos x]$$

$$= -\frac{c_1}{\pi}[\sin x - \cos x] = \sin x - \cos x \qquad \left[\text{taking } \frac{-c_1}{\pi} = 1\right]$$

Thus the required eigenvalues and eigenfunctions are given by

$$\lambda_1 = \frac{1}{\pi}, \qquad g_1(x) = \sin x + \cos x$$

$$\lambda_2 = -\frac{1}{\pi}, \qquad g_2(x) = \sin x - \cos x$$

8.15 Solution of Homogeneous Fredholm Integral Equations of Second Kind with Kernel in the Special Form

In this section we deal with the homogeneous Fredholm equations of second kind with kernel $K(x,t)$ in the special form. For getting the solution of such integral equations, we first reduce the given integral equation into differential equation together with certain boundary conditions. Then, we solve the resulting boundary value problem to determine eigenvalues and eigenfunctions. Following examples will illustrate the procedure :

Example 15 : Find the eigenvalues and eigenfunctions of the homogeneous integral equation

$$g(x) = \lambda \int_0^1 K(x,t) g(t) dt$$

where $K(x,t) = \begin{cases} x(t-1), & 0 \le x \le t \\ t(x-1), & t \le x \le 1 \end{cases}$

Solution : The given integral equation may be rewritten as

$$g(x) = \lambda \left[\int_0^x K(x,t) g(t) dt + \int_x^1 K(x,t) g(t) dt \right]$$

Using the definition of the kerel $K(x,t)$, we have

$$g(x)=\int_0^x \lambda\, t\,(x-1)\, g(t)\,dt+\int_1^x \lambda\, x\,(t-1)\, g(t)\,dt \qquad ...(92)$$

Differentiating (92) w.r.t. 'x' by using Labnitz's rule of differentiation under the integral sign, we get

$$g'(x)=\int_0^x \lambda\, t\, g(t)\,dt+\int_1^x \lambda\,(t-1)\, g(t)\,dt \qquad ...(93)$$

Differentiating (93) w.r.t. 'x; and using Lebnitz's rule as before, we have

$$g''(x)-\lambda\, g(x)=0 \qquad ...(94)$$

which is the derived differential equation to be satisfied by $g(x)$. The relevant boundary condition are

$$g(0)=g(1)=0 \qquad ...(95)$$

Thus we have to solve (94) subject to boundary conditions (95) to determine the eigenvalues and eigenfunctions of the given integral equation.

Now there cases arise :

Case I : Let $\lambda=0$

Then (94) reduces to $g''(x)=0$, whose general solution is

$$g(x)=c_1x+c_2 \qquad ...(96)$$

Using (95) in (96), we obtain $c_1=c_2=0$.

Hence (96) gives $g(x)=0$, which is not an eigenfunction and so $\lambda=0$ is not an eigenvalue.

Case II : Let $\lambda=\mu^2$, where $\mu\neq 0$. Then (94) reduces to

$$g''(x)-\mu^2 g(x)=0$$

whose general solution is

$$g(x)=c_1e^{\mu x}+c_2e^{-\mu x} \qquad ...(97)$$

Putting $x=0$ and $x=1$ in (97) successively and using (95), we obtain

$$\left.\begin{aligned} 0&=c_1+c_2\\ 0&=c_1e^{\mu}+c_2e^{-\mu}\end{aligned}\right\}$$

Solving these equations, we obtain $c_1=c_2=0$. Hence (97) reduces to $g(x)=0$, which is not an eigenfunction and hence $\lambda=\mu^2\ (\mu\neq 0)$ does not give eigenvalues.

Case III : $\lambda=-\mu^2$, where $\mu\neq 0$;

Then (94) reduces to $g''(x)+\mu^2\, g(x)=0$

whose genreal solution is

$$g(x) = c_1 \cos \mu x + c_2 \sin \mu x \qquad ...(98)$$

Putting $x = 0$ and $x = 1$ in (98) successively and using (95), we get

$$c_1 = 0 \text{ and } c_1 \cos \mu + c_2 \sin \mu = 0$$

$$\Rightarrow \qquad c_2 \sin \mu = 0$$

But $c_2 \neq 0$, otherwise $c_2 = 0$ and $c_1 = 0$ will give $g(x) = 0$ and we shall not get on eigenfunction. Hence for existance of eigenfunction, we must have

$$\sin \mu = 0 \text{ so that } \mu = n\pi, \; n = 1,2,3,......$$

Thus the required eigenvalues are given by

$$\lambda_n = \lambda = -\mu^2 = -n^2\pi^2, \; n = 1,2,3,.......$$

From (98), the corresponding eigenfunctions are given by

$$g_n(x) = c_2 \sin n\pi x \qquad [\because c_1 = 0, \mu = n\pi]$$

or $\qquad g_n(x) = \sin \pi n x \qquad$ [taking $c_2 = 1$]

Hence the required eigenvalues and eigenfunctions are given by

$$\lambda_n = -n^2\pi^2, \; g_n(x) = \sin \pi n x, \; n = 1,2,3,....$$

Self-Learning Exercise - III

1. Define eigenvalues of a kernel in the integral equation.
2. Define eigenfunction of a kernel in the integral equation.
3. Define degenerate kernel.
4. Define symmetric kernel of an integral equation.

Fill in the blanks :

5. The eigenvalues of a symmetric kernel are always ....
6. The eigenfunctions of a symmetric kernel are ....

8.16 Summary

In this unit you studied the conversion of initial and boundary value problems to an linear integral equation. Eigenvalues and eigenfunctions were also defined and a result connected with each a these terms were proved. You also studied the method of solving homogeneous Fredholm integral equation of second kind with separable kernels. This method and method for obtaining eigenvalues and eigenfunctions were illustrated by considering a number of solved examples.

8.17 Answers to Self-Learning Exercises

Exercise I

6. Volterra, second
7. Fredholm, second

Exercise II

1. Volterra, second
2. $g(x)=1+\int_0^x g(t)\,dt$
3. Volterra, second
4. Fredholm, second
5. Fredholm, second

Exercise III

5. Real
6. Orthogonal

8.17 Exercise 8 (c)

1. Determine the eigenvalues and the eigenfunction of the following integral equations :

(i) $g(x)=\lambda\int_0^{2\pi}\sin x\cos t\, g(t)\,dt$

[Ans. No eigenvalues**]**

(ii) $g(x)=\lambda\int_{-1}^{1}(1+3xt)\,g(t)\,dt$

[Ans. Eigenvaleus : $\lambda = \frac{1}{2}$; Eigenfunction : $g(x)=1$ and $g(x)=x$ **]**

(iii) $g(x)=\lambda\int_0^{2\pi}\sin x\sin t\, g(t)\,dt$

[Ans. Eigenvalues : $\lambda = \frac{1}{\pi}$, Eigenfunction ; $g(x)=\sin x$ **]**

(iv) $g(x)=\lambda\int_{-1}^{1}(x\cos h\,t - t\cos h\,x)\,g(t)\,dt$

[Ans. No real eigenvalues**]**

(v) $g(x)=\lambda\int_{-1}^{1}(1+t+3xt)\,g(t)\,dt$

[Ans. Eigenvalue : $\lambda = \frac{1}{2}$, Eigenfunction : $g(x)=1$]

2. Determine the eigenvalues and eigenfunctions of the homogenuous integral equation :

$$g(x)=\lambda\int_0^1 K(x,t)\,g(t)\,dt$$

where $K(x,t)=\begin{cases}-e^{-t}\sinh x, & 0\le x\le t\\ -e^{-x}\sinh t, & t\le x\le 1\end{cases}$.

[Ans. $\lambda_n=-1-\mu_n^2$, $g_n(x)=\sin\mu_n x$, $n=1,2,\ldots.$ where μ_n are positive roots of $\tan\mu=-\mu$.]

3. Show that the integral equation $g(x)=\lambda\int_0^{\pi}(\sin x\sin 2t)\,g(t)\,dt$ has no eigenvalues.

4. Find the eigenvalues and eigenfunctions of the following homogenuous integral equation

$$g(x) = \lambda \int_0^{\pi/2} K(x,t)\, g(t)\, dt$$

where $K(x,t) = \begin{cases} \sin x \cos t, & 0 \le x \le t \\ \cos x \sin t, & t \le x \le \pi/2 \end{cases}$

[Ans. $\lambda_n = 4n^2 - 1$; $g_n(x) = \sin 2hx$, $n = 1,2,3,.....$]

5. Define symmetric kernel and prove that every eigenvalue of a symmetric kernel is real and that every eigenfunction corresponding to distrinct eigenvalues are orthogonal.

6. Determine the eigenvalues and eigenfunctions of the following homogeneous integral equations with seprable kernels :

(i) $g(x) = \lambda \int_{-1}^{1} \left(5xt^3 + 4x^2t + 3xt\right) g(t)\, dt$

(ii) $g(x) = \lambda \int_0^1 \left(2xt - 4x^2\right) g(t)\, dt$

[Ans. (i) eigenvalue : $\lambda = \dfrac{1}{4}$; eigenfunction : $g(x) = x^2 + \dfrac{3x}{2}$

(ii) eigenvalue : $\lambda_1 = \lambda_2 = -3$; eigenfunction : $g(x) = x - 2x^2$]

7. Show that the homogenous integral equations

(i) $g(x) = \lambda \int_0^1 \left(t\sqrt{x} - x\sqrt{t}\right) g(t)\, dt$

(ii) $g(x) = \lambda \int_0^1 (3x-2)\, t\, g(t)\, dt$

do not have real eigenvalues and eigenfunctions.

Unit - 9
Solution of General Integral Equations with Special Type of Kernels and by Integral Transform Method

Structure of the Unit

9.0 Objective

In this chapter, we shall discuss the method to obtain eigenvalues and eigen functions of general Fredholm integral equation of second kind with separable kernel $K(x,t)$. We shall also discuss the integral transform method to find the solution of Volterra integral equation of second kind with convolution type kernels and signular integral equations.

9.1 Introduction

There exists an important class of integral equations, which are simply solved by reduction to a system of algebraic equations. We shall call a kernel $K(x,t)$ degenerate (separable) if it consists of the sum of a finite number of terms, each of which in its turn is the product of two factors, one of which depends only on x, and the other on t. In the last unit we considered the solution of homogeneous Fredholm integral equation with separable kernel. In this unit, we shall study the procedure to find solution of general Fredholm integral equation of second kind with separable kernels. We make use of the well known result of linear algebra.

Many interesitng problems of mechanics and physics lead to an integral equation in which the kernel $K(x,t)$ is a function of the difference $(x-t)$ only. The integral transform methods are very convenient in solving integral equations of some special forms. In this chapter, we shall also study the use of Laplace Transform, Fourier transform for the solution of integral equations of some special forms.

9.2 Solution of General Fredholm Integral Equation of Second Kind with Separable Kernels

Consider a Fredholm integral equation of seocnd kind :

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\phi(t)\, dt \qquad ...(1)$$

Since kernel $K(x,t)$ is separable, we take

$$K(x,t) = \sum_{i=1}^{n} g_i(x) h_i(t), \qquad ...(2)$$

where the functions $g_1(x), ... g_n(x)$ are assumed to be lineraly independent, otherwise the number of terms in relation (2) will be reduced. The function $g_i(x)$ and $h_i(t)$ $(i = 1, 2, ... n)$ are assumed to be continuous in square $R: a \le x \le b,\ a \le t \le b$.

Using (2), (1) reduces to

$$\phi(x) = f(x) + \lambda \int_a^b \left[\sum_{i=1}^{n} g_i(x)\ h_i(t)\right]\phi(t)\, dt$$

or $$\phi(x) = f(x) + \lambda \sum_{i=1}^{n} g_i(x) \int_a^b h_i(t)\phi(t)\, dt \qquad ...(3)$$

[interchanging the order of summation and integration]

To solve (3), we assume that

$$\int_a^b h_i(t)\phi(t)\, dt = C_i (i = 1, 2, ... n) \qquad ...(4)$$

Using (4), (3) reduces to

$$\phi(x) = f(x) + \lambda \sum_{i=1}^{n} C_i\ g_i(x) \qquad ...(5)$$

where C_i are unknown constants, since the function $\phi(x)$ is unknown.

Thus (5) is the form of the required solution of the integral equation (1) with separable kernel *i.e.*, the solution of an integral equation (1) is reduced to finding the constants $C_i (i = 1, 2, ... n)$.

We now proceed to evaluate C_i's as follows :

Multiplying both the sides of (5) successively by $h_1(x)$, $h_2(x), ... h_n(x)$ and integrating over the interval (a, b), we have

$$\int_a^b h_1(x)\phi(x)\,dx = \int_a^b h_1(x) f(x)\,dx + \lambda \sum_{i=1}^{\eta} C_i \int_a^b h_1(x) g_i(x)\,dx \qquad ...(\mathrm{A}_1)$$

$$\int_a^b h_2(x)\phi(x)\,dx = \int_a^b h_2(x) f(x)\,dx + \lambda \sum_{i=1}^{\eta} C_i \int_a^b h_2(x) g_i(x)\,dx \qquad ...(\mathrm{A}_2)$$

....

....

$$\int_a^b h_n(x)\phi(x)\,dx = \int_a^b h_n(x) f(x)\,dx + \lambda \sum_{i=1}^{\eta} C_i \int_a^b h_n(x) g_i(x)\,dx \qquad ...(\mathrm{A}_\mathrm{n})$$

Let $\alpha_{ji} = \int_a^b h_j(x) g_i(x)\,dx,\ (i, j = 1, 2, ...n)$ ...(6)

$$\beta_j = \int_a^b h_j(x) f(x)\,dx,\ (j = 1, 2, ...n) \qquad ...(7)$$

Using (4), (6) and (7), (A_1) reduces to

$$C_1 = \beta_1 + \lambda \sum_{i=1}^{\eta} C_i\,\alpha_{1i}$$

or $C_1 = \beta_1 + \lambda\left[C_1\alpha_{11} + C_2\alpha_{12} + ... + C_n\alpha_{1n}\right]$

or $(1-\lambda\alpha_{11})\,C_1 - \lambda\alpha_{12}\,C_2 - .... - \lambda\alpha_{1n}C_n = \beta_1$

Similarly, we may simplify $(\mathrm{A}_2), (\mathrm{A}_3), ...(\mathrm{A}_\mathrm{n})$. Thus, we obtain the following system of linear equations to determine $C_1, C_2, ... C_n$.

$$(1-\lambda\alpha_{11})\,C_1 - \lambda\alpha_{12}\,C_2 - .... - \lambda\alpha_{1n}C_n = \beta_1 \qquad ...(\mathrm{B}_1)$$

$$-\lambda\alpha_{21}\,C_1 + (1-\lambda\alpha_{22})C_2 - .... - \lambda\alpha_{2n}C_n = \beta_2 \qquad ...(\mathrm{B}_2)$$

....

....

and $-\lambda\alpha_{n1}\,C_1 - \lambda\alpha_{n2}\,C_2 - .... + (1-\lambda\alpha_{nn})C_n = \beta_n$ ....(B_n)

The determinant $D(\lambda)$ of the system $(\mathrm{B}_1), (\mathrm{B}_2), ...(\mathrm{B}_\mathrm{n})$ of linear equations is given by

$$D(\lambda) = \begin{vmatrix} 1-\lambda\alpha_{11} & -\lambda\alpha_{12} & \ldots & -\lambda\alpha_{1n} \\ -\lambda\alpha_{21} & 1-\lambda\alpha_{22} & \ldots & -\lambda\alpha_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ -\lambda\alpha_{n1} & -\lambda\alpha_{n2} & \ldots & 1-\lambda\alpha_{nn} \end{vmatrix}, \qquad ...(8)$$

which is a polynomial in λ of degree at most n.

Moreover $D(\lambda) \neq 0$, since, when $\lambda = 0$, $D(\lambda) = 1$.

To discuss the solution of (1), the following situations arise.

Situation (i) : When at least one member of the system $(B_1), \ldots, (B_n)$ is non zero.

Under this situation, following two cases arise :

(a) **If $D(\lambda) \neq 0$**, then the algebraic system $(B_1), \ldots, (B_n)$ has a unique non zero solution (which is obtained by using the Cramer's rule) given by (5).

(b) **If $D(\lambda) = 0$,** then the equation $(B_1), \ldots, (B_n)$ have either no solution or they posses infinite solutions. Hence (1) has either no solution or infinite number of solutions.

Situation (ii) : When $f(x) = 0$, then (7) shows that $\beta_j = 0$ for $i = 1, 2, \ldots, n$. Hence the system of equations $(B_1), \ldots, (B_n)$ reduce to a system of homogeneous linear equations.

Under this situation, following two cases arise ;

(a) If $D(\lambda) \neq 0$, the system of equations $(B_1), (B_2), \ldots, (B_n)$ has only trivial solution $C_1 = C_2 = \ldots C_n = 0$ and so (1) has only unique zero or trivial solution $g(x) = 0$, by (5).

(b) If $D(\lambda) = 0$ at least one of the C_i's can be assigned arbitarily and the remaining C_i's can be determined accordingly. Hence when $D(\lambda) = 0$, infinitely many solution of (1) exists.

Those values of λ for which $D(\lambda) = 0$ are known as the eigenvalues (or characteristic values) and any non zero solution of the homogeneous Fredholm integral equation.

$$g(x) = \lambda \int_a^b K(x,t) g(t)\, dt$$

is known as a corresponding eigenfunction (or characteristic function) of integral equation.

Situation (iii) : When $f(x) \neq 0$, but

$$\int_a^b g_2(x) f(x)\, dx = 0, \int_a^b g_2(x) f(x)\, dx = 0, \ldots, \int_a^b g_n(x) f(x)\, dx = 0$$

i.e. $f(x)$ is orthogonal to all the function $g_1(t), g_2(t), \ldots, g_n(t)$, then (7) shows that $\beta_1 = \beta_2 = \ldots = \beta_n = 0$ and hence the equation $(B_1), \ldots, (B_n)$ reduce to a system of homogeneous linear equations. In this situation following two cases arise ;

(a) If $D(\lambda) \neq 0$, then a unique zero solution $C_1 = C_2 = \ldots\ldots = C_n = 0$ of the system $(B_1), \ldots, (B_n)$ exists and so integral equation (1) has only unique solution $\phi(x) = f(x)$.

(b) If $D(\lambda) = 0$, then the system $(B_1), \ldots, (B_n)$ possess infinite non zero solutions and so (1) has infinite non zero solutions. The solutions corresponding to the eigenvalues of λ are now expressed as the sum of $f(x)$ and arbitrary multiples of eigenfunctions.

Example 1 : Solve $g(x) = e^x + \lambda \int_0^1 2e^x e^t g(t)\, dt$ ...(9)

Solution : The given equation can be written as

$$g(x) = e^x + 2\lambda\, e^x C = e^x(1 + 2\, C\lambda) \quad ...(10)$$

where $C = \int_0^1 e^t g(t)\, dt$ ...(11)

Using (10) in (11), we find that

$$C = \int_0^1 e^t e^t (1 + 2\, C\,\lambda)\, dt = (1 + 2\, C\,\lambda)\left[\frac{e^{2t}}{2}\right]_0^1$$

or $$C = (1 + 2\, C\,\lambda)\,\frac{(e^2 - 1)}{2}$$ or $$C\left[1 - \lambda\,(e^2 - 1)\right] = \frac{1}{2}(e^2 - 1)$$

or $$C = \frac{e^2 - 1}{2\left[1 - \lambda\,(e^2 - 1)\right]},$$ where $$\lambda \neq \frac{1}{e^2 - 1}$$

Putting this value of C in (10), we get

$$g(x) = e^x\left[1 + 2\lambda \frac{e^2 - 1}{2\left[1 - \lambda(e^2 - 1)\right]}\right]$$

or $$g(x) = e^x\ \frac{1 - \lambda(e^2 - 1) + \lambda(e^2 - 1)}{1 - \lambda(e^2 - 1)}$$

or $$g(x) = \frac{e^x}{1 - \lambda(e^2 - 1)},$$ where $$\lambda \neq \frac{1}{e^2 - 1}$$

Example 2 : Solve the Fredholm integral equation of second kind

$$g(x) = x + \lambda \int_0^1 (x\,t^2 + x^2 t)\, g(t)\, dt$$

Solution : We have

$$g(x) = x + \lambda \int_0^1 t^2\, g(t)\, dt + \lambda\, x^2 \int_0^1 t\, g(t)\, dt$$

or $$g(x) = x + \lambda\, x\, C_1 + \lambda\, x^2\, C_2 \quad ...(12)$$

where $C_1 = \int_0^1 t^2\, g(t)\, dt$ ...(13)

and $\quad C_2 = \int_0^1 t\, g(t)\, dt \qquad ...(14)$

Using (12) in (13), we get

$$C_1 = \int_0^1 t^2\left(t + \lambda\, t\, C_1 + \lambda\, t^2 C_2\right) dt$$

or $$C_1 = \left[\frac{t^4}{4} + \frac{\lambda\, C_1 t^4}{4} + \frac{\lambda\, C_2 t^5}{5}\right]_0^1$$

or $$C_1 = \frac{1}{4} + \frac{\lambda\, C_1}{4} + \frac{\lambda\, C_2}{4}$$

or $$(20 - 5\lambda) C_1 - 4\lambda\, C_2 = 5 \qquad ...(15)$$

Similarly, (12) and (14) give

$$C_2 = \int_0^1 t\left(t + \lambda\, t\, C_1 + \lambda\, t^2\, C_2\right) dt$$

or $$C_2 = \left[\frac{t^3}{3} + \frac{\lambda\, C_1 t^3}{3} + \frac{\lambda\, C_2 t^4}{4}\right]_0^1 = \frac{1}{3} + \frac{\lambda\, C_1}{3} + \frac{\lambda\, C_2}{4}$$

or $$-4\lambda\, C_1 + (12 - 3\lambda) C_2 = 4 \qquad ...(16)$$

Solving (15) and (16) for C_1 and C_2, we get

$$C_1 = \frac{60 + \lambda}{240 - 120\,\lambda - \lambda^2}, \qquad C_2 = \frac{80}{240 - 120\,\lambda - \lambda^2}$$

Substituting the values of C_1 and C_2 in (12), we obtain the solution of the integral equation as follows :

$$g(x) = x + \frac{\lambda\, x\, (60 + \lambda)}{240 - 120\lambda - \lambda^2} + \frac{80\,\lambda\, x^2}{240 - 120\lambda - \lambda^2}$$

or $$g(x) = \frac{(240 - 60\,\lambda)\, x + 80\,\lambda\, x^2}{240 - 120\lambda - \lambda^2}$$

Example 3 : Solve the integral equation

$$g(x) = x + \lambda \int_{-\pi}^{\pi} \left(x \cos t + t^2 \sin x + \cos x \sin t\right)\, g(t)\, dt$$

Solution : The given integral equation gives

$$g(x) = x + \lambda\, x\, C_1 + \lambda \sin x\, C_2 + \lambda \cos x\, C_3 \qquad ...(17)$$

where $\quad C_1 = \int_{-\pi}^{\pi} \cos t\, g(t)\, dt \qquad ...(18)$

$$C_2 = \int_{-\pi}^{\pi} t^2\, g(t)\, dt \quad ...(19)$$

and $$C_3 = \int_{-\pi}^{\pi} \sin t\, g(t)\, dt \quad ...(20)$$

Using (17) in (18), we get

$$C_1 = \int_{-\pi}^{\pi} \cos t \left(t + \lambda\, t\, C_1 + \lambda \sin t\, C_2 + \lambda \cos t\, C_3\right) dt$$

or $$C_1 = \left(1 + \lambda\, C_1\right) \int_{-\pi}^{\pi} t \cos t\, dt + \lambda\, C_2 \int_{-\pi}^{\pi} \sin t \cos t\, dt + \lambda\, C_3 \int_{-\pi}^{\pi} \cos^2 t\, dt$$

or $$C_1 = 0 + 0 + 2\,\lambda\, C_3 \int_0^{\pi} \cos^2 t\, dt$$

($\because$ $t\cos t$ and $\sin t \cos t$ are odd functions whereas $\cos^2 t$ in an even function)

or $$C_1 = 2\,\lambda\, C_3 \int_0^{\pi} \frac{1 + \cos 2t}{2}\, dt = \lambda\, C_3 \left[t + \frac{\sin 2t}{2}\right]_0^{\pi}$$

or $$C_1 - \lambda\, \pi\, C_3 = 0 \quad ...(21)$$

Similarly using (17) in (19) and (20), we get

$$C_2 + 4\,\lambda\, C_3\, \pi = 0 \quad ...(22)$$

and $$-2\,\lambda\, \pi\, C_1 - \lambda\, \pi\, C_2 + C_3 = 2\pi \quad ...(23)$$

Thus, we have a system of algebraic equations (21) to (23) for determining C_1, C_2 and C_3. The determinant of this system is

$$D(\lambda) = \begin{vmatrix} 1 & 0 & -\lambda\, \pi \\ 0 & 1 & 4\,\lambda\, \pi \\ -2\,\lambda\, \pi & -\lambda\, \pi & 1 \end{vmatrix} = 1 + 2\,\lambda^2\, \pi^2 \neq 0$$

Thus this system has unique solution :

$$C_1 = \frac{2\,\pi^2 \lambda}{1 + 2\,\lambda^2\, \pi^2}, \qquad C_2 = \frac{-8\,\pi^2\, \lambda}{1 + 2\,\lambda^2\, \pi^2}, \qquad C_3 = \frac{2\pi}{1 + 2\,\lambda^2\, \pi^2}$$

Putting these values of C_1, C_2 and C_3 in (17) the required solution of the given integral equation will be

$$g(x) = x + \frac{2\,\pi^2 \lambda^2 x}{1 + 2\,\lambda^2\, \pi^2} - \frac{8\,\pi^2 \lambda^2 \sin x}{1 + 2\,\lambda^2\, \pi^2} + \frac{2\,\pi\, \lambda \cos x}{1 + 2\,\lambda^2\, \pi^2}$$

or $$g(x) = x + \frac{2\,\pi\, \lambda}{1 + 2\,\lambda^2\, \pi^2} \left(\lambda\, \pi\, x - 4\,\lambda\, \pi \sin x + \cos x\right)$$

Example 4 : Solve the integral equation

$$g(x) = f(x) + \lambda \int_{-1}^{1} (xt + x^2 t^2) g(t)\, dt$$

Also, find its resolvent kernel.

Solution : Given equation is

$$g(x) = f(x) + \lambda x C_1 + \lambda x^2 C_2 \qquad \text{...(24)}$$

where $C_1 = \int_{-1}^{1} t\, g(t)\, dt$ ...(25)

and $C_2 = \int_{-1}^{1} t^2\, g(t)\, dt$ ...(26)

Using (24), (25) reduces to

$$C_1 = \int_{-1}^{1} t\left[f(t) + \lambda C_1 t + \lambda C_2 t^2\right] dt = \int_{-1}^{1} t\, f(t)\, dt + \lambda C_1 \left[\frac{t^3}{3}\right]_{-1}^{1} + \lambda C_2 \left[\frac{t^4}{4}\right]_{-1}^{1}$$

or $C_1 = \int_{-1}^{1} t\, f(t)\, dt + \dfrac{2\lambda C_1}{3}$

or $C_1 \left(1 - \dfrac{2\lambda}{3}\right) = \int_{-1}^{1} t\, f(t)\, dt$

or $C_1 = \dfrac{3}{3 - 2\lambda} \int_{-1}^{1} t\, f(t)\, dt$ ...(27)

Similarly using (24), (26) reduces to

$$C_2 = \frac{5}{5 - 2\lambda} \int_{-1}^{1} t^2\, f(t)\, dt \qquad \text{...(28)}$$

Using (27) and (28) in (24), the required solution is

$$g(x) = f(x) + \frac{3\lambda x}{3 - 2\lambda} \int_{-1}^{1} t\, f(t)\, dt + \frac{5\lambda x^2}{5 - 2\lambda} \int_{-1}^{1} t^2\, f(t)\, dt$$

or $g(x) = f(x) + \lambda \int_{-1}^{1} \left\{ \dfrac{3xt}{3 - 2\lambda} + \dfrac{5x^2 t^2}{5 - 2\lambda} \right\} f(t)\, dt$

The required resolvent kernel $R(x, t; \lambda)$ is given by

$$R(x, t; \lambda) = \frac{3xt}{3 - 2\lambda} + \frac{5x^2 t^2}{5 - 2\lambda}$$

Example 5 : Solve the integral equation and discuss all its possible cases by the method of degenerate kernels

$$g(x) = f(x) = \lambda \int_0^1 (1 - 3xt)\, g(t)\, dt \qquad ...(29)$$

Solution : From the integral equation, we have

$$g(x) = f(x) + \lambda C_1 - 3x\lambda C_2 \qquad ...(30)$$

Where $C_1 = \int_0^1 g(t)\, dt\,, \qquad C_2 = \int_0^1 t\, g(t)\, dt \qquad ...(31)$

Substituting (30) in (31), we have

$$C_1 = \int_0^1 \{ f(t) + \lambda C_1 - 3t\lambda C_2 \}\, dt$$

$$C_2 = \int_0^1 t \{ f(t) + \lambda C_1 - 3t\lambda C_2 \}\, dt$$

or $$C_1 \left[1 - \lambda \int_0^1 dt \right] + 3C_2 \lambda \int_0^1 t\, dt = \int_0^1 f(t)\, dt$$

$$-C_1\lambda \int_0^1 t\, dt + C_2 \left[1 + 3\lambda \int_0^1 t^2\, dt \right] = \int_0^1 t\, f(t)\, dt$$

or $$\left. \begin{array}{l} C_1(1-\lambda) + \dfrac{3}{2}\lambda C_2 = \int_0^1 f(t)\, dt \\ -\dfrac{1}{2} C_1 \lambda + (1+\lambda) C_2 = \int_0^1 t\, f(t)\, dt \end{array} \right] \qquad ...(32)$$

The determinant of the system (32) is given by

$$D(\lambda) = \begin{vmatrix} 1-\lambda & \dfrac{3}{2}\lambda \\ -\dfrac{1}{2}\lambda & 1+\lambda \end{vmatrix} = \frac{1}{4}(4 - \lambda^2)$$

Hence a unique solution of the system (32) exists if and only if $D(\lambda) \neq 0$ i.e. $\lambda \neq \pm 2$ and is obtained by solving (32). By putting the values of C_1 and C_2 so obtained in (30), the repuired solution of (29) follows easily.

In Particular, if $f(x) = 0$ and $\lambda \neq \pm 2$, the only zero solution $C_1 = C_2 = 0$ is obtained from (32) and hence we get trivial solution $g(x) = 0$ for (29). The numbers $\lambda = \pm 2$ are the eigenvalues of the problem.

If $\lambda = 2$, then the equation (32) reduces to

$$\left. \begin{array}{l} -C_1 + 3C_2 = \int_0^1 f(t)\, dt \\ -C_1 + 3C_2 = \int_0^1 t\, f(t)\, dt \end{array} \right] \qquad ...(33)$$

If $\lambda = -2$, then the equation (32) reduces to

$$\left.\begin{aligned} C_1 - C_2 &= \frac{1}{3}\int_0^1 f(t)\, dt \\ C_1 - C_2 &= \int_0^1 t\, f(t)\, dt \end{aligned}\right] \qquad \text{...(34)}$$

Equations (33) and (34) are incompatible (i.e. possess no solution) unless the given function $f(x)$ satisfies the conditon

$$\int_0^1 f(t)\,dt = \int_0^1 t\, f(t)\,dt \Rightarrow \int_0^1 (1-t)\, f(t)\,dt = 0 \qquad \text{...(35)}$$

and $$\frac{1}{3}\int_0^1 f(t)\,dt = \int_0^1 t\, f(t)\,dt \Rightarrow \int_0^1 (1-3t)\, f(t)\,dt = 0 \qquad \text{...(36)}$$

In these cases the corresponding equation pairs (35) and (36) are redundant (i.e. identical and hence possess infinitely many solution)

We now discuss solution of (29). Two cases arise :

Case I : When $f(x) = 0$, then the given integral equation becomes the homogeneous integral equation

$$g(x) = \lambda \int_0^1 (1-3xt)\, g(t)\, dt \qquad \text{...(37)}$$

Then if $\lambda \neq \pm 2$, (29) has only trivial solution $g(x) = 0$, as mentioned above.

For non trivial solution of (37), we have $\lambda = \pm 2$. Hence the eigenvalues are $\lambda = \pm 2$.

To find eigenfunction corresponding to $\lambda = 2$, we use (33) with $f(x) = 0$. Thus pair of equations (33) reduces to $C_1 = 3C_2$ and so (30) becomes

$$g(x) = 2(3C_2 - 3C_2 x) = 6C_2(1-x) = A(1-x)$$

where $A = 6C_2$ is an arbitrary constant.

Thus the function $1-x$ (or any convenient non zero multiple of that function) is the eigenfunction corresponding to the eigenvalue $\lambda = 2$.

Next, to find eigen function corresponding to $\lambda = -2$, we use (34) with $f(x) = 0$. We easily get $C_1 = C_2$ and so (30) becomes

$$g(x) = -2C_1(1-3x) = B(1-3x)$$

where $B = -2C_1$ is arbitrary constant.

Thus the function $1-3x$ (or any convenient non zero multiple of that function) is the eigenfunction corresponding to the eigenvalue $\lambda = -2$.

Case II : Let $f(x) \neq 0$, then (29) is non homogeneous integral equation. Now three cases arise :

(i) **When $\lambda \neq \pm 2$,** (29) possesses a unique solution as explained above.

(ii) **When $\lambda = 2$,** equations (33) show that no solution exists unless $f(x)$ is orthogonal to $1-x$ over the relevant interval $(0,1)$, that is, unless $f(x)$ is orthogonal to the eigenfunction corresponding to $\lambda = 2$. When $f(x)$ satisfies this restriction, equations (33) are identical and these give us

$$C_1 = 3C_2 - \int_0^1 f(t)\,dt$$

Putting this value of C_1 in (30), we get

$$g(x) = f(x) + \lambda\left[3C_2 - \int_0^1 f(t)\,dt\right] - 3xC_2\lambda$$

or $$g(x) = f(x) - 2\int_0^1 f(t)\,dt + 6C_2(1-x) \text{ as } \lambda = 2$$

or $$g(x) = f(x) - 2\int_0^1 f(t)\,dt + A(1-x) \qquad ...(38)$$

where $A = 6C_2$ is an arbitrary constant.

Thus if $\lambda = 2$ and $\int_0^1 (1-t)\, f(t)\, dt = 0$, the given equation (29) possesses infinitely many solutions given by (38).

(iii) **When $\lambda = -2$,** equations (34) show that no solution exists unless $f(x)$ is orthogonal to $1-3x$ over the relevant interval $(0,1)$, that is unless $f(x)$ is orthogonal to the eigenfunction corresponding to $\lambda = -2$. When $f(x)$ satisfies this restriction, equation (34) are identical and these give us

$$C_1 = C_2 + \frac{1}{3}\int_0^1 f(t)\,dt$$

Putting this value of C_1 in (30), we get

$$g(x) = f(x) + \lambda\left[C_2 + \frac{1}{3}\int_0^1 f(t)\,dt\right] - 3xC_2\lambda$$

or $$g(x) = f(x) - \frac{2}{3}\int_0^1 f(t)\,dt - 2C_2(1-3x) \text{ as } \lambda = -2$$

or $$g(x) = f(x) - \frac{2}{3}\int_0^1 f(t)\,dt + B(1-3x) \qquad ...(39)$$

where $B = -2C_2$ is an arbitrary constant.

Thus if $\lambda = -2$ and $\int_0^1 (1-3t)\, f(t)\, dt = 0$, the given equation (29) possesses infinitely many solution given by (39).

Example 6 : Show that the integral equation

$$g(x) = f(x) + \frac{1}{\pi}\int_0^{2\pi} \sin(x+t)\, g(t)\, dt$$

possesses no solution for $f(x) = x$, but that it possesses infinitely many solutions when $f(x) = 1$.

Solution : Given $g(x) = f(x) + \frac{1}{\pi}\int_0^{2\pi} \sin(x+t)\, g(t)\, dt$

or $$g(x) = f(x) + \frac{1}{\pi}\int_0^{2\pi} (\sin x \cos t + \cos x \sin t)\, g(t)\, dt$$

or $$g(x) = f(x) + \frac{\sin x}{\pi}\int_0^{2\pi} \cos t\, g(t)\, dt + \frac{\cos x}{\pi}\int_0^{2\pi} \sin t\, g(t)\, dt$$

or $$g(x) = f(x) + C_1 \frac{\sin x}{\pi} + C_2 \frac{\cos x}{\pi} \qquad ...(40)$$

where $$C_1 = \int_0^{2\pi} \cos t\, g(t)\, dt \qquad ...(41)$$

$$C_2 = \int_0^{2\pi} \sin t\, g(t)\, dt \qquad ...(42)$$

We now discuss two cases as mentioned in the problem.

Case I : Let $f(x) = x$, then (40) reduces to

$$g(x) = x + \frac{C_1 \sin x}{\pi} + \frac{C_2 \cos x}{\pi} \qquad ...(43)$$

Using (43) in (41), we get

$$C_1 = \int_0^{2\pi} \cos t\left[t + \frac{C_1 \sin t}{\pi} + \frac{C_2 \cos t}{\pi}\right] dt$$

or $$C_1 = \int_0^{2\pi} t\cos t\, dt + \frac{C_1}{2\pi}\int_0^{2\pi} \sin 2t\, dt + \frac{C_2}{2\pi}\int_0^{2\pi} (1+\cos 2t)\, dt$$

or $$C_1 = [t\sin t + \cos t]_0^{2\pi} + \frac{C_1}{2\pi}\left[\frac{-\cos 2t}{2}\right]_0^{2\pi} + \frac{C_2}{2\pi}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

or $$C_1 - C_2 = 0 \qquad ...(44)$$

Again using (43), (42) becomes

$$C_1 - C_2 = 2\pi \qquad ...(45)$$

The system of equations (44) and (45) is in consistent and so it possesses no solution.

Hence C_1 and C_2 can not be determined and so (43) shows that the given integral equation possesses no solution when $f(x) = x$.

Case II : Let $f(x) = 1$, then (40) reduces to

$$g(x) = 1 + \frac{C_1 \sin x}{\pi} + \frac{C_2 \cos x}{\pi} \qquad ...(46)$$

Using (46), (41) becomes

$$C_1 = \int_0^{2\pi} \cos t \left[1 + \frac{C_1 \sin x}{\pi} + \frac{C_2 \cos x}{\pi} \right] dt$$

or $$C_2 = \int_0^{2\pi} \cos t \, dt + \frac{C_1}{2\pi} \int_0^{2\pi} \sin 2t \, dt + \frac{C_1}{2\pi} \int_0^{2\pi} (1 + \cos 2t) \, dt$$

or $$C_1 = [\sin t]_0^{2\pi} + \frac{C_1}{2\pi} \left[\frac{-\cos 2t}{2} \right]_0^{2\pi} + \frac{C_2}{2\pi} \left[t + \frac{\sin 2t}{2} \right]_0^{2\pi}$$

or $$C_1 = 0 + 0 + \frac{C_2}{2\pi}(2\pi + 0) \qquad \text{or} \qquad C_1 = C_2 \qquad ...(47)$$

Again using (46), (42) gives

$$C_1 = C_2 \qquad ...(48)$$

From (47) and (48), we see that $C_1 = C_2 = A$ (say), where A is an arbitrary constant. Thus this system has infinite number of solutions $C_1 = A$ and $C_2 = A$. Putting these values in (46), the required solution of given integral equation is $g(x) = 1 + \frac{A}{\pi}(\sin x + \cos x)$ or $g(x) = 1 + C(\sin x + \cos x)$, where $C = \frac{A}{\pi}$ is another arbitrary constant. Since C is an arbitrary constant, we have infinitely many solutions of (40), when $f(x) = 1$.

Example 7 : Solve the equation

$$g(x) = 1 + \lambda \int_0^{\pi/2} \cos(x - t) \, g(t) \, dt$$

and find its eigenvalues.

Solution : The given equation may be written as

$$g(x) = 1 + \lambda \cos x \, C_1 + \lambda \sin x \, C_2 \qquad ...(49)$$

where $$C_1 = \int_0^{\pi/2} \cos t \, g(t) \, dt \qquad ...(50)$$

and $$C_2 = \int_0^{\pi/2} \sin t \, g(t) \, dt \qquad ...(51)$$

Proceeding as in Example 6, we get

$$\left(1-\frac{\lambda\pi}{4}\right)C_1-\frac{\lambda}{2}C_2=1 \qquad ...(52)$$

and $$-\frac{\lambda}{2}C_1+\left(1-\frac{\lambda\pi}{4}\right)C_2=1 \qquad ...(53)$$

On solving system of eqautions (52) and (53), we have

$$C_1=C_2=\frac{1+\frac{\lambda}{2}-\frac{\pi\lambda}{4}}{\left(1-\frac{\pi\lambda}{4}\right)^2-\frac{\lambda^2}{4}}=\frac{1}{1-\frac{\lambda}{4}(\pi+2)}$$

Now putting the values of C_1 and C_2 in (49), we have

$$g(x)=1+\frac{\lambda(\cos x+\sin x)}{1-\frac{\lambda}{4}(\pi+2)}$$

The eigenvalues are given by

$$D(\lambda)=\begin{vmatrix}1-\frac{\lambda\pi}{4} & -\frac{\lambda}{2}\\ -\frac{\lambda}{2} & 1-\frac{\lambda\pi}{4}\end{vmatrix}=0 \text{ or } \left(1-\frac{\lambda\pi}{4}\right)^2-\frac{\lambda^2}{4}=0 \text{ or } \lambda=\frac{4}{\pi\pm 2}$$

For these values of λ, The given non homogeneous integral equation has no solution.

Self Learning Exercise - I

1. State whether the following statements are true or false.

   (i) The eigenfunctions of a symmetric kernel, corresponding to different eigenvalues are not orthogonal.

   (ii) The eigen values of a symmetric kernel are real.

   (iii) We can find the solution of Fredholm integral equation of the second kind with the help of separable kernel.

   (iv) If the kernel $K(x,t)$ is continuous in the rectangle $R: a\le x\le b$, $a\le t\le b$ and the numbers a and b are finite then to every eigenvalue λ, there exists finite number of linearly independent eigenfunctions.

   (v) The number of eigenfunctions is known as index of the eigenvalue.

   (vi) If $\phi(x)$ is an eigenfunction, then $C\phi(x)$ is also an eigenfunction corresponding to same eigenvalue.

2. Define the following

(i) Separable kernel

(ii) Eigenvalues

(iii) Eigenfunctions

(iv) Orthogonal functions

9.3 Exercise 9 (a)

1. Solve the following integral equations :

(i) $g(x) = \sin x + \lambda \int_0^{\pi/2} \sin x \cos t \, g(t) \, dt$

(ii) $g(x) = (1+x)^2 + \int_{-1}^{1} (xt + x^2 t^2) g(t) \, dt$

(iii) $g(x) = x + \lambda \int_0^{\pi} (1 + \sin x \sin t) g(t) \, dt$

(iv) $g(x) = 1 + \int_0^1 (1 + e^{x+t}) g(t) \, dt$

(v) $g(x) = \cos x + \lambda \int_0^{\pi} \sin x \, g(t) \, dt$

(vi) $g(x) = \cos x + \lambda \int_0^{\pi} \sin(x-t) g(t) \, dt$

(vii) $g(x) = \cos x + \lambda \int_{-\pi/4}^{\pi/4} \tan t \, g(t) \, dt$

[**Ans.** (i) $g(x) = \dfrac{2 \sin x}{2 - \lambda}$

(ii) $g(x) = 1 + 6x + \dfrac{25}{9} x^2$

(iii) $g(x) = x + \dfrac{\lambda}{(1 - \lambda\pi)\left(1 - \dfrac{\lambda\pi}{2}\right) + 4\lambda^2} \left[2\lambda\pi + \dfrac{1}{2}\pi^2 \left(1 - \dfrac{1}{2}\lambda\pi \right) + \pi \sin x (1 - 2\lambda\pi) \right]$

(iv) $g(x) = \dfrac{e^2 - 2e - 1 - 2e^x(e-1)}{2(e-1)^2}$

(v) $g(x) = \cos x, \ \lambda \neq \dfrac{1}{2}$

(vi) $g(x)=\dfrac{4\cos x+2\pi\lambda\sin x}{4+\lambda^2\pi^2}$

(vii) $g(x)=\cos x=\dfrac{\pi}{2}$]

2. Solve the integral equation

$$g(x)=f(x)+\lambda\int_0^{2\pi}\sin(x+t)\,g(t)\,dt$$

and discuss all possible solutions.

3. Solve $g(x)=f(x)+\lambda\int_0^1 x\,t\,g(t)\,dt$

[Ans. $g(x)=f(x)+\dfrac{3x\lambda}{3-\lambda}\int_0^1 t\,f(t)\,dt$, where $\lambda\neq 3$**]**

4. Solve $g(x)=x+\lambda\int_0^1(1+x+t)\,g(t)\,dt$

[**Ans.** $g(x)=x+\dfrac{\lambda}{12-24\lambda-\lambda^2}\left[10+(6+\lambda)x\right]$]

5. Solve $g(x)=\dfrac{6}{5}(1-4x)+\lambda\int_0^1(x\log t-t\log x)\,g(t)\,dt$

[**Ans.** $\dfrac{6}{5}(1-4x)+\dfrac{48}{48+29\lambda^2}\left[2\lambda^2x+\left(\lambda+\dfrac{\lambda^2}{4}\right)\log x\right]$

6. Solve the integral equation

$$g(x)=f(x)+\lambda\int_0^1(x+t)\,g(t)\,dt$$

[**Ans.** $g(x)=f(x)+\lambda\int_0^1\dfrac{6(\lambda-2)(x+t)-12\lambda xt-4\lambda}{\lambda^2+12\lambda-12}f(t)\,dt$]

7. Solve $g(x)=\lambda\int_0^1\left(\log\dfrac{1}{t}\right)^p g(t)\,dt=1\quad(p>-1)$

[**Ans.** $g(x)=\dfrac{1}{1-\lambda\,\Gamma(p+1)}$]

9.4 Some Special Types of Integral Equations

9.4.1 Singular Integral Equation

Definition : An integral equation is called singular of either the range of integration is infinite or the kernel is discontinuous.

For example, the singular integral eqaution of first kind are

$$f(x)=\int_0^\infty \sin(xt)\,g(t)\,dt$$

$$f(x)=\int_0^\infty e^{-xt}\,g(t)\,dt$$

In above equations, the range of integration is infinite.

$$f(x)=\int_0^x \frac{g(t)}{\sqrt{x-t}}\,dt$$

In the above equation, the range of integration is finite but the kernel is discontinuous.

Remark : Singular integral equations occur frequently in mathematical physics and possess very unusual properties.

9.4.2 The Abel Integral Equation

One of the simplest form of singular integral equation, which arises in mechanics, is the Abel's integral eqaution

$$f(x)=\int_0^x \frac{g(t)}{(x-t)^\alpha}\,dt\,,\quad 0<\alpha<1$$

where $g(t)$ is an unknown to be determined and $f(x)$ is a known function.

9.4.3 Integro-Differential Equations

An integral equation in which various derivatives of the unknown function $g(x)$ can also be present is said to be integro-differential equation. For example

$$g''(x)=g(x)+\cos x+\int_0^x \sin(x-u)\,g(u)\,du$$

9.4.4 Integral Equation of Convolution Type

The integral equation

$$g(x)=f(x)+\int_0^x K(x-t)\,g(t)\,dt$$

in which the kernel $K(x-t)$ is a function of the difference only, is known as integral equation of the convolution type. Using the definition of convolution, the above integral equation can be written as

$$g(x)=f(x)+K(x)*g(x)$$

9.5 Solution of Volterra Integral Equation of Second kind with Convolution Kernels by Laplace Transform Working Rule

(i) Consider the Volterra integral equation of the first kind

$$f(x) = \int_0^x K(x-t)\,g(t)\,dt \qquad ...(54)$$

or $$f(x) = K(x) * g(x) \qquad ...(55)$$

where the kernel $K(x-t)$ depends only on difference $(x-t)$. Applying the Laplace transform to both sides of equation (55), we get

$$L\{f(x); p\} = L\{K(x) * g(x)\} \qquad ...(56)$$

or $$F(p) = K(p)\;G(p)$$ (by the convolution theorem for Laplace transform)

or $$G(p) = \frac{F(p)}{G(p)} \qquad ...(57)$$

Applying the inverse Laplace transform to both sides of (57), we get

$$g(x) = L^{-1}\left\{\frac{F(p)}{K(p)}; x\right\}$$

(ii) Consider Volterra integral equation of the second kind

$$g(x) = f(x) + \int_0^x K(x-t)\,g(t)\,dt$$

$$= f(x) + K(x) * g(x) \qquad ...(58)$$

Applying the Laplace transform to both sides of (58), we get

$$L\{g(x); p\} = L\{f(x); p\} + L\{K(x) * g(x)\}$$

or $$G(p) = F(p) + K(p)\;G(p)$$

{Using $L\{K(x) * g(x)\} = F(p)\;G(p)$ and the convolution theorem}

or $$G(p)\{1 - K(p)\} = F(p)$$

or $$G(p) = \frac{F(p)}{1 - K(p)} \qquad ...(59)$$

Applying the inverse Laplace transform to both sides of (59), we obtain

$$g(x) = L^{-1}\left\{\frac{F(p)}{1 - K(p)}; x\right\}$$

(iii) Suppose we want the resolvent kernel of (58) in which the kernel $K(x-t)$ depends only on the difference $(x-t)$. By integral transform method, we first show that, if the original kernel $K(x,t)$ is a difference kernel, then so is the resolvent kernel.

The resolvent kernel $R(x,t)$ is given by (refer Art 10.3)

$$R(x,t)=\sum_{m=1}^{\infty}K_m(x,t)=K_1(x,t)+K_2(x,t)+..... \qquad ...(60)$$

[Note that here $\lambda=1$. So we have used symbol $R(x,t)$ in place of the usual symbol $R(x,t;\lambda)$]

The iterated kernels are given by (refer Art 10.2)

$$K_1(x,t)=K(x,t) \qquad ...(61)$$

and $$K_n(x,t)=\int_t^x K(x,z)K_{n-1}(z,t)\,dz, \qquad n=2,3,.... \qquad ...(62)$$

Here by assumption, we have

$K(x,t)=K(x-t)$ therefore by (61), we have

$$K_1(x,t)=K(x,t)=K(x-t) \qquad ...(63)$$

Putting $n=2$ in (62), we have

$$K_2(x,t)=\int_t^x K(x,z)K_1(z,t)\,dz=\int_t^x K(x-z)K(z-t)\,dz$$

$$=\int_0^{x-t} K(x-t-u)K(u)\,du \qquad \text{[putting } z-t=u\text{]}$$

showing that $K_2(x,t)$ depends only on the difference $(x-t)$. Proceeding likewise, we can show that $K_3(x,t)$, $K_4(x,t)$... also depends on the difference $(x-t)$. From (60), it follows that the resolvent kernel will also depend only on the difference $(x-t)$. Therefore we can assume that

$$R(x,t)=R(x-t) \qquad ...(64)$$

The solution of (58) is given by (Refer Art 10.3)

$$g(x)=f(x)+\int_0^x R(x,t)f(t)\,dt$$

or $$g(x)=f(x)+\int_0^x R(x-t)f(t)\,dt \qquad ...(65)$$

Applying the Laplace transform to both sides of (65), we have

$$L\{g(x);p\}=L\{f(x);p\}+L\{R(x)*f(x)\}$$

or $$G(p)=F(p)+R(p)F(p) \qquad ...(66)$$

where $G(p)=L\{g(x);p\}$, $F(p)=L\{f(x);p\}$ and $R(p)=L\{R(x);p\}$ ...(67)

Using (59) in (67), we get

$$\frac{F(p)}{1-K(p)}=F(p)[1+R(p)]$$

or $$R(p)=\frac{1}{1-K(p)}-1=\frac{K(p)}{1-K(p)} \quad ...(68)$$

Applying the inverse Laplace Transform to both sides of (68), we get

$$R(x-t)=L^{-1}\left\{\frac{K(p)}{1-K(p)}\right\} \quad ...(69)$$

Substituting the value of $R(x-t)$ given by (69) in (65) we shall get the desired solution of (58)

Example 8 : Solve the Abel integral equation

(i) $$f(x)=\int_0^x \frac{g(t)}{(x-t)^{\alpha}}\,dt\,,\quad 0<\alpha<1$$

(ii) $$\int_0^x \frac{g(t)}{\sqrt{x-t}}\,dt=1+x+x^2$$

Solution : (i) The given integral equation is of convolution type and therefore the integral equation may be expressed as

$$f(x)=g(x)*x^{-\alpha} \quad ...(70)$$

Taking the Laplace Transform of both sides of (70) and using the convolution theorem, we have

$$L\{f(x);p\}=L\{g(x);p\}\;L\{x^{-\alpha};p\}$$

or $$F(p)=G(p)\frac{\Gamma(1-\alpha)}{p^{1-\alpha}}$$

or $$G(p)=\frac{p^{1-\alpha}F(p)}{\Gamma(1-\alpha)}=\frac{p}{\Gamma(\alpha)\,\Gamma(1-\alpha)}\{\Gamma(\alpha)\;p^{-\alpha}F(p)\}$$

$$=\frac{p}{(\pi/\sin\pi\alpha)}\{\Gamma(\alpha)\;p^{-\alpha}\;F(p)\} \qquad \left(\because\ \Gamma(\alpha)\,\Gamma(1-\alpha)=\frac{\pi}{\sin\pi\alpha}\right)$$

$$=\frac{p\sin\pi\alpha}{\pi}\;L\{x^{\alpha-1}*f(x)\} \qquad \text{(by convolution theorem)}$$

$$= \frac{\sin \pi \alpha}{\pi} \; p \, L\left\{\int_0^x (x-t)^{\alpha-1} f(t)\, dt\right\} \qquad \text{(by definition of convolution)} \qquad ...(71)$$

Let $\quad h(x) = \int_0^x (x-t)^{\alpha-1} f(t)\, dt$ ...(72)

Now $\quad L\{h'(x); p\} = p\, L\{h(x); p\} - h(0) = p\, L\{h(x); p\} \qquad (\because h(0) = 0)$

or $\quad p\, L\left\{\int_0^x (x-t)^{\alpha-1} f(t)\, dt\right\} = L\{h'(x); p\}$ ...(73)

Using (73) in (71), we get

$$G(p) = \frac{\sin \pi \alpha}{\pi} \; L\{h'(x); p\}$$

Inverting, we have

$$g(x) = L^{-1}\{G(p); x\} = \frac{\sin \pi \alpha}{\pi} \frac{d}{dx}\left[\int_0^x (x-t)^{\alpha-1} f(t)\, dt\right]$$

(ii) Rewriting the given equation in convolution form, we have

$$g(x) * x^{-1/2} = 1 + x + x^2 \qquad ...(74)$$

Taking the Laplace transform of both sides of (74) and using the convolution theorem, we have

$$L\{g(x)\}\; L\left\{x^{-1/2}\right\} = L\{1\} + L\{x\} + L\{x^2\}$$

or $\quad G(p)\,\dfrac{\Gamma(1/2)}{p^{1/2}} = \dfrac{1}{p} + \dfrac{1}{p^2} + \dfrac{2!}{p^3}$

or $\quad G(p) = \dfrac{1}{\sqrt{\pi}}\left(\dfrac{1}{p^{1/2}} + \dfrac{1}{p^{3/2}} + \dfrac{2}{p^{5/2}}\right)$ ...(75)

Applying the inverse Laplace Transform to both sides of (75), we get

$$g(x) = \frac{1}{\sqrt{\pi}}\left[L^{-1}\left\{\frac{1}{p^{1/2}}\right\} + L^{-1}\left\{\frac{1}{p^{3/2}}\right\} + 2\, L^{-1}\left\{\frac{1}{p^{5/2}}\right\}\right]$$

$$= \frac{1}{\sqrt{\pi}}\left[\frac{x^{-1/2}}{\Gamma(1/2)} + \frac{x^{1/2}}{\Gamma(3/2)} + \frac{2x^{3/2}}{\Gamma(5/2)}\right]$$

$$= \frac{1}{\sqrt{\pi}}\left[\frac{x^{-1/2}}{\sqrt{\pi}} + \frac{x^{1/2}}{(1/2)\times\sqrt{\pi}} + \frac{2x^{3/2}}{(3/2)\times(1/2)\times\sqrt{\pi}}\right]$$

$$= \frac{1}{\pi}\left[x^{-1/2} + 2x^{1/2} + \left(8/3\right)x^{3/2}\right]$$

Example 9 : Solve the integral equation

$$\sin x = \int_0^x J_0(x-t)\, g(t)\, dt$$

Solution : The given integral equation may be expressed as

$$\sin x = g(x) * J_0(x)$$

Taking the Laplace transform of both sides of the above equation and using the convolution theorem, we have

$$L\{\sin x\} = L\{g(x)\}\ L\{J_0(x)\}$$

or $$\frac{1}{p^2+1} = G(p)\frac{1}{\sqrt{p^2+1}}, \qquad \text{where } G(p) = L\{g(x)\}$$

or $$G(p) = \frac{1}{\sqrt{p^2+1}}$$

Taking the inverse Laplace transform, we have

$$g(x) = L^{-1}\{G(p); x\} = J_0(x)$$

Example 10 : Solve the integral equation

$$g(x) = 1 + \int_0^x \sin(x-t)\, g(t)\, dt$$

and verify your answer.

Solution : The given integral equation can be rewriten as

$$g(x) = 1 + g(x) * \sin x \qquad ...(76)$$

Taking the Laplace transform of both sides of (76) and using the convolution theorem, we get

$$L\{g(x)\} = L\{1\} + L\{g(x)\}\, L\{\sin x\}$$

or $$G(p) = \frac{1}{p} + G(p)\frac{1}{p^2+1}, \qquad \text{where } G(p) = L\{g(x)\}$$

or $$\left(1 - \frac{1}{p^2+1}\right) G(p) = \frac{1}{p}$$

or $$G(p) = \frac{p^2+1}{p^3} = \frac{1}{p} + \frac{1}{p^3} \qquad ...(77)$$

Inverting, (77) reduces to

$$g(x) = 1 + \frac{x^2}{2!} = 1 + \frac{x^2}{2} \qquad ...(78)$$

Verification of Solution (78)

Now we show that the solution of (76) satisfies the given integral equation

$$g(x) = 1 + \int_0^x \sin(x-t)\, g(t)\, dt \qquad ...(79)$$

From (78), we have R.H.S. of (79)

$$= 1 + \int_0^x \sin(x-t)\left(1 + \frac{t^2}{2}\right) dt$$

$$= 1 + \left[\left(1 + \frac{t^2}{2}\right)\cos(x-t)\right]_0^x - \int_0^x t\cos(x-t)\, dt$$

$$= 1 + 1 + \frac{x^2}{2} - \cos x - \left\{-t\sin(x-t)\right\}_0^x + \int_0^x \sin(x-t)\, dt$$

$$= 2 + \frac{x^2}{2} - \cos x - \left\{\cos(x-t)\right\}_0^x$$

$$= 2 + \frac{x^2}{2} - \cos x - (1 - \cos x) = 1 + \frac{x^2}{2} = g(x)$$

$=$ L.H.S. of (79)

Example 11 : Solve the integral equation

$$g(x) = e^{-x} - 2\int_0^x \cos(x-t)\, g(t)\, dt$$

Solution : Rewriting the given integral equation, we have

$$g(x) = e^{-x} - 2\, g(x) * \cos x$$

Applying the Laplace Transform to both sides and using the convolution theorem, we have

$$L\{g(x)\} = L\{e^{-x}\} - 2L\{g(x)\}\; L\{\cos x\}$$

or $\quad G(p) = \dfrac{1}{p+1} - 2\, G(p)\dfrac{p}{p^2+1}, \qquad$ where $G(p) = L\{g(x)\}$

or $\quad G(p)\left\{1 + \dfrac{2p}{p^2+1}\right\} = \dfrac{1}{p+1}$

or $$G(p) = \frac{p^2+1}{(p+1)^3} = \frac{\{(p+1)-1\}^2+1}{(p+1)^3}$$

Inverting it, we get

$$g(x) = L^{-1}\left[\frac{\{(p+1)-1\}^2+1}{(p+1)^3}\right]$$

$$= e^{-x} L^{-1}\left[\frac{(p-1)^2+1}{p^3}\right] \qquad \text{[by first shifting theorem]}$$

$$= e^{-x} L^{-1}\left\{\frac{p^2-2p+2}{p^3}\right\} = e^{-x} L^{-1}\left\{\frac{1}{p} - \frac{2}{p^2} + \frac{2}{p^3}\right\}$$

$$= e^{-x}\left\{1-2x+2.\frac{x^2}{2!}\right\}$$

$$= e^{-x}\left(1-2x+x^2\right)$$

or $$g(x) = e^{-x}(1-x)^2$$

Example 12 : Solve

$$g'(x) = x + \int_0^x g(x-t)\cos t\, dt, \qquad g(0) = 4$$

Solution : The given integral equation can be written as

$$g'(x) = x + g(x) * \cos x \qquad ...(80)$$

Also given that $g(0) = 4$

Applying the Laplace transform to both sides of (80) and using the convolution theorem, we get

$$L\{g'(x)\} = L\{x\} + L\{g(x)\}\ L\{\cos x\}$$

or $$pG(p) - g(0) = \frac{1}{p^2} + G(p)\frac{p}{p^2+1}, \quad \text{where } G(p) = L\{g(x)\}$$

or $$\left(1 - \frac{1}{p^2+1}\right) pG(p) - 4 = \frac{1}{p^2}$$

or $$\frac{p^3}{p^2+1}G(p) = 4 + \frac{1}{p^2} \qquad \text{or} \qquad G(p) = \frac{p^2+1}{p^3}\left(4+\frac{1}{p^2}\right)$$

or $$G(p)=\frac{4(p^2+1)}{p^3}+\frac{p^2+1}{p^5}=\frac{4}{p}+\frac{5}{p^3}+\frac{1}{p^5}$$

Inverting it we get

$$g(x)=4\,L^{-1}\left\{\frac{1}{p}\right\}+5\,L^{-1}\left\{\frac{1}{p^3}\right\}+L^{-1}\left\{\frac{1}{p^5}\right\}$$

or $$g(x)=4+5\times\left(\frac{x^2}{2!}\right)+\left(\frac{x^4}{4!}\right)$$

$$=4+\frac{5x^2}{2}+\frac{x^4}{24}$$

Example 13 : Find the resoltvent kernel of the Volterra integral equation and hence its solution

$$g(x)=f(x)+\int_0^x (x-t)\,g(t)\,dt$$

Solution : The given integral equation can be written as

$$g(x)=f(x)+g(x)*x \qquad ...(81)$$

Applying the Laplace transform to both sides of (81) and using the convolution theorem, we have

$$L\{g(x)\}=L\{f(x)\}+L\{g(x)\}\;L\{x\}$$

or $$G(p)=F(p)+G(p)\frac{1}{p^2}, \qquad \text{where } G(p)=L\{g(x)\};F(p)=L\{f(x)\}$$

or $$\left(1-\frac{1}{p^2}\right)G(p)=F(p)$$

or $$G(p)=\frac{p^2}{p^2-1}\;F(p)$$

Let $R(x-t)$ be the resolvent kernel of the given integral equation. Then we know that the required solution is given by

$$g(x)=f(x)+\int_0^x R(x-t)\,f(t)\,dt$$

or $$g(x)=f(x)+R(x)*f(x) \qquad ...(82)$$

Applying the Laplace transform to both sides of (82) and using the convolution theorem, we have

$$L\{g(x)\}=L\{f(x)\}+L\{R(x)\}\;L\{f(x)\}$$

or $$G(p)=F(p)+R(p)F(p) \qquad \text{where } R(p)=L\{R(x)\}$$

or $$\frac{p^2}{p^2-1}F(p)=F(p)+R(p)F(p)$$

or $$R(p)=\frac{p^2}{p^2-1}-1=\frac{1}{p^2-1}$$

Inverting, $R(x)=L^{-1}\{R(p)\}=\sin h\, x$

so that $R(x-t)=\sin h(x-t)$,

giving required resolvent kernel.

Substituting the above value of $R(x-t)$, the required solution is

$$g(x)=f(x)+\int_0^x \sin h(x-t)\, f(t)\,dt$$

Example 14 : Determine the resolvent kernel and hence solve the integral equation

$$g(x)=f(x)+\int_0^x e^{x-t}\, g(t)\,dt$$

Solution : The given integral equation can be written as

$$g(x)=f(x)+g(x)*e^x \qquad ...(83)$$

Applying the Laplace transform to both sides of (83) and using the convolution theorem, we have

$$L\{g(x)\}=L\{f(x)\}+L\{g(x)\}\; L\{e^x\}$$

or $$G(p)=F(p)+G(p)\frac{1}{p-1}, \quad \text{where} \quad G(p)=L\{g(x)\};\; F(p)=L\{f(x)\}$$

or $$\left(1-\frac{1}{p-1}\right)G(p)=F(p)$$

or $$G(p)=\frac{p-1}{p-2}F(p)$$

Let $R(x-t)$ be the resolvent kernel of the given integral equation. Then we know that the required solution is given by

$$g(x)=f(x)+\int_0^x R(x-t)\, f(t)\,dt$$

or $$g(x)=f(x)+R(x)*f(x) \qquad ...(84)$$

Applying the Laplace transform to both side of (84) and using the convolution theorem, we have

$$L\{g(x)\}=L\{f(x)\}+L\{R(x)\}\; L\{f(x)\}$$

or $\quad G(p) = F(p) + R(p)F(p) \quad$ where $\quad R(p) = L\{R(x)\}$

or $\quad \dfrac{p-1}{p-2}F(p) = F(p)[1 + R(p)]$

or $\quad 1 + R(p) = \dfrac{p-1}{p-2} \quad$ or $\quad R(p) = \dfrac{p-1}{p-2} - 1 = \dfrac{1}{p-2}$

Inverting, $R(x) = L^{-1}\left\{\dfrac{1}{p-2}\right\} = e^{2x}$

$\therefore \quad R(x-t) = e^{2(x-t)}$, giving required resolvent kernel.

Substituting the above value of $R(x-t)$, the required solution

$$g(x) = f(x) + \int_0^x e^{2(x-t)} f(t)\,dt$$

9.7 Solution of Singular Integral Equations by Fourier Transform

The whole procedure will be clear from the following examples :

Example 15 : Solve for $f(x)$ the integral equation

$$\int_0^\infty f(x)\cos px\,dx = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$$

Hence deduce that $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}dt = \frac{\pi}{2}$

Solution : Let $\displaystyle\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos px\,dx = F_c\{f(x)\} = F_c(p)$

Then $\quad F_c(p) = \begin{cases} \sqrt{\dfrac{2}{\pi}}(1-p), & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$

Hence by the Fourier cosine inversion formula, we have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(p)\cos px\,dp$$

$$= \sqrt{\frac{2}{\pi}}\int_0^1 \sqrt{\frac{2}{\pi}}(1-p)\cos px\,dp$$

$$= \frac{2}{\pi}\left[\left\{(1-p)\frac{\sin px}{x}\right\}_0^1 + \int_0^1 \frac{\sin px}{x}\, dp\right]$$

$$= \frac{2}{\pi}\cdot\frac{1}{x^2}(-\cos px)_0^1$$

$$= \frac{2(1-\cos x)}{\pi x^2}, \text{ which is the required solution.}$$

Deduction : Substituting the value of $f(x)$ in the given integral equation, we get

$$\int_0^\infty \frac{2(1-\cos x)}{\pi x^2}\cos px\, dx = \begin{cases} 1-p, & 0 \le p < 1 \\ 0 \quad , & p > 1 \end{cases}$$

Letting $p \to 0$, this equation yields

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos x}{x^2}\, dx = 1$$

or $$\int_0^\infty \frac{2\sin^2(x/2)}{x^2}\, dx = \frac{\pi}{2}$$

Putting $x = 2t$, $dx = 2dt$, we at once get

$$\int_0^\infty \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2}$$

Example 16 : Solve for $f(x)$, the integral equation

$$\int_0^\infty f(x)\sin px\; dx = \begin{cases} 1 & , \quad 0 \le p \le 1 \\ 2 & , \quad 1 \le p < 1 \\ 0 & , \quad p > 2 \end{cases}$$

Solution : Let $\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin px\; dx = F_s\{f(x)\} = F_s(p)$. Then

$$F_s(p) = \sqrt{\frac{2}{\pi}}\begin{cases} 1 & , \quad 0 \le p \le 1 \\ 2 & , \quad 1 \le p < 1 \\ 0 & , \quad p > 2 \end{cases}$$

Hence by the Fourier sine inversion formula, we get

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(p)\sin px\, dp$$

$$= \frac{2}{\pi}\int_0^1 1.\sin px\,dp + \frac{2}{\pi}\int_1^2 2\sin px\,dp + \frac{2}{\pi}\int_2^\infty 0\sin px\,dp$$

$$= \frac{2}{\pi}\left[\frac{-\cos px}{x}\right]_0^1 + \frac{4}{\pi}\left[\frac{-\cos px}{x}\right]_1^2$$

$$= \frac{2}{\pi}\left[-\cos x + 1 + 2\{-\cos 2x + \cos x\}\right]$$

$$= \frac{2}{\pi}(1 + \cos x - 2\cos 2x)$$

Example 17 : Solve : $\int_0^\infty f(x)\cos px\,dx = e^{-p}$.

Solution : Let $\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos px\,dx = F_c\{f(x)\} = F_c(p)$.

Then $F_c(p) = \sqrt{\frac{2}{\pi}}\, e^{-p}$.

Hence by the Fourier cosine inversion formula, we have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(p)\cos px\,dp = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\, e^{-p}\cos px\,dp$$

$$= \frac{2}{\pi}\int_0^\infty e^{-p}\cos px\,dp$$

$$= \frac{2}{\pi}\left[\frac{e^{-p}}{1+x^2}(-\cos px + x\sin px)\right]_0^\infty$$

as $\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$

$$= \frac{2}{\pi(1+x^2)}$$

Thus $f(x) = \frac{2}{\pi(1+x^2)}$

Self-Learning Exercise - II

1. Define following :

   (i) Singular integral equation.

(ii) The Abel integral equation.

(iii) Integro-differential equation.

(iv) Integral equation of convolution type.

2. State whether the following statements are true or false.

(i) An integral of the type $\int_{-\infty}^{\infty} K(p,t)F(t)\,dt$ is defined as the integral transform of $F(t)$ provided it is divergent.

(ii) The resolvent kernel of the non homogeneous integral equation cannot be determined by the method of integral transform.

(iii) The convolution of two functions $G(t)$ and $H(t)$, where $-\infty < t < \infty$, is denoted and derived by $G*H = \int_{-\infty}^{\infty} G(t)H(x-t)\,dt$

(iv) If $L^{-1}\{f(p)\} = F(t)$ and $L^{-1}\{g(p)\} = G(t)$, where $F(t)$ and $G(t)$ are two functions of class A. Then $L^{-1}\{f(p)*g(p)\} = \int_0^1 F(u)G(t-u)\,du = F*G$.

9.7 Summary

In this chapter, we have seen that the kernel is separable, the problem of solving an integral equation of second kind reduces to that of solving an algebraic system of equations. We have discussed the integral transtorm method, a useful tool for the solution of integral equation of some special forms.

9.8 Answers to Self Learning Expercises

Exercise-I

1. (i) False (ii) True (iii) True (iv) True (v) True (vi) True.

2. See text.

Exerciese-II

1. See Text

2. (i) False (ii) False (iii) True (iv) False

9.9 Exercise 9 (b)

1. Solve the Abel integral equation

(i) $$\int_0^x \frac{g(t)}{(x-t)^{1/3}}\,dt = x(1+x)$$

(ii) $$\int_0^x \frac{g(t)}{(x-t)^{1/2}}\,dt = f(x)$$

[**Ans.** (i) $g(x)=\frac{3\sqrt{3}}{4\pi}x^{1/3}(2+3x)$ (ii) $g(x)=\frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{\sqrt{x-t}}dt$]

2. Solve the in homogeneous integral equation

$$g(x)=1-\int_0^x (x-t)g(t)dt$$

Also verify result.

[Ans. $g(x)=\cos x$ **]**

3. Solve

(i) $\int_0^x g(t)\cos(x-t)dt = g'(x) \text{ if } g(0)=1$

(ii) $\int_0^x g'(t)g(x-t)dt = 24x^3, \text{ if } g(0)=0$

[**Ans.** (i) $g(x)=1+\frac{x^2}{2}$ (ii) $g(x)=\pm\frac{16x^{3/2}}{\sqrt{\pi}}$]

4. Solve $\int_0^x g(t)g(x-t)dt = 16\sin 4x$

[**Ans.** $g(x)=\pm 8\, J_0(4x)$]

5. Solve the Volterra integral equation of second kind

$$g(x)=x^2+\int_0^x g(t)\sin(x-t)dt$$

[**Ans.** $g(x)=x^2+\frac{x^4}{12}$]

6. Solve : $g(x)=a\sin x-2\int_0^x \cos(x-t)g(t)dt$

[**Ans.** $g(x)=a\,x\,e^{-x}$]

7. Solve the integral differential equation

$$g'(x)=\sin x+\int_0^x g(t)\cos(x-t)dt \text{ where } g(0)=0$$

[**Ans.** $g(x)=\frac{x^2}{2}$]

Unit - 10
Solution of Integral Equation of Second Kind by Successive Approximation and Substitution

Structure of the Unit

10.0 Objective

In this unit, we shall discuss the solution of Fredholm and Volterra integral equation of the second kind by the method of successive substitution and the method of successive approximation.

10.1 Introduction

We already know that ordinary first order differential equations are solved by the well known Picard method of successive approximation. In this unit, we shall study an iterative scheme based on the same principle for linear integral equations of the second kind. Throughout our discussion we shall assume that the function $f(x)$ and $K(x,t)$ involued in an integral equation are L_2 functions.

10.2 Iterated Kernles or Functions

Definition :

(i) Consider Fredholm integral equation of the second kind

$$g(x) = f(x) + \lambda \int_a^b K(x,t)\, g(t)\, dt$$

Then, the iterated kernels $K_n(x,t)$, $n = 1,2,3,....$ are defined as follows :

$$K_1(x,t) = K(x,t)$$

and $$K_n(x,t) = \int_a^b K(x,z)\; K_{n-1}(z,t)\; dz\,, \qquad n = 2,3,......$$

or $$K_n(x,t) = \int_a^b K_{n-1}(x,z)\ K(z,t)\ dz, \qquad n = 2,3,......$$

(ii) Consider Volterra integral equation of the second kind

$$g(x) = f(x) + \lambda \int_a^x K(x,t)\ g(t)\, dt$$

Then, the iterated kernels $K_n(x,t)$, $n = 1,2,3,.....$ are defines as follows :

$$K_1(x,t) = K(x,t)$$

and $$K_n(x,t) = \int_t^x K(x,z)\ K_{n-1}(z,t)\ dz, \qquad n = 2,3,......$$

or $$K_n(x,t) = \int_t^x K_{n-1}(x,z)\ K(z,t)\ dz, \qquad n = 2,3,......$$

10.3 Resolvent Kernel or Reciprocal Kernel

(i) Suppose solution of Fredholm integral equation of the second kind

$$g(x) = f(x) + \lambda \int_a^b K(x,t) g(t)\, dt \qquad ...(1)$$

takes the form

$$g(x) = f(x) + \lambda \int_a^b R(x,t;\lambda) f(t)\, dt$$

or $$g(x) = f(x) + \lambda \int_a^b \Gamma(x,t;\lambda) f(t)\, dt$$

Then $R(x,t;\lambda)$ or $\Gamma(x,t;\lambda)$ is known as the **resolvent kernel** of (1).

(ii) Suppose solution of volterra integral equation of the second kind

$$g(x) = f(x) + \lambda \int_a^x K(x,t) g(t)\, dt \qquad ...(2)$$

takes the form

$$g(x) = f(x) + \lambda \int_a^x R(x,t;\lambda) f(t)\, dt$$

or $$g(x) = f(x) + \lambda \int_a^x \Gamma(x,t;\lambda) f(t)\, dt$$

Then $R(x,t;\lambda)$ or $\Gamma(x,t;\lambda)$ is known as the **resolvent kernel** of (2).

10.4 Solution of Fredholm Integral Equation of Second Kind by Successive Substitution

Theorem 1 : Let $g(x) = f(x) + \lambda \int_a^b K(x,t) g(t) dt$ ...(3)

be given Fredholm integral equation of the second kind. Suppose that

(i) **Kernel** $K(x,t) \neq 0$ **is real and continuous in the ractangle** R**, for which** $a \leq x \leq b$, $a \leq t \leq b$. **Suppose that** $|K(x,t)| \leq P$**, where** P **is the maximum value of** $|K(x,t)|$ **in** R.

(ii) $f(x) \neq 0$ **is real and continuous in an interval** I ; $a \leq x \leq b$. **Let** $|f(x)| \leq Q$, **where** Q **is the maximum value of** $|f(x)|$ **in the interval** I.

(iii) λ **is a constant such that** $|\lambda| < \dfrac{1}{P(a-b)}$.

Then (3) has a unique continuous solution in I **and this solution is given by the absolutely and uniformly convergent series :**

$$g(x) = f(x) + \lambda \int_a^b K(x,t) f(t) dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1) f(t_1) dt_1 \, dt$$

$$+\lambda^3 \int_a^b K(x,t) \int_a^b K(t,t_1) \int_a^b K(t_1,t_2) \, f(t_2) dt_2 \, dt_1 \, dt + \ldots\ldots$$

Proof : Rewriting (3) as

$$g(x) = f(x) + \lambda \int_a^b K(x,t) \, g(t) dt_1$$

or $$g(t) = f(t) + \lambda \int_a^b K(t,t_1) \, g(t_1) dt_1 \qquad ...(4)$$

Since there exists a continous solution $g(x)$ of (3), so substituting the unkown function $g(t)$ under an integral sign from the relation (4) in (3), we get

$$g(x) = f(x) + \lambda \int_a^b K(x,t) \left\{ f(t) + \lambda \int_a^b K(t,t_1) \, g(t_1) dt_1 \right\} dt$$

or $$g(x) = f(x) + \lambda \int_a^b K(x,t) \, f(t) dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1) \, g(t_1) dt_1 \, dt \qquad ...(5)$$

Rewriting (4), we have

$$g(t) = f(t) + \lambda \int_a^b K(t,t_2) \, g(t_2) dt_2$$

or $$g(t_1) = f(t_1) + \lambda \int_a^b K(t_1,t_2) \, g(t_2) dt_2$$

Substituting the above value of $g(t)$ in (5), we get

$$g(x) = f(x) + \lambda \int_a^b K(x,t) f(t) dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1)$$

$$\left\{ f(t_1) + \lambda \int_a^b K(t_1, t_2)\, g(t_2)\, dt_2 \right\} dt_1\, dt$$

or $$g(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\, dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1) f(t)\, dt_1\, dt$$

$$+\lambda^3 \int_a^b K(x,t) \int_a^b K(t,t_1) \int_a^b K(t_1,t_2)\, g(t_2)\, dt_2\, dt_1\, dt$$

Proceeding likewise, we have

$$g(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\, dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1) f(t_1)\, dt_1\, dt$$

$$+\lambda^3 \int_a^b K(x,t) \int_a^b K(t,t_1) \int_a^b K(t_1,t_2)\, g(t_2)\, dt_2\, dt_1\, dt$$

$$+ \ldots\ldots + \lambda^n \int_a^b K(x,t) \int_a^b K(t,t_1) \ldots \int_a^b K(t_{n-2}, t_{n-1}) f(t_{n-1})\, dt_{n-1} \ldots dt_1\, dt$$

$$+ R_{n+1}(x) \qquad \ldots(6)$$

where $$R_{n+1}(x) = \lambda^{n+1} \int_a^b K(x,t) \int_a^b K(t,t_1) \ldots \int_a^b K(t_{n-1}, t_n)\, g(t_n)\, dt_{n-1} \ldots dt_1\, dt \qquad \ldots(7)$$

Now, let us consider the following infinite series

$$g(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\, dt + \lambda^2 \int_a^b K(x,t) \int_a^b K(t,t_1) f(t_1)\, dt_1\, dt + \ldots \qquad \ldots(8)$$

In view of the assumpition (i) and (ii), each term of the series (8) is continous in I. It follows that the series (8) is also continous in I, provided it converges uniformly in I.

Let $S_n(x)$ denote the general term of the series (8) i.e., let

$$S_n(x) = \lambda^n \int_a^b K(x,t) \int_a^b K(t,t_1) \ldots \int_a^b K(t_{n-2}, t_{n-1}) f(t_{n-1})\, dt_{n-1} \ldots dt_1\, dt$$

Since $|K(x,t)| \le P$ and $|f(x)| \le Q$,

therefore $|S_n(x)| \le |\lambda^n|\, Q P^n (b-a)^n$

which will be convergent if $|\lambda|\, P(b-a) < 1$

or $$|\lambda| < \frac{1}{P(b-a)} \qquad \ldots(9)$$

which hold in view of assumption (iii).

Thus the series (8) converges absolutely and uniforly when the relation (9) is satisfied.

If (3) has a continous solution, clearly it must be expressed by (8). If $g(x)$ is continuous in I, $|g(x)|$ must have a maximum value M. Thus,

$$|g(x)| \le M \qquad ...(10)$$

Now from (7), we have

$$|R_{n+1}(x)| \le |\lambda|^{n+1} MP^{n+1}(b-a)^{n+1} \qquad \text{[using assumption (ii) and (10)]}$$

Since (9) holds, so

$$\lim_{n\to\infty} R_{n+1}(x) = 0$$

It follows that the function $g(x)$ satisfying (6) is the continuous function given by the series (8).

10.5 Solution of Volterra Integral Equation of the Second Kind by Successive Substitutions

Theorem 2 : Let

$$\boldsymbol{g(x) = f(x) + \lambda \int_a^x K(x,t)\, g(t)\, dt} \qquad \textbf{...(11)}$$

be given Volterra integral equation of the second kind. Suppose that conditions (i) and (ii) given with Theorem 1 are satisfied and λ is a non zero numrical parameter.

Then (11) has a unique continuous solution in I and this solution is given by the absolutely and uniformly convergent series.

$$\boldsymbol{g(x) = f(x) + \lambda \int_a^x K(x,t)\, f(t)\, dt + \lambda^2 \int_a^x K(x,t) \int_a^t K(t,t_1)\, f(t_1)\, dt_1\, dt + \dots}$$

Proof : Rewriting (11), we have

$$g(x) = f(x) + \lambda \int_a^x K(x,t)\, g(t_1)\, dt_1$$

or $$g(t) = f(t) + \lambda \int_a^t K(t,t_1)\, g(t_1)\, dt_1 \qquad ...(12)$$

Since there exists a continous solution $g(x)$ of (11), so substituting the unknown function under an integral sign from the equation (12) in (11), we obtain

$$g(x) = f(x) + \lambda \int_a^x K(x,t) \left[f(t) + \lambda \int_a^t K(t,t_1)\, g(t_1)\, dt_1 \right] dt$$

or $$g(x) = f(x) + \lambda \int_a^x K(x,t) f(t)\, dt + \lambda^2 \int_a^x K(x,t) \int_a^t K(t,t_1)\, g(t_1)\, dt_1\, dt \qquad ...(13)$$

Rewriting (12), we have

$$g(t) = f(t) + \lambda \int_a^t K(t,t_2)\, g(t_2)\, dt_2$$

or $$g(t_1)=f(t_1)+\lambda\int_a^t K(t_1,t_2)\, g(t_2)dt_2$$

Substituting this value of $g(t_1)$ in (13), we find that

$$g(x)=f(x)+\lambda\int_a^x K(x,t)\, f(t)dt+\lambda^2\int_a^x K(t,t_1)$$

$$\left[f(t)+\lambda\int_a^{t_1} K(t_1,t_2)\, g(t_2)dt_2\right]dt_1\, dt$$

or $$g(x)=f(x)+\lambda\int_a^x K(x,t)\, f(t)dt+\lambda^2\int_a^x K(x,t)\int_a^t K(t,t_1)\, f(t_1)dt_1\, dt$$

$$+\lambda^3\int_a^x K(x,t)\int_a^t K(t,t_1)\int_a^{t_1} K(t_1,t_2)\, g(t_2)dt_2\, dt_1\, dt$$

Proceeding like wise, we have

$$g(x)=f(x)+\lambda\int_a^x K(x,t)\, f(t)dt+\lambda^2\int_a^x K(x,t)\int_a^t K(t,t_1)\, f(t_1)dt_1\, dt$$

$$+...+\lambda^n\int_a^x K(x,t)\int_a^t K(t,t_1)...\int_a^{t_{n-2}} K(t_{n-2},t_{n-1})\, f(t_{n-1})dt_{n-1}...dt_1\, dt+R_{n+1}(x) \quad ...(14)$$

where

$$R_{n+1}(x)=\lambda^{n+1}\int_a^x K(x,t)\int_a^t K(t,t_1)...\int_a^{t_{n-1}} K(t_{n-1},t_n)\, g(t_n)dt_n...dt_1\, dt \quad ...(15)$$

Now, let us consider the following infinite series

$$g(x)=f(x)+\lambda\int_a^x K(x,t)\, f(t)dt+\lambda^2\int_a^x K(x,t)\int_a^t K(t,t_1)\, f(t_1)dt_1\, dt+... \quad ...(16)$$

In view of the assumptions each term of the series (16) is continuous in I. It follows that the series (16) is also continuous in I, provided it converges unifrormly in I.

Let $S_n(x)$ denote the general term of the series (16) i.e.

$$S_n(x)=\lambda^n\int_a^x K(x,t)\int_a^t K(t,t_1)...\int_a^{t_{n-2}} K(t_{n-2},t_{n-1})\, f(t_{n-1})dt_{n-1}...dt_1\, dt$$

$$|S_n(x)|=\left|\lambda^n\int_a^x K(x,t)\int_a^t K(t,t_1)...\int_a^{t_{n-2}} K(t_{n-2},t_{n-1})\, f(t_{n-1})dt_{n-1}...dt\right|$$

Since $|K(x,t)|\le P$ and $|f(x)|\le Q$, we have

$$|S_n(x)|\le|\lambda|^n Q\, P^n\frac{(b-a)^n}{n!}$$

or $$|S_n(x)|\le|\lambda|^n Q\frac{[P(b-a)]^n}{n!},\ a\le x\le b$$

Clearly, the series, for which the positive constant

$$\frac{|\lambda|^n Q[P(b-a)]^n}{n!}$$ is the general expression for the n^{th} term,

is convergent for all values of λ, $P, Q, (b-a)$. It follows that the series (16) is absolutely and uniformly convergent.

If (11) has a continuous solution, clearly it must be expressed by (14). If $g(x)$ is continuous in I, $|g(x)|$ must have a maximum value M. Thus $|g(x)| \leq M$.

Now from (15), we have

$$|R_{n+1}(x)| = \left|\lambda^{n+1}\int_a^x K(x,t) \int_a^t K(t,t_1)\ldots\int_a^{t_{n-1}} K(t_{n-1},t_n)\, g(t_n)\, dt_n \ldots dt_1\, dt\right|$$

$$\therefore \qquad |R_{n+1}(x)| \leq |\lambda|^{n+1} MP^{n+1}\frac{(x-a)^{n+1}}{(n+1)!}$$

$$\leq |\lambda|^{n+1} MP^{n+1}\frac{(b-a)^{n+1}}{(n+1)!} \qquad (a \leq x \leq b)$$

Hence $\lim\limits_{n\to\infty} R_{n+1}(x) = 0$

It follows that the function $g(x)$ satisfying (14) is the continuous function given by the series (16).

10.6 Solution of Fredholm Integral Equation of Second Kind by Successive Approximation. Iterative Method (Iterative Scheme), Neumann's Series

Consider the Fredholm integral equation of second kind as

$$g(x) = f(x) + \lambda\int_a^b K(x,t)\, g(t)\, dt \qquad \ldots(17)$$

where (i) the kernel $K(x,t) \neq 0$ is real and continuous in the rectangle R for which $a \leq x \leq b$ and $a \leq t \leq b$.

(ii) $f(x) \neq 0$ is real and continuous in an interval I, for which $a \leq x \leq b$

(iii) λ is a non zero numerical parameter.

As a zero order approximation (for the starting approximation) to the desired function $g(x)$, let us take the solution $g_0(x)$, i.e.

$$g_0(x) = f(x) \qquad \ldots(18)$$

The solution $g_0(x)$ is substituted into the RHS of (17) to get the first order approximation $g_1(x)$

i.e. $$g_1(x) = f(x) + \lambda \int_a^b K(x,t)\, g_0(t)\, dt \qquad ...(19)$$

This function, when substituted into (17), yields the second approximation. This process is then repeated; the $(n+1)^{th}$ approximation is obtained by substituting the n^{th} approximation in the right side of (17) which results in the recurrence relation

$$g_{n+1}(x) = f(x) + \lambda \int_a^b K(x,t)\, g_n(t)\, dt \qquad ...(20)$$

We know that the iterated kernels (or iterated functions) $K_n(x,t)(n = 1,2,3,......)$ are defined by

$$K_1(x,t) = K(x,t) \qquad ...(21)$$

and $$K_n(x,t) = \int_a^b K(x,z)\, K_{n-1}(z,t)\, dz \qquad ...(22)$$

Using (18) in (19), we obtain

$$g_1(x) = f(x) + \lambda \int_a^b K(x,t)\, f(t)\, dt \qquad ...(23)$$

Putting $n = 1$ in (20), the second order approximation $g_2(x)$ is given by

$$g_2(x) = f(x) + \lambda \int_a^b K(x,z)\, g_1(z)\, dz \qquad ...(24)$$

Using (23) in (24), we obtain

$$g_2(x) = f(x) + \lambda \int_a^b K(x,z) \left[f(z) + \lambda \int_a^b K(z,t)\, f(t)\, dt \right] dz$$

or $$g_2(x) = f(x) + \lambda \int_a^b K(x,z)\, f(z)\, dz + \lambda^2 \int_a^b K(x,z) \left[\int_a^b K(z,t)\, f(t)\, dt \right] dz \qquad ...(25)$$

or $$g_2(x) = f(x) + \lambda \int_a^b K(x,t)\, f(t)\, dt + \lambda^2 \int_a^b f(t) \left[\int_a^b K(x,z)\, K(z,t)\, dz \right] dt$$

[on changing the order of integration in third term]

or $$g_2(x) = f(x) + \lambda \int_a^b K_1(x,t_1)\, f(t)\, dt + \lambda^2 \int_a^b K_2(x,t) f(t)\, dt$$ (using (21) and (22)]

or $$g_2(x) = f(x) + \sum_{m=1}^{2} \lambda^m \int_a^b K_m(x,t)\, f(t)\, dt \qquad ...(26)$$

Proceeding likewise, we easily obtain by mathematical induction the n^{th} approximate solution $g_n(x)$ of (17) as

$$g_n(x) = f(x) + \sum_{m=1}^{n} \lambda^m \int_a^b K_m(x,t)\, f(t)\, dt \qquad ...(27)$$

Taking the limit as $n \to \infty$, we obtain the so called Neumann series

$$g(x) = \underset{n\to\infty}{Lim}\, g_n(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^b K_m(x,t)\, f(t)\, dt \qquad \text{...(28)}$$

In order to determine the resolvent kernel (or reciprocal kernel) $R(x,t;\lambda)$ or $\Gamma(x,t;\lambda)$ in terms of the iterated kernel $K_m(x,t)$, changing the order of integration and summation in the so called Neumann series (28), we find that

$$g(x) = f(x) + \lambda \int_a^b \left[\sum_{m=1}^{\infty} \lambda^{m-1}\ K_m(x,t) \right] f(t)\, dt \qquad \text{...(29)}$$

Comparing (29) with

$$g(x) = f(x) + \lambda \int_a^b R(x,t;\lambda)\, f(t)\, dt$$

We obtain

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) \ \text{or} \ \sum_{m=0}^{\infty} \lambda^m K_{m+1}(x,t) \qquad \text{...(30)}$$

Obviously

$$g(x) = f(x) + \lambda \int_a^b R(x,t;\lambda)\, f(t)\, dt \qquad \text{...(31)}$$

will be solution of integral equation (17) in terms of the resolvent kernel.

Determination of the condition of convergence of (28)

Consider the partial sum (27) and apply the Schwarz inequality to the general term of this sum, we obtain

$$\left| \int_a^b K_m(x,t)\, f(t)\, dt \right|^2 \le \left(\int_a^b \left| K_m(x,t) \right|^2 dt \right) \int_a^b \left| f(t) \right|^2 dt \qquad \text{...(32)}$$

Let D be the norm of f. Then

$$D^2 = \int_a^b \left| f(t) \right|^2 dt \qquad \text{...(33)}$$

Further, let C_m^2 denote the upper bound of the integral

$$\int_a^b \left| K_m(x,t) \right|^2 dt, \ \text{so that} \ \int_a^b \left| K_m(x,t) \right|^2 dt \le C_m^2$$

Schwarz Inequality : If $\phi(x)$ and $\psi(x)$ and L_2 functions then

$$\left| (\phi, \psi) \right| \le ||\, \phi \,||\ ||\, \psi \,||$$

Hence the equation (32) becomes

$$\left|\int_a^b K_m(x,t) f(t)\,dt\right|^2 \le C_m^2 D^2 \qquad ...(34)$$

We now connect the estimate C_m^2 with the estimate C_1^2. For this purpose by applying the Schwarz inequality to the relation (22), we obtain

$$|K_m(x,t)|^2 \le \int_a^b |K_{m-1}(x,z)|^2\,dz \int_a^b |K(z,t)|^2\,dz$$

which when integrated with respect ot t, yields

$$\int_a^b |K_m(x,t)|^2\,dt \le B^2 C_{m-1}^2 \qquad ...(35)$$

where $B^2 = \int_a^b \int_a^b |K(x,t)|^2\,dx\,dt$ ...(36)

The inequality (35) gives rise to the recurrence relation

$$C_m^2 \le B^{2m-2} C_1^2 \qquad ...(37)$$

From (34) and (37), we have the inequality

$$\left|\int_a^b K_n(x,t) f(t)\,dt\right|^2 \le C_1^2 D^2 B^{2m-2} \qquad ...(38)$$

Thus, the general term of the partial sum (27) has a magnitude less than the quantity $D\,C_1|\lambda|^m B^{m-1}$, and it follows that the infinite series (28) converges faster than the geometric series with common ratio $|\lambda| B < 1$,

or $$|\lambda| < \frac{1}{\left[\int_a^b \int_a^b |K(x,t)|^2\,dx\,dt\right]^{1/2}} \qquad ...(39)$$

is satisfied, then the series (28) will be uniformly convergent.

Uniqueness of solution for a give λ

If possible, let $g_1(x)$ and $g_2(x)$ be two solutions of equation (17). Then we have

$$g_1(x) = f(x) + \lambda \int_a^b K(x,t)\, g_1(t)\,dt \qquad ...(40)$$

and $$g_2(x) = f(x) + \lambda \int_a^b K(x,t)\, g_2(t)\,dt \qquad ...(41)$$

Substracting (41) from (40) and setting $g_1(x) - g_2(x) = \phi(x)$, we have

$$g_1(x) - g_2(x) = \lambda \int_a^b K(x,t)\left[g_1(t) - g_2(t)\right] dt$$

or $$\phi(x)=\lambda\int_a^b K(x,t)\phi(t)\,dt \qquad ...(42)$$

which is a homogeneous integral equation.

Applying the Schwarz inequality to equation (42), we have

$$|\phi(x)|^2 \le |\lambda|^2 \int_a^b |K(x,t)|^2 dt \int_a^b |\phi(t)|^2 dt \qquad ...(43)$$

Intergrating (43) $w.r.t.,\ x$, we obtain

$$\int_a^b |\phi(x)|^2 dx \le |\lambda|^2 \int_a^b \int_a^b |K(x,t)|^2 dx\, dt \int_a^b |\phi(x)|^2 dx$$

or $$\int_a^b |\phi(x)|^2 dx \le |\lambda|^2 B^2 \int_a^b |\phi(x)|^2 dx \qquad \text{[by (36)]}$$

or $$\left(1-|\lambda|^2 B^2\right) \int_a^b |\phi(x)|^2 dx \le 0 \qquad ...(44)$$

In view of the inequality (39), $|\lambda|B<1$, we have

$$\phi(x)=0 \text{ or } g_1(x)-g_2(x)=0 \text{ or } g_1(x)=g_2(x)$$

Thus (27) has a unique solution.

Example 1 : Find the resolvent kernels of the following kernels

(i) $K(x,t)=(1+x)(1-t),\qquad a=-1,\ b=0$

(ii) $K(x,t)=e^{x+t},\qquad a=0,\quad b=1$

Solution : (i) By definition of iterated kernels, we have

$$K_1(x,t)=K(x,t)=(1+x)(1-t)$$

and $$K_n(x,t)=\int_{-1}^0 K(x,z)K_{n-1}(z,t)\,dz,\ n=2,3,\ldots \qquad ...(45)$$

Putting $n=2$ in (45), we have

$$K_2(x,t)=\int_{-1}^0 K(x,z)K_1(z,t)\,dz=\int_{-1}^0 (1+x)(1-z)(1+z)(1-t)\,dz$$

or $$K_2(x,t)=\frac{2}{3}(1+x)(1-t)$$

Now putting $n=3$ in (45), we have

$$K_3(x,t)=\int_{-1}^0 K(x,z)K_2(z,t)\,dz=\int_{-1}^0 (1+x)(1-z)\frac{2}{3}(1+z)(1-t)\,dz$$

or $$K_3(x,t)=\frac{2}{3}K_2(x,t)=\left(\frac{2}{3}\right)^2(1+x)(1-t)$$

an so on, Thus in general, we have

$$K_m(x,t) = \left(\frac{2}{3}\right)^{m-1} (1+x)(1-t)$$

Now, the required resolvent kernel is given by

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) = \sum_{m=1}^{\infty} \lambda^{m-1} \left(\frac{2}{3}\right)^{m-1} (1+x)(1-t)$$

$$= (1+x)(1-t) \sum_{m=1}^{\infty} \left(\frac{2\lambda}{3}\right)^{m-1} \qquad ...(46)$$

But $\displaystyle\sum_{m=1}^{\infty} \left(\frac{2\lambda}{3}\right)^{m-1} = 1 + \frac{2\lambda}{3} + \left(\frac{2\lambda}{3}\right)^2 + \left(\frac{2\lambda}{3}\right)^3 + \ldots$

which is an infinite geometric series with common ratio $\left(\frac{2\lambda}{3}\right)$

$$\therefore \quad \sum_{m=1}^{\infty} \left(\frac{2\lambda}{3}\right)^{m-1} = \frac{1}{1-(2\lambda/3)} = \frac{3}{3-2\lambda} \qquad ...(47)$$

provided that $\left|\frac{2\lambda}{3}\right| < 1$ or $|\lambda| < \frac{3}{2}$ ...(48)

Using (47) and (48), (46) reduces to

$$R(x,t;\lambda) = \frac{3(1+x)(1-t)}{3-2\lambda} \text{ where } |\lambda| < \frac{3}{2}$$

(ii) By definition of iterated kernels, we have

$$K_1(x,t) = K(x,t) = e^{x+t}$$

Putting $n = 2$ in (45), we have

$$K_2(x,t) = \int_0^1 K(x,z) K_1(z,t) dz$$

$$= e^{x+t} \int_0^1 e^{2z} dz = e^{x+t} \left(\frac{e^2-1}{2}\right)$$

Putting $n = 3$ in (45), we have

$$K_3(x,t) = \int_0^1 K(x,z) K_2(z,t) dz$$

$$= e^{x+t} \left(\frac{e^2-1}{2}\right) \int_0^1 e^{2z} dz = e^{x+t} \left(\frac{e^2-1}{2}\right)^2 \text{ and so on.}$$

Thus, in general, we have

$$K_m(x,t) = e^{x+t}\left(\frac{e^2-1}{2}\right)^{m-1}, \qquad m = 1,2,3,\ldots.$$

Now, the required resolvent kernel is given by

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) = \sum_{m=1}^{\infty} \lambda^{m-1} e^{x+t}\left(\frac{e^2-1}{2}\right)^{m-1}$$

$$= e^{x+t} \sum_{m=1}^{\infty} \left[\frac{\lambda(e^2-1)}{2}\right]^{m-1} \qquad \text{...(48)}$$

The involved series is an infinite geometric series with common ratio $\left\{\lambda\left(e^2-1\right)\right\}/2$

$$\therefore \qquad \sum_{m=1}^{\infty} \left[\frac{\lambda(e^2-1)}{2}\right]^{m-1} = \frac{1}{1-\left[\lambda(e^2-1)/2\right]} = \frac{2}{2-\lambda(e^2-1)} \qquad \text{...(49)}$$

provided that $\left|\frac{\lambda(e^2-1)}{2}\right| < 1$ or $|\lambda| < \frac{2}{e^2-1}$

Using (49), (48) reduces to

$$R(x,t;\lambda) = \frac{2\,e^{x+t}}{2-\lambda(e^2-1)} \qquad \text{where} \quad |\lambda| < \frac{2}{e^2-1}$$

Example 2 : Solve the following integral equation by the method of successive approximations :

$$g(x) = \left(e^x - \frac{1}{2}e + \frac{1}{2}\right) + \frac{1}{2}\int_0^1 g(t)\,dt$$

solution : Given $g(x) = \left(e^x - \frac{1}{2}e + \frac{1}{2}\right) + \frac{1}{2}\int_0^1 g(t)\,dt$ ...(50)

Here $f(x) = e^x - \frac{1}{2}e + \frac{1}{2}$, $\lambda = \frac{1}{2}$, $K(x,t) = 1$, we have

$$K_1(x,t) = K(x,t) = 1$$

Putting $n = 2$ in (45), we have

$$K_2(x,t) = \int_0^1 K(x,z)\,K_1(z,t)\,dz = \int_0^1 dz = 1$$

Next, putting $n = 3$ in (45), we have

$$K_3(x,t) = \int_0^1 K(x,z) K_2(z,t)\,dz = \int_0^1 dz = 1$$

and so on.

Thus in general, we have

$$K_m(x,t) = 1,\ m = 1,2,3,.......$$

Now, the resolvent kernel $R(x,t;\lambda)$ is given by

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) = \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{m-1}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + ... = \frac{1}{1-\left(\frac{1}{2}\right)} = 2$$

$\therefore\ R(x,t;\lambda) = 2$

Finally, The required solution of (50) is given by

$$g(x) = f(x) + \lambda \int_0^1 R(x,t;\lambda) f(t)\,dt$$

$$= \left(e^x - \frac{1}{2}e + \frac{1}{2}\right) + \frac{1}{2}\int_0^1 2\left(e^t - \frac{1}{2}e + \frac{1}{2}\right) dt$$

$$= e^x - \frac{1}{2}e + \frac{1}{2} + \left[e^t - \frac{1}{2}et + \frac{1}{2}t\right]_0^1$$

$$= e^x - \frac{1}{2}e + \frac{1}{2} + e - \frac{1}{2}e + \frac{1}{2} - 1$$

or $\quad g(x) = e^x$

Example 3 : Using iterative method, solve

$$g(x) = f(x) + \lambda \int_0^1 e^{x-t} g(t)\,dt$$

Solution : Given $g(x) = f(x) + \lambda \int_0^1 e^{x-t} g(t)\,dt$

Here $\quad K(x,t) = e^{x-t}$

Proceeding as in Example 1 (ii), we find that

$$K_1(x,t) = K(x,t) = e^{x-t} = K_2(x,t) = K_3(x,t) = ...$$

Thus, in general, we have

$$K_m(x,t) = e^{x-t}\ ,\ m = 1,2,3,\dots.$$

Now the resolvent kernel $R(x,t;\lambda)$ is given by

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) = \sum_{m=1}^{\infty} \lambda^{m-1} e^{x-t}$$

$$= e^{x-t}\left(1+\lambda+\lambda^2+\dots..\right)$$

$$\therefore \qquad R(x,t;\lambda) = e^{x-t}\frac{1}{1-\lambda},\ \text{provided that } |\lambda|<1$$

Thus, the required solution is given by

$$g(x) = f(x) + \lambda\int_0^1 R(x,t;\lambda)\, f(t)\, dt$$

or $$g(x) = f(x) + \frac{\lambda}{1-\lambda}\int_0^1 e^{x-t}\, f(t)\, dt, \qquad \text{provided that } |\lambda|<1$$

Example 4 : Solve by the method of successive approximation

$$g(x) = \frac{3}{2}e^x - \frac{1}{2}x\,e^x - \frac{1}{2} + \frac{1}{2}\int_0^1 t\, g(t)\, dt$$

Solution : Given $g(x) = \frac{3}{2}e^x - \frac{1}{2}x\,e^x - \frac{1}{2} + \frac{1}{2}\int_0^1 t\, g(t)\, dt$

Here $f(x) = \frac{3}{2}e^x - \frac{1}{2}x\,e^x - \frac{1}{2},\ \lambda = \frac{1}{2},\ K(x,t) = t$

Iterated kernels $K_m(x,t)$ are given by

$$K_1(x,t) = K(x,t) = t$$

$$K_2(x,t) = \int_0^1 K(x,z)\ K_1(z,t)\ dz = \int_0^1 z\, t\, dz = \frac{1}{2}t,$$

$$K_3(x,t) = \int_0^1 K(x,z)\ K_2(z,t)\ dz = \int_0^1 z\left(\frac{1}{2}\,t\right)dz = \left(\frac{1}{2}\right)^2 t$$

and so on.

Thus in general, we have

$$K_m(x,t) = \left(\frac{1}{2}\right)^{m-1} t$$

Now, the resolvent kernel $R(x,t;\lambda)$ is given by

$$R(x,t;\lambda)=\sum_{m=1}^{\infty}\lambda^{m-1}K_m(x,t)=\sum_{m=1}^{\infty}\left(\frac{1}{2}\right)^{m-1}t$$

$$=t\sum_{m=1}^{\infty}\left(\frac{1}{4}\right)^{m-1}=t\left[1+\frac{1}{4}+\left(\frac{1}{4}\right)^2+....\right]$$

$$=t\,\frac{1}{1-(1/4)}=\frac{4t}{3}$$

Thus, the required solution of the integral equation is given by

$$g(x)=f(x)+\lambda\int_0^1 R(x,t;\lambda)f(t)\,dt$$

$$=\left(\frac{3}{2}e^x-\frac{1}{2}xe^x-\frac{1}{2}\right)+\frac{2}{3}\int_0^1\left(\frac{3}{2}te^t-\frac{1}{2}t^2e^t-\frac{1}{2}t\right)dt$$

$$=\left(\frac{3}{2}e^x-\frac{1}{2}xe^x-\frac{1}{2}\right)+\frac{2}{3}\left[\frac{3}{2}\left(te^t-e^t\right)-\frac{1}{2}\left(t^2e^t-2te^t+2e^t\right)-\frac{t^2}{4}\right]_0^1$$

or $$g(x)=\frac{3}{2}e^x-\frac{1}{2}xe^x-\frac{1}{3}e+1 \qquad \text{(on simplification)}$$

Example 5 : By iterative method, solve

$$g(x)=1+\lambda\int_0^{\pi}\sin(x+t)\,g(t)\,dt$$

Solution : Given $g(x)=1+\lambda\int_0^{\pi}\sin(x+t)\,g(t)\,dt$

Here $f(x)=1,\ K(x,t)=\sin(x+t)$

Now $K_1(x,t)=K(x,t)=\sin(x+t)$,

$$K_2(x,t)=\int_0^{\pi}K(x,z)K_1(z,t)\,dz=\int_0^{\pi}\sin(x+z)\sin(z+t)\,dz$$

$$=\frac{1}{2}\int_0^{\pi}\left[\cos(x-t)-\cos(2z+x+t)\right]dz$$

$$=\frac{1}{2}\left[z\cos(x-t)-\frac{1}{2}\sin(2z+x+t)\right]_0^{\pi}$$

or $$K_2(x,t)=\frac{\pi}{2}\cos(x-t) \quad \text{(on simplification)}$$

Similarly, we have

$$K_3(x,t) = \int_0^\pi K(x,z)\ K_2(z,t)\,dz = \int_0^\pi \sin(x+z).\frac{\pi}{2}\cos(z-t)\,dz$$

$$= \frac{\pi}{4}\int_0^\pi \left[\sin(2z+x-t)+\sin(x+t)\right] dz$$

$$= \frac{\pi}{4}\left[-\frac{1}{2}\cos(2z+x-t)+z\sin(x+t)\right]_0^\pi$$

or $\quad K_3(x,t) = \left(\frac{\pi}{2}\right)^2 \sin(x+t)$, (on simplification),

$$K_4(x,t) = \left(\frac{\pi}{2}\right)^3 \cos(x-t)$$

or $\quad K_5(x,t) = \left(\frac{\pi}{2}\right)^4 \sin(x+t)$

Thus, we find that all odd iterated kernels invlove $\sin(x+t)$ and all even iterated kernels involve $\cos(x-t)$.

Now, the resolvent kernel $R(x,t;\lambda)$ is given by

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$$

$$= K_1(x,t) + \lambda\, K_2(x,t) + \lambda^2 K_3(x,t) + \lambda^3 K_4(x,t) + \ldots.$$

$$= K_1(x,t) + \lambda\, K_3(x,t) + \lambda^4 K_5(x,t)$$

$$+ \ldots. + \lambda\left[K_2(x,t) + \lambda^2 K_4(x,t) + \lambda^4 K_5(x,t) + \ldots\right]$$

$$= \sin(x+t)\left[1+\left(\frac{\lambda\pi}{2}\right)^2+\left(\frac{\lambda\pi}{2}\right)^4+\ldots..\right]$$

$$+\frac{\lambda\pi}{2}\cos(x-t)\left[1+\left(\frac{\lambda\pi}{2}\right)^2+\left(\frac{\lambda\pi}{2}\right)^4+\ldots..\right]$$

$$= \left[\sin(x+t)+\frac{\lambda\pi}{2}\cos(x-t)\right]\left[1+\left(\frac{\lambda\pi}{2}\right)^2+\left(\frac{\lambda\pi}{2}\right)^4+\ldots..\right]$$

$$= \frac{2\sin(x+t)+\lambda\,\pi\cos(x-t)}{2}\cdot\frac{1}{1-\left(\frac{\lambda\pi}{2}\right)^2}$$

provided that $\left|\frac{\lambda\pi}{2}\right|<1$ or $|\lambda|<\frac{2}{\pi}$

or $$R(x,t;\lambda)=\frac{2}{4-\lambda^2\pi^2}\left[2\sin(x+t)+\lambda\,\pi\cos(x-t)\right]$$

Hence the solution of the given integral equation is given by

$$g(x)=f(x)+\lambda\int_0^{\pi}R(x,t;\lambda)\,f(t)\,dt$$

$$=1+\frac{2\lambda}{4-\lambda^2\pi^2}\int_0^{\pi}\{2\sin(x+t)+\lambda\,\pi\cos(x-t)\}\,dt$$

$$=1+\frac{2\lambda}{4-\lambda^2\pi^2}\left[-2\cos(x+t)-\lambda\,\pi\sin(x-t)\right]_0^{\pi}$$

or $$g(x)=1+\frac{4\lambda}{4-\lambda^2\pi^2}\left[2\cos x+\lambda\,\pi\sin x\right]\text{ where }|\lambda|<2/\pi.$$

Example 6 : Find the resolvent kernel of the following integral equation

$$g(x)=1+\lambda\int_0^1(1-3xt)\,g(t)\,dt$$

For what value of λ, the solution does not exist. Also find the solution of the above integral equation.

Solution : Given $g(x)=1+\lambda\int_0^1(1-3xt)\;g(t)\,dt$

Here $f(x)=1$, $K(x,t)=1-3xt$, $\lambda=1$ (say)

Now, we have

$$K_1(x,t)=K(x,t)=1-3xt,$$

$$K_2(x,t)=\int_0^1 K(x,z)\;K_1(z,t)\;dz=\int_0^1(1-3xz)\;(1-3zt)\;dz$$

$$=\int_0^1\left[1-3z(x+t)+9xtz^2\right]dz$$

or $$K_2(x,t)=1-\frac{3}{2}(x+t)+3xt,\text{ (on simplification),}$$

$$K_3(x,t)=\int_0^1 K(x,z)\,K_2(z,t)\;dz$$

$$= \int_0^1 (1-3xz)\left\{1-\frac{3}{2}(z+t)+3zt\right\} dz$$

$$= \int_0^1 \left[\left(1-\frac{3}{2}t\right)-3z\left(\frac{1}{2}-t+x-\frac{3}{2}xt\right)+9xz^2\left(\frac{1}{2}-t\right)\right] dz$$

$$= 1-\frac{3}{2}\,t-\frac{3}{2}\left(\frac{1}{2}-t+x-\frac{3}{2}xt\right)+3x\left(\frac{1}{2}-t\right)$$

$\therefore$ $$K_3(x,t)=\frac{1}{4}(1-3xt)=\frac{1}{4}K_1(x,t),$$

$$K_4(x,t)=\int_0^1 K(x,z)\,K_3(z,t)\,dz$$

$$= \frac{1}{4}\int_0^1 (1-3xz)(1-3zt)\,dz$$

$$= \frac{1}{4}\left[1-\frac{3}{2}(x+t)+3xt\right] \qquad \text{(as before)}$$

or $$K_4(x,t)=\frac{1}{4}K_2(x,t),$$

$$K_5(x,t)=\int_0^1 K(x,z)\,K_4(z,t)\,dz$$

$$= \frac{1}{4}\int_0^1 (1-3xz)\left[1-\frac{3}{2}(z+t)+3zt\right]dz$$

$$= \frac{1}{4}.\frac{1}{4}\,(1-3xt) \quad \text{(as before)}$$

$\therefore$ $$K_5(x,t)=\left(\frac{1}{4}\right)^2(1-3xt)=\left(\frac{1}{4}\right)^2 K_1(x,t)$$

By symmetry, we may write

$$K_6(x,t)=\left(\frac{1}{4}\right)^2 K_2(x,t), \qquad K_7(x,t)=\left(\frac{1}{4}\right)^3 K_1(x,t)$$

and so on.

Hence the resolvent kernel is given by

$$R(x,t;\lambda)=\sum_{m=1}^{\infty}\lambda^{m-1}K_m(x,t)$$

$$=K_1(x,t)+\lambda K_2(x,t)+\lambda^2K_3(x,t)+\lambda^3K_4(x,t)+.....$$

$$=K_1(x,t)+\lambda^2K_3(x,t)+\lambda^4K_5(x,t)+.....$$

$$+\lambda\left[K_2(x,t)+\lambda^2K_4(x,t)+\lambda^4K_6(x,t)+.....\right]$$

$$=K_1(x,t)+\frac{\lambda^2}{4}K_1(x,t)+\frac{\lambda^4}{4^2}K_1(x,t)+.....$$

$$+\lambda\left[K_2(x,t)+\frac{\lambda^2}{4}K_2(x,t)+\frac{\lambda^4}{4^2}K_2(x,t)+.....\right]$$

$$=K_1(x,t)\left[1+\frac{\lambda^2}{4}+\left(\frac{\lambda^2}{4}\right)^2+...\right]+\lambda K_2(x,t)\left[1+\frac{\lambda^2}{4}+\left(\frac{\lambda^2}{4}\right)^2+...\right]$$

$$=\left[K_1(x,t)+\lambda K_2(x,t)\right]\left[1+\frac{\lambda^2}{4}+\left(\frac{\lambda^2}{4}\right)^2+....\right]$$

$$=\left[(1-3xt)+\lambda\left\{1-\frac{3}{2}(x+t)+3xt\right\}\right]\frac{1}{1-(\lambda^2/4)},\lambda^2<4\,or\,|\lambda|<2$$

or $R(x,t;\lambda)=\dfrac{4}{4-\lambda^2}\left[1+\lambda-\dfrac{3}{2}x\lambda-3t\left(x+\dfrac{\lambda}{2}-x\lambda\right)\right]$, where $|\lambda|<2$

Hence the solution of the integral equation is given as

$$g(x)=f(x)+\lambda\int_0^1 R(x,t;\lambda)f(t)dt$$

$$=1+\frac{4\lambda}{4-\lambda^2}\int_0^1 1+\lambda-\frac{3\lambda}{2}x-3t\left(x+\frac{1}{2}\lambda-x\lambda\right)dt$$

$$=1+\frac{4\lambda}{4-\lambda^2}.\frac{4+\lambda-6x\lambda}{4},\text{ (on simplification)}$$

$$=\frac{4+2\lambda(2-3x\lambda)}{4-\lambda^2},\ |\lambda|<2$$

The solution exists only when $|\lambda|<2$.

10.7 Solution of Volterra Intgral Equation of the Second Kind by Successive Approximations. It erative Method, Neumann Series

Consider Volterra integral equation of the second kind

$$g(x)=f(x)+\lambda\int_a^x K(x,t)g(t)\,dt \qquad ...(51)$$

As a zero order approximation to the required solution $g(x)$ let us take $g_0(x)=f(x)$

Further, if $g_n(x)$ and $g_{n-1}(x)$ are the n^{th} and $(n-1)^{th}$ order approximation respectively, these are connected by

$$g_n(x)=f(x)+\lambda\int_a^x K(x,t)g_{n-1}(t)dt \qquad ...(52)$$

Iterated kernels $K_n(x,t)$ are given by

$$K_1(x,t)=K(x,t) \quad \text{and} \quad K_n(x,t)=\int_t^x K(x,z)K_{n-1}(z,t)dz \quad (n=2,3....)$$

Puitting $n=1$, in (52), the first order approximation $g_1(x)$ is given by

$$g_1(x)=f(x)+\lambda\int_a^x K(x,t)g_0(t)dt$$

or
$$g_1(x)=f(x)+\lambda\int_a^x K(x,t)f(t)dt \qquad ...(53)$$

Next putting n = 2 in (52), the second order approximation $g_2(x)$ in given by

$$g_2(x)=f(x)+\lambda\int_a^x K(x,t)g_1(t)dt$$

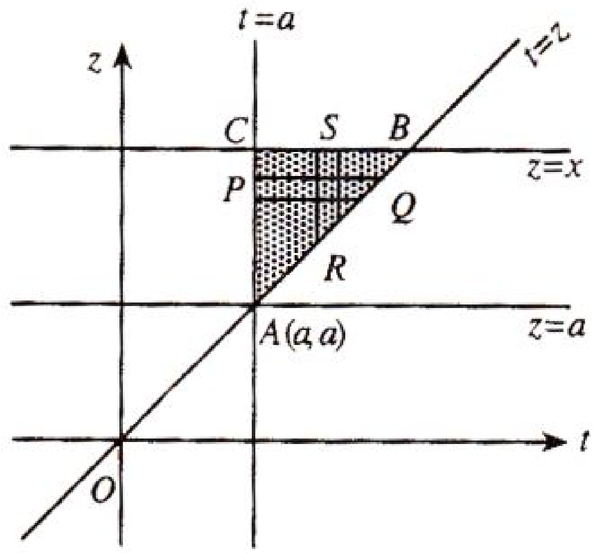
or
$$g_2(x)=f(x)+\lambda\int_a^x K(x,z)g_1(z)dz \qquad ...(54)$$

Substituting $g_1(z)$ from (53) in (54), we get

$$g_2(x)=f(x)+\lambda\int_a^x K(x,z)\left[f(z)+\lambda\int_a^z K(z,t)f(t)dt\right]dz$$

or
$$g_2(x)=f(x)+\lambda\int_a^x K(x,z)f(z)dz+\lambda^2\int_{z=a}^x K(x,z)\left[\int_{t=a}^z K(z,t)f(t,dt)\right]dz \qquad ...(55)$$

Now consider the double integral on the R.H.S. of (53). The limits of integration are given by $t=a$, $t=z$, $z=a$, $t=z$, $z=a$, $z=x$. Clearly the region of integration is the triangle ABC as shown in the Figure. Obviously, strips have been taken parallel to t–axis in this double integral (strip PQ). When we wish to change the order of integration in the above mentioned double integral, we shall take strips parallel to z-axis (strip RS). Then for the same region (triangle ABC), the limits of integration are given by $z=t$, $z=x$; $t=a$, $t=x$.



Figrue 10.1

Thus, we have

$$\int_{z=a}^{x} K(x,z)\left[\int_{t=a}^{z} K(z,t) f(t)\,dt\right]dz = \int_{t=a}^{x} f(t)\left[\int_{z=t}^{x} K(x,z)K(z,t)dz\right]dt \qquad ...(56)$$

Using (56) in (55), we obtain

$$g_2(x) = f(x) + \lambda\int_a^x K(x,z) f(z)dz + \lambda^2 \int_{t=a}^{x} f(t)\left[\int_{z=t}^{x} K(x,z)K(z,t)dz\right] dt$$

or $$g_2(x) = f(x) + \lambda\int_a^x K(x,t) f(t)dt + \lambda^2 \int_a^x f(t)\, K_2(x,t) f(t)dt$$

or $$g_2(x) = f(x) + \lambda\int_a^x K_1(x,t) f(t)dt + \lambda^2 \int_a^x K_2(x,t) f(t)dt$$

or $$g_2(x) = f(x) + \sum_{m=1}^{z} \lambda^m \int_a^x K_m(x,t) f(t)dt$$

Proceeding likewise, we easily obtain by mathematical induction the n^{th} approximate solution $g_n(x)$ of (59) as

$$g_n(x) = f(x) + \sum_{m=1}^{n} \lambda^m \int_a^x K_m(x,t) f(t)dt$$

Taking the limit as $n \to \infty$, we obtain the so called Neumann series

$$g(x) = \underset{n\to\infty}{Lim}\, g_n(x) = f(x) + \sum_{m=1}^{\infty} \lambda^m \int_a^x K_m(x,t)\, f(t)\, dt$$

In order to determine the resolvent kernel (or reciprocal kernel) $R(x,t;\lambda)$ or $\Gamma(x,t;\lambda)$ in terms of iterated kernels $K_m(x,t)$, changing the order of integration and summation, we find that

$$g(x) = f(x) + \lambda \int_a^x \left[\sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) \right] f(t)\, dt \qquad \text{...(57)}$$

Comparing (57) with $g(x) = f(x) + \lambda \int_a^x R(x,t;\lambda) f(t)\, dt$, we obtiain,

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$$

Example 7 : Find the resolvent kernel of the Volterra integral equation with the kernel

$$K(x,t) = \frac{(2+\cos x)}{(2+\cos t)}$$

Solution : Iterated kernels $K_n(x,t)$ are given by

$$K_1(x,t) = K(x,t)$$

and $\quad K_n(x,t) = \int_t^x K(x,z) K_{n-1}(z,t)\, dz, \qquad n = 2,3,......$

Given that $K(x,t) = K_1(x,t) = \dfrac{2+\cos x}{2+\cos t}$

Also, we have

$$K_2(x,t) = \int_t^x K(x,z) K_1(z,t)\, dz = \int_t^x \frac{2+\cos x}{2+\cos z} \cdot \frac{2+\cos z}{2+\cos t}\, dz$$

$$= \frac{2+\cos x}{2+\cos t} \int_t^x dz = \frac{2+\cos x}{2+\cos t}(x-t)$$

Similarly

$$K_3(x,t) = \int_t^x K(x,z) K_2(z,t)\, dz$$

$$= \int_t^x \frac{2+\cos x}{2+\cos z} \cdot \frac{2+\cos z}{2+\cos t}(z-t)\, dz$$

$$= \frac{2+\cos x}{2+\cos t} \int_t^x (z-t)\, dz = \frac{2+\cos x}{2+\cos t}\left[\frac{(z-t)^2}{2}\right]_t^x$$

$$= \frac{2+\cos x}{2+\cos t} \frac{(x-t)^2}{2!}$$

and so on.

Thus is general we have

$$K_n(x,t) = \frac{2+\cos x}{2+\cos t}\,\frac{(x-t)^{n-1}}{(n-1)!}, \qquad n = 1,2,3,\ldots$$

Now, the required resolvent kernel is given by

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$$

$$= K_1(x,t) + \lambda K_2(x,t) + \lambda^2 K_3(x,t) + \ldots\ldots$$

$$= \frac{2+\cos x}{2+\cos t} + \frac{2+\cos x}{2+\cos t}\cdot\frac{\lambda(x-t)}{1!} + \frac{2+\cos x}{2+\cos t}\left[\frac{\lambda(x-t)}{2!}\right]^2 + \ldots.$$

$$= \frac{2+\cos x}{2+\cos t} + \left[1 + \frac{\lambda(x-t)}{1!} + \frac{[\lambda(x-t)]^2}{2!} + \frac{[\lambda(x-t)]^3}{3!} + \ldots\ldots.\right]$$

$$= \frac{2+\cos x}{2+\cos t}\, e^{\lambda(x-t)}$$

Example 8 : By means of resolvent kernel, find the solution of

$$g(x) = e^x \sin x + \int_0^x \frac{2+\cos x}{2+\cos t}\, g(t)\, dt$$

Solution : Given that

$$g(x) = e^x \sin x + \int_0^x \frac{2+\cos x}{2+\cos t}\, g(t)\, dt$$

Comparing with

$$g(x) = f(x) + \lambda \int_0^x K(x,t)\, g(t)\, dt$$

we have

$$f(x) = e^x \sin x\,,\ \lambda = 1\,,\ K(x,t) = \frac{2+\cos x}{2+\cos t}$$

Proceed as in solved Ex. 7 and show that

$$R(x,t;\lambda) = \frac{2+\cos x}{2+\cos t}\, e^{x-t} \qquad [\because \lambda = 1]$$

Hence the required solution is given by

$$g(x) = f(x) + \lambda \int_0^x R(x,t;\lambda)\, f(t)\, dt$$

$$= e^x \sin x + \int_0^x \frac{2+\cos x}{2+\cos t}\, e^{x-t}\, e^t \sin t\, dt$$

$$= e^x \sin x - (2+\cos x)\, e^x \int_0^x \frac{-\sin t}{2+\cos t}\, dt$$

$$= e^x \sin x - e^x (2+\cos x)\left[\log(2+\cos t)\right]_0^x$$

$$= e^x \sin x - e^x (2+\cos x)\log\left(\frac{2+\cos x}{3}\right)$$

$$\therefore \qquad g(x) = e^x \sin x + e^x (2+\cos x)\log\left(\frac{3}{2+\cos x}\right)$$

Example 9 : With the aid of the resolvent kernel, find the solution of the integral equation

$$g(x) = e^{x^2+2x} + 2\int_0^x e^{x^2-t^2} g(t)\, dt$$

Solution : Given that

$$g(x) = e^{x^2+2x} + 2\int_0^x e^{x^2-t^2} g(t)\, dt$$

Thus we have

$$f(x) = x^2 + 2x,\ \lambda = 2,\ K(x,t) = e^{x^2-t^2}$$

Iterated kernels $K_m(x,t)$ are given by

$$K_1(x,t) = K(x,t) = e^{x^2-t^2}$$

and $\quad K_2(x,t) = \int_t^x K(x,z)\, K_1(z,t)\, dz = \int_t^x e^{x^2-z^2}\, e^{z^2-t^2}\, dz$

$$= e^{x^2-t^2}\int_t^x dz = e^{x^2-t^2}(x-t)$$

$$K_3(x,t) = \int_t^x K(x,z)\, K_2(z,t)\, dz = \int_t^x e^{x^2-z^2}\, e^{z^2-t^2}(z-t)\, dz$$

$$= e^{x^2-t^2}\int_t^x (z-t)\, dz = e^{x^2-t^2}\left[\frac{(z-t)^2}{2}\right]_t^x$$

$$= e^{x^2-t^2}\frac{(x-t)^2}{2!}$$

and so on.

Thus in general, we have

$$K_m(x,t) = e^{x^2-t^2}\frac{(x-t)^{m-1}}{(m-1)!},\ m = 1,2,3,\ldots\ldots$$

Now the resolvent kernel is given by

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} \; K_m(x,t)$$

$$= K_1(x,t) + \lambda \, K_2(x,t) + \lambda^2 K_3(x,t) + ...$$

$$= e^{x^2-t^2} + \lambda \, e^{x^2-t^2} \frac{(x-t)}{1!} + \lambda^2 \, e^{x^2-t^2} \frac{(x-t)^2}{2!} + ...$$

$$= e^{x^2-t^2} \left[1 + \lambda \; \frac{(x-t)}{1!} + \frac{\left[\lambda (x-t)^2\right]}{2!} + ... \right]$$

$$= e^{x^2-t^2} \; e^{\lambda(x-t)}$$

$\therefore \qquad R(x,t;\lambda) = e^{x^2-t^2} \; e^{2(x-t)} \qquad (\because \lambda = 2)$

Hence the required solution is given by

$$g(x) = f(x) + \lambda \int_0^x R(x,t;\lambda) f(t) dt$$

$$= f(x) + 2 \int_0^x e^{x^2-t^2} \; e^{2(x-t)} \, e^{t^2+2t} \; dt$$

$$= e^{x^2+2x} + 2 \, e^{x^2+2x} \int_0^x dt$$

$\therefore \qquad g(x) = e^{x^2+2x}(1+2x)$

Example 10 : Solve $g(x) = \cos x - x - 2 + \int_0^x (t-x) g(t) dt$

Solution : Given that

$$g(x) = \cos x - x - 2 + \int_0^x (t-x) g(t) dt$$

Thus, we have

$$f(x) = \cos x - x - 2 \,, \;\; \lambda = 1 \,, \;\; K(x,t) = t - x$$

Iterated kernels $K_m(x,t)$ are given by

$$K_1(x,t) = K(x,t) = t - x$$

and $\quad K_m(x,t) = \int_t^x K(x,z) K_{m-1}(z,t) \, dz \,, \qquad m = 2,3,....$

Now $\quad K_2(x,t) = \int_t^x K(x,z) K_1(z,t) \, dz = \int_t^x (z-x)(t-z) \, dz$

$$= \left[(t-z)\,\frac{(z-x)^2}{2} \right]_t^x - \int_t^x (-1)\,\frac{(z-x)^2}{2}\,dz$$

$$= \frac{1}{2}\int_t^x (z-x)^2\,dz = \frac{1}{2}\left[\frac{(z-x)^3}{3}\right]_t^x \qquad \text{(integrating by parts)}$$

$$\therefore \qquad K_2(x,t) = -\frac{(t-x)^3}{3!}$$

and $\quad K_3(x,t) = \int_t^x K(x,z)\,K_2(z,t)\,dz = \int_t^x (z-x)\left\{-\frac{(t-z)^3}{3!}\right\}dz$

$$= -\frac{1}{3!}\int_t^x (z-x)\,(t-z)^3\,dz = -\frac{1}{3!}\int_t^x \left\{\left[(z-x)\,\frac{(t-z)^4}{-4}\right]_t^x - \int_t^x 1.\frac{(t-z)^4}{(-4)}\,dz\right\}$$

(integrating by parts)

$$= -\frac{1}{4.3!}\int_t^x (t-z)^4\,dz = -\frac{1}{4.3!}\left[\frac{(t-z)^5}{(-5)}\right]_t^x = \frac{(t-x)^5}{5!}$$

and so on.

Thus in general, we have

$$K_m(x,t) = (-1)^{m-1}\,\frac{(t-x)^{2m-1}}{(2m-1)!},\ m = 1,2,3,\ldots.$$

Now, the required resolvent kernel is given by

$$R(x,t;\lambda) = \sum_{m=1}^{\infty}\lambda^{m-1}K_m(x,t) = \sum_{m=1}^{\infty}K_m(x,t) \qquad (\text{ as } \lambda = 1)$$

$$= K_1(x,t) + K_2(x,t) + K_3(x,t) + \ldots..$$

$$= \frac{(t-x)}{1!} - \frac{(t-x)^3}{3!} + \frac{(t-x)^5}{5!} - \ldots.$$

$$\therefore \qquad R(x,t;\lambda) = \sin(t-x)$$

Hence the required solution is given by

$$g(x) = f(x) + \lambda\int_0^x R(x,t;\lambda)\,f(t)\,dt$$

$$= \cos x - x - 2 + \int_0^x \sin(t-x)(\cos t - t - 2)\,dt$$

$$= \cos x - x - 2 + \int_0^x \sin(t-x)\cos t\,dt - \int_0^x t\sin(t-x)\,dt - 2\int_0^x \sin(t-x)\,dt$$

$$= \cos x - x - 2 + \frac{1}{2}\int_0^x [\sin(2t-x) - \sin x]\,dt$$

$$- \int_0^x t\sin(t-x)\,dt - 2\int_0^x \sin(t-x)\,dt$$

$$= \cos x - x - 2 + \frac{1}{2}\left[-\frac{\cos(2t-x)}{2} - t\sin x\right]_0^x - [-t\cos(t-x)]_0^x$$

$$+ \int_0^x [1.\{-\cos(t-x)\}\,dt] - 2[-\cos(t-x)]_0^x$$

$$= \cos x - x - 2 + \frac{1}{2}\left[-\frac{\cos x}{2} - x\sin x + \frac{\cos x}{2}\right]$$

$$- \left\{-x + [\sin(t-x)]_0^x\right\} + 2(1-\cos x)$$

$$= \cos x - x - 2 - \frac{1}{2}x\sin x + x - \sin x + 2 - 2\cos x$$

$$\therefore \qquad g(x) = -\cos x - \sin x - \frac{1}{2}x\sin x$$

Example 11 : By means of resolvent kernel, find the solution of

$$g(x) = 1 + x^2 + \int_0^x \frac{1+x^2}{1+t^2}\,g(t)\,dt$$

Solution : Given $g(x) = 1 + x^2 + \int_0^x \frac{1+x^2}{1+t^2}\,g(t)\,dt$

Now, we have

$$f(x) = 1 + x^2,\ \lambda = 1,\ K(x,t) = \frac{1+x^2}{1+t^2}$$

Iterated kernels $K_m(x,t)$ are given by

$$K_1(x,t) = K(x,t) = \frac{1+x^2}{1+t^2}$$

$$K_2(x,t) = \frac{1+x^2}{1+t^2}\,(x-t)$$

$$K_3(x,t) = \frac{1+x^2}{1+t^2}\,\frac{(x-t)^2}{2!}$$

and so on

Thus is general, we have

$$K_m(x,t) = \frac{1+x^2}{1+t^2}.\frac{(x-t)^{m-1}}{(m-1)!}, \quad m = 1,2,3,....$$

Now, the resolvent kernel $R(x,t;\lambda)$ is given by

$$R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t) = \sum_{m=1}^{\infty} K_m(x,t) \qquad (\because \lambda = 1)$$

$$= K_1(x,t) + K_2(x,t) + ...$$

$$= \frac{1+x^2}{1+t^2} + \frac{1+x^2}{1+t^2}(x-t) + \frac{1+x^2}{1+t^2}\,.\,\frac{(x-t)^2}{2!} + ...$$

$$= \frac{1+x^2}{1+t^2}\left[1 + \frac{(x-t)}{1!} + \frac{(x-t)^2}{2!} + ...\right] = \frac{1+x^2}{1+t^2}\,e^{x-t}$$

Finally, the required solution of our given integral equation is given by

$$g(x) = f(x) + \lambda \int_0^x R(x,t;\lambda)\, f(t)\, dt$$

$$= 1 + x^2 + e^x\left(1+x^2\right)\int_0^x e^{-t}\, dt = 1 + x^2 + e^x\left(1+x^2\right)\left[-e^{-t}\right]_0^x$$

$$\therefore \qquad g(x) = e^x\left(1+x^2\right)$$

Example 12 : Using the method of successive approximation, solve the integral equation

$$g(x) = 1 + \int_0^x (x-t)\, g(t)\, dt, \quad \text{taking } g_0(x) = 0$$

Solution : Given that

$$g(x) = 1 + \int_0^x (x-t)\, g(t)\, dt$$

and $\qquad g_0(x) = 0$

Thus, we have $f(x) = 1$, $\lambda = 1$, $K(x,t) = x - t$

The n^{th} order approximation $g_n(x)$ is given by

$$g_n(x) = f(x) + \lambda \int_0^x K(x,t) g_{n-1}(t)\, dt$$

or $$g_n(x) = 1 + \int_0^x K(x,t) g_{n-1}(t)\, dt$$

Now, $$g_1(x) = 1 + \int_0^x (x-t) g_0(t)\, dt = 1$$

$\therefore$ $$g_1(x) = 1$$

$$g_2(x) = 1 + \int_0^x (x-t) g_1(t)\, dt = 1 + \int_0^x (x-t) dt$$

$$= 1 + \left(xt - \frac{t^2}{2} \right)_0^x$$

$\therefore$ $$g_2(x) = 1 + \frac{x^2}{2} = 1 + \frac{x^2}{2!}$$

and $$g_3(x) = 1 + \int_0^x (x-t) g_2(t)\, dt = 1 + \int_0^x (x-t)\left(1 + \frac{t^2}{2}\right) dt$$

$$= 1 + \int_0^x \left(x + \frac{1}{2} x t^2 - t - \frac{t^3}{2} \right) dt = \left(1 + x t + \frac{x t^3}{6} - \frac{t^2}{2} - \frac{t^4}{8} \right)_0^x$$

$\therefore$ $$g_3(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!}$$

and so on.

Thus in general, we have

$$g_n(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\dots\dots + \frac{x^{2n-2}}{(2n-2)!}$$

Hence the required solution $g(x)$ is given by

$$g(x) = \lim_{n\to\infty} g_n(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\dots\dots + \frac{x^{2n-2}}{(2n-2)!} + \dots\dots$$

$\therefore$ $$g(x) = \cos h\, x$$

Example 13 : Solve

$$g(x) = x.2^x - \int_0^x 2^{x-t} g(t)\, dt, \; g_0(x) = x.2^x$$

by using the method of successive approximations

Solution : On comparing the given integral equation with the standard Volterra integral equation of second kind, we have

$$f(x) = x\,2^{x},\ K(x,t) = 2^{x-t},\ \lambda = -1$$

Now, $$g_1(x) = f(x) + \lambda \int_0^x K(x,t)\, g_0(t)\, dt$$

$$= x\,2^{x} - 2^{x}\int_0^x t\, dt$$

$$= \left(x - \frac{x^2}{2}\right) 2^{x}$$

$$g_2(x) = x.2^{x} - \int_0^x 2^{x-t}\left(t - \frac{t^2}{2}\right) 2^{t}\, dt$$

$$= \left(x - \frac{x^2}{2!} + \frac{x^3}{3!}\right) 2^{x}$$

Proceeding likewise, we finally obtain

$$g_n(x) = \left[x - \frac{x^2}{2!} + \frac{x^3}{3!} - \ldots\ldots + (-1)^{n-1}\frac{x^n}{n!}\right] 2^{x}$$

Now $$g(x) = \lim_{n\to\infty} g_n(x)$$

$$= \left[x - \frac{x^2}{2!} + \frac{x^3}{3!} - \ldots\ldots + (-1)^{n-1}\frac{x^n}{n!}\right] 2^{x}$$

$$= 2^{x}\left(1 - e^{-x}\right)$$

Self-Learning Exercise

1. Define following :
   (i) Iterated kernel
   (ii) Resolvent kernel
   (iii) Reciprocal function
   (iv) Neumann series
2. State whether the following statements are true or false :
   (i) If the sum of infinite series occuring in the formula of resolvent kernel cannot be determined, then in such cases, we may use the method of successive approximation.

(ii) The series $g(x) = f(x) + \sum_{m=1}^{\infty} \int_a^b K_m(x,t)\, g(t)\, dt$ is knwon as Neumann series.

(iii) The n^{th} approximate solution $g_n(x)$ of Fredholm integral equation of second kind $g(x) = f(x) + \int_a^b K_m(x,t)\, g(t)\, dt$ is obtained by

$$g_n(x) = f(x) + \sum_{m=1}^{n} \lambda^m \int_a^b K_m(x,t)\, f(t)\, dt$$

(iv) If $K(x,t)$ is real and continuous in R, then a reciprocal function $K(x,t)$ provided that $M(b-a) > 1$, where $|K(x,t)| < M$ in R.

(v) A function $k(x,t)$ reciprocal to $K(x,t)$ will exist, provided the series

$$-k(x,t) = K_1(x,t) + K_2(x,t) + \ldots.. + K_n(x,t) + \ldots.$$

converges uniformely.

(vi) The iterated kernel of the function $K(x,t) = e^x \cos t$, $a = 0$, $b = \pi$ does not exist.

10.8 Summary

In this unit, we have seen that the solution of integral equation of the second kind can be obtained with the aid of resolvent kernel. If the sum of infinite series occuring in the formula of resolvent kernel cannot be determined, then in such cases, we use to method of successive approximation.

10.9 Answers to Self-Learning Exercise

(1) See Text.

(2) (i) True (ii) False (iii) True (iv) False (v) True (vi) False

10.10 Exercise 10

1. Find the iterated kernel $K_m(x,t)$ of the following kernels for specified values of a and b

(i) $K(x,t) = x - t$; a = 0, $b = 1$

(ii) $K(x,t) = x e^t$; $a = 0$, $b = 1$

[Ans. (i) $K_{2m-1}(x,t) = \left(-\frac{1}{12}\right)^{m-1}(x-t)$, $K_{2m}(x,t) = \left(-\frac{1}{12}\right)^{m-1}\left(\frac{x+t}{2} - \frac{1}{3} - xt\right)$ $m = 1,2,3,\ldots$

(ii) $K_n(x,t) = x e^t$, $n = 1,2,3,\ldots\ldots\ldots$]

2. Construct the resolvent kernels for the following kernels for specified values of a and b:

(i) $K(x,t) = x t + x^2 t^2$; $a = -1$, $b = 1$

(ii) $K(x,t) = \sin x \cos t$; $a = 0$, $b = \pi/2$

[Ans. (i) $R(x,t;\lambda) = \frac{3xt}{3-2\lambda} + \frac{5x^2t^2}{5-2\lambda} : |\lambda| < 3/2$

(ii) $R(x,t;\lambda) = \frac{2\sin x \cos t}{2-\lambda} : |\lambda| < 2$]

3. Define resolvent kernel and find the resolvent kernel of the kernel $K(x,t) = 1 - 3xt$ in (0, 1).

[Ans. $R(x,t;\lambda) = \frac{4}{4-\lambda^2}\left[1+\lambda-\frac{3}{2}x\lambda - 3t\left(x+\frac{\lambda}{2}-x\lambda\right)\right], |\lambda| < 2$]

4. Solve the following Fredholm integral equation of second kind

$$g(x) = \sin x - \frac{\pi}{4} + \frac{1}{4}\int_0^{\pi/2} x\,t\,g(t)\,dt$$

$\left[\right.$ **Ans.** $g(x) = \sin x - \frac{\pi}{4}$ $\left.\right]$

5. Find the resolvent kernel of the following Volterra kernels

(i) $K(x,t) = 1$ (ii) $K(x,t) = e^{x-t}$

(iii) $K(x,t) = 3^{x-t}$ (iv) $K(x,t) = \frac{1+x^2}{1+t^2}$

[Ans. (i) $R(x,t;\lambda) = e^{\lambda(x-t)}$ (ii) $R(x,t;\lambda) = e^{(x-t)(1+\lambda)}$

(iii) $R(x,t;\lambda) = 3^{x-t}e^{\lambda(x-t)}$ (iv) $R(x,t;\lambda) = \frac{1+x^2}{1+t^2}e^{\lambda(x-t)}$]

6. Solve the following integral equation, with the aid of resolvent kernels

(i) $g(x) = 1 + \int_0^x (t-x)\,g(t)\,dt$ (ii) $g(x) = 1 + \lambda\int_0^x e^{3(x-t)}\,g(t)\,dt$

(iii) $g(x) = e^{x^2} + \int_0^x e^{x^2-t^2}\,g(t)\,dt$ (iv) $g(x) = 1 - 2x - \int_0^x e^{x^2-t^2}\,g(t)\,dt$

[Ans. (i) $g(x) = \cos x$ (ii) $g(x) = \frac{1}{3+\lambda}\left(3+\lambda\,e^{(3+\lambda)x}\right)$

(iii) $g(x) = e^{x^2+x}$ (iv) $g(x) = e^{x^2-x} - 2x$]

7. Using the method of successive approximation, solve the integral equation

$$g(x) = x - \int_0^x (x-t)\,g(t)\,dt\,,\; g_0(x) = 0$$

[Ans. $g(x) = \sin x$]

8. Using the method of the successive approximation, solve the integral equation

$$g(x) = 1 + \int_0^x g(t)\,dt \text{ taking } g_0(x) = 0$$

[Ans. $g(x) = e^x$]

Unit - 11
Integral Equations with Symmetric Kernels

Structure of the Unit

11.0 Objective

We have already discussed eigenvalues and eigenfunctions for integral equations in Unit 9. We have established that the eigenvalues of an integral equation are the zeros of certain determinant. In this process we have seen that there are many kernels for which there are no eigenvalues. However, in this unit we shall prove that for a symmetric kernel that is not identically zero, at least one eigenvalue will always exist. This is an important characteristic of symmetric kernels.

11.1 Introduction

The part of Fredholm theory which involves integral operators generated by real symmetric kernels is referred to as the Hilbert-Schmidt theory of integral equation. Owing to the richness of its result, the theory has attracted extensive attention of those as well as those interested in practical applications of integral equations as well as those interested in abstract theory, specially functional analysis. In this unit, we are going to focus our attention to those aspects of the theory which constitute the interface between differential equation and integral equations.

11.2 General Definitions and Inequlities

(a) Regularity Conditions

Definition :

A function $f(x)$ is said to be square integrable if

$$\int_a^b |f(x)|^2 dx < \infty$$

A square integrable function $f(x)$ is called an L_2 -function i.e. a function $f(x)$ is said to be L_2 -function if the following conditions are satisfied.

(i) $\int_a^b \int_a^b |K(x,t)|^2 dx\,dt < \infty \quad \forall\, x \in [a,b],\ \forall\, t \in [a,b]$

(ii) $\int_a^b |K(x,t)|^2 dx < \infty \ ; \ \forall\, x \in [a,b]$

(iii) $\int_a^b |K(x,t)|^2 dx < \infty \ ; \ \forall\, t \in [a,b]$

(b) The Inner or Scalar Product of Two Functions :

The inner or scalar product of two complex L_2 -functions g and h of a real variable x, $a \le x \le b$ is denoted by (g,h). It is defined as

$$(g,h) = \int_a^b g(x)\bar{h}(x)\,dx$$

where $\bar{h}(x)$ is the complex conjugate of $h(x)$.

(c) Orthogonal Functions :

Two functions are called orthogonal if thier inner product is zero, that is, g and h are orthogonal if

$$(g,h) = 0 \text{ if } \int_a^b g(x)\bar{h}(x)\,dx = 0$$

(d) Norm of a Complex Function :

The norm of a complex function $g(x)$ is defined as

$$\|g(x)\| = \left[\int_a^b g(x)\bar{g}(x)\,dx\right]^{1/2} = \left[\int_a^b |g(x)|^2 dx\right]^{1/2}$$

A function $g(x)$ is said to be normalized if $\|g(x)\| = 1$. It follows that a nonnull function (whose norm is not zero) can always be normalized by dividing it by its norm.

(e) Inequalitis

(i) Schwarz Inequality : If $g(x)$ and $h(x)$ are L_2 -functions, then $|(g,h)| \le \|g\|\ \|h\|$.

(ii) Minkowski Inequality : If $g(x)$ and $h(x)$ are L_2 -functions, then $\|g+h\| \le \|g\| + \|h\|$.

(iii) Bessel's Inequality : If $f(x)$ is real and continuous and $g_m(m = 1,2,......n)$ are real, continuous and consisting of a normalized orthogonal set, then Bessel's inquality states that

$$\sum_{m=1}^{\infty} \int_a^b |f(x) g_m(x)|^2 \, dx \le \int_a^b |f(x)|^2 \, dx$$

11.3 Complex Hilbert Space

Definition : A linear space of infinite dimension with inner product (or scalar product) (x, y), which is a complex number is called a complex Hilbert space, if it satisfies the following three axions :

(i) $(x, x) > 0$ for $x \neq 0$

(ii) $(\alpha x_1 + \beta x_2, y) = \alpha (x_1, y) + \beta (x_2, y)$ where α and β are arbitary complex numbers

(iii) $\overline{(x, y)} = (y, x)$,

where the bar denotes the complex conjugate.

Let H be the set of complex valued functions $g(x)$ defined in the interval (a, b) such that $\int_a^b |g(x)|^2 \, dx < \infty$, then H is linear and complex Hilbert space $L_2\ (a, b)$ or L_2.

The norm of function generates the natural metric

$$d(f, g) = \|f - g\| = (f - g, f - g)^{1/2}$$

The concept of completeness is a fundamental concept in the theory of Hilbert space. A metric space is called complete, if every Cauchy sequence of functions in a metric space is convergent. A Hilbert space is an inner product linear space that is complete in its natural metric. The completeness of L_2 - spaces plays an important role in the theory of linear operators such as the Fredholm operater K, defined as

$$Kg = \int_a^b K(x, t) g(t) \, dt \qquad ...(1)$$

The operator adjoint to K is

$$\overline{K} h = \int_a^b \overline{K}(t, x) h(t) \, dt \qquad ...(2)$$

For the operators (1) and (2), we have the following important relation :

$$(K g, h) = (g, \overline{K} h), \qquad ...(3)$$

which can be easily proved as follows :

LHS of (3) $= (K g, h) = \int_a^b \overline{h}(x) \left[\int_a^b K(x, t) g(t) \, dt \right] dx$

(By detinition of inner product of two functions)

$$= \int_a^b g(t) \left[\int_a^b K(x, t) \overline{h}(x) \, dx \right] dt \qquad \text{(changing the order of integration)}$$

$$= \int_a^b g(x)\left[\int_a^b K(t,x)\bar{h}(t)\,dt\right]dx$$

$$= \int_a^b g(x)\left[\int_a^b \overline{K(t,x)h(t)\,dt}\right]dx \qquad ...(4)$$

$$= \left(g, \overline{K}h\right)$$

If the kernel is symmetric, then (4) becomes

$$(Kg,h) = (g,Kh) \qquad ...(5)$$

i.e. a symmetric operator is self addiont.

Further, we know that permutation of factors in a scalar product is equivalent to taking the complex conjugate i.e.

$$(g,Kg) = \left(\overline{Kg,g}\right)$$

Combining this with (5), we find that, for a symmetric kernel, the inner product (Kg,g) is always real. The converse of this is also true.

11.4 Orthogonal System of Functions

System of orthogonal functions play an important role in the theory of integral equations and their applications.

Definition : A finite or an infinite set $\{g_k\}$ is said to be an orthogonal set if

$$\left(g_i, g_j\right) = 0 \text{ or } \int_a^b g_i(x)\,g_j(x)\,dx = 0,\ i \neq j$$

If non of the elements of this set is zero vector, then it is called a proper orthogonal set.

A set $\{g_i\}$ is orthonormal if

$$\left(g_1, g_j\right) = \int_a^b g_i(x)\,g_j(x)\,dx = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

11.5 Gram-Schmidt Method for Construction of Set of Orthonormal Functions

Let $\{h_1, h_2, ..., h_k, ...\}$ is a finite or an infinite independent set of functions. Then we can construct an orthonormal set $\{g_1, g_2, ..., g_k, ...\}$ by the well-known Gram-Schmidt procedure as follows :

Let $\quad g_1 = \dfrac{h_1}{\|h_1\|}$

To obtain g_2, we define

$$w_2(x) = h_2(x) - (h_2, g_1) g_1.$$

The function w_2 is clearly orthogonal to g_1. Hence g_2 is obtained by setting $g_2 = w_2 / \|w_2\|$. Continuing this process, we have

$$w_k(x) = h_k(x) - \sum_{i=1}^{k-1} (h_k, g_i) g_j \text{ where } g_k = w_k / \|w_k\|$$

Now, if we are given a set of orthogonal functions, we can convert it into an orthogonal set simply by dividing each function by its norm.

Starting from an arbitrary orthonormal system, it is possible to construct the theory of Fourier series. Suppose we want to find the best approximation of an arbitrary function $h(x)$ in terms of a linear combination of an orthonormal set $(g_1, g_2, ... g_n)$. The best approximation, means that to choose the coefficient $\alpha_i (i = 1,2,......n)$ in such a manner that

$$\left\| h - \sum_{i=1}^{n} \alpha_i g_i \right\|^2 = \|h\|^2 + \sum_{i=1}^{n} \left|(h - g_i) - \alpha_i\right|^2 - \sum_{i=1}^{n} \left|(h, g_i)\right|^2 \qquad ...(6)$$

Obviously $\left\| h - \sum_{i=1}^{n} \alpha_i g_i \right\|^2$ is minimum when $\alpha_i = (h, g_i) = c_i$ (say)

The number α_i are called the Fourier coefficients of the function $h(x)$ relative to the orthonormal system, $\{g_1\}$. In that case, the relation (6) reduces to

$$\left\| h - \sum_{i=1}^{n} \alpha_i g_i \right\|^2 = \|h\|^2 - \sum_{i=1}^{n} |c_i|^2$$

Since the left side in non-negative, we have

$$\sum_{i=1}^{n} |c_i|^2 \le \|h\|^2$$

which for the infinite set $\{g_i\}$ leads to the Bessel inequality

$$\sum_{i=1}^{\infty} |c_i|^2 \le \|h\|^2$$

Suppose we are given an infinite orthonormal system $\{g_i(x)\}$ in L_2 and a sequence of constants $\{\alpha_i\}$, then the convergence of the series $\sum_{k=1}^{\infty} \left|\alpha_k^2\right|$ is evidently a necessary condition for the existence of an L_2-function $f(x)$ where Fourier coefficients with respect to the system g_i and α_i. Note that this condition is also sufficient by Riesz-Fisher theorem.

11.6 Other Useful Definitions

Definition 1 :

Given a sequence of functions $< f_n(x) >$ and a function $f(x)$ in L_2 - space defined on an interval I, then the sequence $< f_n >$ converges uniformly on I if

$$\underset{x \in I}{Sup} \left| f_m(x) - f_n(x) \right| \to 0 \text{ as } m, n \to \infty$$

Definition 2 :

The sequence $< f_n(x) >$ converges uniformly to $f(x)$ if

$$\underset{x \in I}{Sup} \left| f(x) - f_n(x) \right| \to 0 \text{ as } n \to \infty$$

Definition 3 :

The sequence $< f_n(x) >$ converges in the mean on $[a, b]$ if

$$\int_a^b \left| f_m(x) - f_n(x) \right|^2 dx \to 0 \text{ as } m, n \to 0$$

Also, it converges in mean to $f(x)$ if

$$\lim_{n \to \infty} \int_a^b \left| f(x) - f_n(x) \right|^2 dx = 0$$

Definition 4 :

A series of function $S = \sum_{i=1}^{\infty} f_i(x)$, $\forall\ x \in I$ converges uniformly (or in mean) to $F(x)$, if the sequence of partial sums

$$S_n(x) = \sum_{i=n}^{n+p} f_i(x)$$

converges uniformly (or in the mean) to $F(x)$.

Definition 5 :

The series S, defined above is said to be absolutely convergent if the series

$\sum_{i=1}^{\infty} \left| f_i(x) \right|$ is pointwise convergent.

Remark :

On a finite closed domain, uniform convergence implies convergence in the mean. The converse is not true. For example, as the open interval (0, 1) the sequence $< e^{-nx} >$ convergence in the mean but not uniformly.

11.7 Symmetric Kernels

Definition :

A kernel $K(x,t)$ is called symmetric (or complex symmetric or Hermitian) if it coincides with its own conjugate i.e. if $K(x,t)=\overline{K}(t,x)$, where the bar denotes the complex conjugate. In the case of real kernel, the symmetry reduces to the equality

$$K(x,t)=K(t,x)$$

An integral equation with a symmetric kernel is called a **symmetric equation.**

Theorem 1 : If a kernel is symmeteric, then all its iterated kernels are also symmetric.

Proof : Let kernel $K(x,t)$ be symmetric. Then by definition

$$K(x,t)=\overline{K}(t,x) \qquad ...(7)$$

where the bar denotes the complex conjugate.

The iterated kernels $K_n(x,t)$, $n=1,2,3,....$ are defined as

$$K_1(x,t)=K(x,t) \qquad ...(8)$$

$$K_n(x,t)=\int_a^b K(x,z)K_{n-1}(z,t)\,dz, \qquad n=2,3,...... \qquad ...(9)$$

or $$K_n(x,t)=\int_a^b K_{n-1}(x,z)K(z,t)\,dz, \qquad n=2,3,....... \qquad ...(10)$$

We shall use mathematical induction to prove the required result.

Putting $n=2$ in (9), we obtain

$$K_2(x,t)=\int_a^b K(x,z)K_1(z,t)\,dz=\int_a^b K(x,z)K(z,t)\,dz \qquad \text{[by (8)]}$$

$$=\int_a^b \overline{K}(z,x)\overline{K}(t,z)\,dz \qquad \text{[by (9)]}$$

$$=\int_a^b \overline{K}(t,z)\overline{K}(z,x)\,dz$$

$$=\int_a^b \overline{K}(t,z)\overline{K}_1(z,x)\,dz \qquad \text{[by(8)]}$$

$$=\overline{K}_2(t,x) \qquad \text{[by (9)]}$$

Thus $$K_2(x,t)=\overline{K}_2(t,x)$$

showing that $K_2(x,t)$ is symmetric by definition. Hence the required result is true for $n=1$ and $n=2$.

Let the result be true for $n = m$.

i.e $\quad K_m(x,t) = \overline{K}_m(t,x)$

We shall now prove that the result is also true for $n = m+1$.

i.e. $\quad K_{m+1}(x,t) = \overline{K}_{m+1}(t,x)$

Putting $n = m+1$ in (9), we have

$$K_{m+1}(x,t) = \int_a^b K(x,z)\, K_m(z,t)\, dz$$

$$= \int_a^b \overline{K}(z,x)\, \overline{K}_m(t,z)\, dz$$

$$= \int_a^b \overline{K}_m(t,z)\, \overline{K}(z,x)\, dz = \overline{K}_{m+1}(t,x)$$

showing that $K_{m+1}(x,t)$ is symmetric. Hence by mathematical induction $K_n(x,t)$ is symmetric for $n = 1,2,3.....$

11.8 Fundamental Properties of Eigenvalues and Eigenfunctions for Symmetric Kernels

Consider the symmetric integral equation

$$\lambda \int_a^b K(x,t)\, g(t)\, dt = f(x) \quad \text{or } \lambda\, Kg = f\ ;\ K(x,t) = \overline{K}(t,x) \qquad ...(A)$$

Now we establish certain properties for (A) contained in following theorems.

Theorem 2 : The eigenvalues of a symmetric kernel are real.

Proof : Let λ be an eigenvalue of the kernel $K(x,t)$ and corresponding eigenfunction is $g(x)$. Then by definition of the eigenfunction, we have

$$g(x) = \lambda \int_a^b K(x,t)\, g(t)\, dt \qquad ...(11)$$

Multiplying (11) by $\overline{g}(x)$ and integrating with respect to x in (a,b), we obtain

$$\int_a^b g(x)\, \overline{g}(x)\, dx = \lambda \int_a^b \overline{g}(x) \left[\int_a^b K(x,t)\, g(t)\, dt \right] dx$$

or $\quad \left|\left|g(x)\right|\right|^2 = \lambda (Kg, g)$

or $$\lambda = \frac{\left|\left|g(x)\right|\right|^2}{(Kg, g)} \qquad ...(12)$$

Since both numerator and denominator of RHS of (12) are real, therefore λ is also real.

Remark : For another proof of this theorem, refer Theorem of Art 8.13.

Theorem 11.8.2 : The eigenfunctions of a symmetric kernel corresponding to distinct eigenvalues are orthogonal.

or

If $K(x,t)$ is symmetric and $g_1(x)$, $g_2(x)$ are fundamental function of $K(x,t)$ for λ_1 and λ_2 respectively $(\lambda_1 \neq \lambda_2)$, then $g_1(x)$ and $g_2(x)$ are orthogonal on the interval (a,b)

i.e. $$\int_a^b g_1(x)\bar{g}_2(x)\,dx = 0$$

Remark : For proof of the Theorem, refer Theorem 2 of Art. 8.14.

11.9 Expansion of Eigenfunctions and Bilinear Form

Let $K(x,t)$ be a nonnull, symmetric kernel which has a finite or an infinite number of eigenvalues (always real and non zero). Consider these eigenvalues, in a sequence

$$\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ldots \lambda_n \qquad \text{...(18)}$$

in such a way that each eigenvalues is repeated as many times as its multiplicity. We further denumerate these eigenvalues in the order that corresponding to their absolute values i.e.

$$0 < |\lambda_1| \le |\lambda_2| \le \ldots\ldots\ldots \le |\lambda_n| \le |\lambda_{n+1}|$$

Let $$g_1(x),\ g_2(x), \ldots\ldots g_n(x) \qquad \text{...(19)}$$

be the sequence of eigenfunctions corresponding to the eigenvalues given by the sequence (18). These eigenfunctions are no longer repeated and are linearly independent, corresponding to the same eigenvalue. Thus to each eigenvalue λ_k in (18) there corresponds just one eigenfunction $g_k(x)$ in (19), suppose that these eigenfunctions have been orthonormalized.

Suppose that a symmetric L_2-kernel has at least one eigenvalue, say λ_1. Then $g_1(x)$ is the corresponding eigenfunction. It follows that the second 'truncated' symmetric kernel

$$K^{(2)}(x,t) = K(x,t) - \frac{g_1(x)\,\bar{g}_1(t)}{\lambda_1} \qquad \text{...(20)}$$

is non null and it will also have at least one eigenvalue λ_2 (we choose the smallest if there are more than one eigenvalues) with corresponding normalized eigenfunction $g_2(x)$. The function $g_1(x) \neq g_2(x)$ even if $\lambda_1 = \lambda_2$, since

$$\int_a^b K^{(2)}(x,t)\,g_1(t)\,dt = \int_a^b K(x,t)\,g_1(t)\,dt - \frac{g_1(x)}{\lambda_1}\int_a^b g_1(t)\,\bar{g}_1(t)\,dt$$

$$= \frac{g_1(x)}{\lambda_1} - \frac{g_1(x)}{\lambda_1} = 0$$

Similarly, the third "truncated" kernel

$$K^{(3)}(x,t) = K^{(2)}(x,t) - \frac{g_2(x)\ \bar{g}_k(t)}{\lambda_2} = K(x,t) - \sum_{k=1}^{2} \frac{g_k(x)\ \bar{g}_2(t)}{\lambda_k} \qquad ...(21)$$

gives the third eigenvalue λ_3 and the corresponding normalized eigenfunction $g_3(x)$.

Continuing in this way, we finally arrive at the two following possibilities :

(i) The above process terminates after n steps, that is $K^{(n+1)}(x,t) = 0$ and the kernel $K(x,t)$ is a degenerate kernel,

$$K(x,t) = \sum_{k=1}^{n} \frac{g_k(x)\ \bar{g}_k(t)}{\lambda_k} \qquad ...(22)$$

(ii) The above process can be continued identifinitely and there are infinite number of eigenvalues and eigenfunctions.

Remark 1 : We have denoted the least eigenvalue and the corresponding eigenfunction of $K^{(n)}(x,t)$ as λ_n and g_n, which are the n^{th} eigenvalue and the n^{th} eigenfunction in the sequences (18) and (19). This fact can be justified with help of Theorem 11.9.2, which is given below.

Theorem 4 : Let the sequence $\{g_k(x)\}$ be all the eigenfunctions of a symmetric L_2-kernel with $\{\lambda_k\}$ as the corresponding eigenvalues. Then the series

$$\sum_{n=1}^{\infty} \frac{|g_n(x)|^2}{\lambda_n^2}$$

converges and its sum is bounded by C_1^2, which is an upper bound of the integral $\int_a^b |K^2(x,t)|\ dt$.

Proof : The Fourier coefficients a_n of the function $K(x,t)$ with fixed x, with respect to the orthonormal system $\{\bar{g}_n(x)\}$ are given by

$$a_n = \int_a^b K(x,t)\, g_n(t)\ dt = \frac{g_n(x)}{\lambda_n}$$

Using Bessel's inequality, we now obtain

$$\sum_{n=1}^{\infty} \frac{|g_n(x)|^2}{\lambda_n^2} \leq \int_a^b |K(x,t)|^2\, dt \leq C_1^2$$

Theorem 5 : Let the sequence $\{g_n(x)\}$ be all the eigenfunctions of a symmetric kernel $K(x,t)$, with $\{\lambda_n\}$ as the corresponding eigenvalues. Then the truncated kernel

$$K^{(n+1)}(x,t)=K(x,t)-\sum_{m=1}^{n}\frac{g_m(x)\,\bar{g}_m(t)}{\lambda_m}$$

has the eigenvalues $\lambda_{n+1},\lambda_{n+2},......$ to which corresponds the eigenfunctions $g_{n+1}(x)$, $g_{n+2}(x)$,.... The kernel $K^{n+1}(x,t)$ has no other eigenvalues or eigenfunctions.

Proof : (i) We begin with fact that the integral equation

$$g(x)-\lambda\int_a^b K^{(n+1)}(x,t)\,g(t)\,dt=0$$

$$\Rightarrow\qquad g(x)-\lambda\int_a^b K(x,t)\,g(t)\,dt+\lambda\sum_{m=1}^{n}\frac{g_m(x)}{\lambda_m}(g,g_m)\,dt=0 \qquad ...(23)$$

Setting $\lambda=\lambda_j$ and $g(x)=g_j(x)$, $j\ge n+1$ on LHS of (23)

and using the orthogonality condition, we obtain

$$g_j(x)-\lambda_j\int_a^b K(x,t)\,g_j(t)\,dt=0$$

which means that $g_j(x)$ and λ_j for $j\ge n+1$ are the eigenfunctions and eigenvalues of the kernel $K^{n+1}(x,t)$.

(ii) Let λ and $g(x)$ be an eigenvalue and eigenfunction of the kernel $K^{n+1}(x,t)$. Then

$$g(x)-\lambda\int_a^b K(x,t)\,g(t)\,dt+\lambda\sum_{m=1}^{n}\frac{g_m(x)}{\lambda_m}(g,g_m)=0 \qquad ...(24)$$

Taking the scalar product of (24) with $g_j(x)$, $j\le n$ and using the orthonormality of the $g_j(x)$, we have

$$(g,g_j)-\lambda\,(Kg,g_j)+\frac{\lambda}{\lambda_j}(g,g_j)=0 \qquad ...(25)$$

But $\quad (Kg,g_j)=(g,Kg_j)=\lambda_j^{-1}(g,g_j)$

Hence (25) becomes

$$(g,g_j)+\frac{\lambda}{\lambda_j}\{(g,g_j)-(g,g_j)\}=(g,g_j)=0 \qquad ...(26)$$

In view of (26) we find that the least term in the left side of equation (24) vanistes and hence (24) reduces to

$$g(x)-\lambda\int_a^b K(x,t)\,g(t)\,dt=0$$

This means that λ and $g(x)$ are eigenvalue and eigenfunction of the kernel $K(x,t)$ and that $g \neq g_j$, $j \leq n$. In fact, we see that g is orthogonal to all $g_j, j \leq n$, and $g(x)$ and λ are surely contained in the sequences $\{g_k(x)\}$ and $\{\lambda_k\}$, $k \geq n+1$ respectively.

Remark : Combining the results of the above two Theorem 11.9.1 and 11.9.2, we easily find that, if the symmetric kernel $K(x,t)$ has only a finite number of eigenvalues, then it must be separable. The proof follows by noting that $K^{n+1}(x,t)$ has no eigenvalues and therefore it must be null. Hence, we must have

$$K(x,t) = \sum_{m=1}^{n} \frac{g_m(x)\ \overline{g}_m(t)}{\lambda_m}$$

11.10 Hilbert-Schmidt Theorem

Statement : If $\phi(x)$ can be written in the form

$$\phi(x) = \int_a^b K(x,t)\, h(t)\ dt$$

where $K(x,t)$ is a symmetric L_2-kernel and $h(t)$ is an L_2-function, then $\phi(x)$ can be expended in an absolutely and uniformely convergent Fourier series with respect to the orthonormal system of eigenfunctions $g_1(x), g_2(x), \ldots.g_n(x)$ of the kernel $K(x,t)$:

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n\ g_n(x)$$

where $\phi_n = (\phi, g_n)$

The Fourier coefficients ϕ_n of the function $\phi(x)$ are related to the Fourier coefficients h_n of the function $h(x)$ by the relations

$$\phi_n = \frac{h_n}{\lambda_n}$$

and $\quad h_n = (h, g_n)$

where λ_n are the eigenvalues of the kernel $K(x,t)$.

Proof : Let $K(x,t)$ be a non null, symmetric kernel which has a finite or an infinite number of eigenvalues (always real and non zero). Consider these eigenvalues, in a sequence

$$\lambda_1, \lambda_2, \ldots\ldots \lambda_n, \ldots.. \qquad \ldots(27)$$

in such a way that each eigenvalue is repeated as many times as its multiplicity. We further elenumerase there eigenvalues such that

$$0 \le |\lambda_1| \le |\lambda_2| \le .... \le |\lambda_n| \le |\lambda_{n+1}| \le .....$$

Let $\quad g_1(x), g_2(x), ... g_n(x), ...$ ...(28)

be the sequence of eigenfunctions corresponding to the eigenvalues given by the sequence (27). These eigenfunctions are no longer repeated and are linearly independent corresponding to the same eigenvalue. Thus, to each eigenvalue λ_k in (27), there corresponds just one eigenfunction $g_k(x)$ in (28). Further, suppose that these eigenfunctions have been orthonormalized.

Now, the Fourier coefficients of the function $\phi(x)$ with regards to the orthonormal system $\{g_n(x)\}$ are given by

$$\phi_n = (\phi, g_n) = (Kh, g_n) = (h, Kg_n) \qquad (\because \text{ symmetric operator is self ajoint})$$

$$= \frac{1}{\lambda_n}(h, g_n) = \frac{h_n}{\lambda_n} \qquad \left\{ \because g_n(x) = \lambda_n \int_a^b K(x,t) g_n(t) \Rightarrow g_n = \lambda_n K g_n \right.$$

$$\left. \therefore (h, Kg_n) = \left(h, \frac{g_n}{\lambda_n}\right) = \frac{1}{\lambda_n}(h, g_n) \right\}$$

Hence the Fourier series for $\phi(x)$ is given by

$$\phi(x) - \sum_{n=1}^{\infty} \phi_n \, g_n(x) = \sum_{n=1}^{\infty} \frac{h_n}{\lambda_n} g_n(x) \qquad ...(29)$$

We now estimate the remainder term of the series (29). We have

$$\left| \sum_{k=n+1}^{n+p} h_k \frac{g_k(x)}{\lambda_k} \right|^2 \le \sum_{k=n+1}^{n+p} h_k^2 \sum_{k=n+1}^{n+p} \frac{|g_k(x)|^2}{\lambda_k^2}$$

$$\le \sum_{k=n+1}^{n+p} h_k^2 \sum_{k=1}^{\infty} \frac{|g_k(x)|^2}{\lambda_k^2} \qquad ...(30)$$

Now the Fourier coefficients a_n of the function $K(x,t)$ with respect to orthonormal system $\{g_n(x)\}$ are given by

$$a_n = \int_a^b K(x,t) g_n(t) \, dt = \frac{g_n(x)}{\lambda_n}$$

Using Bessel's inequality, we get

$$\sum_{n=1}^{\infty} \frac{|\phi_n(x)|^2}{\lambda_n^2} \le \int_a^b |K(x,t)|^2 \, dt \le C_1^2 \qquad \left[if \; |K(x,t) \le C_1| \right]$$

Hence inner series in (30) is bounded.

Moreover since $h(x)$ is a L_2-function, the series $\sum_{k=1}^{\infty} h_k^2$ is convergent and the partial sum $\sum_{k=n+1}^{n+p} h_k^2$ can be made arbitrary small. Hence, the series (29) converges absolutely and uniformly.

Now, we prove that series (29) converges to $\phi(x)$. For this prupose, we denote its partial sum as

$$S_n(x) = \sum_{m=1}^{n} \frac{h_m}{\lambda_m} g_m(x)$$

and estimate the value of $\left|\left|\phi(x) - S_n(x)\right|\right|$

Now, $$\phi(x) - S_n(x) = Kh - \sum_{m=1}^{n} \frac{h_m}{\lambda_m} g_m(x)$$

$$= Kh - \sum_{m=1}^{n} \frac{(h, g_m)}{\lambda_m} g_m(x)$$

$$= K^{(n+1)} h$$

where $K^{(n+1)}$ is the truncated kernel.

But $$\left|\left|\phi(x) - S_n(x)\right|\right|^2 = \left|\left|K^{n+1} h\right|\right|^2 = \left(K^{(n+1)} h, K^{(n+1)} h\right)$$

$$= \left(h, K^{(n+1)} K^{(n+1)} h\right) = \left(h, K_2^{(n+1)} h\right) \qquad ...(30)$$

where we have used the self-adjointness of the kernel $K^{(n+1)}$ and also the relation

$$K^{(n+1)} K^{(n+1)} = K_2^{(n+1)}$$

We know that the set of eigenvalues of the second iterated kernel coincide with the set of squares of the eigenvalues of the given kernel. Thus, we see that the least eigenvaluse of the kernel $K_2^{(n+1)}$ is equal to λ_{n+1}^2 and, we obtain

$$\frac{1}{\lambda_{n+1}^2} = \max\left[\frac{\left(h, K_2^{(n+1)} h\right)}{(h, h)}\right], \qquad ...(31)$$

where we have omitted the modules sign from the scalar product $\left(h, K_2^{(n+1)} h\right)$, because it is a positive quantity.

Using (31) in (30), we have

$$\left|\left|\phi(x) - S_n(x)\right|\right|^2 = \left(h, K_2^{(n+1)} h\right) \le \frac{(h, h)}{\lambda_{n+1}^2} \qquad ...(32)$$

Since $\lambda_{n+1} \to \infty$, (32) gives

$$\left\| \phi(x) - S_n(x) \right\|^2 \to 0 \text{ as } n \to \infty \quad \text{...(33)}$$

Again $\left\| \phi(x) - S(x) \right\| \le \left\| \phi(x) - S_n(x) \right\| + \left\| S_n(x) - S(x) \right\| \quad \text{...(34)}$

where $S(x)$ is the limit of the series $S_n(x)$.

Since the series (29) converges uniformly, we have, for an arbitrary small $\in > 0$, $\left| S_n(x) - S(x) \right| < \in$ where n is sufficiently large,

$$\therefore \quad \left\| S_n(x) - S(x) \right\| < \in (b-a)^{1/2}$$

and hence $\left\| S_n(x) - S(x) \right\| \to 0 \quad \text{...(35)}$

Using (35) and (33) in (34), we have

$$\phi(x) = S(x) \text{ as required.}$$

11.11 Schmidt's Solution of Non Homogeneous Fredholms Integral Equation of Second Kind

Consider Fredholm integral equation of second kind

$$g(x) = f(x) + \lambda \int_a^b K(x,t)\, g(t)\, dt \quad \text{...(36)}$$

where $K(x,t)$ is continuous, real and symmetric and λ is not an eigenvalue.

We shall require the Hilbert-Schmidt theorem, stated in the following modified form :

Let $Y(x)$ be generated from a continuous function $g(x)$ by the operator

$$\lambda \int_a^b K(x,t)\, g(t)\, dt$$

where $K(x,t)$ is continous, real and symmetric, so that

$$Y(x) = \lambda \int_a^b K(x,t)\, g(t)\, dt$$

Then $Y(x)$ can be represented over interval (a,b) by a linear combination of the normalized eigenfunctions of homogeneous integral equation

$$g(x) = \lambda \int_a^b K(x,t)\, g(t)\, dt$$

having $K(x,t)$ as its kernel.

Procedure of solution of (36)

Now let $g(x)$ be continuous solution of (36), then

$$g(x)-f(x)=\lambda\int_a^b K(x,t)\,g(t)\,dt \qquad ...(37)$$

If the equation (36) possesses a continuous soluton $g(x)$, then the function $\left[g(x)-f(x)\right]$ is generated by (37). Hence it can be represented by a linear combination of the normalized characteristic function $\phi_m(x)$ of the form

$$g(x)-f(x)=\sum_{m=1}^{n} a_m\,\phi_m(x),\qquad a\le x\le b \qquad \text{[By Hilbert-Schmidt theorem]} \quad ...(38)$$

Let $\lambda_m(m=1,2,3,........)$ be the eigenvalues corresponding to the eigenfunction $\phi_m(x)$. Let

$$\lambda\neq\lambda_m\ \ \forall\ m=1,2,3,......$$

Since $\phi_m(x)$, $(m=1,2,3,.......)$ are normalized, therefore we have

$$\int_a^b \phi_m(x)\,\phi_n(x)\,dx=\begin{cases}0, & m\neq n\\ 1, & m=n\end{cases} \qquad ...(39)$$

Multiplying both sides of (38) by $\phi_m(x)$ and then integrating with respect to 'x' from a to b, we get

$$\int_a^b g(x)\phi_m(x)dx-\int_a^b f(x)\phi_m(x)dx=a_1\int_a^b \phi_1(x)\phi_m(x)\,dx$$

$$+.....+\ a_m\int_a^b \phi_m(x)\phi_m(x)\,dx+.....+\ a_n\int_a^b \phi_m(x)\phi_n(x)\,dx \qquad ...(40)$$

Suppose that

$$C_m=\int_a^b g(x)\phi_m(x)dx \qquad ...(41)$$

and $$f_m=\int_a^b f(x)\phi_m(x)dx \qquad ...(42)$$

Making use of (39), (41) and (42) in (40), we obtain

$$C_m-f_m=a_m \qquad ...(43)$$

Now multiplying both sides of (36) $\phi_m(x)$ and then integrating with respect to 'x' from a to b, we have

$$\int_a^b g(x)\phi_m(x)dx=\int_a^b f(x)\phi_m(x)dx+\lambda\int_a^b\left\{\int_a^b K(x,t)\,g(t)dt\right\}\phi_m(x)dx$$

or $$C_m=f_m+\lambda\int_a^b g(t)\left\{\int_a^b K(x,t)\phi_m(x)dx\right\}dt \quad \text{(by changing the order of integration)}$$

or $$C_m=f_m+\lambda\int_a^b g(t)\left\{\int_a^b K(t,x)\phi_m(x)dx\right\}dt \quad \text{(since } K(x,t) \text{ is symmetric)} \qquad ...(44)$$

Since $\phi_m(x)$ is eigenfunction corresponding to the eigenvalue λ_m, therefore we have

$$\phi_m(x) = \lambda_m \int_a^b K(x,t)\phi_m(t)\,dt$$

or $$\phi_m(x) = \lambda_m \int_a^b K(x,z)\phi_m(z)\,dz$$

or $$\phi_m(t) = \lambda_m \int_a^b K(t,z)\phi_m(z)\,dz$$

or $$\phi_m(t) = \lambda_m \int_a^b K(t,x)\phi_m(x)\,dx$$

or $$\int_a^b K(t,x)\phi_m(x)\,dx = \frac{\phi_m(t)}{\lambda_m} \qquad \text{...(45)}$$

Using (45), (44) reduces to

$$C_m = f_m + \lambda \int_a^b \frac{g(t)\,\phi_m(t)}{\lambda_m}\,dt$$

or $$C_m = f_m + \frac{\lambda}{\lambda_m} \int_a^b g(x)\phi_m(x)\,dx$$

or $$C_m = f_m + \frac{\lambda\,C_m}{\lambda_m}, \qquad \text{[using (41)]} \qquad \text{...(46)}$$

From (43), we find that

$$C_m = a_m + f_m \qquad \text{...(47)}$$

Eliminating C_m from (46) and (47), we get

$$a_m + f_m = f_m + \frac{\lambda}{\lambda_m}(a_m + f_m)$$

or $$a_m\left(1 - \frac{\lambda}{\lambda_m}\right) = \frac{\lambda}{\lambda_m} f_m$$

or $$a_m = \frac{\lambda}{\lambda_m - \lambda} f_m \qquad \text{...(48)}$$

where $\lambda \neq \lambda_m$ and so a_m is well defined.

Now putting the value of a_m from (48) in (38), we get

$$g(x) - f(x) = \sum_{m=1}^{n} \frac{\lambda\,f_m}{\lambda_m - \lambda}\,\phi_m(x)$$

or $$g(x) = f(x) + \lambda \sum_{m=1}^{n} \frac{f_m}{\lambda_m - \lambda} \phi_m(x) \qquad ...(49)$$

which is the required solution of the integral equation (36).

From (42), we have

$$f_m = \int_a^b f(t)\phi_m(t)\,dt \qquad ...(50)$$

Using (50), (49) may be rewritten as

$$g(x) = f(x) + \lambda \sum_{m} \frac{\phi_m(x)}{\lambda_m - \lambda} \int_a^b f(t)\phi_m(t)\,dt$$

or $$g(x) = f(x) + \lambda \int_a^b \left[\sum_{m} \frac{\phi_m(x)\,\phi_m(t)}{\lambda_m - \lambda} \right] f(t)\,dt$$

or $$g(x) = f(x) + \lambda \int_a^b R(x,t;\lambda) f(t)\,dt \qquad ...(51)$$

where the resolvent kernel $R(x,t;\lambda)$ is given by

$$R(x,t;\lambda) = \sum_{m} \frac{\phi_m(x)\phi_m(t)}{\lambda_m - \lambda} \qquad ...(52)$$

Three Important cases arise :

Case I : Unique Solution : If $\lambda \neq \lambda_m$, (48) gives well defined value of a_m for substituting in (38). Thus solution (49) exists uniquely if λ does not take on an eigenvalue.

Case II : Infinitely many solution exists : Let $\lambda = \lambda_k$, where λ_k is the k^{th} eigenvalue and also let $f_k = 0$ that is $\int_a^b f(x)\phi_k\,dx = 0$

It follows that the function $f(x)$ is orthogonal to all eigenfunction $\phi_k(x)$ belonging to the eigenvalue λ_k.

Then (46) reduces to

$C_k = 0 + \frac{\lambda}{\lambda} C_k$ or $C_k = C_k$ which is a trivial identity and hence imposes no restriction on C_k. From (48), it then follows that the coefficient a_k of $\phi_k(x)$ in (49), which formally assumes the form $\frac{0}{0}$ in truely arbitrary. Hence, we rewrite solution (49) as follows :

$$g(x) = f(x) + A\,\phi_k(x) + \sum_{m}{}' \frac{f_m}{\lambda_m - \lambda} \phi_m(x) \qquad ...(53)$$

where dash implies that we should neglect $m = k$ in the summation and A is an arbitrary constant. (53) shows that equation (39) has an infinitely many solutions.

Case III : No Solution : Let $\lambda = \lambda_k$, where λ_k is the k^{th} eigenvalue and also let $f_k \neq 0$ that is

$$\int_a^b f(x)\phi_k(x)dx \neq 0$$

i.e. eigenfunction $\phi_k(x)$ is not orthogonal to $f(x)$. Then, because of the presence of the term

$$\frac{f_k\,\phi_k(x)}{\lambda_k - \lambda} \qquad ...(54)$$

in (49), we find that no solution exists, since the term (54) is undefined.

Example 1 : Solve the symmetric integral equation.

$$g(x) = (x+1)^2 + \int_{-1}^{1}(xt + x^2t^2)g(t)dt$$

by using Hilbert - schmidt theorem.

Solution : Given integral equation is

$$g(x) = (x+1)^2 + \int_{-1}^{1}(xt + x^2t^2)g(t)dt$$

Comparing with standard Fredholm integral equation,

$$K(x,t) = xt + x^2t^2 \text{ and } a = -1,\ b = 1$$

$$f(x) = (x+1)^2, \text{ and } \lambda = 1$$

We begin with determining eigenvalues and the correspanding normalized eigenfunctions of homogenous integral equation.

$$g(x) = \lambda\int_{-1}^{1}(xt + x^2t^2)g(t)dt$$

or $$g(x) = \lambda\, x\int_{-1}^{1} tg(t)dt + \lambda x^2\int_{-1}^{1} t^2 g(t)dt$$

or $$g(x) = \lambda\, c_1 x + \lambda\, c_2 x^2, \qquad ...(55)$$

where $$c_1 = \int_{-1}^{1} tg(t)dt \qquad ...(56)$$

and $$c_2 = \int_{-1}^{1} t^2 g(t)\,dt \qquad ...(57)$$

Equations (55) and (56) gives

$$c_1 = \int_{-1}^{1} t(\lambda c_1 t + \lambda c_2 t^2)\,dt = c_1\lambda\left[\frac{t^3}{3}\right]_{-1}^{1} + c_2\lambda\left[\frac{t^4}{4}\right]_{-1}^{1}$$

or $$c_1 = \frac{2c_1\lambda}{3} + 0 \text{ or } c_1\left(1 - \frac{2\lambda}{3}\right) + 0.c_2 = 0 \qquad ...(58)$$

Similarly using (55) in (57), we set

$$0.c_1 + \left(1 - \frac{2\lambda}{5}\right)c_2 = 0 \qquad ...(59)$$

The non trivial solution of the equatin (58) and (59) can be obtained if

$$D(\lambda) = \begin{vmatrix} 1-(2\lambda/3) & 0 \\ 0 & 1-(2\lambda/5) \end{vmatrix} = 0$$

or $$\left(1 - \frac{2\lambda}{3}\right)\left(1 - \frac{2\lambda}{5}\right) = 0$$

Thus the required eigenvalues are

$$\lambda_1 = \frac{3}{2} \text{ and } \lambda_2 = \frac{5}{2}$$

Putting $\lambda = \lambda_1 = \frac{3}{2}$ in (58) and (59), we obtain

$$c_1.0 + 0.c_2 = 0 \text{ and } 0.c_1 + \frac{2}{5}c_2 = 0$$

giving $c_2 = 0$ and c_1 is arbitrary. Putting these values in (55) the required eigenfunction $g_1(x)$ is given by

$$g_1(x) = \left(\frac{3}{2}\right)c_1 x$$

Setting $(3/2)c_1 = 1$, we may take $g_1(x) = x$.

Now, the corresponding normalized eigenfunction $\phi_1(x)$ is given by

$$\phi_1(x) = \frac{g_1(x)}{\left[\int_{-1}^{1}[g_1(x)]^2 dx\right]^{1/2}} = \frac{x}{\left[\int_{-1}^{1} x^2 dx\right]^{1/2}} = \frac{x}{\left[\left[\frac{x^3}{3}\right]_{-1}^{1}\right]^{1/2}}$$

or $$\phi_1(x) = \frac{x}{\sqrt{2/3}} = x.\sqrt{\frac{3}{2}} = \frac{x\sqrt{6}}{2} \qquad ...(60)$$

Again, putting $=\lambda = \lambda_2 = \frac{5}{2}$ in (58) and (59), we obtain

$$-\frac{2}{3}c_1 + 0.c_2 = 0 \text{ and } 0.c_1 + 0.c_2 = 0,$$

giving $c_1 = 0$ and c_2 is arbitrary. Putting these values in (55) the required eigenfunction $g_2(x)$ is given by

$$g_2(x) = (5/2)c_2 x^2$$

setting $(5/2)c_2 = 1$, we may take $g_2(x) = x^2$.

Now , the correspanding normalized eigenfunction $\phi_2(x)$ is given by

$$\phi_2(x) = \frac{g_2(x)}{\left[\int_{-1}^{1}[g_2(x)]^2 dx\right]^{1/2}} = \frac{x^2}{\left[\int_{-1}^{1} x^4 dx\right]^{1/2}} = \frac{\sqrt{10}}{2}x^2 \qquad \text{....(61)}$$

Again $f_1 = \int_{-1}^{1} f(x)\phi_1(x)\,dx = \int_{-1}^{1}(x+1)^2\left(\frac{\sqrt{6}}{2}x\right)dx$ [by (60)]

$$= \frac{\sqrt{6}}{2}\int_{-1}^{1} x(x^2+2x+1)dx = \frac{\sqrt{6}}{2}\left[\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2}\right]_{-1}^{1}$$

or $\quad f_1 = 2\sqrt{6}/3$

and $\quad f_2 = \int_{-1}^{1} f(x)\phi_2(x)dx = \int_{-1}^{1}(x+1)^2\left(\frac{\sqrt{10}}{2}x^2\right)dx$ [by (61)]

$$= \frac{\sqrt{10}}{2}\left[\frac{x^5}{5} + \frac{2x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1} = \frac{8}{15}\sqrt{10}$$

Now $\lambda = 1$. Also $\lambda_1 = 3/2$ and $\lambda_2 = 5/2$

Therefore $\lambda\,\lambda_1$ and $\lambda \neq \lambda_2$

Hence the given integral equation has a unique solution as

$$g(x) = f(x) + \lambda\sum_{m=1}^{2}\frac{f_m}{\lambda_m - \lambda}\phi_m(x)$$

or $\quad g(x) = (x+1)^2 + \frac{f_1\phi_1(x)}{\lambda_1 - 1} + \frac{f_2\phi_2(x)}{\lambda_2 - 1}$

or $$g(x)=(x+1)^2+\frac{\left\{\left(\frac{2}{3}\right)\sqrt{6}\right\}\left\{\left(\frac{\sqrt{6}}{2}\right)x\right\}}{(3/2)-1}+\frac{\left\{\left(\frac{8}{15}\right)\sqrt{10}\right\}\left\{\frac{\sqrt{10}}{2}x^2\right\}}{(5/2)-1}$$

or $$g(x)=(x+1)^2+4x+(16/9)x^2=x^2+2x+1+4x+(16/9)x^2$$

or $$g(x)=(25/9)x^2+6x+1$$

Example 2 : Using Hilbert schmidt theorem, find the solution of the symmetric integral equation.

$$g(x)=x^2+1+\frac{3}{2}\int_{-1}^{1}\left(xt+x^2t^2\right)g(t)\,dt$$

Solution : Given integral equation is

$$g(x)=x^2+1+\frac{3}{2}\int_{-1}^{1}\left(xt+x^2t^2\right)g(t)\,dt$$

Comparing this integral equation with standard Fredholm integral equation

$$g(x)=f(x)+\lambda\int_{-1}^{1}\left(xt+x^2t^2\right)g(t)\,dt$$

we have

$$f(x)=x^2+1,\ \lambda=\frac{3}{2},\ a=-1,\ b=1 \text{ and } K(x,t)=xt+x^2t^2$$

We begin with determining eigenvalues and the corresponding normalized eigenfunctions of homogeneous integral equation

$$g(x)=\lambda\int_{-1}^{1}\left(xt+x^2t^2\right)g(t)\,dt$$

Now, proceeding as in Ex.1, we obtain

$$\lambda_1=3/2,\ \lambda_2=5/2,\ \phi_1(x)=\left(x\sqrt{6}\right)/2$$

$$\phi_2(x)=\left(x^2\sqrt{10}\right)/2$$

$$f_1=0 \text{ and } f_2=8\frac{\sqrt{10}}{15}$$

Here $\lambda=3/2=\lambda_1$ and $\lambda\neq\lambda_2$

Since $\lambda=\lambda_1$ and $f_1=0$, hence infinitely many solution of given integral equation exist and are given by (refer case II, Art. 11.11)

$$g(x)=f(x)+A\phi_1(x)+\lambda\sum_{m=1}^{2}{}'\frac{f_m}{\lambda_m-\lambda}\phi_m(x) \qquad ...(62)$$

where dash in the above sum means that the term with $m = 1$ must be ngelected.

$\therefore$ (62) takes the form

$$g(x) = f(x) + A\,\phi_1(x) + \lambda\,\frac{f_2}{\lambda_2 - \lambda}\,\phi_2(x)$$

or $$g(x) = x^2 + 1 + A\left(\frac{x\sqrt{6}}{2}\right) + \frac{3}{2} \times \frac{\left(8\sqrt{10}/15\right)}{\left(5/2\right) - \left(3/2\right)} \times \frac{x^2\sqrt{10}}{2}$$

or $$g(x) = x^2 + 1 + Cx + 4x^2$$

or $$g(x) = 5x^2 + Cx + 1$$

where $C = \left(A\sqrt{6}/2\right)$ is an arbitrary constant.

Example 3 : Solve the following symmetric integral equation with the help of Hilbert-Schmidt theorem,

$$g(x) = 1 + \lambda \int_0^{\pi} \cos(x + t)\, g(t)\, dt$$

Solution : Consider the corresponding homogeneous integral equation as

$$g(x) = \lambda \int_0^{\pi} \cos(x + t)\, g(t)\, dt$$

or $$g(x) = \lambda \cos x \int_0^{\pi} \cos t\, g(t)\, dt - \lambda \sin x \int_0^{\pi} \sin t\, g(t)\, dt$$

or $$g(x) = \lambda\, C_1 \cos x - \lambda\, C_2 \sin x \qquad \text{...(63)}$$

where $$C_1 = \int_0^{\pi} \cos t\, g(t)\, dt \qquad \text{...(64)}$$

and $$C_2 = \int_0^{\pi} \sin t\, g(t)\, dt \qquad \text{...(65)}$$

Using (63), (64) becomes

$$C_1 = \int_0^{\pi} \cos t \left(\lambda\, C_1 \cos t - \lambda\, C_2 \sin t\right) dt$$

or $$C_1 = \frac{\lambda\, C_1}{2} \int_0^{\pi} (1 + \cos 2t)\, dt - \frac{\lambda\, C_2}{2} \int_0^{\pi} \sin 2t\, dt$$

or $$C_1 = \frac{\lambda\, C_1}{2} \left[t + \frac{\sin 2t}{2}\right]_0^{\pi} + \frac{\lambda\, C_2}{4} \left[\cos 2t\right]_0^{\pi}$$

or $$C_1 (2 - \lambda\, \pi) + 0.C_2 = 0 \qquad \text{...(66)}$$

Again, using (63), (65), we get

$$C_2 = \int_0^{\pi} \sin t \left(\lambda C_1 \cos t - \lambda C_2 \sin t \right) dt$$

or $$C_2 = -\frac{\lambda C_1}{2} [\cos 2t]_0^{\pi} - \frac{\lambda C_2}{2} \left[t - \frac{\sin 2t}{2} \right]_0^{\pi}$$

or $$0.C_1 + (2 + \lambda \pi) C_2 = 0 \qquad ...(67)$$

Equation (66) and (67) have a non trivial solution if

$$D(\lambda) = \begin{vmatrix} 2 - \lambda \pi & 0 \\ 0 & 2 + \lambda \pi \end{vmatrix} = 0 \text{ or } (2 - \lambda \pi)(2 + \lambda \pi) = 0$$

Thus the required eigenvalues are

$$\lambda_1 = 2/\pi \text{ and } \lambda_2 = -2/\pi$$

Putting $\lambda = \lambda_1 = 2/\pi$ in (66) and (67), we obtain

$$0.C_1 + 0.C_2 = 0 \text{ and } 0.C_1 + 4C_2 = 0$$

Giving $C_2 = 0$ and C_1 is arbitrary. Putting these values in (63), the required eigenfunction $g_1(x)$ given by

$$g_1(x) = (2/\pi) C_1 \cos x$$

Setting $(2/\pi) C_1 = 1$, we may take $g_1(x) = \cos x$.

The corresponding normalized eigenfunction $\phi_1(x)$ is given by

$$\phi_1(x) = \frac{g_1(x)}{\left[\int_0^{\pi} \{g_1(x)\}^2 dx \right]^{1/2}} = \frac{\cos x}{\left[\int_0^{\pi} \cos^2 x \, dx \right]^{1/2}} = \frac{\cos x}{\left[\int_0^{\pi} \frac{1 + \cos 2x}{2} dx \right]^{1/2}}$$

or $$\phi_1(x) = \frac{\cos x}{\left\{ \left[\frac{1}{2} \left(x + \frac{\sin 2x}{2} \right) \right]_0^{\pi} \right\}^{1/2}} = \frac{\cos x}{\sqrt{(\pi/2)}} = \sqrt{\frac{2}{\pi}} \cos x \qquad ...(68)$$

Again, putting $\lambda = \lambda_2 = -2/\pi$ in (66) and (67), we get

$$4C_1 + 0.C_2 = 0 \text{ and } 0.C_1 + 0.C_2 = 0$$

giving $C_1 = 0$ and C_2 is arbitrary. Putting these values in (63), the eigenfunction $g_2(x)$ is given by

$$g_2(x) = -(-2/\pi)\, C_2 \sin x = (2/\pi)\, C_2 \sin x$$

setting $(2/\pi)\ C_2 = 1$, we can take $g_2(x) = \sin x$

and the corresponding normalized eigenfunction $\phi_2(x)$ is given by

$$\phi_2(x) = \sqrt{\frac{2}{\pi}}\ \sin x \qquad \text{...(69)}$$

Also $\quad f_1 = \int_0^{\pi} f(x)\ \phi_1(x)\ dx = \int_0^{\pi} 1.\cos x \sqrt{\frac{2}{\pi}}\ dx$

or $\quad f_1 = \sqrt{\frac{2}{\pi}}\ [\sin x]_0^{\pi} = 0 \qquad \text{...(70)}$

and $\quad f_2 = \int_0^{\pi} f(x) \phi_2(x)\ dx = \int_0^{\pi} 1.\sin x \sqrt{\frac{2}{\pi}}\ dx$

or $\quad f_2 = \sqrt{\frac{2}{\pi}}\ [-\cos x]_0^{\pi} = 2\sqrt{\frac{2}{\pi}} \qquad \text{...(71)}$

Three Cases Arise :-

Case I : Let $\lambda \neq \lambda_1$ and $\lambda \neq \lambda_2$, then the integral equation will possess unique solution given by

$$g(x) = f(x) + \lambda \sum_{m=1}^{2} \frac{f_m}{\lambda_m - \lambda}\ \phi_m(x)$$

or $\quad g(x) = 1 + \frac{\lambda}{\lambda_1 - \lambda}\ f_1\ \phi_1(x) + \frac{\lambda}{\lambda_2 - \lambda} f_2\ \phi_2(x)$

or $\quad g(x) = 1 + \left[\frac{\lambda\, \phi_1(x)}{(2/\pi) - \lambda}\right].\ 0 + \frac{\lambda}{(-2/\pi) - \lambda}\ .\ 2 \left(\frac{2}{\pi}\right)^{1/2} . \left(\frac{2}{\pi}\right)^{1/2} \sin x$

or $\quad g(x) = 1 - \frac{4\lambda \sin x}{2 + \lambda\pi} \qquad \text{...(72)}$

Case II : Let $\lambda = \lambda_2 = -2/\pi$. Since $f_2 \neq 0$, so integral equation possesses no solution.

Case III : Let $\lambda = \lambda_1 = 2/\pi$. Since $f_1 = 0$, there exists infinitely many solution given by

$$g(x) = f(x) + A\,\phi_1(x) + \lambda \sum_{m=1}^{2}{}' \frac{f_m}{\lambda_m - \lambda}\ \phi_m(x) \qquad \text{...(73)}$$

where dash implies that we should neglect $m = 1$ in the summation and A is an arbitrary constant. Accordingly (73) reduces to

$$g(x) = f(x) + A\,\phi_1(x) + \frac{\lambda}{\lambda_2 - \lambda}\ f_2\ \phi_2(x)$$

or $$g(x)=1+A\left(\frac{2}{\pi}\right)^{1/2}\cos x+\frac{\left(\frac{2}{\pi}\right)}{-\left(\frac{2}{\pi}\right)-\left(\frac{2}{\pi}\right)}2.\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{2}{\pi}\right)^{1/2}\sin x$$

or $$g(x)=1+c\cos x-\frac{2\sin x}{\pi}$$

where $c\left(=\frac{A\sqrt{2}}{\sqrt{\pi}}\right)$ is an arbitrary constant.

Example 4 : Determine the eigenvalues and the corresponding eigenfunctions of the equation

$$g(x)=f(x)+\lambda\int_0^{2\pi}\sin(x+t)\,g(t)dt \qquad ...(74)$$

where $f(x)=x$. Obtain the solution of this equation when λ is not an eigenvalue.

Solution : We begin with determining eigenvalues and the corresponding normalized eigenfunctions of homogeneous integral equation.

$$g(x)=\lambda\int_0^{2\pi}\sin(x+t)\,g(t)dt$$

Proceeding as in Example 3, we find that

or $$g(x)=\lambda c_1\sin x+\lambda c_2\sin x \qquad ...(74)$$

where $$c_1=\int_0^{2\pi}\cos t\ g(t)dt \qquad ...(75)$$

and $$c_2=\int_0^{2\pi}\sin t\ g(t)dt \qquad ...(76)$$

$$c_1-\lambda\pi c_2=0 \qquad ...(77)$$

or $$\lambda\pi c_1-c_2=0 \qquad ...(78)$$

Equations (77) and (78) have a nontrivial solution only if

$$D(\lambda)=\begin{vmatrix}1 & -\lambda\pi\\ \lambda\pi & -1\end{vmatrix}=0 \text{ or } -1+\lambda^2\pi^2=0 \text{ so that } \lambda=\frac{1}{\pi}\ or\ \frac{-1}{\pi}$$

Hence the required eigenvalues are

$\lambda_1=\frac{1}{\pi}$ and $\lambda_2=\frac{-1}{\pi}$ and corresponding eigenfunctions are

$$g_1(x)=\frac{c_1}{\pi}(\sin x+\cos x),\ g_2(x)=\frac{c_1}{\pi}(\sin x-\cos x)$$

Setting $\frac{c_1}{\pi} = 1,$ we may take $g_1(x) = \sin x + \cos x$, $g_2(x) = \sin x - \cos x$

Hence the corresponding normalized eigenfunction $\phi_1(x)$ is given by

$$\phi_1(x) = \frac{g_1(x)}{\left[\int_0^{2\pi} \{g_1(x)\}^2 dx\right]^{1/2}}$$

or $$\phi_1(x) = \frac{\sin x + \cos x}{\sqrt{2\pi}} \text{ (on simplification).} \qquad ...(79)$$

Similarly $$\phi_2(x) = \frac{\sin x - \cos x}{\sqrt{2\pi}} \qquad ...(80)$$

Also $$f_1 = \int_0^{2\pi} f(x)\phi_1(x)dx = \int_0^{2\pi} \frac{x(\sin x + \cos x)}{\sqrt{2\pi}} dx = -\sqrt{2}\,\pi$$

and $$f_2 = \int_0^{2\pi} f(x)\phi_2(x)dx = \int_0^{2\pi} \frac{x(\sin x - \cos x)}{\sqrt{2\pi}} dx = -\sqrt{2}\,\pi$$

Given that $\lambda \neq \lambda_1$ and $\lambda \neq \lambda_2$. Hence (74) will possess unique solution given by

$$g(x) = f(x) + \lambda \sum_{m=1}^{2} \frac{f_m}{\sqrt{\lambda_m - \lambda}} \phi_m(x)$$

or $$g(x) = x + \frac{\lambda f_1 \phi_1(x)}{\lambda_1 - \lambda} + \frac{\lambda f_2 \phi_2(x)}{\lambda_2 - \lambda}$$

or $$g(x) = x + \frac{\lambda(-\sqrt{2\pi})(\sin x + \cos x)}{\{(1/\pi) - \lambda\}\sqrt{2\pi}} + \frac{\lambda(-\sqrt{2\pi})(\sin x - \cos x)}{\{-(1/\pi) - \lambda\}\sqrt{2\pi}}$$

or $$g(x) = x - \lambda\pi \sin x\left(\frac{1}{1-\lambda\pi} - \frac{1}{1+\lambda\pi}\right) - \lambda\pi \cos x\left(\frac{1}{1-\lambda\pi} + \frac{1}{1+\lambda\pi}\right)$$

or $$g(x) = x - \frac{2\lambda^2\pi^2 \sin x}{1-\lambda^2\pi^2} - \frac{2\lambda\pi \cos x}{1-\lambda^2\pi^2}$$

Example 5 : Using Hilbert-Schmidt method, solve the integral equation

$$g(x) = x + \lambda \int_0^1 K(x,t)\, g(t)dt \qquad ...(81)$$

where $$K(x,t) = \begin{cases} x(t-1), & 0 \le x \le t \\ t(x-1), & t \le x \le 1 \end{cases} \qquad ....(82)$$

Solution : Consider the corresponding homogeneous integral equation as

$$g(x)=\lambda\int_0^1 K(x,t)\,g(t)\,dt$$

or $$g(x)=\lambda\left[\int_0^x K(x,t)\,g(t)\,dt+\int_x^1 K(x,t)\,g(t)\,dt\right]$$

or $$g(x)=\lambda\int_0^x \lambda t(x-1)g(t)dt+\int_x^1 \lambda x(t-1)\,g(t)dt \qquad \text{(using (82))}$$

Differentiating both sides w.r.t. 'x' and using Leibnitz rule, we have

or $$g'(x)=\int_0^x \lambda t g(t)dt+\int_x^1 \lambda(t-1)\,g(t)dt$$

Differentiating both sides again w.r.t. 'x' and using Leibnitz rule, we have

$$g''(x)=\lambda g(x) \text{ or } g''(x)-\lambda g(x)=0 \qquad ...(83)$$

Also, we have $g(0)=0, g(1)=0$ ...(84)

The required function $g(x)$ is a solution of the differential equation (83) together with boundary conditions (84).

Three Cases Arise :

Case I : Let $\lambda=0$. Then (83) takes the form $g''(x)=0$, whose general solution is

$$g(x)=Ax+B \qquad ...(85)$$

From (84) and (85), we have

$$B=0 \text{ and } A+B=0\Rightarrow A=0$$

Thus we have $g(x)=0$, which is not an eigenfunction.

Hence $\lambda=0$ is not an eigenvalue.

Case II : Let $\lambda=\mu^2$, where $\mu\neq 0$. Then equation (83) reduces to $g''(x)-\mu^2 g(x)=0$, whose general solution is

$$g(x)=A\,e^{\mu x}+B\,e^{-\mu x} \qquad ...(86)$$

From (84) and (86) we have

$$A+B=0 \text{ and } A\,e^{\mu}+B\,e^{-\mu}=0$$

or $$A=B=0$$

Thus we have $g(x)=0$, which is not an eigenfunction.

Case III : Let $\lambda = -\mu^2$, where $\mu \neq 0$. Then equation (83) reduces to $g''(x) + \mu^2 g(x) = 0$, whose general solution is

$$g(x) = A\cos\mu x + B\sin\mu x \qquad \text{...(87)}$$

From the boundary condition (84), we have

$$A = 0 \text{ and } 0 = A\cos\mu + B\sin\mu$$

or $$B\sin\mu = 0 \qquad \text{...(88)}$$

Now, we must take $B \neq 0$, otherwise $A = 0$ and $B = 0$ will give $g(x) = 0$ as before and hence we shall not get eigenfunction. Since $B \neq 0$ (88) reduces to

$$\sin\mu = 0$$

$\therefore$ $\mu = n\pi$ where n is any integer.

or $$\lambda = -\mu^2 = -n^2\pi^2, \quad n = 1,2,3,.......$$

Hence the required eigenvalues λ_n are given by

$$\lambda_n = -n^2\pi^2, \; n = 1,2,3,....... \qquad \text{...(89)}$$

Putting $A = 0$ and $\mu = n\pi$ in (87), we obtain

$$g(x) = B\sin n\pi\mu$$

Let $B = 1$, the required eigenfunctions are

$$g_n(x) = \sin n\pi\mu, \; n = 1,2,3,........ \qquad \text{...(90)}$$

The normalized eigenfunctions $\phi_n(x)$ are given by

$$\phi_n(x) = \frac{g_n(x)}{\left[\int_0^1 \{g_n(x)\}^2 dx\right]^{1/2}} = \frac{\sin n\pi x}{\left[\int_0^1 \sin^2 n\pi x\, dx\right]^{1/2}}$$

$$= \frac{\sin n\pi x}{\left[\int_0^1 \frac{1-\cos 2n\pi x}{2} dx\right]^{1/2}} = \frac{\sin n\pi x}{\left\{\left[\frac{1}{2}\left(x - \frac{\sin 2n\pi x}{2n\pi}\right)\right]_0^1\right\}^{1/2}}$$

or $$\phi_n(x) = \frac{\sin n\pi x}{1/\sqrt{2}} = \sqrt{2}\sin n\pi x \qquad \text{...(91)}$$

Now $f_n = \int_0^1 f(x)\phi_n(x)\, dx = \int_0^1 (x).\left(\sqrt{2}\sin n\pi x\right) dx$

$$= \sqrt{2}\left\{\left[x\left(\frac{-\cos n\pi x}{n\pi}\right)\right]_0^1 - \int_0^1\left(\frac{-\cos n\pi x}{n\pi}\right) dx\right\}$$

$$= \sqrt{2}\left\{\frac{-\cos n\pi}{n\pi} + \frac{1}{n\pi}\int_0^1 \cos n\pi x\, dx\right\}$$

$$= \sqrt{2}\left\{-\frac{(-1)^n}{n\pi} + \frac{1}{n^2\pi^2}\left[\sin n\pi x\right]_0^1\right\}$$

or $$f_n = \frac{(-1)^{n+1}\sqrt{2}}{n\pi} \qquad ...(92)$$

Now two cases arise :

Case (i) : Let λ be not an eigenvalue, that is $\lambda \neq \lambda_n$, $n = 1,2,3,......$ Then the given integral equation will possess unique solution given by

$$g(x) = f(x) + \lambda \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n - \lambda}\phi_n(x)$$

or $$g(x) = x + \lambda \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{2}\sqrt{2}\sin n\pi x}{n\pi\left(-n^2\pi^2 - \lambda\right)}$$

or $$g(x) = x + \frac{2\lambda}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n \sin n\pi x}{n\left(n^2\pi^2 + \lambda\right)}$$

Case (ii) : Let $\lambda = \lambda_n = -n^2\pi^2$, $n = 1,2,3,...$ Then from (92), $f_n \neq 0$ for $n = 1,2,3,...$ Hence the given integral equation will possess no solution.

Self-Learning Exercise :

1. Define the following :

   (i) Orthogonal functions

   (ii) Schwarz inequality

   (iii) Orthonormal set

   (iv) L_2 -function

   (v) Hilbert-Schmidt Theorem

   (vi) Norm of a function.

2. State whether the following statements are true or false :

(i) A function $g(x)$ is said to be normalized if $\left\|g(x)\right\| = 0$

(ii) A kernel $K(x,t)$ is said to be symmetric if $K(x,t) = \overline{K}(t,x)$.

(iii) If a kernel is symmetric then all its iterated kernels are also symmetric.

(iv) The eigenvalues of a symmetric kernel are real.

(v) The eigenfunctions of a symmetric kernel, corresponding to different eigenvalues are orthogonal.

(vi) A non zero function, with non zero norm can always be normalized by dividing it by its norm.

(vii) A function $f(x)$ is said to be square integrable if

$$\int_a^b |f(x)|^2 dx < \infty$$

11.12 Summary

In this unit you have studied eigenvalues and eigenfunctions of Fredholm equations for symmetric kernels. Properties of symmetric kernels have been discussed. Schmidt's solution of the non homogeneous Fredholm integral equation of second kind have been studied.

11.13 Answers to Self-Learning Exercise

1. See Text

2.

(i)	False	(ii)	True	(iii)	True	(iv)	True
(v)	True	(vi)	True	(vii)	True		

11.14 Exercise

1. State and prove Hilbert-Schmidt Theorem.

2. Discuss the solution of the integral equation

$$g(x) = f(x) + \lambda \int_a^b K(x,t)\, g(t)\, dt$$

by schmidt's method and mention the nature of the kernel also discuss the cases of unique solution, no solution and infinitely many solution of the integral equation.

3. Using Hilbert-Schmidt theorem, solve the following symmetric integral equations :

(i) $g(x) = x + \lambda \int_0^{2\pi} \sin(x+t)\, g(t)\, dt$

(ii) $g(x) = \cos 3x + \lambda \int_0^{\pi} \cos(x+t)\, g(t)\, dt$

[Ans. (i) $$g(x) = x - \frac{2\lambda\pi\sqrt{\pi}}{1-\lambda^2\pi^2}(\lambda\pi\sin x + \cos x)$$

(ii) $$g(x) = 1 - \frac{4\lambda\sin x}{2+\lambda\pi} \text{ (unique solution)}$$

$$g(x) = 1 + c\cos x - \frac{2}{\pi}\sin x \text{ (infinitely many solution)]}$$

4. Using Hilbert-Schmidt theorem, solve the integral equation

$$g(x) = \cos\pi x = \lambda\int_0^1 K(x,t)\,g(t)\,dt,$$

where $K(x,t) = \begin{cases} (\lambda+1)t, & 0 \le x \le t \\ (t+1)x, & t \le x \le 1 \end{cases}$

[Ans.

$$g(x) = \cos\pi x + \lambda\left[\frac{1+e}{\pi^2+1}\cdot\frac{e^x}{\lambda-1} - \frac{\pi(\sin\pi x + \pi\cos\pi x)}{2(\lambda+\pi^2)} + A(\sin n\pi x + n\pi\cos n\pi x)\right]$$

Unit - 12
Classical Fredholm Theory

Structure of the Unit

12.0 Objective

In the theory of integral equations, the well known theorem of linear algebra which is related to solution of the system of linear algebraic equations play a leading role. In this unit, we shall discuss the solution of the non-homogenaeas Fredholm integral equation of second kind by replacing the integral, appearing in the equation, with a sum which reduces the equation to a system of linear equations and assuming the number of terms of the sum tends to infinity.

12.1 Introduction

In Unit 10, we have derived the solution of the Fredholm integral equation of second kind

$$g(x) = f(x) + \lambda \int_a^b K(x,t)\, g(t)\, dt \qquad ...(1)$$

as a uniformly convergent power series in the parameter λ for suitable small value of $|\lambda|$. Fredholm in fact obtained the solution of integral equation (1) in general form which is valid for all values of the parameter λ. His solutions are contained in three theorems, which are known as Fredholm's first, second and third fundamental theorems.

In this Unit, we shall study equation (1) when the functions $f(x)$ and the Kernel $K(x,t)$ are any integrable functions. Furthermore, the present method enables us to get explicit formulas for the solution in terms of certain determinants. There are three Fredholm theorems out of which we give details of Fredholm's first theorem and illustrates it with the help of solved examples. The second and third theorems can be found in any standard text books on Integrals equations.

12.2 Fredholm's First Theorem

Statement : The non-homogeneous Fredholm integrals equation of the second kind

$$g(x) = f(x) + \lambda \int_a^b K(x,t) g(t)\, dt \qquad ...(2)$$

where the function $f(x)$ and $K(x,t)$ are integrable, has a unique solution

$$g(x) = f(x) + \lambda \int_a^b R(x,t;\lambda) f(t)\, dt \qquad ...(3)$$

where the resolvent Kernel $R(x,t;\lambda)$ is a meromorphic function of the parameter λ defined by

$$R(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)}, \qquad D(\lambda) \neq 0 \qquad ...(4)$$

$D(x,t;\lambda)$ adn $D(\lambda)$ are entire functions of parameter λ defined by Fredholm series of the form

$$D(x,t;\lambda) = K(x,t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \int \int K\begin{pmatrix} z, & z_1, \ldots\ldots z_m \\ t, & z_1, \ldots\ldots z_m \end{pmatrix} dz_1 \ldots\ldots dz_m \qquad ...(5)$$

and $$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \int \ldots\ldots \int K\begin{pmatrix} z_1, \ldots\ldots z_m \\ z_1, \ldots\ldots z_m \end{pmatrix} dz_1 \ldots\ldots dz_m \qquad ...(6)$$

both of which converges for all values of λ. In particular, the solution of homogeneous integral equation is identically zero.

Also note that

$$\begin{vmatrix} K(x_1,t_1) & K(x_1,t_2) & \ldots & K(x_1,t_n) \\ K(x_2,t_1) & K(x_2,t_2) & \ldots & K(x_2,t_n) \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ K(x_n,t_1) & K(x_n,t_2) & \ldots & K(x_n,t_n) \end{vmatrix} = K\begin{pmatrix} x_1, x_2, \ldots, x_n \\ t_1, t_2, \ldots, t_n \end{pmatrix} \qquad ...(7)$$

is known the Fredholm determinant.

Proof : We divide the internal (a,b) into n equal parts,

$$x_1 = t_1 = a,\ x_2 = t_2 = a+h, \ldots, x_n = t_n = a + (n-1)h \qquad ...(8)$$

where $h = (b-a)/n$. Thus, we get the approximate formula,

$$\int_a^b K(x,t) g(t)\, dt = h \sum_{j=1}^{n} K(x, x_j) g(x_j) \qquad ...(9)$$

Hence (2) reduces to

$$g(x) = f(x) + \lambda h \sum_{j=1}^{n} K(x, x_j) g(x_j) \qquad ...(10)$$

which holds for all values of x in the internal (a,b). In particular it must be satisfied at the n points of division $x_i (i = 1,2,.......n)$. We thus, obtain following system of equations

$$g(x_i) = f(x_i) + \lambda h \sum_{j=1}^{n} K(x_i, x_j) g(x_i), (i = 1,2,.....n) \qquad ...(11)$$

Let us introduce the following symbls;

$$g(x_i) = g_i, \; f(x_i) = f_i, \; K(x_i, x_j) = K_{ij} \qquad ...(12)$$

Then (11) gives an approximation for (2) in terms of the system of n linear equations with n unknowns $g_1, g_2, ...g_n$ as

$$g_i - \lambda h \sum_{j=1}^{n} K_{ij} \, g_j = f_i \; (i,1,2,......n) \qquad ...(13)$$

Re-writing (13), we have

$$\left.\begin{array}{l} (1-\lambda h K_{11}) g_1 - \lambda h K_{12} g_2 - \quad ... \; - \lambda h K_{1n} \, g_n = f_1 \\ -\lambda h K_{21} \, g_1 + (1-\lambda h K_{22}) \, g_2 - \quad ... \; - \lambda h K_{2n} \, g_n = f_2 \\ \quad . \qquad\qquad . \qquad\qquad . \\ \quad . \qquad\qquad . \qquad\qquad . \\ \quad . \qquad\qquad . \qquad\qquad . \\ -\lambda h K_{n1} \, g_1 - \lambda h K_{n2} \, g_2 - \quad ... \; + (1-\lambda h K_{nn}) g_n = f_n \end{array}\right] \qquad ...(14)$$

The solutions $g_1, g_2 ....g_n$ obtained by solving the algebraic system of equations (14) may be expressed in the form of the ratios of certain determinants, with the resolvent determinant $D_n(\lambda)$ of the algebraic system (14), where

$$D_n(\lambda) = \begin{vmatrix} 1-\lambda h K_{11} & -\lambda h K_{12} .... -\lambda h K_{1n} \\ -\lambda h K_{21} & 1-\lambda h K_{22} .... -\lambda h K_{2n} \\ . & . \qquad\qquad . \\ . & . \qquad\qquad . \\ . & . \qquad\qquad . \\ -\lambda h K_{n1} & -\lambda h K_{n2} .... 1-\lambda h K_{nn} \end{vmatrix} \qquad ...(15)$$

provided that $D_n(\lambda) \neq 0$

The approximate eigenvalues can be obtained by setting this determinant to zero. Now expanding $D_n(\lambda)$ in powers of the quantity $(-\lambda h)$, we find that the first term not containing this factor is equal to unity. The term containing $(-\lambda h)$ in the first power is the sum of all the determinants containing only one column $-\lambda h K_{rs}, r = 1,.....n$. Considering the contribution from all the colomns $s = 1,.......n$. we find that the total contribution is $-\lambda h \sum_{s=1}^{n} K_{ss}$.

The term containing the factor $(-\lambda h)^2$ is the sum of all the determinants having two columns with that factor. This gives rise to the determinants of the form

$$(-\lambda h)^2 \begin{vmatrix} K_{pp} & K_{pq} \\ K_{qp} & K_{qq} \end{vmatrix}$$

where (p,q) is an arbitrary pair of integers taken from the sequence 1,......n with $p < q$.

Similarly, the term containing the factor $(-\lambda h)^3$ is the sum of the determinants of the form

$$(-\lambda h)^3 \begin{vmatrix} K_{pp} & K_{pq} & K_{pr} \\ K_{qp} & K_{qq} & K_{qr} \\ K_{rp} & K_{rr} & K_{rr} \end{vmatrix}$$

where (p,q,r) is an arbitrary triplet of integers selected from the sequence $1,2,.......n$ with $p < q < r$.

Proceeding likwise we may obtain the remaining terms in the explansion of $D_n(\lambda)$. Thus the determinant (15) may be expressed in the form

$$D_n(\lambda) = 1 - \lambda h \sum_{s=1}^{n} K_{ss} + \frac{(-\lambda h)^2}{2!} \sum_{p,q=1}^{n} \begin{vmatrix} K_{pp} & K_{pq} \\ K_{qp} & K_{qq} \end{vmatrix}$$

$$+ \frac{(-\lambda h)^3}{3!} \sum_{p,q,r=1}^{n} \begin{vmatrix} K_{pp} & K_{pq} & K_{pr} \\ K_{qp} & K_{qq} & K_{qr} \\ K_{rp} & K_{rq} & K_{rr} \end{vmatrix} + ... + \frac{(-\lambda h)^n}{n!} \sum_{p_1,p_2,...,p_n=1}^{n} \begin{vmatrix} K_{p_1p_1} & K_{p_1p_2} \ldots K_{p_1p_n} \\ K_{p_2p_1} & K_{p_2p_2} \ldots K_{p_2p_n} \\ . & . & . \\ . & . & . \\ . & . & . \\ K_{p_np_1} & K_{p_np_2} & K_{p_np_n} \end{vmatrix}$$

$$= 1 - \lambda h \sum_{m=1}^{n} K\left(x_p, x_p\right) + \frac{(-\lambda h)^2}{2!} \sum_{p,q=1}^{n} K\begin{pmatrix} x_p, x_q \\ x_p, x_q \end{pmatrix} + \ldots \quad \text{(by (7))} \qquad \ldots(16)$$

Since $\underset{n\to\infty}{Lim}\ h = \underset{n\to\infty}{Lim} \frac{(b-a)}{n} = 0$ and each term of the sum (16) tends to some single, double, triple integral etc, therefore we have

$$D(\lambda) = 1 - \lambda \int_a^b K(x,x)\, dx + \frac{\lambda^2}{2!} \int_a^b \int_a^b K\begin{pmatrix} x_1, x_2 \\ x_1, x_2 \end{pmatrix} dx_1\, dx_2$$

$$-\frac{\lambda^3}{3!} \int_a^b \int_a^b \int_a^b K\begin{pmatrix} x_1, x_2, x_3 \\ x_1, x_2, x_3 \end{pmatrix} dx_1\, dx_2\, dx_3 + \ldots. \qquad \ldots(17)$$

where $D(\lambda)$ is called the Fredholm's determinant and the series occuring on RHS of (17) is called the Fredholm's first series.

It may be noted that Hilbert gave a rigorous proof of the fact that the sequence $D_n(\lambda) \to D(\lambda)$ in the limit and Fredholm proved the convergence of the series (17) for all values of λ by using the fact that the kernel $K(x,t)$ is bounded and integrable function. Thus, $D(\lambda)$ is an entire function of the complex parameter λ.

If $R(x,t;\lambda)$ be the resolvent kernel, then we wish to find the solution of (2) in the form given by (3) where we expect $R(x,t;\lambda)$ to be the quotient

$$R(x,t;\lambda) = D(x,t;\lambda)/D(\lambda),$$

where $D(x,t;\lambda)$ is the sum of certain functional series and is yet to be determined. We know that the resolvent kernel $R(x,t;\lambda)$ satisfies the following relations;

$$R(x,t;\lambda) = K(x,t) + \lambda \int_a^b K(x,z)\, R(z,t;\lambda) dz \qquad \ldots(18)$$

From (4) and (18), it follows that

$$\frac{D(x,t;\lambda)}{D(\lambda)} = K(x,t) + \lambda \int_a^b K(x,z) \frac{D(z,t;\lambda)}{D(\lambda)} dz, \qquad \{D(\lambda) \neq 0\}$$

$$D(x,t;\lambda) = K(x,t)\, D(\lambda) + \lambda \int_a^b K(x,z)\, D(z,t;\lambda) dz \qquad \ldots(19)$$

The form of the series (17) for $D(\lambda)$ suggests that we seek the solution of equation (19) in the form of a power series in the parameter λ i.e.

$$D(x,t;\lambda) = B_0(x,t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x,t) \qquad ...(20)$$

For this purpose, rewriting the series (17) as

$$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \mu_m \qquad ...(21)$$

where $\mu_m = \int_a^b ... \int_a^b K\begin{pmatrix} x_1, x_2, ..., x_m \\ x_1, x_2, ..., x_m \end{pmatrix} dx_1 ... dx_m$ ...(22)

Now, substituting the series for $D(x,t;\lambda)$ and $D(\lambda)$ from (20) and (21) in (19) and comparing the coefficients of equal powers of λ, we obtain the following recursion relations :

$$B_0(x,t) = K(x,t) \qquad ...(23)$$

and $B_m(x,t) = \mu_m K(x,t) - m \int_a^b K(x,z)\, B_{m-1}(z,t)\, dz$ ...(24)

Now, we shall prove that for each $m (m = 1,2,3,....)$,

$$B_m(x,t) = \int_a^b .... \int_a^b K\begin{pmatrix} x, z_1, z_2, ..., z_m \\ t, z_1, z_2, ..., z_m \end{pmatrix} dz_1 .... dz_m \qquad ...(25)$$

First, observe that for $m = 1$, (24) takes the form

$$B_1(x,t) = \mu_1 K(x,t) - \int_a^b K(x,z)\, B_0(z,t) dz$$

$$= K(x,t) \int_a^b K(z,z)\, dz - \int_a^b K(x,z)\, B_0(z,t) dz$$

$$= \int_a^b K\begin{pmatrix} x, z \\ t, z \end{pmatrix} dz \qquad ...(26)$$

showing that (25) holds for $m = 1$

To prove that (25) holds for general m, we expend the determinant under the integral sign by the relation :

$$K\begin{pmatrix} x, z_1, z_2, ..., z_m \\ t, z_1, z_2, ..., z_m \end{pmatrix} = \begin{vmatrix} K(x,t) & K(x_1,z_1) & ... K(x_1,z_m) \\ K(z,t) & K(z_1,z_1) & ... K(z_1,z_m) \\ .... & .... & .... \\ .... & .... & .... \\ K(z_m,t) & K(z_m,z_1) & ... K(x_m,z_m) \end{vmatrix} \qquad ...(27)$$

with respect to the elements of the given row, transposing in turn the first column one place to the right, integrating both sides and using (22), proof of (25) follows by mathematical induction.

From (21), (23) and (25), we arrive at the so called Fredholm's second series :

$$D(x,t;\lambda) = K(x,t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \int_a^b \int_a^b K\begin{pmatrix} x, z_1, ...z_m \\ t, z_1, ...z_m \end{pmatrix} dz_1, ...dz_m \qquad ...(28)$$

The series (28) converges for all values of λ. In the end we show that the solution in the form obtained by Fredholm is unique and is given by (3). Before doing this, we find that the integral equation (18) satisfied by $R(x,t;\lambda)$ is valid for all values of λ for which $D(\lambda) \neq 0$. From Unit10, we already know that (18) holds for $|\lambda| < B^{-1}$, where

$$B = \left[\int_a^b \int_a^b |K(x,t)\ dx\ dt| \right]^{1/2}$$

Since both sides of (18) are thus found to be mermorphic, the result follows.

To establish the uniqueness of the solution of (2), we assume that $g(x)$ is a solution of (2), provided that $D(\lambda) \neq 0$. Rewriting (2) as

$$g(z) = f(z) + \lambda \int_a^b K(z,t)\ g(t)\ dt \qquad ...(29)$$

Multiplying both sidess of (29) by $R(x,z;\lambda)$ and then integrating both sides with respect to 'z' from a to b, we get

$$\int_a^b R(x,z;\lambda)\ g(z)\ dz = \int_a^b R(x,z;\lambda)\ f(z)\ dz + \lambda \int_a^b \left[\int_a^b R(x,z;\lambda)\ K(z,t)\ dz \right] g(t)\ dt \qquad ...(30)$$

Using (18), we have

$$R(x,t;\lambda) = K(x,t) + \lambda \int_a^b R(x,z;\lambda)\ K(z,t)\ dz$$

or $$\lambda \int_a^b R(x,z;\lambda)\ K(z,t)\ dz = R(x,t;\lambda) - K(x,t) \qquad ...(31)$$

From (30) and (31), we have

$$\int_a^b R(x,z;\lambda)\ g(z)\ dz = \int_a^b R(x,z;\lambda)\ f(z)\ dz + \int_a^b \left[R(x,t;\lambda) - K(x,t) \right] g(t)\ dt$$

or $$\int_a^b R(x,t;\lambda)\ g(t)\ dt = \int_a^b R(x,t;\lambda)\ f(t)\ dt$$

$$+ \int_a^b R(x,t;\lambda) g(t)\ dt - \int_a^b K(x,t)\ g(t)\ dt$$

or $$\int_a^b K(x,t)\, g(t)\, dt = \int_a^b R(x,t;\lambda)\, f(t)\, dt \qquad ...(32)$$

From (2), we have

$$\int_a^b K(x,t)\, g(t)\, dt = \frac{g(x) - f(x)}{\lambda} \qquad ...(33)$$

From (32) and (33), we have

$$\frac{g(x) - f(x)}{\lambda} = \int_a^b R(x,t;\lambda)\, f(t)\, dt$$

or $$g(x) = f(x) + \lambda \int_a^b R(x,t;\lambda)\, f(t)\, dt$$

but this form is unique.

12.3 Working Rule for Calculating the Resolvent Kernel and Solution of Fredholm Integral Equation of Second Kind by Using Fredholm First Theorem

For the given Fredholm integral equation

$$g(x) = f(x) + \lambda \int_a^b K(x,t) g(t) dt\,, \qquad ...(34)$$

the resolvent Kernel is given by

$$R(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)}\,, \qquad ...(35)$$

provided that $D(\lambda) \neq 0$

Here $$D(x,t;\lambda) = K(x,t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!}\, B_m(x,t) \qquad ...(36)$$

and $$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!}\, \mu_m \qquad ...(37)$$

where coefficients are given by

$$B_n(x,t) = \int_a^b \cdots\cdots \int_a^b \begin{vmatrix} K(x,t) & K(x,z_1) & \ldots & K(x,z_n) \\ K(z_1,t) & K(z_1,z_1) & \ldots & K(z_1,z_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ K(z_n,t) & K(z_n,z_1) & \ldots & K(z_n,z_n) \end{vmatrix}\, dz_1 \ldots dz_n \qquad ...(38)$$

$$B_0(x,t) = K(x,t) \quad ...(39)$$

and $$\mu_n = \int_a^b \cdots \int_a^b \begin{vmatrix} K(z_1,z_1) & K(z_1,z_2) & \ldots K(z_1,z_n) \\ K(z_2,z_1) & K(z_2,z_2) & \ldots K(z_2,z_1) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ K(z_n,z_1) & K(z_n,z_2) & \ldots K(z_n,z_n) \end{vmatrix} dz_1 \ldots dz_n \quad ...(40)$$

The function $D(x,t;\lambda)$ is called the Fredholm minor and $D(\lambda)$ is called the Fredholm determinant.

After getting $R(x,t;\lambda)$ the unique and continuous solution of (2) is given by

$$g(x) = f(x) + \lambda \int_a^b R(x,t;\lambda)\, f(t)\, dt \quad ...(41)$$

Alternative Method for Calculating $B_m(x,t)$ and μ_m

We have $\mu_0 = 1$, $\mu_n = \int_a^b B_{n-1}(s,s)\, ds$ ...(42)

and $$B_n(x,t) = \mu_n K(x,t) - n \int_a^b K(x,z)\, B_{n-1}(z,t)\, dz, \; m \geq 1 \quad ...(43)$$

Since $\mu_0 = 1$ and $B_0(x,t) = K(x,t)$, we can use formulas (42) and (43) to find in succession $\mu_1, B_1(x,t)$; $\mu_2, B_2(x,t)$ and so on. Continuiting in this way, all the coefficient can be calculated.

Example 1 : For the integral equation

$$g(x) = f(x) + \lambda \int_a^b K(x,t)\, g(t) dt$$

find $D(\lambda)$ and $D(x,t;\lambda)$ for the kernel.

$$K(x,t) = \sin x \; ; \; a = 0, \; b = \pi$$

Solution : Here $K(x,t) = \sin x$, $a = 0$, $b = \pi$

Now, we know that

$$D(x,t;\lambda) = K(x,t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x,t) \quad ...(44)$$

and $$D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \mu_m \quad ...(45)$$

$$\mu_0 = 1$$

$$B_0(x,t) = K(x,t) = \sin x$$

and $\quad \mu_p = \int_0^1 B_{p-1}(s,s)\,ds\,,\ p > 1 \qquad ...(46)$

and $\quad B_p(x,t) = \mu_p K(x,t) - p\int_0^1 K(x,z)\ B_{p-1}(z,t)\ dz,\ p \geq 1 \qquad ...(47)$

Putting $p = 1$, in (46), we obtain

$$\mu_1 = \int_0^{\pi} B_0(s,s)\,ds = \int_0^{\pi} \sin s\,ds = 2$$

Putting $p = 1$ in (47), we obtain

$$B_1(x,t) = \mu_1 K(x,t) - \int_0^{\pi} K(x,z)\ B_0(z,t)\ dz$$

$$= 2\sin x - \int_0^{\pi} \sin x \sin z\ dz$$

$$= 2\sin x - \sin x\,[-\cos z]_0^{\pi}$$

$$= 2\sin x - 2\sin x = 0$$

Since $B_1(x,t) = 0$, therefore we have

$$B_p(x,t) = 0 \quad \text{and} \quad \mu_p = 0 \ \text{ for all } p \geq 2$$

Substituting the above value in (44) and (45), we get

$$D(x,t;\lambda) = K(x,t) = \sin x, \qquad D(\lambda) = 1 - \lambda\,\mu_1 = 1 - 2\lambda$$

Example 2 : Using the recurrence relations, find the resolvent kernels of the following kernels :

(i) $\quad K(x,t) = \sin x \cos t;\ 0 \leq x \leq 2\pi,\ 0 \leq t \leq 2\pi$

(ii) $\quad K(x,t) = 4\,xt - x^2;\ 0 \leq x \leq 1,\ 0 \leq t \leq 1$

Solution : (i) Here $K(x,t) = \sin x \cos t$

The resolvent kernel $R(x,t;\lambda)$ is given by

$$R(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)}, \qquad ...(48)$$

where $D(x,t;\lambda)$ and $D(\lambda)$ are given by (44) and (45) respectively. Also we have

$$B_0(x,t) = K(x,t) = \sin x \cos t \quad \text{...(49)}$$

$$\mu_0 = 1 \text{ and } \mu_p = \int_0^{2\pi} B_{p-1}(s,s)\,ds,\ p \geq 1 \quad \text{...(50)}$$

and $$B_p(x,t) = \mu_p K(x,t) - p\int_0^{2\pi} K(x,z)\,B_{p-1}(z,t)\,dz,\ p \geq 1 \quad \text{...(51)}$$

Putting $p=1$ in (50), we have

$$\mu_1 = \int_0^{2\pi} B_0(s,s)\,ds = \int_0^{2\pi} \sin s \cos s\,ds$$

$$= \frac{1}{2}\int_0^{2\pi} \sin 2s\,ds = \frac{1}{2}\left[\frac{-\cos 2s}{2}\right]_0^{2\pi} = 0$$

Putting $p=1$ in (51), we obtain

$$B_1(x,t) = \mu_1 K(x,t) - \int_0^{2\pi} K(x,z)\,B_0(z,t)\,dz$$

$$= -\int_0^{2\pi} (\sin x \cos z)(\sin z \cos t)\,dz$$

$$= -\sin x \cos t \int_0^{2\pi} \sin z \cos z\,dz = 0$$

Since $B_1(x,t) = 0$, therefore

$$B_p(x,t) = 0 \text{ and } \mu_p = 0 \text{ for all } p \geq 2$$

Substituting these values in (44) and (45), we have

$$D(x,t;\lambda) = K(x,t) = \sin x \cos t \quad \text{and} \quad D(\lambda) = 1.$$

Hence $R(x,t;\lambda) = \sin x \cos t$

(ii) Here $K(x,t) = 4xt - x^2$

We have $B_0(x,t) = K(x,t) = 4xt - x^2$

$$\mu_0 = 1 \text{ and } \mu_p = \int_0^{1} B_{p-1}(s,s)\,ds,\ p \geq 1 \quad \text{...(52)}$$

and $$B_p(x,t) = \mu_p K(x,t) - p\int_0^{1} K(x,z)\,B_{p-1}(z,t)\,dz,\ p \geq 1 \quad \text{...(53)}$$

Putting $p=1$ in (52), we have

$$\mu_1 = \int_0^1 B_0(s,s)\,ds = \int_0^1 (4s^2 - s^2)\,ds = \left[s^3\right]_0^1 = 1$$

Putting $p=1$ in (53), we obtain

$$B_1(x,t)=\mu_1 K(x,t)-\int_0^1 K(x,z)\ B_0(z,t)\,dz$$

$$=4\,xt-x^2-\int_0^1 (4\,xz-x^2)\ (4zt-z^2)\,dz$$

$$=4\,xt-x^2-\int_0^1\left[-4xz^3+z^2(x^2+16xt)-4x^2tz\right]\,dz$$

$$=4\,xt-x^2-\left[-xz^4+\frac{z^3}{3}(x^2+16xt)-2x^2t\,z^2\right]_0^1$$

$$=4\,xt-x^2-\left[-x+\frac{1}{3}(x^2+16xt)-2x^2t\right]$$

$$=2x^2t-\frac{4}{3}x^2+x-\frac{4}{3}xt$$

Again, putting $p=2$ in (52), we obtain

$$\mu_2=\int_0^1\ B_1(s,s)\,ds=\int_0^1\ \left(2s^3-\frac{4}{3}s^2+s-\frac{4}{3}s^2\right)ds$$

$$=\left[\frac{s^4}{2}-\frac{4}{9}s^3+\frac{s^2}{2}-\frac{4s^3}{9}\right]_0^1=\frac{1}{9}$$

Next, putting $p=2$ in (53), we obtain

$$B_2(x,t)=\mu_2\ K(x,t)-2\int_0^1\ K(x,z)\ B_1(z,t)\,dz$$

$$=\frac{1}{9}(4xt-x^2)-2\ \int_0^1\ (4xz-x^2)\left[2z^2t-\frac{4}{3}z^2+z-\frac{4}{3}zt\right]\,dz$$

$$=\frac{1}{9}(4xt-x^2)-2\ \int_0^1\ (4xz-x^2)\left[z^2\left(2t-\frac{4}{3}\right)+z\left(1-\frac{4t}{3}\right)\right]\,dz$$

$$=\frac{1}{9}(4xt-x^2)$$

$$-2\ \int_0^1\left[4x\left(2t-\frac{4}{3}\right)z^3+z^2\left\{4x\left(1-\frac{4t}{3}\right)-x^2\left(2t-\frac{4}{3}\right)\right\}-x^2\left(1-\frac{4t}{3}\right)z\right]\,dz$$

$$=\frac{1}{9}(4xt-x^2)$$

$$-2\left[x\left(2t-\frac{4}{3}\right)z^4+\frac{z^3}{3}\left\{4x\left(1-\frac{4t}{3}\right)-x^2\left(2t-\frac{4}{3}\right)\right\}-\frac{x^2}{2}\left(1-\frac{4t}{3}\right)z^2\right]_0^1$$

$$=\frac{1}{9}\left(4xt-x^2\right)$$

$$-2\left[x\left(2t-\frac{4}{3}\right)+\frac{1}{3}\left\{4x\left(1-\frac{4t}{3}\right)-x^2\left(2t-\frac{4}{3}\right)\right\}-\frac{x^2}{2}\left(1-\frac{4t}{3}\right)\right]=0$$

Since $B_2(x,t)=0$, therefore $B_p(x,t)=0$ and $\mu_p=0$ for all $p\geq 3$.

Substituting these values in (44) and (45) we have

$$D(x,t;\lambda)=K(x,t)-\lambda\,B_1(x,t)=4xt-x^2-\lambda\left(2x^2t-\frac{4}{3}x^2+x-\frac{4}{3}xt\right)$$

and $\quad D(\lambda)=1-\lambda\,\mu_1+\dfrac{\lambda^2}{2!}\mu_2=1-\lambda+\dfrac{\lambda^2}{18}$

$$\therefore \qquad R(x,t;\lambda)=\frac{4xt-x^2-\lambda\left(2x^2t-\frac{4}{3}x^2+x-\frac{4}{3}xt\right)}{1-\lambda+\left(\lambda^2/18\right)}$$

Example 3 : Find the resolvent kernel and solution of

$$g(x)=f(x)+\lambda\int_0^1(x+t)\,g(t)\,dt \qquad ...(54)$$

Solution : Comparing (54) with $g(x)=f(x)+\lambda\int_0^1 K(x,t)\,g(t)\,dt$,

we have $K(x,t)=x+t$

Now, $\quad B_0(x,t)=K(x,t)=x+t$

$$\mu_0=1 \text{ and } \mu_p=\int_0^1 B_{p-1}(s,s)\,ds,\ p\geq 1 \qquad ...(55)$$

and $\quad B_p(x,t)=\mu_pK(x,t)-p\int_0^1 K(x,z)\,B_{p-1}(z,t)\,dz \qquad ...(56)$

Putting $p=1$ in (55), we get

$$\mu_1=\int_0^1 B_0(s,s)\,ds=\int_0^1 2s\,ds=\left[s^2\right]_0^1=1$$

Putting $p=1$ in (56), we obtain

$$B_1(x,t) = \mu_1 K(x,t) - \int_0^1 K(x,z)\, B_0(z,t)\; dz$$

$$= (x+t) - \int_0^1 (x+z)(z+t)\; dz$$

$$= (x+t) - \int_0^1 \left(z^2 + z(x+t) + xt\right)\; dz$$

$$= (x+t) - \left[\frac{1}{3}z^3 + \frac{z^2}{2}(x+t) + xt^2\right]_0^1$$

$$= (x+t) - \frac{1}{3} - \frac{1}{2}(x+t) - xt$$

$$= \frac{1}{2}(x+t) - xt - \frac{1}{3}$$

Also putting $p = 2$ in (55), we obtain

$$\mu_2 = \int_0^1 B_1(s,s)\; ds = \int_0^1 \left[\frac{1}{2}(s+s) - s^2 - \frac{1}{3}\right] ds$$

$$= \left[\frac{s^2}{2} - \frac{s^3}{3} - \frac{1}{3}s\right]_0^1 = -\frac{1}{6}$$

Next, putting $p = 2$ in (56), we obtain

$$B_2(x,t) = \mu_2 K(x,t) - 2\int_0^1 K(x,z)\; B_1(z,t)\; dz$$

$$= -\frac{1}{6}(x+t) - 2\int_0^1 (x+z)\left[\frac{1}{2}(z+t) - zt - \frac{1}{3}\right] dz$$

$$= -\frac{1}{6}(x+t) - 2\int_0^1 (x+z)\left[z\left(\frac{1}{2} - t\right) + \frac{1}{2}t - \frac{1}{3}\right] dz$$

$$= -\frac{1}{6}(x+t) - 2\int_0^1 \left[z^2\left(\frac{1}{2} - t\right) + z\left(\frac{1}{2}t - \frac{1}{3} + \frac{x}{2} - xt\right) + x\left(\frac{1}{2}t - \frac{1}{3}\right)\right] dz$$

$$= -\frac{1}{6}(x+t) - 2\left[\frac{z^3}{3}\left(\frac{1}{2} - t\right) + \frac{z^2}{2}\left(\frac{1}{2}t - \frac{1}{3} + \frac{x}{2} - xt\right) + x^2\left(\frac{1}{2} - \frac{1}{3}\right)\right]_0^1$$

$$= -\frac{1}{6}(x+t) - 2\left[\frac{1}{3}\left(\frac{1}{2} - t\right) + \frac{1}{2}\left(\frac{1}{2}t - \frac{1}{3} + \frac{x}{2} - xt\right) + x\left(\frac{t}{2} - \frac{1}{3}\right)\right]$$

$= 0$ (on simplication)

Since $B_2(x,t) = 0$, it follows from (55) adn (56), that

$$B_p(x,t) = 0,\ \mu_p = 0 \text{ for } p \geq 3$$

Substituting these values in (44) and (45), we have

$$D(x,t;\lambda) = K(x,t) - \lambda\, B_1(x,t)$$

$$= (x+t) - \lambda \left[\frac{1}{2}(x+t) - xt - \frac{1}{3}\right]$$

and $$D(\lambda) = 1 - \lambda\,\mu_1 + \frac{\lambda^2}{2!}\mu_2 = 1 - \lambda - \frac{1}{12}\lambda^2$$

Thus $$R(x,t;\lambda) = \frac{x + t - \lambda\left[\frac{1}{2}(x+t) - xt - \frac{1}{3}\right]}{1 - \lambda - \left(\lambda^2/12\right)}$$

Hence the required solution of the integral equation is given by

$$g(x) = f(x) + \lambda \int_0^1 R(x,t;\lambda) f(t)\, dt$$

or $$g(x) = f(x) + \lambda \int_0^1 \frac{(x+t) - \lambda\left[\frac{1}{2}(x+t) - xt - \frac{1}{3}\right]}{1 - \lambda - \left(\lambda^2/12\right)} f(t)\, dt$$

Example 4 : Using Fredholm determinants, find the resolvent kernel, when $K(x,t) = xe^t$, $a = 0$, $b = 1$

Solution : Here $K(x,t) = xe^t$

We have $B_0(x,t) = K(x,t) = xe^t$

Also $$B_1(x,t) = \int_0^1 \begin{vmatrix} K(x,t) & K(x,z_1) \\ K(z,t) & K(z_1,z_1) \end{vmatrix} dz_1$$

$$= \int_0^1 \begin{vmatrix} x\,e^t & x\,e^{z_1} \\ z_1\,e^t & z_1\,e^{z_1} \end{vmatrix} dz_1$$

$= 0$ (since two columns of the determinant under the integral sign are identical)

Similarly $B_2(x,t)=\int_0^1\int_0^1\begin{vmatrix}K(x,t) & K(x,z_1) & K(x,z_2)\\ K(z_1,t) & K(z_1,z_1) & K(z_1,z_2)\\ K(z_2,t) & K(z_2,z_1) & K(z_2,z_2)\end{vmatrix}dz_1dz_2$

$$=\int_0^1\int_0^1\begin{vmatrix}x\,e^t & x\,e^{z_1} & x\,e^{z_2}\\ z_1e^t & z_1e^{z_1} & z_1e^{z_2}\\ z_2e^t & z_2e^{z_1} & z_2e^{z_2}\end{vmatrix}dz_1dz_2=0$$

Since $B_1(x,t)=B_2(x,t)=0$, it follows that $B_n(x,t)=0$, for $n\geq 1$.

Thus, we have from equation (40) Art. 12.3,

$$\mu_1=\int_0^1 K(z_1,z_1)\,dz_1=\int_0^1 z_1e^{z_1}dz_1$$

$$=\left[z_1e^{z_1}\right]_0^1-\int_0^1 e^{z_1}dz_1$$

$$=e-\left[e^{z_1}\right]_0^1=e-(e-1)=1$$

and $\mu_2=\int_0^1\int_0^1\begin{vmatrix}z_1e^{z_1} & z_1e^{z_2}\\ z_2e^{z_1} & z_2e^{z_2}\end{vmatrix}dz_1\ dz_2$

$$=0$$

Obviously $\mu_m=0$ for all $m\geq 2$

Now, $D(x,t;\lambda)=K(x,t)+\sum_{m=1}^{\infty}\frac{(-\lambda)^m}{m!}B_m(x,t)$

$$=K(x,t)-\lambda\,B_1(x,t)+\frac{\lambda^2}{2!}B_2(x,t)$$

$=xe^t$ (by substituting values of $B_1(x,t)$ and $B_2(x,t)$ etc)

and $D(\lambda)=1+\sum_{m=1}^{\infty}\frac{(-\lambda)^m}{m!}\mu_m$

$$=1-\lambda\,\mu_1+\frac{\lambda^2}{2!}\mu_2$$

$=1-\lambda$ (by substituting the values of μ_1 and μ_2)

Thus the Fredholm resolvent kernel is given by

$$R(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)} = \frac{xe^t}{1-\lambda}$$

Example 5 : Using Fredholm theory, solve

$$g(x) = \cos 2x + \int_0^{2\pi} \sin x \cos t \; g(t)\, dt$$

Solution : Comparing given integral equation with

$$g(x) = f(x) + \lambda \int_0^{2\pi} K(x,t)\, g(t)\, dt$$

we have $f(x) = \cos 2x$, $K(x,t) = \sin x \cos t$, $\lambda = 1$

We use Fredholm determinant method.

Here $\quad B_0(x,t) = K(x,t) = \sin x \cos t$

Also $\quad B_1(x,t) = \int_0^{2\pi} \begin{vmatrix} K(x,t) & K(x,z_1) \\ K(z_1,t) & K(z_1,z_1) \end{vmatrix} dz_1$

$$= \int_0^{2\pi} \begin{vmatrix} \sin x \cos t & \sin x \cos z_1 \\ \sin z_1 \cos t & \sin z_1 \cos z_1 \end{vmatrix} dz_1$$

$= 0$ (since the determinant under the integral sign vanish)

$\therefore \quad B_p(x,t) = 0$ for all $p \geq 2$

Next, $\quad \mu_1 = \int_0^{2\pi} K(z_1,z_1)\, dz_1 = \int_0^{2\pi} \sin z_1 \cos z_1 \; dz_1 = 0$

Hence $\quad \mu_p = 0$ for all $p \geq 2$

Thus, we have

$$D(x,t;\lambda) = K(x,t) + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} B_m(x,t)$$

$$= K(x,t) - \lambda\, B_1(x,t) + \ldots.$$

$$= \sin x \cos t$$

and $\quad D(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} \mu_m$

$$= 1 - \lambda\, \mu_1 + \ldots\ldots\ldots = 1,$$

The Fredholm resolvent kernel is given by

$$R(x,t;\lambda)=\frac{D(x,t;\lambda)}{D(\lambda)}=\sin x\cos t$$

Hence the required solution of the integral equation is

$$g(x)=f(x)+\lambda\int_0^{2\pi} R(x,t;\lambda)\, f(t)\,dt$$

$$=\cos 2x+\lambda\int_0^{2\pi}\sin x\cos t\,\cos 2t\,dt$$

$$=\cos 2x+\sin x\int_0^{2\pi}\cos t\,\cos 2t\,dt$$

$$=\cos 2x+\sin x.0$$

$$g(x)=\cos 2x$$

Example 6 : Solve the following integral equation

$$g(x)=x+\lambda\int_0^1\left(4xt-x^2\right)g(t)\,dt$$

Solution : Comparing given integral equation with

$$g(x)=f(x)+\lambda\int_a^b K(x,t)\,g(t)\,dt,$$

we have $a=0, b=1$, $f(x)=x$ and $K(x,t)=4xt-x^2$

Proceeding as in solved example 2 (ii) and obtain

$$R(x,t;\lambda)=\frac{4xt-x^2-\lambda\left(2x^2t-\frac{4}{3}x^2+x-\frac{4}{3}xt\right)}{1-\lambda+\left(\lambda^2/18\right)}$$

Hence the required solution of given integral equation is

$$g(x)=f(x)+\lambda\int_0^1 R(x,t;\lambda)\,f(t)\,dt$$

$$=x+\lambda\int_0^1\frac{4xt-x^2-\lambda\left(2x^2t-\frac{4}{3}x^2+x-\frac{4}{3}xt\right)}{1-\lambda+\left(\lambda^2/18\right)}\,t\,dt$$

$$=x+\frac{18\lambda}{\lambda^2-18\lambda+18}\int_0^1\left[4xt^2-x^2t-\lambda\left(2x^2t^2-\frac{4}{3}x^2t+xt-\frac{4}{3}xt^2\right)\right]dt$$

$$= x + \frac{18\lambda}{\lambda^2 - 18\lambda + 18}\left[\frac{4}{3}xt^3 - \frac{x^2t^2}{2} - \lambda\left(\frac{2}{3}x^2t^3 - \frac{2}{3}x^2t^2 + \frac{xt^2}{2} - \frac{4}{9}xt^3\right)\right]_0^1$$

$$= x + \frac{18\lambda}{\lambda^2 - 18\lambda + 18}\left[\frac{4}{3}x - \frac{x^2}{2} - \lambda\left(\frac{2}{3}x^2 - \frac{2}{3}x^2 + \frac{x}{2} - \frac{4}{9}x\right)\right]$$

$$= \frac{\left(\lambda^2 - 18\lambda + 18\right)x + \lambda\left(24x - 9x^2 - \lambda x\right)}{\lambda^2 - 18\lambda + 18}$$

$$= \frac{3x\left(2\lambda - 3\lambda x + 6\right)}{\lambda^2 - 18\lambda + 18}$$

Example 7 : Find $D(\lambda)$ and $D(x,t;\lambda)$ and solve the integral equation

$$g(x) = x + \lambda\int_0^1\left[xt + \sqrt{xt}\,\right] g(t)\, dt$$

Solution : On comparing given equation with

$$g(x) = f(x) + \lambda\int_0^1 K(x,t)\, g(t)\, dt\,,$$

we have $f(x) = x$, $K(x,t) = xt + \sqrt{xt}$

The resolvent kernel $R(x,t;\lambda)$ is given by

$$R(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)}$$

where $D(x,t;\lambda) = K(x,t) + \sum_{m=1}^{\infty}\frac{(-\lambda)^m}{m!}\, B_m(x,t)$

and $D(\lambda) = 1 + \sum_{m=1}^{\infty}\frac{(-\lambda)^m}{m!}\,\mu_m$

Now, we have

$$B_0(x,t) = K(x,t) = xt + \sqrt{xt},\ \mu_0 = 1$$

and $\mu_p = \int_0^1 B_{p-1}(s,s)\, ds,\ p \geq 1$ ...(57)

and $B_p(x,t) = \mu_p K(x,t) - p\int_0^1 K(x,z)\, B_{p-1}(z,t)\, dz,\ p \geq 1$ ...(58)

Putting $p = 1$ in (57), we obtain

$$\mu_1 = \int_0^1 B_0(s,s)\,ds = \int_0^1 (s^2+s)\,ds = \left(\frac{s^3}{3}+\frac{s^2}{2}\right)_0^1 = \frac{5}{6}$$

Also $B_1(x,t) = \mu_1 K(x,t) - \int_0^1 K(x,z)\,B_0(z,t)\,dz$

$$= \frac{5}{6}\left\{xt+\sqrt{xt}\right\} - \int_0^1 \left(xz+\sqrt{xz}\right)\left(zt+\sqrt{zt}\right)dz$$

$$= \frac{5}{6}\left\{xt+\sqrt{xt}\right\} - \int_0^1 \left\{xtz^2 + x\sqrt{t}\,z^{3/2} + t\sqrt{x}\,z^{3/2} + z(xt)^{1/2}\right\}dz$$

$$= \frac{5}{6}\left\{xt+\sqrt{xt}\right\} - \left[\frac{xtz^3}{3} + \left(x\sqrt{t}+t\sqrt{x}\right)\frac{z^{5/2}}{5/2} + \frac{z^2}{2}(xt)^{1/2}\right]_0^1$$

$$= \frac{5}{6}\left\{xt+\sqrt{xt}\right\} - \left[\frac{xt}{3} + \frac{2}{5}\left(x\sqrt{t}+t\sqrt{x}\right) + \frac{1}{2}(xt)^{1/2}\right]$$

$$= \frac{1}{2}xt + \frac{1}{3}\sqrt{xt} - \frac{2}{5}\left(x\sqrt{t}+t\sqrt{x}\right)$$

Again $\mu_2 = \int_0^1 B_1(s,s)\,ds$

$$= \int_0^1 \left[\frac{1}{2}s^2 + \frac{1}{3}s - \frac{2}{5}\left(s\sqrt{s}+s\sqrt{s}\right)\right]ds$$

$$= \int_0^1 \left(\frac{1}{2}s^2 + \frac{1}{3}s - \frac{4}{5}s^{3/2}\right)ds$$

$$= \left[\frac{s^3}{6} + \frac{s^2}{6} - \frac{4}{5}\times\frac{s^{5/2}}{5/2}\right]_0^1 = \frac{1}{75}$$

Next, putting $p=2$ in (57), we obtain

$$B_2(x,t) = \mu_2 K(x,t) - 2\int_0^1 K(x,z)B_1(z,t)\,dz$$

$$= \frac{1}{75}\left\{xt+\sqrt{xt}\right\} - 2\int_0^1 \left\{xz+\sqrt{xz}\right\}\left\{\frac{1}{2}zt + \frac{1}{3}\sqrt{zt} - \frac{2}{5}\left(z\sqrt{t}+t\sqrt{z}\right)\right\}dz$$

$$= \frac{1}{75}\left\{xt+\sqrt{xt}\right\} - 2\int_0^1 \left\{\frac{1}{2}xtz^2 + \frac{x\sqrt{t}}{3}z^{3/2} - \frac{2x\sqrt{t}}{5}z^2\right.$$

$$-\frac{2xt}{5}z^{3/2}+\frac{1}{2}t\sqrt{xz}^{3/2}+\frac{1}{3}z(xt)^{1/2}-\frac{2}{5}(xt)^{1/2}z^{3/2}-\frac{2}{5}t\sqrt{xz}\Big\}\,dz$$

$$=\frac{1}{75}\left\{xt+\sqrt{xt}\right\}-2\left[\frac{xtz^3}{6}+\frac{2x\sqrt{t}}{15}z^{5/2}-\frac{2x\sqrt{t}\,z^3}{15}\right.$$

$$\left.-\frac{4xt\,z^{5/2}}{25}+\frac{t\sqrt{x}\,z^{5/2}}{5}+\frac{(xt)^{1/2}z^2}{6}-\frac{4(xt)^{1/2}}{25}z^{5/2}-\frac{t\sqrt{x}\,z^2}{5}\right]_0^1$$

$$=\frac{1}{75}\left\{xt+\sqrt{xt}\right\}$$

$$-2\left[\frac{xt}{6}+\frac{2x\sqrt{t}}{15}-\frac{2x\sqrt{t}}{15}-\frac{4xt}{25}+\frac{t\sqrt{x}}{5}+\frac{(xt)^{1/2}}{6}-\left(\frac{4}{25}\right)(xt)^{5/2}-t\sqrt{x}\right]$$

$$=0 \text{ (on simplification)}$$

Since $B_2(x,t)=0$, therefore we easily see that

$$B_p(x,t)=0,\ \mu_p=0 \quad \text{for } p\geq 3$$

Thus we find that

$$D(x,t;\lambda)=K(x,t)-\lambda B_1(x,t)$$

$$=(xt)+(xt)^{1/2}-\lambda\left\{\frac{1}{2}xt+\frac{1}{3}(xt)^{1/2}-\frac{2}{5}\left(x\sqrt{t}+t\sqrt{x}\right)\right\}$$

and $$D(\lambda)=1-\lambda\mu_1+\frac{\lambda^2}{2!}\mu_2$$

$$=1-\frac{5}{6}\lambda+\frac{1}{15}\lambda^2$$

Therefore

$$R(x,t;\lambda)=\frac{xt+(xt)^{1/2}-\lambda\left\{\frac{1}{2}xt+\frac{1}{3}(xt)^{1/2}-\frac{2}{5}\left(x\sqrt{t}+t\sqrt{x}\right)\right\}}{1-(5/6)\lambda+(1/50)\lambda^2}$$

Hence the required solution is

$$g(x)=f(x)=\lambda\int_0^1 R(x,t;\lambda)f(t)\,dt$$

or $$g(x)=x+\lambda\int_0^1 \frac{xt+(xt)^{1/2}-\lambda\left\{\frac{1}{2}xt+\frac{1}{3}(xt)^{1/2}-\frac{2}{5}\left(x\sqrt{t}+t\sqrt{x}\right)\right\}}{1-(5/6)\lambda+(1/150)\lambda^2}\, t\,dt$$

$$=x+\frac{\lambda}{1-(5/6)\,\lambda+(1/150)\,\lambda^2}\left[\frac{xt^3}{3}+\frac{2\sqrt{x}t^{5/2}}{5}-\frac{\lambda xt^3}{6}-\frac{2\sqrt{x}\lambda\,t^{5/2}}{15}-\frac{4x\lambda\,t^{5/2}}{25}-\frac{2\sqrt{x}\lambda\,t^3}{15}\right]_0^1$$

$$=x+\frac{\lambda}{1-\left(5/6\right)\,\lambda+\left(1/150\right)\,\lambda^2}\left[\frac{x}{3}+\frac{2\sqrt{x}}{5}-\frac{\lambda x}{6}-\frac{2\lambda\sqrt{x}}{15}-\frac{4x\lambda}{25}-\frac{2\lambda\sqrt{x}}{15}\right]$$

$$=\frac{150x+\lambda\left(60\sqrt{x}-75x\right)+21x\lambda^2}{\lambda^2-125\lambda+150}$$

Example 8 : Solve the integral equations

$$g(x)=1+\lambda\int_0^{\pi}\sin(x+t)\,g(t)\,dt$$

Solution : Comparing the given equation with the standard Fredholm integral equation of second kind, we find that

$$a=0,\ b=\pi,\ f(x)=1,\ K(x,t)=\sin(x+t)$$

Now, $B_0(x,t)=K(x,t)=\sin(x+t)$

$$\mu_0=1,\qquad \mu_p=\int_0^{\pi}B_{p-1}(s,s)\,ds,\ \ p\geq 1 \qquad ...(59)$$

and $$B_p(x,t)=\mu_pK(x,t)-p\int_0^{\pi}\sin(x+z)\,B_{p-1}(z,t)\,dz,\qquad p\geq 1 \qquad ...(60)$$

Taking $p=1$ in (59), we obtain

$$\mu_1=\int_0^{\pi}B_0(s,s)\,ds=\int_0^{\pi}\sin 2s\,ds=\left[\frac{-\cos 2s}{2}\right]_0^{\pi}=0$$

Also on taking $p=1$ in (60), we obtain

$$B_1(x,t)=\mu_1K(x,t)=\int_0^{\pi}\sin(x+z)\,B_0(z,t)\,dz$$

$$=-\int_0^{\pi}\sin(x+z)\,\sin(z+t)\,dz$$

$$=-\frac{1}{2}\int_0^{\pi}\{\cos(x-t)-\cos(x+t+2z)\}\,dz$$

$$= -\frac{1}{2}\int_0^{\pi}\left[z\cos(x-t) - \frac{1}{2}(x+t+2z)\right]_0^{\pi}$$

$$= -\frac{\pi}{2}\cos(x-t) \qquad \text{(on simplication)}$$

Again, putting $p = 2$ in (59), we obtain

$$\mu_2 = \int_0^{\pi} B_1(s,s)\, ds = \int_0^{\pi} (-\pi/2)\, ds$$

or $\quad \mu_2 = -\pi^2/2$

Next, putting $p = 2$ in (59), we obtain

$$B_2(x,t) = \mu_2 K(x,t) - 2\int_0^{\pi} K(x,z)\, B_1(z,t)\, dz$$

$$= \frac{-\pi^2}{2}\sin(x+t) - 2\int_0^{\pi}\sin(x+z)\left\{\frac{-\pi}{2}\cos(z-t)\right\} dz$$

$$= \frac{-\pi^2}{2}\sin(x+t) + \frac{\pi}{2}\int_0^{\pi}\left\{\sin(2z+x-t) + \sin(x+t)\right\} dz$$

$$= \frac{-\pi^2}{2}\sin(x+t) + \frac{\pi}{2}\left[\frac{-\cos(2z+x-t)}{2} + z\sin(x+t)\right]_0^{\pi}$$

$$= \frac{-\pi^2}{2}\sin(x+t) + \frac{\pi}{2}\left[-\frac{1}{2}\cos(x-t) + \frac{1}{2}\cos(x-t) + \pi\sin(x+t)\right]$$

$$= 0$$

Since $B_2(x,t) = 0$, it follows (59) and (60) that

$$B_p(x,t) = 0, \quad \mu_p = 0 \quad \text{for} \quad p \ge 3 \qquad \text{...(61)}$$

Now, $\quad D(x,t;\lambda) = K(x,t) + \sum_{m=1}^{\infty}\frac{(-\lambda)^m}{m!} B_m(x,t)$

$$= K(x,t) - \lambda\, B_1(x,t)$$

$$= \sin(x+t) + \frac{\pi}{2}\lambda\, \cos(x-t)$$

and $\quad D(\lambda) = 1 + \sum_{m=1}^{\infty}\frac{(-\lambda)^m}{m!}\mu_m$

$$= 1 - \lambda \mu_1 + \frac{\lambda^2}{2!} \mu_2$$

$$= 1 - (\pi^2/4)\lambda^2$$

Therefore $R(x,t;\lambda) = \frac{D(x,t;\lambda)}{D(\lambda)}$

$$= \frac{\sin(x+t) + (\lambda \pi/2)\cos(x-t)}{1 - (\pi^2/4)\lambda^2}$$

Thus the required solution is

$$g(x) = f(x) + \lambda \int_0^\pi R(x,t;\lambda) f(t)\, dt$$

$$= 1 + \lambda \int_0^\pi \frac{\sin(x+t) + (\lambda \pi/2)\cos(x-t)}{1 - (\pi^2/4)\lambda^2}\, dt$$

$$= 1 + \frac{4\lambda}{4 - \pi^2\lambda^2}\left[-\cos(x+t) - \frac{\pi\lambda}{2}\sin(x-t)\right]_0^\pi$$

$$= 1 + \frac{4\lambda}{4 - \pi^2\lambda^2}\left[\cos x + \frac{1}{2}\pi\lambda \sin x + \cos x + \frac{1}{2}\pi\lambda\sin x\right]$$

or $$g(x) = 1 + \frac{4\lambda}{4 - \pi^2\lambda^2}(2\cos x + \pi\lambda \sin x)$$

Sel-Learning Exercise

1. State whether the following statements are true of False :

(i) Fredholm's first theorem hold when λ is a root of the equation $D(\lambda) = 0$

(ii) The unique and continuous solution of the Fredholm integral equation of second kind

$$g(x) = f(x) + \lambda \int_a^b K(x,t)\, g(t)\, dt$$

is given by

$$g(x) = f(x) + \lambda \int_a^b R(x,t;\lambda)\, g(t)\, dt$$

where $R(x,t;\lambda)$ is resolvent kernel.

(iii) The resolvent kernel $R(x,t;\lambda)$ satisfies the following relations :

$$R(x,t;\lambda) = K(x,t) + \lambda \int_a^b K(x,z) R(z,t;\lambda)\, dz$$

(iv) The series $D(\lambda)$ is an absolutely and uniformely converging power series in λ.

(v) The series $D(x,t;\lambda)$ is not absolutely but uniformely converging power series in λ.

(vii) The coefficients μ_n and the function $B_n(x,t)$ satisfy the following recurrrence relations

(a) $\mu_0 = 0$, $\mu_n = \int_a^b B_{n-1}(s,s)\, ds$

(b) $B_n(x,t) = \mu_n K(x,t) + n \int_a^b K(x,z)\, B_{n-1}(z,t)\, dz$, $n \geq 1$

2. Define following

(i) Fredholm determinant

(ii) Fredholm first series

(iii) Fredholm resolvent kernel.

(iv) Fredholm minor.

12.4 Summary

In this Unit, we have seen that the Fredholm first theorem enables us to get explicit formula for the solution of Fredholm integral equation of second kind in term of certain determinant. We have also seen that the Fredholm resolvent kernel of the integral equation can be found from the recurrence relations.

12.5 Answer's to Self Learning Exercise

1. (i) False (ii) False (iii) True

(iv) True (v) False (vi) False

2. See text.

12.6 Exercise 12

1. Use Fredholm determinants to find the resolvent kernel

$$R(x,t;\lambda) = D(x,t;\lambda) / D(\lambda)$$

of the kernel $K(x,t) = xe^t$ under the limits of integral $a = 0,\ b = 1$. Hence solve the integral equation

$$g(x) = e^{-x} + \lambda \int_0^1 x\, e^t g(t)\, dt$$

[Ans. $R(x,t;\lambda) = \dfrac{x e^t}{1-\lambda}$, solution $g(x) = e^{-x} + \dfrac{\lambda\, xt}{1-\lambda}$, $\lambda \neq 1$]

2. State Fredholm's first fundamental theorem. Using Fredholm'm theory, solve

$$g(x)=e^{x}+\lambda\int_{0}^{1}x\,t\,g(t)\,dt$$

[Ans. $g(x)=e^{x}+\dfrac{3\lambda x}{3-10^{3}\lambda}\left(1+9e^{10}\right)$]

3. Show by using the Fredholm's theory that the resolvent kernel for the integral equation with kernel $K(x,t)=1-3xt$ in the interval $(0,1)$ is

$$R(x,t;\lambda)=\left[\frac{4}{4-\lambda^{2}}\right]\left[1+\lambda-\frac{(x+t)}{2}-3(1-\lambda)xt\right],\ \lambda\neq\neq 2$$

4. State and prove first and second series for non homogeneous Fredholm integral equation of second kind.

5. For the integral equation

$$g(x)=f(x)+\lambda\int_{0}^{1}K(x,t)\,g(t)\,dt$$

Compute $D(\lambda)$ and $D(x,t;\lambda)$ for the following kernels for the sepcified limits a and

(i) $K(x,t)=e^{x-t},\ a=0,\ b=1$

(ii) $K(x,t)=\sin(x+t),\ a=0,\ b=\pi$

(iii) $K(x,t)=2e^{x}e^{t},\ a=0,\ b=1$

(iv) $K(x,t)=t,\ a=4,\ b=10$

[Ans. (i) $D(\lambda)=1-\lambda,\ D(x,t;\lambda)=e^{x-t}$

(ii) $D(\lambda)=1-(\pi^{2}/4)\lambda^{2},\ D(x,t;\lambda)=\sin(x+t)+\dfrac{\pi}{2}\lambda\cos(x-t)$

(iii) $D(\lambda)=1-\lambda(e^{2}-1),\ D(x,t;\lambda)=2e^{x}e^{t}$

(iv) $D(\lambda)=1-50\lambda,\quad D(x,t;\lambda)=t$]

6. Determine the resolvent kernel and hence solve the following integral equation

(i) $g(x)=\sin x+\lambda\int_{4}^{10}x\,g(t)\,dt$

(ii) $g(x)=e^{x}+\lambda\int_{a}^{1}2e^{x}e^{t}\,g(t)\,dt$

(iii) $g(x)=\sec^{2}x+\lambda\int_{0}^{1}g(t)\,dt$

(iv) $g(x)=1+\int_0^1 (1-3xt)g(t)\,dt$

[Ans. (i) $R(x,t;\lambda)=x/(1-42\lambda)$

$$g(x)=\sin x+\frac{2\lambda x\sin 7\sin z}{1-42\lambda}$$

(ii) $R(x,t;\lambda)=\dfrac{2e^x e^t}{1-\lambda(e^2-1)}$

$$g(x)=e^x/1-\lambda(e^2-1)$$

(iii) $R(x,t;\lambda)=\dfrac{1}{1-\lambda}$

$$g(x)=\sec^2 x+\frac{\lambda}{1-\lambda}\tan 1$$

(iv) $R(x,t;\lambda)=\dfrac{2}{3}[4-3(x+t)]$

$$g(x)=\frac{8-6x}{3}\ ]$$

7. Using the Fredholm determinants, find the resolvent kernel of the following kernels :

(i) $2x-t$, $0\le x\le 1$ $0\le t\le 1$

(ii) $\sin x\cos t$ $0\le x\le 2\pi$ $0\le 1\le 2\pi$

[Ans. (i) $R(x,t;\lambda)=\dfrac{2x-t-\lambda\left(\frac{2}{3}-x-t+2xt\right)}{1-\left(\lambda/2\right)+\left(\lambda^2/6\right)}$

(ii) $R(x,t;\lambda)=\sin x\cos t$]

8. Using the recurrence relation, find the resolvent kernel of the following kernels

(i) $K(x,t)=x+t+1$, $-1\le x\le 1$, $-1\le t\le 1$

(ii) $K(x,t)=\sin x$, $0\le x\le \pi$

[Ans. (i) $R(x,t;\lambda)=\dfrac{x+t+1+2\lambda\left(xt+1/3\right)}{1-2\lambda-\left(4\lambda^2/3\right)}$

(ii) $R(x,t;\lambda)=\dfrac{\sin x}{1-2\lambda}$]