MA/MSc MT-07

Vardhaman Mahaveer Open University, Kota

Viscous Fluid Dynamics

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Vardhaman Mahaveer Open University, Kota

Viscous Fluid Dynamics

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PREFACE

The present book entitled **"Viscous Fluid Dynamics"** has been designed so as to cover the unit-wise syllabus of Mathematics-07 course for M.A./M.Sc. (Final) students of Vardhaman Mahaveer Open University, Kota. It can also be used for competitive examinations. The basic principles and theory have been explained in a simple, concise and lucid manner. Adequate number of illustrative examples and exercises have also been included to enable the students to grasp the subject easily. The units have been written by various experts in the field. The unit writers have consulted various standard books on the subject and they are thankful to the authors of these reference books.

UNIT - 1

Basic Concepts

Structure of the Unit

1.0 Objectives

In this unit, our object is to be aware about the basic concepts required in the development of the theory of viscous flow. We will also study about the fundamental equations for the viscous compressible fluid. The governing equations for the compressible and incompressible fluids in various coordinate systems are given for ready reference.

1.1 Introduction

The subject of viscous fluid flow is of great significance to the mankind, the passage of blood through veins, the falling of rain through the atmosphere and the current in the oceans are few examples of the flow. Therefore it is interesting to study the viscous fluid flow in order to utilize and control its effects for the benefit of the society. This unit deals with the basic concepts of viscous flow and fundamental equations for the flow. This unit also deals the general theory of stress and rate fo strain.

1.2 Fluid

By fluid we mean that a substance which is capable of flowing and it yields to a pressure however small it may be. The fluids are classified in two forms ideal (perfect) and real (actual) fluid.

1.2.1 Ideal Fluid

A fluid is said to be ideal or perfect if it does not exert any shearing stress however small. In ideal fluids, there are no tangential forces between the adjoining layers of the fluid but only normal stresses are present. The pressure at every point of an ideal fluid is equal in all directions, whether the fluid be at rest or in motion.

1.2.2 Real Fluid

The fluid which actually exist in nature are considered real or actual fluid. These fluids possess all the five physical properties i.e. density, volume, temperature, pressure and viscosity. Real fluids are divided into two categories viz liquids and gases. We generally regard liquids as incompressible fluid and gases as compressible fluids.

If the density of the fluid be constant then it is called incompressible fluid and if density be a function of hydrostatic pressure then it is called compressible fluid. Generally water and air are considered an incompressible and compressible fluid respectively.

1.3 Density

The The mass density or simply density ρ at any point is defined as $\rho = \lim_{\delta\upsilon \to 0} \frac{\delta m}{\delta \upsilon}$; where $\delta\upsilon$ is the volume element around a point in the fluid and δm is the mass of the fluid contained within $\delta\upsilon$. The unit of density in MKS system is kilogram / meter^3 i.e. ML^{-3}.

1.4 Viscosity

Viscosity of a fluid is that characteristics of real fluids due to which they exhibit a certain resistance to alternation of form or exerting internal resistance to a change in shape. Viscosity is also known as an internal friction.

Consider the motion of a fluid between two parallel plates AB and CD at a distance d apart. The lower plate CD is at rest while the plate AB is moving with uniform velocity U parallel to itself as shown in fig. 1.1. Here we suppose that there is no slip on the surface when the fluid is in contact with a solid. The velocity will decrease as we go downwards from AB to zero in contact with CD. In order to maintain the motion of the plate AB, a horizontal force proportional to U/d per unit area of AB is required. Thus we have the force in the form of tangential or shear stress τ given by $\tau = \mu \frac{U}{d}$ where μ is a constant of proportionality and independent of U and d. It depends only on the nature of the fluid. this constant μ is a measure of the viscosity of the fluid and is called the "coefficient of viscosity" or "coefficient of shear viscosity."

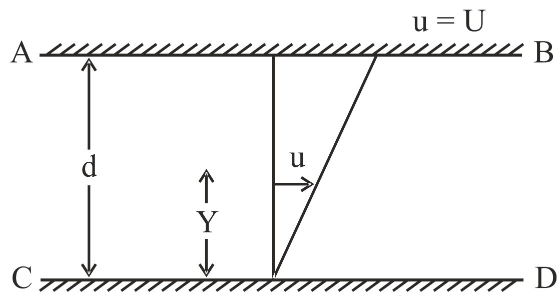


Fig. 1.1

For ordinary fluids, since there is no slip on the walls and the fluid is displaced in such a manner that the various layers of fluid slides uniformaly over one another then the velocity u of a layer of the fluid at a distance y from the lower plate is $u = U\frac{y}{d}$. It may be seen that if $\frac{U}{d}$ is replaced by the velocity gradient $\frac{du}{dy}$, we obtain Newton's law of viscosity as $\tau = \mu \frac{du}{dy}$.

The dimensions of the coefficient of viscosity μ can be found as -

$$\mu = \frac{\text{shearing stress}}{\text{velocity gradient}} = \frac{\text{force/area}}{\text{velocity/length}} = ML^{-1}T^{-1}$$

Hence the unit of μ is kilogram per meter second and 1kg per meter second is equal to 10 poise. Poise is the practical unit of coefficient of viscosity. If we write $\upsilon = \frac{\mu}{\rho}$ i.e. the ratio of μ to the density ρ, υ is called the 'kinematic viscosity.'

The coefficient of viscosity is very small for water, gases, alcohol but not negligible and it is very large in case of oil, glycerine. Some typical values of μ and υ are given below in C.G.S. units at 15°C and under atmospheric pressure.

Gases / Liquids	μ	υ
Air	0.00018	0.15
Oxygen	0.0002	0.15
Hydrogen	0.00009	1.5
Water	0.0114	0.0114
Mercury	0.016	0.0012
Glycerine	13	10
Pitch	10^{10}	10^{10}

For liquids the viscosity coeffiecient μ is nearly independent of pressure but decreases rapidly with increasing temperature also for gases it is independent of pressure but increases with temprature.

1.5 Most General Motion of a Fluid Element

In this article we shall prove that the general motion of a fluid particle consists of three parts a translation, a rotation and a deformation. We shall show this by considering the relative motion between two neighbouring points of a fluid element.

Consider the small movement of a fluid pareticle from P to Q.

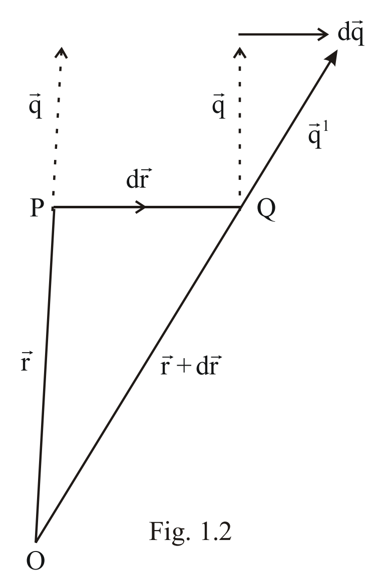


Fig. 1.2

Let $\vec{q}$ be the velocity at P(x, y, z) and $\vec{q}^1 = \vec{q} + d\vec{q}$ be the velocity of a neighbouring point Q. Also let $\vec{r}$ and $\vec{r} + d\vec{r}$ be the position vectors of P and Q respectively. Then we have

$$\vec{q}^1 = \vec{q} + d\vec{q}$$

$$\vec{q}^1 = \vec{q} + \left(\frac{\partial \vec{q}}{\partial x}dx + \frac{\partial \vec{q}}{\partial y}dy + \frac{\partial \vec{q}}{\partial z}dz\right)$$

$$= \vec{q} + \hat{i}\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\right) + \hat{j}\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz\right) + \hat{k}\left(\frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz\right)$$

where $\hat{q}\,[u(x,y,z),\ v\,(x,y,z),\ w(x,y,z)]$ Rewriting right hand side of above, we get

$$\vec{q} = \vec{q} + \hat{i}\left[\left\{\frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial \omega}{\partial x}\right)dz - \frac{1}{2}\left(\frac{\partial \nu}{\partial x} - \frac{\partial u}{\partial y}\right)dy\right\} + \left\{\frac{\partial u}{\partial x}dx + \frac{1}{2}\left(\frac{\partial \nu}{\partial x} + \frac{\partial u}{\partial y}\right)dy + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial \omega}{\partial x}\right)dz\right\}\right]$$

$$+ \hat{j}\left[\left\{\frac{1}{2}\left(\frac{\partial \nu}{\partial x} - \frac{\partial u}{\partial y}\right)dx - \frac{1}{2}\left(\frac{\partial \omega}{\partial y} - \frac{\partial \nu}{\partial z}\right)dz\right\} + \left\{\frac{\partial \nu}{\partial y}dy + \frac{1}{2}\left(\frac{\partial \nu}{\partial x} + \frac{\partial u}{\partial y}\right)dx + \frac{1}{2}\left(\frac{\partial \omega}{\partial y} + \frac{\partial \nu}{\partial z}\right)dz\right\}\right]$$

$$+ \hat{k}\left[\left\{\frac{1}{2}\left(\frac{\partial \omega}{\partial y} - \frac{\partial \nu}{\partial z}\right)dy - \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial \omega}{\partial x}\right)dx\right\} + \left\{\frac{\partial \omega}{\partial z}dz + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial \omega}{\partial x}\right)dx + \frac{1}{2}\left(\frac{\partial \omega}{\partial y} + \frac{\partial \nu}{\partial z}\right)dy\right\}\right]$$

$$\therefore \quad \vec{q}^{\,1} = \vec{q} + \vec{\omega} \times d\hat{r} + \vec{D} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where $\vec{\omega}$ is the angular velocity given by

$$\vec{\omega} \times d\vec{r} = \hat{i}\left[\left\{\frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial \omega}{\partial x}\right)dz - \frac{1}{2}\left(\frac{\partial \nu}{\partial x} - \frac{\partial u}{\partial y}\right)dy\right\}\right]$$

$$+ \hat{j}\left[\left\{\frac{1}{2}\left(\frac{\partial \nu}{\partial x} - \frac{\partial u}{\partial y}\right)dx - \frac{1}{2}\left(\frac{\partial \omega}{\partial y} - \frac{\partial \nu}{\partial z}\right)dz\right\}\right]$$

$$+ \hat{k}\left[\left\{\frac{1}{2}\left(\frac{\partial \omega}{\partial y} - \frac{\partial \nu}{\partial z}\right)dy - \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial \omega}{\partial x}\right)dx\right\}\right] \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and $$\vec{D} = \hat{i}\left[\frac{\partial u}{\partial x}dx + \frac{1}{2}\left(\frac{\partial \nu}{\partial x} + \frac{\partial u}{\partial y}\right)dy + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial \omega}{\partial x}\right)dz\right]$$

$$+ \hat{j}\left[\frac{\partial \nu}{\partial y}dy + \frac{1}{2}\left(\frac{\partial \nu}{\partial x} + \frac{\partial u}{\partial y}\right)dx + \frac{1}{2}\left(\frac{\partial \omega}{\partial y} + \frac{\partial \nu}{\partial z}\right)dy\right]$$

$$+ \hat{k}\left[\frac{\partial \omega}{\partial z}dy + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial \omega}{\partial x}\right)dx + \frac{1}{2}\left(\frac{\partial \omega}{\partial y} + \frac{\partial \nu}{\partial z}\right)dy\right]$$

$$\vec{D} = \hat{i}\left(\in_{xx} dx + \in_{xy} dy + \in_{xz} dz\right) + \hat{j}\left(\in_{yy} dy + \in_{yx} dx + \in_{zy} dz\right) + \hat{k}\left(\in_{zz} dz + \in_{zx} dx + \in_{zy} dy\right)$$

$$\vec{D} = \hat{i}\left(\in_x d\vec{r}\right) + \hat{j}\left(\in_y d\vec{r}\right) + \hat{k}\left(\in_z d\vec{r}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where $\in_x = \hat{i}\in_{xx} + \hat{j}\in_{xy} + \hat{k}\in_{xz}$

$\in_y = \hat{i}\in_{yx} + \hat{j}\in_{yy} + \hat{k}\in_{yz}$

$\in_z = \hat{i}\in_{zx} + \hat{j}\in_{zy} + \hat{k}\in_{zz}$

are the strain -rate tractions of the fluid elements in the x,y and z directions respecitvely.

Equation (1) respesents the most general motion of a fluid motion. The first term $\vec{q}$ respresents the translation velocity vector and it represents the linear motion of all parts of the fluid element without changing the shape of the element. Hence the first term represents the pure translatoty part of the motion.The second term $\vec{w} \times d\vec{r}$ represants the pure rotation of the fluid element. The third term $\vec{D}$ represents the rate of strain term and so the third term D gives the deformation of the fluid element. Due to this term, this velocity of a fluid element differs from a solid.

Thus we see that the most general motion of a fluid element can be expressed as the combination of translation, rotation and deformation of the fluid element.

1.6 Strain Analysis

A body is said to be strained when the various parts of the body undergo a relative displacement under the action of some impressed force. It is a non-dimensional deformation measuring the change in the relative positions of various parts due to any cause. However if the whole body undergoes a displacement i.e. translation or rotation etc. without any change in the relative positions of different parts of the body it is not a strain. There are two types of strain.

1.6.1 Normal Strain

It is the rate of the change in length of a part of body to the initial length, where the element is taken to be a straight line. If in the unstrained state the length of a line element ℓ_1 and in the strained state ℓ_2, then normal strain $= \dfrac{\ell_2 - \ell_1}{\ell_1}$

1.6.2 Shearing Strain

When two elements lying on a straight line undergo a relative displacement, the change in the angle between them before and after the displacement is known as shearing starin. If in the unstrained state the elements are A and B, in the strained state they take the positions A^1 and B^1, then the angle between the straight lines AB and A^1B^1 is the shearing strain.

1.7 Stress Analysis

1.7.1 Body and Surface Forces

In the study of fluid dynamics, we distinguish between two types of forces acting on a fluid element, namely body forces and surface forces. The body forces are distributed throughout the volume of the body and expressed as force per unit mass of the element. Examples of such forces are (i) force due to gravity (ii) electromagnetic force when the fluid is electrically conducting and moving in the presence of the magnetic field and (iii) if the coordinate system is accelerating or decelerating, centrifugal and Coriolis forces, may have to be included among the body forces.

In the space occupied by a fluid in motion or at rest, imagine a surface enclosing some part of the fluid. The portions of the fluid close to the surface on its two sides, internal and external, exert forces on each other which are in the nature of actions and reactions which are called internal forces. Since they act across a surface that is imagined to separate the fluid , they are called surface forces. These forces are expressed as "force per unit surface area of a fluid element"

1.7.2 Stress and Stress Vector

Let us consider a point P in the fluid and take an infinitesimal area δs surrounding the point P. Let (x,y,z) be the coordinate of point P referred to a set of fixed axes OX, OY, OZ. Also let $\hat{n}$ be the unit vector in the direction of the normal to the plane area δs and consider the forces exerted across δs by the portion of the fluid which lies on the side of $\hat{n}$. The fluid may be either in motion or at rest. These surface forces are not, in general, distributed uniformaly across δs. These, by the principle of statics, can be combined into a single force $\delta \vec{F}$ through P and a single couple of moment $\delta \vec{C}$ about some axis. If we gradually shrink the area of the plane surface to the point P both $\delta \vec{F}$

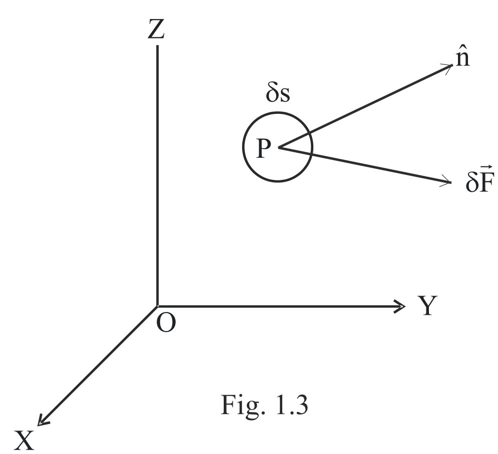


Fig. 1.3

and $\delta\vec{C}$ tend to zero. However, for a vanishingly small area $\delta s \to 0$, the equivalent surface force $\delta\vec{F}$ ay be assumed to be proportional to the surface area.

We know that in the case of inviscid fluid, $\delta\vec{F}$ is along the direction of $\hat{n}$, so that there is only normal stress. On the other hand, in the case of viscous fluid, frictional forces are called into play between the surface and the fluid so that $\delta\vec{F}$ will now possess normal and tangential components $\delta\vec{F}_{nn}$ and $\delta\vec{F}_{ns}$. The normal and shear stresses are defind as follows.

The normal stress $= \lim\limits_{\delta s \to o} \dfrac{\delta F_{nn}}{\delta s}$

and the shear stress $= \lim\limits_{\delta s \to o} \dfrac{\delta F_{ns}}{\delta s}$

Now $\delta\vec{F}/\delta s$ tends to a definite number as $\delta s \to 0$. This number will depend not only on the position of the point P but also on the orientation of the area δs. Hence it is presented by the vector symbol $\vec{F}_n$ the subscript $\hat{n}$ indicates the direction of the normal to δs at P as discussed earlier.

$\vec{F}_n$ so defind, is called the stress vector or suface traction at P corresponding to the orientation $\hat{n}$ of the area. Thus we have

Stress vector $\vec{F}_n = \lim\limits_{\delta s \to 0} \dfrac{\delta\vec{F}}{\delta s}$.

1.7.3 Components of Stress Tensor

Let σ_{nx}, σ_{ny}, σ_{nz} be the Cartesian components of $\vec{F}_n$ and $\hat{i}$, $\hat{j}$, $\hat{k}$ be the unit vectors parallel to the axes. Then we have

$$\vec{F}_n = \sigma_{nx}\hat{i} + \sigma_{ny}\hat{j} + \sigma_{nz}\hat{k}$$

In particular, if the direction $\hat{n}$ is parallel to x - axis, we have

$$\vec{F}_x = \sigma_{xx}\hat{i} + \sigma_{xy}\hat{j} + \sigma_{xz}\hat{k}$$

Similary $\vec{F}_y = \sigma_{yx}\hat{i} + \sigma_{yy}\hat{j} + \sigma_{yz}\hat{k}$

and $\vec{F}_z = \sigma_{zx}\hat{i} + \sigma_{zy}\hat{j} + \sigma_{zz}\hat{k}$

In this way nine quantities are defined at a point, which may be arranged as follows.

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

The above mentioned nine quantities σ_{ij} constitute the components of the stress tensor of order two. It is expressed by τ_{ij} also.

We have used the double subscript notation for stress components. The first subscript denotes the direction of the normal to the plane on which the stress acts and the second subscript denotes the direction of the force producing the stress. It follows that normal stresses have repeated subscripts. The diagonal elements σ_{xx}, σ_{yy}, σ_{zz} are said to be normal stresses and the remaining six elements σ_{xy}, σ_{xz}, σ_{yx}, σ_{yz}, σ_{zx}, σ_{zy} are said to be shearing stresses. The matrix is said to be stress matrix.

1.8 Symmetry of Stress Tensor

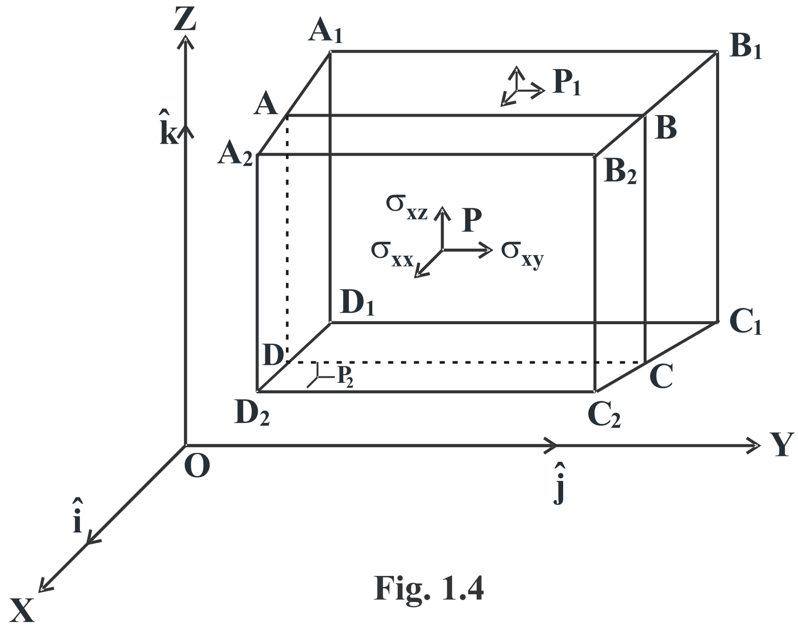


Fig. 1.4

In general, the motion of a fluid element can be separated into an instantaneous translation and an instantaneous rotation. Construct a parallelopiped whose edges of length $\delta x,\ \delta y, \delta z$ parallel to coordinate axes and P(x, y, z) be its centre as shown in fig. 1.4. We consider the motion of the above parallelopiped of viscous fluid.

Here we suppose that the fluid mass of the element $P\ \delta x\ \delta y\ \delta z$ remains constant and the coordinates of P_1 and P_2 be $\left(x-\frac{1}{2}\delta x,\ y,\ z\right)$ and $\left(x+\frac{1}{2}\delta x,\ y,\ z\right)$ respectively

At P, the force components on the face ABCD parallel to coordinate axes OX, OY, OZ are

$$\left[\sigma_{xx}\ \delta y\ \delta z,\ \sigma_{xy}\ \delta y\ \delta z,\ \sigma_{xz}\ \delta y\ \delta z,\right]$$

At P_2 the force components on the face $A_2B_2C_2D_2$ parallel to ABCD of area $\delta y\,\delta z$ parallel to axes are $\left[\left(\sigma_{xx}+\frac{\delta x}{2}.\frac{\partial\sigma_{xx}}{\partial x}\right)\delta y\delta z,\ \left(\sigma_{xy}+\frac{\delta x}{2}.\frac{\partial\sigma_{xy}}{\partial x}\right)\delta y\delta z,\ \left(\sigma_{xz}+\frac{\delta x}{2}.\frac{\partial\sigma_{xz}}{\partial x}\right)\delta y\delta z\right]$ where $\hat{\imath}$ is the unit normal vector measured outward.

At P_1, since $-\hat{\imath}$ is the unit normal vector measured outwards from the fluid the corresponding force components on the rectangular surfce $A_1B_1C_1D_1$ parallel to ABCD of area $\delta y\delta z$ are

$$\left[-\left(\sigma_{xx}-\frac{\delta x}{2}\frac{\partial\sigma_{xx}}{\partial x}\right)\delta y\delta z,-\left(\sigma_{xy}-\frac{\delta x}{2}.\frac{\partial\sigma_{xy}}{\partial x}\right)\delta y\delta z-\left(\sigma_{xz}-\frac{\delta x}{2}.\frac{\partial\sigma_{xz}}{\partial x}\right)\delta y\delta z\right]$$

Hence, the force on the parallel planes $A_2B_2C_2D_2$ and $A_1B_1C_1D_1$ passing thourgh P_2 and P_1 are equivalent to a single force at P with the components of force as

$$\left[\frac{\partial\sigma_{xx}}{\partial x}\delta x\,\delta y\,\delta z,\ \frac{\partial\sigma_{xy}}{\partial x}\delta x\,\delta y\,\delta z,\ \frac{\partial\sigma_{xz}}{\partial x}\delta x\,\delta y\,\delta z,\right]$$

together with couples whose moments are

$-\sigma_{xz}\delta x\ \delta y\ \delta z$ and $\sigma_{xy}\delta x\ \delta y\ \delta z$ along OY and OZ respectively.

Similarly, the components of force on the parallel planes perpendicular to the y-axis are

$$\left[\frac{\partial\sigma_{yx}}{\partial y}\delta x\,\delta y\,\delta z,\ \frac{\partial\sigma_{yy}}{\partial y}\delta x\,\delta y\,\delta z,\ \frac{\partial\sigma_{yz}}{\partial y}\delta x\,\delta y\,\delta z\right]$$

together with couple whose moments are

$-\sigma_{yx}\delta x\,\delta y\,\delta z$ and $\sigma_{yz}\delta x\,\delta y\,\delta z$ about OZ and OX respectively.

Similarly, the components of force on the parallel planes perpendicular to the z-axis are

$$\left[\frac{\partial\sigma_{zx}}{\partial z}\delta x\,\delta y\,\delta z,\ \frac{\partial\sigma_{zy}}{\partial z}\delta x\,\delta y\,\delta z,\ \frac{\partial\sigma_{zz}}{\partial z}\delta x\,\delta y\,\delta z\right]$$

together with couple whose moments are

$-\sigma_{zy}\delta x\,\delta y\,\delta z$ and $\sigma_{zx}\delta x\,\delta y\,\delta z$ about OX and OY respectively.

Thus, the surface forces on all the six faces of the rectangular parallelopiped are equivalent to a single force at P as

$$\left[\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}\right)\delta x\,\delta y\,\delta z,\ \left(\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zy}}{\partial y}\right)\delta x\,\delta y\,\delta z,\ \left(\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}\right)\delta x\,\delta y\,\delta z\right]$$

together with a vector couple having components as

$$\left[(\sigma_{yz}-\sigma_{zy})\delta x\,\delta y\,\delta z,\ (\sigma_{zx}-\sigma_{xz})\delta x\,\delta y\,\delta z,\ (\sigma_{xy}-\sigma_{yx})\delta x\,\delta y\,\delta z\right]$$

Now let X,Y and Z are the components of external body force perunit mass at P then components of the total body force on the parallelopiped are.

$$(X\rho\,\delta x\,\delta y\,\delta z,\ Y\rho\,\delta x\,\delta y\,\delta z,\ Z\rho\,\delta x\,\delta y\,\delta z,)$$

Taking moments about the OX through P, we get

total moment of forces = (moment of inertia about OX). (angular acceleration)

$$\Rightarrow (\sigma_{yz}-\sigma_{zy})\delta x\,\delta y\,\delta z+O_4=O_5$$

where O_4 and O_5 represent quantities of 4th and 5th order of smallness in $\delta x\,\delta y\,\delta z$. Hence, to the third order of smallness in $\delta x\,\delta y\,\delta z$, it reduces to

$$(\sigma_{yz}-\sigma_{zy})\delta x\,\delta y\,\delta z=0$$

or $\sigma_{yz}-\sigma_{zy}=0$

or $\sigma_{yz}=\sigma_{zy}$

similarly

$\sigma_{zx}=\sigma_{xz}$

and $\sigma_{xy}=\sigma_{yx}$

Thus, this shows that the stress tensor is symmetirc.

1.9 State of Stress at A Point

The state of stress at a point in the fluid is said to be completely known if the direction and magnitude of the stress vector at the point is known or can be determined from the known data for every possible orientation of area.

Theorem : The state of stress at a point is completely known if the nine components of stress tensor at that point are known.

Consider the motion of a small tetrahedron OABC.

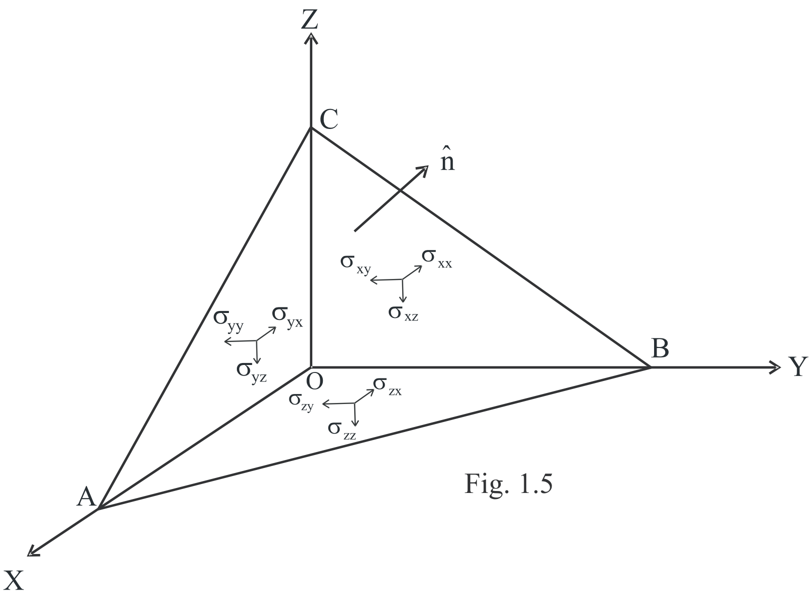


Fig. 1.5

Taking the faces of the tetrahedron OABC along the coordinate planes and face ABC has the area Δ. Let $\ell, \mathrm{m}, \mathrm{n}$ be the direction cosines of normal $\hat{n}$ to ABC drawn outwords. All the possible stresses on the fluid element of viscous fluid are shown in fig. 1.5.

Since the tetrahedron is small, the stress across every face may be taken to be uniform. Let the stress vectors on faces OBC, OCA, OAB and ABC are $\vec{F}_x, \vec{F}_y, \vec{F}_z$ and $\vec{F}_n$ respectively. Let $\vec{B}$ the body force per unit mass acting on the fluid element and $\vec{a}$ the acceleration of the element. By using Newton's second law of motion, the equation of motion of the tetrahedron gives

$$\Delta\vec{F}_n - \ell\,\Delta\vec{F}_x - m\Delta\vec{F}_y - n\,\Delta\vec{F}_z + \frac{1}{3}\rho\Delta\, þ\vec{B} = \frac{1}{3}\rho\Delta þ\vec{a} \text{ ....(1)}$$

Where þ is the perpendicular on ABC from O and $\frac{1}{3}\rho\Delta p$ the mass of tetrahedron, $-\Delta\ell - \Delta m - \Delta n$ are the areas of the faces BOC, COA and AOB respectively. Since the outward normal on the faces are in the negative directions of the axes, which follows that the direction cosine of the outward normal $\hat{n}$ with respect to the other three outward normals are $-\ell, -m, -n$.

Dividing (1) by Δ and assuming that the plane ABC approaches O moving parallel to itself so that $þ \to O$ and $\Delta \to o$ then we have

$$\vec{F}_n = \ell\,\vec{F}_x + m\vec{F}_y + n\vec{F}_z \text{ ....(2)}$$

and we know that

$$\vec{F}_n = \hat{\imath}\sigma_{nx} + \hat{j}\sigma_{ny} + \hat{k}\sigma_{nz}$$

$$\vec{F}_x = \hat{\imath}\sigma_{xx} + \hat{j}\sigma_{xy} + \hat{k}\sigma_{xz}$$

$$\vec{F}_y = \hat{\imath}\sigma_{yx} + \hat{j}\sigma_{yy} + \hat{k}\sigma_{yz}$$

$$\vec{F}_z = \hat{\imath}\sigma_{zx} + \hat{j}\sigma_{zy} + \hat{k}\sigma_{zz} \text{ ....(3)}$$

from (2) and (3), we obtain

$$\sigma_{nx} = \ell\sigma_{xx} + m\sigma_{yx} + n\sigma_{zx}$$

$$\sigma_{ny} = \ell\sigma_{xy} + m\sigma_{yy} + n\sigma_{zy}$$

$$\sigma_{nz} = \ell\sigma_{xz} + m\sigma_{yz} + n\sigma_{zz} \text{ ....(4)}$$

Which shows that, the state of stress at a point is completely known if the nine stress tensor components are known. We also express the equations (4) in the following matrix form

$$\begin{bmatrix} \sigma_{nx} \\ \sigma_{ny} \\ \sigma_{nz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix} \qquad ....(5)$$

We know from art 1.8 that the stress tensor is symmetric i.e.

$\sigma_{xy} = \sigma_{yx}$, $\sigma_{yz} = \sigma_{zy}$ and $\sigma_{xz} = \sigma_{zx}$.

Hence it follows that six components are sufficient to determine the state of stress at a point rather than nine components.

1.10 Plane Stress, Principal Stresses and Principal Directions

If we consider the state of stress in which

$$\sigma_{xy} = \sigma_{xz} = \sigma_{zz} = 0$$

then relation (4) of art. 1.9 will be reduced to

$$\sigma_{nx} = \ell\sigma_{xz} + m\sigma_{yx}$$

$$\sigma_{ny} = \ell\sigma_{xy} + m\sigma_{yy}$$

and $\sigma_{nz} = 0$ ....(1)

which is known as the plane stress. Let $\hat{n}$ be the normal to XY plane as shown in fig. 1.6. Here normal $\hat{n}$ inclined at angle θ with *x*-axis then

$\ell = \cos\theta$; $m = \sin\theta$

then (1) reduces to

$$\sigma_{nx} = \sigma_{xx}\cos\theta + \sigma_{yx}\sin\theta$$

$$\sigma_{ny} = \sigma_{xy}\cos\theta + \sigma_{yy}\sin\theta$$

$$\sigma_{nz} = 0 \qquad ....(2)$$

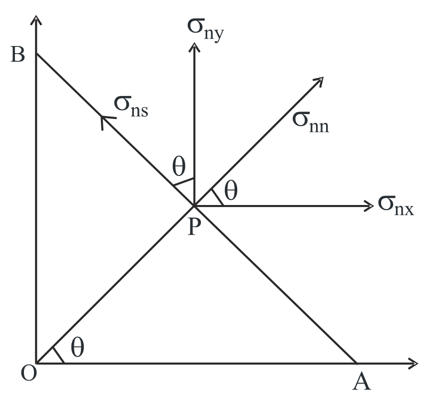


Fig. 1.6

when resolving the stress components along and perpendicular to the normal $\hat{n}$, we obtain normal component of stress

$$\sigma_{nn} = \sigma_{nx}\cos\theta + \sigma_{ny}\sin\theta$$

$$= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta$$

$$= \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta + \sigma_{xy}\sin 2\theta \qquad ....(3)$$

and tangential component of stress

$$\sigma_{ns} = \sigma_{ny}\cos\theta - \sigma_{nx}\sin\theta$$

$$= (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \sigma_{xy}\cos^2\theta - \sigma_{yx}\sin^2\theta$$

$$= -\frac{1}{2}(\sigma_{xx} - \sigma_{yy})\sin 2\theta + \sigma_{xy}\cos 2\theta \qquad ....(4)$$

If $\sigma_{ns} = 0 \Rightarrow \tan 2\theta = \dfrac{2\sigma_{xy}}{(\sigma_{xx} - \sigma_{yy})}$ ....(5)

If $\sigma_{xy} = 0$ and $\sigma_{xx} = \sigma_{yy}$ then θ be indeterminate.

then from (3) and (4) we have

$$\sigma_{nn} = \sigma_{xx} = \sigma_{yy}$$

and $\sigma_{ns} = 0$

This state is knwon as uniform plane stress.

If σ_{xy} =0 and $\sigma_{xx} \neq \sigma_{yy}$ then $\tan 2\theta = \tan 2\left(\frac{\pi}{2} + \theta\right)$.

Hence for any state of stress at a point, there are two mutually perpendicular directions, corresponding to which the tangential components of stress vanishes. These two directions given in relation (5) are knwon as the principal directions of stress at the point and the normal stress corresponding to them are called the pricipal stresses. The principal stresses are denoted by $\sigma_1, \sigma_2, \sigma_3$ and

$\sigma_1 = \sigma_{xx},\ \sigma_2 = \sigma_{yy}$

then $\sigma_{nn} = \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta$

$$\sigma_{ns} = \frac{1}{2}(\sigma_2 - \sigma_1)\sin 2\theta \quad \text{..........(6)}$$

1.11 Stress in a Fluid at Rest

When the fluid is at rest then the tangential stresses do not exist which states that the stress vector at any point of the fluid is normal to any plane surface passing through the point.

In this case the stress tensor are given by

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}$$

For such a state of stress, considering equilibrium of an infinitesimal tetrahedron, we may see that the magnitude of $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ at the point is the same for all elemental planes passing through the point. If σ denotes the stress at a point for time being and σ is positive then σ_{ij} represents a tensile stress, which is contrary to the experience, since in the interior of the fluid not any tensile stresses ocuur. This means that the nature of σ_{ij} is of compression and it will be appropriate to replace σ by $-þ$. Then the stress in a fluid at rest is given by

$$\sigma_{ij} = \begin{bmatrix} -þ & 0 & 0 \\ 0 & -þ & 0 \\ 0 & 0 & -þ \end{bmatrix}$$

or $\sigma_{ij} = -p\delta_{ij}$ $\left(\delta_{ij} = \begin{matrix} 0 & i \neq j \\ 1 & i = j \end{matrix}\right)$

for all orientations of the coordinate axes. The scalar þ is called the hydro-static pressure at the point.

1.12 Stress in a Fluid in Motion

When the fluid is in motion then both the tangential and normal stresses occur and the state of stress in a moving fluid can be expressed as

$\sigma_{ij} = -\text{þ}\,\delta_{ij} + \tau_{ij}$

where þ resprsents pressure, which is similar to but not identical with the hydrostatic pressure and τ_{ij} represents viscous or frictional stresses.

We know that the viscous stresses are assumed to be proportional to the rates of strain occuring at the point considered and proportionality constant, known as the viscosity coefficient and it depends on the nature of the fluid. Thus the viscous stresses occur only when the fluid is in non-uniform motion and the viscous stresses disappear and leaving the stress tensor as that of a uniform pressure in all directions in uniform motion. If the fluid is at rest then the viscous stresses become zero and the uniform pressure is known as the hydro static pressure. It is due to behaviour of τ_{ij} which is known as the viscous stress tensor.

1.13 Relation between Stress and Rate of Strain Components

Stokes made the following assumptions in order to find the relation between stress and rate of strain components.

(i) The stress components are linear functions of the rate of strain components.

(ii) The relation between stress components and rate of strain components are invariant to orientation of the coordinate axes.

(iii) When the velocity gradients are zero, the stress components must reduce to hydrostatic pressure.

We know that there are six independent stress components and six rate of strain components. To derive the relation between stress and rate of strain it is convinient to start with the principal axes of the two quadrics at a point, because of isotropy the principal axes of the stress quadric coincide with those of rate of strain quadric at every point in the continuum.

It is clear that referred to the principal axes $\sigma_{ij} = 0$ and $\in_{ij} = 0$ when $i \neq j$. In view of the first and third assumptions the non zero components of the stress tensor $\sigma_{11}, \sigma_{22}, \sigma_{33}$ are related to the non-zero components of the rate of strain tensor $\in_{11}, \in_{22}, \in_{33}$ in the following manner

$$\sigma_{11} = -\text{þ} + a_{11}\in_{11} + a_{12}\in_{22} + a_{13}\in_{33}$$

$$\sigma_{22} = -\text{þ} + a_{21}\in_{11} + a_{22}\in_{22} + a_{23}\in_{33}$$

$$\sigma_{33} = -\text{þ} + a_{31}\in_{11} + a_{32}\in_{22} + a_{33}\in_{33} \quad \text{.........................(1)}$$

Where the a_{ij} are the constants to be determined.

In view of second assumption, any permutation of the $\in$'s must effect the same permutation of σ's.

Now, permute the $\in_{11}, \in_{22}, \in_{33}$ to $\in_{22}, \in_{33}, \in_{11}$ (rotation of axes) and re-arrange to obtain.

$$\sigma_{22} = -\text{þ} + a_{13}\in_{11} + a_{11}\in_{22} + a_{12}\in_{33}$$

$$\sigma_{33} = -\text{þ} + a_{23}\in_{11} + a_{21}\in_{22} + a_{22}\in_{33}$$

$$\sigma_{11} = -\text{þ} + a_{33}\in_{11} + a_{31}\in_{22} + a_{32}\in_{33} \quad \text{..(2)}$$

Also, by interchanging the axes 1 and 2, in the first relation of set of relation (1), we have

$$\sigma_{22} = -\text{þ} + a_{11}\in_{22} + a_{12}\in_{11} + a_{13}\in_{33} \quad \text{..(3)}$$

on comparing relations (1), (2) adn (3), we obtain

$a_{12} = a_{21} = a_{13} = a_{23} = a_{32} = a_{31} = \lambda$ (say)

and $a_{11} = a_{22} = a_{33} = \lambda + 2\mu$ (say)..(4)

where λ and μ are the moment numbers whose physical meaning have to be obtained

On using (4), the set of equatin (1) become

$$\sigma_{11} = -\text{Þ} + 2\mu \in_{11} + \lambda\left(\in_{11} + \in_{22} + \in_{33}\right)$$

$$\sigma_{22} = -\text{Þ} + 2\mu \in_{22} + \lambda\left(\in_{11} + \in_{22} + \in_{33}\right)$$

$$\sigma_{33} = -\text{Þ} + 2\mu \in_{33} + \lambda\left(\in_{11} + \in_{22} + \in_{33}\right) \qquad \text{.............(5)}$$

Now these equation (5) plus the six equations implict in the fact that $\sigma_{ij} = \in_{ij} = 0$ when $i \neq j$, may all be combined into a single tensor equations as

$$\sigma_{ij} = -\text{Þ}\,\delta_{ij} + 2\mu \in_{ij} + \lambda \in_{kk} \delta_{ij} \qquad \text{.............(6)}$$

which is the required relationship between the stress components and rate of strain components for arbitrary choice of the coordinate axes.

We know that the state of stress in a moving fluid given by

$$\sigma_{ij} = -\text{Þ}\,\delta_{ij} + \tau_{ij} \qquad \text{.............(7)}$$

From (6) and (7) ; we have the relationship between the components of the viscous stress tensor and rate of strain tensor given by

$$\tau_{ij} = 2\mu \in_{ij} + \lambda \in_{kk} \delta_{ij} \qquad \text{.............(8)}$$

If considered a state of shearing motion ; the velocity field be $\vec{V} = \left(v_1(x_2), 0, 0\right)$ then we have

$$\in_{12} = \in_{21} = \frac{1}{2}\frac{dv_1}{dx_2} \qquad \text{.............(9)}$$

Thus ; from (8) and (9) we obtain

$$\tau_{12} = \tau_{21} = \mu\frac{dv_1}{dx_2} \qquad \text{.............(10)}$$

and all the other viscous stresses are zero. μ be the coefficient of viscosity and λ is important only in the case of compressible fluids because in an incompressible fluid it does not paly any part.

1.14 Stoke's Law of Friction

We know that the relationship between the components of the viscous stress tensor and rate of strain tensor is given by

$$\tau_{ij} = 2\mu \in_{ij} + \lambda \in_{kk} \delta_{ij} \qquad \text{.............(1)}$$

It can be written as

$$\tau_{ij} = -\frac{2}{3}\mu \in_{kk} \delta_{ij} + 2\mu \in_{ij} + \left(\lambda + \frac{2}{3}\mu\right) \in_{kk} \delta_{ij} \qquad \text{.............(2)}$$

Clearly $\tau_{ii} = \left(3\lambda + 2\mu\right)e_{ii}$ ....(3)

$$\text{or} \quad \frac{\tau_{ii}/3}{\in_{ii}} = \left(\lambda + \frac{2}{3}\mu\right) = K(\text{say})$$

Where K is the bulk viscosity. The bulk viscosity defined as the ratio of the mean normal viscous stress to the rate of volumetric strain in a state of pure dilation. Stokes assumed that bulk viscosity K=0 then

$$\lambda = -\frac{2}{3}\mu \qquad \text{.............(4)}$$

Using (4) in equation (1), we have

$$\tau_{ij} = 2\mu \in_{ij} - \frac{2}{3}\mu \in_{kk} \delta_{ij} \quad \text{...(5)}$$

which is the required relationship between viscous stress tensor and rate of strain tensor and is known as "Stoke's law of friction".

1.15 Thermal Conductivity

The study of heat transfer has great importance in different branches of science and technology, In all types of substances viz solid, liquid and gases, the temperature difference reduces with the lapse of time flowing heat from the region of higher temperature to the region of lower temperature.

Basically, there are three modes of heat transfer viz conduction, convection and radiation. In solids, the process of heat transfer takes place by the mode of conduction, while in liquid and gases the process of heat transfer takes place by the mode of conduction, convection and radiation simultaneously.

The process of heat transfer takes place in solid due to transfer of internal energy from one molecule to another, known as conduction. Fourier's law of heat conduction state that the conductive heat flow per unit area is proportional to the temperature gradient.

$$q = -k\frac{dT}{dy}$$

Where k is the constant of proportionality and is known as the coefficient of thermal conductivity and the negative sign show that heat flow is in the direction of decreasing temperature.

1.16 Generalized Law of Heat Conduction

In an isotropic medium in which the temperature varies in all three direction then Fourier's law of heat conduction for each of the coordinate directions :

$$q_1 = -k\frac{\partial T}{\partial x_1}, \qquad q_2 = -k\frac{\partial T}{\partial x_2}, \qquad q_3 = -k\frac{\partial T}{\partial x_3}$$

We may write these three relations in Cartesian tensor notation as

$$q_i = -k\frac{\partial T}{\partial x_i}$$

which is the three dimensional form of Fourier's law. It states that the heat flux vector $\vec{q}$ is proportional to the temperature gradient ∇T and is oppositely directed. The ratio of thermal conductivity k to the product of density ρ and specific heat C_p is known as the thermal diffusivity, which is usually denoted by a and given by

$$a = \frac{k}{\rho C_p}$$

The unit of thermal diffusivity a is the same as that of kinematic viscosity i.e.

$$[a] = \frac{\text{meter}^2}{sec.} = L^2T^{-1}$$

1.17 Specific Heat

The specific heat C of a fluid is defined as the amount of heat required to raise the temperature of a unit mass of the fluid by one degree. Thus $C = \frac{\partial Q}{\partial T}$, where Q is the quantity of heat added per unit mass of the fluid.

The specific heat in fact depends on the process in which the heat is added. Usually the process considered is either at constant volume or at constant perssure, thus we have

Specific heat at constant volume $C_v = \left(\frac{\partial Q}{\partial T}\right)_V$

and specific heat at constant pressure $C_p = \left(\frac{\partial Q}{\partial T}\right)_p$

The ratio of the two specific heats is usually denoted by the symbol $\gamma = \frac{C_p}{C_v}$ and is known as the "adiabatie exponent" of the gas.

Self Learning Exercise

1. Fill in the blanks in following
   (a) Fluid which obeys Newton's law of viscosity is known as..................................
   (b) The ratio of the mean normal viscous stress to the rate of volumetric strain is known as ....
   (c) The ratio of thermal conductivity to the product of density and specific heat is known as ....
   (d) The dimensions of the coefficient of viscosity is ....
   (e) the coefficient of viscosity depends only on ....
2. Define normal and shearing strain.
3. What do you mean by stress vector ?
4. What is viscous stress tensor ?
5. Write down the Stoke's law of friction.

Example . 1

The stress tensor at a point P is

$$\sigma_{ij} = \begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ 2 & 0 & 4 \end{pmatrix}$$

Deternime the stress vector on the plane at P whose unit normal is $\hat{n} = \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$.

Solution :

We know that $\hat{n} = \ell\,\hat{i} + m\,\hat{j} + n\,\hat{k}$

then here $\ell = \frac{2}{3}$, $m = -\frac{2}{3}$ and $n = \frac{1}{3}$ then we have

$$\begin{bmatrix} \sigma_{nx} \\ \sigma_{ny} \\ \sigma_{nz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \ell \\ m \\ n \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \sigma_{nx} \\ \sigma_{ny} \\ \sigma_{nz} \end{bmatrix} = \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2/3 \\ -\frac{2}{3} \\ 1/3 \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} \sigma_{nx} \\ \sigma_{ny} \\ \sigma_{nz} \end{bmatrix} = \begin{bmatrix} 4 \\ -\frac{10}{3} \\ 0 \end{bmatrix}$$

Hence the stress vector $\vec{F}_n$ is given by

$$\vec{F}_n = \hat{i}\,\sigma_{nx} + \hat{j}\,\sigma_{ny} + \hat{k}\,\sigma_{nz}$$

$$\vec{F}_n = 4\,\hat{i} - \frac{10}{3}\,\hat{j}$$

Which is the required stress vector.

Example : 2

What type of the motion do the following velocity components constitute ?

$u = a + by - cz$; $v = d - bx + ez$

and $w = f + cx - ey$ where a, b, c, d, e, f are arbitrary constants.

Solution :

Let $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$ be the velocity at a point P and $q^1 = q + dq$ be the velocity at a neighbouring point Q. Then we known from article. 1.5 that

$$\vec{q}^{\,1} = \vec{q} + \vec{W} \times d\vec{r} + \vec{D}$$

here given that $u = a + by - cz$; $v = d - bx + ez$

and $w = f + cx - ey$

$$\vec{W} = \frac{1}{2}\left[\hat{i}\left(\frac{\partial \omega}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial \omega}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]$$

$$\vec{W} = \frac{1}{2}\left[-2e\,\hat{i} - 2c\hat{j} - 2b\hat{k}\right] = -\left(e\hat{i} + c\hat{j} + b\hat{k}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\therefore \; \vec{W} \times d\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -e & -c & -b \\ dx & dy & dz \end{vmatrix}$$

$$= \hat{i}\left(b\,dy - c\,dz\right) + \hat{j}\left(e\,dz - b\,dx\right) + \hat{k}\left(c\,dx - e\,dy\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now $\in_{xx} = \frac{\partial u}{\partial x} = 0$; $\in_{yy} = \frac{\partial v}{\partial y} = 0$; $\in_{zz} = \frac{\partial \omega}{dz} = 0$

and $\in_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(b - b\right) = 0$

$$\in_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial \omega}{\partial y}\right) = \frac{1}{2}(e - e) = 0$$

$$\in_{zx} = \frac{1}{2}\left(\frac{\partial \omega}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{1}{2}(c - c) = 0$$

Hence $D = 0$

Hence the motion of fluid of motion be translatory motion with velocity $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$ and rotational motion with velocity $\vec{W} \times d\vec{r}$. The rate of strain $\vec{D} = 0$ which means that motion is free from deformation.

Hence the rigid body motion

Example : 3

Velocity field at point is given by 1+2y-3z, 4-2x+5z, 6+ 3x -5y. Show that it represent a rigid body motion.

Solution :

Here given that $v = 4 - 2x + 5z$, $w = 6 + 3x - 5y$ and $u = 1 + 2y - 3z$

Here $\vec{W} = \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}\right]$

$$\vec{W} = -\left(5\hat{i} + 3\hat{j} + 2\hat{k}\right)$$

$$\therefore \quad \vec{W} \times d\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & -3 & -2 \\ dx & dy & dz \end{vmatrix}$$

or $\quad \vec{W} \times d\vec{r} = (2dy - 3dz)\hat{i} + (5dz - 2dx)\hat{j} + (3dx - 5dy)\hat{k}$ ....(2)

and $\quad \in_{xx} = \frac{\partial u}{\partial x} = 0 \quad ; \quad \in_{yy} = \frac{\partial v}{\partial y} = 0 \quad ; \quad \in_{xx} = \frac{\partial w}{\partial z} = 0$

$$\in_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}(2 - 2) = 0$$

$$\in_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}(5 - 5) = 0$$

$$\in_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{1}{2}(3 - 3) = 0$$

Hence $\quad \vec{D} = 0$

Hence the motion of a fluid element is made up of only two parts viz pure translation and pure rotation without any deformation. So the given velocity distribution represents a rigid body motion.

1.18 Summary

This unit is devoted to the study of strain and stress. We have studied about stress tensor, stokes law of friction, thermal conductivity and specific heat also.

1.19 Answer to self learning exercise

1. (a) Newtonian fluid

   (b) Bulk Viscosity

   (c) Thermal diffusivity

   (d) $ML^{-1}T^{-1}$

   (e) Nature of the Fluid.

2. See article 1.6

3. $\vec{F}_n = \lim_{\delta s \to 0} \frac{\delta \vec{F}}{\delta s}$

4. See article 1.12

5. See article 1.14

1.20 Exercise

1. Write short notes on

   (a) Viscosity

   (b) Thermal conductivity

2. Define the stress at a point in a fluid and show that it is a symmetric second order tensor

3. Define Stoke's law of friction

4. Distinguish between body and surface force. Define stress at a point

5. What do you mean by thermal conductivity

6. Show that the erate of strain tensor is a symmetric tensor

7. Show that the following velocity components represent a rigid body motion

   $u = a + by - cz \quad ; \quad v = d - bx + ez \quad ; \quad w = f + cx - ey$

   where a, b, c, d e and f are arbitrary constants.

UNIT - 2

Fundamental Equations of the Flow of Viscous Fluids

Structure of the Unit

2.0 Objectives

This unit provides a general overview about fundamental equations of the flow of viscous fluids. After reading this unit you will be able to understand about the equation of state, equation of continuity, equation of motion and equation of energy. Here in this unit you will also consider the vorticity and circulation in a viscous incompressible fluid motion.

2.1 Introduction

The fundamental equations of the flow of viscous compressible fluids are

(a) Equation of state ; (one)
(b) Equation of continuity ; (one)
(c) Equation of motion ; (three)
(d) Equation of Energy ; (one)

These equations are mathematical expressions of basic physical concepts. These are six in number and therefore determine the six unknowns of the fluid motion i.e. velocity components (3), the temperature, the pressure and the density, which all are the functions of both space coordinates and time.

2.2 Equation of State

The equation of state of a substance is a relation between the pressure, temperature and the density. It is an experimental fact that a relationship between these three thermodynamic variables exist so there exist an equation of state corresponding to a given homogenous substance, solid, liquid or gas. The relationship may be expressed as

$f(p, \rho, T) = 0$ ....(1)

which is known as the "Equation of state". The exact nature of the function f is, in general, very complicated and varies with fluid. For gases, at high temperature or low pressure the relation (1) can be written as

$$\frac{p}{\rho RT} = 1 + \rho B(T) + C(T)\rho^2 + \text{..}(2)$$

where B(T), C (T), ... are the function of temperature only and R is the gas constant. For the perfect gas or an ideal gas, the equation of state is given as

$$p = \rho RT \text{ ....}(3)$$

and it is called Boyle's Law. The equation of state of a viscous compressible fluid will be taken as the equation of state of a perfect gas. If the fluid be incompressible then the equation of state is simply

ρ = constant

2.3 Equation of continuity

The equation of continuity aims at expressing the law of conservation of mass in mathematical form. the law of conservation of mass states that fluid mass can neither be created nor destroyed.

Thus, in a continuous motion, the equation of continuity expresses the fact the increase in the mass of the fluid within any closed surface drawn in the fluid in any time must be equal to the excess of the mass that flows in over the mass that flows out.

2.3.1 Equation of continuity in Cartesian tensor notation :

Let us consider a closed surface S, enclosing a fixed volume V in the region occupied by the moving fluid. If η_j be the normal unit vector in the outward direction to the elementary surface ds of the closed surface S and v_j be the velocity of the fluid at the poiint, then the inward normal velocity is $-v_j\eta_j$.

Thus the mass of the fluid entering per unit time through the element dS is $-\rho v_j \eta_j \, dS$.

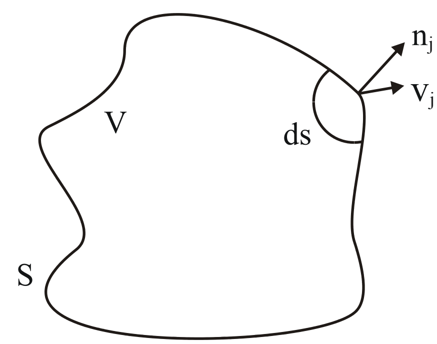


Fig. 2.1

Hence the total mass of the fluid entering per unit time through the surface S is

$$-\int_S \rho v_j \eta_j \, dS \text{ ....}(1)$$

and the mass of the fluid within the closed surface S is

$$-\int_V \rho \, dv \text{ ....}(2)$$

Therefore the rate of mass increases within surface S is simply

$$\frac{\partial}{\partial t}\int_V \rho \, dv \quad \text{or} \quad \int_V \frac{\partial \rho}{\partial t} dv \text{ ....}(3)$$

Here the differentiation and integration being interchangeable because a fixed volume is considered. Now, by the law of conservation of the fluid mass, the rate of increase the mass of fluid through S must be equal to the total rate of mass flowing into V.

$$\text{Hence} \quad \int_V \frac{\partial \rho}{\partial t} dv = -\int_S \rho v_j \eta_j ds \text{ ....}(4)$$

On applying Gauss's theorem, we have

$$\int_V \frac{\partial \rho}{\partial t} dv = -\int_V \frac{\partial(\rho v_j)}{\partial x_j}.dv \quad \text{or} \quad \int_V \left\{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_j)}{\partial x_j}\right\} dv = 0$$

Since V is an arbitrary chosen volume, we deduce that

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v_j)}{\partial x_j}=0$$

which is the required equation of continuity in Cartesian tensor notation.

2.3.2 Equation of Continuity in Vector form

Let S be an arbitrary small closed surface drawn in the compressible fluid enclosing a volume V and let S be taken fixed in space. Let δs denote the element of the surface S enclosing P and $\hat{n}$ be the unit normal outward drawn at δs and $\vec{q}$ be the fluid velocity at P.

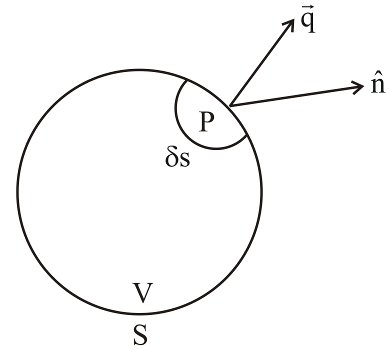


Fig. 2.2

The rate of mass flow across $\delta s = \rho(\hat{n}.\vec{q})\delta s$

Then the total rate of mass flow across the surface S is

$$= \int_S \rho(\hat{n}.\vec{q})dS$$

On using Gauss-divergence theorem, the total rate of mass flow across surface

$$S = \int_V \nabla.(\rho\vec{q})dV$$

The total rate of mass flow into volume $V = -\int_V \nabla.(\rho\vec{q}).dV$ ....(1)

Again the mass of the fluid within S at time $t = \int_V \rho dV$

Hence the total rate of mass increase within surface $S = \frac{\partial}{\partial t}\int_V \rho dV \;=\; \int_V \frac{\partial \rho}{\partial t}dV$ ....(2)

By using concept of continuity, (using (1) and (2) we have

$$\int_V \frac{\partial \rho}{\partial t}dV \;=\; -\int_V \nabla.(\rho\vec{q})dV$$

or $$\int_V\left[\frac{\partial \rho}{\partial t}+\nabla.(\rho\vec{q})\right]dV=0$$

or $$\frac{\partial \rho}{\partial t}+(\nabla.\rho\vec{q})=0 \quad ....(3)$$

$$\frac{D\rho}{Dt}+\rho \quad \text{div } \vec{q}=0$$

where $\frac{D}{Dt} \cong \frac{\partial}{\partial t}+(\vec{q}.\nabla)$ is known as the material derivative or differential following the motion.

In the case of steady compressible flow when $\frac{\partial}{\partial t}=0$ the equation of continuity reduces to

$$\text{div } (\rho\vec{q})=0$$

If ρ is homogeneous and has the same constant value throughout the fluid, the equation of continuity reduces to

$$\text{div } \vec{q}=0 \quad ....(5)$$

2.4 Navier - Stokes Equations of Motion

The equations of motion are derived from Newton's second law of motion which states that the rate of change of linear momentum = total applied force

Let us consider a closed surface S, enclosing a volume V in the ragion occupied by the moving fluid. The rate of change in momentum along the element ds is $v_i(-\rho v_j \eta_j)ds$ Therefore, the rate of change in momentum enters the controlled surface S is

$$-\int_S v_i(\rho v_j \eta_j \, ds) \quad \text{...(1)}$$

The rate at which the momentum increases in the enclosed volume V is

$$\frac{\partial}{\partial t}\int_V \rho v_i \, dv \quad \text{...(2)}$$

From relation (1) and (2) the rate of change in linear momentum is given by

$$\frac{\partial}{\partial t}\int_V \rho v_i \, dv + \int_S v_i(\rho v_j \eta_j)ds \quad \text{..........................(3)}$$

In the fluid motion, there are two forces (i) force acting throughout the mass of the body of fluid, such as gravitational forces, known as body forces and (ii) forces acting on the boundary, the fluid stresses and are known as surface stresses. If f_i be the body forces per unit mass and P_i be the force on the boundary per unit area then the total applied force is given by

$$\int_V \rho f_i \, dv + \int_S P_i \, ds \quad \text{..(4)}$$

Where the stress vector P_i is given by

$$P_i = \sigma_{ij} n_j \quad \text{and} \quad \sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

Using (3) to (5) the equation of motion can be written as

$$\frac{\partial}{\partial t}\int_V \rho v_i \, dv + \int_S v_i(\rho v_j \eta_j)ds = \int_V \rho f_i \, dv + \int_S P_i \, ds$$

On using Gauss-divergence theorem, it reduces to

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial t}(\rho v_i v_j) = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$$

or $$\rho\frac{\partial v_i}{\partial t} + v_i\frac{\partial \rho}{\partial t} + v_i\frac{\partial(\rho v_j)}{\partial x_j} + \rho v_j\frac{\partial v_i}{\partial x_j} = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$$

using $\frac{\partial(\rho v_j)}{\partial x_j} = -\frac{\partial p}{\partial t}$ from equation of continuity we have

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \quad \text{...(6)}$$

which is valid for all continuous fluid medium.

Now, to use these equation to determine the velocity distribution, we insert the expression for the viscous stresses in terms of velocity gradients and fluid properties. For isotropic Newtonian fluid these expressions are given by the constitutive equation

$$\tau_{ij} = 2\mu \in_{ij} - \frac{2}{3}\mu \in_{kk} \delta_{ij} \qquad \text{...(7)}$$

where $$\in_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad \text{..(8)}$$

Using (7), (8) in equation (6), we finally get

$$\rho\left\{\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right\} = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}\right)\right] \text{.........(9)}$$

Equations (9) are knwon as Navier- Stoke's equations for the motion of a viscous compressible fluid and are three in number. Taking μ as constant and

$$\frac{\partial}{\partial x_j}\left(\frac{\partial v_j}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}\right) = 0$$

it reduces for incompressible fluids to

$$\rho\left(\frac{\partial v_j}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) = \rho f_i - \frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j} \qquad \text{..(10)}$$

Equation (10) in vector notation can be written as

$$\rho\frac{D\vec{q}}{Dt} = \rho\vec{F} - \nabla p + \mu\nabla^2\vec{q} \qquad \text{...(11)}$$

where $\frac{D\vec{q}}{Dt} \cong \frac{\partial}{\partial t} + (\vec{q}.\nabla)$ is the "material derivative" as defined earlier.

2.5 Equation of Energy

Consider the motion of a viscous compressible Newtonian fluid. Here we consider the conservation of energy on the basis of the first law of thermodynamics. The conservation of energy requires that, the difference in the rate of supply of energy to a controlled surface S enclosing a volume V in the region occupied by a moving fluid and rate at which the energy goes out through S must be equal to the net rate of increase of energy in the enclosed volume V.

This can be easily written in an equation form as the rate of heat which is produced by external sources + the rate at which heat is produced by the work of the surface stresses — the rate of energy loss by heat – the rate of energy loss by heat convection = the rate of increase of energy in the enclosed volume.

or

$$\int_V \frac{\partial Q}{\partial t}dv + \int_S (\sigma_{ij}\eta_j)ds - \int_S E_t\,\rho v_j \eta_j\,ds - \int_S q_j\eta_j\,ds = \frac{\partial}{\partial t}\int_V \rho E_t dv \text{.......................................(1)}$$

where E_t be the total energy of the system per unit mass. If K and I are potential energy and internal energy then

$$E_t = \frac{1}{2} v_i v_i + K + I \qquad \text{...(2)}$$

The heat flux vector q_j is given by the generalized heat conduction law

$$q_j = -\kappa \frac{\partial T}{\partial x_j} \qquad \text{..(3)}$$

Using (3) in equation (1), and changing the surface integral into volume integral by Gauss's divergence theorem and taking V to be an arbitrary volume, then we get the qeuation of energy as :

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x_j}\left(v_i \sigma_{ij}\right) - \frac{\partial}{\partial x_j}\left(E_t \rho v_j\right) + \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j}\right) - \frac{\partial}{\partial t}\left(E_t \rho\right) \text{......................(4)}$$

To simplify the energy equation (4) we are assuming following relations

(i) Using the material derivative $\frac{D}{Dt} \cong \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x_j}$ and combining the third and fifth terms of equation (4) we have

$$\frac{\partial}{\partial x_j}\left(E_t \rho v_j\right) + \frac{\partial}{\partial t}\left(E_t \rho\right) = E_t \left\{\frac{\partial \rho}{dt} + \frac{\partial}{\partial x_j}\left(\rho v_j\right)\right\} + \rho\left(\frac{\partial E_t}{\partial t} + v_j \frac{\partial E_t}{\partial x_j}\right) \text{.........(5)}$$

We know that the equation of continuity in the cartesian tensor notation is

$$\frac{\partial \rho}{\partial t} + \frac{\partial\left(\rho v_j\right)}{\partial x_j} = 0 \qquad \text{..(6)}$$

Using equation (6) in (5) we obtain

$$\frac{\partial}{\partial x_j}\left(E_t \rho v_j\right) + \frac{\partial}{\partial t}\left(E_t \rho\right) = \rho \frac{DE_t}{Dt} \qquad \text{......................(7)}$$

Again using relation (2) in R.H.S. of relation (7) we have

$$\rho \frac{DE_t}{Dt} = \rho\left[v_i \frac{Dv_i}{Dt} + \frac{DI}{Dt} + \frac{\partial k}{\partial t} + v_j \frac{\partial k}{\partial x_j}\right]$$

Since potential energy is independent of time and depends only on space coordinates hence $\frac{\partial k}{\partial t} = 0$

$$\Rightarrow \rho \frac{DE_t}{Dt} = \rho\left[v_i \frac{Dv_i}{Dt} + \frac{DI}{Dt} + v_j \frac{\partial k}{\partial x_j}\right] \text{...........................(8)}$$

(ii) The equation of continuity is

$$\rho \frac{Dv_i}{Dt} = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \qquad \text{..(9)}$$

and $\quad \sigma_{ij} = -p\delta_{ij} + \tau_{ij}$ ....(10)

Using (10) in (9), it reduces to

$$\rho\frac{Dv_i}{Dt} = \rho f_i + \frac{\partial}{\partial x_j}(\sigma_{ij})$$

Here Kis potential energy then $f_i = -\frac{\partial K}{\partial x_i}$

Hence $\frac{\partial}{\partial x_j}(\sigma_{ij}) = \rho\left(\frac{Dv_i}{Dt} + \frac{\partial k}{\partial x_i}\right)$ ....(11)

Now the second term in equation (4) i.e. $\frac{\partial}{\partial x_j}(v_i\sigma_{ij})$ can be written as

$$\frac{\partial}{\partial x_j}(v_i\sigma_{ij}) = v_i\frac{\partial}{\partial x_j}(\sigma_{ij}) + \sigma_{ij}\frac{\partial v_i}{\partial x_j}$$

$$= \rho v_i\left(\frac{Dv_i}{Dt} + \frac{\partial k}{\partial x_i}\right) + \left(-p\delta_{ij} \_ \tau_{ij}\right)\frac{\partial v_i}{\partial x_j}$$

$$= \rho v_i\left(\frac{Dv_i}{Dt} + \frac{\partial k}{\partial x_i}\right) - p\frac{\partial v_i}{\partial x_j} + \tau_{ij}\frac{\partial v_i}{\partial x_j}$$

$$= \rho v_i\left(\frac{Dv_i}{Dt} + \frac{\partial k}{\partial x_i}\right) - p\frac{\partial v_i}{\partial x_j} + \phi \quad ....(12)$$

where $\phi = \tau_{ij}\frac{\partial v_i}{\partial x_j}$ ....(13)

We know that the constitutive equation for an isotropic Newtonian fluid is

$$\tau_{ij} = 2\mu \in_{ij} - \frac{2}{3}\mu \in_{kk} \delta_{ij} \quad ....(14)$$

Now using (14) in (13) we have

$$\phi = 2\mu\left\{\left(\in_{ij} - \frac{1}{3}\in_{kk}\delta_{ij}\right)\right\}\frac{\partial v_i}{\partial x_j}$$

$$\phi = \mu\left\{\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right\}\frac{\partial v_i}{\partial x_j} \quad ....(15)$$

is the heat generated due to frictional forces and known as dissipations function.

Hence the equation of energy (4), with the help of equations (7), (8) and (12) can be simplified to

$$\frac{\partial Q}{\partial t} + \rho v_i\frac{Dv_i}{Dt} + \rho v_i\frac{\partial k}{\partial x_j} - p\frac{\partial v_i}{\partial x_i} + \phi - \rho v_i\frac{Dv_i}{Dt} - \rho\frac{DI}{Dt} - \rho v_j\frac{\partial k}{\partial x_j} + \frac{\partial}{\partial x_j} + \left(\kappa\frac{\partial T}{\partial x_j}\right) = 0$$

or $\frac{\partial Q}{\partial t} - p\frac{\partial v_i}{\partial x_i} - \rho\frac{DI}{Dt} - \frac{\partial}{\partial x_j}\left(\kappa\frac{\partial T}{\partial x_j}\right) + \phi = 0$ ....(16)

Again using the equation of continuity, we have

$$p\frac{\partial v_i}{\partial x_i} = p\left(-\frac{1}{\rho}\frac{D\rho}{Dt}\right) = \frac{p}{\rho}\cdot\frac{D\rho}{Dt} = p\rho.\frac{D}{Dt}\left(\frac{1}{\rho}\right)$$

Thus, the energy equation (16) reduces to

$$\rho\left[\frac{DI}{Dt} + p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right] = \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x_j}\left(\kappa\frac{\partial T}{\partial x_j}\right) + \phi \quad \text{.............................(17)}$$

which is the energy equations in terms of internal energy I and fluid temperature T.

For the perfect gas, we know that

$$p = \rho RT \text{ and } C_p - C_v = R$$

then the internal energy I is given as

$$I = C_v T = (C_p - R)T$$

or $$I = C_p T - \frac{p}{\rho} \quad \text{..(18)}$$

On substituting the value I from (18) in (17) it reduces to

$$\rho\left[\frac{D}{Dt}(C_p T) - \frac{D}{Dt}\left(\frac{p}{\rho}\right) + p\frac{D}{Dt}\left(\frac{1}{\rho}\right)\right] = \frac{\partial Q}{\partial T} + \frac{\partial}{\partial x_j}\left(\kappa\frac{\partial T}{\partial x_j}\right) + \phi$$

or $$\rho\frac{D}{Dt}(C_p T) = \frac{Dp}{Dt} + \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x_j}\left(\kappa\frac{\partial T}{\partial x_j}\right) + \phi \quad \text{............................(19)}$$

For incompressible fluid

$$\frac{\partial v_i}{\partial x_i} = 0 \quad ; \quad I = C_v T,$$

The energy equation (16) with constant viscosity and heat conductivity, becomes

$$\rho C_v \frac{DT}{Dt} = \frac{\partial Q}{\partial t} + \kappa\frac{\partial}{\partial x_i}\left(\frac{\partial T}{\partial x_i}\right) + \phi$$

where $$\phi = \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\frac{\partial v_i}{\partial x_j} \quad \text{..(20)}$$

2.6 Boundary Conditions

The solution of the fundamental equations of the flow of viscous fluids becomes fully determined physically when the boundary and initial conditions are specified. The initial condition will be studied in the flow problems and the boundary conditions are studied in geometrical considerations together with the no slip conditions.

2.7 Vorticity

The Navier-Stokes equations for a viscous incompressible fluid motion may be interpreted as the vorticity transport equations, if we assume that the external forces are conservative then they can be derived from a force potential V_f such that

$$\vec{F} = -\nabla V_f \quad \text{...(1)}$$

Using Lagrange's vector identity

$$(\vec{q}.\nabla)\vec{q} = \nabla\left(\frac{1}{2}\vec{q}^2\right) - \vec{q}\times\vec{\Omega} \quad \text{..............................(2)}$$

Where $\vec{\Omega} = \nabla \times \vec{q}$ ....(3)

$\vec{\Omega}$ is the vorticity vector. The Navier Stokes. equations may be written as

$$\frac{\partial \vec{q}}{\partial t} - \left(\vec{q}\times\vec{\Omega}\right) = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}\vec{q}^2 + v_f\right) + \nu\nabla^2\vec{q}$$

Taking the curl to both sides of the equation and keeping in view that curl of a gradient is zero then we obtain

$$\frac{\partial \vec{\Omega}}{\partial t} - \nabla\times\left(\vec{q}\times\vec{\Omega}\right) = \nu\nabla^2\vec{\Omega} \quad \text{......................................(4)}$$

Using the vector identity

$$\nabla\times\left(\vec{q}\times\vec{\Omega}\right) = \left(\vec{\Omega}.\nabla\right)\vec{q} - \left(\vec{q}.\nabla\right)\vec{\Omega} + \vec{q}\nabla.\vec{\Omega} - \vec{\Omega}\nabla.\vec{q}$$

$$\nabla\times\left(\vec{q}\times\vec{\Omega}\right) = \left(\vec{\Omega}.\nabla\right)\vec{q} - \left(\vec{q}.\nabla\right)\vec{\Omega} \quad \text{...............................(5)}$$

Because $\nabla.\vec{q} = 0$ and $\nabla.\vec{\Omega} = 0$

On using (5) in (4) it reduces to

$$\frac{\partial \vec{\Omega}}{\partial t} + \left(\vec{q}.\nabla\right)\vec{\Omega} = \left(\vec{\Omega}.\nabla\right)\vec{q} + \nu\nabla^2\vec{\Omega}$$

$$\frac{D\vec{\Omega}}{Dt} = \left(\vec{\Omega}.\nabla\right)\vec{q} + \upsilon\nabla^2\vec{\Omega} \quad \text{.............................(6)}$$

which is known as vorticity transport equation. The term $\left(\vec{\Omega}.\nabla\right)\vec{q}$ represents the rate at which $\vec{\Omega}$ varies for a given particle when the vortex lines move with the fluid, the strengths of the vortices remaining constant. The term $\nu\nabla^2\vec{\Omega}$ represents the rate of dissipation of vorticity through friction.

For the two dimensional motion, if $\vec{q} = u\hat{i} + v\hat{j}$ then $\left(\vec{\Omega}.\nabla\right)\vec{q} = 0$ then equation (6) reduced to

$$\frac{D\vec{\Omega}}{Dt} = \nu\nabla^2\vec{\Omega} \quad \text{...(7)}$$

This equations is of the same form as the equation of heat conduction in the liquid. Hence vorticity diffuses through a liquid in almost the same way as heat does. By analogy it follows that vorticity cannot be generated within the interior of a viscous fluid.

2.8 Circulation

The circulaton is defined as the line integral of the velocity along a closed curve.

Thus $\Gamma = \oint_C \vec{q}.d\vec{r}$ ....(1)

The time rate of change of circulaton if the closed curve,drawn in a viscous incompressible fluid, moves with the fluid we have

$$\frac{D\Gamma}{Dt} = \oint_C \left[\left\{ \frac{D\vec{q}}{Dt}.d\vec{r} + \vec{q}.\frac{D}{Dt}(d\vec{r}) \right\} \right] \text{.................(2)}$$

Here, $\frac{D}{dt}(d\vec{r})$ is the difference of the velocities at the end of the line vector $(d\vec{r})$, so that

$$\frac{D}{dt}(d\vec{r}) = (d\vec{r}.\nabla)\vec{q} \text{ ....(3)}$$

If the external forces are conservative, then the equation of motion may be written as.

$$\frac{D\vec{q}}{Dt} = -\nabla\left(\frac{p}{\rho} + V_f\right) - \nu \ \text{curl}\,\vec{\Omega} \text{.................................(4)}$$

where on using the following vector identity

$$\text{curl}\ \vec{\Omega} = \nabla \times (\nabla \times \vec{q}) = \nabla(\nabla.\vec{q}) - \nabla^2\vec{q} = -\nabla^2\vec{q} \text{ ....(5)}$$

and using equations (3) and (4) in equation (2) it reduces to

$$\frac{D\Gamma}{Dt} = -\oint_C \nabla\left\{\frac{p}{\rho} + V_f - \frac{1}{2}\vec{q}^2\right\}.d\vec{r} - \nu\oint_C(\text{curl}\ \vec{\Omega}).d\vec{r}$$

$$= \oint_C d\left(\frac{p}{\rho} + v_f - \frac{1}{2}\vec{q}^2\right) - \nu\oint_C(\text{curl}\,\vec{\Omega}).d\vec{r}$$

$$\frac{D\Gamma}{Dt} = -\nu\oint_C(\text{curl}\vec{\Omega}).dr \text{ ....(6)}$$

Hence the rate of change of circulation in a closed curve, drawn in a viscous incompressible fluid, moving with the fluid depends only on the kinematic viscosity and on the space rate of change of the vorticity components at the contour.

If $\nu = 0$ i.e. if the fluid is taken as in viscid , we get the well known "Kelvin's circulation theorem" viz, the circulation round any closed curve moving with the fluid does not change with time, provided the fluid is inviscid, the field of force is conservative and pressure is a single valued function of density only .

Self Learning Exercise

1. Write down the equation of state for the incompressible viscous fluid.
2. Define conservation of mass.
3. State Boyle's law
4. Write vorticity transport equation
5. State Kelvin's circulation theorem.

2.9 Tables on The Fundamental Equations

In this article, we now present the tables of the basic fundamental equations for in compressible fluids in Cartesian tensors and in three orthogonal coordinate systems.

Table 2.1

Fundamental equations ofa viscous incompressible fluid in Cartesian tensors.

Equation of State $\quad p = \rho RT$

Equation of continuity $\quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) = 0$

Equations of Motion $\quad$ Navier- Stokes equations

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}.\delta_{ij}\right)\right\}$$

which are three in number

Equation of Energy $\quad \rho \frac{D}{Dt}\left[C_p T\right] = \frac{D\rho}{Dt} + \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x_i}\left(\kappa \frac{\partial T}{\partial x_i}\right) + \phi$

where $\quad \phi = \mu\left\{\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right)\right\}\frac{\partial v_i}{\partial x_j}$

and $\quad \frac{D}{Dt} \cong \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x_j}$

The coefficient of viscosity μ and thermal conductivity k depend on temperature and this dependency is significant for the flow of compressible fluids.Then there are eight unknown v_1, v_2, v_3, p, ρ, T, μ and κ instead of six and therefore two more equations are required. These two equations are

$\mu = \mu(T)$ and $\kappa = \kappa(T)$

Table 2.2

Fundamental equations of a viscous incompressible fluid motion with constant fluid properties in cartesian coordinates (x, y, z)

Equation fo continuity $\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z} = 0$

Equation of Motion

x - component $\quad \rho \frac{Du}{Dt} = \rho f_x - \frac{\partial p}{\partial x} + \mu \nabla^2 u$

y - component $\quad \rho \frac{Dv}{Dt} = \rho f_y - \frac{\partial p}{\partial y} + \mu \nabla^2 v$

z - component $\quad \rho \frac{D\omega}{Dt} = \rho f_z - \frac{\partial p}{\partial z} + \mu \nabla^2 \omega$

Equation of Energy $\quad \rho C_v \frac{DT}{Dt} = \frac{\partial Q}{\partial t} + \kappa \nabla^2 T + \phi$

where $\quad \frac{D}{Dt} \cong \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and $$\phi = 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+\mu\left[\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2\right]$$

The components of the viscous stress tensors are

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} \quad : \quad \tau_{xy}=\tau_{yx} = \mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right);$$

$$\tau_{yy} = 2\mu\frac{\partial v}{\partial y} \quad : \quad \tau_{yz}=\tau_{zy} = \mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right);$$

$$\tau_{zz} = 2\mu\frac{\partial w}{\partial z} \quad : \quad \tau_{zx}=\tau_{xz} = \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right);$$

The components of the heat-flux vector are :

$$q_x = -\kappa\frac{\partial T}{\partial x} \quad ; \quad q_y = -\kappa\frac{\partial T}{\partial y} \quad \text{and} \quad q_z = -\kappa\frac{\partial T}{\partial z}$$

Table 2.3

Fundamental equations of a viscous incompressible fluid with constant fluid properties in cylindrical polar coordinates (r, θ, z)

Equation of continuity
$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}=0$$

Equations of Motion

r - component

$$\rho\left(\frac{Dv_r}{Dt}-\frac{v_\theta^2}{r}\right) = \rho f_r-\frac{\partial p}{\partial r}+\mu\left(\nabla^2 v_r-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right)$$

θ - component

$$\rho\left(\frac{Dv_\theta}{Dt}-\frac{v_r v_\theta}{r}\right) = \rho f_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left(\nabla^2 v_\theta-\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta}{r^2}\right)$$

z - component

$$\rho\frac{Dv_z}{Dt} = \rho f_z-\frac{\partial p}{\partial z}+\mu\left(\nabla^2 v_z\right)$$

Equation of Energy
$$\rho C_v\frac{DT}{Dt}=\frac{\partial Q}{\partial t}+\kappa\nabla^2 T+\phi_c$$

where $$\phi_c=\mu\left[2\left\{\left(\frac{\partial v_r}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right)^2+\left(\frac{\partial v_z}{\partial z}\right)^2\right\}\right.$$

$$\left.+\left\{r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial \theta}\right\}^2+\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta}+\frac{\partial v_\theta}{\partial z}\right)^2+\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)^2\right]$$

$$\frac{D}{Dt}\cong\frac{\partial}{\partial t}+v_r\frac{\partial}{\partial r}+\frac{v_\theta}{r}\frac{\partial}{\partial \theta}+v_z\frac{\partial}{\partial z} \quad \text{and} \quad \nabla^2\cong\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}$$

The components of the viscous stress tensor are :

$$\tau_{rr} = 2\mu \frac{\partial v_r}{\partial r} \quad : \quad \tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right],$$

$$\tau_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) \quad : \quad \tau_{\theta z} = \tau_{z\theta} = \mu\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right],$$

and $$\tau_{zz} = 2\mu \frac{\partial v_z}{\partial z} \quad ; \quad \tau_{zr} = \tau_{rz} = \mu\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right],$$

The components of the heat flux vector are :

$$q_r = -k\frac{\partial T}{\partial r} \; ; \; q_\theta = -k\frac{1}{r}\frac{\partial T}{\partial \theta} \quad \text{and} \quad q_\phi = -k\frac{\partial T}{\partial z}$$

Tabel 2.4

Fundamental equations of a viscous incompressible fluid with constant fluid properties in spherical polar coordinates (r, θ, ϕ)

Equation of Continuity :

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} = 0$$

Equation of Motion

r - component ,

$$\rho\left(\frac{Dv_r}{Dt} - \frac{v_\theta^2 + v_\phi^2}{r}\right) = \rho f_r - \frac{\partial p}{\partial r} + \mu\left[\nabla^2 v_r - \frac{2v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2v_\theta \cot\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right]$$

θ - component ,

$$\rho\left(\frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot\theta}{r}\right) = \rho f_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right]$$

ϕ - component ,

$$\rho\left(\frac{Dv_\phi}{Dt} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi \cot\theta}{r}\right) = \rho f_\phi - \frac{1}{r\sin\theta} - \frac{\partial p}{\partial \phi} + \mu\left[\nabla^2 v_\phi - \frac{v_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \phi}\right]$$

Equation of Energy :

$$\rho C_v \frac{DT}{Dt} = \frac{\partial Q}{\partial t} + \kappa\nabla^2 T + \phi_s$$

where

$$\phi_s = \mu\left[2\left\{\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}\right)^2\right\}\right.$$

$$\left. + \left\{r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right\}^2 + \left\{\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right\}^2 + \left\{\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right\}^2\right]$$

$$\frac{D}{Dt} \cong \frac{\partial}{\partial t} + v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

and $$\nabla^2 \cong \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

The components of the viscous stress tensor are :

$$\tau_{rr} = 2\mu\frac{\partial v_r}{\partial r}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]$$

$$\tau_{\theta\theta} = 2\mu\left[\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right]$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right]$$

$$\tau_{\phi\phi} = 2\mu\left[\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\theta}+\frac{v_r}{r}+\frac{v_\theta\cot\theta}{r}\right]$$

$$\tau_{\phi r} = \tau_{r\phi} = \mu\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi}+r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]$$

The component of the heat flux vector are :

$$q_r = -\kappa\frac{\partial T}{\partial r}\ ;\ q_\theta = -\kappa\frac{1}{r}\frac{\partial T}{\partial\theta} \quad \textbf{and} \quad q_\phi = -\kappa\frac{1}{r\sin\theta}\frac{\partial T}{\partial\phi}$$

2.10 Summary

In this unit, we have learnt about the fundamental equations of the flow of viscous fluid. The equation of state, equation of motion and equation of energy have been derived. We also studied about vorticity transport equation and Kelvin's circulation theorem. These equations in different co-ordinate system have been given in tabular form for ready reference with constant fluid properties.

2.11 Answer to Self Learning Exercise

1. ρ = constant
2. The fluid mass can neither be created nor destroyed.
3. $p = \rho RT$
4. See Article. 2.7
5. See Article. 2.8

2.12 Exercise

1. Obtain Navier-Stokes equations of motion of a fluid in Cartesian coordinates
2. Obtain Navier - Stokes equations of motion in Cartesian coordinates for two dimensional incompressible viscous flow.
3. Obtain equation of continuity in Cartesian coordinate system
4. Deduce Kelvin's circulation theorem.
5. Define circulation. Show that the time rate of change of circulation in a closed circuit. drawn in a viscous incompressible fluid under the action of conservative forces, moving with the fluid depends only on the kinematic viscosity and on the space rates of change of the vorticity components at the contour.
6. Prove that the vorticity $\vec{\Omega}$ satisfies the differential equation

$$\frac{D\vec{\Omega}}{Dt} = (\vec{\Omega}.\nabla)\vec{q} + \nu\nabla^2(\vec{\Omega})$$

UNIT - 3

Dynamical Similarity and Inspection and Dimensional Analysis

Structure of the Unit

3.0 Objectives

This unit provides a general overview of dymanical similarity and dimensional analysis, non-dimensional parameters. After reading this unit, you will be able to learn non-dimensional parameters viz Reynolds number, Froude number, Mach number, Prandtl number, Eckert number, Grashoff number, Brinkmann number, and non-dimensional coefficients viz lift and drag coefficients, Skin friction, Nusselt number, recovery factor and their importance in the study of problems in fluid dynamics.

3.1 Introduction

In previous units we have studied the fundamental equations of the flow of viscous fluids. There is no known general method to solve these equations because these equations have non-linear character. There are few particular cases which have the exact solutions under restricted conditions. In this unity and in the following units, we shall discuss some flows using the approximation based on smallness and largeness of certain non-dimenssional numbers. But first we shall discuss how to obtain non-dimensional number and what are they. These non dimensional quantities are very useful when in experiments we use prototype (geometrically similar but reduced in size) of actual bodies, there comes the need for dynamic similarity.

3.2 Dynamical similarity (Reynold's Law)

Two fluid motions are said to be 'dynamically similar' if with geometrically similar boundaries the flow patterns are geometrically similar.

Now we discuss the conditions under which the fluid motions are dynamically similar. In other words we have to find out those parameters which characterise a flow problem. There are two methods for finding these parameters viz (i) dimensional analysis and (ii) inspection analysis. In inspection analysis, we reduce the fundamental equations into a non-dimensional form and obtain the non-dimensional parameters from the resulting equations. In dimensional analysis, we form non dimensional parameters from the physical quantities occurring in a problem, even when the knwoledge of the governing equation is missing

3.3 Inspection Analysis

In this analysis, firstly the governing equations reduce in to dimensionaless forms then obtain the non-dimensional parameters from the equations. To understand it, we take an example as follows: The governing equations of a viscous compressible fluid are

$$p = \rho RT \qquad \text{..........(1)}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) = 0 \qquad \text{..........(2)}$$

and $$\rho\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) = \rho f_i - \frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_j} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right)\right\} \qquad \text{..........(3)}$$

where physical quantities have their usual meanings

Now, on introducing following non-dimensional quantities to reduce the above equations in a non-dimensional form

$$x_i^* = \frac{x_i}{L_o}, \quad v_i^* = \frac{v_i}{U_o} \quad t^* = \frac{tu_o}{L_o} \quad \rho^* = \frac{\rho}{\rho_o} \quad \mu^* = \frac{\mu}{\mu_o}$$

$$p^* = \frac{p}{p_o}, \quad f_i^* = \frac{f_i}{f_o}, \quad T^* = \frac{T}{T_o} \quad ; \delta_{ij}^* = \delta_{ij} \qquad \text{..........(4)}$$

where the quantities with subscripte 'o' are certain reference values associated with the flow.

Thus the governing equations (1) to (3) reduced in the non-dimensional form as :

$$p^* = \rho^* R^* T^* \qquad \text{..........(5)}$$

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial}{\partial x_j^*}(\rho^* v_i^*) = 0 \qquad \text{..........(6)}$$

$$\rho\left(\frac{\partial v_j^*}{\partial t^*} + v_j^*\frac{\partial v_j^*}{\partial x^*}\right) = \frac{f_0 L_0}{u_0^2}\rho^* f_i^* - \frac{p_o}{\rho u_0^2}\frac{\partial p^*}{\partial x_i^*}$$

$$+\frac{\mu_o}{\rho_o U_o L_o}\frac{\partial}{\partial x_j}\left\{\mu^*\left(\frac{\partial v_i^*}{\partial x_j^*} + \frac{\partial v_j^*}{\partial x_i^*} - \frac{2}{3}\frac{\partial v_k^*}{\partial x_k^*}\delta_{ij}^*\right)\right\} \qquad \text{..........(7)}$$

It is seen that the solution of above equation depends on the following non-dimensional quantities :

$$\frac{\mu_o}{\rho_o U_o L_o}, \quad \frac{f_o L_o}{U_0^2}, \quad \frac{p_o}{\rho_o U_0^2}$$

Hence, for the complete dynamical similarity of the flows of viscous compressible fluid past geometrically similar bodies, when the body force is the gravitational force only, we must have the same dimensionless quantities

3.4 Dimensional Analysis

Every physical problem innolves some physical quantities which can be measured in different units. But the physical problem itself should not depend on the units used for measuring these quantities. Now the question arises whether the units of each physical quantity is independent or can the units of one physical quantity be expressed in terms of the units of other physical quantities. The answer is that we can express the units of one physical quantity in terms of units of other physical quantities.

In dimensional analysis of any problem, we write down the dimensions of each physical quantity in terms of fundamental units. Then by dividing and rearranging the different units, we get some non-dimensional numbers.

In fluid dynamics there are four fundamental units, viz., length, mass, time and temperature in which the dimensions of all the physical quantities occuring in such a flow problem can be expressed. We shall denote the dimensions of these fundamental units by [L] [M] [T] and [θ] repectively.

3.5 Buckingham π - Theorem

The important theorem about the non-dimensional numbers is the π - theorem.

Statement :

If there are n variables in a given physical problem and if there are m fundamental dimensions, then there will be (n – m) independent dimensionless parameters.

In other words, if Q_1, Q_2, Q_3........................Q_n be n physical quantities involved in a physical phenomenon and if there are m independent fundamental units in this system .

$\phi(Q_1, Q_2,......................Q_n) = 0$

is equivalent to the relation

$f(\pi_1, \pi_2,.......................\pi_{n-r}) = 0$

where $\pi_1, \pi_2,.......................\pi_{n-r}$ are the dimensionless quantities formed by the Q_n's and r is the rank of the dimensional matrix of the physical quantities.

The proof of the above π-theorem is based on the following theorem of matrix algebra on the solution of linear algebraic equations.

"If we have m homogenous equations with n unknown, then the number of independent solutions is n – r, where r is the rank of the matrix of coefficients and any other solution can be expressed asa linear combination of these linearly independent solutions."

Let $Q_1, Q_2,...........Q_n$ be the physical quantities and let their dimensions be expressed in terms of m fundamental units $u_1, u_2, ...u_m$, in the following manner :

$$[Q_1] = \left[u_1^{a_{11}}\ u_2^{a_{21}}.......................u_m^{a_{m1}}\right]$$

$$[Q_2] = \left[u_1^{a_{12}}\ u_2^{a_{22}}.......................u_m^{a_{m2}}\right]$$

....

....

$$[Q_n] = \left[u_1^{a_{1n}}\ u_2^{a_{2n}}.......................u_m^{a_{mn}}\right]$$

So that a_{ij} is the exponent of u_i in the dimension of Q_j. The dimensional matrix of the given physical quantities is written as the following mxn matrix

$$A = \begin{array}{c} \\ u_1 : \\ u_2 : \\ \\ u_m : \end{array} \begin{array}{c} \begin{array}{cccc} Q_1 : & Q_1 : & .... & Q_n : \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & \\ a_{m1} & a_{m2} & .... & a_{mn} \end{bmatrix} \end{array}$$

Now, let us form a product π of powers of Q_1, Q_2,..................Q_n, as

$\pi = Q_1^{x_1} Q_2^{x_2} ....Q_n^{x_n}$

then

$$[\pi] = \left[\left(u_1^{a_{11}}\ u_2^{a_{21}} ....u_m^{a_{m1}}\right)^{x_1} \left(u_1^{a_{12}}\ u_2^{a_{22}} ....u_m^{a_{m12}}\right)^{x_2} ......\left(u_1^{a_{1n}}\ u_2^{a_{2n}} ....u_m^{a_{mn}}\right)^{x_n} \right]$$

If the product π is dimensionless then

$a_{11}x_1 + a_{12}\, x_2 + ...+ a_{1n}x_n = 0$

$a_{21}x_1 + a_{22}\, x_2 + ...+ a_{2n}x_n = 0$

....

$a_{m1}x_1 + a_{m2}\, x_2 + ...+ a_{mn}x_n = 0$ or $AX = 0$

which is a set of m homogenous equations in n unknowns.

Hence the number of linearly independent solutions of this equation are n–r. Thus corresponding to each independent solution of X, we have a dimensionless product π and the number of dimension less products in a complete set will be n–r.

3.6 Method of Finding Out the π – Products

We may find the π - product of a complete set in following manner :

(i) Write down the dimensional matrix of n physical quantities, involving the physical phenomenon, having m independent units.

(ii) Determine the rank of the dimensional matrix. If the rank of the matrix is r then the number of π's will be $n - r$.

(iii) Select r quantities out of the n physical quantities as the base quantities, which have different non zero dimensions.

(iv) Express π_1, π_2,....................π_{n-r} as power products of these r quantities raised to arbitrary integer exponents and one of the excluded, but different in different π's, $(n - r)$ quantities.

(v) Equate to zero the total dimension of each fundamental unit in each π- product to get the integer exponents.

3.7 Application of π – Theorem to Viscous compressible fluid Motion

In the fluid dynamics, the physical quantities involved are

$L, U, \rho, \mu, \kappa, g, p, C_p, T$

and the fundamental units in which the dimensions of all these quantities can be expressed are length, mass, time and temperature.

(i) The dimensional matrix in the present problem is

$$\begin{array}{c} \\ L : \\ M : \\ t : \\ \theta : \end{array} \begin{array}{c} \begin{array}{ccccccccc} L & U & \rho & \mu & \kappa & g & p & C_p & T \end{array} \\ \begin{bmatrix} 1 & 1 & -3 & -1 & 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & -3 & -2 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

(ii) The rank of the above matrix is 4 then the number of independent dimension less product be $9-4=5$.

(iii) Now, taking L, U, ρ and κ as the base quantities.

(iv) Now, let

$$\pi_1 = L^{x_1} \quad U^{x_2} \quad \rho^{x_3} \quad \kappa^{x_4} \quad \mu\,;$$
$$\pi_2 = L^{x_5} \quad U^{x_6} \quad \rho^{x_7} \quad \kappa^{x_8} \quad g\,;$$
$$\pi_3 = L^{x_9} \quad U^{x_{10}} \quad \rho^{x_{11}} \quad \kappa^{x_{12}} \quad p\,;$$
$$\pi_4 = L^{x_{13}} \quad U^{x_{14}} \quad \rho^{x_{15}} \quad \kappa^{x_{16}} \quad C_p\,;$$

$$[\pi_1] = \left[\left(L^{x_1}\right)\left(Lt^{-1}\right)^{x_2}\left(L^{-3}M\right)^{x_3}\left(LMt^{-3}\theta^{-1}\right)^{x_4}\left(L^{-1}Mt^{-1}\right)\right]$$

$$\pi_5 = L^{x_{17}} \quad U^{x_{18}} \quad \rho^{x_{19}} \quad \kappa^{x_{20}} \quad T;$$

$$= \left[L^{x_1+x_2-3x_3+x_4-1} + M^{x_3+x_4+1}t^{-x_2-3x_4-1}\theta^{-x_3}\right]$$

If π_1 is dimension less, then we must have

$$x_1 + x_2 - 3x_3 + x_4 - 1 = 0\,;$$
$$x_3 + x_4 + 1 = 0\,;$$
$$-x_2 - 3x_4 - 1 = 0\,;$$
$$-x_4 - 1 = 0\,;$$

Therefore : $x_1 = -1, \quad x_2 = -1, \quad x_3 = -1, \quad$ and $\quad x_4 = 0$

hence $\pi_1 = L^{-1}U^{-1}\rho^{-1}\mu = \dfrac{\mu}{UL\rho}$

Similarly $\pi_2 = \dfrac{Lg}{U^2}$; $\pi_3 = \dfrac{p}{\rho U^2}$; $\pi_4 = \dfrac{LU\rho C_p}{\kappa}$ and $\pi_5 = \dfrac{\kappa T}{LU^3\rho}$

From these dimension less products, we can construct the five dimension less numbers as

$$Re = \frac{1}{\pi_1}, \quad F_r = \frac{1}{\pi_2}, \quad p_r = \pi_1\pi_4$$

$$\frac{\gamma-1}{\gamma} = \frac{\pi_3}{\pi_4\pi_5} \quad \text{and} \quad Ma^2 = \frac{1}{\gamma\pi_3}$$

Hence with viscous fluid dynamics there are only five independent dimensionless groups.

3.8 Physical Importance of non-dimensional Parameters

We know that the inertia force always exists in all flow problems. Besides the inertia force, there always exist some additional forces which are responsible for fluid motion. The required conditions for dynamic similarity can always be obtained by considering the ratio of the inertia force and any one of the remaining forces. Since ratios of two forces will be considered, we obtain some dimensionless number as discussed below.

3.8.1 Reynold's Number

The ratio of inertial forces and viscous forces is termed as Reynolds Number and is given by

$$Re = \frac{\text{Inertia forces}}{\text{Viscous forces}} = \frac{\rho U^2/L}{\mu U/L^2} = \frac{UL}{\nu}$$

Where U, L, ρ and μ are some charecteristic values of the velocity, length, density and coefficient of viscosity respectively and $\nu = \mu/\rho$ in kinematic viscosity.

The British Scientist Osborne Reynolds, demonstrated the importance of Reynolds number in the dynamics of viscous fluid. For small Reynold number (Re<<1) the viscous forces will be predominant and effect of viscosity will be felt in the whole flow field. On the contrary, for large Reynold number (Re>>1), the inertia forces will be predominant and effect of viscosity can be considered to be confined in a thin layer near a solid body known as boundary layer. For a large value of Re, the flow ceases to be laminar and become turbulent. The value of Reynolds Number, when the nature of flow changes from laminar to turbulent, is called the critical Reynolds number.

3.8.2 Froude Number :

The ratio of inertia force to the gravity force is termed as Froude number and given by

$$Fr = \frac{\text{Inertia force}}{\text{gravity force}} = \frac{\rho U^2 / L}{\rho g} = \frac{U^2}{gL}$$

where L and U denote the characteristic length and characteristic velocity respectively. It is important only when there is a free surface i.e. in an open channel flow problem. In such cases too the force due to gravity may be neglected in comparison to the inertia force.

3.8.3 Mach Number :

The ratio of the flow velocity to the velocity of sound is known as Mach Number and given by

$$Ma = \frac{\text{Material Velocity}}{\text{Sound Velocity}} = \frac{U}{C}$$

where U is the velocity of flow and C be the velocity of sound, Mach number is also expressed in terms of the ratio of inertia force and the elastic force. It is a measure of the compressibility of the fluid. When the Mach number is small (Ma<<1), the fluid can be taken as incompressible and if mach Number is nearly one or greater then one, the fluid will be compressible. However, for large Mach numbers the effect of compressibility must be taken into account. According to the magnitude of the Mach number the flows are. generally classified as follows.

Mach Number	Type of Flow
$Ma < 1$	Subsonic
$Ma \simeq 1$	Transsonic
$Ma = 1$	Sonic
$1 < Ma \le 6$	Supersonic
$Ma > 6$	Hypersonic

3.8.4 Prandtl Number

The Prandtl number is a dimensioless parameter which is the ratio of the kinematic viscosity to the thermal diffusivity and is given by

$$Pr = \frac{\text{kinematic vis cos ity}}{\text{thermal diffusivit y}} = \frac{\nu}{a} = \frac{\mu / \rho}{\kappa / \rho C_p} = \frac{\mu C_p}{\kappa}$$

where $\upsilon\left(=\frac{\mu}{\rho}\right)$ is the kinematic viscosity, k the thermal conductivity, C_p the specific heat at constant pressure and a the thermal diffusivity. The ratio of these two quantities should express the relative magnitude of diffusion of momentum and heat in the fluid. It is a measure of the relative importance of heat conduction and viscosity of the fluid.

The Prandtl number is a material property fo the fluid and varies with fluid. For liquid metals the Prandtl number is very small i.e. for the Mercury Pr = 0.44 but for highly viscous fluids it may be very large e.g. for glycerine Pr = 7250. Prandtl number for air is 0.733 while for water it is 7.0 at 60°F.

3.8.5 Eckert Number

The dimensionless parameter Eckert number is defind as $E_C = \dfrac{U^2}{C_p T}$

where U, C_p and T are the velocity, specific heat at constant pressure and some reference value of the temperature respectively. In compressible fluids it determines the relative rise in temperature of the fluid due to adiabatic compression. It can also be retained in incompressible flow, if the frictional heat is to be considered. In high speed flow, for gases the Eckert number becomes equivalent to Mach number and is given by

$$E_C = (\gamma - 1)Ma^2$$

Where Ma and $\gamma = C_p / C_v$ are the Mach number and ratio of the specific heats respectively.

3.8.6 Grashoff Number

Grashoff number is a dimensionless parameter representing the raito of the buoyancy forces to the viscous forces in the free convection flow system. It is given by

$$Gr = \frac{g\beta(T_\omega - T_\infty)L^2}{\nu^2}$$

where g the gravitational acceleration, β the volumetric coefficient of thermal expansion, T_ω the temperature of the wall, T_∞ the free stream temperature, L the distance from the wall. It has a role similar to that played by the Reynolds number in forced convection flow field and is the primary paramoter used as a criterion for transition from laminar to turbulent boundary layer flow.

3.8.7 Pe'clet Number

In the theory of heat transfers, a non-dimensional parameter Pe'clet number is defined as the ratio of UL to the thermal diffusivity and is given by

$$Pe = \frac{UL}{a} = \frac{UL}{\nu} . \frac{\nu}{a} = Re.Pr$$

Hence the Pe'clet number is the product of Reynolds number and Prandtl number. It plays an important role when the viscous force is small while thermal force is large as compared to inertia force.

3.8.8 Brinkman Number

The dimensionless parameter Brinkman number is defined as

$$Br = \frac{\mu U^2}{\kappa(T_2 - T_1)}$$

where μ, U, κ, T_1 and T_2 are some reference value of the viscosity, velocity, conductivity and two different temperatures respectively. It is a measure of the extent to which viscous heating is important relative to the heat flow resulting from the impressed temperature difference (T_2–T_1).

3.8.9 The Ratio of Specific Heats

The ratio of specific heat at constant pressure C_p to that at constant volume C_v is usually designated as γ therefore $\gamma = \dfrac{C_p}{C_v}$. It is a measure of the relative complexity of the gas moleucles.

3.8.10 Euler Number

The ratio of pressure force to inertia force is known as Euler number and is given by

$$Eu = \frac{\text{Pressure force}}{\text{Inertia force}} = \frac{P}{V^2\rho}$$

Where P, V are the characteristics pressure and velocity respectively. When the pressure force is the predominating force, Euler's number must be the same for dynamical similarity of two flows.

3.9 Non-dimensional coefficients in the Dynamics of Viscous Fluids

In order to complete the studies of non-dimensional quantities, which occur in the dynamics of viscous fluids, let us mention some important non-dimensional coefficients which are usually calculated in the analysis and their values are compared with experimental results.

3.9.1 Lift and Drag Coefficients

If $\vec{F}$ is the force on an obstacle placed in an otherwise undisturbed stream, due to the system of stresses over its surface, then the component of F in the direction of the undisturbed stream is called the drag force and denoted by D, and the component at right angle to this called the lift and denoted by L.

If S represents a typical area associated with the obstacle, then the drag coefficient C_D and lift coefficient C_L are given as

$$C_D = \frac{D}{\rho U^2 S/2} \quad \text{and} \quad C_L = \frac{L}{\rho U^2 S/2}$$

where notation have their nusal meaning.

3.9.2 Coefficient of skin friction :

The dimensionless shearing stress on the surface of a body due to a fluid motion is known as coefficient of skin - friction and is given by

$$C_f = \frac{\tau_\omega}{\rho U^2/2}$$

where ρ the density,

U the characteristic velocity and

τ_ω the shearing stress on the surface of the body

3.9.3 Nusselt Number

The rate of heat transfer at the surface of the body is defined in terms of a non-dimensional parameter, which is known as Nusselt number and denoted by Nu. The heat exchanged between the body and the fluid can be calculated with the help of a coefficient of heat transfer α (x), which is defined by Newton's cooling law as given by

$$q(x) = \alpha(x)\,(T_w - T_\infty) \quad \text{.........(1)}$$

where q (x) is the quantity of heat exchanged between the wall and the fluid perunit area per unit time at a point x, T_w the wall temperature and T_∞ the free stream temperature.

According the Fourier's law, the heat exchanged between the fluid and the body due to conduction are given by

$$q(x) = -\kappa\left(\frac{\partial T}{\partial y}\right) y = 0 \quad \text{.........(2)}$$

where κ is the thermal conductivity and y is the normal direciton to t he surface of the body.

On using Fourier's law of heat conduction and Newton's cooling law, the rate of heat transfer in terms of Nusselt number is given by

$$Nu = \frac{\alpha(x).L}{\kappa} = -\frac{L}{(T_\omega - T_\infty)}\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

where negative sign shows the decrease in temperature and L be characteristic length.
This number is very important in the problems where heat transfer is in consideration.

3.9.4 Temperature Recovery Factor

The temperature which a surface assumes under the influence of internal friction is called they recovery temperature or adiabatic wall temperature. The dimension less temperature recovery factor is given by

$$r = \frac{T_r - T_\infty}{U^2/2C_p}$$

It is important in the high speed flow in which the frictional heat plays an important role.

Self Learning Exercise

1. State the Buckingham π-theorem.
2. What do you mean by critical Reynolds number ?
3. Which dimensionless parameter is product of Reynolds and Prandtl numbers ?
4. Define Newton's law of cooling
5. Define Fourier's law of heat exchange.

Example - 1

An oil of specific gravity 0.85 is flowing through a pipe of 5 cm. diameter at the rate of 3 liter/sec. Find the type of flow, if the viscosity for the oil is 3.8 Poise.

Solution :

Velocity of oil = $V = \frac{\text{Discharge}}{\text{Area}} = \frac{3000}{\pi(5/2)^2} = 152.8\,\text{cm/sec}.$

Diameter of pipe = L = 5 cm.

$\mu = 3.8$ and $\rho = 0.85$

Hence $$Re = \frac{UL\rho}{\mu} = \frac{152.8 \times 5 \times 0.85}{3.8} = 171$$

Since Re = 171 < 2000 : it follows that the flow must be laminar .

Example - 2

A 1:20 model of an air-duct is to be tested in water which is 45 times more viscous and 850 times more dense than air. What should be the pressure drop in the prototype if the pressure drop is 3 kg/cm^2 in the model when tested under hydrodynamically similar conditions ?

Solution :

Here we have for dynamic similerty

$$Re = \frac{V_p L_p \rho_p}{\mu_p} = \frac{V_m L_m \rho_m}{\mu_m}$$

where p is subscript and is considered for the prototype and m for the model

$$\Rightarrow \frac{V_p}{V_m} = \frac{\rho_m}{\rho_p} \times \frac{L_m}{L_p} \times \frac{\mu_p}{\mu_m}$$

$$\frac{V_p}{V_m} = \frac{850}{20 \times 45} = \frac{17}{18}$$

and $E_u = \frac{P_p}{\rho_p V_p^2} = \frac{P_m}{\rho_m V_m^2}$

$$\Rightarrow P_p = P_m \times \frac{\rho_p}{\rho_m} \times \left(\frac{V_p}{V_m}\right)^2$$

$$P_p = 3 \times \frac{1}{850} \times \left(\frac{17}{18}\right)^2$$

$$P_p = 3.4 \times 10^{-3} \text{ kg/cm}^2$$

which gives the required presure drop.

3.10 Summry

After studying this unit, you are able to reduce any fundamental governing equations into a non-dimensional form of it. You are also capable to understand the dimension less parameters and their physical importance in dynamics of viscous fluids and its practical utility.

3.11 Answers to self learning exercise

1. See article 3.5
2. The value of Reynolds number, when the nature of flow changes from laminar to turbulent is called critical Reynolds number.
3. Pe = Re.Pr
4. See article 3.9.3
5. See article 3.9.3

3.12 Exercise

1. Explain the principal of dynamic similarity
2. State and prove Buckingham π-theoram
3. What are the dimensions of coefficient of viscosity and kinematic viscosity ?
4. Find out the complete set of π-products when the physical quantities involved in a phenomenon are L, U, ρ, μ, g, p, C_p and T. Symbols have their usual meanings.
5. Explain the physical significance of the Reynold number, Mach number, Prandtl number and Froude number
6. Define following non-dimensional coefficients
   (a) Lift and drag coefficient
   (b) Skin friction coefficient
   (c) Nusselt number
   (d) Recovery factor.

UNIT - 4

Exact Solutions of The Navier- Stoke's Equations

Structure of the Unit

4.0 Objectives

This units provides some exact solutions of the Navier - Stoke's equations for steady incompressible flow with constant fluid properties by changing them to solvable differential equation under certain boundary conditions for symmetrical channels.

4.1 Introduction

The Navier-Stoke's equations are second order non linear partial differential equations. There is no any known general method to solve these equations. Only in few special cases exact solution can be obtained with certain assumptions about the state of the fluid and configuration of the flow pattern. In this unit, we propose to study some useful real problems for which exact solution are possible, viz; the steady incompressible flow with constant fluid properties. Now we present some solvable viscous flow problems by analytical methods.

4.2 Steady Incompressible Flow with Constant Fluid Properties

If at various points of the flow field all quantities such as velocity, density, pressure associated with the flow field remains unchanged with time then the motion is said to be steady . If the said quantities depend on time then motion is said to be unsteady. In steady motion the various quantities of flow field are the functions of the space coordinate and independent of time. In the incompressible fluid motion, the density of the fluid remains unchanged or constant throughout the flow field. In this unit we shall study only some exact solutions for steady incompressible fluid motions.

4.3 Flow between Parallel Plates

A very simple solution of the fundamental governing equations can be obtained for the flow between two parallel plates which are kept at a finite distance apart.

Consider a steady laminar flow of viscous incompressible fluid between two infinite parallel plates which are kept at a finite distance h.

Let the x-axis be along the direction of the flow, the y-axis is taken at right angle to it and the width of the plates, parallel to z-axis, be large compared to the distance between the plates. Here we use the word "infinite" implies that the width of the plates is large compared to the distance between them.

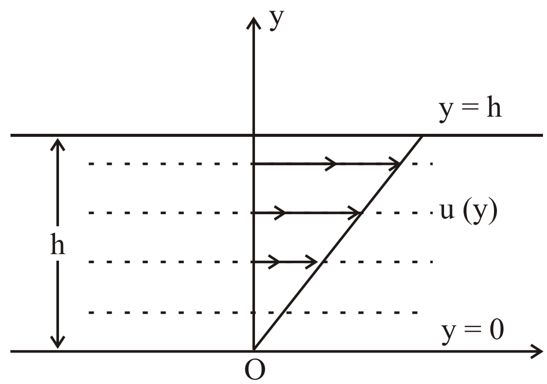


Fig. 4.1

Hence the motion is two dimensional and therefore all the variables will be independent of z-coordinate The motion of the fluid between the two plate is caused due to difference in pressure at different points in x-direction, i.e. the motion is due to a pressure gradient and the motion takes place only in x-direction. Thus

$$\frac{\partial}{\partial z}(\ ) \equiv 0 \quad ; \quad u = u(x, y) \quad ; \quad v = 0 \quad ; \quad w = 0 \quad \text{and} \quad p = p\,(x, y) \text{ ....(1)}$$

where u, v, w are velocity component in the directions of x, y and z - axis.

In the absence of body force, the Navier - Stokes equations (Ref table 2.2) becomes

Equation of continuity : $\frac{\partial u}{\partial x} = 0$ ....(2)

Equations of motion : $0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$ ....(3)

and $0 = -\frac{\partial p}{\partial y}$ ....(4)

Equations (2) and (4) respectively show that u is the function of y only and pressure p is a function of x only. Therefore, the equation (3) becomes a total differential equation

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} \text{ ....(5)}$$

Differentiating both sides w.r. to x, we have

$$0 = \frac{1}{\mu}\frac{d^2p}{dx^2} \quad \text{or} \quad \frac{d}{dx}\left(\frac{dp}{dx}\right) = 0$$

so that $\frac{dp}{dx}$ = constant.................................(6)

On integrating equation (5), we find

$$\frac{du}{dx} = \frac{1}{\mu}.\frac{dp}{dx}.y + A$$

Again inlegrating, we have

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + Ay + B$$

where A and B are arbitrary constants to be determined by the boundary conditions for the different type of flows.

4.3.1 Plane Couette Flow

The flow between two parallel infinite plates one of which is at rest and the other moving with a uniform velocity U in its own plane and the pressure gradient be zero then it is known as Plane Couette flow. In this flow the flow is due to motion of the bounding plate which transmits the motion in successive layer of the fluid

If stationary plate is taken in the direction of x-axis and the distance between the plates be h and upper plate has been given a velocity U parallel to x-axis then the boundary conditions are

$y = 0\ ;\ u = 0$

$y = h\ ;\ u = U$..(8)

Hence the velocity distribution in this flow in absence of a pressure gradient $\frac{dp}{dy}$ is obtain from equation (7) and is given by

$$u = Ay + B$$

On using boundary conditions to find constants

We have $A = \frac{U}{h}$ and $B = 0$

Hence $u = \frac{U}{h}y$

or $\frac{u}{U} = \frac{y}{h}$ ....(9)

which is the velocity distribution in non-dimenssional form and which is linear as shown in fig 4.1. The graph shown there is called the velocity profile.

The volume rate of flow Q per unit width per unit time, at any normal section is given by

$$Q = \int_0^h u\ dy \text{ ....(10)}$$

Substituting the value of u from equation (9) in (10), we have

$$Q = \frac{Uh}{2} \text{ ....(11)}$$

The coefficient of skin friction in the present case is given by

$$C_f = \frac{\tau_\omega}{\rho U^2/2} \text{ ....(12)}$$

where shearing stress τ_ω is given by

$$\tau_\omega = \left(\mu \frac{du}{dy}\right)_{y=0}$$

$$\tau_\omega = \frac{\mu U}{h} \quad \text{...(13)}$$

On substituting the value of τ_ω from (13) in (12), we find

$$C_f = \frac{\mu U/h}{\rho U^2/2} = 2\frac{\mu}{\rho U h} = \frac{2}{Re}$$

$$C_f = \frac{2}{Re} \text{ (Re is Reynolds number} = \frac{\rho U h}{\mu}\text{)}$$

Hence the value of skin friction is fixed.

4.4 Plane Poiseuille Flow

Consider the steady laminar flow of viscous incompressible fluid between two infinite stationary parallel plates at distance 2b apart. Let x-axis be taken in the middle of the channel parallel to the plate. Let x-axis be the direction of flow and y-axis in the direction perpendicular to the flow. The width of the plates is parallel to z-axis.

The width of the plates be large compared to the distance between the plates. The motion is now two dimensional and therefore all the variables will be independent of z-coordinate.

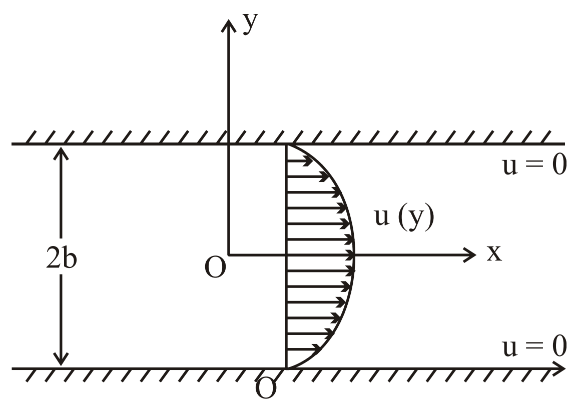


Fig. 4.2

Hence

$$\frac{\partial}{\partial z}(\;) \cong 0, \quad u = u(x, y), \nu = 0 \;;\; \omega = 0 \;;\; \text{and } p = p(x, y) \text{ ....(1)}$$

Further, the equation of continuity and the equation of motion reduce to

$$\frac{\partial u}{\partial x} = 0 \text{ ....(2)}$$

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \text{ ....(3)}$$

$$0 = -\frac{\partial p}{\partial y} \text{ ....(4)}$$

From above governing equations,we conclude that u will be a functin of y only and p will be a function of x only thus these equations can be written as

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} \text{ ....(5)}$$

Differentiating both sides, w.r. to x, we find

$$0=\frac{1}{\mu}\frac{d^2p}{dx^2} \Rightarrow \frac{d}{dx}\left(\frac{dp}{dx}\right)=0 \Rightarrow \frac{dp}{dx}=\text{const.},\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

On integrating equation (5), we have

$$\frac{du}{dy}=\frac{1}{\mu}\frac{dp}{dx}y+A$$

Again integrating , we have

$$u=\frac{1}{2\mu}\frac{dp}{dx}y^2+Ay+B \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

The boundary conditions are

$$y=\pm b\ ; \quad u=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

On using boundary conditions in equation (7), we find

$$0=\frac{1}{2\mu}\frac{dp}{dx}b^2+B+Ab$$

and $0=\frac{1}{2\mu}\frac{dp}{dx}b^2+B-Ab$

So that A $=0$ and $B=-\frac{1}{2\mu}\frac{dp}{dx}b^2$

Hence the velocity distribution are given by

$$u=-\frac{1}{2\mu}\frac{dp}{dx}\left(b^2-y^2\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

which is parabolic and these velocity profiles are as shown in fig 4.2 and the maximum velocity occurs in the middle of the channel (at $y=0$) which is

$$u_{max}=-\frac{1}{2\mu}\frac{dp}{dx}b^2=-\frac{b^2}{2\mu}\frac{dp}{dx}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Hence the Non-dimensional velocity distribution in a plane Poiseuille flow is given by

$$\frac{u}{u_{max}}=\left(1-\frac{y^2}{b^2}\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

The average velocity distribution for the present flow is given by

$$u_a=\frac{1}{2b}\int_{-b}^{b}u\ dy$$

$$=\frac{1}{2b}u_{max}\int_{-b}^{b}\left(1-\frac{y^2}{b^2}\right)dy$$

$$=\frac{u_{max}}{2b}.\left[y-\frac{y^3}{3b^2}\right]_{-b}^{b}$$

$$u_a = \frac{2}{3}u_{max} \quad \text{...} (12)$$

The shearing stress at the lower plate (y = –b) is

$$\tau_\omega = \left(\mu \frac{du}{dy}\right)_{y=-b}$$

$$= \mu\left(-\frac{2y}{b^2}u_{max}\right)_{y=-b}$$

$$= \frac{2\mu u_{max}}{b} \quad \text{...} (13)$$

4.5 Generalized Plane Couette Flow

Consider the steady laminar flow of viscous incompressible fluid between two infinite parallel plates one of which is at rest and the other is moving with uniform velocity U in its own plane and the pressure gradient is non zero. If the x-axis is taken along the stationary plate and the distance between the plates be denoted by h, then the boundary conditions are

$$y = 0 \quad ; \quad u = 0$$

$$y = h \quad ; \quad u = U \quad \text{...} (1)$$

The motion is two dimensional and therefore all the variables will be independent of z-coordinate.

Hence $\frac{\partial}{\partial z}(\ ) \cong 0 \quad ; \quad u = u(x, y), \quad v = 0, \quad w = 0 \quad \text{and} \quad p = p\,(x, y)$..............(2)

Further the equation of continuity and equations of motion reduce to

$$\frac{\partial u}{dx} = 0 \quad \text{...} (3)$$

$$0 = -\frac{\partial p}{dx} + \mu\frac{\partial^2 u}{dy^2} \quad \text{...} (4)$$

and

$$0 = -\frac{\partial p}{dy} \quad \text{...} (5)$$

From above equations, we conclude that u is the function of y only and p is function of x only, therefore these equaton can be written as

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} \quad \text{...} (6)$$

where $\frac{dp}{dx}$ is constant.

On integrating, we have

$$u = \frac{1}{2\mu}\frac{dp}{dx}.y^2 + Ay + B \quad \text{...} (7)$$

where A and B are arbitrary constants to be determined by the boundary condutions (1). Using (1) in (7), we have.

$$A = \frac{U}{h} - \frac{dp}{dx} \cdot \frac{h}{2\mu} \quad ; \quad B = 0$$

Hence

$$u = \frac{y}{h} U - \frac{h^2}{2\mu} \frac{dp}{dx} \cdot \frac{y}{h}\left(1 - \frac{y}{h}\right) \quad \text{.......(8)}$$

Let us introduce the dimension less pressure gradient as.

$$P = \frac{h^2}{2\mu U}\left(-\frac{dp}{dx}\right) \quad \text{.......(9)}$$

Then the velocity distribution in a generalized plane couette flow, in non-dimensional form, is given by

$$\frac{u}{U} = \frac{y}{h} + P\frac{y}{h}\left(1 - \frac{y}{h}\right) \quad \text{.......(10)}$$

From equation (10) it is clear that the velocity field will depend on the nature of the non-dimensional pressure gradient P. There are three possible different cases for the nature of P.

- **Case I $P > 0$**

  When the pressure is decreasing in the direction of flow then the velocity distribution is positive over the entire width between the plates, as shown fig. 4.3

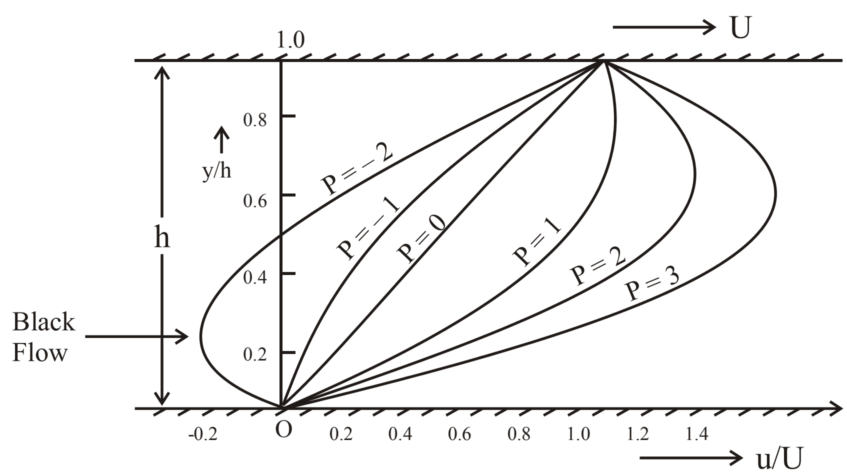


Fig. 4.3

- **Case II $P = 0$**

  When the pressure gradient is zero or when the pressure is constant throughout flow then the velocity distribution is linear which is clearly the case of plane couctte flow.

- **Case III $P < 0$**

  When the pressure is increasing in the direction of the flow or in other words, for an adverse pressure gredient, we find that a back flow may occur near the stationary plate at $P < -1$. It is due to the influence of the adverse pressure gradient which surpasses the dragging action of the faster layer on the fluid particles in that ragion.

4.5.1 Volume rate of flow

The volume rate of flow Q per unit width per unit time at any normal section of the channel is given by

$$Q = \int_0^h u \, dy \quad \text{.......(11)}$$

Substituting the value of u from the equation (10) in equation (11), we have

$$Q = U\int_{o}^{h}\left[\frac{y}{h} + P\frac{y}{h}\left(1-\frac{y}{h}\right)\right]dy$$

$$= U\left[\frac{y^2}{2h} + P\left(\frac{y^2}{2h} - \frac{y^3}{2h^2}\right)\right]_0^h$$

$$= \frac{Uh}{2} + \frac{PUh}{6} \quad \text{...(12)}$$

If $P = -3$, then the volume rate of flow becomes zero. This means that there is no net flow across any section perpendicular to the direction of motion.

4.5.2 Coefficient of skin Friction

The shearing stress on the stationary plate is given by

$$\tau_\omega = \left(\mu\frac{du}{dx}\right)_{y=0} = \mu\left[u\left\{\frac{1}{h} + \frac{P}{h}\left(1-\frac{2y}{h}\right)\right\}\right]_{y=0}$$

$$\tau_\omega = \frac{\mu U}{h}(P+1)$$

Hence the coefficient of skin friction in the present case at stationary plate is

$$C_f = \frac{\tau_\omega}{\rho U^2/2} = \frac{\mu U(P+1)}{h\rho U^2/2} = \frac{2\mu(P+1)}{h\rho U} = \frac{2(1+P)}{Re}$$

where $Re = \dfrac{h\rho U}{\mu}$ is Reynolds number. Clearly C_f is positive if P is positive and it will be negative if $P < -1$ in the case of back flow and zero if $P = -1$

4.6 Flow in a circular pipe (Hagen - Poiseuille Flow)

Consider the steady laminar flow, without body force of an incompressible fluid through an infinite circular pipe of radius R with axial symmetry as shown in fig. 4.4. Such a motion is maintained by the presence of a pressure gradient along the axis of the pipe.

Now let the axis of the pipe be taken as z-axis along which the flow takes place and r denotes the radial distance measured outward from the z-axis. Due to axial symmetry of flow $\frac{\partial}{\partial\theta} \cong 0$

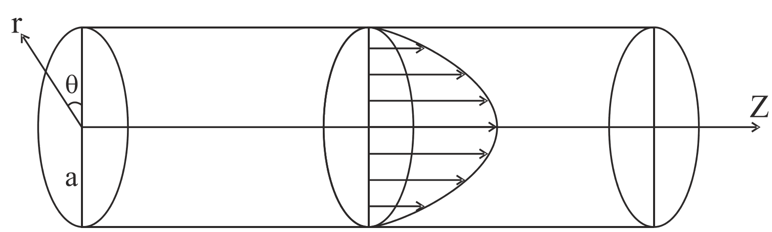


Fig. 4.4

or in other words, all the variables will be independent of θ. Also the only non-zero component of velocity is V_z, Hence the governing equations from table No.2.3 of unit 2 in cylindrical coordinates reduce to

$$\frac{\partial V_z}{\partial z} = 0 \quad \text{..(1)}$$

$$\frac{\partial p}{\partial r} = 0 \text{ ..,..,...(2)}$$

and $$\mu\left(\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r}\right) = \frac{\partial p}{\partial z} \text{ .,..................................(3)}$$

Hence from equations (1) and (2), it is clear that V_z is a function of r only and p is the function of z only. Therefore the equation (3) can be written as

$$\mu\left(\frac{d^2 V_z}{dr^2} + \frac{1}{r}\frac{dV_z}{dr}\right) = \frac{dp}{dz} \text{ ....(3)}$$

Since p is not a function of r and V_z is not a function of Z then equation (4) can be valid only when $\frac{dp}{dz}$ is a constant.

Equation (4) is now written as

$$\frac{d}{dr}\left[r\frac{dV_z}{dr}\right] = \frac{r}{\mu}\frac{dp}{dz}$$

On integrating, we have

$$r\frac{dV_z}{dr} = \frac{r^2}{2\mu}\frac{dp}{dz} + A$$

or $$\frac{dV_z}{dr} = \frac{r}{2\mu}\frac{dp}{dz} + \frac{A}{r}$$

Again integrating, we have

$$V_z = \frac{dp}{dz} \cdot \frac{r^2}{4\mu} + A\log r + B \text{ ....(5)}$$

Where A and B are arbitrary constents to be determinied by the following boundary conditions.

$$r = 0 \quad ; \quad \frac{dV_z}{dr} = 0 \qquad \text{(due to symmetry)}$$

$$r = R \quad ; \quad V_z = 0 \qquad \text{(No slip condition).............(6)}$$

Therefore, the equation (5) and equations (6) give

$$A = 0 \qquad B = -\frac{1}{4}\frac{dp}{dz}.\frac{R^2}{\mu}$$

Hence equation (5) becomes

$$V_z = -\frac{R^2}{4\mu}\frac{dp}{dz}\left[1 - \left(\frac{r}{R}\right)^2\right] \text{ ....(7)}$$

which is the form of a paraboloid of revolution as shown fig. 4.4. The pressure gradient $\frac{dp}{dz} = \frac{p_2 - p_1}{L}$ where L be the distance between two sections of the pipe where the pressures p_1 and p_2 are measured. The maximum velocity occurs on the axis of the pipe and is given by

$$(V_z)_m = -\frac{R^2}{4\mu}\frac{dp}{dz}$$

$$(V_z)_m = \frac{p_1 - p_2}{L}\frac{R^2}{4\mu} \quad \text{...(8)}$$

Hence the velocity distribution in a non-dimensional form in Hagen-Poiseuille flow is given by

$$\frac{V_z}{(V_z)_{max}} = 1 - \left(\frac{r}{R}\right)^2 \quad \text{...(9)}$$

The average velocity over a croos section can be obtained as

$$\overline{V}_z = \frac{\int_O^R V_z . 2\pi r\, dr}{\pi R^2}$$

$$= -\frac{R^2}{8\mu}\frac{dp}{dz} = \frac{1}{2}(V_z)_{max} \quad \text{..(10)}$$

The volume rate of flow Q given by

$$Q = \pi R^2 \overline{V}_z$$

$$= \pi R^2 . \left(\frac{-R^2}{8\mu}\frac{dp}{dz}\right)$$

$$= \frac{\pi R^4}{8\mu L}(p_1 - p_2) \quad \text{...(11)}$$

4.6.1 Coefficient of Skin Friction

The shearing stress on the wall of the pipe is given by (table 2.3 of unit 2)

$$(\tau_{rz})_\omega = -\left(\mu\frac{dV_z}{dr}\right)_{r=R} = -\frac{R}{2}\frac{dp}{dz}$$

$$= \frac{4\mu}{R}\overline{V}_z \quad \text{..(12)}$$

Hence the coefficient of skin-friction on the wallof the pipe is given by

$$C_f = \frac{(\tau_{rz})_\omega}{\rho(\overline{V}_z)^2/2}$$

$$C_f = \frac{\frac{4\mu}{R}\overline{V}_z}{\rho(\overline{V}_z)^2/2} = \frac{8\mu}{\rho R\overline{V}_z} \quad \text{...(13)}$$

If Reynolds number $Re = \frac{2R\rho\overline{V}_z}{\mu}$ then equation (13) reduces to

$$C_f = \frac{16}{Re} \quad \text{...(16)}$$

Showing that skin friction can be obtained from the knowledge of Reynold number. The above formula is used to determine energy losses in pipe flow.

4.7 Flow in tube of Uniform Cross-Section

In usual practice the pipes of different shapes are employed in order to transport a given fluid. Consider the steady flow of a viscous incompressible fluid through a tube of arbitrary, but uniform cross section. Let z-axis be parallel to the generators of the tube. The only non-zero component of velocity is the velocity along z-axis, therefore.

$$u = v = 0 \quad \text{.....(1)}$$

and the equation of continuity reduces to

$$\frac{\partial \omega}{\partial z} = 0 \quad \text{.....(2)}$$

which shows that ω is a function of x and y only.

The Navier-stokes equations of motion in Cartesian coordinates, in the absence of any external force are

$$\frac{\partial p}{\partial x} = 0 \quad \text{.....(3)}$$

$$\frac{\partial p}{\partial y} = 0 \quad \text{.....(4)}$$

and $$\mu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) = \frac{\partial p}{\partial z} \quad \text{.....(5)}$$

Since it is clear from equations (3) and (4) that ω is independent of z and p is independent of x and y , then equation (5) takes the form

$$\mu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) = \frac{dp}{dz} \quad \text{.....(6)}$$

Diffrentiating both sides of equation (6). w.r. to z we get.

$$O = \frac{d}{dz}\left(\frac{dp}{dz}\right)$$

giving $$\frac{dp}{dz} = \text{constant} = -P \text{ (say)}$$

Hence the problem reduces to the solution of the equation

$$\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} = -\frac{P}{\mu} \quad \text{.....(7)}$$

with the boundary condition $\omega = 0$ on the surface of the tube.

Thus the problem reduces to solving Poission's equation (7) with the boundary cordition $\omega = 0$ on the surface of the tube. Direct solution of it is not easy. So to simplify the solution we convert equation (7) into a Laplace equation by the transformation

$$\omega = \psi - \frac{P}{4\mu}\left(x^2 + y^2\right) \quad \text{.....(8)}$$

which satisfy the two dimensional Laplace equation

$$\frac{\partial^2\psi}{dx^2}+\frac{\partial^2\psi}{dy^2}=0 \qquad \text{.......(9)}$$

with the boundary condition

$$\psi=\frac{P}{4\mu}\left(x^2+y^2\right) \qquad \text{.......(10)}$$

on the surface of the tube.

We will now discuss some of the solutions of the equation (9) subject to condition (10), for tubes of different cross section and shall calculate the volume rate of flow in each case.

4.7.1 Tube of Circular Cross-Section

Let the cross-section of the tube a circle

$$x^2+y^2\le a^2 \qquad \text{.......(11)}$$

Since on the surface of the tube $r=a$,

the suitable solution of the Laplace equation (9) is

$$\psi=A(\text{const.})=\frac{Pa^2}{4\mu} \qquad \text{.......(12)}$$

with the boundary condition (10). Hence the velocity distribution is given by

$$\omega=\frac{P}{4\mu}\left(a^2-r^2\right) \qquad \text{.......(13)}$$

The volume rate of flow Q is given by

$$Q=\int_0^a \omega.2\pi r\ dr$$

$$=\int_0^a \frac{P}{4\mu}\left(a^2-r^2\right)2\pi r\ dr=\frac{P\pi}{2\mu}\int_0^a\left(a^2-r^2\right)r.dr$$

$$=\frac{P\pi}{2\mu}.\frac{a^4}{4}=\frac{P\pi}{8\mu}a^4$$

The results for velocity and volume rate are identical to those obtained for Hagen-Poiseuille flow.

4.7.2 Tube of Annular Cross Section

The suitable solution of the Laplace equation (9) for this type of flow is

$$\psi=A\log r+B \qquad \text{.......(15)}$$

The boundary conditions on ψ are

$$r=a\ ;\quad \psi=\frac{P}{4\mu}a^2$$

$$r=b\ ;\quad \psi=\frac{P}{4\mu}b^2 \qquad \text{.......(16)}$$

with the help of (16), we determine the constants A and B.

On subsitiluting boundary conditions (16) in equation (15) we have.

$$A \log a + B = \frac{P}{4\mu} a^2$$

and $$A \log b + B = \frac{P}{4\mu} b^2$$

which gives that

$$A(\log a - \log b) = \frac{P}{4\mu}(a^2 - b^2)$$

$$A = \frac{P}{4\mu} \frac{(a^2 - b^2)}{\log(a/b)} \quad \text{...(17)}$$

$$B = \frac{P}{4\mu} a^2 - \frac{P}{4\mu} \frac{\log a.(a^2 - b^2)}{\log(a/b)} \quad \text{..(18)}$$

Substituting these value in (15), we obtain

$$\psi = \frac{P}{4\mu} \frac{(a^2 - b^2)}{\log(a/b)} \log(r/a) + \frac{P}{4\mu} a^2 \quad \text{..(19)}$$

Hence the velocity distribution in the annular region between two concentric cylinders of radii a and $(b < a)$ will be obtain on substituting eq (19) in the equation $\omega = \psi - \frac{P}{4\mu} r^2$

$$\omega = \frac{P}{4\mu}\left[(a^2 - r^2) + (a^2 - b^2)\frac{\log(r/a)}{\log(a/b)}\right] \quad \text{...(20)}$$

The volume rate of flow Q is given by

$$Q = \int_b^a \omega.2\pi r.dr = -2\pi \int_a^b \frac{P}{4\mu}\left[(a^2 - r^2)r + \frac{(a^2 - b^2)\log(r/a)}{\log(a/b)}.r\right]dr$$

$$= -\frac{2\pi P}{4\mu}\left[\left(\frac{a^2 r^2}{2} - \frac{r^4}{4}\right) + \frac{(a^2 - b^2)}{\log(a/b)} \cdot \left(\frac{r^2}{2}\log\frac{r}{a} - \frac{1}{4}r^2\right)\right]_a^b$$

$$= -\frac{\pi P}{2\mu}\left[\left(\frac{a^2 r^2}{2} - \frac{r^4}{4}\right)_a^b + \frac{(a^2 - b^2)}{\log(a/b)}\left(\frac{r^2}{2}\log\frac{r}{a} - \frac{1}{4}r^2\right)_a^b\right]$$

$$= -\frac{\pi P}{2\mu}\left[\frac{a^2}{2}(b^2 - a^2) - \frac{1}{4}(b^4 - a^4) + \frac{(a^2 - b^2)}{\log(a/b)}\left\{\frac{b^2}{2}\log\frac{b}{a} - \frac{1}{4}(b^2 - a^2)\right\}\right]$$

$$= -\frac{\pi P}{2\mu}\left[\frac{1}{4}(b^4 - a^4) + \frac{(a^2 - b^2)^2}{4\log(a/b)}\right]$$

Hence $$Q = \frac{\pi P}{8\mu}\left[(a^4 - b^4) - \frac{(a^2 - b^2)^2}{\log(a/b)}\right] \quad \text{...(21)}$$

Then the average velocity in the annulus is given by

$$\omega_a = \frac{Q}{\pi(a^2 - b^2)}$$

Hence $$\omega_a = \frac{P}{8\mu}\left[(a^2 + b^2) - \frac{(a^2 - b^2)}{\log(a/b)}\right] \qquad \text{.......(22)}$$

The shearing stress is obtained on using equation (20) as

$$\tau_{rz} = \mu\frac{d\omega}{dr} = \frac{P}{4}\left[-2r + \frac{(a^2 - b^2)}{\log(a/b)}\cdot\frac{1}{(r/a)}\cdot\frac{1}{a}\right]$$

$$= \frac{P}{4}\left[\frac{(a^2 - b^2)}{\log(a/b)r} - 2r\right] \qquad \text{.......(23)}$$

Hence the shearing stress at the inner and outer surface of the annulus are respectively given by

$$(\tau_{rz})_b = \frac{P}{4}\left[\frac{(a^2 - b^2)}{b.\log(a/b)} - 2b\right] \qquad \text{.......(24)}$$

and $$(\tau_{rz})_a = \frac{P}{4}\left[\frac{(a^2 - b^2)}{a.\log(a/b)} - 2a\right] \qquad \text{.......(25)}$$

which shows that the shearing stress at both walls are positive.

4.7.3 Tube of Elliptic Cross-Section

Let the cross-section of the tube be an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{.......(26)}$$

Let $$\psi = A(x^2 - y^2) + B \qquad \text{.......(27)}$$

be a suitable solution of the Laplace equation for the present tube then

$$\omega = A(x^2 - y^2) + B - \frac{P}{4\mu}(x^2 + y^2) \qquad \text{.......(28)}$$

Since on the boundary of the pipe $\omega = 0$

then $$\frac{P}{4\mu}(x^2 + y^2) = A(x^2 - y^2) + B$$

or $$\frac{1}{B}\left(\frac{P}{4\mu} - A\right)x^2 + \frac{1}{B}\left(\frac{P}{4\mu} + A\right)y^2 = 1 \qquad \text{.......(29)}$$

Equation (29) must be identical to ellipse (26)

If $$\frac{1}{B}\left(\frac{P}{4\mu} - A\right) = \frac{1}{a^2} \quad \text{and} \quad \frac{1}{B}\left(\frac{P}{4\mu} + A\right) = \frac{1}{b^2}$$

on solving, we have

$$A = \frac{P}{4\mu}\frac{(a^2 - b^2)}{(a^2 + b^2)} \quad \text{and} \quad B = \frac{P}{2\mu}\frac{a^2b^2}{(a^2 + b^2)} \qquad \text{.......(30)}$$

Hence the velocity distribution in an elliptic cylinder is given by

$$w = \frac{P a^2 b^2}{2\mu\left(a^2+b^2\right)}\left[1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right] \qquad \text{.....(31)}$$

The volume rate of flow Q is given by

$$Q = \iint w\, dx\, dy$$

$$= \frac{Pa^2b^2}{2\mu\left(a^2+b^2\right)}\iint\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dx\ dy$$

$$= \frac{Pa^2b^2}{2\mu\left(a^2+b^2\right)}\left[\pi ab-\frac{1}{a^2}\pi ab.\frac{a^2}{4}-\frac{1}{b^2}\pi ab.\frac{b^2}{4}\right]$$

$$= \frac{\pi P a^3 b^3}{4\mu\left(a^2+b^2\right)} \qquad \text{.....(32)}$$

4.7.4 Tube of Equilateral Triangular Cross-Section

Let the each side of the triangle be of length $2a\sqrt{3}$, the z-axis passes through the centre of gravity of the section and the axes of x and y are perpendicular to the two sides as shown. The equation to the boundary in present case will be

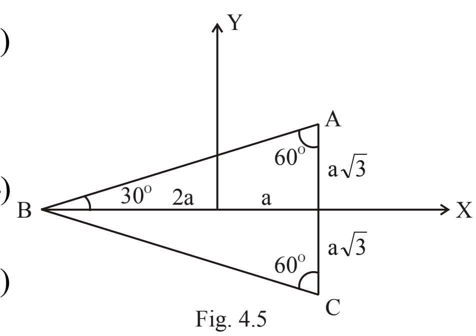


Fig. 4.5

$$(x-a)\left(x-\sqrt{3}y+2a\right)\left(x+\sqrt{3}y+2a\right)=0 \qquad \text{.....(33)}$$

The suitable solution of the Laplace equation is

$$\psi = A\left(x^3-3xy^2\right)+B \qquad \text{.....(34)}$$

then $w = A\left(x^3-3xy^2\right)+B-\frac{P}{4\mu}\left(x^2+y^2\right)$ .....(35)

On the boundary of the pipe w = 0 and if x = a is a part of the boundry, we have

$$\frac{P}{4\mu}\left(a^2+y^2\right) = A\left(a^3-3ay^2\right)+B$$

Therefore

$$Aa^3+B=\frac{Pa^2}{4\mu}$$

and $\quad -3aA = \frac{P}{4\mu}$

which gives $\quad A=-\frac{P}{12\mu a}$ and $B=\frac{Pa^2}{3\mu}$ .....(36)

Using (36) in (35), we otbain the velocity distribution in an equilateral triangular cylinder as

$$w=-\frac{P}{12\mu a}\left(x^3-3xy^2\right)+\frac{Pa^2}{3\mu}-\frac{P}{4\mu}\left(x^2+y^2\right)$$

$$= -\frac{P}{12\mu a}(x-a)(x-\sqrt{3}y+2a)(x+\sqrt{3}y+2a) \qquad \text{..........................(37)}$$

The volume rate of flow Q is given by

$$Q = \iint w \, dx \, dy$$

$$Q = -\frac{P}{12\mu a} \int_{x=-2a}^{a} \int_{y=-(x+2a)/\sqrt{3}}^{\frac{(x+2a)}{\sqrt{3}}} \left(x^3 - 3xy^3 + 3ax^2 + 3ay^2 - 4a^3\right) dx \; dy$$

$$= -\frac{P}{6\sqrt{3}\mu a} \int_{-2a}^{a} \left(\frac{2}{3}x^4 + \frac{10}{3}ax^3 + 4a^2x^2 - \frac{8}{3}a^3x - \frac{10a^4}{3}\right) dx$$

$$Q = \frac{27}{20\sqrt{3}} \frac{Pa^4}{\mu} \qquad \text{..(38)}$$

The average flow over an equilateral traangular cross - section is given by

$$\text{Average flow} = \frac{\text{Flux}}{\text{Area}}$$

$$= \frac{27}{20\sqrt{3}} \frac{Pa^4}{\mu} \Big/ \left(\frac{1}{2}.3a.2a\sqrt{3}\right)$$

$$= \frac{3}{20} \cdot \frac{Pa^2}{\mu} \qquad \text{...(39)}$$

4.7.5 Tube of Rectangular Cross - Section

Consider the flow through a rectangular pipe whose cross-section is bounded by the planes $x = \pm a$ and $y = \pm b$. Then the problem is to solve the equation

$$\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} = -\frac{P}{\mu} \qquad \text{...(40)}$$

with the boundary conditions

$$x = \pm a \quad ; \quad \omega = 0 \quad -(\mathrm{I})$$

$$x = \pm b \quad ; \quad \omega = 0 \quad -(\mathrm{II}) \text{....................................(41)}$$

Let the particular integral of the equation (40) satisfying the first boundary condition be

$$\omega_1 = \frac{P}{2\mu}\left(a^2 - x^2\right) \qquad \text{...(42)}$$

which is not satisfies by the II boundary condition.

Now we take a suitable solution ω_2 for the Laplace equation, so that

$$\omega_1 = \omega_1 + \omega_2 \qquad \text{...(43)}$$

satisfies the equation (40) and all the boundary conditions (41)

for the symmetry of the cross-section, it is obvious that ω must be an even function of x and y. Since ω_1 is the even function of x and ω_2 should be an even function in x and y.

Now take $\omega_2 = X(x) . Y(y)$

as a solution to the equation $\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} = 0$

Therefore $\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$

or $-\frac{1}{X}\frac{d^2x}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2}$ ....(42)

L.H.S. of (42) is a function of x alone whereas R.H.S. is a function of y only. So equation (42) is valid only if each side is a constant say C_n^2 so it gives

$$-\frac{1}{X}\frac{d^2X}{dx^2} = C_n^2 = \frac{1}{Y}\frac{d^2Y}{dy^2}$$

or $\frac{d^2X}{dx^2} + C_n^2 X = 0$ and $\frac{d^2Y}{dy^2} - C_n^2 Y = 0$ ....(43)

On solving (43), We have

$$X = A\cos(C_n x) + B\sin(C_n x)$$

and $Y = C\cosh(C_n y) + D\sinh(C_n y)$

Since w_2 is an even function of x and y, we must have $B = D = 0$

so that the terms sin $(C_n x)$ and $\sinh(C_n y)$ must be zero when $x = \pm a$ and $y = \pm b$

Hence $\omega_2 = \sum_{n=0}^{\infty} A_n \cos(C_n x)\cosh(C_n y)$ ....(44)

Now from equation (42) , we have

$$\omega = \omega_1 + \omega_2$$

$$\omega = \frac{P}{2\mu}\left(a^2 - x^2\right) + \sum_{n=0}^{\infty} A_n \cos(C_n x)\cosh(C_n y) \text{ ....(45)}$$

subject to the boundary conditions (41)

The first boundary conditions of (41) gives that

$$\cos(C_n a) = 0$$

$$\Rightarrow C_n = \frac{(2n+1)\pi}{2a} \text{ ....(46)}$$

and the second boundary condition, now requires

$$-\frac{P}{2\mu}\left(a^2 - x^2\right) = \sum_{n=0}^{\infty} An \frac{\cos(2n+1)\pi}{2a} \cosh\frac{(2n+1)\pi b}{2a} \text{ ....(47)}$$

for $-a \le x \le a$

Multiplying both sides by $\cos(2n+1)\frac{\pi x}{2a}$ and integrating between the limits $-a$ to a and noting that

$$\int_{-a}^{a} \cos\frac{(2m+1)\pi}{2a} \quad \cos\frac{(2n+1)\pi a}{2a}\, dx \quad = \begin{matrix} o \text{ if } n \neq m \\ a \text{ if } n = m \end{matrix}$$

then we find

$$An = -\frac{P(-1)^n.32a^2}{2\mu\ \pi^3\ (2n+1)^3\ \cosh\left[(2n+1).\ \frac{\pi b}{2a}\right]}.$$

Hence the velocity distributioin in a channel of rectangular cross- section is given by

$$w = \frac{P}{2\mu}\left[(a^2-x^2)-\frac{32a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\frac{\pi x}{2a}\cosh(2n+1)\frac{\pi b}{2a}}{(2n+1)^3\cosh(2n+1)\frac{\pi b}{2a}}\right] \text{......................(48)}$$

And the volume rate of flow Q is given by

$$Q = \int_{-b}^{b}\int_{-a}^{a} \omega\ dx\ dy$$

$$= \frac{P}{2\pi}\left[\frac{8a^3b}{3}-\frac{512a^4}{\pi^5}\sum_{n=0}^{\infty}\frac{\tanh\ (2n+1)\frac{\pi b}{2a}}{(2n+1)^5}\right] \text{...........................(49)}$$

4.8 Flow between Two Concentric Rotating Cylinders (Couette Flow)

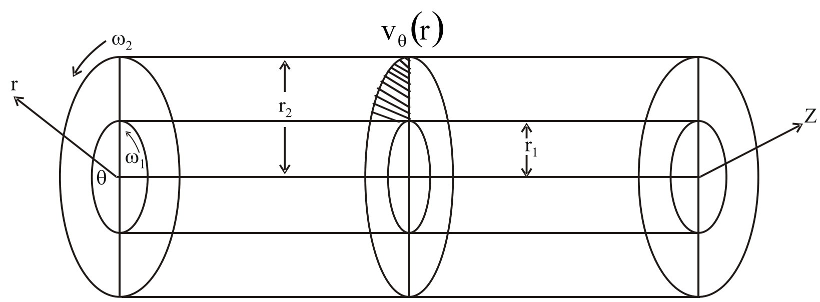


Fig. 4.6

Let us consider the flow between two concertric rotating cylinders. Let r_1, ω_1 and r_2, ω_2 be the radius and angular velocity of the inner and of the outer cylinder respectively.

Assuming that z-axis is along the common axis of the cylinders and r denotes the radial distance measured outward from the z-axis. In this case the non-zero component of velocity is V_θ and there is no pressure grdient in the θ-direction. Hence the equation of continuity and the equations of motion (table 2.3 of unit 2) in cylindrical polar coordinates reduces to

$$\frac{\partial v_\theta}{\partial \theta} = 0 \text{ .... (1)}$$

which shows that v_θ is the function of r.

$$\rho\frac{V_\theta^2}{r} = \frac{dp}{dr} \text{ .... (2)}$$

$$\frac{d^2V_\theta}{dr^2}+\frac{d}{dr}\left(\frac{V_\theta}{r}\right)=0 \text{ .... (3)}$$

on integrating equation (3), we obtain

$$v_\theta = Ar + \frac{B}{r} \text{ ....(4)}$$

where A and B are constents of integration to be determined by using the boundary condition.

$$r = r_1 \quad ; \qquad v_\theta = r_1\omega_1$$

$$r = r_2 \quad ; \qquad v_\theta = r_2\omega_2 \text{ ....(5)}$$

On using (5) in (4) we have

$$r_1\omega_1 = A\,r_1 + \frac{B}{r_1}$$

$$r_2\omega_2 = A\,r_2 + \frac{B}{r_2}$$

On solving these, we have

$$A = \frac{\omega_2 r_2^2 - \omega_1 r_1^2}{r_2^2 - r_1^2}$$

and $$B = -\frac{r_1^2 r_2^2}{r_2^2 - r_1^2}(\omega_2 - \omega_1)$$

Substituting these value in equation (4), we obtain the velocity distribution as.

$$v_\theta = \frac{1}{\left(r_2^2 - r_1^2\right)}\left[\left(\omega_2 r_2^2 - \omega_1 r_1^2\right) - \left(\omega_2 - \omega_1\right)\frac{r_1^2 r_2^2}{r}\right]^2$$

The radial pressure distribution may, be calculated from equation (2).
Equation (2) may be written as

$$\frac{dp}{dr} = \frac{\rho}{r}V_\theta^2$$

or $$\frac{dp}{dr} = \frac{\rho}{r}\cdot\frac{1}{\left(r_2^2 - r_1^2\right)^2}\left[\left(\omega_2 r_2^2 - \omega_1 r_1^2\right)r - \left(\omega_2 - \omega_1\right)\frac{r_1^2 r_2^2}{r}\right]^2$$

$$= \frac{\rho}{\left(r_2^2 - r_1^2\right)^2}\left[\left(\omega_2 r_2^2 - \omega_1 r_1^2\right)^2 r - \frac{2r_1^2 r_2^2}{r}\left(\omega_2 r_2^2 - \omega_1 r_1^2\right).\left(\omega_2 - \omega_1\right) + \frac{r_1^4 r_2^4}{r^3}\left(\omega_2 - \omega_1\right)^2\right]$$

On integrating, we have

$$p = \frac{\rho}{\left(r_2^2 - r_1^2\right)^2}\left[\left(\omega_2 r_2^2 - \omega_2 r_1^2\right)^2\frac{r^2}{2} - 2r_1^2 r_2^2\left(\omega_2 r_2^2 - \omega_1 r_1^2\right).\left(\omega_2 - \omega_1\right)\log r - \frac{r_1^4 r_2^4}{2r^2}\left(\omega_2 - \omega_1\right)^2\right] + C \text{ ....(7)}$$

where C is the constant of integration to be determined by taking $p = p_1$ at $r = r_1$ then

$$C = p_1\,\frac{\rho}{\left(r_2^2 - r_1^2\right)^2}\left[\left(\omega_2 r_2^2 - \omega_1 r_1^2\right)^2\frac{r_1^2}{2} - 2r_1^2 r_2^2\left(\omega_2 r_2^2 - \omega_1 r_1^2\right).\left(\omega_2 - \omega_1\right)\log r_1 - \frac{r_1^4 r_2^4}{2r_1^2}\left(\omega_2 - \omega_1\right)\right]$$

Hence on substituting the value of C in equation (7) we obtain the pressure distribution as

$$p = p_1 + \frac{\rho}{\left(r_2^2 - r_1^2\right)^2}\left[\left(\omega_2 r_2^2 - \omega_1 r_1^2\right)^2 \left[\frac{r^2 - r_1^2}{2}\right]\right.$$

$$\left. - 2r_1^2 r_2^2\left(\omega_2 r_2^2 - \omega_1 r_1^2\right)\left(\omega_2 - \omega_1\right) \log \frac{r}{r_1} - \frac{r_4^4 r_2^4}{2}\left(\omega_2 - \omega_2\right)^2\left(\frac{1}{r^2} - \frac{1}{r_1^2}\right)\right] \qquad \text{.............................(8)}$$

If the inner cylinder is at rest then $\omega_1 = 0$ and in that particular case, the velocity components v_θ is given by

$$v_\theta = \frac{\omega_2 r_2^2}{\left(r_2^2 - r_1^2\right)}\left(r - \frac{r_1^2}{r}\right) \qquad \text{..(9)}$$

The only viscous stress in the fluid is $\tau_{r\theta}$ which is given by

$$\tau_{r\theta} = \mu r \frac{d}{dr}\left(\frac{v_\theta}{r}\right)$$

$$\tau_{r\theta} = \frac{2\mu\omega_2 r_1^2 r_2^2}{\left(r_2^2 - r_1^2\right)r^2} \qquad \text{..(10)}$$

when the inner cylinder is taken at rest.

4.8.1 TORQUE

The torque is the force which is required to turn the outer cyulinder. This may now be easily claculated as the product of the force and the arm of the couple as

$$M = \pi r_2 L \left(\tau_{r\theta}\right)_{r = r_2} .2r_2$$

$$= 4\pi\mu L \frac{r_1^2 r_2^2}{\left(r_2^2 - r_1^2\right)}\omega_2 \qquad \text{..(11)}$$

where L is the length of the cylinder.

Self Learning Exercise

1. Define plane Couette flow
2. What is the difference between plane Couette flow and plane Poiseuille flow ?
3. How do you explain a back flow in case of generalized Couette flow ?
4. Define volume rate of flow

4.9 Answers to Self Learning Exercise

1. See article. 4.3.1
2. See article. 4.3. & 4.4
3. See article. 4.5
4. See article. 4.5.1

4.10 Summary

In this unit, you have studied about the exact solutions of the Navier-Stoke's equations for the steady viscous incompressible fluid motion between two parallel plate, between two concentric rotating cylinder and flow in tubes of uniform cross-section. You have also studied about volume rate of flow, coefficient of skin friction and torque in the fluid motions.

Thus you are now aware that the Navier-Stoke's equations can have exact solutions for certain flow through channels of simple geometry.

4.11 Exercise

1. Discuss the flow of an incompressible viscous fluid between two parallel plates taking the fluid properties to be constant when one of the plated is given a constant velocity in its own plane.
2. Discuss the plane Poiseuille flow between two parallel plates.
3. Discuss the generalized plane Couette flow. Derive the results for various characteristic for plane Couette flow taking that as a particular case.
4. A viscous incompressible fluid moves in a steady flow under constant pressure gradient P parallel to the axis in the annular space between two coaxial cylinders of radii a and b (b < a). Show that the volume rate of flow is given by

$$Q = \frac{\pi P b^4}{8\mu}\left[\left(n^4 - 1\right) - \frac{\left(n^2 - 1\right)^2}{\log n}\right]$$

where $n = a/b$

5. Show that the volume rate of flow is given by $Q = \frac{27 P a^4}{20\sqrt{3}\mu}$ in the steady flow of a viscous incompressible fluid through a tube with uniform equilateral triangular cross section.
6. Find the velocity distribution for the steady flow of a viscous incompressible fluid in the annular region between two concentric cylinders.
7. Obtain the viscous stress in the flow between two concentric rotating cylinder when the inner cylinder being at rest. Also find the torque

UNIT - 5

Stagnation point flow and flow due to a rotating disc

Structure of the Unit

5.0 Objectives

The purpose of this unit is to find some more, to be precise two exact solutions of Navier-Stokes equations. The two problems are (i) Stagnation point flow (Hiemenz flow) and (ii) Flow due to a rotating circular disc (Kar man flow)

5.1 Introduction

In this unit we shall get exact solutions of Navier-Stokes equations for two problems of different geometry.

First problem popularly known as Hiemenz flow discuss the flow in the neighbourhood of a stagnation point in two dimensions. In this problem the velocity distribution shows the presence of a boundary layer, about which we will learn later, of small thickness for small kinematic viscosity.

The second problem is about a flow due to a rotating disc, popularly known as Karmans problem. In this flow also we will have an effect due to boundary layer whose thickness is again small for small kinematic viscosity.

In both the problems, velocity distributions have been shown graphically.

5.2 Definitions

5.2.1 Stagnation Point

Stagnation Point is the point where the velocity is zero in the potential flows i.e. the flow of an ideal fluid.

5.2.2 Boundary Layer :

It is a small layer near wall in which all the viscous effects are supposed to be confined and out of this layer the flow is treated as potential flow.

5.3 Stagnation Point Flow (Hiemenz Flow)

An exact solution of the flow of a viscous incompressible fluid in the neighbourhood of a stagnation point in a plane may be obtained by considering the flow at a large distance from the stagnation point to be the potential flow i.e. the flow of an ideal or non viscous fluid.

The velocity and pressure in a potential flow in the neighbourhoods of the stagnation point considered as pole $x = 0$, $y = 0$ in a plane arc.

$U = bx$, $V = -by$..(1A)

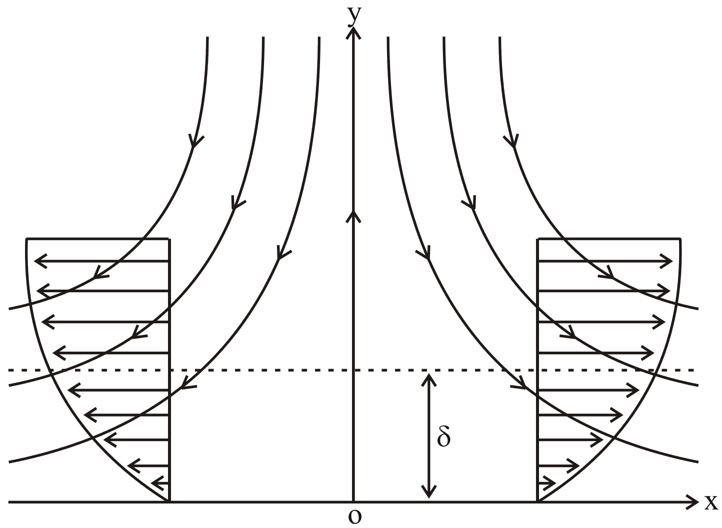


Fig. 5.1

Bernoulli equation

and $\frac{p}{\rho}+\frac{1}{2}q^2=\frac{p_0}{P}$

or $\frac{p}{\rho}+\frac{1}{2}\left(U^2+V^2\right)=\frac{p_0}{\rho}$

where p_0 is the pressure at stagnation point.

Thus $p_0-p \;=\frac{1}{2}\rho b^2\left(x^2+y^2\right)$.. (1B)

When viscosity is included, we take the following forms of the velocity and pressure distribution for the flow

$u=xf^1(y),\quad v=-f(y)$...(2)

and $p_0-p=\frac{\rho}{2}b^2\left[x^2+F(y)\right]$...(3)

where prime denote differential with respect to y

We know that Navier-Stokes equations in two dimensional steady motion are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\;=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 y}{dx^2}+\frac{\partial^2 u}{dy^2}\right)\text{.............................(4)}$$

and $$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\;=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{dx^2}+\frac{\partial^2 v}{dy^2}\right)\text{.............................(5)}$$

using (2), (3), in (4) and (5) we get

$f'^2-ff''=b^2+\upsilon f'''$...(6)

and $ff^1\;=\frac{1}{2}b^2F^1-\nu f^{11}$ ....(7)

Boundary conditions are :

$y\;=\;0,\qquad u\;=\;v\;=\;0$

i.e. $f\;=\;f^1=0$ [(by using (2) & (3)]....................(8)

and at origin $p=p_0$ i.e. $F=0$...(9)

and at a large distance, $U=u$ as $y\to\infty$

So $bx = xf^1(\infty)$

i.e. $f^1 = b$ as $y \to \infty$ ....(10)

So in all, the boundary condition are

$$\left.\begin{array}{l} y = 0, \quad f = f^1 = F = 0 \\ y \to 0, \quad f^1 = b \end{array}\right\} \qquad \text{...(11)}$$

in order to solve the equation (6) for f, we make the following transformation

$$\eta = \sqrt{\frac{b}{\nu}}.y \qquad \text{and} \qquad f(y) = \sqrt{b\nu}\,\phi(\eta)$$

Therefore

$$f^1(y) = \frac{df}{d\eta}.\frac{d\eta}{dy} = \sqrt{b\nu}\,\phi'(\eta)\sqrt{\frac{b}{\nu}} = b\phi^1(\eta) \text{... (12)}$$

and

$$f^{11}(y) = b\phi^{11}(\eta).\sqrt{\frac{b}{\nu}} \text{.. (13)}$$

$$f^{111}(y) = \frac{b^2}{\nu}\phi^{111}(\eta) \text{.. (14)}$$

Using (12), (13), (14) in (6) with B.C.S (11) equation (6) becomes

$\phi^{111} + \phi\phi^{11} - \phi^{12} + 1 = 0$

$$\text{i.e.} \quad \frac{d^3\phi}{d\eta^3} + \phi\frac{d^2\phi}{d\eta^2} - \left(\frac{d\phi}{d\eta}\right)^2 + 1 = 0 \text{.. (15)}$$

with corresponding boundary condition for ϕ as

$$\left.\begin{array}{l} \eta = 0, \quad \phi = 0, \quad \phi^1 = 0 \\ \eta \to \infty, \quad \phi^1 = 1 \end{array}\right\} \qquad \text{.. (16)}$$

here a prime indicates differentiation with respect to η

Equation (15) was first solved numerically by Hiemenz and the solution was later improved by L. Howarth. The dimension less velocity in the x-direction is given by

$$\frac{u}{U} = \frac{xf^1(y)}{bx} = \frac{f^1(y)}{b} = \phi^1(\eta) \quad \text{[from (12)] .... (17)}$$

The unknown function F(y) occuring in the expression for pressure in equation (3) will be obtained by integrating equation (7)

$$b^2F = 2\nu f^1 + f^2$$

$$= 2\nu.b\phi^1 + b\nu\phi^2 \quad \text{[from (12)]}$$

$$\text{or } F = \frac{\nu}{b}\left(\phi^2 + 2\phi^1\right) \text{.................................(18)}$$

substitute the value of F in equation (3) we get the required pressure distribution

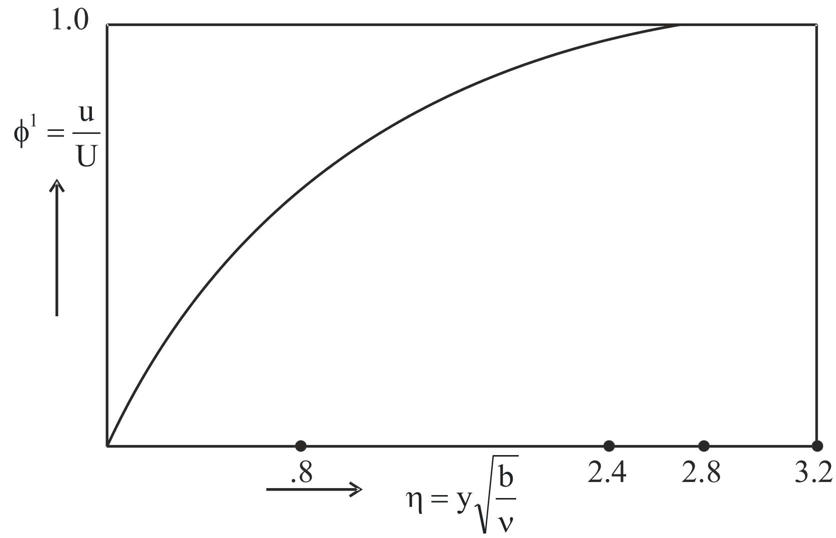


Fig. 5.2

from the fig.5.2 it is seen that for $\eta = 2.4$ maximum velocity is reached and the corresponding value of y obtained from the relation

$\eta = y\sqrt{\frac{b}{\nu}}$, we symbolise it as δ

so that

$$2.4 = \delta\sqrt{\frac{b}{\nu}}$$

$$\Rightarrow \delta = 2.4\sqrt{\frac{\nu}{b}} \quad ....(19)$$

If ν is small then δ will be small and it can be said that the viscous effects are confined in a very thin layer near the wall and the thickness of the layer is proportional to $\sqrt{\nu}$

5.4 Flow due to a rotating disc (Kármán Flow)

Let us confider the flow due to a disc which rotates with an angular velocity w about an axis perpendicular to its plane in a fluid other wise at rest. In order to avoid the edge effect the disc is considered to be of infinite radius. Due to the action of centrifugal forces the fluid near the disc will be thrown outward so the radial component v_r exists. Due to rotation of disc azimuthal component v_θ also exists and so is the axial component v_z.

Thus in this case all the three components of velocity in the cylindrical polar coordinates exist.

The boundary conditions for the motion are

at $z = 0$, $v_r = 0$, $v_\theta = r\omega$, $v_z = 0$

and

as $z \to \infty$, $v_r = 0$, $v_\theta = 0$

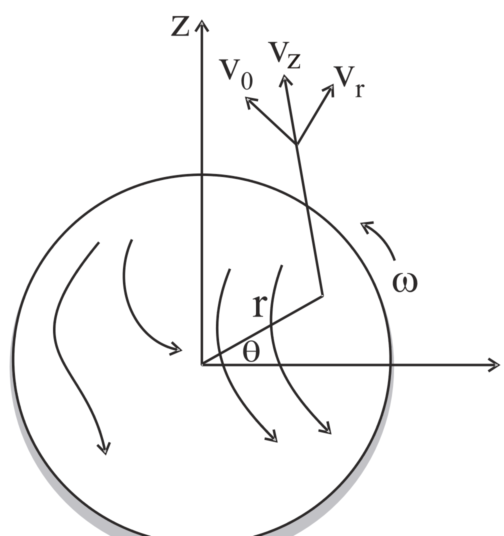


Fig. 5.3

Since the motion is steady and symmetrical in θ-direction, so we have

$$\frac{\partial(\ )}{\partial t}=0 \quad \text{and} \quad \frac{\partial(\ )}{\partial \theta}=0$$

therefore reduced flow equation are

equaiton of continuity : $\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{\partial v_z}{\partial z}=0$

Navier-Stokes equations for the fluid motion

r - Component -

$$\rho\left(v_r\frac{\partial v_r}{dr}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\theta^2}{r}\right)=-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}+\frac{\partial^2 v_r}{\partial z^2}-\frac{v_r}{r^2}\right)$$

θ - Component -

$$\rho\left(v_r\frac{\partial v_\theta}{\partial r}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_r v_\theta}{r}\right)=\mu\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}+\frac{\partial^2 v_\theta}{\partial z^2}-\frac{v_\theta}{r^2}\right)$$

z - Component -

$$\rho\left(v_r\frac{\partial v_z}{\partial r}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}+\frac{\partial^2 v_z}{\partial z^2}\right)$$

Now we take the following form of velocity and pressure distribution.

$$\begin{aligned} v_r &= \omega r f(z)\\ v_\theta &= \omega r g(z)\\ v_z &= (\nu\omega)^{1/2}h(z)\\ p &= \rho\upsilon\omega P(z)\end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

substituting (1) in the equations of motion and equation of continuity, we have

$$\frac{1}{r}\frac{\partial}{\partial r}(\omega r^2 f)+\sqrt{\nu\omega}\ h^1=0$$

$$\text{i.e. } 2\omega f+\sqrt{\nu\omega}\ h^1=0 \Rightarrow 2f+\left(\frac{\nu}{\omega}\right)^{1/2}h^1=0 \ldots\ldots\ldots\ldots(2)$$

$$f^2-g^2+\left(\frac{\nu}{\omega}\right)^{1/2}hf^1=\left(\frac{\nu}{\omega}\right)f^{11} \ldots\ldots\ldots\ldots(3)$$

$$2fg+\left(\frac{\nu}{\omega}\right)^{1/2}hg^1=\left(\frac{\nu}{\omega}\right)g^{11} \ldots\ldots\ldots\ldots(4)$$

$$hh^1=-P^1+\left(\frac{\nu}{\omega}\right)^{1/2}h^{11} \ldots\ldots\ldots\ldots(5)$$

where a prime denotes differentiation w.r.t.z.

In order to remove the coefficient $(\nu/\omega)^{1/2}$ and (ν/ω), we make the following transformation

$$\eta=(\omega/\nu)^{1/2}.\ z$$

$$f(z)=f(\eta),\; g(z)=G(\eta),\; h(z)=H(\eta)$$

$$\text{and } P(z)=p(\eta) \qquad \ldots\ldots (6)$$

Using (6) in equations (2) to (5) we get

$$2F+H^{1}=0$$

$$F^{2}-G^{2}+F^{1}H=F^{11} \qquad \ldots\ldots (7)$$

$$2FG+G^{1}H=G^{11}$$

and $HH^{1}=-p^{1}+H^{11}$

with the boundary conditions

$$\eta=0,\quad F=0,\quad G=1,\quad H=0$$

and

$$\eta\to\infty,\quad F=0,\quad G=0 \qquad \ldots\ldots (8)$$

where a prime now denotes differentiation w.r.t. to η.

The solution of first three equation of (7) can be obtained with the help of the boundary condition in (8) and then the last of equation(7) will give P in terms of H. Van Karman was the first to obtain a solution of this system of equations by an approximate method which was later improved by Cochran and other workers.

It is evident from figure 5.4 that the value of F,G and H-tend asymptotically to their limiting values. However all of these limiting values are attained approximately about η=5. We have considered η in terms of z and if we consider this value of $\eta = 5$ corresponding to some $z = \delta$, we have

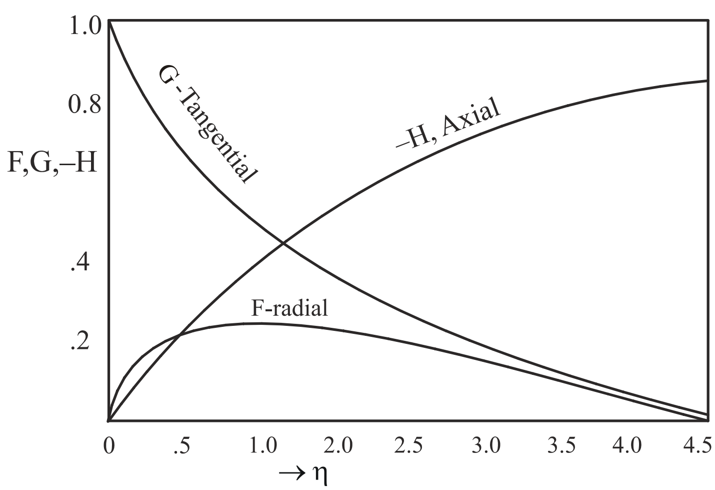


Fig. 5.4

$$\delta \simeq 5(\nu/\omega)^{1/2} \qquad \ldots\ldots (9)$$

If (ν/ω) is small then δ will also be small and once again we will have a boundary layer type of flow. The circumfrential shearing stress on the plane on the plate will be

$$\tau_{z\theta}=\mu\left(\frac{\partial v_{\theta}}{\partial z}\right)_{z=0}$$

$$=\ \mu\omega r\left(\frac{w}{v}\right)^{1/2} G'(0) \qquad \ldots\ldots (10)$$

and the frictional moment on one side of the disc is

$$m\ =\ -\int_{0}^{a} r\tau_{z\theta}\,2\pi r\,dr$$

$$=\ -\frac{1}{2}\pi\rho a^{4}\left(vw^{3}\right)^{1/2} G'(0) \qquad \ldots\ldots (11)$$

5.5 Self Learning Exercise

1. .... is a point where the velocity is zero in potential flow.
2. $\phi^{111} + \phi\phi^{11} - \phi^{1^2} + 1 = 0$ equation correspond to ....flow.
3. Flow due to a rotating disc is also known as.......................

5.6 Summary

In this unit we have discussed two important problems which give exact solution to Navier-Stokes equations.

In the first one we considered a flow which is near a stagation point, defined as the point with zero velocity in potential flow.

In the second problem. a flow is considered which is due to rotation of a disc and the flow is above the infinite disc. An important aspect of this is that in this flow all the three velocity components exist in cylindrical polar coordinates .

5.7 Answer to Self Learning Exercise

1. Stagnation Point
2. Hiemanz
3. Karman flow

5.8 Exercise

1. Discuss stagnation point flow of an incompressible, viscous fluid (Hiemanz flow)
2. A viscous incompressible fluid is bounded on one side (Z > 0) by a circular disc of infinite radius and lying at z = 0 and rotating about its axis r = 0. Verify that the steady flow is given by

$$v_r = \omega r F(\eta), \quad v_\theta = \omega r G(\eta), \quad v_z = (\nu\omega)^{1/2} H(\eta)$$

and $\quad p = \rho\nu\omega P(\eta)$

where ω is the angular velocity of the plate and

$\eta = \left(\dfrac{\omega}{\nu}\right)^{1/2} z$ with other symbols have their usual meanings

UNIT - 6

Unsteady Motion of Fluids

Structure of the Unit

6.0 Objectives

In this unit you will learn about unsteady motion of a fluid . In unsteady flows the velocity mainly is considered to be depending on time. This type of flow has many important applications in the fields of Engineering

6.1 Introduction

Unsteady motion is the study of fluid motion whenever the flow is time dependent. We start with "Flow due to a plane-wall suddenly set in motion "which is known as Stokes first problem and follow it with the "Flow due to an oscillating plane wall" which is known as Stokes second problem. Sometime it is also called as Rayleigh's problem.

6.2 Definitions

Unsteady Motion : If the velocity changes with time then we call the motion to be an unsteady motion.

Startup Flows : When at $t = 0$ i.e. initially entire fluid is at rest.

6.3 Concept of Unsteady Flow

Exact solutions of the unsteady Navier Stokes equations exist when there already exist exact solutions of the corresponding steady flow

Suppose the infinite long flat plate is considered in the X-direction.

Initially both the plate and the fluid are at rest, suddenly the plate is jerked into motion in its own plane with a constant velocity U.

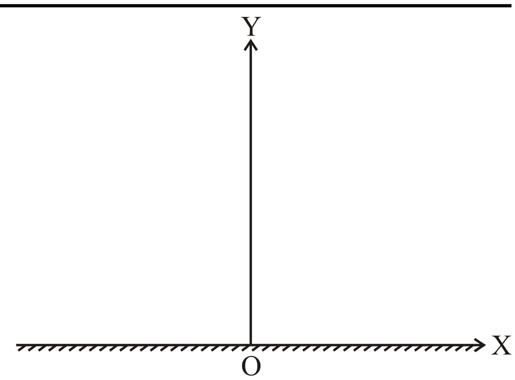


Fig. 6.1

Because the motion of the boundary is in X-direction two components in y-and z-directions v and w respectively are zero. So the only non zero component of velocity is u which is a function of y and t only. The pressure is uniform at every point in the fluid over the wall, hence it is assumed to be constant. Hence, the Navier-Stokes equations in cartesian co-ordinate reduce to

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

This is the governing equation for unsteady motion of the fluid over and due to moving plate.

6.4 Stoke's First Problem

We consider the flow close to a wall which is suddenly set into motion with a constant velocity U_0 in its own plane. this problem was first solved by G.G. Stokes (1856) in his famous treatment of the pendulum.

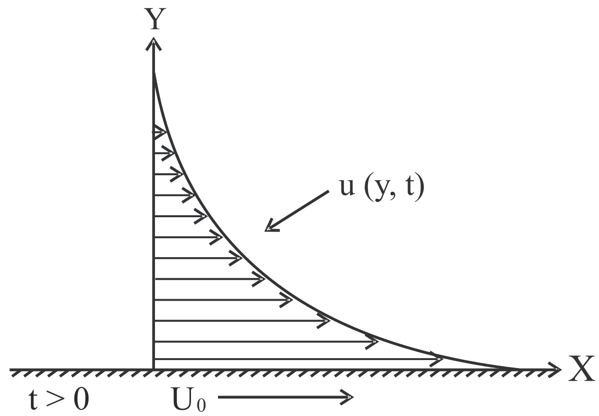


Fig. 6.2

Here the motion is due to plate suddenly started with a constant velocity U_0 in x-direction in its own-plane. When plate moves with a velocity U_0 the fluid adjescent to it also moves with velocity U_0. The initial and boundary conditions on u (y, t) are

$$\begin{aligned} &t \le 0 \quad u(y,0) \quad (\text{initially}) \\ &t > 0 \quad y = 0 \quad u(0,t) = U_0 \\ &\qquad\quad y \to \propto \quad U(\propto, t) = 0 \end{aligned} \qquad \text{.......(1)}$$

The reduced Navier Stokes equation in cartesion coordinates as given in (6.3) is

$$\frac{\partial u}{\partial t} = \frac{\nu \partial^2 u}{\partial y^2} \qquad \text{.......(2)}$$

To solve equation (2) we make the following substitution

$$u = U_0 f(\eta) \quad \text{and} \quad \eta = \frac{y}{2\sqrt{\nu t}} \qquad \text{.......(3)}$$

so that

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}.\frac{\partial \eta}{\partial t} = U_0 f^1(\eta).\frac{y}{2\sqrt{\nu}}\left(-\frac{1}{2}t^{-3/2}\right)$$

$$= U_0 f^1(\eta).\frac{y}{2\sqrt{\nu t}} \times -\frac{1}{2}t$$

$$= \frac{-U_0 f^1(\eta)}{2t}.\eta$$

and

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}.\frac{\partial \eta}{\partial y} = U_0 f^1(\eta).\frac{1}{2\sqrt{\nu t}}$$

$$\& \quad \frac{\partial^2 u}{\partial y^2} = \frac{U_0}{4\nu t} f^{11}(\eta) \qquad \text{.......(4)}$$

substituting (4) in (2), we have.

$$\frac{-U_0}{2t}.f^1(\eta).\eta = \frac{U_0}{4t} f^{11}(\eta)$$

$$\text{or } f^{11}(\eta) + 2\eta f^1(\eta) = 0 \quad \text{.....................................(5)}$$

where prime denotes differentiation w.r.t. to η.
The corresponding boundary conditions are then

$$\left.\begin{array}{l} \eta = 0, \quad f(0) = 1 \\ \eta \to \infty, \; f(\infty) = 0 \end{array}\right\} \quad \text{...(6)}$$

Integrating equation (5), we have

$$f^1(\eta) = C_1 e^{-\eta 2}$$

Hence

$$f(\eta) = C_1 \int_0^\eta e^{-\eta 2} d\eta + C_2$$

at $\quad \eta=0, \quad f=1 \Rightarrow \quad 1=C_2$

Also $\quad \eta \to \infty, \qquad f = 0,$ so that

$$0 = C_1 \int_0^\infty e^{-\eta^2} d\eta + 1$$

or

$$C_1 = \frac{-1}{\int_0^\infty e^{-\eta^2} d\eta} = \frac{-2}{\sqrt{\pi}}$$

Hence $\quad f(\eta) = \frac{-2}{\sqrt{\pi}} \int_0^\eta e^{-\eta^2} d\eta + 1$

$$= -\text{erf}(\eta) + 1$$

$$= \text{erfc}(\eta)$$

Hence the velocity u is given by

$$u = U_0 \; f(\eta) \; = \; U_0 \text{erfc}(\eta)$$

where erf and erfc are known standard functions error function and compumentary error function. Numerical values for different η are known.
The velocity profile are shown in the figure 6.3
They continuously decrease as $\eta \to \infty$ i.e. they asymptotically reach their limiting value zero
For all practical purpose when the value reaches at $\eta = 2$ and therefore the corresponding value of y, which we call as δ, be comes $\delta = 4\sqrt{\nu t}$.
This distance δ is called the penetration depth with standard condition. This penetration depth is proportional to the square root of the product of viscosity & time.

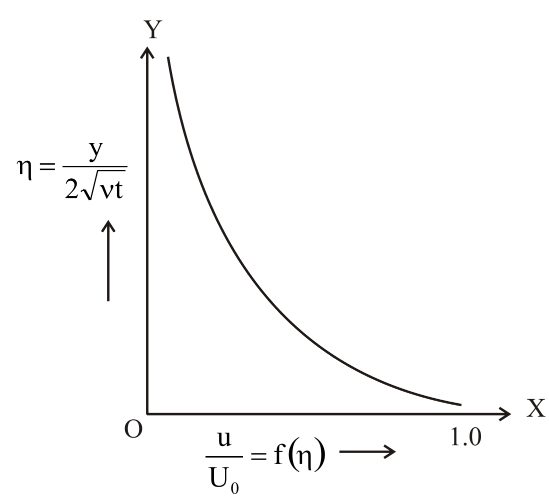


fig. 6.3

6.5 Stokes Second Problem

Another simple unsteady flow is that in which a plane wall oscillates with a prescribed velocity $U_0 \cos nt$ when initially plane was at rest where U_0 is the amplitude and n is the frequency of the oscillation of the wall. This flow was first studied by Stokes and later by Lord Rayleigh. In the literature it is known either as Stokes second problem or simply Rayleigh problem.

Due to the presence of the fluid, amplitude of the fluid motion will be function of y and frequency will remain unchanged. Hence for the flow here

$u = f(y)\cos nt \;=$ Real part $\left(f(y)e^{int}\right)$..(1)

Now the reduced Navier Stokes equation for unsteady motion is (refer 6.3)

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \text{ ....(2)}$$

Substituting (1) in (2) we get the differential equation

$$ine^{int} f(y) = \nu . e^{int} f^{11}(y)$$

or $$f^{11}(y) - \frac{in}{\nu} f(y) = 0 \text{ ....(3)}$$

$$\left(\frac{d^2}{dy^2} - \frac{in}{\nu}\right) f(y) = 0$$

with boundary conditions

$$u(0,t) = f(0)\, e^{int} = U_0 e^{int}$$

$$\Rightarrow f(0) = U_0$$

$$U(\propto, t) = f(\propto)\, e^{int} = 0$$

or $$f(\propto) \to 0$$

So $$\begin{matrix} y = 0 & ; & f(0) = U_0 \\ y \to \infty & , & f(\infty) \Rightarrow 0 \end{matrix} \text{ ....(4)}$$

So solution of equation is

$$A_1 e^{(1+i)\sqrt{\frac{n}{2\nu}}.y} + A_2 e^{-(1+i)\sqrt{\frac{n}{2\nu}}.y} \text{ ....(5)}$$

when $y \to \infty$ then $f(\infty) \to 0$ so we have A_1=0

Therefore $f(y) = A_2 e^{-(1+i)\sqrt{\frac{n}{2\nu}}.y}$

at $y = 0$; $f(0) = U_0 \;\Rightarrow\; A_2 = U_0$

so from (5)

$$f(y) = U_0 e^{-(1+i)\sqrt{\frac{n}{2\nu}}.y} \text{ ....(6)}$$

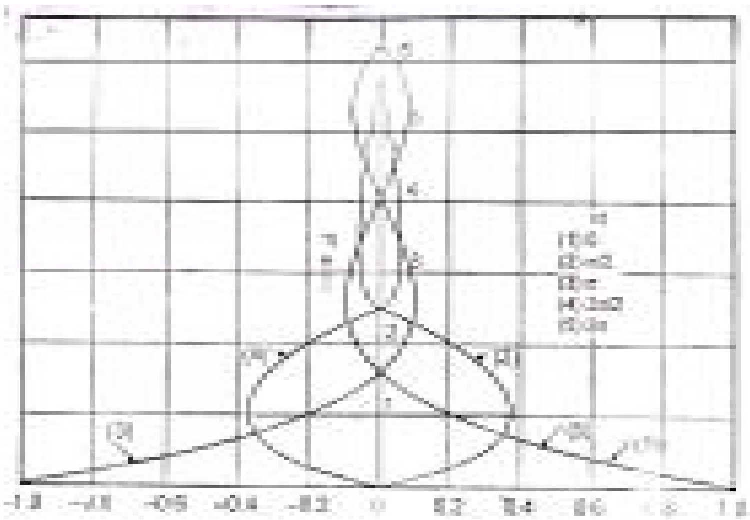
from (1) and (6)

$$u = \text{real part}\left[U_0 e^{int} e^{-(1+i)\sqrt{\frac{n}{2\nu}}.y}\right] = U_0 e^{-\sqrt{\frac{n}{2\nu}}.y} \cos\left(nt - \sqrt{\frac{n}{2\nu}} y\right)$$

If we put $\sqrt{\frac{n}{2\nu}}y = \eta$ then $\frac{u}{U_0} = e^{-\eta}\cos(nt-\eta)$

This shows that u is periodic in both y and t.

The velocity profiles, for this flow are shown in fig.6.4 for different values of nt.



$\longrightarrow \frac{u}{U_0}$

fig. 6.4

6.6 Self Learning Exercise

1. Flow due to a plane - wall suddenly set in motion is known as...................
2. Flow due to an oscillating plane wall is known as......................................
3. Reduce Navier Stokes equation are ....

6.7 Answer Self Learning Exercise

1. Stokes first problem
2. Stokes second problem or Rayleigh problem
3. $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$

6.8 Exercise

1. Discuss the flow due to a plane wall suddenly set in motion in its own plane in an infinite mass of viscous incompressible fluid, which is otherwise at rest.
2. Viscous incompressible fluid occupies the region $y > 0$ on one side of an infinite plate $y = 0$. The plate oscillates with a velocity U_0cos nt in the x-direction. Show that the velocity distribution of the fluid motion is given by $u = U_0 e^{-n\cos(nt-\eta)}$

$$\eta = \left(\frac{n}{2\nu}\right)^{1/2} y$$

UNIT - 7

Starting Flow and Suction / Injection Through Porous Walls

Structure of the Unit

7.0 Objectives

The purpose of this unit is to discuss starting flow in plane-Couette motion and suction/injection through porous walls. These are two different types of fluid motions. While one is a study of an unsteady motion and other has practical utility

7.1 Introduction

In this unit two types of motion are considered, one is a typical unsteady problem where in time is measured at the initial level when the plane wall is started to cause the flow in the channel. So even if the flow at a later stage becomes a Couette flow the velocity profiles are changed with time.
In another problem the boundaries have been considered porous and to cause the change in motion of the fluid which is moving with a constant velocity the same fluid is injected and sucked at the two plates with equal constant velocities.

7.2 Definitions

7.2.1 Suction : When the fluid is drawn out through porous boundaries the process is called suction.

7.2.2 Injection : When the fluid is pushed in through the porous boundary the process is called injection.

7.2.3 Starting flow : An unsteady flow in which the time is measured from the moment the fluid is given a motion.

7.3 Starting Flow in Plane-Couette Motion

Consider a plate placed along x-axis which is suddenly set in motion in its own plane with a constant velocity $u = U_0$ in the presence of another plate which is at rest and is parallel to the lower plate at a distance 'h'. Take the x-axis along the lower plate and the y-axis is taken normal to the plates. The fluid between the plates is at rest before starting motion of lower plate. The governing equations of motion of viscous incompressible fluid between parallel plates is given

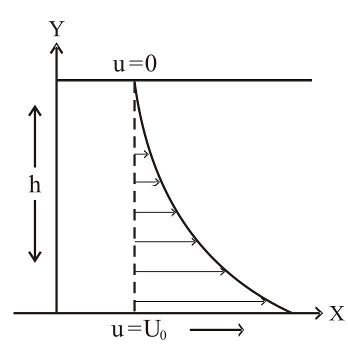


Fig. 7.1

by $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$ ....(1)

where u is the velocity component along x-axis, t is the time and ν is the kinematic viscosity.
This equation is the outcome of the consideration of Navier-Stokes and equation of continuity with the geometry of the problem in two dimensions. The initial and boundary conditions are :
Initial condition :

$$t \le 0 \qquad u = 0 \qquad \text{for} \quad 0 \le y \le h$$

Boundary conditions are

$$\left.\begin{array}{l} t > 0 \ : \ u = U_0 \quad \text{when } y = 0 \\ \qquad\quad u = 0 \quad \text{when } y = h \end{array}\right] \ldots\ldots\ldots\ldots(2)$$

To solve let us introduce the transformation

$$u = U_0 f(\eta) \ldots\ldots\ldots\ldots(3)$$

where $\eta = \dfrac{y}{2\sqrt{\nu t}}$ is a dimensionless quantity

From above

$$\frac{\partial u}{\partial t} = -\frac{U_0 f^1(\eta)}{2t}\eta$$

and $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{U_0}{4\nu t} f^{11}(\eta)$

Hence from (1) we have

$$f^{11}(\eta) + 2\eta f^1(\eta) = 0 \ldots\ldots\ldots\ldots (4)$$

where prime denotes differentiation w.r.t. to η.
Now the boundary conditions are reduces to

$$\left.\begin{array}{ll} \eta = 0 & f(\eta) = 1 \\ \eta = \eta_1 & f(\eta) = 0 \end{array}\right] \ldots\ldots\ldots\ldots (5)$$

where $\eta_1 = \dfrac{h}{2\sqrt{\nu t}}$ (say)

Solving equation (4) , we have

$$f(\eta) = 1 - \mathrm{erf}(\eta) = \mathrm{erfc}\,\eta \ldots\ldots\ldots\ldots (6)$$

as the particular solution, which satisfies the first boundary condition,, where

$$\mathrm{erf}\eta = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2 d\eta}$$

If f(η) is a solution of (4) then $f(\alpha\eta_1 \pm \eta)$ is also a solution of it, where α is an arbitrary constant. Thus the solution of equation (4), which satisfies both the boundary conditions in (5), can be taken as

$$\frac{u}{U_0} = \sum_{\eta=0}^{\infty} f(2\eta\eta_1 + \eta) - \sum_{\eta=0}^{\infty} f[2(\eta+1)\eta_1 - \eta]$$

$$= \mathrm{erfc}(\eta) - \mathrm{erfc}(2\eta_1 - \eta) + \mathrm{erfc}(2\eta_1 + \eta) - \mathrm{erfc}(4\eta_1 - \eta) + \mathrm{erfc}(4\eta_1 + \eta) \ldots\ldots\ldots\ldots(7)$$

The velocity profiles for different values of $\sqrt{\dfrac{\nu t}{h}}$ in fig.7.2 show as to how the stationary upper wall effect the velocity variation.
As expected the velocity tends asymptotically to the linear distribution of the steady state as the time gets larger to approach infinite value.

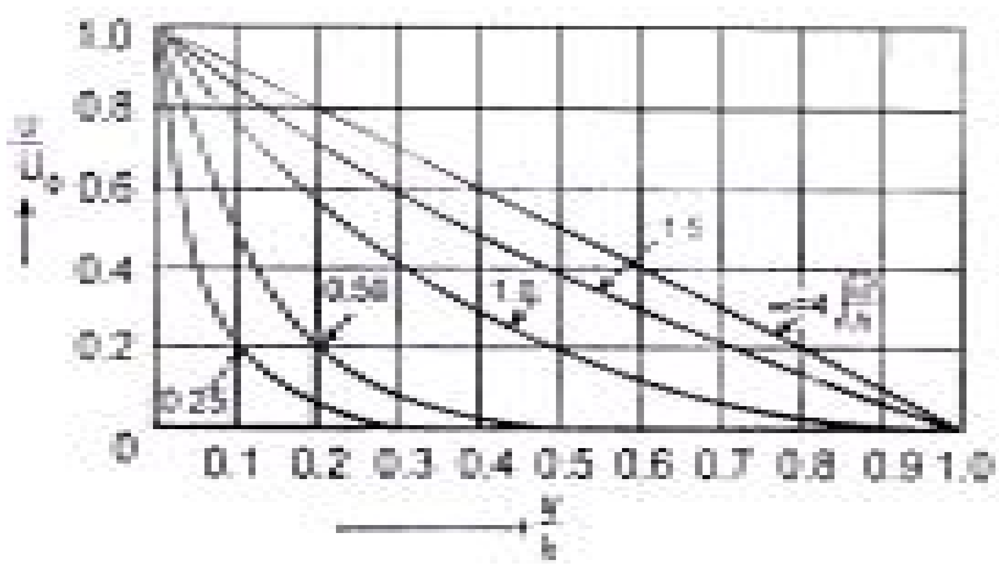


fig. 7.2

7.4 Suction / Injection Through Porous Walls

In this section we will discuss the problems which also give exact solution to Navier-Stokes equations. In this we are going to consider boundaries which are porous through which fluid can be sucked and or can be injected in. By porous boundaries we mean that the boundary has very fine holes distributed un infinity all along the boundary.

We will here discuss two problems (i) flow between two parallel porous plates, and (ii) plane Couette flow with porous walls.

7.4.1 Flow between two parallel Porous plates.

Here we take two parallel infinite plates at $y = -h$ and $y = h$ and the main flow is along x-axis and the same fluid as the one which is flowing in the channel is pressed in (injected) through lower porous wall at velocity v_0 and through the upper porous wall the fluid is taken out (sucked) with some velocity v_0 thus suction and injection are both in the direction of y-axis as shown in the fig.7.3. All the physical quantities are independent of z, hence this flow can be treated as two dimensional flow.

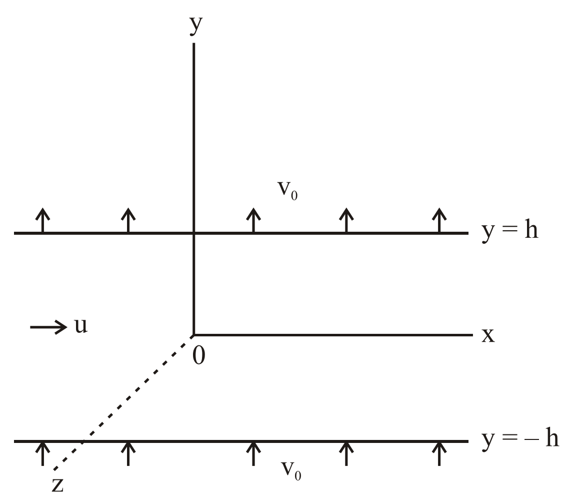


Fig. 7.3

Now, with above geometry, the equation of continuity and Navier-Stokes equation reduce to

$$\frac{\partial v}{\partial y} = 0 \quad \text{...(1)}$$

so that v is constant and hence $v = v_0$

and $$v_0 \frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \quad \text{.................................(2)}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad \text{..(3)}$$

so that p is independent of y

and hence $$\frac{\partial p}{\partial x} = -p \quad \text{.......................................(4)}$$

From (2) therefore we here the equation

$$\frac{d^2u}{dy^2} - \frac{v_0}{v}\frac{du}{dy} = -\frac{P}{\rho v} \quad \text{.......................(5)}$$

Solving (5) $$u = A + \frac{P}{\rho v_0} y + Be^{\frac{v_0 y}{v}} \quad \text{................................(6)}$$

under the boundary conditions

$$y = \pm h \,.\; u = 0 \quad \text{...(7)}$$

which gives $$\left.\begin{aligned} A &= -\frac{Ph}{\rho v_0}\coth\frac{v_0 h}{v} \\ B &= -\frac{Ph}{\rho v_0}\operatorname{cosech}\frac{v_0 h}{v} \end{aligned}\right] \quad \text{...................(8)}$$

From (6) and (8) we get the required velocity expression

7.4.2 Plane Couette flow with porous walls

In this flow there is no pressure gradient and hence $-\frac{\partial p}{\partial x} = P = 0$ and the upper plate is moving with a constant velocity U in its own plane. With these minor changes equation (5) becomes

$$\frac{d^2u}{dy^2} = \frac{v_0}{v}\frac{du}{dy} \quad \text{..(9)}$$

under the boundary conditions

$$\left.\begin{aligned} y &= -h, & u &= 0 \\ y &= h, & u &= U \end{aligned}\right] \quad \text{.. (10)}$$

Solution of (9) under (10) is

$$\frac{u}{U} = \frac{\left(e^{\frac{v_0 y}{v}} - e^{-\frac{v_0 h}{v}}\right)}{\left(e^{\frac{v_0 h}{v}} - e^{-\frac{v_0 h}{v}}\right)} \quad \text{.. (11)}$$

This is the expression for the velocity distribution for the flow in plane Coutte flow with porous boundaries.

7.5 Self Learning Exercise

1. What is meant by porous boundaries ?
2. How the starting flow is an unsteady motion ?
3. $\operatorname{erf}\eta =$ ....

7.6 Summary

In this unit three problems have been discussed. One is a representative problem of an unsteady flow and other two are problems when the boundaries have been treated to be porous

In the first problem a form of Couette flow is considered when the time is measured with the start of motion of the upper plate. Velocity distribution is calculated in terms of error function.

In second and third problems the boundaries have been considered to be porous and there is injection of the same fluid from one boundary while there is suction at other boundary. Two problems correspond to plane Poiseuille flow and plane Couette flow.

7.7 Answers to Self Learning Exercise

1. The boundary has very fine holes distributed uniformly all along the boundary.
2. In such flow problems initial velocity consideration are made so that all the subsequent motion becomes time dependent
3. $\frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2}\,d\eta$

7.8 Exercise

1. Discuss the starting flow in plane Couette Motion (see 7.3)
2. Obtain an expression for the flow between two parallel Porous plates (see 7.4.1)

UNIT - 8

Temperature Distribution in Fluid Motion

Structure of the Unit

8.0 Objectives

After studying this unit, you should be able to know application of temperature distribution in various simple physical phenomena of fluid flow . You will get an idea of temperature distribution in parallel plates, in a pipe, between two concentric rotating cylinders and plane Couette flow with Transpiration cooling.

8.1 Introduction

In the study of fluid flows it is not only important to discuss velocity and related characteristics but it is also useful and important to know as to how much heat is exchanged between the fluid and the body in contact, which can be in the form of boundaries.

It will be useful to learn about the heat transfer in the cases of flow through the channels of various geometries. Here we will discuss the heat transfer problems through channels of simple cross section like flow between parallel plates. in circular cylinder or pipes, between concentric rotating cylinders. An important dimensionless coefficient Nusselt number is a measure of heat conduction which has also been calculated in various problems.

8.2 Flow between Parallel Plates (Temperature distribution)

The equation of energy for the steady flow between two parallel plates without heat addition, becomes

$$PC_v u \frac{\partial T}{\partial x} = K \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{\partial u}{\partial y} \right)^2 \text{ ....(1)}$$

We K and μ are taken to be constants, if the plates are kept at constant temperature then $\frac{\partial T}{\partial x} = 0$

So equation (1) becomes

$$K\frac{\partial^2 T}{\partial y^2} = -\mu\left(\frac{\partial u}{\partial y}\right)^2 \quad \ldots\ldots(2)$$

From equation (2) we will calculate the temperature distribution for different situations between parallel plates.

8.2.1 Plane Couette Flow

You have in earlier unit, obtained the velocity distribution for the plane Couette flow as

$$\frac{u}{U} = \frac{y}{h}$$

Therefore

$$\frac{du}{dy} = \frac{U}{h}$$

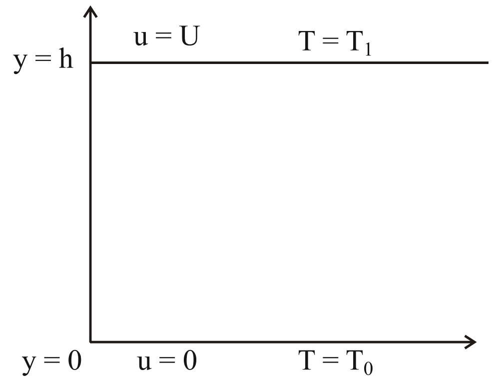


Fig. 8.1

where h is the distance between the plates and U is the velocity of upper plane in its own plane. So in case of plane Couette flow equation (2) becomes

$$K\frac{d^2 T}{dy^2} = -\mu\frac{U^2}{h^2} \quad \ldots\ldots(3)$$

We consider that the plates are kept at different temperture so that the boundary condition for the temperature are.

$$\begin{aligned} y = 0 \;&;\; T = T_0 \\ y = h \;&;\; T = T_1 \end{aligned} \quad \ldots\ldots(4)$$

where $T_1 > T_0$

Integrating equation (3) twice, we find that

$$T = -\frac{\mu U^2}{2Kh^2}y^2 + a_1 y + a_2 \quad \ldots\ldots(5)$$

where a_1 and a_2 are constants

Using (4) in (5) we get

$$a_2 = T_0$$

and $a_1 = \left[(T_1 - T_0) + \frac{\mu U^2}{2k}\right]\frac{1}{h}$

Hence

$$T = -\frac{\mu U^2}{2kh^2}y^2 + \left[(T_1 - T_0) + \frac{\mu U^2}{2k}\right]\frac{1}{h}\times y + T_0$$

$$T - T_0 = (T_1 - T_0)\frac{y}{h} + \frac{\mu U^2 y}{2kh}\left[1 - \frac{y}{h}\right]$$

$$\Rightarrow \frac{T - T_0}{T_1 - T_0} = \frac{y}{h} + Ec.Pr.\frac{y}{h}\times\frac{1}{2}\left[1 - \frac{y}{h}\right] \quad \ldots\ldots(6)$$

Where $Ec = \dfrac{U^2}{C_p(T_1 - T_0)}$ (Eckert number)

$$Pr = \frac{\mu C_p}{k} \quad \text{(Prandtl number)}$$

The dimensionless coefficient of heat transfer viz. Nusselt number at the uppere plate is given by

$$Nu = -\frac{h}{(T_1 - T_0)}\left(\frac{\partial T}{\partial y}\right)_{y=h} \quad \text{...(7)}$$

Subsituting value of $\dfrac{\partial T}{\partial y}$ from (7) in (6) we find

$$Nu = \frac{Ec.Pr}{2} - 1 \quad \text{..(8)}$$

Thus the Nusselt number will be positive if EcPr>2 and in this case the heat will be transferred from fluid to the upper plate. If Ec.Pr.<2 then the Nusselt number will be negative i.e. the reversal in the heat transfer will take place and the heat will be transferred from upper plate to the fluid. if Ec. Pr = 2 there will be no transfer of heat between the fluid and upper plate.

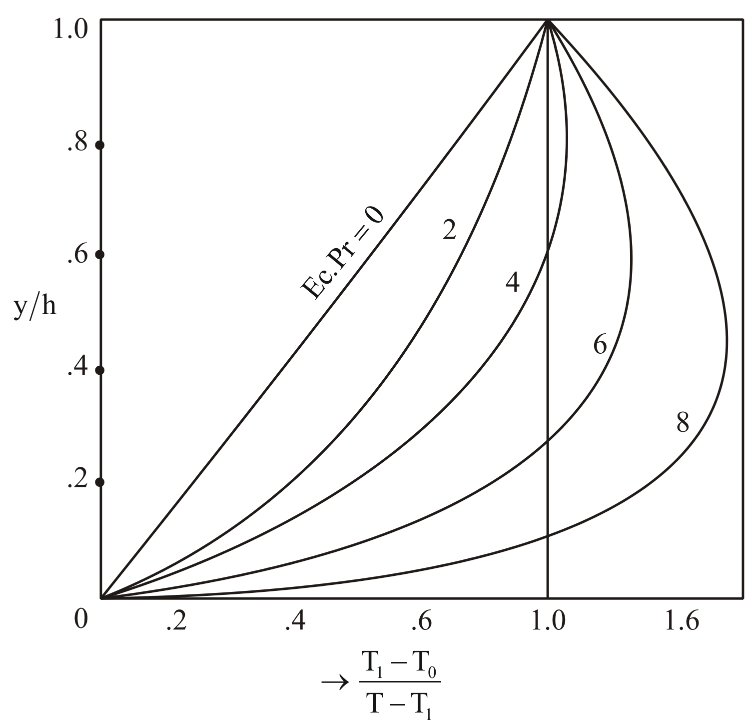


Fig. 8.2

If both the plates are kept at the same constant temperature T_o then the boundary conditions are

$$\begin{aligned} y = 0 \;&;\; T = T_0 \\ y = h \;&;\; T = T_0 \end{aligned} \quad \text{...(9)}$$

Now integrating (3) twice we get

$$T = -\frac{\mu}{2k}\frac{U^2}{h^2}y^2 + a_1 y + a_2$$

Using (9) we get
$$a_2 = T_0$$
$$a_1 = \frac{-\mu U^2}{2kh}$$

$$T - T_0 = \frac{\mu U^2}{2k}\frac{y}{h}(1 - y/h) \quad \text{.......................................(10)}$$

For maximum value of T

$$\frac{dT}{dy} = 0 \quad \Rightarrow \quad y = h/2$$

Thus the max. temperature exist in the middle of the channel

$$T_m - T_0 = \frac{\mu U^2}{8k} \quad \text{..(11)}$$

so the $\frac{T - T_0}{T_m - T_0} = \frac{4y}{h}(1 - 4/h)$

which is parabolic in nature (fig. 8.3)

Also the Nusselt number at the lower plate is defined as

$$N_u = \frac{-h}{(T_0 - T_m)}\left(\frac{\partial T}{\partial y}\right)_{y=0} = 4$$

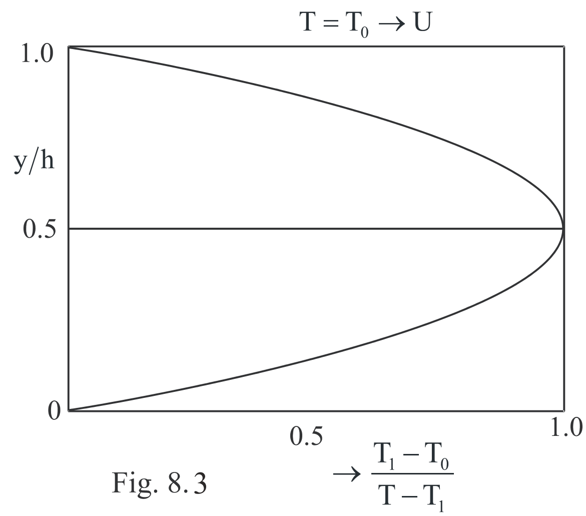


Fig. 8.3

If we assume that at one of the plate, say the stationary plate, no heat transfer takes place (adiabatic wall) then B.C's are

$$y = 0, \quad \frac{\partial T}{\partial y} = 0$$
$$y = h, \quad T = T_1 \quad \text{...(12)}$$

Using (12) in (3) we get

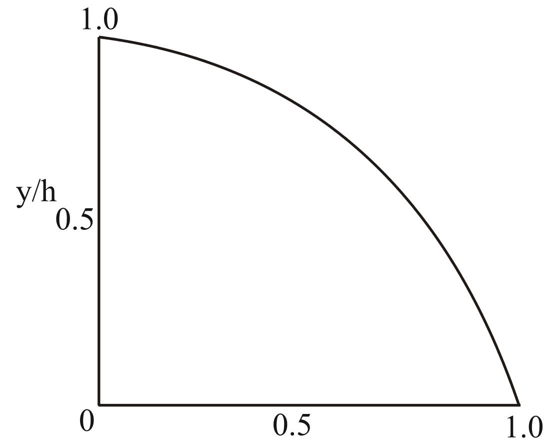
$$T - T_1 = \frac{\mu U^2}{2k}\left(1 - \frac{y^2}{h^2}\right) \quad \text{...(13)}$$

The temperature which an insulated surface assumes under the influence of internal friction is known as recovery temperature (Tr). The difference between recovery temperature and the temperature of the upper plate is given by (13) as

$$T_r - T_1 = \frac{\mu U^2}{2k}$$

So that $\frac{T - T_1}{T_r - T_1} = \left(1 - \frac{y^2}{h^2}\right)$ which is paraboic in nature (fig. 8.4). The recovery factor in a plane Couette flow is defined as

$$r = \frac{T_r - T_1}{U^2/2Cp} = Pr \quad \text{...(15)} \quad \text{(Prandtl number)}$$



$$\frac{T - T_1}{T_r - T_1}$$

Fig. 8.4

8.2.2 Plane Poiseuille Flow

From the earlier unit we know that the velocity distribution for the plane-Poiseuille flow is given by

$$\frac{u}{u_m} = 1 - \frac{y^2}{b^2}$$

where distance between two plates is 2 b and u_m is the maximum velocity in the mid plane

So $\frac{du}{dy} = -\frac{2u_m y}{b^2}$

Now with this on using equation (3) for the heat conduction we get

$$\frac{d^2T}{dy^2} = -\mu\left(\frac{dx}{dy}\right)^2 = -\frac{4\mu u_m^2 y^2}{b^4} \quad \text{...(16)}$$

Let both the plates be kept at the same constant temperature T_0, therefore, the boundary conditions are

$$y = \pm b, \; T = T_0 \quad \text{...(17)}$$

Hence the solution of equation (16) is given by

$$T = -\frac{\mu U_m^2}{3kb^4} y^4 + Ay + B \quad \text{..............................(18)}$$

Using (17) in (18) we get

$$A = 0 \quad \text{and} \quad B = T_0 + \frac{\mu u_m^2}{3k}$$

Thus (18) becomes

$$T - T_0 = \frac{\mu u_m^2}{3k}\left(1 - \frac{y^4}{b^4}\right) \quad \text{..............................(19)}$$

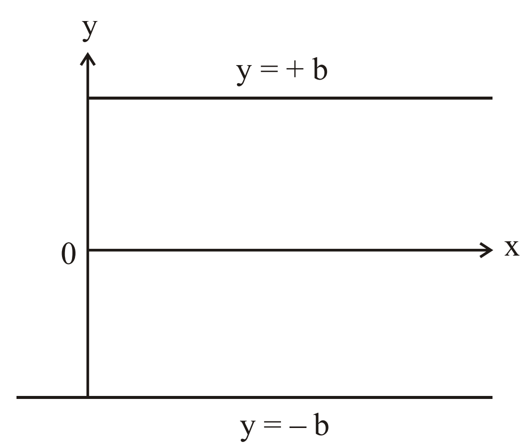


Fig. 8.5

Further the maximum temperature occurs, when $\frac{dT}{dy} = 0$

which gives y = 0

Therefore the maximum temperature exists in the middle of the channel y = 0 and is given by

$$T_m - T_0 = \frac{\mu u_m^2}{3k} \quad \text{..............................(20)}$$

Also $\dfrac{T - T_0}{T_m - T_0} = 1 - \dfrac{y^4}{b^4}$. This is shown in fig. 8.6

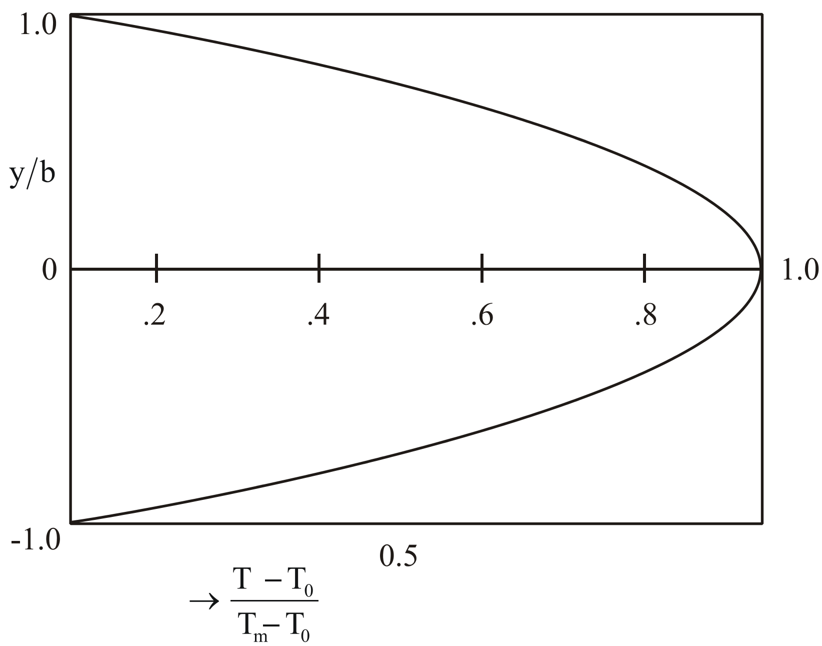


Fig. 8.6

8.2.3 Generalized Plane Couette Flow

The velocity distribution in the generalized Couette flow is given by as we have obtained it in earlier unit

$$u = \frac{yU}{h} + \frac{PUy}{h}\left(1 - \frac{y}{h}\right) = \frac{U}{h}\left[(1+P)y - \frac{Py^2}{h}\right]$$

so that $\quad \dfrac{du}{dy} = \dfrac{U}{h}\left[(1+P) - \dfrac{2Py}{h}\right]$

and therefore the equation (3) becomes

$$k\frac{d^2T}{dy^2} = -\mu\frac{U^2}{h^2}\left[(1+P)^2 + \frac{4P^2y^2}{h^2} - 4P(1+P)\frac{y}{h}\right] \quad \text{............................(21)}$$

Let both the plates be kept at the same constant temperature T_0 . Here the boundary conditions are

$$\begin{aligned} y = 0 \;\; &; \;\; T = T_0 \\ y = h \;\; &; \;\; T = T_0 \end{aligned} \quad \text{..(22)}$$

The solution of equation (21) with the B.C's (22) is

$$T - T_0 = \frac{\mu U^2}{6K}\left[3(1+P)^2\left(1 - \frac{y}{h}\right) - 4P(1+P)\left(1 - \left(\frac{y}{h}\right)^2\right) + 2P^2\left(1 - \left(\frac{y}{h}\right)^3\right)\right]\frac{y}{h} \quad \text{...................(23)}$$

The temperature gradient at the lower plate is given by

$$\left(\frac{dT}{dy}\right)_{y=0} = \frac{\mu U^2}{6K}\left[2+(1+P)^2\right] \quad \text{..(24)}$$

This shows that the heat will always be transferred from the fluid to the lower plate irrespective to the sign of P.

8.3 Temperature distribution in a Pipe

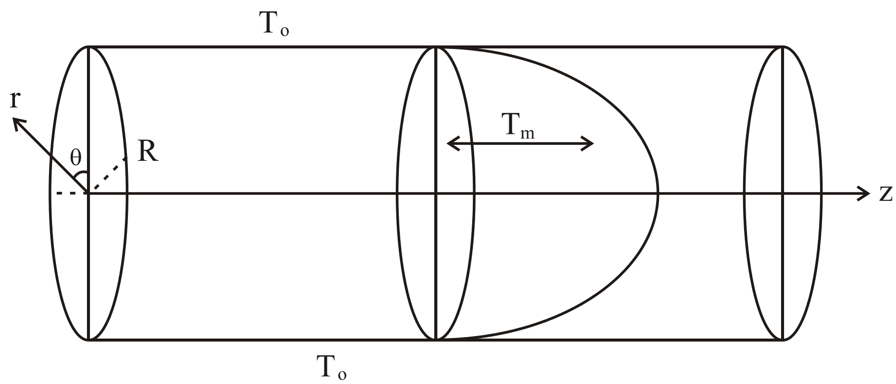


Fig. 8.7

The steady laminar flow through a long straight circular pipe, without body forces the velocity distribution is given by the relation

$$v_z = -\frac{R^2}{4\mu}\frac{dp}{dz}\left[1-\left(\frac{r}{R}\right)^2\right] = \frac{R^2}{4\mu}P\left[1-\frac{r^2}{R^2}\right] \qquad \text{where } P=-\frac{dp}{dz} \text{......................(1)}$$

which has been obtained in an earlier unit as Hagen-Poiseuille flow and where v_z is velocity component in z-direction and r denotes the radial distance measured outward from the z-axis and R is the radius of the pipe

The non zero component of velocity is v_z and the energy equation for steady flow of a viscous incompressible fluid through the pipe without addition of external heat becomes

$$\rho c_v v_z \frac{\partial T}{\partial z} = K\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)+\mu\left(\frac{\partial v_z}{\partial r}\right)^2 \text{ ....(2)}$$

here μ and K are taken to be constants.

Here we consider two cases

(i) when the wall of the pipe is kept at a constant temperature

(ii) when the wall of the pipe is kept at a uniform temperature gradient.

8.3.1 Wall at Constant Temperature

If the wall of the pipe is kept at a constant temperature then $\frac{\partial T}{\partial z} = 0$ and equation (2) then becomes

$$K\left(\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}\right) = -\mu\left(\frac{\partial v_z}{\partial r}\right)^2 = -\mu\left[-\frac{2rP}{4\mu}\right]^2 = -\frac{4\mu}{R^4}\left[(v_z)_m^2 . r^2\right] \text{.............................(3)}$$

$$(v_z)_{max} = (v_z)_m = \frac{R^2 P}{4\mu} \quad \text{[Max. velocity occur on the axis of the pipe]}$$

The boundary condition are

$$\left. \begin{array}{ll} r = 0, & T = \text{finite} \\ r = R, & T = T_0 \end{array} \right] \text{..(4)}$$

Solution of equation (3) can be obtained by putting

$\frac{dT}{dr} = V$ so that (3) be comes

$$K\left(\frac{\partial V}{\partial r} + \frac{1}{r}V\right) = -\frac{4\mu}{R4}\left[(v_z)_m\right]^2 \times r^2$$

which is a linear differential equations with integreting factor = r

Hence the solution is

$$V.r = \int \frac{-4\mu}{KR^4}\left[(v_z)_m\right]^2 \times r^3 dr + A$$

$$= \frac{-4\mu}{KR^4}\left[(v_z)_m\right]^2 \times \frac{r^3}{4} + \frac{A}{r}$$

$$\frac{dT}{dr} = \frac{-4\mu}{KR^4}\left[(v_z)_m\right]^2 \times \frac{r^3}{4} + \frac{A}{r}$$

Integrating again we get

$$T = \frac{\mu}{KR^4}\left[(v_z)_m\right]^2 \times \frac{1}{4} r^4 + A \log r + B$$

or

$$T = -\frac{\mu\left[(v_z)_m\right]^2}{4KR^4} r^4 + A \log r + B \text{...(5)}$$

Applying the B.C's (4) we get

$$A = 0 \text{ and } B = T_0 + \frac{\mu (v_z)_m^2}{4k}$$

Hence equation (5) becomes

$$T = -\frac{\mu\left[(v_z)_m\right]^2}{4kR^4} r^4 + T_0 + \frac{\mu (v_z)_m^2}{4K}$$

$$T - T_0 = \frac{\mu (v_z)_m^2}{4k}\left[1 - \frac{r^4}{R^4}\right] \text{..(6)}$$

The maximum temperature exists on the axis of the pipe i.e. at $r = 0$

so $$T_m - T_0 = \frac{\mu (v_z)_m^2}{4k} \text{..(7)}$$

where T_m = Maximum Temperature

Hence the non-dimensional tempereature distribution is given by

$$\frac{T-T_0}{T_m-T_0}=1-\frac{r^4}{R^4} \quad \text{...(8)}$$

The mean temperature over a cross section is given by

$$T_{mean}=\frac{\int_0^R T.2\pi r\,dr}{\pi r^2}=T_0+\frac{1}{6}\frac{\mu}{K}(v_z)_m^2 \quad \text{......................................(9)}$$

and the rate of heat transfer in terms of the Nusselt number at the wall is given by

$$N_u=-\frac{2R}{(T_{mean}-T_0)}\left(\frac{\partial T}{\partial r}\right)_{r=R}=12 \quad \text{..(10)}$$

8.3.2 Wall at uniform Temperature gradient

Let the wall of the pipe is kept at a constant temperature gradient i.e. $\frac{\partial T}{\partial z}=A$ (Constant)

We may assume the solutjion of equation (2) in the form $T=Az+g(r)$..................(11)

But $v_z=\frac{R^2P}{4\mu}\left[1-\frac{r^2}{R^2}\right]$

& $(v_z)_{max}=\frac{R^2P}{4\mu}$ so $\frac{v_z}{(v_z)_{max}}=\left[1-\frac{r^2}{R^2}\right]$

so $v_z=\left[1-\frac{r^2}{R^2}\right](V_z)_{max}$ ....(12)

use (11) & (12) in (2) and and neglecting the dissipation term we get

$$\rho C_v(v_z)_{max}\left[1-\frac{r^2}{R^2}\right].A=K\left(\frac{d^2g}{dr^2}+\frac{1}{r}\frac{dg}{dr}\right)$$

or $$\frac{d^2g}{dr^2}+\frac{1}{r}\frac{dg}{dr}=\frac{\rho C_v(v_z)_{max}.A}{K}\left[1-\frac{r^2}{R^2}\right]$$

Let $B=\frac{\rho C_v(v_z)_{max}.A}{K}$ then

$$\frac{d^2g}{dr^2}+\frac{1}{r}\frac{dg}{dr}=B\left[1-\frac{r^2}{R^2}\right] \quad \text{...(13)}$$

Then corresponding B.C.'s are

$$\left.\begin{array}{ll} r=0\ ; & g \text{ is finite} \\ \text{and} \quad r=R\ ; & g=0 \end{array}\right] \quad \text{..(14)}$$

Solution of (13) is given by the following method

Let $\frac{dg}{dR}=G$

$$\frac{dG}{dR}+\frac{G}{r}=B\left[1-\frac{r^2}{R^2}\right]$$

which is linear different equation with integrating factor $= r$

Hence the solution

$$Gr=\int B\left[1-\frac{r^2}{R^2}\right]r\,dr+C_1$$

$$=B\left[\frac{r^2}{2}-\frac{r^4}{4R^2}\right]+C_1$$

so $$G=B\left[\frac{r}{2}-\frac{r^3}{4R^2}\right]+\frac{C_1}{r}$$

$$\Rightarrow \frac{dg}{dr}=B\left[\frac{r}{2}-\frac{r^3}{4R^2}\right]+\frac{C_1}{r}$$

Integrating we get

$$g=B\left[\frac{r^2}{4}-\frac{r^4}{16R^2}\right]+C_1\log r+C_2 \quad\text{...................................(15)}$$

using (14) in (15) we get

$r=0$ then $C_1=0$

when $r=R$ then $C_2=-\frac{3}{16}BR^2$

So from (15)

$$g=B\left[\frac{r^2}{4}-\frac{r^4}{16R^2}\right]-\frac{3}{16}BR^2$$

So from (11)

$$T=Az+B\left[\frac{r^2}{4}-\frac{r^4}{16R^2}\right]-\frac{3}{16}BR^2$$

$$=Az-BR^2\left[\frac{3}{16}-\frac{1}{4}\frac{r^2}{R^2}+\frac{1}{16}\frac{r^4}{R^4}\right]$$

or $$T=Az-\frac{\rho C_v(v_z)_{max}.AR^2}{K}\left[\frac{3}{16}-\frac{1}{4}\left(\frac{r}{R}\right)^2+\frac{1}{16}\left(\frac{r}{R}\right)^4\right] \quad\text{...........................(16)}$$

Max. temperature T_m exists on the axis of the pipe.

$$T_m=Az-\frac{\rho C_v(v_z)_{max}.AR^2}{K}\times\frac{3}{16} \quad\text{...(17)}$$

Now we calculate the unweighted mean temperature (T_{mean}) and weightedmean temperature (T_{mean}) with respect to the velocity i.e. the temperature which is measured in fluid which is mixed after passing through the pipe, respectively.

$$T_{mean} = \frac{\int_0^R T.2\pi r\,dr}{\pi R^2} = Az - \frac{\rho c_v (v_z)_{max} AR^2}{K} \times \frac{1}{12} \quad \text{.........................(18)}$$

$$T_{mean} = \frac{\int_0^R T.v_z 2\pi r\,dr}{\int_0^R v_z.2\pi r\,dr} = Az - \frac{11}{96}\frac{\rho c_v (v_z)_m AR^2}{K} \quad \text{..............................(19)}$$

The Nusselt number, based on the un-weighted mean temperature T_{mean} is given by

$$Nu = -\frac{2R}{(T_{mean} - T_w)}\left(\frac{\partial T}{\partial r}\right)_{r=R} = 6 \quad \text{......................................(20)}$$

where $T_w = Az$

When Nusselt number is based on the weighted mean temperature T_{Mean}, we have

$$Nu = -\frac{2R}{(T_{mean} - T_w)}\left(\frac{\partial T}{\partial r}\right)_{r=R} = \frac{48}{11} \quad \text{......................................(21)}$$

8.4 Temperature Distribution between Two Concentric Rotating Cylinders

The equation of energy for steady flow of a viscous incompressible fluid between two concertric rotating cylinders without addition of external heat in cylindrical polar coordinates reduces to

$$o = K\frac{1}{r}\frac{d}{dt}\left(r\frac{dt}{dr}\right) + \mu\left[r\frac{d}{dr}\left(\frac{v_\theta}{r}\right)\right]^2 \quad \text{.............................. (1)}$$

The boundary condition are

$$r = r_1 \;; \qquad T = T_1$$
$$r = r_2 \;; \qquad T = T_2 \quad \text{.. (2)}$$

For the flow between two concertric cylinder velocity distribution is given by, which we have already derived in earlier unit.

$$v_\theta = \frac{1}{r_2^2 - r_1^2}\left[\left(w_2 r_2^2 - w_1 r_1^2\right)r - \left(w_2 - w_1\right)\frac{r_1^2 r_2^2}{r}\right] \text{........... (3)}$$

where r_1,w_1 and r_2,w_2 are the radius and angular velocity of the inner and of the outer cylinders respectively.

Putting the value of v_θ from (3) in equation (1), we get

$$0 = \frac{K}{r}\frac{d}{dt}\left(r\frac{dT}{dr}\right) + \frac{4\mu(w_2 - w_1)^2 \mu_1^4 r_1^4}{\left(r_2^2 - r_1^2\right)^2} \times \frac{1}{r^4}$$

or $$\frac{d}{dt}\left(r\frac{dT}{dr}\right) = -\frac{4\mu}{K}\frac{(w_2 - w_1)^2 r_1^4 r_2^4}{\left(r_2^2 - r_1^2\right)^2} \times \frac{1}{r^3} \quad \text{.............................(4)}$$

Integrating (4) w.r.t.r we have

$$r\frac{dT}{dr} = \frac{2\mu}{K}\frac{(w_2 - w_1)^2 r_1^4 r_2^4}{(r_2^2 - r_1^2)^2}\times\frac{1}{r^2} + A$$

or $$\frac{dT}{dt} = \frac{2\mu}{K}\frac{(w_2 - w_1)^2 r_1^4 r_2^4}{(r_2^2 - r_1^2)^2}\times\frac{1}{r^3} + \frac{A}{r}$$

Again integrating we have

$$T = -\frac{\mu}{K}\frac{(w_2 - w_1)^2 r_1^4 r_2^4}{(r_2^2 - r_1^2)^2}\times\frac{1}{r^2} + A\log r + B \quad \text{.............................(5)}$$

Applying the boundary condition (2) we finaly get

$$\frac{T - T_1}{T_2 - T_1} = \frac{N(r^2 - r_1^2)r_2^2}{(r_2^2 - r_1^2)r^2} + (1 - N) + \frac{\log(r/r_1)}{\log(r_2/r_1)} \quad \text{........................(6)}$$

where $N = \frac{\mu(w_2 - w_1)^2 r_1^2 r_2^2}{(r_2^2 - r_1^2)(T_2 - T_1)}$

where Nis a non-dimensional parameter

Equation (6) gives the required temperature distribution in the fluid between two concentric rotating cylinders.

8.5 Temperature Distribution of Plane-Couette flow with Transpiration Cooling

Consider steady flow of a viscous incompressible fluid between two parallel plates placed at a distance 'h' apart with lower plate placed along x-axis.

The y-axis is taken normal to the plates. The lower plate is at rest and some fluid is injected at the rate v_0 through it and the upper plate is moving with velocity U in its own plane i.e. parallel to x-axis and same fluid is withdrawn (suction) at the same rate v_0 through it.

Since the plates are infinite, then non zero component of velocity u will be function of y only.

Hence the equation continuity and the navier Stokes equation be come

$$\frac{dv}{dy} = 0 \quad \text{...(1)} \qquad \therefore \text{ v is independent of y}$$

and

$$\rho v\frac{du}{dy} = \mu\frac{d^2u}{dy^2} \quad \text{................................. (2)}$$

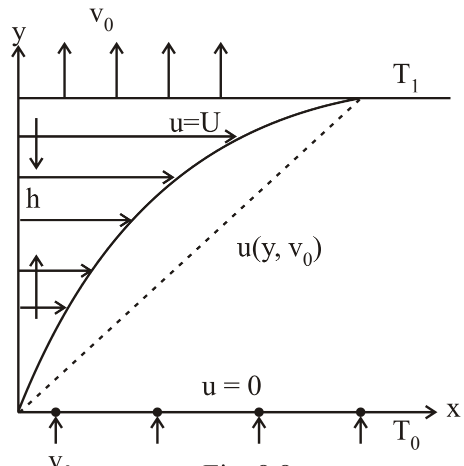


Fig. 8.8

The boundary condition being

$$\begin{aligned} y = 0 \;&;\; u = 0, \quad v = v_0 \\ y = h \;&;\; u = U, \quad v = v_0 \end{aligned} \quad \text{.................. (3)}$$

Hence from (1) $v = v_0$

$$\therefore \text{ (2) becomes } \quad \frac{d^2u}{dy^2} = \frac{\rho v_0}{\mu}\frac{du}{dy}$$

Solving (2) subject to boundary conditioning (3) we get

$$\frac{u}{U}=\frac{e^{\lambda\eta}-1}{e^{\lambda}-1} \quad \text{...(4)}$$

Where $\eta=\frac{y}{h}$ and $\lambda=\frac{v_0 h}{\nu}$ which is injection parameter

Let the temperature at lower plate be T_0 and that at upper plate be T_1, both T_1, T_2 being considered constants. So the governing equation for steady flow of a viscous incompressible fluid in the absence of external heat in present case is

$$\rho c_p v_0 \frac{dT}{dy}=k\frac{d^2T}{dy^2}+\mu\left(\frac{du}{dy}\right)^2 \quad \text{................(5)}$$

where ρ is the density , c_p is the specific heat at constant pressure and K is the thermal conductivity

The boundary canditions are

$$\begin{aligned} y=0 \;;\; T=T_0 \\ y=h \;;\; T=T_1 \end{aligned} \quad \text{.....................................(6)}$$

Introducing following non-dimensional parameters

$$T^*=\frac{T-T_0}{T_1-T_0}$$

$$E_C=\frac{U^2}{c_p(T_1-T_0)} \qquad \text{(Eckert Number)}$$

$$P_r=\frac{\mu c_p}{K} \qquad \text{(Prandtl Number)}$$

$$P_C^1=\lambda.p_r \qquad \text{(Peclet Number)}$$

The equation (5) with the help of equation (4) can be written as

$$\frac{d^2T^*}{d\eta^2}-P_C^1\frac{dT^*}{d\eta}=-Ec.Pc^1\frac{e^{2\lambda n}}{\left(e^{\lambda}-1\right)^2} \quad \text{.........................(7)}$$

and the corresponding B.C's are

and $$\begin{aligned} \eta=0 \;;\; T^*=0 \\ \eta=1 \;;\; T^*=1 \end{aligned} \quad \text{...(8)}$$

The solution of the equation (7), Subject to the boundary condition is

$$T^*=\frac{Ec.P_c^1}{\left(e^{\lambda}-1\right)^2}\left[\frac{e^{P_c^1}\left\{1-e^{(2\lambda-P_c^1)\eta}\right\}}{\left(2\lambda-P_c^1\right)}-\frac{e^{P_c^1}\left\{1-e^{(2\lambda-P_c^1)\eta}\right\}}{\left(2\lambda-P_c^1\right)}\times\frac{\eta P_c^1-1}{e^{P_c^1}-1}\right]+\frac{\eta P_c^1-1}{e^{\eta P_c^1}-1} \quad \text{..................(9)}$$

If heat generated due to internal friction is neglected i.e. if the Ec is taken to be zero then (9) becomes.

$$T^*=\frac{e^{\eta P_C^1}-1}{e^{\eta P_C^1}-1} \quad \text{...(10)}$$

In order to see the heat transfer at the stationary plate, let us calculate the dimensionless coefficient of heat transfer (Nusselt number.)

With present notations.

$$Nu = \left(\frac{\partial T^*}{\partial \eta}\right)_{\eta=0} \qquad \text{...(11)}$$

putting the values of T^* from (10) in (11) and after simplification, we find

$$Nu = \frac{P_c^1}{e^{P_c^1} - 1} \qquad \text{..(12)}$$

when $\lambda = 0$ i.e. $P_c^1 = 0$, the value of Nusselt number is unity and it goes on decreasing as the value of P_c^1 increases, which shows the cooling of stationary plate with the injection process.

8.6 Self Learning Exercise

1. Write down the temperature distribution equation in plane Couette flow
2. Write down dimensionaless temperature distribution in Hagen Poiseville flow.
3. Write down the energy equation in plane Coutte flow with transpriration cooling.

8.7 Answer to Self Learngin Exercise

1. $$\frac{T - T_0}{T_1 - T_0} = \frac{y}{h} + \frac{1}{2} Ec.Pr.\frac{y}{h}\left(1 - \frac{y}{h}\right)$$

2. $$\frac{T - T_0}{T_m - T_0} = 1 - \frac{r^4}{R^4}$$

3. $$\rho c_p v_0 \frac{dT}{dy} = k\frac{d^2T}{dy^2} + \mu\left(\frac{du}{dy}\right)^2$$

8.8 Exercise

1. Discuss the temperature distribution in plane-Couette flow.
2. Discuss the temperature distribution in plane Poiseuille flow.
3. Discuss the temperature distribution in Generalised Couette flow.
4. Discuss the temperature distribution in pipe.
5. Discuss the temperature distribution between two concentric rotating cylinders.
6. Discuss the temperature distribution of plane-Couette flow with transpiration cooling.

UNIT - 9

Theory of very Slow Motion

Structure of the Unit

9.0 Objectives

In this unit you will study

1. Navier-Stokes equations deduction to the case of slow motion.
2. Stokes flow past a sphere.
3. Oseen flow past a sphere.

9.1 Introduction

In the past units you have gone through Navier-Stokes equation and a few exact solutions admissible by these equations for some simple configurations.

The exact solutions obtained hither to are valid for all values of Reynold number, Re except some critical Re values. The cases Re<<1 which corresponds to very slow motion and $\mathrm{Re} \to \infty$ which leads to turbulent flow are of special type and have been instrumental in the development and understanding of fluid mechanics. These two cases give rise to altogether different simplifications to Navier -Stokes equations. This unit discusses the case of Re<<1 which is useful in understanding the flow phenomenon of slow motion of fluid past a sphere and cylinder. The case $\mathrm{Re} \to \infty$ giverise to boundary layer theory which has been dealt with in seperate units.

The present unit entails the slow motion of a fluid past a sphere wherein Stokes flow and Oseen Flow would be discussed. The theory of slow motion finds application in lubrications theory as well. Note that when Re is quite small (Re<<1) that is viscosity of the fluid is large or the characteristic length and velocity of the body are small, then the viscous forces will be apprecibly larger then the inertia forces. Hence for Re<<1 the inertia terms may be neglected from the Navier-Stokes equation as a first approximation. These reduced equations are known as Stokes equations and contrary to non linear Navier Stokes equations, these are ordinary differential equations which are easily amenable to solution.

A fundamental point to note is that order of Stokes equations and that of Navier-Stokes equations is the same.

It was Stokes, who first used the simplified Stokes equation to examine the slow flow past a sphere.

9.2 Stoke's Equation for Slow Motion

If the fluid velocity is very low, then quantities containing squares of the velocity are negligible in comparison to other quantities, hence the above equations for slow motion reduce to

$$\nabla.\vec{V} = 0 \quad \text{...(3)}$$

$$\frac{\partial \vec{V}}{\partial t} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{V} \quad \text{...(4)}$$

We take divergence of (4) and make use of (3) to yield with $\nabla^2 p = 0$ which is Laplacian equation

We now take up the discussion on Stoke's flow past a sphere.

9.3 Stoke's Flow Past a Sphere

Let us consider steady flow with uniform stream velocity U_∞ past a solid sphere of radius a. The sphere is kept fixed at its position. The fluid motion is considered to be very slow (Re << 1).

A Cartesian coordinate system is considered as shown in the figure (9.1)

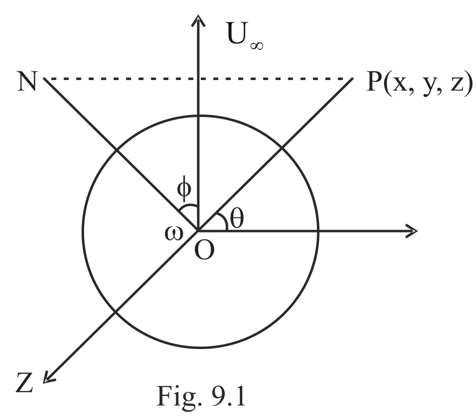


Fig. 9.1

Thus the Stokes equation for the setup are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{(equation of continuity) ....(5)}$$

$$\nabla^2 u = \frac{1}{\mu}\frac{\partial p}{\partial x} \quad \text{...(6)}$$

$$\nabla^2 v = \frac{1}{\mu}\frac{\partial p}{\partial y} \quad \text{...(7)}$$

$$\nabla^2 w = \frac{1}{\mu}\frac{\partial p}{\partial z} \quad \text{...(8)}$$

when $\vec{v} = (u, v, w)$ is the velocity

The boundary conditions are

$r = a : u = v = w = 0$

$r \to \infty : u = U_\infty, \quad v = w = 0$

and $p = 0$ ....(9)

Now we have to prescribe pressure in view of the physical set up. Following points are worth noting.

1. The pressure is harmonic function

2. The pressure on negative side of the sphere (x < 0) i.e. the side of the sphere facing the approaching flow is higher as compared to other of the side of the sphere (x > 0) ultimately pressure vanishes at infinity.

In view fo this, we prescribe, pressure of the form $p = -\frac{Ax}{r^3}$ ....(10)

which simply satisfies above condition. A being constant to be evaluated.

$$\Rightarrow \frac{\partial p}{\partial x} = -\frac{A}{r^3} - Ax\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right)$$

$$= A\left(\frac{3x^2}{r^5} - \frac{1}{r^3}\right)$$

$$\frac{\partial p}{\partial y} = \frac{3xyA}{r^5}, \qquad \frac{\partial p}{\partial z} = \frac{3Axz}{r^5} \text{ ....(11)}$$

Thus, we have

$$\nabla^2 u = \frac{A}{\mu}\left(\frac{3x^2}{r^5} - \frac{1}{r^3}\right)$$

$$\nabla^2 v = \frac{A}{\mu}\frac{3xy}{r^5}$$

$$\nabla^2 w = \frac{A}{\mu}\frac{3xz}{r^5} \text{ ....(12)}$$

The equations (12) are linear partial differential equations and their particular integrals can be determined by a closer look on the symmetry and a slice of mathematical endeavour. Following points are to be noted.

1. The flow is symmetrical about yz-plane vis-a vis x-axis
2. u is even function of x, y, z
3. v is odd about x and y and even in z
4. w is even in y but odd in x, z

Above facts lead to conclude that u, v, w have particular integrals respectively as

$$\frac{-A}{2\mu}\frac{x^2}{r^3}, \quad -\frac{-A}{2\mu}\frac{xy}{r^3}, \quad \frac{-A}{2\mu}\frac{xz}{r^3}$$

In order to obtain complete solution, we need to solve

$$\nabla^2 u = 0, \qquad \nabla^2 v = 0. \quad \nabla^2 w = 0$$

For this, suitable solutions (harmonic functions) of these equations are added to particular integrals to give rise to complete solutions so that the conditions on u,v,w are not violated even after adding particular integral

Quantities comprising

u_∞, $\frac{1}{r}$, $\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right)$ are added to u

$\frac{\partial^2}{\partial x\,\partial y}\left(\frac{1}{r}\right)$ is added to v

$\frac{\partial^2}{\partial x\,\partial z}\left(\frac{1}{r}\right)$ is added to w

Thus, the complete solution is obtained as

$$u = -\frac{A}{2\mu}\frac{x^2}{r^3} + U_\infty + \frac{B}{r} + C\left(\frac{3x^2}{r^5} - \frac{1}{r^3}\right)$$

$$v = -\frac{A}{2\mu}\frac{xy}{r^3} + 3D\frac{xy}{r^5}$$

$$w = -\frac{A}{2\mu}\frac{xz}{r^3} + 3D\frac{xz}{r^5}$$

where B, C, D are constants to be determined. Having known u, v, w as above, further note that equation of continuity must be satisfied. Hence

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\left[15Cx^2 + 15D\left(y^2 + z^2\right)\right]\frac{x}{r^7} + (9C + 6D)\frac{x}{r^5} - \frac{A}{2\mu}\frac{x}{r^3} - \frac{Bx}{r^3} \text{ ....(13)}$$

Here it is noticeable that if C=D, then first two terms cancel out and if $B = -\frac{A}{2\mu}$ then last two terms cancel each other and consequently equation of continuity is satisfied. Now on putting $B = -\frac{A}{2\mu}$ in the expression for u and on making use of boundary conditions, we get

$$A = \frac{3}{2}\mu a U_\infty, \quad C = D = \frac{Aa^2}{6\mu} \text{ ....(14)}$$

Consequently, we obtain

$$u = \left[\frac{3}{4}\frac{ax^2}{r^3}\left(\frac{a^2}{r^2} - 1\right) + 1 - \frac{a}{4r}\left(3 + \frac{a^2}{r^2}\right)\right]U_\infty \text{ ....(15)}$$

$$v = \frac{3}{4}\frac{axy}{r^3}\left(\frac{a^2}{r^2} - 1\right)U_\infty \text{ ....(16)}$$

$$w = \frac{3}{4}\frac{axz}{r^3}\left(\frac{a^2}{r^2} - 1\right)U_\infty \text{ ....(17)}$$

$$p = -\frac{3}{2}\frac{\mu U_\infty a x}{r^3} \text{ ....(18)}$$

9.3.1 Stresses on the surface of the sphere :

The pressure p as derived above is the difference of the actual pressure from the pressure of undisturbed stream.

(i) Now, the pressure at the point $x = -a, a$ (leading and trailing stagnation points respectively) are

$$(p)_{-a} = \frac{3}{2}\frac{\mu U_\infty}{a}, \quad (p)_{+a} = -\frac{3}{2}\frac{\mu U_\infty}{a} \text{ ....(19)}$$

(ii) The normal stresses on the surface of the sphere are given by

$$(\sigma_{rr})_{r=a} = (-p + \tau_{rr})_{r=a} = \left(-p + 2\mu\frac{\partial v_r}{\partial r}\right)_{r=a} \text{ ....(20)}$$

(iii) Tangential stress is

$$(\sigma_{r\theta})_{r=a} = (\tau_{r\theta})_{r=a} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]_{r=a} \text{ ....(21)}$$

Where v_r, v_θ are the radial, tangential components of the velocity in a meridian plane as shown in the figure (9.2), when

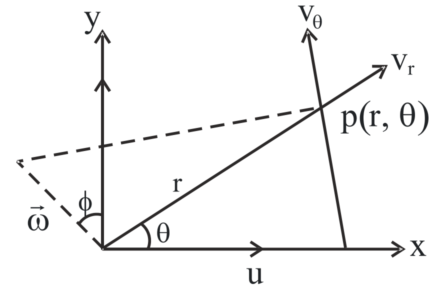


Fig. 9.2

$$x = r\cos\theta$$
$$y = r\sin\theta\cos\phi$$
$$z = r\sin\theta\sin\phi$$

Hence, we compute

$$v_r = u\cos\theta + (v\cos\phi + w\sin\phi)\sin\theta$$
$$= u\frac{x}{r} + v\frac{y}{r} + w\frac{z}{r}$$
$$= \left(1 - \frac{3}{2}\frac{a}{r} + \frac{1}{2}\frac{a^2}{r^3}\right)U_\infty w\cos\theta \quad \text{.....(22)}$$

$$v_\theta = (v\cos\phi + w\sin\phi)\cos\theta - u\sin\theta$$
$$= v\frac{xy}{\vec{w}r} + w\frac{zx}{\vec{w}r} - \frac{u\vec{w}}{r}$$

when $\vec{w} = r\sin\theta$, $\vec{w} = y^2 + z^2$

$$v_\theta = -\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right)u_\infty \sin\theta \quad \text{.....(23)}$$

This gives

$$(\sigma_{rr})_{r=a} = \frac{3}{2}\mu\frac{U_\infty}{a}\cos\theta$$
$$(\sigma_{r\theta})_{r=a} = -\frac{3}{2}\mu\frac{U_\infty}{a}\sin\theta \quad \text{..... (24)}$$

9.3.2 Drag on the surface of the sphere

Having determined the stresses, we now compute the drag on the sphere as follows

drag
$$D = \int_0^\pi (\sigma_{rr})_{r=a}\cos\theta(2\pi a\sin\theta)a\,d\theta + \int_0^\pi (\sigma_{r\theta})_{r=a}\cos\left(\frac{\pi}{2}+\theta\right)(2\pi a\sin\theta)\,a\,d\theta$$
$$= 2\pi a\mu U_\infty + 4\pi a\mu U_\infty$$
$$= 6\pi a\mu U_\infty \quad \text{.....(25)}$$

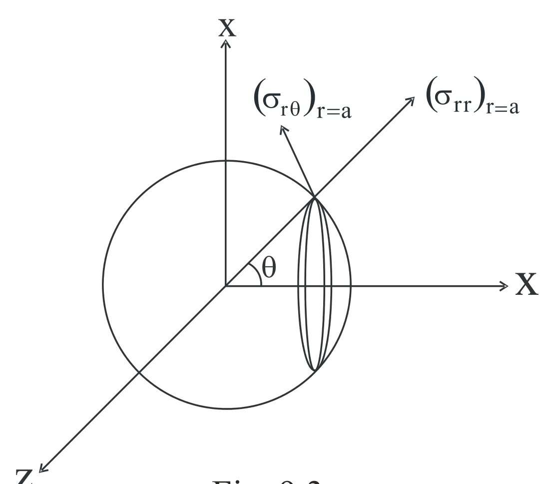


Fig. 9.3

Thus $D = 6\pi\mu U_\infty a$ is the Stokes expression for the drag on the sphere. Note that out of the total drag $\frac{1}{3}$rd of the drag $(2\pi a\mu U_\infty)$ is due to normal stresses and $\frac{2}{3}$rd i.e. $4\pi\mu U_\infty$ is due to the shear stress.

9.3.3 Stoke's Stream Function:

Stream function describes the flow pattern. The stream function ψ is given as

$$\psi = -\int_0^{\theta} (r^2 \sin\theta) v_r d\theta$$

when $\quad v_r = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta} \quad$ Thus $\quad \psi = -\frac{U_\infty}{2}\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right) r^2 \sin^2\theta$

Note -1 Stream function ψ as obtained above can be viewed as composed of two parts ψ_1 and ψ_2

where $$\psi_1 = -\frac{1}{2} U_\infty r^2 \sin^2\theta \left[1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right]$$

$$\psi_2 = \frac{3}{4} U_\infty a r \sin^2\theta$$

Physically ψ_1 signifies irrotational flow past a doublet (a singularity) situated at the origin. Note that ψ_1 contributes nothing to the total force on the sphere.

Physically ψ_2 represents rotationed flow and has a singularity at $r = 0$ known as "Stokeslet" Stokeslet can be summarised physically as a force applied to the fluid at a point. The dgrag $6\pi\mu U_\infty a$ experienced by the sphere is purely due to the Stokeslet.

Note -2 A case of a sphere moving uniformly through a viscous fluid can be made out if we superimpose on the flow field a velocity $-U_\infty$ in the direction of x. The stream function of the superimposed flow is given by

$$\psi = \frac{1}{2} U_\infty r^2 \sin^2\theta$$

Note -3 The stream lines ψ =constant have neen depicted in the following figure. The stream lines indicate the flow pattern.

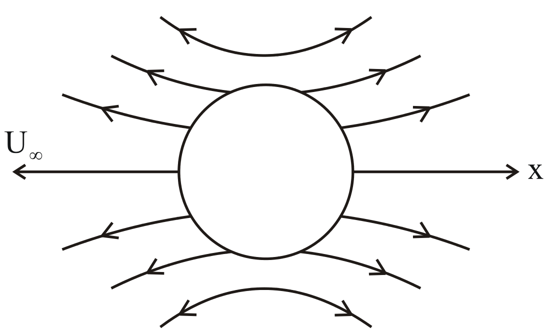


Fig. 9.4

9.4 Self Learning Exercise – 1

1. The condition for very slow motion is

   (a) Re >1 (b) Re <1 (c) Re >>1 (d) Re <<1

2. In the theory of very slow motion, which of the following is true for the pressure p

   (a) $\nabla p = 0$ (b) $\nabla p \neq 0$ (c) $\nabla^2 p = 0$ (d) $\nabla^2 p \neq 0$

3. In Stoke's flow past a sphere, the sphere of radius a (where notations have their usual meanings)

   (a) Experiences no drag

   (b) Experiences drag of magnitude $6\mu U_\infty$

   (c) Experiences dgar of magnitude $6\mu U_\infty a$

   (d) Experiences dgar of magnitude $6\mu U_\infty a\pi$

9.5 Oseen Flow

9.5.1 Oseen Equations

Oseen analyzed the validity of Stoke's equations for the slow motion and extended pertinent submissions. He pointed out Stoke's assumptions are into valid at large distance from the body. He argued that

(i) In deducing Stoke's equation form the Navier equations, the inertia term $(\vec{v}.\nabla)\vec{v}$ were neglected and the viscous terms $\nu\nabla^2\vec{v}$ were taken into account. He reasoned that the order of ratio of inertial term to the viscous term at a distance r is

$$\frac{U^2/r}{\frac{\nu U}{r^2}} = \frac{Ur}{\nu} = \frac{UL}{\nu}\frac{r}{L} = R_e \frac{r}{L}$$

[Note that order of inertia term at distance r is $\frac{U^2}{r}$ and that of viscous term is $\frac{\nu U}{r^2}$]

The above ratio unequivocally says that stoke's equations are valid when both Re and $\frac{r}{L}$ are small.

That means that Stoke's equation would describe the flow "accuratly" in the neighborhood of the body and when Re is small. He questioned the situation what happens when Re is small but r is quite large. Oseen improved the situation and suggested that for larger and small Re, we may retain only those inertia terms which are of comparable magnitudes.

With the viscous terms and at large distance r, an appropriate approximation can be made to the effect that one may regard the flow as a small perturbation (departrue) from the uniform flow.

That is velocity $\vec{v}$ can be taken as

$$\vec{v} = \vec{U}_\infty + \vec{u}^*$$

where $\vec{U}_\infty$ = unoform stream velocity at infinity

$\vec{u}^*$ = small perturbation in $\vec{U}_\infty$

Substituting $\vec{v} = \vec{U}_\infty + \vec{u}^*$ in the Navier - Stoke's equations, we get

$$\nabla.\vec{u}^* = 0 \quad \text{...(1)}$$

$$\frac{\partial \vec{u}^*}{\partial t} + (\vec{U}\cdot\nabla)\vec{u}^* + (\vec{u}^*\cdot\nabla)\vec{u}^* = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{u}^* \quad \text{.............(2)}$$

Acknowledging the lesser contribution of $(\vec{u}^*.\nabla)\vec{u}^*$

and involved mathematical difficulty due to non linearity of this term, Oseen neglected this term and carried out his analysis with the following improved equations.

$$\nabla.\vec{u}^* = 0 \quad \text{...(3)}$$

$$\frac{\partial \vec{u}^*}{\partial t} + (\vec{U}\cdot\nabla)\vec{u}^* = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{u}^* \quad \text{.............................(4)}$$

These are called Oseen's equations.

Note -1

In the neighbourhood of the body, Oseen equations are exactly the same as those of Stokes equations since the term $(\vec{U}_\infty.\nabla)\vec{u}^*$ would be negligible in comparison to the viscous term.

Note - 2

Oseen' equations are valid for any large r and for any large Re. Further it should be noted that if Re is small and r is large, then the equations are valid in the whole flow region since they are valid at large distance whereas in finite region they are slight departure from the Navier-Stokes equations by negligible inertia terms.

Having gone through Oseen' reasoning we, now, move to our main problem, that is, Oseen' flow past a sphere which is an improvement on Stokes solution.

9.5.2 Oseen' flow past a sphere

Let us consider a steady flow with uniform stream $\vec{U}_\infty$ past a solid sphere of radius a held fixed. A Cartesian coordinate system is chosen in such a way that the origin is at the centre of the sphere and the x-axis is in the direction of the flow.

Thus, the Oseen' equations for the steady flow reduce to

$$\frac{\partial u^*}{\partial x}+\frac{\partial v^*}{\partial y}+\frac{\partial w^*}{\partial z}=0 \quad \text{...(5)}$$

$$\rho U_\infty \frac{\partial u^*}{\partial x}+\frac{\partial p}{\partial x}=\mu\nabla^2 u^* \quad \text{...(6)}$$

$$\rho U_\infty \frac{\partial v^*}{\partial x}+\frac{\partial p}{\partial y}=\mu\nabla^2 v^* \quad \text{...(7)}$$

$$\rho U_\infty \frac{\partial w^*}{\partial x}+\frac{\partial p}{\partial z}=\mu\nabla^2 w^* \quad \text{...(8)}$$

$$\left(\text{where} \quad \vec{v}=u^*\hat{i}+v^*\hat{j}+w^*\hat{k}\right)$$

and the boundary conditions are

$$r=a, \quad u^*=-U_\infty, \quad v^*=0, \quad w^*=0 \quad \text{.............................(9)}$$

$$\nabla^2 p=0 \quad \text{...(10)}$$

[Recall it, we have obtained this expression in the case of Stokes flow]

Let ϕ be harmonic function i.e. $\nabla^2\phi=0$, ...(11)

then particular solution to momentum equations (6 - 8) can be obtained if we express

$$p=\rho U_\infty \frac{\partial\phi}{\partial x} \quad \text{..(12)}$$

and $$u^*=-\frac{\partial\phi}{\partial x}, \quad v^*=-\frac{\partial\phi}{\partial y}, \quad w^*=-\frac{\partial\phi}{\partial z}, \quad \text{..................(13)}$$

Hence, complete solution of the equation (6 - 8) can be obtained as,

$$\left.\begin{aligned} u^* &= -\frac{\partial\phi}{\partial x}+u_0 \\ v^* &= -\frac{\partial\phi}{\partial y}+v_0 \\ w^* &= -\frac{\partial\phi}{\partial z}+w_0 \end{aligned}\right] \quad \text{...(14)}$$

where (u_0, v_0, w_0) constitute the solution of the following equations,

$$\left.\begin{aligned}\rho U_\infty \frac{\partial u_0}{\partial x} &= \mu\nabla^2 u_0 \\ \rho U_\infty \frac{\partial v_0}{\partial x} &= \mu\nabla^2 v_0 \\ \rho U_\infty \frac{\partial w_0}{\partial x} &= \mu\nabla^2 w_0\end{aligned}\right] \quad \text{...(15)}$$

and follow the continuity equation

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial z} = 0 \quad \text{...(16)}$$

Now, note that if ξ is the function such that

$$\frac{\partial \xi}{\partial x} = \frac{\nu}{U_\infty} \nabla^2 \xi \quad \text{...(17)}$$

Then, we can determine the solutions of (15) as

$$\left.\begin{aligned}u_0 &= \frac{\partial \xi}{\partial x} - \frac{U_\infty}{\nu}\xi \\ v_0 &= \frac{\partial \xi}{\partial y} \\ w_0 &= \frac{\partial \xi}{\partial z}\end{aligned}\right] \quad \text{...(18)}$$

Now, note that the equation (17) can be redesigned as

$$\left(\nabla^2 - \alpha^2\right) e^{-\alpha x}\xi = 0 \quad \text{...(19)}$$

where $\alpha = \frac{U_\infty}{2\nu}$ ....(20)

(19) has the solution of the form

$$\xi = \frac{A}{r} e^{-\alpha(r-x)} \quad \text{..(21)}$$

For small αr, we have, $\xi = A\left[\frac{1}{r} - \alpha + \frac{\alpha x}{r} + \ldots\ldots\ldots\right]$ ....(22)

We, now, come back to equation (14) and (18).
Making use of with (20) with some simplifications we obtain

$$u^* = -\frac{\partial \phi}{\partial x} + \frac{\partial \xi}{\partial x} - 2\alpha\xi$$

$$v^* = -\frac{\partial \phi}{\partial y} + \frac{\partial \xi}{\partial y}$$

$$w^* = -\frac{\partial \phi}{\partial z} + \frac{\partial \xi}{\partial z}$$

equation (21) is completely determined if we have expression for unknown quantity ϕ. Note that α, ξ are available with us. For this we prescribe ϕ as follow on taking note of the fact that ϕ must have only zonal harmonics of negative degree

$$\phi = B_0 + B_1 \frac{\partial}{\partial x}\left(\frac{1}{r}\right) + B_2 \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + B_3 \frac{\partial^3}{\partial x^3}\left(\frac{1}{r}\right) + \ldots \ldots \ldots \quad (24)$$

Using (22), (24) in u^* – component of (23) making use of boundary conditions (9) and equating zero the coefficients of different power of x, we ultimately obtain.

$$B_0 = A = \frac{3a\nu}{2} \quad B_1 = -\frac{U_\infty a^3}{4}$$

where (note that) $a\alpha = \frac{U_\infty a}{2\nu} = \frac{Re}{4}$ is taken to be small.

Consequent upon the above analysis, the velocity components are obtained as

$$u = U_\infty + u = U_\infty\left[\frac{3}{4}\frac{ax^2}{r^3}\left(\frac{a^2}{r^2} - 1\right) - \frac{a}{4r}\left(3 + \frac{a^2}{r^2}\right) + 1\right]$$

$$v = v^* = \frac{3}{4}\frac{axy}{r^3}\left(\frac{a^2}{r^2} - 1\right)U_\infty \ldots \ldots \ldots \quad (25)$$

$$w = w^* = \frac{3}{4}\frac{axz}{r^3}\left(\frac{a^2}{r^2} - 1\right)U_\infty$$

Note that equation (25) are the same as were obtained by Stoke's.Consequently the drag coefficient in Oseen' analysis comes out to be the same as we obtained in Stoke's flow

9.5.3 Stream Function :

Stream function provides the flow pattern, hence, now we devise the formula for it.
The stream function ψ is given as

$$\psi = -\int_0^\theta \left(r^2 \sin\theta\right) v_r^* d\theta \ldots \ldots \ldots \quad (26)$$

where v_r^* is the radial velocity which is given as

$$v_r^* = -\frac{\partial\phi}{\partial r} + \frac{\partial\xi}{\partial r} - 2\alpha\xi\cos\theta \ldots \ldots \ldots \quad (27)$$

Thus $$\psi = r^2 \int_0^\theta \left(\frac{\partial\phi}{\partial r} - \frac{\partial\xi}{\partial r} + 2\alpha\xi\cos\theta\right)\sin\theta\, d\theta \ldots \ldots \ldots \quad (28)$$

on making use of ϕ, ξ as obtained above, we obtain

$$\psi = \frac{3}{4}\nu a(1 + \cos\theta)\left[1 - e^{-\alpha r(1-\cos\theta)}\right] - \frac{U_\infty a^3}{4r}\sin^2\theta \ldots \ldots \ldots \quad (29)$$

For small values of αr, we have

$$\psi = \frac{3}{4}U_\infty ar\left(1 - \frac{a^2}{3r^2}\right)\sin^2\theta \ldots \ldots \ldots \quad (30)$$

Note that expression (30) is the same as we obtained in the Stoke's flow analysis.
The distinctions of Oseen' flow from Stoke's flow is exhibited in the flow pattern when we plot the steam lines. The stream lines ψ =constant are depicted in the figure (9.5)
The figure shows that in Oseen's flow the stream lines are different in front of and behind the sphere. In fact behind the sphere we come a cross a wake.

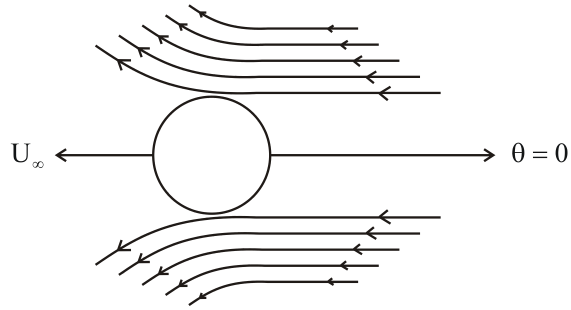


Fig. 9.5

9.6 Self Learning Exercise - 2

1. Oseen's equations for slow motion are valid
   (a) in the neighbourhood of the body only
   (b) not in the neighbourhood of the body
   (c) in the neigbourhood of the body but not at large distance
   (d) at any distance from the body.
2. In Oseen slow motion analysis for the flow past a sphere perturbation is assumed in
   (a) viscosity
   (b) pressure
   (c) density
   (d) velocity

9.7 Summary

In this unit one has discussed the theory of very slow motion which is the case of very small Reynolds number values. For small Re values, the governing non-linear Navier-Stoke's equations can be simplified to give rise to ordinary differential equations which are rather amenable to analytical solution. You have seen how Stoke's deduced his equations for the slow motion and devised solution for the flow past a sphere. He devised expressions for quantities of interest such as drag and the stream function. Oseen' put in efforts to remove the short comings of Stokes analysis. Oseen' method is valid for every distance from body whereas that of Stoke's is true in the neighbourhood of the body.

9.8 Answer to Self-Learning Exercise

Self Learning Exercise -1

1. (d) 2. (c) 3. (d)

Self Learning Exercise -2

1. (d) 2. (d)

9.9 Exercise

1. Write a short note on the theory of very slow motion with reference to Stoke's flow past a sphere
2. Show that drag on a sphere of radius r for the Stoke's flow past the sphere is $6\pi r\mu U_\infty$ where notations have their usual meanings
3. Explain Stoke's flow past a sphere
4. Explain Oseen's flow past a sphere
5. How Oseen's method is an improvement on the method by Stokes

UNIT - 10

Concept of Boundary Layer Theory

Structure of the Unit

10.0 Objectives

After studying this unit you will be able to understand

1. The notion of boundary layer, its genesis and applications.
2. Various characteristics of boundary layer

10.1 Introduction

Boundary layer theory propounded by Ludwig Prandtl, a German Physicist, in 1904 in his seminal paper at Heidelberg entitled "On the Motion of fluid with very Little Friction" proved to be revolutionary in the development of fluid mechanics. Prandtl's eight page paper and 10 minutes presentation gave the world a key to many unresolved problems in fluid mechanics at the beginning of the 20th century.

This theory emphasized the importance of viscosity in large Reynolds number flows. The theory bridged the gap of then prevalent classical hydrodynamics and the hydraulics. The former dealt with the theoretical analysis of the flow but did not have answer to many practical flow problems e.g. drag experienced by a body flowing through fluid, fluid flowing past a body, pressure loss in tubes due to fluid motion etc., the latter dealt with practical flow problems and their solutions based on the experimental data which mostly covered the engineering aspect of fluid motion. Prandtl's boundary layer theory showed the way to overcome similar challenges. His theory is the original example of the use of the singular perturbation method which he applied to governing partial differential equations. Note that the boundary layer theory was all about the flow with very low viscosity fluids (such as air, water) for which Re(Reynolds number) $\rightarrow \infty$. It may be noted that $\mathrm{Re} = \infty$ is considered to be corresponding to ideal fluid flow i.e. zero viscosity or non viscous fluids. Here it is worth to keep in mind that real fluid flows have large or moderate but finite Re values. Prandtl's boundary layer theory is a theory to determine the asymptotic behaviors of flows for high Reynolds number or in other words high Reynolds number flows which are small perturbation from the limiting case $R \rightarrow \infty$. This is what Prandtl did.

10.2 Boundary Layer

In his theory, Prandtl theorized that when low viscosity (however small) fluid flows past a body, the viscous effects such as stresses and forces due to viscosity, diffusion of vorticity etc. are significant and comparable in magnitude with convection and other inertia forces in a very thin fluid layer adjacent to the surface in contact with the fluid, .This thin layer is called the boundary layer. The flow field outside this layer may be regarded as in viscid. Prandtl used no slip condition while devising the boundary layer theory. No slip-condition means that the effect of friction is to cause the fluid immediately adjacent to the surface to stick to surface. The boundary layer to hypothesis supports the intuitive expectations that for the small viscosity fluid flows the effects of viscosity on the flow are unimportant over most of the flow field but the no slip condition at the surface must be satisfied for even vanishingly small viscosity.

The physical significance of the no slip condition is that there is zero relative velocity between the surface and the fluid adjacent to it. Since the viscosity is very small and if the fluid path along the surface is not too long, then the velocity changes appreciably over very short distance normal to the surface of the immersed body in a fluid flow.

Thus boundary layer region is the region of very large velocity gradients. If we recall Newton's shear stress law which states that shear stress is proportional to velocity gradient. Thus local shear stress can be very significant within the boundary layer. This theory solved the dilemma of zero drag resulted from the potential flow theory which was in practice prior to Prandtl's work.

One may recall that prior to Prandtl, potential flow (incompressible) irrotational flow theory was successfully used in many very high Reynolds number flow problems where complete negation of viscons effects served as a good approximation but resulted a zero drag. This phenomenon remained unexplained then. Further, potential flow theory failed to satisfy the no-slip condition contrary to what was observed in real situations. Prandtl's theory was outcome of theoretical and experimental investigations. It was shows that the flow past a body can be partitioned into two regions

(i) A very thin layer (boundary layer) adjacent to the surface where viscosity effects dominate. [note that the boundary-layer thickness is supposed everywhere to be small compared with distances parallel to the boundary over which the flow velocity change appreciably]

(ii) Outside this layer, the fluid may be regarded as inviscid or may be treated as a potential flow.

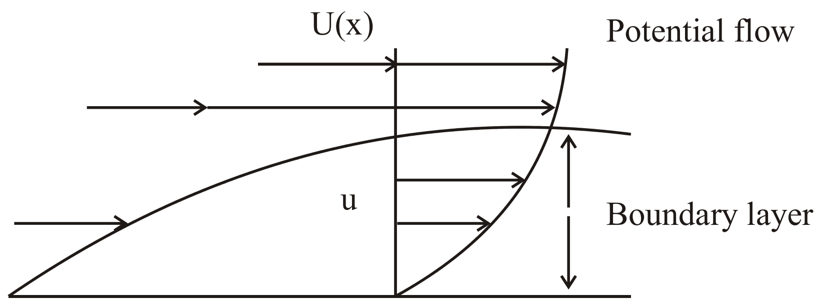


Fig. 10.1

10.2.1 Applications

Initially the theory was formulate for laminar flow of an incompressible fluid which successesfully agreed with the experimental investigations. This simulated the growth of the fluid mechanics and the idea was exetended to compressible and turbulent flows. With the advent and challenges of flight technology and urge to design optimal equipment, boundary layer theory proved to be good simulation tool. The aerodymanic shapes of present day four wheelers are due to boundary layer analysis. Now, we briefly outline the applications of boundary layer theory

1. It served a basis to many branches of fluid and mechanics aerodynamic such as airfoil theory and gas dynamics
2. It helped to compute frictional drag of bodies in a flow whether body is in motion in the fluid or the fluid is in motion past the fixed body
3. It extended explanation to reverse flow situations in many flow regimes. It facilitated the understanding of separation of flow from the body and the formation of eddies at the back of the body.
4. Like the boundary layer equations for momentum. boundary layer approximations for thermal and solutal regime have been devised.

The boundary layer theory has got so much significance that all its different area have emerged as full fledged branches in themselves.

10.3 Prandtl's Boundary Leyer Theory

Uptil now you have had a fair idea of boundary layer. and its historic background. In this section we sharpen our understanding of the notion

First we consider flow over an airfoil shaped body as shown in the figere 10.2. The body experiences a net aerodynamic force due to the fluid pressure and shear stress.

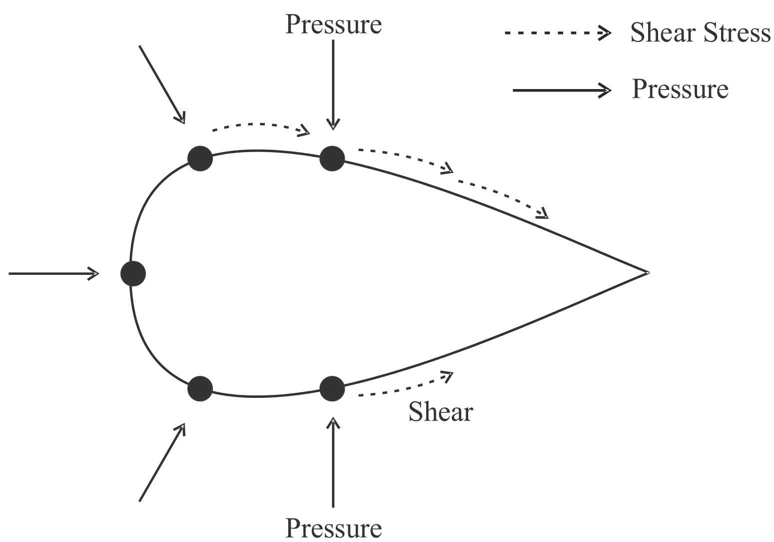


Fig. 10.2

Intuitively you can reason that to get the net aerodynamic force both the pressure distribution and shear stress distribution first be determined and then integrated over the whole surface of the airfoil. The computation of pressure is rather easier as compared to shear stress, simply because in computing the pressure we may assume that the fluid is inviscid. But this cannot be done for shear computation. For it, one has to take internal friction into account and the complexity begins! This is one example where in Navier-Stokes equations are hard to solve.

10.4 Characteristicsc Boundary Layer parameters

We now present some characteristic parameters of boundary layer whose computation provide vital insight into the phenomena and are important in many practical problems.

(i) Boundary layer Thickness 'δ'

Boundary layer thickness is the distance in which the velocity in boundary layer approaches to the potential flow velocity asymptotically. Boundary layer thickness is the distance from the wall where the fluid velocity u in boundary layer differs from the potential flow velocity U by 1% i.e.

boundary layer thickness

$$\delta = (y)_u = 99\%U$$

(ii) Displacement Thickness 'δ'$_1$

This is more sensible measure of boundary layer thickness and is defined as

$$\delta_1 = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$

Physically it can be thought of as the distance through which stream lines just outside the boundary layer are displaced laterally by the fluid retardation in the boundary layer.

(iii) Momentum thickness δ_2

The momentum thickness δ_2 is defined as

$$\delta_2 = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right) dy$$

(iv) Skin Friction

The shearing stress on the surface y = 0 or the skin friction is given by

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

10.5 Self Learning Exercies

1. Boundary layer theory was formulated by
   (a) Reynolds
   (b) Sakiadis
   (c) Blasius
   (d) Prandtl
2. Boundary layer theory formulation considers
   (a) Slip condition
   (b) No - slip condition
   (c) Variable pressure
   (d) Variable temperature
3. In the pressure in the boundary layer is same...
4. The bounary layer flow the viscous effect of the fluid is...

10.6 Summary

In this unit an introductory note to the motion of boundary layer was presented. The motion is applicable to describe the mechanics of vanishingly small viscosity fluid flow. The theory has been a great success to serve as s key tool to devise state of art also dynamic designs of automobiles, missile technology, high speed airplanes, sophisticated war airplanes to name a few.

It has also helped in dealing with problem inclusive of drag like in case of ship sailing.

The applications of boundary layer theory are so varied that it has led to grow various branches in fluid mechanics undoubtedly advent of commercial software, development of numerical software techniques have helped to analyse many boundary layer equations for laminar and turbulent flow.

10.7 Answers to Self Learning Exercise

1. (d)
2. (b)
3. As that of at the edge of boundary layer
4. Confined in a thin layer adjacent to the wall.

10.8 Exercise

1. Write a note on boundary layer theory
2. Write a note on characteristic parameters of boundary layer theory

UNIT - 11

Velocity and Thermal Boundary Layer in Two Dimensional Flow

Structure of the Unit

11.0 Objectives

After studying this unit you will be able to

1. derive the velocity boundary layer equations for two dimensional flow
2. derive the thermal boundary layer equations for two dimensional flow

11.1 Introduction

In previous unit we have learnt about the existence and importance of the boundary layer. We have learnt that how Prandtl presented a theory which could answers to some unresolved practical problems.

In this chapter we will understand the existence and the altered Navier-Stokes equations in the boundary layer above which there is potential flow.

The velocity boundary layer equations have been obtained through two approaches viz. order of magnitude approach and asymptotic approach alongwith necessary boundary condition.

It is also seen in the previous unit that a thermal boundary layer also exists which is an elegans to momentum boundary layer. In this unit is has been suggested that boundary layer approximations can be defined through the order of magnitude approach.

11.2 Velocity Boundary layer Equations in Two Dimensional Form

We will derive boundary layer equatons for the flow past a solid plane wall by two different approaches viz order of magnitude approach and the asymptotic approach.

Let us consider a two dimensional flow of a viscous incompressible fluid over a plane solid wall. A Cartesian co ordinate system is considered as shown in the figure.

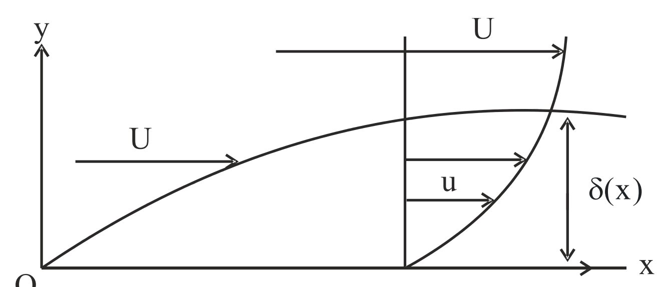


Fig. 11.1

Two dimensional Navier-Stokes equations and equation of continuity are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \text{ ....(1)}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \text{....................(2)}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \text{....................(3)}$$

and the boundary condtions are

$$y=0 \quad ; \quad u=v=0$$

$$y\to\infty \quad ; \quad u\to U(x,\ t)=U_\infty \text{ ....(4)}$$

where (u, v) are the velocity components in (x, y) directions and U_∞ is the free stream (potential) velocity. Note that since the wall is solid, hence due to no slip condition $u=v=0$ at $y=0$. Further it should be noted that velocity component u which is zero at the wall, grows rapidly in the boundary layer to match the free stream velocity at the edge of the boundary layer [i.e. $y\to\infty$; $u\to U_\infty$]

Let us assume that the boundary layer thickness is δ. Note that in this thin boundary layer region $\delta << L$ where L is the characteristic length. For convenience, in this discussion we take $L=1$. In the boundary layer region, viscous effects are significant.

11.2.1 Order of magnitude approach

We now use order of magnitude here for each terms in the governing equations and it is denoted by 0 ( ).

Let us consider quantities t, x and u of 0 (1) and y of $0(\delta)$ where $\delta<<1$

We observe that the quantities

$$\frac{\partial u}{\partial t},\ \frac{\partial u}{\partial x},\ \frac{\partial^2 u}{\partial x^2} \quad \text{are each of } 0(1)$$

Further, we note that the quantities $\frac{\partial u}{\partial y}$ and $\frac{\partial^2 u}{\partial y^2}$ are of $0(\delta^{-1})$ and $0(\delta^{-2})$ respectively.

It is pertinent to note that since y is of 0 (δ) therefore the normal velocity component v is of 0 (δ).

$$\left[\because\ v=\frac{\partial y}{\partial t},\ 0\ (y)=\delta,\ 0(t)=1\right]$$

$\Rightarrow$ v is of $0(\delta)$

Like wise we conclude that

$\frac{\partial v}{\partial y}$ is of 0 (1) and the quantities $\frac{\partial v}{\partial t},\ \frac{\partial v}{\partial x},\ \frac{\partial^2 v}{\partial x^2}$ are of 0 (δ) each and $\frac{\partial^2 v}{\partial y^2}$ is of $0\left(\frac{1}{\delta}\right)$

This order of magnitude analysis enables us to determine the significant terms to be retained or we can say the terms which can be omitted. Thus we see that the term $\frac{\partial^2 u}{\partial x^2}$ which is of 0 (I) is negligible in comparision to $\frac{\partial^2 u}{\partial y^2}$ since $0\left(\frac{\partial^2 u}{\partial y^2}\right)$ is $\frac{1}{\delta^2}$ which is large. With these consideration equation (2) becomes

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \quad ....(3)$$

Further it is presumed that the viscous term enjoys the same order as that of the inertia term, that is of 0 (1). This leads to conclude that ν is $0(\delta^2)$. This reasoning has been validated by some exact solutions.

In view of the above analysis, the eq. (3) suggests that

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} \quad \text{is of } O(\delta)$$

Physically this means that the pressure in the boundary layer grows with O (δ^2) and thus may be neglected. This finding is of utmost importance to conclude that the pressure is constant in the normal direction and may be taken equal to the pressure at the edge of the boundary layer where it is computed by the inviscid free stream flow.

$$\text{Thus } -\frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}$$

where U is the potential flow velocity. In view of the above analysis, the boundary layer equations for the unsteady two dimensional incompressible flow over a solid plane wall are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+v\frac{\partial^2 U}{\partial y^2} \quad ....(4)$$

together with the boundary conditions.

$$y = 0 \quad ; \quad u = 0 = v$$

$$y \to \infty \quad ; \quad u \to U(x,t) = U_\infty$$

11.2.2 Asymptotic Approach

The boundary layer equations derived above may alternatively be obtained by asymptotic approach. Infact, the boundary layer equations are asymptotic form of Navier-Stokes equations at large Reynolds number.

Let us again consider the equations (1) - (3)

We introduce the following non dimensional quantities.

$$\bar{t} = \frac{t}{T},\ \bar{x} = \frac{x}{X},\ \bar{y} = \frac{y}{Y},\ \bar{u} = \frac{u}{U},\ \bar{v} = \frac{v}{V},\ \bar{p} = \frac{p}{P}$$

where T , X , Y, U, V, P are the characteristic measure of the corresponding quantities.

On putting these quantitics in the equations (1) - (3) we find,

$$\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{XV}{YU}\frac{\partial \bar{v}}{\partial \bar{y}} = 0 \quad ....(5)$$

$$\frac{X}{UT}\frac{\partial\bar{u}}{\partial\bar{t}}+\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{XV}{YU}\frac{\partial\bar{u}}{\partial\bar{y}}=-\frac{P}{\rho U^2}\frac{\partial\bar{p}}{\partial\bar{x}}+\frac{\upsilon}{XU}\frac{\partial^2\bar{u}}{\partial\bar{x}^2}+\frac{\upsilon X}{Y^2U}\frac{\partial^2\bar{u}}{\partial\bar{y}^2} \text{ ....(6)}$$

and $$\frac{X}{UT}\frac{\partial\bar{v}}{\partial\bar{t}}+\bar{u}\frac{\partial\bar{v}}{\partial\bar{x}}+\frac{XV}{YU}\bar{v}\frac{\partial\bar{v}}{\partial\bar{y}}$$

$$=-\frac{PX}{\rho YUV}\frac{\partial\bar{p}}{\partial\bar{y}}+\frac{\upsilon}{XU}\frac{\partial^2\bar{v}}{\partial\bar{x}^2}+\frac{\upsilon X}{Y^2U}\frac{\partial^2\bar{v}}{\partial\bar{y}^2} \text{ ....(7)}$$

considering X and U as the fundamental units, we have

$$T=\frac{X}{U},\quad P=\rho U^2$$

and $Re=\frac{XU}{\upsilon}$ (Reynolds number)

the units of measurement of Y and V are computed taking note of the fact that the equations (5)-(7) ought to have a single flow parameter i.e. Reynolds number Re. Consequently, we take

$$\frac{XV}{YU}=1 \quad\text{and}\quad \frac{\upsilon X}{Y^2U}=1$$

$$\Rightarrow Y=\frac{X}{\sqrt{Re}},\quad V=\frac{U}{\sqrt{Re}} \text{ ....(8)}$$

in view of the above reasoning, the equation (5) - (7) take the form

$$\frac{\partial\bar{u}}{\partial\bar{x}}+\frac{\partial\bar{v}}{\partial\bar{y}}=0 \text{ ....(9)}$$

$$\frac{\partial\bar{u}}{\partial\bar{t}}+\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{u}}{\partial\bar{y}}=-\frac{\partial\bar{p}}{\partial\bar{x}}+\frac{1}{Re}\frac{\partial^2\bar{u}}{\partial\bar{x}^2}+\frac{\partial^2\bar{u}}{\partial\bar{y}^2} \text{ ....(10)}$$

$$\frac{1}{Re}\left(\frac{\partial\bar{v}}{\partial\bar{t}}+\bar{u}\frac{\partial\bar{v}}{\partial\bar{x}}+\bar{v}\frac{\partial\bar{v}}{\partial\bar{y}}\right)=-\frac{\partial\bar{p}}{\partial\bar{y}}+\frac{1}{Re^2}\frac{\partial^2\bar{v}}{\partial\bar{x}^2}+\frac{1}{Re}\frac{\partial^2\bar{v}}{\partial\bar{y}^2} \text{ ....(11)}$$

We have seen that boundary layer equations hold good for large Re values. Thus when Re is large, then small parameter $\frac{1}{\sqrt{Re}}$ is pertinent parameter in boundary layer analysis. In order to have the solution of the above equatons we introduce the following expansions

$$\bar{u}=u_0+\frac{1}{\sqrt{Re}}u_1+\frac{1}{\left(\sqrt{Re}\right)^2}u_2\text{.......... (12)}$$

$$\bar{v}=v_0+\frac{1}{\sqrt{Re}}v_1+\frac{1}{\left(\sqrt{Re}\right)^2}v_2\text{..........(13)}$$

$$\bar{p}=p_0+\frac{1}{\sqrt{Re}}p_1+\frac{1}{\left(\sqrt{Re}\right)^2}p_2\text{..........(14)}$$

Making use of these perturbations is the equation (9)-(11) and then comparing the terms independent of Re

we obtain

$$\frac{\partial u_0}{\partial\bar{x}}+\frac{\partial\bar{v}_0}{\partial\bar{y}}=0 \text{ ....(15)}$$

$$\frac{\partial u_0}{\partial \bar{t}} + u_0 \frac{\partial u_0}{\partial \bar{x}} + v_0 \frac{\partial u_0}{\partial \bar{y}} = -\frac{\partial p_0}{\partial \bar{x}} + \frac{\partial^2 u_0}{\partial \bar{y}^2} \quad \text{.........................(16)}$$

$$0 = -\frac{\partial p_0}{\partial \bar{y}} \quad \text{.........................(17)}$$

On converting these equations in dimensional form and dropping the index zero, we get the same equations as we got through order of magnitude approach. The equation $-\frac{\partial p}{\partial y}$ is not taken into account since it is assumed that the pressure is independent of y in the boundary layer and infact the same as at the outer edge of the boundary layer.

11.3 Self Learning Exercise

1. Velocity components in the boundary layer adjescent to stationary plane wall
   (a) remain same throughout
   (b) dimension as one goes towards the potential flow.
   (c) grows rapidly as one goes towards the potential flow

2. Boundary layer equation are ....(1)........................ form of Navier - Stokes equation at large...........................(2)..........................

11.4 Thermal Boundary Layer

The concept of thermal boundry layer is analogous to the momentum boundry layer. When a fluid flows over a heated / cooled body then transfer of heat is experinced . As we know there are three modes of heat transfer conduction, convection and radiation. Radiative heat transfer is significant if the thermal regime involves high temperature. In the present text we will not discuss the radiation aspects. Coming back to the central issue of the thermal boundary layer, it is seen that at high Reynolds number, the thermal regime also exhibits boundary layer character. That means temperature field can be divided into two regions (i) The region close to the wall where thermal conductivity k has a key role and (ii) the region in which k can be neglected.

This unit is restricted to the analysis when the density and viscosity are consent i.e. not dependent on temperature and pressure. This pre condition is ensured by the assumption that temperature and pressure difference within the boundary layer are small. Here it is pertinent to remind that in general momentum boundary layer thickness and thermal bounary layer thickness are not the same.

Further, it is to be remembered of that we derived the boudnary layer equaiton for momentum for fluids having small viscosity. For small thermal conductivity fluids, the energy equation can be simplified to yield thermal boundary layer equations.

We known that energy equation for two dimensional steady flow of a viscous incompressible fluid is

$$\rho C_p \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \right) = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \phi \quad \text{....................................(1)}$$

where (u, v) are velocity components in (x, y) directions, T is the temperature, ρ is the density and C_p is specific heat at constent pressure, k is the thermal conductivity and ϕ is the viscous dissipation given by

$$\phi = 2\mu\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right) + \mu\left(\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right) \quad \text{............(2)}$$

The boundary layer approximation for the eq. (1) can be derived by order of magnitude as we have done earlier for the momentum equations.

Let δ_T denote the thermal boundary layer thickness and δ is the thickness of velocity boundary layer then v is of o(δ).

11.5 Summary

In this unit, you have learnt to derive the boundary layer equations for momentum and the energy regimes. We have seen that same momentum boundary layer equation are derived by order of magnitude approach and asymptotic approach (i.e. $Re \rightarrow \infty$). The analysis was made for the flow of viscous incompressible fluid past a thin plate. Similarly, formulation for the thermal boundary layer was also made through order of magnitude approach.

11.6 Answer to Self Learning Exercise.

1. (c)
2. (1) asymptotic

   (2) Reynolds number

11.7 Exercise

1. Derive two dimensional boundary layer equation for the viscous incompressibel fluid flow past a thin plate
2. Derive two dimensional thermal boundary layer equation for the viscous in compressible fluid flow past a thin plate.

UNIT - 12

Blasius - Topfer Solution

Structure of the Unit

12.0 Objectives

In this unit you learn the derivation of

1. boundary layer solution of viscous incompressible fluid flow over a flat plate. This solution has been initiated by Blasius
2. a simple solutions of thermal boundary layer flow over a flat plate in a particular case when Prandtl number has been taken as unity.

12.1 Introduction

The boundary layer flow along a thin flat plate is the simplest example of boundary layer theory. This case was infact the first example of the boundary layer theory proposed by Ludwig Prandtl (1904). The analysis presented here is part of doctoral thesis of H. Blasius (1908).
In this unit we will also discuss the solution of a thermal boundary layer problem of a forced convection laminar boundary layer flow past a flat plate for a particular value of Prandtl number which has been taken as unity.

12.2 Boundary Layer Flow on Flat Plate

Boundary layer on a flat plate (Blasius Topfer solution)

Blasius produced a solution to the steady boundary layer flow on a flat plate with the help of the similarity solution. The partial differential equations were reduced to ordinary differential equation. The resultant system is amenable to the solution.
Here we consider a steady flow of a viscous incompressible fluid over a very thin solid flat plate. It is assumed that the plate is semi infinite in length. A Cartesian coordinate system is considerd as shown in the figure wherein x-axis is taken along the flat plate and the y-axis perpendicular to it with origin at the edge of the plate. The plate starts at x = 0 and extends along the x-axis. The fluid flows parallel to the plate with free stream velocity U_∞

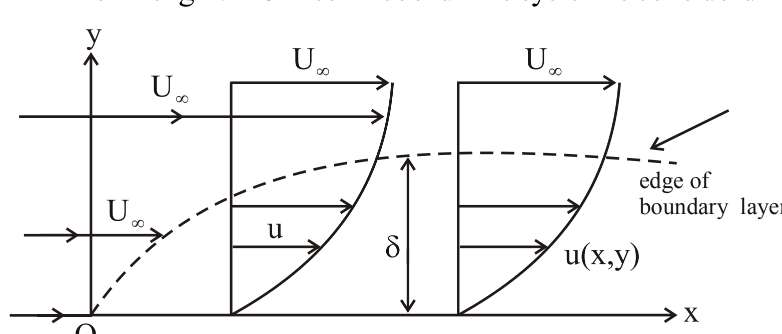


Fig. 12.1

In this case, the potential flow velocity is uniform i.e. U_∞ hence there is no pressure gradient along the x-axix. Thus

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = U_\infty \frac{dU_\infty}{dx} = 0$$

Thus the equation of continuity and boundary layer equations for the two dimensional flow and for the flow considered here become.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \text{ ....(1)}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon \frac{\partial^2 u}{\partial x^2} \text{ ....(2)}$$

together with the boundary conditions

$$y = 0 \ ; \ u = v = 0 \text{ ....(3)}$$

$$y \to \infty \ ; \ u \to U_\infty \text{ ....(4)}$$

It is pertinent to note that we can hope for similar solution since the setup has no characteristic length, therefore we can presume that the velocity profiles at different distances from the leading edge (i.e. $x = 0$) are similar to one another. In the foregoing analysis we would explore the possibility of simlar solution so that the equation (2) is converted to ordinary differential equation.

Seeking Similar Solutions :

By dimensional considerations and reasoning we find that

$$\delta(x) \sim \sqrt{\frac{x\upsilon}{U_\infty}}$$

Hence we can have similarity $\eta \sim \dfrac{y}{\delta(x)}$

$$\eta = y \Big/ \sqrt{\frac{x\upsilon}{U_\infty}} \quad = \quad y\sqrt{\frac{U_\infty}{\upsilon x}} \text{ ....(5)}$$

Let ψ be stream function so that the velocities take the form

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \text{ ....(6)}$$

The equation suggests that the dimension of ψ is the same as that of $\sqrt{\upsilon U_\infty x}$

Hence we can set $\psi = \sqrt{\upsilon U_\infty}\, f(\eta)$...(7)

η being a dimensionless quantity

In view of (6) and (7) the velocity components are obtained as

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}.\frac{\partial \eta}{\partial y} = U_\infty f^1(\eta) \text{ ....(8)}$$

$$v = -\frac{\partial \psi}{\partial x} = -\left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial x}\right) = \sqrt{\frac{\upsilon U_\infty}{x}}\left(\eta f^1 - f\right) \text{ ....(9)}$$

where f^1 denots derivative of f with respect to η. Similarly

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(U_\infty f^1\right)=-\frac{U_\infty}{x}\eta f^{11}$$

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(U_\infty f^1\right)=U_\infty\sqrt{\frac{U_\infty}{\upsilon x}}f^{11}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(U_\infty\sqrt{\frac{U_\infty}{\upsilon x}}f^{11}\right)=U_\infty\sqrt{\frac{U_\infty}{\upsilon x}}f^{111}$$

Susbstituting the values of u and v and the partial derivaties as obtained above,in the equation (2) we get

$$2f^{111}+ff^{11}=0 \text{ ....(9)}$$

and the corresponding boundary conditions (3), (4) take the form

$$\left.\begin{array}{lll}\eta=0 & ; & f=0,\ f^1=0\\ \eta\to\infty & ; & f^1\to 1\end{array}\right\} \text{ ....(10)}$$

The equation (9) along with boundary condition (10) is known as Blasius equation. Note that (9) does not have closed form solution since it is a non-linear differential equation. It can be solved numerically. However, Blasius himself succeeded in presenting a series solution to it subject to the boundary conditions (10)

12.2.1 Blasius Series Solution

Blasius idea for obtaining a series solution was to obtain series expansions for $f(\eta)$ about $\eta=0$ and for large η and to join these two expansions at a suitable value of these two expansions at a suitable value of η.

Series Solution about $\eta=0$

Let us consider

$$f(\eta)=a_0+a_1\eta+\frac{a_2}{\lfloor 2}\eta^2+\frac{a_3}{\lfloor 3}\eta^3+\text{................}$$

So that

$$f^1(\eta)=a_1+a_2\eta+\frac{a_3}{\lfloor 2}\eta^2+\frac{a_4}{\lfloor 3}\eta^3+\text{................}$$

$$f^{11}(\eta)=a_2+a_3\eta+\frac{a_4}{\lfloor 2}\eta^2+\frac{a_5}{\lfloor 3}\eta^3+\text{................} \quad \text{...(11)}$$

$$f^{111}(\eta)=a_3+a_4\eta+\frac{a_5}{\lfloor 2}\eta^2+\frac{a_6}{\lfloor 3}\eta^3+\text{................}$$

where primes denote differential with respect to η. Since for

$$\eta=0\ ,\quad f=0,\quad f^1=0$$

we have clearly

$$a_0=0=a_1 \text{ ....(12)}$$

We now substitute the above expansions in the equation (9), and on simplification we obtain

$$2a_3+2a_4\eta+\left(a_2^2+2a_5\right)\frac{\eta^2}{2}+\left(4a_2a_3+2a_6\right)\frac{\eta^3}{6}+\text{......}=0 \text{ ....(13)}$$

Note that (13) is an identity which holds for every value of η. Hence coefficients of different powers of η must vanish identically. This leads to

$$a_3=0=a_4=a_6=a_7,\qquad a_5=-\frac{a_2^2}{2},\ a_8=-\frac{11}{4}a_2^3 \text{ etc.}$$

Thus we see that the coefficients in the expensions (11) are either zero or can be expressed in terms of a_2
In view of the above analysis, we find that

$$f(\eta) = \frac{a_2}{\lfloor 2}\eta^2 - \frac{a_2^2}{2\lfloor 5}\eta^5 + \frac{11a_2^3}{4\lfloor 8}\eta^8 - \text{.................................}(14)$$

Series Solution for large η (i.e, $\eta \to \infty$)

$$\frac{f^{111}}{f^{11}} = -\frac{f}{2} \text{ on integration } f^{11} = Ae^{-\int \frac{f}{2} d\eta} \text{ where A is constant of integration}$$

On integrating again and taking the boundary conditon $\eta \to \infty, \quad f^1 \to 1$ into account, we get

$$f^1 = 1 + A\int_{\infty}^{\eta}\left(e^{\int -\frac{f}{2}d\eta}\right)d\eta$$

Integrating again,

$$f = \eta - B + A\int_{\infty}^{\eta}\left(\int_{\infty}^{\eta} e^{-\int \frac{f}{2}d\eta} d\eta\right)d\eta \text{...}(15)$$

where B is also a constant of integration
As a first approximation as $\eta \to \infty$, Blasius set

$$f = \eta - B \text{...}(16)$$

using (16) in integral (15), we get

$$f(\eta) = \eta - B + A\int_{\infty}^{\eta}\left[\int_{\infty}^{\eta} e^{-\int \frac{(\eta - B)d\eta}{2}} d\eta\right]d\eta \quad \text{or} \quad f(\eta) = \eta - B + A\int_{\infty}^{\eta}\left[\int_{\infty}^{\eta} e^{-\int \frac{(\eta-B)}{4}d\eta} d\eta\right]d\eta$$

Note that f(η) as obtained above satisfies the condition $\eta \to \infty, \quad f'(\eta) \to 1$
However the constants A and B are still to be evaluated. Infact, the constants a_2, A and B are chosen in such a way that $f(\eta)$, $f'(\eta)$ and $f''(\eta)$ are continuous when the expeansions are joined. Sufficient number of terms in these expansions are taken into account to get the desired accuracy. The values obtained by Blasius himself are

a_2=0.332, B = 1.73, A = 0.231

12.3 Thermal Boundary Layer : Simple Solution for P_r=1

Here we will consider a particular problem of flow past a flat plate **[forced convection laminar boundary layer flow past a flat plate for P_r=1]**

Let us consider a steady flow of a viscous incompressible fluid over a thin semi-infinite flat plate. The insulated flat plate is considered at temperature T_ω. The free stream velocity is U_∞ and tempreture T_∞ plate is assumed to be along the direction of fluid stream. A Cartesian coordinate system is considered and the origin is taken at the leading edge of the plate and axis of x along the plate. Thus the two dimensional flow is governed by the equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \text{...}(1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2} \text{..}(2)$$

and $$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho C_p}\left(\frac{\partial u}{\partial y}\right)^2 \text{.....................}(3)$$

with the boundary conditions

$$\left.\begin{array}{lll} y = 0 & ; & u = v = 0, \quad T = T_{\omega} \\ y \to \infty & ; & u \to U_{\infty}, \quad T \to T_{\infty} \end{array}\right\} \quad \ldots\ldots\ldots\ldots(4)$$

The solution of the above equation (3) can easily be obtained for the case $P_r=1$ Note that when $P_r = \frac{vC_p\rho}{K}$ then is means that $\upsilon = \frac{K}{\rho Cp}$. This is the special situation when the equation (2) and (3) seem is to be identical in the sense that u, T may be interchanged to get the either equation with the boundary conditions (4)

$$\frac{T - T_{\omega}}{T_{\infty} - T_{\omega}} \text{ is replaced by } \frac{u}{U_{\infty}}$$

Hence we take $$\frac{T - T_{\omega}}{T_{\infty} - T_{\omega}} = \frac{u}{U_{\infty}} \quad \text{or} \quad 1 - \frac{T - T_{\omega}}{T_{\infty} - T_{\omega}} = 1 - \frac{u}{U_{\infty}}$$

or $$\frac{T - T_{\infty}}{T_{\omega} - T_{\infty}} = 1 - \frac{u}{U_{\infty}} \quad \ldots\ldots\ldots\ldots(5)$$

(5) is known as Croccos first integral for $P_r=1$
which gives the solution for temperature distribution in terms of wall temperature, uniform stream temperature, velocity distribution in the boundary layer and uniform stream velocity.

12.4 Self Learning Exercise

1. Boundary layer flow on a flat plate is also known as ....
2. Blasius equation have closed form situation
   True or False
3. $P_r=1$ implies v =
4. What is Croccos first integral ?

12.5 Summary

In this unit we have discussed two problems. One is a fluid boundary layer flow problem over a flat plate. This leads to Blasius Topfer solution in a non linear differential equation form. This solution of this form was carried and by Blasius which has been given here. Another problem is of thermal boundary layer in a fluid moving over an insulated plate at a constant temperature. which is different from the uniform flow temperature. This solution leads to Croccos first integral for $P_r=1$ i.e. Prandtl number equal to unity.

12.6 Answer to Self Learning Exercise

1. Basius-Topfer solution
2. False
3. $\frac{K}{\rho C_p}$
4. $\frac{T - T_{\infty}}{T_{\omega} - T_{\infty}} = 1 - \frac{u}{U_{\infty}}$

12.7 Exercises

1. Discuss the boundary layer flow over a flat plate.
2. Obtain Crocco's first integral for $P_r=1$

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