MA/MSc MT-05

Vardhaman Mahaveer Open University, Kota

Mechanics

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Course Introduction

*The Present book entitled **"Mechanics"** has been designed so as to cover the unit-wise syllabus of Mathematics-Fifth paper for M.A./M.Sc. (Previous) students of Vardhaman Mahaveer Open University, Kota. It can also be used for competitive examinations. The basic principles and theory have been explained in a simple, concise and lucid manner. Adequate number of illustrative examples and exercises have also been included to enable the students to grasp the subject easily. The units have been written by various experts in the field. The unit writers have consulted various standard books on the subject and they are thankful to the authors of these reference books.*

UNIT - 1
D'Alembert's Principle

Structure of the unit

1.0 Objective

This unit provides a general overview of D'Alembert's Principle. Moment of Inertia and product of Inertia of some bodies, After reading this unit you will be able to learn

1. About rigid body and effective forces, impressed forces
2. D'Alembert's Principle
3. Motion of Centre of Inertia
4. Motion relative to the centre of Inertia

1.1 Introduction

D'Alembert's (1717-1783) was a French Mathematician. He is remembered for D'Alembert's Principle for the solution of wave equation.

D'Alembert's suggested an important method by which the equations of motion of a rigid body may be obtained without writing the equations of motion of several particles and without knowing the unknown forces. By using D'Alembert's Principle we will be able to convert a dynamical problem into a statical one. Also we will apply D'Alembert's Principle to derive the general equations of motion of a rigid body for finite forces.

1.2 Definitions

Impressed Forces :

The external forces acting on a body are called impressed forces. For example, if a body is tied to string then tension in the string is an impressed force. Similarly weight of a body is also an impressed force.

Effective Forces :

The force possessed by an accelerated mass particle is known as effective force i.e. product of mass and its acceleration of moving particle is called its effective force. For examples, if m be the mass of a particle and its acceleration be f, then mf is called effective force of this particle. Thus $m\frac{d^2x}{dt^2}$ is the effective force parallel to the axis of x. Similarly $m\frac{d^2s}{dt^2}$ and $m\frac{v^2}{\rho}$ are effective forces on the particle along the tangent and normal respectively.

Also $-m\frac{d^2x}{dt^2}$, $-m\frac{d^2s}{dt^2}$, $-m\frac{v^2}{\rho}$ etc. are called reversed effective forces of a particle parallel to x-axis, tangent and normal respectively.

Newton's Second Law of Motion :

Newton's second law state that, Applied force = effective force

If F is the applied force or external force and m be mass of particle, f being the acceleration, then $F = mf$

Equations of Motion :

If (x, y, z) be the coordinates of a moving particle of mass m, at any time t and F_1, F_2, F_3 be the components of forces parallel to three axes, then by Newton's Second Law, the equations of motion of the particle are $m\frac{d^2x}{dt^2} = F_1$, $m\frac{d^2y}{dt^2} = F_2$, $m\frac{d^2z}{dt^2} = F_3$. Thus for a particle motion, the impressed force and reversed effective force are in equilibrium.

Rigid Body : Definition :

A rigid body is formed by material particles whose mutual distances do not change under all circumstances of motions. For a rigid body, we assume that

(i) The action between its two particles act along the straight line joining them, and

(ii) The action and reaction between them are equal and opposite.

1.3 Moment of Inertia and Product of Inertia of some bodies

In the study of motion of rigid body, in general linear and rotational motion, two very important entities moment of inertia (M.I.) and product of inertia (P.I.) are needed, these two entities change with the changing shape and size of the body even if its mass is not altered. Now we shall define moment of Inertia and product of Inertia. Also we shall write results of moment of Inertia and product of Inertia of some bodies.

Moment of Inertia :

Definition 1 : Let m be the mass of an elementary particle of a body of mass m and r be the distance of this element from the given line, then $\sum mr^2$ is called the moment of Inertia of the given body about the **given line,** where the summation is taken for all the particle of the body. M.I. of a body is clearly a scalar quantity.

Definition 2 : For mass of M of the body, we can always find a lenght k such that $\sum mr^2 = Mk^2 \Rightarrow k^2 = \frac{\sum mr^2}{M}$.

This length k is called the "radius of gyration" of the body about the given line.

The radius of gyration of a body rotating about a given axis is equal to the distance from the axis of rotation, the square of which when multiplied by the total mass of the body, will give the moment of Inertia of the body about the axis, Moment of Inertia of a body is always positive or zero only.

Similarly, we can define M.I. of a body with respect to a plane or a point.

Some Importants Symbols :

If (x, y, z) be the coordinates of a point P, then its distance from x-axis is $PA = \sqrt{y^2 + z^2}$, as its distance from $xy-$ plane is $PB(=z)$ and its distance from origin $= OP = \sqrt{x^2 + y^2 + z^2}$, then the moments of Inertia of the body about the coordinates ox, oy and oz are generally denoted by A, B and C.

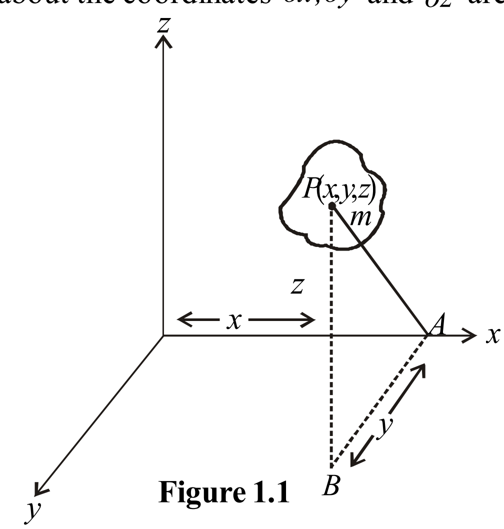


Figure 1.1

Thus $A = \sum m(PA)^2 = \sum m(y^2 + z^2) =$ M.I. about ox

$B = \sum m(z^2 + x^2) =$ M.I. about oy

$C = \sum m(x^2 + y^2) =$ M.I. about oz

Similarly, M.I. about origin (about a point) is denoted by H and is defined as $H = \sum m(OP)^2 = \sum m(x^2 + y^2 + z^2)$

Moment of Inertia of body about yz plane, zx plane and xy planes, are defined by $A' = \sum mx^2$, $B' = \sum my^2$, $C' = \sum mz^2$ respectively.

Important result about a Plane lamina:

The Moment of Inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of the M.I.'s about any two perpendicular axes in the plane which intersects the first axis. Let the plane of the lamina be xy-plane (as is Fig. 1.1) so that $z=0$ for all points lying on the plane. Any axis perpendicular to the plane be chosen as axis of z.

Then M.I. about z-axis is $= C = \sum m\left(x^2+y^2\right)$

M.I. about x-axis is $A = \sum m\left(y^2+z^2\right) = \sum m\left(y^2+0\right) = \sum m\, y^2$

M.I. about y-axis is $B = \sum m\left(z^2+x^2\right) = \sum m\, x^2$

then clearly $C = A+B$

Product of Inertia :

Definition : : Let m be the mass of elementary particle of a body situated at the point (x, y) referred to two mutually perpendicular lines ox and oy, then $\sum mxy$ is called the product of Inertia (P.I.) of the body with respect to the lines ox and oy.

Symbols : If a system of rectangular axes ox, oy, oz be taken in space and if coordinates of any element of mass m of the body referred to these axes be (x, y, z), then the quantities $\sum myz, \sum mzx$ and $\sum mxy$ are called the **Products of Inertia** with respect to the pair of axes (oy, oz), (oz, ox) and (ox, oy) respectively, and are usually denoted by D, E and F respectively, thus

$$D = \sum myz, \quad E = \sum mzx, \quad F = \sum mxy$$

Product of Inertia of any particle may be positive, negative or zero. For example, if the system consists of only two particles, each of mass m, at the points $(2, 1, -4)$ and $(3, 2, 2)$

then $\quad D = m(-4 \times 1) + m(2 \times 2) = -4m + 4m = 0$

$$E = m(2 \times -4) + m(3 \times 2) = -8m + 6m = -2m$$

$$F = m(2 \times 1) + m(3 \times 2) = 2m + 6m = 8m$$

Moment of Inertia in some Standard Cases :

1. M.I. of a uniform rod of length $2a$ and mass M, about an axis through an extermity and perpendicular to it is $= \frac{4}{3} Ma^2$

i.e., M.I. of rod AB about line LAL' is $\frac{4}{3} Ma^2$

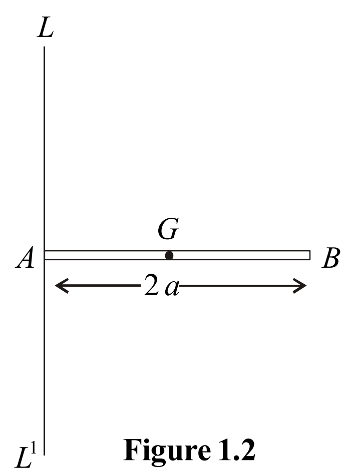


Figure 1.2

2. M.I. of a uniform rod of length $2a$ and mass M about an axis through the mid point and perpendicular to it is $= \frac{1}{3} M a^2$

i.e. M.I. of rod AB about line LGL' is $\frac{1}{3} M a^2$

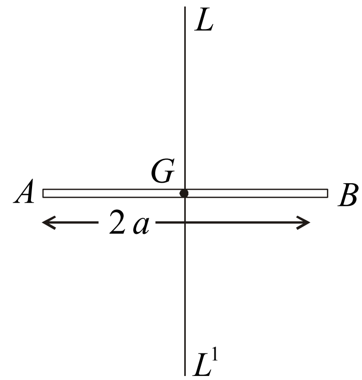


Figure 1.3

3. M.I. of a uniform rod of length $2a$ and mass M about an axis through one extremity and making an angle θ with the rod is $= \frac{4}{3} M a^2 \sin^2 \theta$

i.e. M.I. of rod AB about line LAL' $\frac{4}{3} M a^2 \sin^2 \theta$

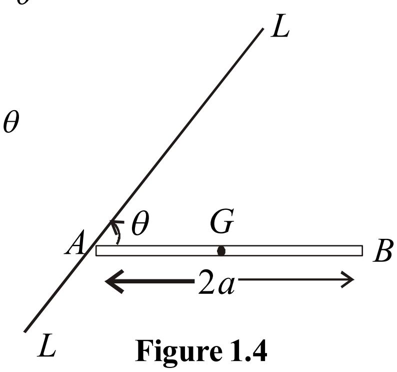


Figure 1.4

4. M.I. of a uniform rectangular lamina of mass M and sides of length $2a$ and $2b$ about a line through its centre and

(i) Parallel to side $2a = \frac{1}{3} M b^2$

i.e. M.I. of lamina $ABCD$

about $ox = \frac{1}{3} M b^2$

(ii) Parallel to side $2b = \frac{M}{3} a^2$

i.e. M.I. of lamina $ABCD$ about $oy = \frac{M}{3} a^2$

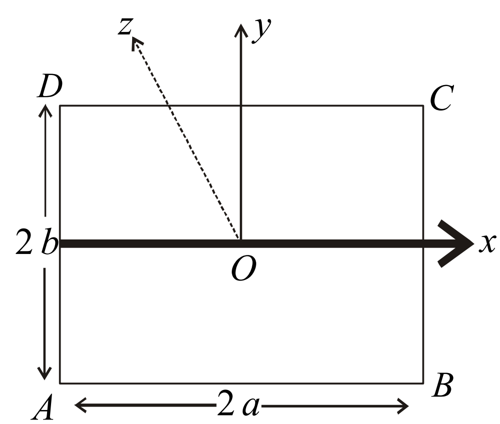


Figure 1.5

(iii) Perpendicular to its plane $= \frac{1}{3} M \left(a^2 + b^2\right)$

i.e. M.I. of lamina $ABCD$ about $oz = \frac{1}{3} M \left(a^2 + b^2\right)$

5. M.I. of a rectangular parallelopiped of mass M and length of edges $2a, 2b, 2c$ about an axis through the centre and parallel to

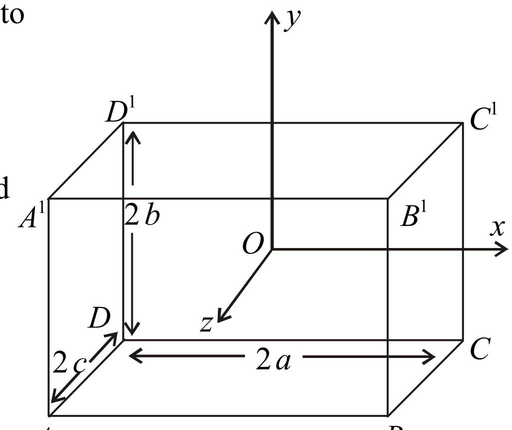


Figure 1.6

(i) edge of length $2a = \frac{M}{3}(b^2 + c^2)$

i.e. M.I. of rectangular parallelopiped

about $ox = \frac{M}{3}(b^2 + c^2)$

(ii) edge of length $2b = \frac{M}{3}(a^2 + c^2)$

i.e. M.I. of rectangular parallelopiped about $o\,y = \frac{M}{3}(a^2 + c^2)$

(iii) edge of length $2c = \frac{M}{3}(a^2 + b^2)$

i.e. M.I. of rectangular parallelopiped about $o\,z = \frac{M}{3}(a^2 + b^2)$

6. M.I. of a circular ring of mass M and radius a about

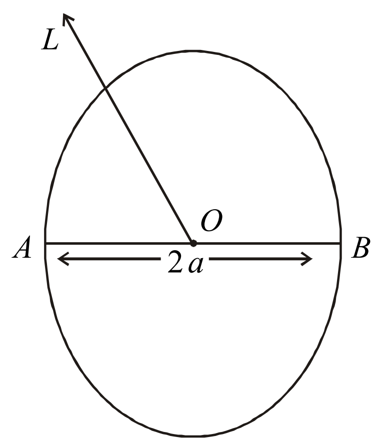


Figure 1.7

(i) a diameter is $= \frac{1}{2} M a^2$

i.e. M.I. about diameter AB

$= \frac{M a^2}{2}$

(ii) an axis through centre perpendicular

to its plane is $= M a^2$

i.e. M.I. of circular ring about $OL = M a^2$

7. M.I. of a circular disc of mass M and radius a about

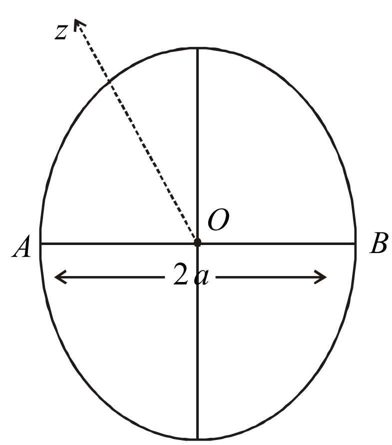


Figure 1.8

(i) a diameter is $= \frac{1}{4} M a^2$

i.e. M.I. of circular disc

about diameter $AB = \frac{M}{4} a^2$

(ii) about an axis through the centre and perpendicular

to its plane $= \dfrac{M a^2}{2}$

i.e. M.I. of circular disc about oz is $= \dfrac{M a^2}{2}$

8. M.I. of an elliptic disc of mass M and major and minor axes of length $2a$ and $2b$ about

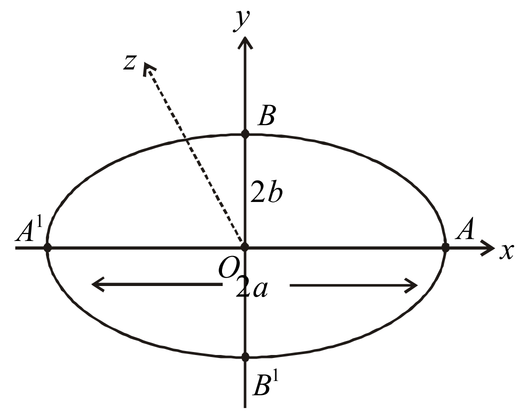


Figure 1.9

(i) Major axis $= \dfrac{M}{4} b^2$

i.e. M.I. about $ox = \dfrac{M}{4} b^2$

(ii) Minor axis $= \dfrac{M}{4} a^2$

i.e. M.I. about $oy = \dfrac{M}{4} a^2$

9. M.I. of a hollow sphere (spherical shell) of mass M and radius a about one of its diameter $= \dfrac{2}{3} M a^2$

i.e. M.I. of hollow sphere about diameter

AB (or CD) is $\dfrac{2}{3} M a^2$

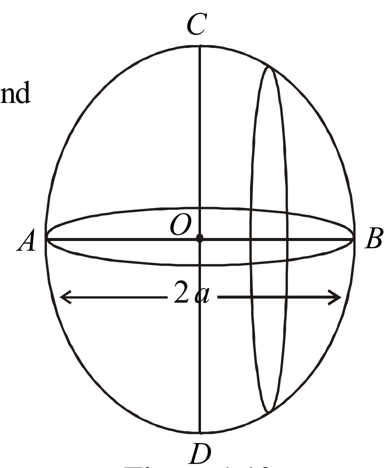


Figure 1.10

10. M.I. of a solid sphere of mass M and radius a about one of its diameter

is $\dfrac{2}{5} M a^2$

i.e. M.I. of solid sphere about diameter $AB = \dfrac{2}{5} M a^2$

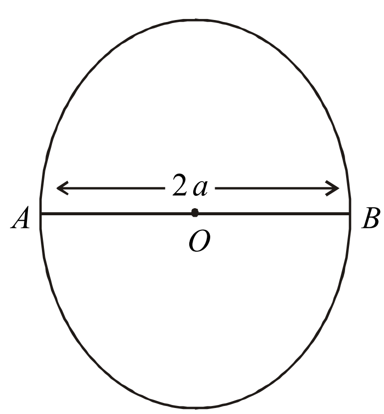


Figure 1.11

11. M.I. of a solid ellipsoid of mass M and length of axes $2a$, $2b$ and $2c$ about

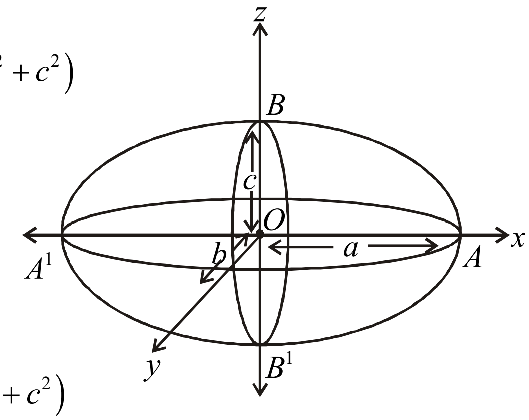


Figure 1.12

(i) The axis of length $2a$ is $= \frac{M}{5}\left(b^2 + c^2\right)$

i.e. M.I. of solid ellipsoid

about $ox = \frac{M}{5}\left(b^2 + c^2\right)$

(ii) the axis of length $2b$ is $= \frac{M}{5}\left(a^2 + c^2\right)$

i.e. M.I. of solid ellipsoid

about $oy = \frac{M}{5}\left(a^2 + c^2\right)$

(iii) the axis of length $2c$ is $= \frac{M}{5}\left(a^2 + b^2\right)$

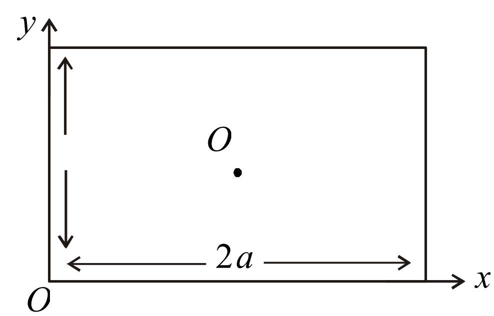
i.e. M.I. of solid ellipsoid

about $oz = \frac{M}{5}\left(a^2 + b^2\right)$

Product of Inertia of some Standard Cases :

1. Product of Inertia of rectangular plate of mass M and length of sides $2a, 2b$

about its sides ox and oy is $= Mab$



Figur 1.13

2. P.I. of an elliptic quadrantal disc of mass M and major and minor axis of lengths $2a$, $2b$ with respect to its axes (major and minor)

ox, oy is $= \frac{Mab}{2\pi}$

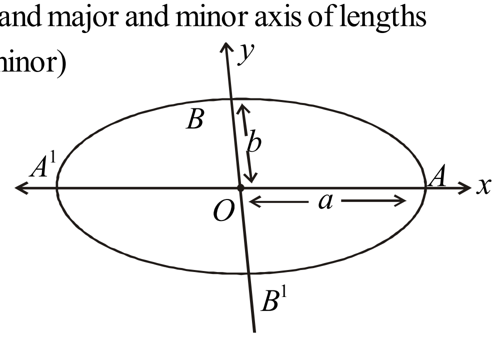


Figure 1.14

3. P.I. of a semi circular disc of mass M and radius a about the diameter (OA) and the tangent at its end (OY) is $= \frac{4}{3\pi} M a^2$

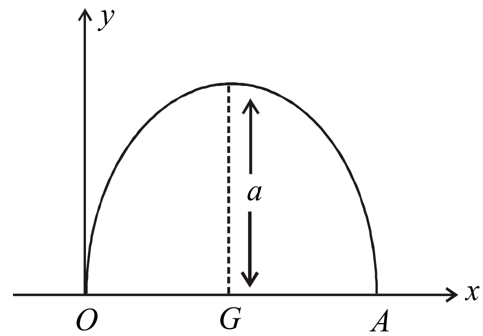


Figure 1.15

4. P.I. of a semi circular wire of mass M and radius a about the diameter and tangent at its extremity $= \frac{2 M a^2}{\pi}$

1.4 Theorem of Parallel Axes

1. For Moment of Inertia : The M.I. of a body about any given axis is equal to the M.I. about the parallel axis through the centre of gravity of the body together with the M.I. of whole mass of the body placed at C.G. taken about the given axis.

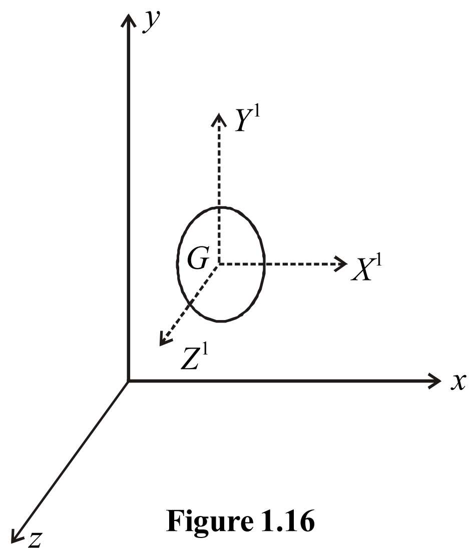


Figure 1.16

i.e. M.I. of body about ox = M.I. about Gx' + M.I. of whole mass placed at G about ox

Similarly for oy and oz.

2. For Product of Inertia : The P.I. of a body with respect to a pair of given rectangular axes is equal to the P.I. with respect to the pair of parallel axes through the C.G. of body together with the P.I. of whole mass placed at C.G. with respect to the given axes.

As in Figure 1.16

P.I. about ox, oy = P.I. about Gx', Gy' + P.I. of mass M placed at G, about ox, oy.

Similarly for oy, oz and oz, ox.

Moment of Inertia about any line in space :

The moment of a body about three mutually perpendicular axes ox, oy, oz at a point o are A, B, C respectively and the Products of Inertia with respect to the pair of axes $oy, oz\ ;\ oz, ox\ ;\ ox, oy$ be D, E, F, then M.I. of body about line POQ, whose direction cosines are l, m, n is

$$Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Flm$$

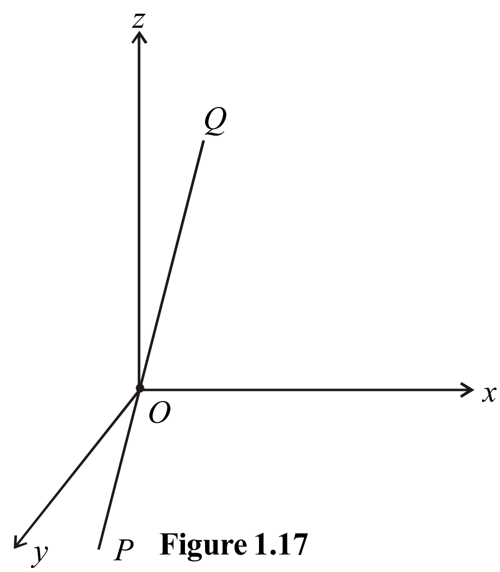


Figure 1.17

Principal Axes and Principal Moments :

It is always possible to find an orientation of the coordinates axes for the body such that all the three products of Inertia vanish simultaneously. Then the three mutually perpendicular axes of coordinates axes are known as principal axes and the correspoinding Moment of Inertia as principal moments.

Self Learning Exercise - I :

1. Is Moment of Inertia of a body is always positive or zero?
2. Is P.I. of a body always positive or zero?
3. Write M.I. of a uniform rod of mass M and length $2a$ about an axis through an extremity and perpendicular to it.
4. Give two examples of impressed forces.

1.5 D'Alembert's Principle

Statement : "Under all circumstances of motion, the reversed effective forces acting on each particle of the body and the external forces form a system in equilibrium among themselves".

Proof : Suppose a rigid body be in motion. At time t, let $\vec{r}$ be position vector of a particle of mass m. It $\vec{F}$ be the resultant of external forces and R be the resultant of internal forces (i.e. mutual actions) acting on it, then by Newton's Second Law

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F} + \vec{R}$$

or $\qquad \vec{F} + \vec{R} - m\dfrac{d^2\vec{r}}{dt^2} = 0 \qquad$ ...(1)

This equation shows that the three forces $\vec{F}$, $\vec{R}$ and $\left(-m\dfrac{d^2r}{dt^2}\right)$ are in equilibrium.

Similarly we can argue for different particles of the rigid body. Hence for the rigid body, we have

$$\sum\vec{F} + \sum\vec{R} + \sum\left(-m\frac{d^2\vec{r}}{dt^2}\right) = 0 \qquad ...(2)$$

But the internal forces i.e. mutual actions of a body are in equilibrium i.e. $\sum\vec{R} = 0$, then from eqn (2), we have

$$\sum\vec{F} + \sum\left(-m\frac{d^2\vec{r}}{dt^2}\right) = 0 \qquad ...(3)$$

where $\sum\left(-m\dfrac{d^2r}{dt^2}\right)$ is reversed effective force

therefore eqn (3) shows that, the reversed effective forces acting at each particle of the system (rigid body) and the external (impressed) forces on the system are in equilibrium.

This proves the D'Alembert's Principle.

Angular Momentum of a System of Prticles : Let $\vec{r}$ be the position vector, of a particle of mass m relative to a point o, then the vector sum H (say),

$H = \sum\vec{r}\times m\vec{v} = \sum m\,\vec{r}\times\vec{v}$ is called angular momentum of the system about the point o, where $\vec{v}$ is the velocity or moment of momentum is called angular momentum.

Centroid of System : Let $\vec{r}$ be the position vector of any particle of mass m of the system (rigid body) relative to a point o, then the point with position vector

$\vec{r} = \dfrac{\sum m\vec{r}}{\sum m}$ is defined as the centroid of the system.

Also, if $\vec{v}$ be the velocity of the particle of mass m and $\vec{V}$ be the velocity of the centroid, then

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{\sum m\dfrac{dr}{dt}}{\sum m} = \frac{\sum m\vec{v}}{\sum m}$$

1.6 General Equations of Motion

To deduce the general equations of motion of a rigid body from D'Alembert's Principle. (when froces are finite)

Let $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$ be the components of external (impressed) forces parallel to the axes of coordinates respectively acting on a particle of mass m whose coordinates are (x, y, z) at any time t.

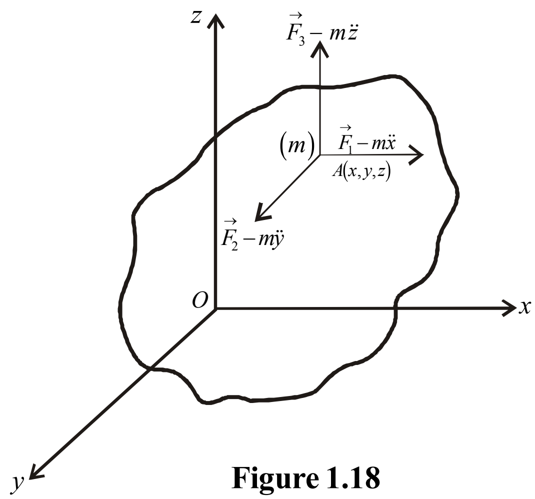


Figure 1.18

Then the force whose components are

$$\left(\vec{F}_1 - m\frac{d^2x}{dt^2}\right) = \left(\vec{F}_1 - m\ddot{x}\right), \left(\vec{F}_2 - m\ddot{y}\right), \text{and} \left(\vec{F}_3 - m\ddot{z}\right)$$ at a point $A\ (x, y, z)$ together with similar forces acting at each point of the body forming a system in equilibrium, thus we get

$$\sum\ (F_1 - m\ddot{x}) = 0,\ \sum\ (F_2 - m\ddot{y}) = 0,\ \sum\ (F_3 - m\ddot{z}) = 0$$

or $$\sum F_1 = \sum m\ddot{x} \qquad ...(1)$$

$$\sum F_2 = \sum m\ddot{y} \qquad ...(2)$$

$$\sum F_3 = \sum m\ddot{z} \qquad ...(3)$$

these equations state that the sum of the components of the effective forces parallel to the coordinate axes, are respectively equal to the components of impressed (external) forces parallel to the same axes.

Refer fig. 1.18 amd taking moment about ox, we get

$$\sum\left[y\ (F_3 - m\ddot{z}) - z\ (F_3 - m\ddot{y})\right] = 0$$

or $$\sum m\ (-\ y\ddot{z} + z\ddot{y}) + \sum(y F_3 - z F_2) = 0$$

or $$\sum m\ (y\ddot{z} - z\ddot{y}) = \sum(y F_3 - z F_2) \qquad ...(4)$$

Similarly, taking moment about $o\,y$ and $o\,z$, we get

$$\sum m\left(z\ddot{x}-x\ddot{z}\right)=\sum\left(zF_1-xF_3\right) \quad ...(5)$$

and $$\sum m\left(x\ddot{y}-y\ddot{x}\right)=\sum\left(xF_2-yF_1\right) \quad ...(6)$$

these equation {(4) to (6)} state that the sum of the moments of the effective forces about the coordinate axes are respectively equal to the sum of the moments of the external forces about the same axes.

The set of equations (1) to (3) and (4) to (6) are the general equations of motion of a rigid body.

Self Learning Exercise - 2

1. Write the statement of D'Alembert's Principle.
2. Define angular momentum.
3. Write three general equations of motion of a rigid body for moments.

1.7 Motion of Centre of Inertia

(Motion of Translation)

To show that the centre of Inertia of a body moves as if all the mass of the body were collected at it and as if all the external forces were acting at it in directions parallel to those in which they act.

Let (x, y, z) be the coordinates of a particle of mass m. Further let $(\bar{x}, \bar{y}, \bar{z})$ be the coordinates of Centre of Inertia of body of mass M, (i.e. $\sum m = M$), then $M\bar{x}=\sum mx$, throughout the motion.

Now we shall find that how the equations of motion (eq^n (1) to (3) of above article) can be simplifed by a proper choice of coordinates. We shall be interested in the resolved parts of momentum and the resolved part of the effective force of a system in any direction. Let the chosen direction be x-axis.

The resolved part of its momentum in x-direction

$$= m\frac{dx}{dt}$$

So that the resolved part of the momentum of whole system

$$= \sum m\frac{dx}{dt}$$

$$\therefore \quad M\frac{d\bar{x}}{dt}=\sum m\frac{dx}{dt} \quad ...(1)$$

Similarly in y and z directions

$$M\frac{d\bar{y}}{dt}=\sum m\frac{dy}{dt} \text{ and } M\frac{d\bar{z}}{dt}=\sum m\frac{dz}{dt} \qquad ...(2)$$

Differentiating (1) and (2) with respect to t, we have

$$M\frac{d^2\bar{x}}{dt^2}=\sum m\frac{d^2x}{dt} \qquad ...(3)$$

$$M\frac{d^2\bar{y}}{dt^2}=\sum m\frac{d^2y}{dt^2} \qquad ...(4)$$

$$M\frac{d^2\bar{z}}{dt^2}=\sum m\frac{d^2z}{dt^2} \qquad ...(5)$$

But from general equation of motion

$$\sum m\frac{d^2x}{dt^2}=\sum F_1,\ \sum m\frac{d^2y}{dt^2}=\sum F_2,\ \sum m\frac{d^2z}{dt^2}=\sum F_3$$

Hence,

$$M\frac{d^2\bar{x}}{dt^2}=\sum F_1 \qquad ...(6)$$

$$M\frac{d^2\bar{y}}{dt^2}=\sum F_2 \qquad ...(7)$$

$$M\frac{d^2\bar{z}}{dt^2}=\sum F_3 \qquad ...(8)$$

Equation (6), (7) and (8) are the equations of motion of a particle of mass M placed at the centre of Inertia acted upon by forces $\sum F_1, \sum F_2, \sum F_3$ parallel and equal to external forces acting at different points of the body. Equations (6) to (8) are called equations of translation motion.

1.8 Motion Relative to the Centre of Inertia

(Motion of Rotation)

To show that the motion of a body about its centre of Inertia is the same as it would be if the centre of Inertia were fixed and the same forces acted on the body.

Let $(\bar{x}, \bar{y}, \bar{z})$ be the coordinates of the centre of gravity G of the body with respect to axes ox, oy and oz. Let P be a particle (mass $=m$) of the body, whose coordinates be (x, y, z). Let coordinats of P with respect to the axes Gx', Gy' and Gz' be (x', y', z'), so that

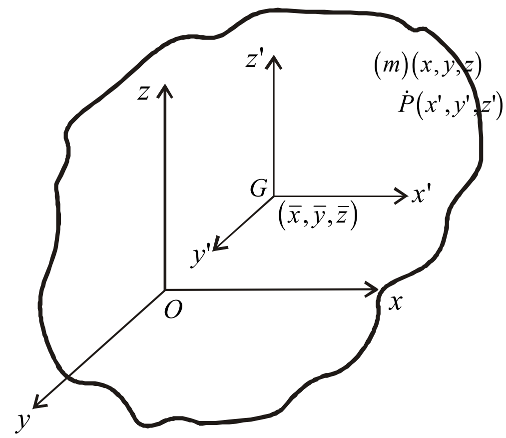


Figure 1.19

$$x = \bar{x} + x', \; y = \bar{y} + y', \; z = \bar{z} + z' \qquad ...(1)$$

On differentiating these twice with respect to t, we get

$$\ddot{x} = \ddot{\bar{x}} + \ddot{x}', \; \ddot{y} = \ddot{\bar{y}} + \ddot{y}', \; \ddot{z} = \ddot{\bar{z}} + \ddot{y}' \qquad ...(2)$$

Now we have [From equation (4) to (6) of article 1.6]

$$\sum m\left(y\ddot{z} - z\ddot{y}\right) = \sum\left(yF_3 - zF_2\right) \qquad \text{etc.} \qquad ...(3)$$

Using values from (2) in eqn (3), we have

$$\sum m\left[\left(\bar{y}+y'\right)\left(\ddot{\bar{z}}+\ddot{z}'\right) - \left(\bar{z}+z'\right)\left(\ddot{\bar{y}}+\ddot{y}'\right)\right] = \sum\left[\left(\bar{y}+y'\right)F_3 - \left(\bar{z}+z'\right)F_2\right]$$

or $$\sum m\left[\bar{y}\ddot{\bar{z}} + \bar{y}\ddot{z}' + y'\ddot{\bar{z}} + y'\ddot{z}' - \bar{z}\ddot{\bar{y}} - \bar{z}\ddot{y}' - z'\ddot{\bar{y}} - z'\ddot{y}'\right]$$

$$= \sum\left[\left(\bar{y}F_3 - \bar{z}F_2\right) + \left(y'F_3 - z'F_2\right)\right]$$

or $$\sum m\left[\left(\bar{y}\ddot{\bar{z}} - \bar{z}\ddot{\bar{y}}\right) + \left(y'\ddot{z}' - z'\ddot{y}'\right) + \left(\bar{y}\ddot{z}' - \bar{z}\ddot{y}' + y'\ddot{\bar{z}} - z'\ddot{\bar{y}}\right)\right]$$

$$= \sum\left(\bar{y}F_3 - \bar{z}F_2\right) + \sum\left(y'F_3 - z'F_2\right)$$

or $$\sum m\left(\bar{y}\ddot{\bar{z}} - \bar{z}\ddot{\bar{y}}\right) + \sum m\left(y'\ddot{z}' - z'\ddot{y}'\right) + \sum m\left(\bar{y}\ddot{z}' - \bar{z}\ddot{y}' + y'\ddot{\bar{z}} - z'\ddot{\bar{y}}\right)$$

$$= \sum\left(\bar{y}F_3 - \bar{z}F_2\right) + \sum\left(y'F_3 - z'F_2\right) \qquad ...(4)$$

Where $\sum m = M =$ whole mass of the body

or $$M\left(\bar{y}\ddot{\bar{z}} - \bar{z}\ddot{\bar{y}}\right) + \sum m\left(y'\ddot{z}' - z'\ddot{y}'\right) + \bar{y}\left(\sum m\ddot{z}'\right) - \bar{z}\sum m\ddot{y}'$$

$$+ \ddot{\bar{z}}\left(\sum m y'\right) - \ddot{\bar{y}}\left(\sum m z'\right) = \sum\left(\bar{y}F_3 - \bar{z}F_2\right) + \sum\left(y'F_3 - z'F_2\right) \qquad ...(5)$$

Since coordinates of G with respect to Gx', Gy' and Gz' are (0, 0, 0), so we have

$$\frac{\sum m x'}{\sum m} = 0 \Rightarrow \sum m x' = 0 \qquad ...(6)$$

$$\frac{\sum m y'}{\sum m} = 0 \Rightarrow \sum m y' = 0 \qquad ...(7)$$

$$\frac{\sum m z'}{\sum m} = 0 \Rightarrow \sum m z' = 0 \qquad ...(8)$$

also on differentiating these equations twice with respect to t

$$\sum m \ddot{x}' = 0 \qquad ...(9)$$

$$\sum m \ddot{y}^1 = 0 \qquad ...(10)$$

$$\sum m \ddot{z}^1 = 0 \qquad ...(11)$$

Further from equation (6) to (8) [art 1.7]

$$M \ddot{\bar{x}} = \sum F_1 \qquad ...(12)$$

$$M \ddot{\bar{y}} = \sum F_2 \qquad ...(13)$$

$$M \ddot{\bar{z}} = \sum F_3 \qquad ...(14)$$

multiplying eqn (13) by $\bar{z}$ and eqn (14) by $\bar{y}$ and subtracting, we have

$$M \left(\bar{y}\ddot{\bar{z}} - \bar{z}\ddot{\bar{y}}\right) = \sum \left(\bar{y} F_3 - \bar{z} F_2\right) \qquad ...(15)$$

Now on using eqn (7), (8); (10), (11) and (15) in eqn (5) we have

$$\sum \left(\bar{y} F_3 - \bar{z} F_2\right) + \sum m \left(y' \ddot{z}' - z' \ddot{y}'\right) + \bar{y} \times 0 - \bar{z} \times 0 + \ddot{\bar{z}} \times 0$$

$$- \bar{y} \times 0 = \sum \left(\bar{y} F_3 - \bar{z} F_2\right) + \sum \left(y' F_3 - z' F_2\right)$$

or $$\sum m \left(y' \ddot{z}' - z' \ddot{y}'\right) = \sum \left(y' F_3 - z' F_2\right) \qquad ...(16)$$

Similarly two more equations can be obtained as

$$\sum m\left(z' \ddot{x}' - x' \ddot{z}'\right) = \sum \left(z' F_1 - x' F_3\right) \qquad ...(17)$$

and $$\sum m\left(x' \ddot{y}' - y' \ddot{x}'\right) = \sum \left(x' F_2 - y' F_1\right) \qquad ...(18)$$

These eqn [(16) to (18)] would have been obtained if we had taken G as origin. These are equations of rotational motion of the rigid body about a fixed point.

Remark :

In the above two articles we have shown that motion of translation and the motion of rotation can be considered independently. So these two articles prove the independence of translation and rotation.

Illustrative Examples :

Example - 1 : A cricular board is placed on a smooth horizontal plane, and a boy runs round the edge of it at a uniform rate, what is the motion of the board?

Solution : Let mass of circular board be M and mass of boy be m.

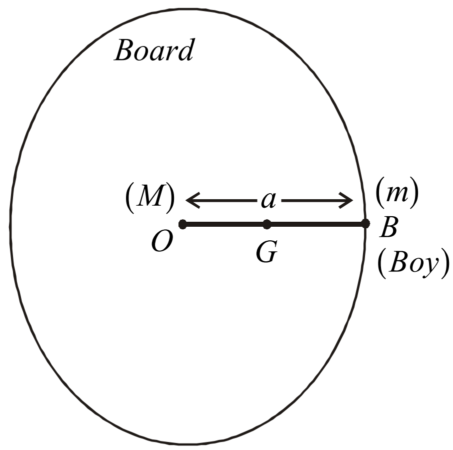


Figure 1.20

Let G be centre of Inertia of the system. Let a be radius of board. Now there is no external force acting on the system in the horizontal direction so by D'Alembert's Principle the centre of Inertia G of the system will remain at rest. Let x be distance of $C.G.$ G from point o, than

$$OG = x = \frac{M.o + m.a}{M+m} = \frac{ma}{M+m},\ a \text{ constant}$$

Thus we observe that G is fixed and OG is constant so that the centre of board o describes a circle with G as centre and OG radius.

Example - 2 : A uniform rod OA, of length $2a$, free to turn about its end 0, revolves with uniform angular velocity w about its vertical oz through o, and is inclined at a constant angle α to oz, show that the value of α is either zero or $\cos^{-1}\left(\frac{3g}{aw^2}\right)$.

Solution : OA is rod of length $2a$ and oz be vertical. Consider an element $PQ = \delta x$ at a distance

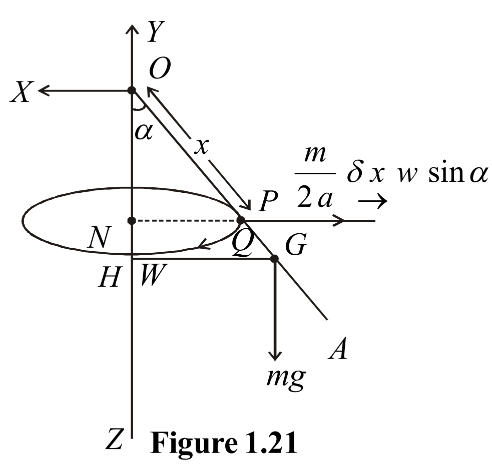


Figure 1.21

$x\ (= oP)$ from 0, whose mass will be $\frac{m}{2a}\delta x$. Draw PN perpendicular to oz. Then element PQ describes a circle of radius PN about N. Hence the only effective force on the element PQ is $\frac{m}{2a}\delta x\, w^2 x \sin\alpha$ along PN (by formula mrw^2). Hence reversed effective force is $\left(\frac{m}{2a}\delta x\, w^2\, x \sin\alpha\right)$ along NP. The only external forces acting on the rod are its weight mg and reaction at o. By D'Alembert's Principle the reversed effective forces and external forces are in equilibrium. To avoid the reactions, we take moments about the line through o, perpendicular to the plane of the figure, we have

$$mg\,(a\sin\alpha) = \sum\left(\frac{m}{2a}\delta x\, w^2\, x \sin\alpha\right)(x\cos\alpha)$$

[$\because$ in ΔONP, $ON = x\cos\alpha$ and in ΔOHG, $HG = a\sin\alpha$]

or $$mga\sin\alpha = \frac{mw^2}{2a}\sin\alpha\cos\alpha\int_0^{2a} x^2 dx$$

$$= \frac{mw^2\ \sin\alpha\cos\alpha}{2a}\cdot\left(\frac{8a^3}{3}\right)$$

$$= \frac{4a^2 w^2\ \sin\alpha\cos\alpha}{3}$$

or $$mga\sin\alpha\left(1-\frac{4w^2a}{3g}\cos\alpha\right) = 0$$

above gives us either $\sin\alpha = 0$, i.e. $\alpha = 0$

or $$1-\frac{4w^2a}{3g}\cos\alpha = 0 \Rightarrow \cos\alpha = \frac{3g}{4aw^2} \Rightarrow \alpha = \cos^{-1}\left(\frac{3g}{4aw^2}\right)$$

Remark : If $w^2 < \frac{3g}{4a}$, then $\frac{3g}{4aw^2} > 1 \Rightarrow \cos\alpha > 1$ which is not possible. Hence in this case $\alpha = 0$ only will be possible.

Example - 3 : A rod, of length $2a$ is suspended by a string, of length l, attached to one end; if the string and rod revolve about the vertical with uniform angular velocity, and their inclinations to the vertical be θ and ϕ respectively, show that

$$\frac{3l}{a} = \frac{(4\tan\theta - 3\tan\phi)\sin\phi}{(\tan\phi - \tan\theta)\sin\theta}$$

Solution : Suppose a rod AB of length $2a$ and mass m be suspended by a string OA of length l.

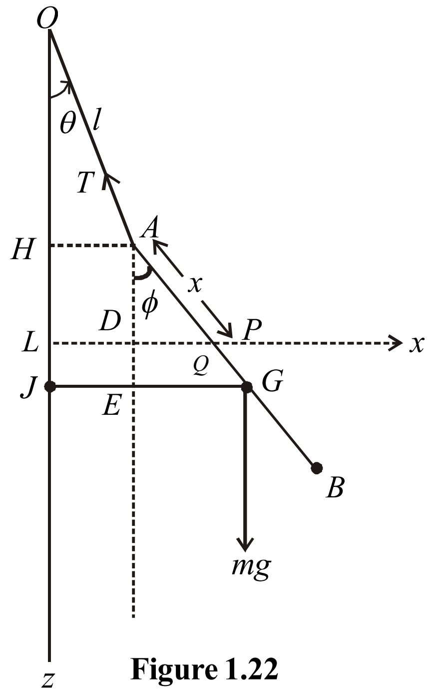


Figure 1.22

OZ be vertical through O. Consider an element $PQ\ (= \delta x)$ of rod AB at a distance x from end A. Then from the fig. 1.22, $OH = l\cos\theta$, $HA = l\sin\theta$

$$\left.\begin{aligned} LP &= LD + DP \\ &= HA + DP = l\sin\theta + x\sin\phi \\ EG &= a\sin\phi,\ AE = a\cos\phi \\ HJ &= AE, OJ = OH + HJ = l\cos\theta + a\cos\phi \end{aligned}\right\} \quad ...(1)$$

If the rod revolves with angular velocity (say) w, then by dynamics of particle, the reversed effective force on P is

$$\frac{m}{2a}\,\delta x\, w^2\ LP = \frac{m}{2a}\,\delta x\, w^2\,(l\sin\theta + x\sin\phi) \text{ along } LP \quad ...(2)$$

The external forces on the rod are tension T (in string OA) and weight mg acting at G.

To avoid tension T, we take moments about lines through A and point O, perpendicular to the plane of the figure respectively.

First taking moment about A, we get

$$m g a\sin\phi = \sum \frac{m}{2a}\,\delta x\, w^2\,(l\sin\theta + x\sin\phi)\ .\ x\cos\phi$$

$$= \int_0^{2a} \frac{m}{2a}\, w^2\,(l\sin\theta + x\sin\phi)\, x\cos\phi\, dx$$

or $$g a\sin\phi = w^2\left(a\, l\sin\theta\,\cos\phi + \frac{4a^2}{3}\sin\phi\,\cos\phi\right) \quad ...(3)$$

Also by taking moment about point o, we get

$$mg\left(a\sin\phi + l\sin\theta\right) = \int_0^{2a} \frac{mw^2}{2a}\left(l\sin\theta + x\sin\phi\right)\left(l\cos\theta + x\cos\phi\right)dx$$

$$= \int_0^{2a} \frac{mw^2}{2a}\left[l^2\sin\theta\cos\theta + lx\sin\phi\cos\theta + lx\sin\theta\cos\phi + x^2\sin\phi\cos\phi\right]dx$$

$$= \frac{mw^2}{2a}\left[l^2\sin\theta\cos\theta\,(x)_0^{2a} + l\sin\phi\cos\theta\left(\frac{x^2}{2}\right)_0^{2a}\right.$$

$$\left. + l\sin\theta\cos\phi\left(\frac{x^2}{2}\right)_0^{2a} + \sin\phi\cos\phi\left(\frac{x^3}{3}\right)_0^{2a}\right]$$

or $$g\left(a\sin\phi + l\sin\theta\right) = w^2\left(l^2\sin\theta\cos\theta + al\cos\theta\sin\phi\right)$$

$$+ w^2\left(al\sin\theta\cos\phi + \frac{4a^2}{3}\sin\phi\cos\phi\right) \qquad ...(4)$$

Subtracting (3) from (4), we get

$$gl\sin\theta = w^2\left(l^2\sin\theta\cos\theta + al\cos\theta\sin\phi\right) \qquad ...(5)$$

Now, dividng eqn (3) by eqn (5), we get

$$\frac{a\sin\phi}{l\sin\theta} = \frac{al\sin\theta\sin\phi + \frac{4a^2}{3}\sin\phi\cos\phi}{l^2\sin\theta\cos\theta + al\cos\theta\sin\phi} = \frac{\left(al\sin\theta + \frac{4a^2}{3}\sin\phi\right)\cos\phi}{\left(l^2\sin\theta + al\sin\phi\right)\cos\theta}$$

$$\frac{\sin\phi}{\sin\theta} = \frac{\left(l\sin\theta + \frac{4a}{3}\sin\phi\right)\cos\phi}{\left(l\sin\theta + a\sin\phi\right)\cos\theta}$$

$$\Rightarrow \sin\phi\cos\theta\left(l\sin\theta + a\sin\phi\right) = \left(l\sin\theta + \frac{4a}{3}\sin\phi\right)\sin\theta\cos\phi$$

$$l\left(\sin\phi\cos\theta - \sin\theta\cos\phi\right)\sin\theta = \frac{a}{3}\left(4\sin\theta\cos\phi - 3\sin\phi\cos\theta\right)\sin\phi$$

on multiplying by 3 and dividing by $\cos\theta\cos\phi$, we get

$$3l\left(\tan\phi - \tan\theta\right)\sin\theta = a\left(4\tan\theta - 3\tan\phi\right)\sin\phi$$

$$\Rightarrow \frac{3l}{a}\left(\frac{4\tan\theta - 3\tan\phi}{\tan\phi - \tan\theta}\right)\frac{\sin\phi}{\sin\theta},$$ which is the required result.

Example 4 : A thin heavy disc can turn freely, about an axis in its own plane, and this revolves horizontalwith a uniform angular velocity w about a fixed point on itself. Show that the inclination θ of the plane of the disc to the vertical is $\cos^{-1}\left(\frac{gh}{k^2w^2}\right)$, where h is the distance of the Center of Inertia of the disc from the axis and k is the radius of gyration of the disc about the axis.

If $w^2 < \frac{gh}{k^2}$ then the plane of the disc is vertical.

Solution : Let OA is the horizontal axis (Fig - 1.23) in the plane of the disc and AZ be vertical

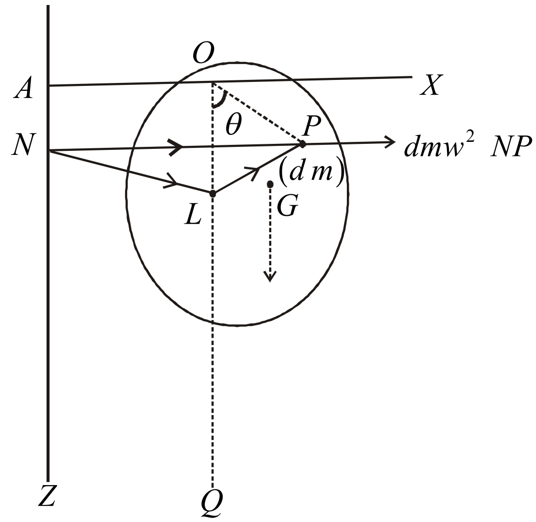


Figure 1.23

line through A. Let OQ be vertical line through O. When the axis OA revolves horizontally about O, the plane of disc will be slightly raised and suppose θ is the inclination of the plane of disc to the vertical.

Also distance of Centre of Inertia G from axis OA is given to be h $(i.e.\ OG = h)$ and the plane of disc makes an angle θ with the vertical, so that the distance of Mg from the axis OA is $h\sin\theta$.

Consider an element dm at point P and draw PN and PL perpendiculars to the verticals through A and O (PO is perpendicular from P on the axis OA). When the axis AO revolves horizontally about O, the element dm describes a cricle of radius PN about AZ with N as centre. The reversed effective force on element dm is $dm w^2 NP$ along NP. Applying triangle law of forces, this force can be thought of as equivalent to two components forces $dm w^2 NL$ along NL and $dm w^2 LP$ along LP.

The force $dm w^2 NL$ along NL is parallel to line OA and as such its moment about OA vanishes.

Now, taking moment about OA, we get

$$Mgh\sin\theta = \sum dm w^2 \ LP.OL$$

$$= w^2 \sum dm(OP\sin\theta)(OP\cos\theta)$$

$$= w^2 \sin\theta\cos\theta \sum dm(OP)^2$$

$$= w^2 \sin\theta\cos\theta \left(M k^2\right) \qquad \left[\because \sum dm OP^2 = M k^2\right]$$

where k is radius of gyration of disc about axis.

or $\quad g h \sin\theta = k^2 w^2 \sin\theta\cos\theta$

or $\quad \sin\theta\left(g h - k^2 w^2 \cos\theta\right) = 0$

$\Rightarrow$ *either* $\sin\theta = 0$ i.e. $\theta = 0$

or $\quad \cos\theta = \dfrac{g h}{k^2 w^2}$ or $\theta = \cos^{-1}\left(\dfrac{g h}{k^2 w^2}\right)$

when $w^2 < \dfrac{g h}{k^2} \Rightarrow \dfrac{g h}{k^2 w^2} > 1 \Rightarrow \cos\theta > 1$, which gives an impossible value of θ. In that $\cos\theta = 0$ is the only possible value and thus the disc will remain vertical.

Example 5 : Two uniform spheres, each of mass M and radius a, are firmly fixed to the ends of two unifrom thin rods, each of mass m and length l, and the other ends of the rods are freely hinged at a point O. The whole system revolves, as in the governor of a steam-engine, about a vertical line through O with angular velocity w. Show that when the motion is steady, the rods are inclined to the vertical at an angle θ given by the equation

$$\cos\theta = \frac{g}{w^2} \cdot \frac{M(l+a) + \frac{1}{2} l m}{M(l+a)^2 + \frac{1}{3} m l^2}$$

Solution : Let OA and OB be two rods each of length l and mass m. The spheres, each of radius a and mass M are fixed at the ends A and B. We take an element $PQ\ (= \delta x)$ at a distance x from point O on either rod.

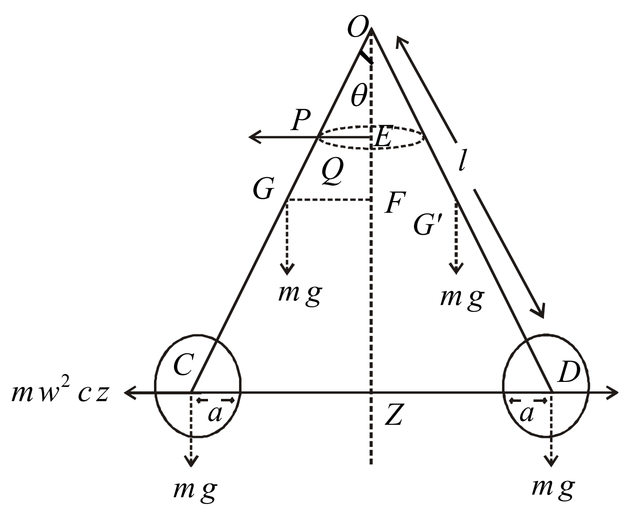


Figure 1.24

Let OZ be vertical through O and PE and CZ be perpendiculars on the vertical OZ. Let C

be centre of one of the spheres. As the whole systems revolves as in the governor of a steam engine about OZ, the element PQ will describes a circle of radius PE with E as centre. Then the reversed effective force on the rod at P is $\left(\frac{m}{l}\,\delta x\, w^2 x \sin\theta\right)$ along EP, where θ is angle which the rod makes with vertical. By similar reasoning the reversed effective force on the sphere of the same side is $M\,w^2\,(ZC)$ along ZC.

Here from the figure, $OP = x$ in ΔOPE, $PE = x\sin\theta$, $OE = x\cos\theta$

$$OG = l/2,\ \text{in } \Delta OGF,\ GF = \frac{l}{2}\sin\theta,\ OF = \frac{l}{2}\cos\theta$$

in ΔOCZ, $OZ = (l+a)\cos\theta$, $CZ = (l+a)\sin\theta$

Keeping the symmetry in consideration and taking moment about an axis through O perpendicular to plane COD, we have

$$M\,w^2\,(l+a)\sin\theta\,.\,(l+a)\cos\theta + \int_o^l \frac{m w^2}{l}\{x\sin\theta\,.\,x\cos\theta\}\,dx$$

$$= M\,g(a+l)\sin\theta + m g\,.\,\frac{l}{2}\sin\theta$$

or $$M\,w^2\,(l+a)^2\sin\theta\cos\theta + \frac{m w^2}{l}\sin\theta\cos\theta\left(\frac{x^3}{3}\right)_o^l$$

$$= M\,g(a+l)\sin\theta + \frac{m g l}{2}\sin\theta$$

or $$\sin\theta\cos\theta\left[M\,w^2\,(l+a)^2 + \frac{m w^2 l^2}{3}\right] = \sin\theta\left[M\,g(a+l) + \frac{m g l}{2}\right]$$

or $$\sin\theta\left[w^2\left\{M\,(a+l)^2 + \frac{ml^2}{3}\right\}\cos\theta - g\left\{M\,(a+l) + \frac{ml}{2}\right\}\right] = 0$$

$\therefore$ either $\sin\theta = 0 \Rightarrow \theta = 0$

or $$w^2\cos\theta\left\{M\,(a+l)^2 + \frac{ml^2}{3}\right\} - g\left\{M\,(a+l) + \frac{ml}{2}\right\} = 0$$

$$\Rightarrow \cos\theta = \frac{g}{w^2}\,.\,\frac{M\,(a+l) + \frac{ml}{2}}{M\,(a+l)^2 + \frac{ml^2}{3}}$$

Example 6 : A plank of mass M is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizontal, and a man of mass M', starting from the upper end, walks down the plank so that is does not move; show that he gets to the other end in time

$$\sqrt{\frac{2M'a}{(M+M')\,g\sin\alpha}}$$

where a is the length of the plan k

Solution : Let ox be horizontal plane and Let AB be the plank of mass M, resting along the line of greatest slope of the inclined plane OL. Let G be $C.G.$ of plank, so weight of plank is acting through G, such that $AG = \frac{a}{2}$.

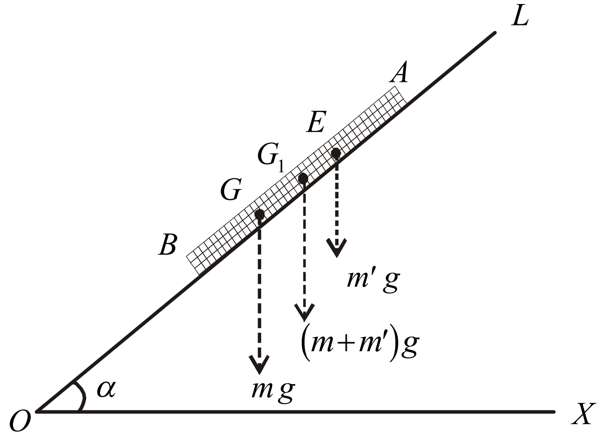


Figure 1.25

Let a man come down a distance x in time t, starting from the upper and A of plank. $AE = x$, E is position of man after a time t, so its weight $M^1 g$ is acting at E. If $\bar{x}$ be the distance ($x = AG$, G, being $C.G.$ of system) of the centre of gravity G_1, of the plank with man on it from end A, then

$$\bar{x} = \frac{\frac{Ma}{2} + M'x}{M+M'}$$

or $$(M+M')\,\bar{x} = \left(\frac{Ma}{2} + M'x\right) \qquad \text{...(1)}$$

Differentiating this relation twice with respect to t, we get

$$(M+M')\,\ddot{\bar{x}} = M'\ddot{x} \qquad \text{...(2)}$$

which gives acceleration of $C.G.$ of the system due to the motion of man.

Now the plank does not move so its $C.G.$ is fixed but man is moving downward, so there is a translation motion. We should consider simply the motion of its $C.G.$, supposing as if all the external forces act on it.

Thus the equation of motion of the $C.G.$ along the plane gives

$$(M+M')\,\ddot{\bar{x}} = Mg\sin\alpha + M'g\sin\alpha \qquad \text{...(3)}$$

Using eqn (2), we get

$$M'\ddot{x} = (M + M')g\sin\alpha$$

or $$\frac{d^2x}{dt^2} = \frac{M + M'}{M'}\; g\sin\alpha$$

Integrating it with respect to t, we get

$$\frac{dx}{dt} = \left(\frac{M + M'}{M'}\right) g\sin\alpha \,.\, t + C_1 \text{, where } C_1 \text{ is constant of integration} \qquad ...(4)$$

but when $t = 0$, $\frac{dx}{dt} = 0$

$\therefore$ $O = O + C_1 \Rightarrow C_1 = O$, then eqn (4) reduces to

$$\frac{dx}{dt} = \frac{M + M'}{M'}\; g\; \sin\alpha\; t$$

again integrating with respect to t, we get

$$x = \frac{M + M'}{M'}\; g\; \sin\alpha\; \frac{t^2}{2} + C_2 \qquad ...(5)$$

but again initially at A, $x = 0$. when $t = 0$, then

$\therefore$ from (5), $$x = \frac{M + M'}{2M'}\; g\; \sin\alpha\; t^2$$

$$\Rightarrow t^2 = \frac{2M'x}{(M + M')\; g\; \sin\alpha} \qquad ...(6)$$

In order to get the time to reach the other end B of plank we put $x = a$ in eqn (6), we get

$$t^2 = \frac{2M'x}{(M + M')\; g\; \sin\alpha} \qquad \Rightarrow t = \sqrt{\frac{2M'x}{(M + M')\; g\; \sin\alpha}}$$

Example 7 : A thin circular disc of mass M and radius a can turn freely about a thin axis OA, which is perpendicular to its plane and passes through a point O of its circumference. The axis OA is compelled to move in a horizontal plane with angular velocity w about its end A. Show that the inclination θ to the vertical of the radius of the disc through O is $\cos^{-1}\left(\frac{g}{aw^2}\right)$ unless $w^2 < \frac{g}{a}$ and then θ is zero.

Solution : Let the plane of the disc be a vertical plane and AZ be vertical line through A and OL

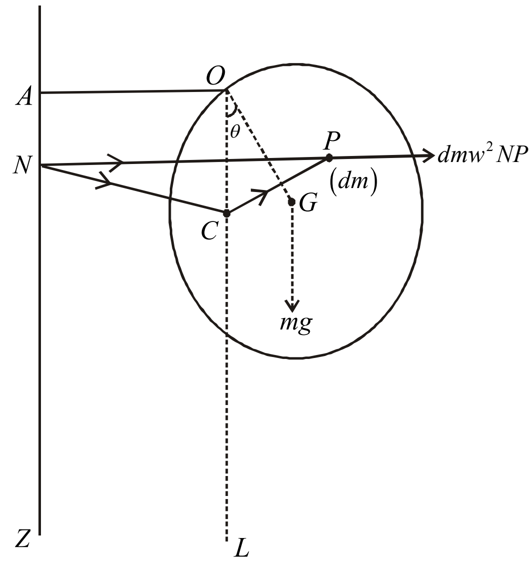


Figure 1.26

be vertical line through O. When the axis OA moves horizontaly round A, the disc will be raised in its vertical plane and suppose that θ is the angle which the radius OG makes with the vertical in this position. Take an element dm at P and draw PN and PC perpendicular to the verticals through A and O respectively. P will describe a circle of radius PN with constant angular velocity w about N as centre. Then the effective force is $d\,m\,w^2 PN$ along PN. The reversed effective force is $d\,m\,w^2 PN$ along NP, which is equivalent to component forces $d\,m\,w^2 NC$ along NC and $d\,m\,w^2 CP$ along CP, by triangle law of forces. But $d\,m\,w^2 NC$ along NC is parallel to OA, so its moment about OA vanishes.

Hence taking moments about OA

$$\sum d\,m\,w^2 CP \,.\, OC = M\,g(a\sin\theta) \qquad (\because OG = a) \qquad ...(1)$$

But $\sum d\,m\;CP\,.\,OC =$ Product of Inertia of the disc with respect to vertical and horizontal lines through O as axes.

$=$ P.I. of the disc about parallel lines through $G + M\,(a\sin\theta)\,(a\cos\theta)$ [By parallel axes theorem]

$$\therefore \qquad \sum d\,m\,CP\,.\,OC = O + M\,a^2\sin\theta\cos\theta \qquad ...(2)$$

Hence on using (2) in (1), we get

$$w^2 M\,a^2\sin\theta\cos\theta = M\,g\,a\sin\theta$$

or $\sin\theta\,(a\,w^2\cos\theta - g) = 0$

which gives $\sin\theta = 0$, i.e., $\theta = 0$

or $a\,w^2\cos\theta - g = 0 \Rightarrow \cos\theta = \left(\dfrac{g}{a\,w^2}\right) \Rightarrow \theta = \cos^{-1}\left(\dfrac{g}{a\,w^2}\right)$

if $w^2 < \dfrac{g}{a} \Rightarrow \left(\dfrac{g}{a\,w^2}\right) > 1 \Rightarrow \cos\theta > 1$. which is not possible, and in that $\cos\theta = o$ is only possibility.

1.9 Summary

In this unit you have studied about M.I. and P.I. of some standard cases, about effective forces, external forces, D'Alembert's Principle, Motion of Centre of Inertia, Motion relative to the centre of Inertia. Some illustrative examples have also been considered.

1.10 Exercise

1. A rough uniform board, of mass m and length $2a$, rests on a smooth horizontal plane and a man of mass M walks on it from one end to the other. Find the distance through which the board moves in this time. $\left(Ans.: \dfrac{2ma}{M+m} \right)$

2. State and prove D'Alembert's Principle.

3. Derive the general equation of motionof a rigid body using D'Alembert's Principle.

4. Derive the equation of translation motion.

5. A rod of length $2a$ revolves with uniform angular velocity w about a vertical axis through a smooth joint at one extremity of the rod so that it describes a cone of semi-vertical angle α, show that

$$w^2 = \frac{3g}{4a\cos\alpha}$$

Prove also that the direction of reaction at the hinge makes with the vertical an angle $\tan^{-1}\left(\frac{3}{4}\tan\alpha\right)$.

Answer to Self learning Exercise

Self Learning Exercise - I

1. Yes (2) No. (3) $\frac{4}{3}Ma^2$

(4) Tension in the string, weight of a body

Self Learning Exercise - II

1. See 1.13 (2) See 1.14

3. $\sum m\left(y\ddot{z} - z\ddot{y}\right) = \sum\left(yF_3 - zF_2\right)$

$$\sum m\left(z\ddot{x} - x\ddot{z}\right) = \sum\left(zF_1 - xF_3\right)$$

$$\sum m\left(x\ddot{y} - y\ddot{x}\right) = \sum\left(xF_2 - yF_1\right)$$

□□□□

UNIT - 2
Motion About a Fixed Axis

Structure of the unit

2.0 Objective

This unit provides a general overview of motion about a fixed axis, moment of momentum, moment of effective forces, Kinetic energy about the axis of rotation. Principle of angular momentum, Principle of conservation of energy, the compound pendulum, centre of percussion of a rigid body. After reading this unit you will be able to understand

1. About moment of momentum of a rigid body about axis of rotation
2. About moment of effective forces about axis of rotation
3. About Kinetic Energy about the axis of rotation
4. About Principle of angular momentum
5. About Principle of conservation of energy
6. About compound pendulum
7. About centre of percussion of a rigid body

2.1 Introduction

We often find the rigid bodies rotating about a fixed axis. In this unit we shall consider the case where some line in the body is fixed and the body rotates about this line considered as a fixed axis. We shall derive expressions for moment of momentum, moment of effective forces, kinetic energy about axis of rotation, equation of motion about the axis of rotation. In the end of the unit we shall study about compound pendulum and centre of percussion of a body.

Some Important results recollected :

In polar coordinates (r,θ), when a particle moves along a curve then at any time t when particle is at $P\ (r,\theta)$

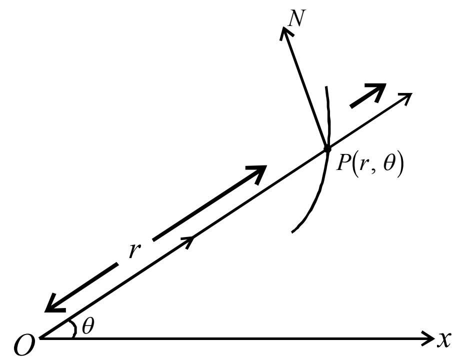


Figure 2.1

(i) Radial acceleration

$= \left(\ddot{r} - r\dot{\theta}^2\right)$, along OP (In direction of r increasing)

where $\dot{r} = \dfrac{dr}{dt}$, $\dot{\theta} = \dfrac{d\theta}{dt}$

(ii) Transverse acceleration $=\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)$, perpendicular to OP (Positive in the sense of θ increasing) i.e. in the direction of PN.

(iii) Radial velocity $= \dot{r}$

(iv) Transverse velocity $= \dot{r}\,\dot{\theta}$

Particular Case :

In case the particle is moving along a circle of radius a, then $r = a$ (Constant) so that

$\dot{r} = 0$, $\ddot{r} = 0$

hence radial acceleration $= \left(-a\dot{\theta}^2\right)$ along OP

$= a\dot{\theta}^2$ along PO

and transverse acceleration $= a\dot{\theta}$ along PN

Radial velocity $= 0$, transverse velocity $= a\dot{\theta}$, along PN

2.2 Moment of effective forces

A rigid body is rotating about a fixed axis, to find the moment of the effective forces about the axis of rotation.

Let us choose OZ as the axis of rotation and a plane $COAB$ be fixed in the space and it may be

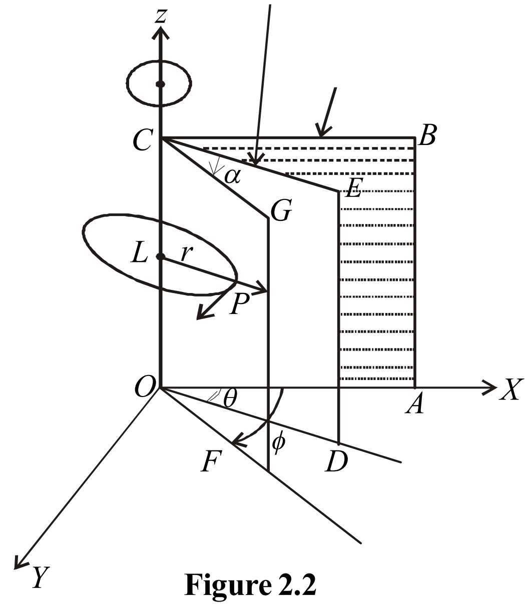


Figure 2.2

taken as the plane of reference. Let any other plane $CODE$ be fixed in the body making an angle θ with the plane $COAB$ i.e. $\angle AOD = \theta$

Consider a particle of mass m of the body at P, and take a plane $COFG$ through P and which makes an angle ϕ with the plane $COAB$, i.e. $\angle AOF = \phi$ Let $\angle ECG = \alpha$, then α is the angle between plane $CODE$ fixed in the body and plane $COFG$. This angle α will remain constant as the body rotates about OZ. From Fig, we have $\phi = \theta + \alpha$. Differentating with respect to t

$$\dot{\phi} = \dot{\theta} + 0 \text{ and also } \quad \ddot{\phi} = \ddot{\theta} \qquad \text{...(1)}$$

i.e. the rate of change of ϕ is same as rate of change of θ.

Let $PL = r$ and point P describes a circle of radius r about L as centre. Hence the acceleration of P along PL and perpendicular to PL are

$$-\left\{-r\left(\frac{d\phi}{dt}\right)^2\right\} = r\frac{d\theta}{dt}, \text{ along } PL \quad [\because -r\dot{\phi} \text{ is along } LP]$$

and $\quad r\dfrac{d^2\phi}{dt^2} = r\dfrac{d^2\theta}{dt^2}$, perpendicular to PL

Hence effective forces on particle P of mass m

are $\quad mr\left(\dfrac{d\theta}{dt}\right)^2$, along PL

and $\quad mr\dfrac{d^2\theta}{dt^2}$, perpendicular to PL.

The effective force $mr\left(\dfrac{d\theta}{dt}\right)^2$ along PL cut OZ at L, so its moment about OZ is zero, whereas the moment of other effective force $mr\,\frac{d^2\theta}{dt^2}$ about OZ is

$$r\,.\left(mr\,\frac{d^2\theta}{dt^2}\right)=mr^2\,\frac{d^2\theta}{dt^2}$$

therefore the moment of all the effective forces on the body

$$=\sum mr^2\,\frac{d^2\theta}{dt^2}=\frac{d^2\theta}{dt^2}\left(\sum mr^2\right),$$

because $\frac{d^2\theta}{dt^2}$ is same for all the particles of the body

$\therefore$ moment of all the effective forces $=\frac{d^2\theta}{dt^2}$ (M.I. of body about OZ)

$=\frac{d^2\theta}{dt^2}\,.\,Mk^2$, where k is the radius of gyration of body about OZ.

Hence Moment of all the effective forces about OZ (axis of rotation) $=Mk^2\,\frac{d^2\theta}{dt^2}$

2.3 Some Important Results

2.3.1 Kinetic Energy of the body about a fixed axis :

Velocity of the particle of mass m at P (See fig.) is $r\,\frac{d\phi}{dt}=r\,\frac{d\theta}{dt}$

$\therefore$ Kinetic energy (K.E.) of particle P of mass m $=\frac{1}{2}\,m\left(r\,\frac{d\theta}{dt}\right)^2$

$\therefore$ Kinetic energy of the whole body $=\sum\frac{1}{2}\,m\,r^2\left(\frac{d\theta}{dt}\right)^2$

$$=\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2\left(\sum mr^2\right)$$

$$=\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2\text{ (M.I. of the body about fixed axis } OZ\text{)}$$

$$=\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2\,.\,Mk^2$$

2.3.2 Moment of Momentum about the axis of rotation : Principle of angular Momentum

Since the velocity of the particle P of mass m (See fig.) is $r\,\dfrac{d\phi}{dt} = r\,\dfrac{d\theta}{dt}$, in a direction perpendicular to PL and $\dot{r}$ is along LP, therefore the momentum of particle of mass m is $m\left(r\,\dfrac{d\theta}{dt}\right)$ and $m\dot{r}$

Hence moment of momentum of particle of mass m about OZ are

$r\,.\left(mr\,\dfrac{d\theta}{dt}\right)$ and $o\,.\left(m\dot{r}\right) = 0$ [because $m\dot{r}$ intersect OZ in L]

so moment of momentum of the whole body about OZ is

$$\sum m\,r^2\,\frac{d\theta}{dt} = \frac{d\theta}{dt}\left(\sum mr^2\right) = \frac{d\theta}{dt}\ \text{(M.I. of body about } OZ\text{)}$$

$$= \frac{d\theta}{dt}\left(M\,k^2\right)$$

$\therefore$ Angular momentum $= M\,k^2\,\dfrac{d\theta}{dt}$

2.3.3 Equation of motion about the axis of rotation :

If a rigid body rotation about a fixed axis then the impressed forces are the external forces and reaction on the axis of rotation. To eliminate the reactions, we take moments about the axis of rotation. By D'Alembert's principle the moment of effective forces is equal to the moment of the impressed forces, about any line. Therefore if L be the moment of impressed forces about the axis of rotation then by the above principle we have

$$M\,k^2\,\frac{d^2\theta}{dt^2} = L \qquad \text{(This is called Principle of angular momentum)}$$

this equation is the differential equation of the motion of the body.

On integrating above equation we will get $\dot{\theta}$ and θ in terms of t. The constant of integrations are evaluated by the known conditions on θ and $\dot{\theta}$ with reference to reference plane fixed in space through OZ.

If somehow L, the moment of external forces (called torque) vanishes, then integration provides $M\,k^2\dot{\theta}$ (angular momentum) as constant. This shows that if moment of external force vanish about an axis, then angular momentum of the body is conserved about that axis, this is known as Principle of conservation of angular momentum.

Remark :

In the case of impulsive forces if w_1, w_2 be the angular velocities of the body just before and just after the action of impulses then the equation of motion is $M\,k^2\left(w_2 - w_1\right) = L$. Where L is the moment of impulses.

2.4 Principle of Conservation of Energy

In general, the Principle of Conservation of Energy is applicable to any system of particles forming a dynamical system under motion due to conservative forces. In our present context, we use it when a rigid body is rotating about a fixed axis.

Kinetic Energy + Potential Energy = Constant

or $$\frac{1}{2} M k^2 \dot{\theta}^2 + V = \text{constant.}$$

where V is the potential energy of the system.

Another Form :

Since change in potential energy is equal to the work done by the conservative forces, so change in K.E. = work done by conservative forces. This equation is used to solve the problems.

Self Learning Exercise - 1

1. In polar coordinates (r, θ), when the particle is moving along a circle of radius a, then give expressions for radial and transverse accelerations.
2. When rigid body is rotating about a fixed axis, then write expression for moment of effective force.
3. Write expression for moment of momentum about axis rotation.

Illustrative Examples :

Example 2.1 : Two unequal masses M and M' rest on two rough planes inclined at angles α and β to the horizon, they are connected by a fine string passing over a small pulley of mass m and radius a, which is placed at the common vertex of two planes, show that the acceleration of either mass is

$$g\left[M(\sin\alpha - \mu\cos\alpha) - M'(\sin\beta + \mu'\cos\beta)\right] \div \left[M + M' + m\frac{k^2}{a^2}\right]$$

where μ and μ' are the coefficients of friction of two planes, k is the radius of gyration of the pulley about its axis and mass M moves downwards.

Solution : Let PQ be horizontal plane and PC and QC are two rough planes inclined at angles α and β to PQ, respectively.

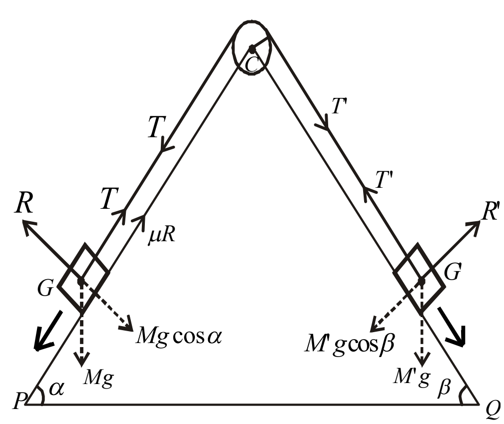


Figure 2.3

Let mass M descends along plane CP a distance (say) x so that the distance moved by mass M^1 on plane QC (up the plane) is also x, in time t (say). If during this time the pulley with centre C has rotated through an angle θ about its axis, then $x = a\,\theta$ ...(1)

then differentiating (1) with respect to t

$$\dot{x} = a\,\dot{\theta} \qquad ...(2)$$

and $$\ddot{x} = a\,\ddot{\theta} \qquad ...(3)$$

Then the equation of motion of pulley $\left(M\,k^2\,\ddot{\theta} = L\right)$ is

$$M\,k^2\,\ddot{\theta} = T\,.\,a - T'\,.\,a \qquad ...(4)$$

where a is radius of pulley and T and T' are tensions in the part of strings GC and $G'C$ $(T > T')$

or $$M\,k^2\,\frac{\ddot{x}}{a^2} = T - T' \qquad ...(5)$$

Also equations of motion for masses M and M' are

$$M\,\ddot{x} = M\,g\sin\alpha - \mu\,R - T$$

or $$M\,\ddot{x} = M\,g\sin\alpha - \mu\,.\,M\,g\,\cos\alpha - T \qquad (\because R = M\,g\cos\alpha) \qquad ...(6)$$

and $$M'\,\ddot{x} = -\,M'\,g\sin\beta - \mu'\,R' + T'$$

or $$M'\,\ddot{x} = -\,M'\,g\sin\beta - \mu'\,M'\,g\cos\beta + T' \qquad (\because R' = M'\,g\cos\beta) \qquad ...(7)$$

Adding equations (5), (6) and (7), we get

$$m\,\frac{k^2}{a^2}\,\ddot{x} + M\,\ddot{x} + M'\,\ddot{x} = (T - T') + (M\,g\sin\alpha - \mu\;M\,g\cos\alpha - T)$$

$$+\,(-\,M'\,g\sin\beta - \mu'\,M'\,g\cos\beta + T')$$

or $$\ddot{x}\left(m\,\frac{k^2}{a^2}\;M + M^1\right) = g\left[M\,(\sin\alpha - \mu\,\cos\alpha) - M'\,(\sin\beta + \mu'\,\cos\beta)\right]$$

or $$\ddot{x} = \frac{g\left[M\,(\sin\alpha - \mu\,\cos\alpha) - M'\,(\sin\beta + \mu'\,\cos\beta)\right]}{\left(m\,\frac{k^2}{a^2}\;M + M'\right)}$$

which is the acceleration of either mass.

Example 2.2 : A uniform rod, of mass m and length $2\,a$, can turn freely about one end which is fixed, it is started with angular velocty w from the position in which it hangs vertically, find its angular velocity at any instant.

If w be such that the angular velocity vanishes when $\theta = \pi$, then prove that the time of describing an angle θ is

$$2\sqrt{\left(\frac{a}{3g}\right)}\log\tan\left(\frac{\pi}{4}+\frac{\theta}{4}\right)$$

Solution : Let OA be a rod of length $2a$ and mass m (See fig.) and G be its Centre of Inertia $\therefore OG = a$; the vertical line OLZ is the line fixed in the space and OA is the line fixed in the body. Here the axis of rotation is the horizontal line through O.

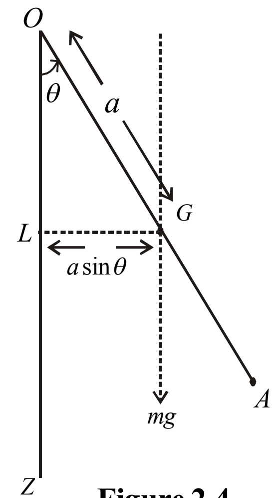


Figure 2.4

Initially the rod is in vertical position $\left(t = 0,\ \theta = 0,\ \dot{\theta} = w\right)$ and let after time t, it makes an angle θ with it. The external forces acting on the body are its weight mg and reaction at O. In $\Delta\, OLG$

$$LG = a\sin\theta,\ OL = a\cos\theta$$

Now taking moment about O, we have (by eqn $M k^2\ddot{\theta} = L$)

$$M k^2\ddot{\theta} = -\,m g a\sin\theta$$

or $$\frac{4}{3}a^2\ddot{\theta} = -\,g a\sin\theta \qquad \left(\because k^2 = \frac{4}{3}a^2 \textit{ for the rod}\right)$$

or $$\ddot{\theta} = -\frac{3g}{4a}\sin\theta \qquad \text{... (1)}$$

Multiplying both sides by $\dfrac{2d\theta}{dt}$ and integrating, we get

$$\int 2\dot{\theta}\ddot{\theta}\,dt = -\int\frac{3g}{4a}\sin\theta\times\frac{2d\theta}{dt}\,.\,dt + C_1$$

or $$\dot{\theta}^2 = -\frac{3g}{2a}\left(-\cos\theta\right) + C_1 \qquad \text{...(2)}$$

Initially, when $\theta = 0$, $\frac{d\theta}{dt} = w$, then from (2)

$$w^2 = \frac{3g}{2a} + C_1 \Rightarrow C_1 = \left(w^2 - \frac{3g}{2a}\right),$$

$$\therefore \quad \dot{\theta}^2 = \frac{3g}{2a}\cos\theta + w^2 - \frac{3g}{2a}$$

or $$\dot{\theta}^2 = w^2 - \frac{3g}{2a}(1 - \cos\theta) \qquad ...(3)$$

this gives the angular velocity of rod at anytime t. If w be such that $\dot{\theta} = 0$ when $\theta = \pi$, then from eqn (3)

$$o = w^2 - \frac{3g}{2a}(1 - \cos\pi) \Rightarrow o = w^2 - \frac{3g}{a}$$

$$\therefore \quad w = \sqrt{\frac{3g}{a}} \qquad ...(4)$$

this is the least value of w for the rod in its lowest position so it just make complelte revolutions. Now using this value of w in (3), we get

$$\dot{\theta}^2 = \frac{3g}{a} - \frac{3g}{2a}(1 - \cos\theta) = \frac{3g}{2a}(1 + \cos\theta) = \frac{3g}{2a} \,.\, 2\cos^2\frac{\theta}{2}$$

$$\therefore \quad \dot{\theta} = \sqrt{\frac{3g}{a}} \,.\, \cos\frac{\theta}{2} \qquad ...(5)$$

Let t be time of describing an angle θ, then on integrating (5), we get

$$\int_0^\theta \frac{d\theta}{\cos\frac{\theta}{2}} = \int_0^t \sqrt{\left(\frac{3g}{a}\right)} \,.\, dt$$

or $$2 \log \tan\left(\frac{\pi}{4} + \frac{\theta}{4}\right) = \sqrt{\frac{3g}{a}} \,.\, (t)$$

or $$t = 2\sqrt{\frac{a}{3g}} \,.\, \log \tan\left(\frac{\pi}{4} + \frac{\theta}{4}\right).$$

Example 2.3: A uniform rod AB is freely movable on a rough inclined plane whose inclination to the horizon is i and whose coefficient of friction is μ, about a smooth pin fixed through the end A; the rod is held in the horizontal position in the plane and allowed to fall from this position. If θ be the angle through which it falls from rest, show that

$$\frac{\sin\theta}{\theta} = \mu \cot i$$

Solution : Let $PQTS$ be a rough inclined plane whose inclination to the horizon is i. Let AO be a

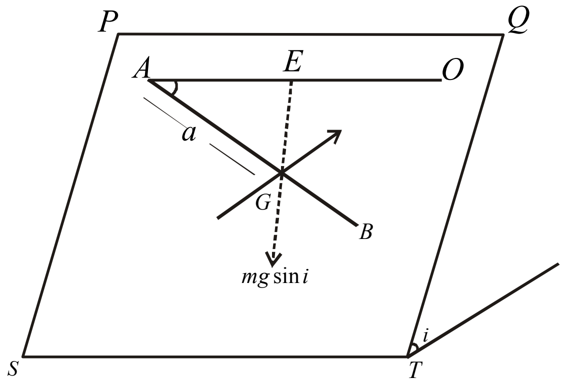


Figure 2.5

rod of mass m and length $2a$, G be its middle point, so $AG = a$ AO is the initial position of rod (when $t = 0, \theta = 0, \dot{\theta} = 0$), it turn about a pin (a fixed axis) at A through an angle θ in time t. AB is its position after lime t, so $\angle OAB = \theta$.

The external forces are mg acting vertically down ward through G, reaction R perpendicular to rod AB, and reaction at A.

The component of mg along the plane is $mg\sin i$ acting through G and $mg\cos i$ perpendicular to the plane (See Fig. 2.5 (a))

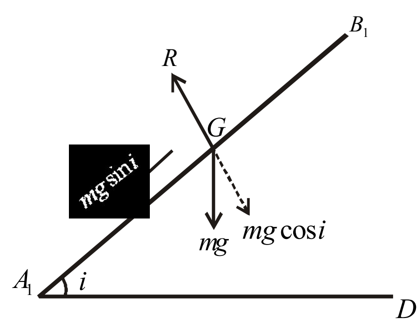


Figure 2.5 (a)

$$\therefore\ R = mg\cos i \qquad ...(1)$$

Force of friction $= \mu R = \mu \, . \, mg \cos i$

In order to avoid reaction at A, we take moment about axis (pin) through

A $\left(by\ equation\ \ M k^2\ddot{\theta} = L\right)$

$$M k^2\ddot{\theta} = mg\sin i\left(AE\right) - \mu\ R\ .\left(AG\right)$$

or $$M k^2\ddot{\theta} = mg\sin i\left(a\cos\theta\right) - \mu\left(mg\cos i\right).\ a$$

or $$k^2\ddot{\theta} = ga\sin i\cos\theta - \mu\ ga\cos i \qquad ...(2)$$

Multiplying both sides by $2\dot{\theta}$ and integrating, we get

$$k^2 \dot{\theta}^2 = 2 a g \sin i \sin\theta - 2 \mu a g \cos i \,\theta + C_1 \qquad ...(3)$$

where C_1, is constant of integration

initially $\theta = 0, \dot{\theta} = 0, t = 0$, then from (3) $C_1 = 0$

$$\therefore \quad k^2 \dot{\theta}^2 = 2 a g \sin i \sin\theta - 2 \mu a g \cos i \,.\, \theta \qquad ...(4)$$

Since the rod turns through an angle θ i.e. it comes to rest after falling an angle θ and hence $\dot{\theta} = 0$, then from (4)

$$o = 2 a g \sin i \sin\theta - 2 \mu a g \cos i \,\theta$$

or $$\frac{\sin\theta}{\theta} = \mu \cot i$$

Example 2.4: The door of a railway carriage stands open at right angles to the length of the train when the latter starts to move with an acceleration f ; the door being supposed to be smoothly hinged to the carriage and to be uniform and of breadth $2a$, show that its angular velocity, when it turned through an angle θ is $\sqrt{\left\{\frac{3f}{2a}\sin\theta\right\}}$

Solution : Let AB be the axis of rotation of the door $ABEF$ of mass M (say). When the train

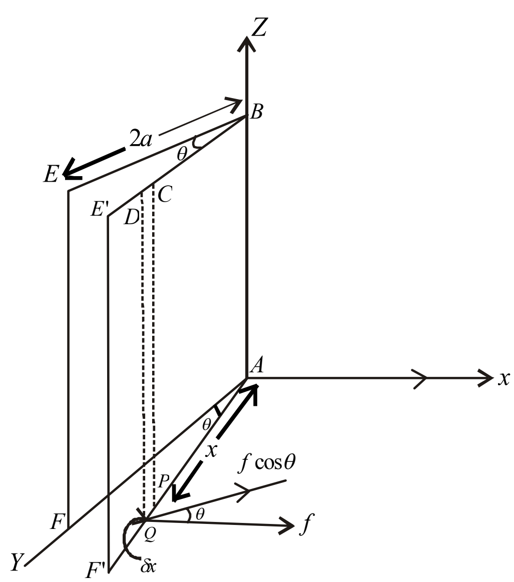


Figure 2.6

moves with an acceleration f in direction Ax (say), then every element of the door $ABEF$ will have the same acceleration f parallel to the rails (i.e. parallel to Ax). Let the door has turned through an angle θ about AB in time t, then component of f perpendicular to the door in its position at time t will be $f\cos\theta$ an shown in the fig.

Consider an elementary strip $PQDC$ of with dx at distance x from A $(AP = x)$

then mass of strip $= \left(\frac{M}{2a} \, dx \right)$

Now, taking moment about AB $\left(by\ eqn\ M k^2 \ddot{\theta} = L \right)$

$$M \frac{4a^2}{3} \ddot{\theta} = \int_0^{2a} \frac{M}{2a} \, dx \, . \, f \cos\theta . x$$

$$= \frac{M}{2a} f \cos\theta \int_0^{2a} x \, dx$$

$$= M a f \cos\theta$$

$\therefore \quad \ddot{\theta} = \frac{3}{4a} f \cos\theta$ ...(1)

Multiplying both sides by $2\dot{\theta}$ and integrating, we get

$$\int 2 \dot{\theta} \ddot{\theta} dt = \int \frac{3f}{4a} \cos\theta \left(2\dot{\theta} \right) dt + C_1$$

or $\quad \dot{\theta}^2 = \frac{3f}{4a} \times 2 \sin\theta + C_1$ ...(2)

but initially $\theta = 0, \dot{\theta} = 0$

$\therefore \quad C_1 = 0$

hence

$$\dot{\theta}^2 = \frac{3f}{2a} \sin\theta \Rightarrow \dot{\theta} = \sqrt{\left\{ \frac{3f}{2a} \sin\theta \right\}}$$

Example 2.5 : A uniform vertical circular plate, of radius a ; is capable of revolving about a smooth horizontal axis through its centre; a rough perfectly flexible chain, whose mass is equal to that of the plate and whose length is equal to its circumference, hangs over its rim in equilibrium, if one end be slightly displaced, show that the velocity of chain, when the end reaches the plate is $\sqrt{\left(\frac{\pi a g}{6} \right)}$.

Solution : Let m be the mass of chain or pulley and let x be the distance moved by the ends A and

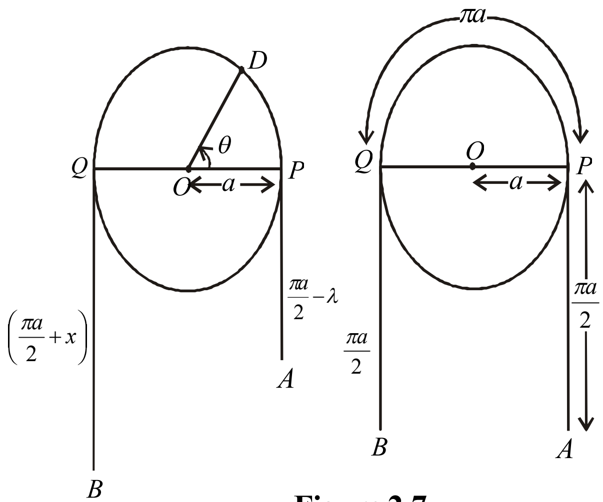


Figure 2.7

B in time t, during which pulley has turned through an angle θ, so by $(arc) = (radius) \times (angle)$

$$\left.\begin{aligned} x &= a\theta \\ \therefore \quad \dot{x} &= a\dot{\theta} \end{aligned}\right\} \qquad \text{...(1)}$$

$$\text{K.E. of the chain} = \frac{1}{2} m\dot{x}^2 \qquad \text{...(2)}$$

$$\text{K.E. of pulley} = \frac{1}{2} mk^2\dot{\theta}^2 = \frac{1}{2} m.\frac{a^2}{2}.\frac{\dot{x}^2}{a^2} = \frac{1}{4} m\dot{x}^2 \qquad \text{...(3)}$$

$$\left(\because k^2 = \frac{a^2}{2} \text{ for a circular plate}\right)$$

$$\therefore \quad \text{total K.E.} = \frac{1}{2} m\dot{x}^2 + \frac{1}{4} m\dot{x}^2 = \frac{3}{4} m\dot{x}^2 \qquad \text{...(4)}$$

Now, we shall find the work done by the chain

work done = mg . (distance moved by $C.G.$ of chain)

Let us find the depth of the $C.G.$ of the chain below PQ consisting of three parts PA, PQ and QB, where PQ is in contact with pulley.

If ρ be the weight per unit length of chain, then

$$\text{weight of part of chain } PA = \left(\frac{1}{2}\pi a - x\right)\rho$$

then depth of its $C.G.$ below $PQ = \frac{1}{2}\left(\frac{\pi a}{2} - x\right)$, similarly

$$\text{weight of part } QB \text{ of chain} = \left(\frac{\pi a}{2} + x\right)\rho$$

then depth of its $C.G.$ below $PQ = \frac{1}{2}\left(\frac{\pi a}{2} + x\right)$

Also, weight of semi circular chain PQ (which is in contact with pulley) $= (\pi a)\,\rho$, and height of its centre of gravity above $PQ = \frac{2a}{\pi}$

Then $\left(by \text{ using } \bar{x} = \frac{\sum m x}{\sum m}\right)$ depth of $C.G.$ of whole chain below PQ (say)

$$\bar{x} = \left\{ \frac{\left(\frac{\pi a}{2} - x\right)\rho \cdot \frac{1}{2}\left(\frac{\pi a}{2} - x\right) + \left(\frac{\pi a}{2} + x\right)\rho \cdot \frac{1}{2}\left(\frac{\pi a}{2} + x\right) + \pi a \rho \cdot \left(-\frac{2a}{\pi}\right)}{\left(\frac{\pi a}{2} - x\right)\rho + \left(\frac{\pi a}{2} + x\right)\rho + \pi a \rho} \right\}$$

or $$\bar{x} = \frac{1}{2\pi a}\left[\frac{\pi^2 a^2}{4} + x^2 - 2a^2\right] \qquad ...(5)$$

Putting $x = 0$, the depth of $C.G.$ below PQ in initial position is

(Say) $$\bar{x}_1 = \frac{1}{2\pi a}\left[\frac{\pi^2 a^2}{4} - 2a^2\right] \qquad ...(6)$$

so distance moved by $C.G.$ of chain (in down ward direction)

$$= (\bar{x} - \bar{x}_1) = \frac{1}{2\pi a}\,x^2 \qquad ...(7)$$

Therefore the work doen by chain

$$= m g \,.\, (\bar{x} - \bar{x}_1) = m g \,.\, \frac{x^2}{2\pi a} \qquad ...(8)$$

Hence the energy equation gives

$$\frac{3}{4} m \dot{x}^2 = m g \,.\, \frac{x^2}{2\pi a} \qquad \text{\{From eqn (4) and (8)\}}$$

or $$\dot{x}^2 = \left(\frac{2 g x^2}{3\pi a}\right) \qquad ...(9)$$

The above relation gives the velocity of the chain after the end has moved a distance x. The end A will reach the plate if $x = \frac{\pi a}{2}$, and hence putting $x = \frac{\pi a}{2}$ in eqn (9), the velocity of chain is given by

$$\dot{x}^2 = \frac{2g}{3\pi a} \cdot \left(\frac{\pi a}{2}\right)^2 = \frac{\pi a g}{6} \qquad \therefore \dot{x} = \sqrt{\frac{\pi a g}{6}}$$

Example 2.6 : Two unequal masses m_1 and m_2 $(m_1 > m_2)$ are suspended by a light string passing over a circular pulley of mass M and radius a. There is no slipping and the friction of the axis may be neglected. If f be the acceleration, show that this is constant; and if k be the radius of gyration of the pulley about the axis, show that

$$k^2 = a^2\{(g-f)\, m_1 - (g+f)\, m_2\} \big/ (M f)$$

Calculate the pressure on the axle.

Solution : Let x be distance moved downwards by the mass m_1 and same distance x moved upward by the mass m_2, and let θ be the angle through which the radius of pulley OD has turned in time t.

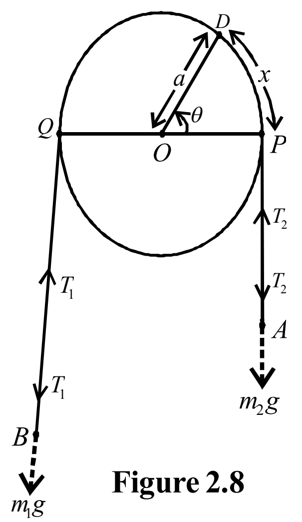


Figure 2.8

Then $x = a\,\theta$ (since there is no slipping). On differentiating with respect to t.

$$\dot{x} = a\,\dot{\theta}, \; \ddot{x} = a\,\ddot{\theta} \Rightarrow \ddot{\theta} = \frac{\ddot{x}}{a} \qquad \text{...(1)}$$

Now equation of motion of the masses m_1 and m_2 are

$$m_1\,\ddot{x} = m_1\,g - T_1 \qquad \text{...(2)}$$

and $$m_2\,\ddot{x} = T_2 - m_2\,g \qquad \text{...(3)}$$

where T_1 and T_2 are tensions in the parts of strings QB and PA. Also the equation of motion of pulley is (taking moment about the axle through O)

$$M\,k^2\ddot{\theta} = T_1\,a - T_2\,a \qquad \text{...(4)}$$

or $$M\,k^2\,\frac{\ddot{x}}{a^2} = T_1 - T_2 \qquad \text{\{from eqn (1)\}} \qquad \text{...(5)}$$

Now adding eqns (2), (3) and (5), we get

$$m_1 \ddot{x} + m_2 \ddot{x} + M k^2 \frac{\ddot{x}}{a^2} = (m_1 g - T_1) + (T_2 - m_2 g) + (T_1 - T_2)$$

or $$\left(m_1 + m_2 + M \frac{k^2}{a^2}\right) \ddot{x} = (m_1 - m_2) g$$

or $$\ddot{x} = \text{acceleration} = f = \frac{(m_1 - m_2) g}{\left(m_1 + m_2 + M \frac{k^2}{a^2}\right)}, \text{ which is constant.} \qquad ...(6)$$

Now from eqn (6)

$$f \left(m_1 + m_2 + M \frac{k^2}{a^2}\right) = m_1 g - m_2 g$$

or $$k^2 \left(\frac{M f}{a^2}\right) = m_1 (g - f) - m_2 (g + f)$$

or $$k^2 = \frac{a^2}{M f} \{m_1 (g - f) - m_2 (g + f)\} \text{ which is the required resnit.} \qquad ...(7)$$

Lastly to find pressure on the axle from eqn (2) and (3) (using $\ddot{x} = f$), we get

$$m_1 f = m_1 g - T_1 \Rightarrow T_1 = m_1 (g - f)$$

$$m_2 f = T_2 - m_2 g \Rightarrow T_2 = m_2 (f + g)$$

adding these, $T_1 + T_2 = m_1 g - m_1 f + m_2 f + m_2 g$

or $$(T_1 + T_2) = (m_1 + m_2) g + (m_2 - m_1) f$$

which is the pressure on the axle.

2.5 The Compound Pendulum

First we shall define simple pendulum and shall show that the period of oscillation in a simple pendulum depends only on the length of the string and not upon the quantity of mass attached.

If a particle is attached to one end of a light inextensible string of length l and other end of which is fixed and is allowed to oscillate in a vertical plane through a small angle, such a system is called simple pendulum.

Let B be the position of the particle of mass m after time t which is attached by light string of length l $(OA = l)$. Let s be length of arc i.e. arc $AB = s$

so that $s = l\theta$, $\therefore \dot{s} = l\dot{\theta}$, $\ddot{s} = l\ddot{\theta}$ ...(1)

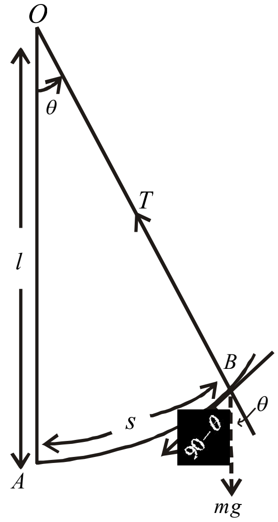


Figure 2.9

Resolving the forces along tangent at B, we get

$$m\frac{d^2s}{dt^2} = -\,mg\sin\theta$$

or $$ml\,\frac{d^2\theta}{dt^2} = -\,mg\sin\theta \qquad \{\text{from }(1)\}$$

or $$\frac{d^2\theta}{dt^2} = -\,\frac{g}{l}\,\theta \qquad (\because \text{ when } \theta \text{ is small then } \sin\theta = \theta) \qquad ...(2)$$

this equation shows that the motion is S.H.M. and the period of oscillation is given by,

$$\frac{2\pi}{\sqrt{\frac{g}{l}}} = 2\pi\sqrt{\frac{l}{g}}$$

$$\therefore \text{ time period } = 2\pi\sqrt{\frac{l}{g}} \qquad ...(3)$$

From eqn (3) it is clear that the period of oscillation in simple pendulum depends only on the length of the string and not upon the quantity of the mass attached.

Definition : A rigid body of any shape or size which is free to turn about a fixed horizontal axis, the external forces being the gravity and the reaction of the fixed axis is called a compound pendulum.

To prove that the time of complete oscillation of a compound pendulum is $2\pi\sqrt{\frac{k^2}{gh}}$, where k is the radius of gyration of the body about a fixed axis and h is distance of the centre of inertia of the body from the fixed axis.

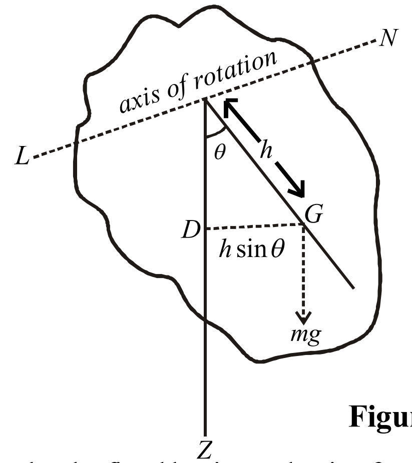


Figure 2.10

Let LON be the fixed horizontal axis of rotation and the vertical plane through LO on the plane of reference and the plane through the axis and centre of gravity G of the rigid body on the plane fixed in the body. Let vertical line through O is OZ, $OG = h$, fig. 2.9 represents a section perpendicular to the axis of rotation through G, cutting the axis of rotation at O. Let $\angle ZOG = \theta$, where θ is the angle which a plane fixed in body makes with a fixed plane in space through LO.

The external forces on the body are its weight Mg acting vertically through G and the reaction of the axis at O. In order to avoid the reaction at O, we take moments about the horizontal axis through O. The moment of Mg about O is - $Mg\,(h\sin\theta)$, which is negative because it has a tendency to decrease angle θ and moment of the reaction at O is zero about O. Then from the equation of motion $\left(M k^2 \ddot{\theta} = L\right)$

i.e. moment of effective forces about axis = moment of external forces

$$\therefore \qquad M k^2 \frac{d^2\theta}{dt^2} = -\, M g h \sin\theta$$

$$\text{or} \qquad \frac{d^2\theta}{dt^2} = -\frac{gh}{k^2}\sin\theta \qquad ...(1)$$

If θ is small, then replacing $\sin\theta$ by θ in eqn (1), we get

$$\frac{d^2\theta}{dt^2} = -\frac{gh}{k^2}\,\theta = -\,\mu\,\theta \qquad \left(\mu = \frac{gh}{k^2}\right) \qquad ...(2)$$

which show that the motion is S.H.M.

Hence time of complete oscillation $T = \dfrac{2\pi}{\sqrt{\mu}} = 2\pi\sqrt{\dfrac{k^2}{gh}}$ ...(3)

2.6 Simple Equivalent Pendulum

Definition :- A simple pendulum having the same periodic time as that of a compound pendulum is called simple equivalent pendulum.

The time of a complete oscillation of a simple pendulum of length l is $2\pi\sqrt{\dfrac{l}{g}}$. The time of a complete oscillation of a compound pendulum is $2\pi\sqrt{\dfrac{k^2}{gh}}$.

If the simple pendulum is equivalent to compound pendulum, then

$$2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{k^2}{gh}} \qquad \therefore\ l = \frac{k^2}{h} \qquad ...(1)$$

Therefore, in the case of a compound pendulum, the length $\left(\dfrac{k^2}{h}\right)$ is called the length of a simple equilvalent pendulum.

2.7 Centre of Suspension and Centre of Oscillation

Centre of Suspension : The point O (fig. 2.10) where the axis of rotation cuts the plane perpendicular to it through centre of gravity G of the body, is called the centre of suspension.

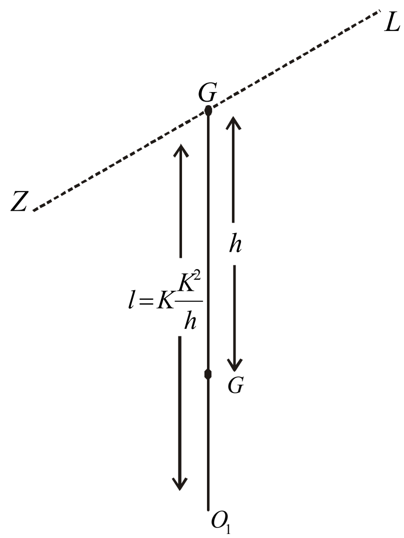


Figure 2.11

Centre of Oscillation : Produce OG to a point O_1 such that $OO_1 = \dfrac{k^2}{h}$ = length of simple equivalent pendulum, then this point O_1 is called the centre of oscillation of the body.

Thus we observe that if the whole mass of the body were collected at centre of oscillation O_1 and hanged from the centre of suspension O by a string of length $\dfrac{k^2}{h}$, the time of oscillation of the compound pendulum will be same as the time of oscillation of the simple equivalent pendulum

We know that in the case of a simple pendulum the period of oscillation depends only on the length of the string and not on the mass attached.

Thus in the case of a simple equivalent pendulum if besides the mass of the body an additional mass is attached at the centre of oscillation the period of oscillation remains same as prior to attaching of the additional mass.

To show that the centre of suspension and centre of oscillation are convertible (or interchangeable)

Let M be mass of body and O the centre of suspension, O_1 be centre of oscillation. Let G be

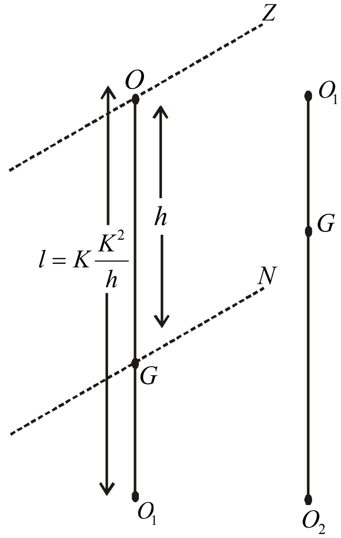


Figure 2.12

Centre of Inertia of body.

$$OG = h,\; OO_1 = l = \frac{k^2}{h} \qquad ...(1)$$

where k is radius of gyration of body about axis (OZ) through O. Let K be the radius of gyration of the body about axis (GN) through G parallel to axis of rotation, then by parallel axis theorem

$$M k^2 = M K^2 + M\,.(OG)^2$$

[M.I. about any axis through O = (M.I. about parallel axis GN through G) + $M(OG)^2$]

$$\therefore \quad k^2 = K^2 + (OG)^2 \qquad ...(2)$$

now $\quad l = OO_1 = \dfrac{k^2}{h} = \dfrac{K^2 + (OG)^2}{OG}$

or $\quad OO_1 . OG = K^2 + (OG)^2$

or $\quad K^2 = OO_1 .\, OG - (OG)^2$

$$= OG\,(OO_1 - OG) = OG\,.\,GO_1$$

or $\quad K^2 = OG\,.\,GO_1 \qquad ...(3)$

Now we shall show that if O_1 be the centre of suspension then the body will swing about point O as its centre of oscillation.

Now, Suppose that when O_1 is the centre of suspension, then O_2 is the centre of oscillation (fig.

2.12). Then arguing as above, we have

$$K^2 = O_1G \,.\, GO_2$$

or $$K^2 = GO_2 \,.\, GO_1 \qquad ...(4)$$

Comparing equations (3) and (4), we observe that the point O_2 is the same as O i.e. the centre of oscillation will be O, when the centre of suspension is O_1.

Therefore, both the centre of suspension and oscillation are convertible.

Minimum time of oscillation of a compound pendulum : To find the minimum time of oscillation of compound pendulum, minimum length of simple equivalent pendulum. Hence deduce that the length of simple equivalent pendulum is infinite then there by the time of oscillation is also infinite.

Let T be time period of a compound pendulum, then

$$T = 2\pi\sqrt{\frac{k^2}{gh}} \qquad ...(1)$$

where $k^2 = K^2 + h^2$, if l be the length of simple equivalent pendulum, then $l = \frac{k^2}{h}$ or

$$l = \frac{K^2 + h^2}{h} \quad (\because k^2 = K^2 + h^2)$$

or $$l = h + \frac{k^2}{h} \qquad ...(2)$$

where $h = OG$ and K is the radius of gyration about an axis through G parallel to axis of rotation.

Then T will be minimum if l is minimum

or T will be minimum if $l = \frac{k^2}{h} = \frac{K^2 + h^2}{h} = h + \frac{k^2}{h}$ is minimum, for that

$$\frac{dl}{dh} = 0 \Rightarrow 1 - \frac{K^2}{h^2} = 0 \Rightarrow h = K \qquad ...(3)$$

now $\frac{d^2 l}{dh^2} = \left(O + \frac{2K^2}{h^3}\right)$ and $\frac{d^2 l}{dh^2} > O$, for $h = K$

so l is minimum for $h = K$

the mimimum value of $l = \left(h + \frac{K^2}{h}\right)_{h=k} = K + \frac{K^2}{K} = 2K$

$\therefore$ Minimum value of $l = 2K \qquad ...(4)$

then minimum value of time period of compound pendulum, from (1) is

$$T = 2\pi \sqrt{\frac{2K}{g}} \quad ...(5)$$

Therefore the period is minimum when the distance between the axis of suspension and the centre of gravity is equal to the radius of gyration (K) about a parallel axis through the centre of gravity.

Again from eqn (2), we observe that if $h = 0$ or $h = \infty$, then $l = O + \frac{K^2}{O} = \infty$ and $l = \infty + \frac{K^2}{\infty} = \infty + O = \infty$, then in both the cases, l the length of simple equivalent pendulum is infinite and consequently the time of oscillation T is also infinite.

Remark : In the above, $h = 0$ corresponds to the position when the axis of rotation passes through the centre of gravity and $h = \infty$ corresponds to the position when axis of rotation is at an infinite distance from the $C.G.$, then l being infinite, then time of oscillation T is infinite.

Self Learning Exercise - 2

1. Write formula for time period of simple pendulum of length l.
2. Define compound pendulum.
3. Write formula for time of complete oscillation of compound pendulum.
4. Define simple equivalent pendulum.

Illustrative Examples :

Example 2.7 : Find the length of simple equivalnet pendulum in the following cases, the axis horizontal :

(i) Circular disc; axis a tangent to it;

(ii) Hemispehere; axis a diameter of the base;

(ii) Cube of side $2a$; axis being diagonal of one face.

Solution : (i) In the fig. 2.13 we have a circular disc of mass M and radius $OG = a$, OZ is tangent to it, which is axis. Let l be length of simple equivalent pendulum, then

$$l = \frac{k^2}{h} \quad ...(1)$$

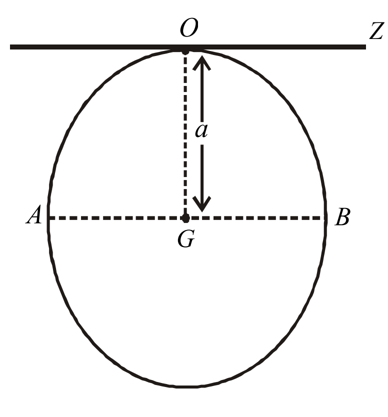


Figure 2.13

where $h = OG = a$

and k is radius of gyration about axis OZ. Then by parallel axis theorem,

(M.I. about OZ) = (M.I. about parallel axis AGB through C.G.) + mass $(OG)^2$

$$\therefore \quad M k^2 = M K^2 + M(a)^2$$

$$\text{or} \quad M k^2 = M.\frac{a^2}{4} + M a^2$$

$$\text{or} \quad k^2 = \frac{a^2}{4} + a^2 \Rightarrow k^2 = \frac{5}{4} a^2 \quad ...(2)$$

and $h = OG = a$, then from eqn (1)

$$l = \frac{\frac{5}{4} a^2}{a} = \left(\frac{5}{4} a\right)$$, which is the length of simple equivalent pendulum.

(ii) In the fig. 2.14, we have shown a hemisphere of mass M and radius $OA = a$ (Say) if G be $C.G.$ of hemisphere then $OG = \frac{3}{8} a$ ...(1)

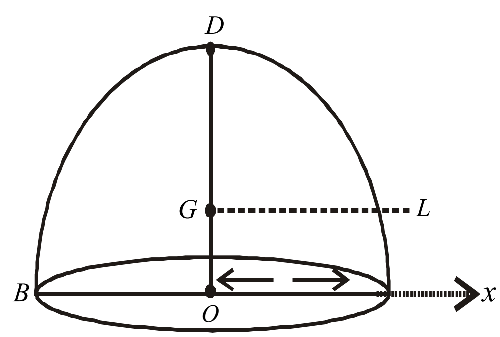


Figure 2.14

BOA is diameter of base, which is axis here. Let l be length of simple equivalent pendulum then

$$l = \frac{k^2}{h} \quad ...(2)$$

where $h = OG = \frac{3a}{8}$, and k be radius of gyration about axis ox.

M.I. about $BOA = \left(\frac{2}{5} M a^2\right) = M k^2$

$$\therefore \quad k^2 = \frac{2}{5} a^2 \quad ...(3)$$

$$\therefore \quad \text{from eqn (2), } l = \frac{\left(\frac{2}{5}a^2\right)}{\left(\frac{3a}{8}\right)} = \left(\frac{16}{15}a\right)$$

which is length of simple equivalent pendulum.

(iii) In the fig. 2.15, we have shown a cube of side $2a$ and mass M. G be its $C.G.$ Let OD be one diagonal of face $OADB$, which is axis here.

then $h = LG = a$ ...(1)

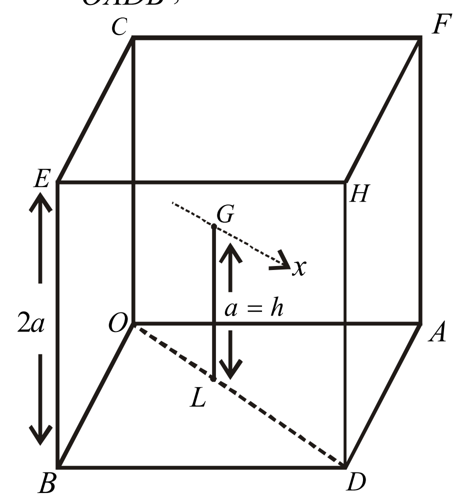


Figure 2.15

Here M.I. about the axis through G parallel to the sides $= \frac{2}{3} M a^2$

Now, M.I. about axis GX, which is parallel to the diagonal OD of face $OADB$, which is inclined at 45° to side OA of cube.

$$= \left(A \sin^2 45 + B \cos^2 45 - 2F \sin 45 \cos 45\right) = \frac{2}{3} M a^2 \left(\frac{1}{2} + \frac{1}{2}\right) - 0 = \frac{2}{3} M a^2 \quad ...(2)$$

[$\because$ M.I. of a body about a line through C.G. inclined at an angle θ to x-axis is

$$= A\cos^2\theta + B\sin^2\theta - 2F\sin\theta\cos\theta$$

here $A =$ M.I. about x-axis $= \frac{2}{3} M a^2$

$B =$ M.I. about y-axis $= \frac{2}{3} M a^2$

$F =$ P.I. $= 0$]

$\therefore$ M.I. about $OD =$ (M.I. about GX) + M.I. of M at G about OD

$$= \frac{2}{3} M a^2 + M(GL)^2 = \frac{2}{3} M a^2 + M a^2$$

$$M\,k^2 = \frac{5}{3}\,M\,a^2 \;\Rightarrow\; k^2 = \frac{5}{3}\,a^2 \quad ...(3)$$

and $\quad h = a \quad ...(4)$

then length of simple equivalnet pendulum

$$l = \frac{k^2}{h} = \frac{\left(\frac{5}{3}a^2\right)}{a} = \frac{5}{3}\,a$$

Example 2.8 : A solid homogeneous cone, of height h and vertical angle 2α, oscillates about a horizontal axis through its centre. Show that the length of the simple equivalent pendulum is $\frac{h}{5}\left(4 + \tan^2\alpha\right)$.

Solution : Let M be mass of the cone of vertical angle 2α and height h, radius of base be r, then from ΔODB, $\tan\alpha = \frac{r}{h}$

$$\Rightarrow r = h\,\tan\alpha \quad ...(1)$$

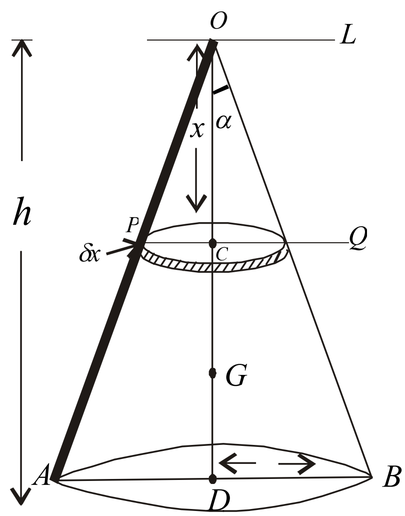


Figure 2.16

Let ρ be mass per unit volume then

$M =$ mass of cone = volume $\times\,\rho$

or $\quad M = \frac{1}{3}\left(\pi r^2 h\right) \times \rho$

or $\quad M = \left(\frac{\pi}{3}\,h^2\tan^2\alpha\;\rho\right) \quad ...(2)$

The solid cone oscillates about horizontal axis OL through the vertex O. Now we shall find M.I. of cone about OL. Consider a circular disc at a distance x from O whose width be δx and radius $CQ = x\tan\alpha$, Then mass of circular disc $=$ Area $\times$ width $\times$ ρ

$$= \pi \left(x \tan\alpha\right)^2 \times \delta x \times \rho \qquad ...(3)$$

Using value of ρ from (2) in (3), we get

$$\text{mass of circular disc} = \left(\pi x^2 \tan^2\alpha \; \delta x\right) \times \frac{3M}{\pi h^3 \tan^2\alpha} = \left(\frac{3M}{h^3} x^2 \delta x\right)$$

Now M.I. of circular disc about diameter PCQ of disc

$$= \frac{(mass)\,(radius)^2}{4} = \frac{3M}{h^3} x^2 \delta x \left(\frac{x^2 \tan^2\alpha}{4}\right)$$

Hence M.I. of circular disc about a parallel axis through O (i.e. about OL), where $OC = x$, is

$$= \frac{3M}{h^3} x^2 \delta x \left(\frac{x^2 \tan^2\alpha}{4}\right) + \left(\frac{3M}{h^3} x^2 \delta x\right)(OC)^2$$

$$= \frac{3M}{h^3} x^2 \delta x \left(\frac{x^2 \tan^2\alpha}{4} + x^2\right)$$

$$= \frac{3M}{h^3} \left(\frac{\tan^2\alpha + 4}{4}\right) x^4 \delta x \qquad ...(4)$$

Therefore M.I. of the cone about OL is

$$= \frac{3M}{h^3} \left(\frac{\tan^2\alpha + 4}{4}\right) \int_o^h x^4 \delta x$$

$$= \frac{3M}{20} \left(\tan^2\alpha + 4\right) h^2$$

$$= M\left[\frac{3}{20} h^2 \left(\tan^2\alpha + 4\right)\right] = M k^2$$

$$\therefore \qquad k^2 = \frac{3}{20} h^2 \left(\tan^2\alpha + 4\right) \qquad ...(5)$$

Also $h' =$ distance of $C.G.$ of cone from $O = OG = \dfrac{3h}{4}$. Hence length of simple equivalent pendulum $l = \left(\dfrac{k^2}{h'}\right)$

or $$l = \frac{\frac{3}{20} h^2 \left(\tan^2\alpha + 4\right)}{\left(\frac{3h}{4}\right)} = \frac{h}{5}\left(\tan^2\alpha + 4\right)$$

Example 2.9 : A bent lever, whose arms are of lengths a and b, the angle between them being α, makes small oscillations in its own plane about the fulcrum; prove that the length of the corresponding simple pendulum is

$$\frac{2}{3}\ \frac{a^3 + b^3}{\sqrt{a^4 + 2a^2 b^2 \cos\alpha + b^4}}$$

Solution : Let AOB be the lever bent at O, and is hanging about horizontal axis through O, to

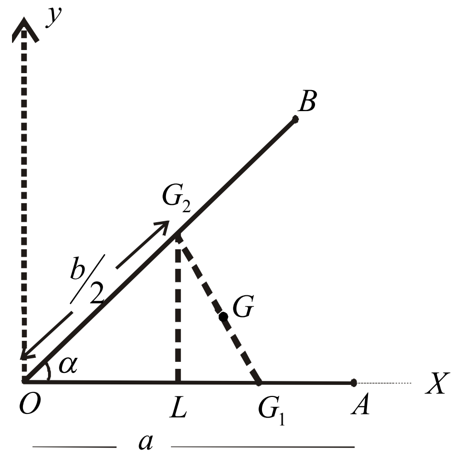


Figure 2.17

swing as a componnd pendulum.

OA and OB are the arms of the lever of length a and b, let G_1 and G_2 be centres of gravity of arms OA and OB respectively, then coordinates of G_1 and G_2 are $\left(\frac{a}{2}, 0\right)$ and $\left(OL = \frac{b}{2}\cos\alpha,\ LG = \frac{b}{2}\sin\alpha\right)$ referred to OA and OY as axes. Let ρ be the weight per unit length of arms. Let $(\bar{x}, \bar{y})$ be the coordinates of $C.G.$, G of the lever, thus, we have

$$\bar{x} = \frac{a\rho \cdot \frac{a}{2} + b\rho \cdot \frac{b}{2}\cos\alpha}{a\rho + b\rho}$$

or $$\bar{x} = \frac{a^2 + b^2\cos\alpha}{2(a+b)} \qquad ...(1)$$

and $$\bar{y} = \frac{a\rho \cdot 0 + b\rho \cdot \frac{b}{2}\sin\alpha}{a\rho + b\rho} = \frac{b^2\sin\alpha}{2(a+b)} \qquad ...(2)$$

Now, $h =$ distance of $C.G.$ of lever from the fulcrum O

$$\therefore \quad h = \sqrt{(\bar{x}^2 + \bar{y}^2)} = \frac{1}{2(a+b)}\sqrt{(a^4 + b^4 + 2a^2b^2\cos\alpha)} \qquad ...(3)$$

Again if k be the radius of gyration of the lever about the axis of rotation through O, then

$$(a\rho + b\rho)\, k^2 = a\rho \cdot \frac{4}{3}\left(\frac{a}{2}\right)^2 + b\rho \cdot \frac{4}{3}\left(\frac{b}{2}\right)^2$$

[$\because$ M.I. of a rod of mass m and length $2a$ about an axis $\perp$ to rod and through an end $= \frac{4}{3} M a^2$]

or $$k^2 = \frac{1}{3} \frac{(a^3 + b^3)}{(a+b)} \qquad ...(4)$$

$\therefore$ length of simple equivalent pendulum $l = \dfrac{k^2}{h}$

or $$l = \frac{\frac{1}{3}\left(\frac{a^3+b^3}{a+b}\right)}{\frac{1}{2(a+b)} \cdot \sqrt{a^4+b^4+2a^2b^2\cos\alpha}} = \frac{2}{3} \frac{a^3+b^3}{\sqrt{a^4+b^4+2a^2b^2\cos\alpha}}$$

Example 2.10: A uniform triangular lamina ABC can oscillate in its own plane about an axis perpendicular to the plane of the lamina through the point A. Prove that the length of the simple equivalent pendulum is

$$\frac{3(b^2+c^2) - a^2}{4\sqrt{\{2(b^2+c^2) - a^2\}}}$$

Solution : Let ABC be a triangular lamina of mass m, this lamina oscillates in its own plane about a line ZAZ^1 which is perpendicular to the plane of the lamina. Side $BC = a$, $AB = c$,

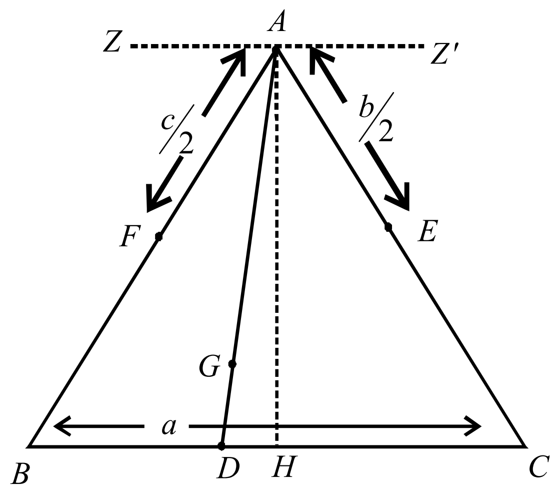


Figure 2.18

$AC = b$. Let AD be median. D, E, F are midpoints of sides BC, AC and AB. Replace the mass of the triangle by three equal particles each of mass $\frac{m}{3}$ placed at mid points D, E and F.

Then $AD^2 = AB^2 + BD^2 - 2\,AB\,.\,BD\cos B$ (In $\Delta\,ABD$)

$$AD^2 = c^2 + \left(\frac{a}{2}\right)^2 - 2\,.\,c\,.\,\frac{a}{2}\left(\frac{c^2+a^2-b^2}{2ac}\right)$$

or $$AD^2 = \left(\frac{2b^2+2c^2-a^2}{4}\right)$$

or $$AD = \frac{1}{2}\sqrt{2b^2+2c^2-a^2} \qquad ...(1)$$

Also Distance of E from $A = AE = \frac{b}{2}$

Distance of F from $A = FA = \frac{c}{2}$

$\therefore$ mk^2 = M.I. of triangle ABC about AZ'

$$= \frac{m}{3}\left[(AD)^2 + (AE)^2 + (AF)^2\right] = \frac{m}{3}\left[\frac{2b^2+2c^2-a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4}\right]$$

or $$mk^2 = \frac{m}{12}\left(3b^2+3c^2-a^2\right) \Rightarrow k^2 = \left(\frac{3b^2+3c^2-a^2}{12}\right) \qquad ...(2)$$

Also $h =$ distance of $C.G.$ from axis of rotation AZ'

$$h = AG = \frac{2}{3}(AD) = \frac{2}{3}\cdot\frac{1}{2}\sqrt{(2b^2+2c^2-a^2)} = \frac{1}{3}\sqrt{(2b^2+2c^2-a^2)} \qquad ...(3)$$

Now, length of the simple equivalent pendulum

$$l = \frac{k^2}{h} = \frac{\left(\frac{3b^2+3c^2-a^2}{12}\right)}{\left(\frac{1}{3}\sqrt{2b^2+2c^2-a^2}\right)} = \frac{3b^2+3c^2-a^2}{4\left[\sqrt{2b^2+2c^2-a^2}\right]} \text{ [from eqn (2) \& (3)]}$$

or $$l = \frac{3\left(b^2+c^2\right) - a^2}{4\sqrt{\left\{2\left(b^2+c^2\right) - a^2\right\}}}$$

Example 2.11: A simple circular pendulum is formed of a mass M suspended from a fixed point by a weightless wire of length l; if a mass m, very small compared with M, be knotted on to the wire at a distance a from the point of suspension, show that the time of a small vibration of the pendulum is approximately diminished by $\frac{m}{2M}\frac{a}{l}\left(1-\frac{a}{l}\right)$ of itself.

Solution : Let T_1 be time of oscillation of simple circular pendulum of mass M fixed by a weightless wire of length l, then $T_1 = 2\pi\sqrt{\frac{l}{g}}$ ...(1)

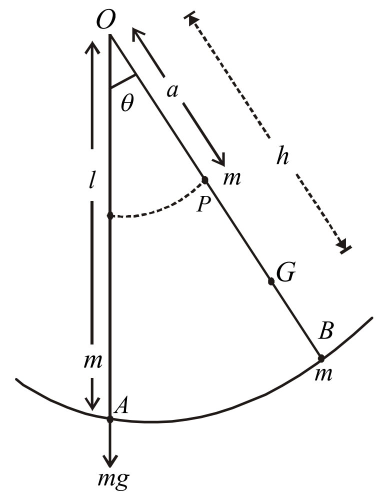


Figure 2.19

Now, let the mass m be knotted at P such that $OP = a$ on the wire. Let k be radius of gyration about axis of rotation through O (i.e. about OZ) and let h be distance of $C.G.$ (of two masses m and M) from O, then

$$h = \frac{Ml + ma}{M+m} \qquad \left(by\ formula\ \bar{x} = \frac{\sum mx}{\sum m}\right) \qquad ...(2)$$

and $(m + M)\, k^2 = ma^2 + Ml^2$

[(M.I. about OZ) = M.I. of mass m about OZ + M.I. of mass M about OZ]

$$\therefore \quad k^2 = \frac{Ml^2 + ma^2}{m+M} \qquad ...(3)$$

If T_2 be time of oscillation for the compound pendulum consisting of masses m and M, then

$$T_2 = 2\pi\sqrt{\frac{k^2}{gh}} = 2\pi\sqrt{\frac{ma^2 + Ml^2}{g(ma + Ml)}} \qquad \text{[eqn (2) and (3)]} \qquad ...(4)$$

Hence, loss in the time period is given by

$$T_1 - T_2 = T_1\left(1 - \frac{T_2}{T_1}\right) \qquad ...(5)$$

now $$\frac{T_2}{T_1} = \left\{\frac{ma^2 + Ml^2}{l(ma + Ml)}\right\}^{1/2} = \left\{\frac{Ml^2\left(1 + \frac{ma^2}{Ml^2}\right)}{l.Ml\left(1 + \frac{ma}{Ml}\right)}\right\}^{1/2}$$

$$= \left(1 + \frac{ma^2}{Ml^2}\right)^{1/2} . \left(1 + \frac{Ma}{Ml}\right)^{-1/2}$$

$$= \left(1 + \frac{ma^2}{2Ml^2} + ...\right)\left(1 - \frac{1}{2}\frac{ma}{Ml} + ...\right)$$

$$= \left(1 - \frac{1}{2}\frac{ma}{Ml} + \frac{ma^2}{2Ml^2} - \frac{m^2a^3}{4m^2l^3} + ...\right)$$

$$\frac{T_2}{T_1} = \left(1 - \frac{1}{2}\frac{ma}{Ml} + \frac{ma^2}{2Ml^2}\right)$$

$$\left(\text{since } \frac{m}{M} \text{ is small, so neglecting higher powers of } \frac{m}{M}\right)$$

$$\therefore \quad \left(1 - \frac{T_2}{T_1}\right) = \left(\frac{1}{2}\frac{ma}{Ml} - \frac{1}{2}\frac{ma^2}{Ml^2}\right) = \frac{m}{2M} . \frac{a}{l}\left(1 - \frac{a}{l}\right) \qquad ...(6)$$

using (6) in eqn (5)

$$T_1 - T_2 = \left\{\frac{m}{2M} . \frac{a}{l}\left(1 - \frac{a}{l}\right)\right\} T_1 \qquad ...(7)$$

Hence time period of pendulum is approximately diminished by

$\frac{m}{2M} . \frac{a}{l}\left(1 - \frac{a}{l}\right)$ of itself.

Remark : When the particle of mass m in attached at point A i.e. $a = l$, then from eqn (7), $T_1 - T_2 = 0 \Rightarrow T_1 = T_2$ then the time of oscillation remains unchanged. In other words the time of oscillation of a simple pendulum depends on the length of the string and not on the mass attached.

Example 2.12: A sphere of radius a, is suspended by a fine wire from a fixed point at a distance l from its centre, show that the time of a small oscillation is given by

$$2\pi\sqrt{\left(\frac{5l^2+2a^2}{5gl}\right)}\left\{1+\frac{1}{4}\sin^2\frac{\alpha}{2}\right\}$$

where α represents the amplitdue of the vibration.

Solution : Let G be the centre of the sphere of mass M and the axis of rotation be OZ passing through $O\cdot$ $OC = l$ (given) ...(1)

Let k be radius of gyration about axis of rotation (OZ) through O.

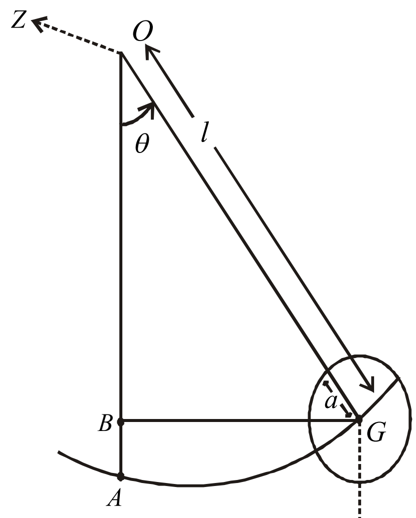


Figure 2.20

$\therefore$ $$Mk^2 = \frac{2M}{5}a^2 + Ml^2$$ (by parallel axis theoram)

or $$k^2 = \left(\frac{2}{5}a^2 + l^2\right)$$ ..(2)

Also, given amplitade α, i.e. $\dot{\theta} = 0$, when $\theta = \alpha$ ...(3)

Now, the equation of motion $\left(Mk^2\ddot{\theta} = L\right)$ gives

$$Mk^2\ddot{\theta} = -Mgl\sin\theta$$ (by taking moment about the horizontal axis through O)

Multiplying both sides by $2\dot{\theta}$ and integrating, we get

$$\int 2\ddot{\theta}\left(\frac{2}{5}a^2 + l^2\right)\dot{\theta}dt = \int -2gl\sin\theta\dot{\theta}dt + C_1$$

or $$\left(\frac{2}{5}a^2 + l^2\right)\dot{\theta}^2 = 2gl\cos\theta + C_1$$ ...(4)

to find value of C_1, using $\dot{\theta} = 0$, when $\theta = \alpha$

$$O = 2gl\cos\alpha + C_1 \Rightarrow C_1 = -2gl\cos\alpha$$ ...(5)

then from (4), we have

$$\left(\frac{2a^2+5l^2}{5}\right)\dot{\theta}^2 = 2gl\cos\theta - 2gl\cos\alpha$$

or $$\dot{\theta} = \sqrt{\frac{2gl\times 5}{2a^2+5l^2}(\cos\theta-\cos\alpha)} = \sqrt{\frac{10gl}{2a^2+5l^2}}\ .(\cos\theta-\cos\alpha)^{1/2}$$

or $$\dot{\theta} = \frac{d\theta}{dt} = \sqrt{\frac{10gl}{2a^2+5l^2}}\left[1-2\sin^2\frac{\theta}{2}-\left(1-2\sin^2\frac{\alpha}{2}\right)\right]^{1/2}$$

or $$\frac{d\theta}{dt} = \sqrt{\frac{10gl}{2a^2+5l^2}}\times\sqrt{2}\left(\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}\right)^{1/2}$$

or $$\frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}}} = \sqrt{2}\sqrt{\frac{10gl}{2a^2+5l^2}}\,dt$$

Integrating for θ, from $\theta = 0$ to $\theta = \alpha$ and let time t be $t = 0$ to $t = t_1$

$$\int_0^{\alpha}\frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}}} = \int_0^{t_1}\sqrt{2}\sqrt{\frac{10gl}{2a^2+5l^2}}\,dt = \int_0^{t_1}\sqrt{2}\,\sqrt{2}\ .\lambda\,dt \qquad ...(6)$$

where $\lambda^2 = \dfrac{5gl}{2a^2+5l^2}$

For integration, put $\sin\dfrac{\theta}{2} = \sin\dfrac{\alpha}{2}\sin\psi$ ...(7)

$$\therefore \quad \frac{1}{2}\cos\frac{\theta}{2}\,d\theta = \sin\frac{\alpha}{2}\cos\psi\,d\psi$$

For limit, when $\theta = 0 \Rightarrow \psi = 0$

and when $\theta = \alpha$, then $\sin\psi = 1 \Rightarrow \psi = \dfrac{\pi}{2}$

then from eqn (6), we get

$$\int_0^{\frac{\pi}{2}}\frac{1}{\sin\frac{\alpha}{2}\sqrt{1-\sin^2\psi}}\times\frac{2\sin\frac{\alpha}{2}\ .\cos\psi\,d\psi}{\cos\frac{\theta}{2}} = 2\lambda\,(t)_0^{t_1}$$

or $$\int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\psi}} = \lambda\,(t_1)$$

or $$t_1 = \frac{1}{\lambda}\int_0^{\frac{\pi}{2}} \left(1-\sin^2\frac{\alpha}{2}\sin^2\psi\right)^{1/2} d\psi$$

$$= \frac{1}{\lambda}\int_0^{\frac{\pi}{2}} \left(1+\frac{1}{2}\sin^2\frac{\alpha}{2}\sin^2\psi + \ldots.\right) d\psi$$

$$= \frac{1}{\lambda}\left[(\psi)_0^{\frac{\pi}{2}} + \frac{1}{2}\sin^2\frac{\alpha}{2}\cdot\frac{1}{2}\frac{\pi}{2}\right] = \frac{1}{\lambda}\left[\frac{\pi}{2}+\frac{1}{8}\sin^2\frac{\alpha}{2}\cdot\pi\right]$$

$$\left[\because \int_0^{\frac{\pi}{2}} \sin^2\psi\, d\psi = \frac{1}{2}\cdot\frac{\pi}{2}\right]$$

or $$t_1 = \frac{\pi}{2}\cdot\frac{1}{\lambda}\left(1+\frac{1}{4}\sin^2\frac{\alpha}{2}\right)$$

Hence the time of complete oscillation $= 4\,t_1$

or time of oscillation $= 2\pi\sqrt{\dfrac{2a^2+5l^2}{5gl}}\left(1+\dfrac{1}{4}\sin^2\dfrac{\alpha}{2}\right)$

Example 2.13: Find the time of oscillation of a compound pendulum consisting of a rod of mass m and length a carrying at one and a sphere of mass m_1 and diameter $2b$, the other end of the rod being fixed.

Solution : Let m be mass of rod OA of length $a\ (= OA)$ and m_1 be mass of sphere with centre at G and radius $= b$. The end O of the rod is fixed and axis of rotation (OZ) passes through O.

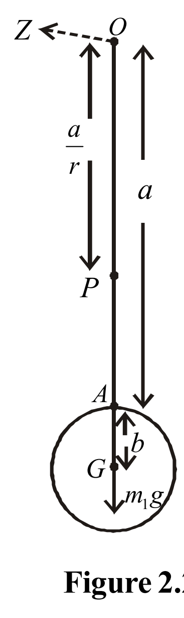


Figure 2.21

We know that M.I. of rod of length $2a$ about an axis through its extremity

$$= \frac{4}{3} m a^2 = \frac{m}{3}(2a)^2 = \frac{m}{3}(length\ of\ rod)^2$$

and hence M.I. of a rod OA of length a about an axis (OZ) through its extremity is $\frac{ma^2}{3}$ ...(1)

Again M.I. of a sphere of mass m_1 about an axis through its centre is $\left(\frac{2}{5} m_1 b^2\right)$ and hence its M.I. about a parallel axis (OZ) through O, (where $OG = a+b$), is $m_1\left[\frac{2}{5}b^2 + (a+b)^2\right]$ ...(2)

If k be the radius of gyration about an axis through O (i.e. about OZ), then

$$(m + m_1)\, k^2 = m\frac{a^2}{3} + m_1\left[\frac{2}{5}b^2 + (a+b)^2\right] \qquad \text{[from (1) and (2)]}$$

$$\therefore \qquad k^2 = \frac{m\frac{a^2}{3} + m_1\left[\frac{2}{5}b^2 + (a+b)^2\right]}{m + m_1} \qquad ...(3)$$

Again let h be the distance of $C.G.$ of the combined body from O, then

$$h = \frac{m\frac{a}{2} + m_1(a+b)}{m + m_1} \qquad ...(4)$$

Hence $$l = \frac{k^2}{h} = \frac{m\frac{a^2}{3} + m_1\left[\frac{2}{5}b^2 + (a+b)^2\right]}{m\frac{a}{2} + m_1(a+b)} \qquad \text{[from eqn (3) and (4)]} \quad ...(5)$$

therefore, period of a small oscillation

$$T = 2\pi\sqrt{\frac{k^2}{gh}}$$

or $$T = 2\pi\sqrt{\frac{m\frac{a^2}{3} + m_1\left[\frac{2}{5}b^2 + (a+b)^2\right]}{g\left\{m\frac{a}{2} + m_1(a+b)\right\}}}$$

Example 2.14: A rectangular plate swings in a vertical plane about one of its corners. If its period is one second, find the length of the diagonal.

Solution : Considering a rectangular plate $ABCD$ of mass m and having length of sides $2a$, $2b$. Let G be $C.G.$ of plate

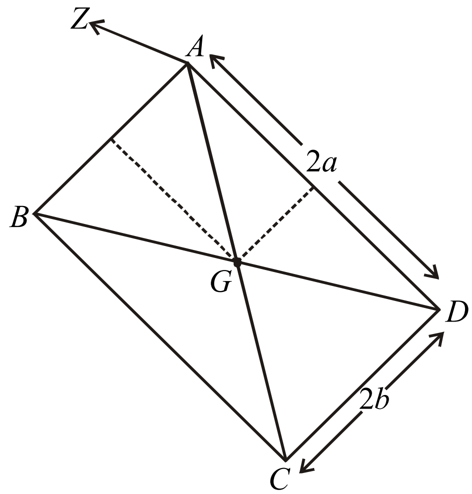


Figure 2.22

Let $AG = h = \sqrt{a^2 + b^2}$ ...(1)

Now, let plate be suspended from corner A and allowed to swing about a horizontal axis AZ, so that plane of plate remains vertical. Let k be the radius of gyration of the plate about axis AZ, through A.

Then $$mk^2 = m\left(\frac{a^2 + b^2}{3}\right) + m\,(AG)^2$$

$$= m\left(\frac{a^2 + b^2}{3}\right) + m\,(a^2 + b^2)$$

or $$mk^2 = \frac{4}{3}m\,(a^2 + b^2) \Rightarrow k^2 = \frac{4}{3}(a^2 + b^2) \quad ...(2)$$

or $$k^2 = \frac{4}{3}h^2 \quad ...(3)$$

$\therefore$ length of simple equivalent pendulum

$$l = \frac{k^2}{h} = \frac{k^2}{AG} = \frac{\frac{4}{3}h^2}{h} = \frac{4}{3}h \quad ...(4)$$

Therefore time period $T = 2\pi\sqrt{\frac{k^2}{gh}} = 1$ (given) or $\quad 4\pi^2 \dfrac{k^2}{gh} = 1$

$$4\pi^2 \times \frac{4}{3}\frac{h}{g} = 1 \qquad \left[\because k^2 = \frac{4h^2}{3}\right]$$

$$\Rightarrow h = \frac{3g}{16\pi^2} \qquad ...(5)$$

Now length of diagonal $= AC = 2\,AG = 2h = 2\left(\dfrac{3g}{16\pi^2}\right) = \dfrac{3g}{8\pi^2}$.

Example 2.15: An ellipse of axes a and b and a circle of radius b are cut from the same sheet of thin uniform metal and are superposed and fixed together with their centres coincident. The figure is free to move in its own vertical plane about one end of the major axis; show that the length of the equivalent simple pendulum is

$$\frac{5a^2 - ab + 2b^2}{4a}$$

Solution : Let ρ be the density of the sheet per square area and m_1 and m_2 be masses of ellipse and circular sheet, then

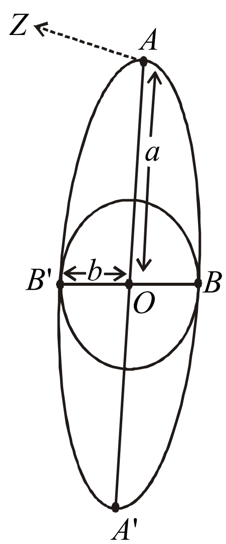


Figure 2.23

$$m_1 = (\pi a b)\,\rho \qquad ...(1)$$

and $\quad m_2 = (\pi b^2)\,\rho \qquad ...(2)$

M.I. of circular disc about an axis through centre O perpendicular to its plane $= m_2 \dfrac{b^2}{2}$

Hence its M.I. about an axis through A (i.e. about AZ) perpendicular to its plane is

$$= m_2 \left(\frac{b^2}{2} + a^2 \right) \qquad (\because AO = a) \qquad ...(3)$$

Similarly M.I. of elliptic disc about an axis through A

$$= m_1 \left(\frac{a^2 + b^2}{4} + a^2 \right) \qquad \text{(by parallel axis theorem)} \qquad ...(4)$$

Hence if k be the radius of gyration of body about an axis (AZ) through A perpendicular to the lamina, then

$$(m_1 + m_2)\, k^2 = m_1 \left(\frac{a^2 + b^2}{4} + a^2 \right) + m_2 \left(\frac{b^2}{2} + a^2 \right) \qquad \text{[from eqn (3) and (4)]}$$

or $$\pi b \rho \,(a + b)\, k^2 = \pi a b \rho \,\frac{5a^2 + b^2}{4} + \pi b^2 \rho \left(\frac{b^2 + 2a^2}{2} \right) \qquad \text{[from (1) and (2)]}$$

$$\therefore \qquad k^2 = \frac{a\,(5a^2 + b^2) + b\,(2b^2 + 4a^2)}{4\,(a + b)} = \frac{5a^2\,(a + b) - a^2 b + a b^2 + 2\,b^3}{4\,(a + b)}$$

$$= \frac{5a^2\,(a + b) - ab\,(a + b) + 2ab^2 + 2\,b^3}{4\,(a + b)}$$

$$= \frac{5a^2\,(a + b) - ab\,(a + b) + 2b^2\,(a + b)}{4\,(a + b)}$$

$$\therefore \qquad k^2 = \left(\frac{5a^2 - ab + 2b^2}{4} \right) \qquad ...(5)$$

Also $h =$ depth of $C.G.\ O$ below $A = a$

$\therefore \qquad l =$ length of simple equivalent pendulum

$$= \frac{k^2}{h} = \left(\frac{5a^2 - ab + 2b^2}{4a} \right)$$

2.8 Reaction of the axis of rotation

To find the reaction of the axis of rotation when a rigid body rotates about it under the action of external forces such that both the body and forces are symmetrical with respect to the plane through the centre of gravity perpendicular to the axis, and let gravity be the only external force.

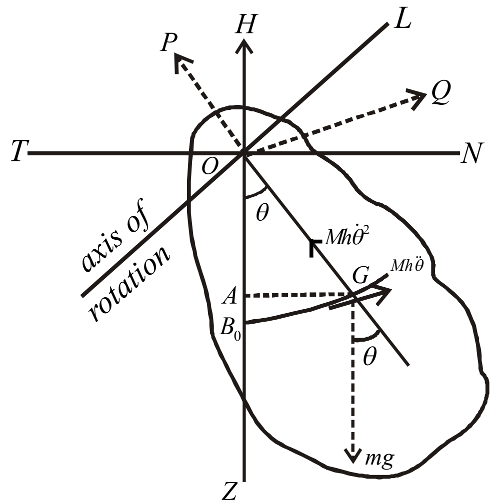


Figure 2.24

Due to symmetry of the rigid body, the reaction of the axis of rotation OL reduces to a single force acting at point O in the plane of paper. Let the components of this single force along and perpendicular to GO are P and Q, where G is centre of gravity of body. Here $OG = h$, Now as body swings about O, G describes a circular path with O as centre and $h\ (= OG)$ as radius as shown in fig. 2.22.

$\therefore$ Radial acceleration of G along GO is $= h\,\dot{\theta}^2$

and transverse acceleration of G perpendicular GO is $= h\,\ddot{\theta}$

Let M be mass of body and weight $M\,g$ is acting at G in downward direction. Then equaton of motion of $C.G.$, G are

$$M\,h\left(\frac{d\,\theta}{d\,t}\right)^2 = P - M\,g\cos\theta \qquad \text{(along } GO\text{)} \qquad ...(1)$$

and $$M\,h\,\frac{d^2\,\theta}{d\,t^2} = Q - M\,g\sin\theta \qquad \text{(perpendicular to } GO\text{)} \qquad ...(2)$$

Now taking moment abot the axis of rotation OL

we have $$M\,k^2\,\frac{d^2\,\theta}{d\,t^2} = -\,M\,g\,.\,h\,\sin\theta \qquad \left(OG = h,\ so\ AG = h\,\sin\theta\right) \quad ...(3)$$

where k is the radius of gyration of the body about OL. Also moment of $M\,g$ about OL is $-\,M\,g\,h\,\sin\theta$, which is negative because tendency of Mg is to decrease θ. In eqn (3) moment of reaction components P and Q vanish, because these are acting at O.

From equations (2) and (3), on eliminating $\frac{d^2\,\theta}{d\,t^2}$ we shall get the value of reaction component Q,

and again on integrating eqn (3) $\left(by\ multiplying\ 2\,\frac{d\,\theta}{d\,t}\right)$, we shall get $\left(\frac{d\,\theta}{d\,t}\right)^2$ and putting in eqn (1), we shall get P. (Constant of integration can be determined by initial conditions)

If R be the resultant reaction, then

$$R = \sqrt{P^2 + Q^2} \qquad ...(4)$$

and its direction will make an angle $= \tan^{-1}\left(\frac{Q}{P}\right)$

with the direction of reaction component P i.e. with GO.

Remark : (i) Horizontal component of reaction will be $= (P\sin\theta - Q\cos\theta)$ $(along\ OT)$

(ii) Vertical component of reaction will be $= P\cos\theta + Q\sin\theta$ $(along\ OH)$

Illustrative Examples :

Example 2.16: A thin uniform rod has one end attached to a smooth hinge and is allowed to fall from a horizontal position, Show that the horizontal strain on the hinge is greatest when the rod is inclined at an angle 45° to the vertical, and that the vertical strain is then $\frac{11}{8}$ times the weight of the rod.

Solution : Let OA be a rod of mass M and length $2a$. G be its centre of gravity. Then $OG = a$, initially rod was horizontal i.e. in direction of OX (fig. 2.23), so that

$$t = 0, \dot{\theta} = 0 \text{ when } \theta = \frac{\pi}{2} \qquad ...(1)$$

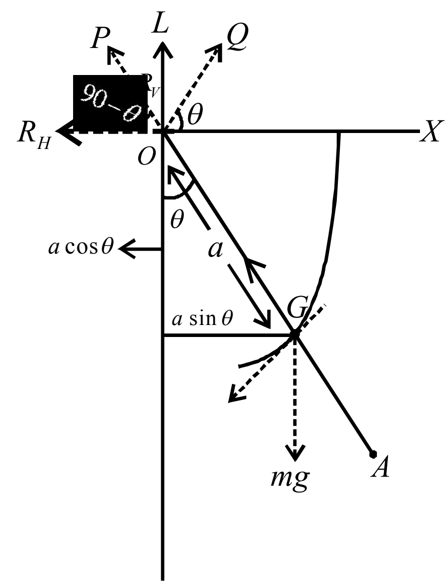


Figure 2.25

where θ is the angle which rod makes with vertical OZ at time t. When rod falls from horizontal, then centre of gravity G is describing a circle of radius a $(= OG)$ about O as centre. Let P and Q be components of reaction at O in direction of GO and perpendicular to GO.

Let R_H and R_V be horizontal and vertical strains on the hinge.

Then equation of motion of G along GO and perpendicular to GO are

$$M a \left(\frac{d\theta}{dt}\right)^2 = P - M g \cos\theta \qquad (along\ GO) \qquad ...(2)$$

and $$M a \frac{d^2\theta}{dt^2} = Q - M g \sin\theta \qquad (\perp\ to\ GO) \qquad ...(3)$$

Let k be radius of gyration of rod OA about axis through O

$$M k^2 = \frac{4}{3} M a^2 \Rightarrow k^2 = \frac{4}{3} a^2 \qquad ...(4)$$

Now, the moment equation about O gives

$$M k^2 \frac{d^2\theta}{dt^2} = - M g a \sin\theta \qquad \text{(Moments of } P \text{ and } Q \text{ will be zero)} \qquad ...(5)$$

or $$\frac{4}{3} a^2 \frac{d^2\theta}{dt^2} = - g a \sin\theta$$

or $$\frac{d^2\theta}{dt^2} = - \frac{3g}{4a} \sin\theta \qquad ...(6)$$

Multiplying by $\frac{2d\theta}{dt}$ and integrating, we get

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{3g}{4a} \cos\theta + C_1 \qquad ...(7)$$

we know

$$\dot{\theta} = 0, \ \theta = \frac{\pi}{2} \qquad \therefore C_1 = 0$$

$\therefore$ from (7), $$\left(\frac{d\theta}{dt}\right)^2 = \frac{3g}{2a} \cos\theta \qquad ...(8)$$

to find P, using value of $\dot{\theta}^2$ in eqn (2) from (8), we get

$$M a \cdot \frac{3g}{2a} \cos\theta = P - M g \cos\theta \Rightarrow P = \frac{5}{2} M g \cos\theta \qquad ...(9)$$

and using value of $\frac{d^2\theta}{dt^2}$ from (6) in (3), we get

$$Ma\cdot\left(-\frac{3g}{2a}\sin\theta\right)=Q-Mg\sin\theta\Rightarrow Q=\frac{1}{4}Mg\sin\theta \quad ...(10)$$

Now, the horizontal strain $=R_H=$ sum of components of P and Q in horizontal direction OH

$$=P\sin\theta+(-Q\cos\theta)$$

or $$R_H=P\sin\theta-Q\cos\theta \quad ...(11)$$

$$=\left(\frac{5}{2}Mg\cos\theta\right)\sin\theta-\left(\frac{1}{4}Mg\sin\theta\right)\cos\theta \quad \text{[from eqn (9) and (10)]}$$

or $$R_H=\frac{9}{8}Mg\sin 2\theta \quad ...(12)$$

which is maximum when $\sin 2\theta=1$

$$\sin 2\theta=1=\sin\frac{\pi}{2}\Rightarrow 2\theta=\frac{\pi}{2}\Rightarrow\theta=\frac{\pi}{4}$$

Hence the horizontal strain is maximum when the rod is inclined at an angle 45° to the vertical.

Also vertical strain $=R_V=$ sum of components of P and Q in vertical direction OL

$$=P\cos\theta+Q\sin\theta$$

or $$R_V=\left(\frac{5}{2}Mg\cos\theta\right)\cos\theta+\left(\frac{1}{4}Mg\sin\theta\right)\sin\theta \quad \text{[from (9) and (10)]}$$

or $$R_V=\left(\frac{5}{2}Mg\cos^2\theta+\frac{1}{4}Mg\sin^2\theta\right)$$

$$=\frac{5}{2}Mg\cos^2\frac{\pi}{4}+\frac{1}{4}Mg\sin^2\frac{\pi}{4} \quad \left(when\ \theta=\frac{\pi}{4}\right)$$

or $$R_V=\frac{11}{8}Mg=\frac{11}{8}\ \text{(weight of rod)}$$

Example 2.17: A uniform semi circular arc of mass m and radius a, is fixed at its ends to two point in the same vertical line, and is rotating with constant angular velocity w. Show that the horizontal thrust on the upper end is $m\cdot\dfrac{g+w^2a}{\pi}$

Solution : Suppose the uniform semi circular arc (wire) with centre at O, mass m and radius $(OD=OE)$ a, rotates about $DE=(=2a)$ with constant angular velocity w. Let G be $C.G.$ of arc such that

$$OG=\frac{2a}{\pi} \quad ...(1)$$

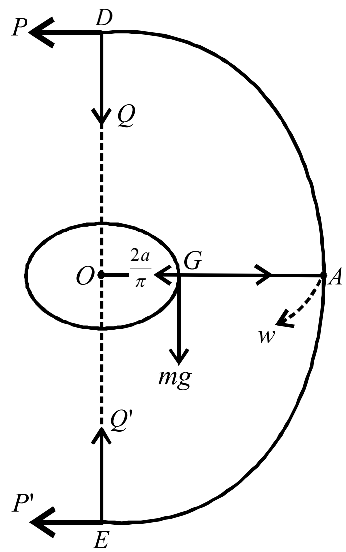


Figure 2.26

Than, as wire rotates, the point G will describe a circle of radius $OG\left(=\frac{2a}{\pi}\right)$ about O as centre with constant angular velocity w. i.e. $\frac{d\theta}{dt}=w$ (constant), $\therefore \frac{d^2\theta}{dt^2}=0$.

Hence the effective force on G is only $m\cdot\frac{2a}{\pi}\left(\frac{d\theta}{dt}\right)^2$ acting along GO.

Let P and Q be horizontal and vertical components of reaction at D, P' and Q' be horizontal and vertical components at point E as shown in fig. To find the horizontal thrust $(=P)$ at upper and D, taking the moment of forces about lower end E, we get

Moment of effective forces = moment of external forces

$$\therefore \qquad m\frac{2a}{\pi}w^2\,.\,(OE)=P\,.\,2a+mg\left(-\frac{2a}{\pi}\right)$$

or $$\frac{2am}{\pi}w^2\,.\,a=2aP-\frac{2amg}{\pi}$$

or $$P=\frac{m\left(g+aw^2\right)}{\pi}$$

Example 2.18: A heavy homogeneous cube, of weight W, can swing about an edge which is horizontal; it starts from rest being displaced from its unstable position of equilibrium; when the per pendicular from the centre of gravity upon the edge has turned through an angle θ, show that the components of the action at the hinge along the rightangles to this perpendicular are

$$\frac{1}{2}W(3-5\cos\theta) \text{ and } \frac{1}{4}W\sin\theta$$

Solution : Let G' be the initial position of $C.G.$ of cube in the figure.

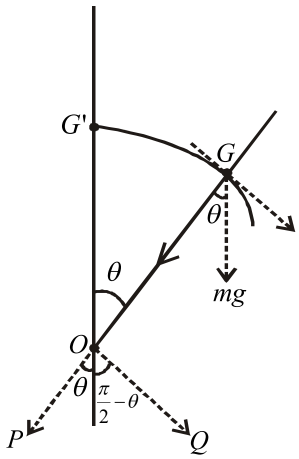


Figure 2.27 (a)

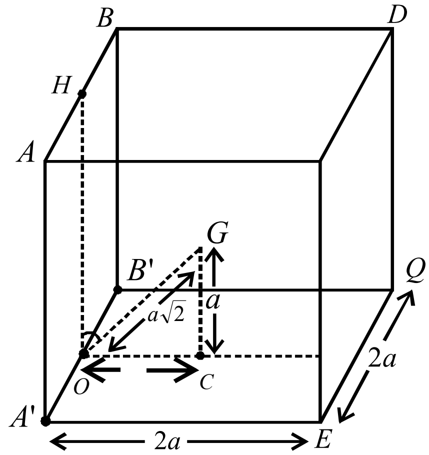


Figure 2.27 (b)

Let G be position of $C.G.$ when $G'O$ has turned through an angle θ about the horizontal edge $A'B'$ i.e. $\angle G'OG = \theta = \angle HOG$. Let $2a$ be the length of an edge of the cube, then

$$OG = \sqrt{OC^2 + CG^2} \qquad \text{[from fig. 2.27 (b)]}$$

or $$OG = \sqrt{a^2 + a^2} = a\sqrt{2} \qquad ...(1)$$

When cube swing about edge $A'B'$, then G describes a circle of radius $OG\ (= a)$ about O as centre. Let P and Q are the components of reaction of axis along and perpendicular to the direction GO.

If $Mk^2 =$ M.I. of cube about an edge

then $$Mk^2 = \frac{2}{3}ma^2 + M\left(a\sqrt{2}\right)^2 = \frac{8}{3}Ma^2$$

[$\because$ M.I. of cube of edge of length $2a$ about any line through $C.G. = \frac{2}{3}Ma^2$]

$\therefore$ $$k^2 = \frac{8}{3}a^2 \qquad ...(2)$$

Now, equation of motion of G along and perpendicular to GO

are $$Ma\sqrt{2}\,\dot{\theta}^2 = P + Mg\cos\theta \qquad ...(3)$$

and $$Ma\sqrt{2}\,\ddot{\theta} = Q + Mg\sin\theta \qquad ...(4)$$

Now taking moments about O, we get

$$Mk^2\ddot{\theta} = Mga\sqrt{2}\,\sin\theta \qquad ...(5)$$

or $\quad M\frac{8}{3}a^2\ddot{\theta} = M g\sqrt{2}\sin\theta$ [from eqn (2)]

or $\quad \ddot{\theta} = \frac{3\sqrt{2}}{8a} g \sin\theta$ ...(6)

using value of $\ddot{\theta}$ from (6) in (4), we get

$$M a\sqrt{2}\left(\frac{3\sqrt{2}}{8a} g \sin\theta\right) = Q + M g \sin\theta$$

or $\quad Q = -\frac{M g}{4}\sin\theta = -\frac{W}{4}\sin\theta$ ...(7)

Here $-ve$ sign show that Q is opposite direction to what is shown in the figure. $(\because M g = W)$

Now integrating eqn (6) by multiplying $2\dot{\theta}$, we get

$$\dot{\theta}^2 = -\frac{3\sqrt{2}}{4a} g\cos\theta + C_1 \quad ...(8)$$

when $v = 0$, $\dot{\theta} = 0$ (initial condition)

$$\therefore \quad o = -\frac{3\sqrt{2}}{4a} g + C_1 \Rightarrow C_1 = \frac{3\sqrt{2}\,g}{4a} \quad ...(9)$$

Hence from (8)

$$\dot{\theta}^2 = -\frac{3\sqrt{2}}{4a} g\cos\theta + \frac{3\sqrt{2}\,g}{4a} C_1 = \frac{3\sqrt{2}}{4a} g(1-\cos\theta) \quad ...(10)$$

Using value of $\dot{\theta}^2$ from (10) in (3), we get

$$M a\sqrt{2}\left[\frac{3\sqrt{2}\,g}{4a}(1-\cos\theta)\right] = P + M g\cos\theta$$

or $\quad P = M g\left[\frac{3}{2}(1-\cos\theta) - \cos\theta\right] = \frac{M g}{2}(3 - 5\cos\theta)$

or $\quad P = \frac{W}{2}(3 - 5\cos\theta) \qquad (\because W = M g)$

Example 2.19: A right cone, of vertical angle 2α, can turn freely about an axis, passing through the centre of its base and perpendicular to the axis, if the cone starts from rest with its axis horizontal, show that when the axis is vertical, the thrust on the fixed axis is to weight of the cone as

$$1 + \frac{1}{2}\cos^2\alpha \text{ to } 1 - \frac{1}{3}\cos^2\alpha$$

Solution : Let M be mass of cone, a be radius of base $(a = OE)$ and h be height of cone $(h = OD)$, then $OG = \frac{h}{4}$, where G is $C.G.$ of cone. The cone turn round a horizontal diameter AB of the base of cone.

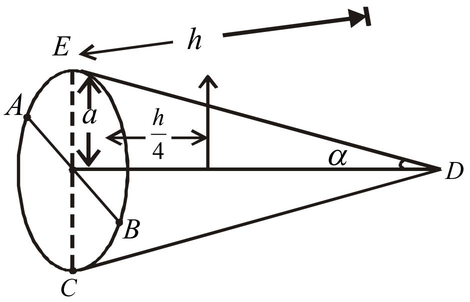


Figure 2.28 (a)
(Initial Position)

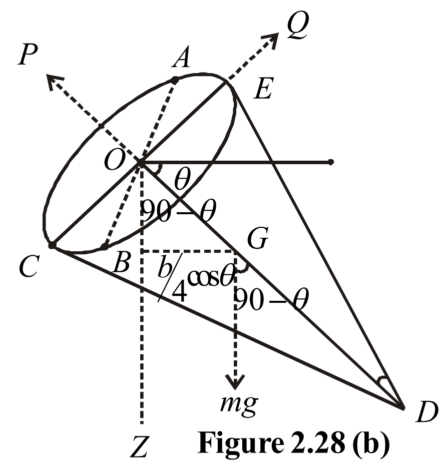


Figure 2.28 (b)
(Displaced Position)

When cone turns, its $C.G.$, G will describe a circle of radius $OG = \frac{h}{4}$ about O as centre. At any time t let axis of cone makes an angle θ with horizontal OL. Initially

$$t = 0,\ \theta = 0,\ \dot{\theta} = 0 \qquad \text{...(1)}$$

Now, M.I. of cone about an axis through $C.G.$ and $\perp$ to the axis of cone (i.e. about GY as in fig. 2.28 (a)) is

$$\frac{3M}{80}\left(h^2 + 4a^2\right) \qquad \text{...(2)}$$

If Mk^2 be the M.I. of cone about axis of rotation (AB), then

$$Mk^2 = \text{M.I. about } GY + M\,(OG)^2$$

$$= \frac{3M}{80}\left(h^2 + 4a^2\right) + M\left(\frac{h}{4}\right)^2$$

or $$Mk^2 = \frac{M}{80}\left(3h^2 + 12a^2 + 5h^2\right) \Rightarrow k^2 = \frac{1}{20}\left(2h^2 + 3h^2\tan^2\alpha\right)$$

$$\left(\because In\ \Delta EOD,\ \tan\alpha = \frac{a}{h} \Rightarrow a = h\tan\alpha\right)$$

or $$k^2 = \frac{h^2}{20}\left(2 + 3\tan^2\alpha\right) \qquad \text{...(3)}$$

Now, equations of motion of G along and perpendicular to GO are

$$M\left(\frac{h}{4}\right)\dot{\theta}^2 = P - M\,g\sin\theta \qquad \text{(along } GO) \qquad ...(4)$$

and $$M\left(\frac{h}{4}\right)\ddot{\theta} = M\,g\cos\theta - Q \qquad (\perp \text{ to } GO) \qquad ...(5)$$

where P and Q are components of reactions at O along GO and perpendicular to GO. In eqn (5), $M\,g\cos\theta$ is positive as it is in the sense of θ increasing and Q is negative.

Also taking moment about O, we get

$$M\,k^2\,\ddot{\theta} = M\,g\,.\,\frac{h}{4}\cos\theta$$

or $$k^2\,\ddot{\theta} = \frac{g\,h}{4}\cos\theta \qquad ...(6)$$

Integrating by multiplying both sides by $2\,\dot{\theta}$ w.r. to t

$$k^2\,\dot{\theta}^2 = \frac{g\,h}{2}\sin\theta + C_1 \qquad ...(7)$$

to determine C_1, using eqn (1), $\theta = 0$, $\dot{\theta} = 0$

$C_1 = 0$

$\therefore$ from eqn (7)

$$k^2\,\dot{\theta}^2 = \frac{g\,h}{2}\sin\theta \qquad ...(8)$$

Now, using (6) in eqn (5), we get

$$\frac{M\,h}{4}\left(\frac{g\,h}{4\,k^2}\cos\theta\right) = M\,g\,\cos\theta - Q$$

$$\therefore \qquad Q = M\,g\,\cos\theta\left[1 - \frac{h^2}{16\,k^2}\right]$$

Clearly, when the axis is vertical, then $\theta = 90^\circ$, $\therefore\ Q = 0$

Now, using value of $\dot{\theta}^2$ from (8) in (4), we get

$$P = M\,g\,\sin\theta + M\,.\,\frac{h}{4}\,\frac{g\,h}{2\,k^2}\sin\theta = M\,g\,\sin\theta\left[1 + \frac{h^2}{8\,k^2}\right]$$

when $\dot{\theta} = 90^\circ$, then $P = M\,g\left[1 + \frac{h^2}{8\,k^2}\right]$

or $$\frac{P}{Mg} = \left[1 + \frac{1}{8} \cdot \frac{20}{2 + 3\tan^2\alpha}\right] \qquad \left(\because \textit{from } (3), \frac{h^2}{k^2} = \frac{20}{2 + 3\tan^2\alpha}\right)$$

$$= \frac{16 + 24\tan^2\alpha + 20}{8(2 + 3\tan^2\alpha)} = \frac{3 + 6\sec^2\alpha}{6\sec^2\alpha - 2} = \frac{3\cos^2\alpha + 6}{6 - 2\cos^2\alpha}$$

or $$\frac{P}{Mg} = \frac{\frac{1}{2}\cos^2\alpha + 1}{1 - \frac{1}{3}\cos^2\alpha} \Rightarrow \frac{P}{W} = \frac{1 + \frac{1}{2}\cos^2\alpha}{1 - \frac{1}{3}\cos^2\alpha}$$

2.9 Centre of Percussion

Definition : If a body, rotating about a given axis, is so struck that there is no impulsive pressure on the axis, then any point on the line of acting of the force is called a centre of percussion. The line of action of the force is called the line of percussion.

Axis of spontaneous rotation : If the line of action of the blow is given, then the axis about which the body begins to turn is called the axis of spontaneous rotation. Evidently it coincides with the position of the fixed axis in the first case.

To find the centre of percussion of a rod suspended freely from one end struck by a blow.

Consider a rod OA of length $2a$ and mass M, which is freely suspended from one end O and struck by a horizontal blow (say) F at a point B of the rod such that $OB = x\ (say)$

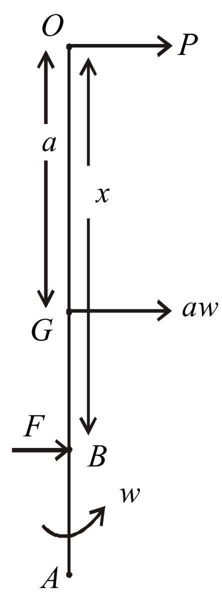


Figure 2.29

When the rod is struck by blow F, let P be the impulsive action at O and w be the angular velocity communicated to the rod, then velocity of the $C.G.$ G of rod after the blow is $a\,w$.

Hence equating the change in the momentum in the horizontal direction to the total external impulse in that direction, we have

$$M\,(a\,w - 0) = F + P, \qquad \text{i.e. } M\,a\,w = F + P \qquad ...(1)$$

Also we know that change in the moment of momentum about axis of rotation through O is equal to moment of external impulses, so we get

$$M k^2 (w - 0) = F \,.\, x + P \,.\, o \qquad (\because \text{ before blow } w = 0)$$

or $$M k^2 w = F x \qquad ...(2)$$

where k is radius of gyration of rod about axis of rotation through O,

$\therefore$ $$k^2 = \frac{4}{3} a^2 \qquad ...(3)$$

If the blow F has been given through the centre of percussion, then there should be no impulse on the axis of rotation, so $P = 0$, then eqn (1) reduces to

$$M a w = F \qquad ...(4)$$

To determine the position of the centre of percussion we shall eliminate F between the equation (2) and (4). From (2)

$$F = \frac{M k^2 w}{x},$$

Now, from (4)

$$M a w = \frac{M k^2 w}{x} \Rightarrow x = \frac{k^2}{a} \qquad ...(5)$$

Which is the length of simple equivalent pendulum.. therefore in this case the centre of persucssion coincides with the centre of oscillation.

Self Learning Exercise - 3

1. Write expressions for horizontal reaction and vertical reaction.
2. Define centre of percussion.
3. Define axis of spontaneous rotation.

Illustrative Examples :

Example 2.20: Find the position of centre of percussion in the following cases :

(i) Uniform circular plane, axis is horizontal tangent.

(ii) A uniform rod with one end fixed.

Solution : (i) Here $\bar{x}$ = distance of $C.G.$ from horizontal tangent AX is $= a$

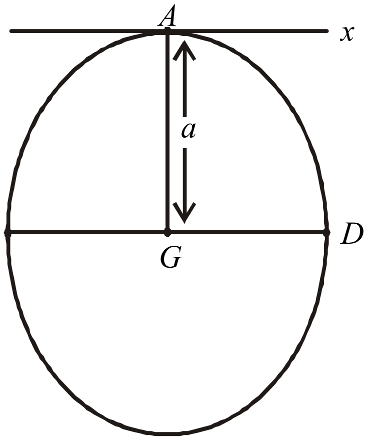


Figure 2.30

k be radius of gyration about horizontal tangent AX

$$\therefore\ k^2 = \frac{a^2}{4} + a^2 = \frac{5}{4}a^2 \qquad \left(\because \text{ M.I. about diameter } CO = \frac{Ma^2}{4}\right)$$

$\therefore$ distance of centre of percussion below the highest point

$$= \frac{k^2}{\bar{x}} = \frac{\left(\frac{5}{4}a^2\right)}{a} = \left(\frac{5}{4}a\right)$$

(ii) In this case, let length of rod $= 2a$, then

$\bar{x}$ = distance of $C.G.$ of rod from fixed end $= a$, and

$$k^2 = \frac{4}{3}a^2$$

(M.I. of rod of length $2a$, about an axis through an end and perpendicular to the rod is $\frac{4}{3}Ma^2$)

$\therefore$ distance of centre of percussion below the highest point

$$= \frac{k^2}{\bar{x}} = \frac{\frac{4}{3}a^2}{a} = \left(\frac{4}{3}a\right)$$

Example 2.21: Find the centre of percussion of a triangle ABC which is free to move about its side BC.

Solution : Let M be mass of triangle ABC. Let D, E, F be middle points of sides BC, CA and AB respectively of triangle ABC. Draw perpendicular AL from A on side BC. Also draw median AD. Let $AL = h$.

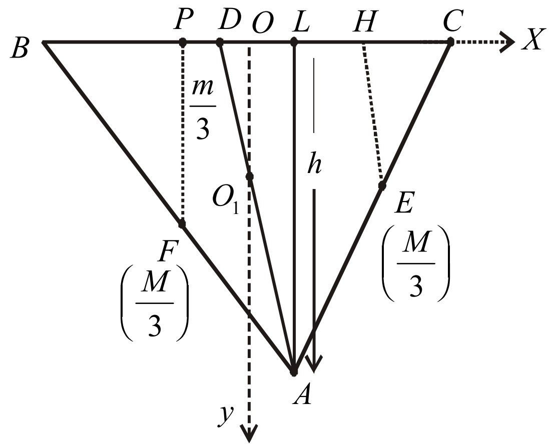


Figure 2.31

Take O as middle point of DL, OX as x-axis (along side BC) with oy perpendicular to BC as another coordinate axis. Then we can prove that side BC (or axis ox) is principal axis at O.

The triangle ABC is kinetically equivalent to three particles each of mass $\frac{M}{3}$ placed at D, E and F.

$\therefore$ P.I. about OX and oy of trinagle = P.I. of three particles D, E and F

$$= \frac{1}{3} M \,.\, OH \,.\, \frac{h}{2} + \frac{1}{3} M\,(-\,PO)\,.\,\frac{h}{2} + \frac{M}{3}\,.\,O$$

(for E) (for F) (for D)

($\because$ coordinates of E and F are $\left(OH, \frac{h}{2}\right)$, are $\left(-\,OP, \frac{h}{2}\right)$ with respect to ox, oy)

or P.I. of $\Delta\, ABC = \frac{Mh}{6}\left[OH - OP\right] = \frac{Mh}{6}\left[OL + LH - (OD + DP)\right]$

$$= \frac{Mh}{6}\left[LH - DP\right] \qquad (\because OD = OL)$$

or P.I. of $\Delta\, ABC$ about $ox, oy = \frac{Mh}{6}\left[\frac{1}{2} LC - \left(BD - BP\right)\right]$

$$= \frac{Mh}{6}\left[\frac{1}{2} b\cos C - \frac{a}{2} + \frac{c}{2}\cos B\right]$$

$$= \frac{Mh}{12}\left[b\cos C - c\cos B - a\right]$$

$$= \frac{Mh}{12}\left[a - a\right] = 0$$

Now $M k^2 =$ M.I. of ΔABC about side BC

= M.I. of three particles D, E and F each of mass $\frac{M}{3}$

$$M k^2 = \frac{M}{3}\left(\frac{h}{2}\right)^2 + \frac{M}{3}\left(\frac{h}{2}\right)^2 + \frac{M}{3}(0)^2 = \frac{1}{6} M h^2 \quad ...(1)$$

or $$k^2 = \frac{h^2}{6}$$

$\bar{x} =$ the depth of $C.G.$ of the centre of percussion below BC along a vertical through O

$$= \frac{k^2}{\bar{x}} = \frac{\left(h^2/6\right)}{\left(h/3\right)}$$

$$= \left(\frac{h}{2}\right)$$

this will be point O_1, which is point of intersection of axis oy and median AD i.e. it is mid point of median AD.

2.10 Summary

1. In this unit you have studied about moment of momentum, moment of effective forces, kinetic energy about axis of rotation, principle of angular momentum, the compound pendulum, centre of percussion.
2. Some results of velocities and accelerations, formula for finding length of simple equivalent pendulum and formula for finding position of centre of percussion will help the students to easily understand various results obtain in this unit.

2.11 Exercise

1. A uniform chain of length 20 metre and mass 40 kg. hangs in equal lengths over a solid circular pulley of mass 10 kg. and small radius, the axis of the pulley being horizontal. Masses of 40 kg. and 35 kg are attached to the ends of the chain and motion takes place, show that the time taken by the smaller mass to reach the pulley is $\frac{5\sqrt{6}}{7}\log_e\left(9 + 4\sqrt{5}\right)$ sec, where $g = 9.8\ m/\sec^2$.

2. A uniform disc of mass M, is free to turn about a horizontal axis through its centre perpendicular to its plane. A particle of mass m is attached to a point in the edge of the disc. If the motion starts from the upward in which radius to the particle makes an angle α with the upward vertical, find the angular velocity when m is in its lowest position.

$$\left[W = 2\cos\frac{\alpha}{2}\sqrt{\frac{2mg}{a(2m+M)}}, \text{ a being radius of the disc}\right]$$

3. A rigid body can turn freely about an axis fixed in the body and in space. To find the moment of effective forces and kinetic energy about the axis of rotation.

$$\left[\text{moment of effective forces } = M k^2 \ddot{\theta};\ K.E. = \frac{1}{2} M k^2 \dot{\theta}^2\right]$$

4. A flat circular disc of radius a has a hole in it of radius b whose centre is at a depth c from the centre of disc $(c < a - b)$. The disc is free to oscillate in a vertical plane about a smooth horizontal circular rod of radius b passing through the hole. Show that the length of the equivalent simple pendulum is

$$c + \frac{1}{2} \frac{a^4 - b^4}{a^2 c}$$

5. Three equal particles are attached to a weightless rod at equal distance apart. The system is suspended from, and is free to turn about, a point of the rod distance x from the middle particle. Find the time of a small oscillation and show that it is least when $x = (0.82\ a)$ nearly.

$$\left[T = 2\pi \sqrt{\frac{3x^2 + 2a^2}{3xg}},\ 2\pi \sqrt{\frac{2a\sqrt{b}}{3g}}\right]$$

6. Find the lengths of simple equivalnet pendulums in the following cases, the axis being horizontal

(i) A circular wire, axis is a tangent to it

(ii) A cube of side $2a$, axis being an edge. $\left[(i)\ l = \frac{3a}{2},\ (ii)\ l = \frac{4}{3} a\sqrt{2}\right]$

7. A circular area can turn freely about a horizontal axis which passes through a point O of its circumference and is perpendicular to its plane. If motion commences when the diameter through O is vertically above O, show that, when the diameter has turned through an angle θ, the components of the diameter has turned perpendicular to this diameter are respectively

$\frac{1}{3} W (7\cos\theta - 4)$ and $\frac{1}{3} W \sin\theta$.

8. A uniform rod of length $2a$ and weight W is turning about its end O and starts from the position in which it was vertically above O. When it has turned through an angle θ, show that the horizontal and vertical reaction are $\frac{3}{4} W \sin\theta\,(2 - 3\cos\theta)$ and $\frac{1}{4} W (1 - 3\cos\theta)^2$.

9. A pendulum is constructed of a solid sphere of mass M and radius a which is attached to the end of a rod of mass m and length b. Show that there will be no strains on the axis if the pendulum be struck at a distance $\left[M\left\{\frac{2}{5}a^2 + (a+b)^2\right\} + \frac{1}{3} mb^2\right] \div \left[M(a+b) + \frac{1}{2} m b^2\right]$ from the axis.

10. Find the position of the centre of percussion of a sector of a circle, axis in the plane of sector

perpendicular to its symmetrical radius and passing through the centre of circle.

$$\left[\frac{3a\,(\alpha + \sin\alpha\,\cos\alpha)}{8\sin\alpha},\ a = \text{radius of circle},\ 2\alpha = \text{angle of sector}\right]$$

Answers

Self Learning Exercise - 1

1. Radial acceleration $\left(-\,a\dot{\theta}^2\right)$ along OP, where O is pole and P is $(r,\,\theta)$ point, and transverse acceleration $= a\,\ddot{\theta}$, perpendicular to radius vector OP.

2. $M\,k^2\,\ddot{\theta}$, where k is radius of gyration about axis of rotation.

3. $M\,k^2\,\dfrac{d\theta}{dt}$

Self Learning Exercise - 2

1. $T = 2\pi\sqrt{\dfrac{l}{g}}$ 2. See art. 2.7

3. $T = 2\pi\sqrt{\dfrac{k^2}{gh}}$, where k is the radius of gyration of body about a fixed axis and h is the distance of the centre of inertia from fixed axis.

4. See art. 2.8

Self Learning Exercise - 3

1. $R_H = P\sin\theta - Q\cos\theta,\ R_V = P\cos\theta + Q\sin\theta$

2. See art. 2.12 3. See art. 2.12

□□□□

UNIT - 3
Motion of a Rigid Body in Two Dimensions Under Finite Forces and Impulsive Forces

Structure of the unit

3.0 Objective

This unit provides a general overveiw of motion of a rigid body in two dimensions under finite forces and under impulsive forces. After reading this unit you will be able to learn

1. About equations of motion in two dimensions under finite forces and impulsive forces
2. About K.E. of a rigid body in a two dimensional motion in terms of motion of centre of inertia and motion relative in to centre of inertia. Also change in K.E. due to action of impulsive forces.
3. About angular momentum about origin, Rolling and Slinding frictions.
4. Rolling of sphere on a rough inclined plane and sliding of a rod, sliding and rolling of a sphere on an inclined plane.
5. About rolling and sliding of a sphere on a fixed sphere, unstable euqilibrium between two smooth spheres.
6. About rolling of a solid cylinder, inside a rough hollow cylinder
7. About sliding and rolling of a rotating disc on a rough horizontal plane.
8. About impact of a rotating elastic sphere on a fixed horizontal rough plane.

3.1 Introduction

First we shall study here the motion in two diemnsions when the impressed forces are finite in nature. The motion of rigid body takes place in three dimenstions, the study of its motion is very much simplified when all particles of the body move parallel to a given fixed polane. Such a motion of the rigid body is called two dimensional motion or plane motion. In such a case the motion can be considered as a translation motion parallel to a given fixed plane plus a rotation about a suitable axis perpendicular to the plane. This axis is often choosen to pass through the centre of mass of the body under consieration.

In this unit, secondly we shall study the motion in two dimension under impulsive forces where the effect of finite forces on the body are neglected.

3.2 Equations of Motion in Two Dimensions when the forces are finite

We know that the motion of a rigid body consists of two independent motion :

(i) The motion of the centre of gravity (motion of translation)

(ii) The motion about centre of gravity (motion of rotation)

The motion of centre of gravity : By this we mean that the total mass M of rigid body is supposed to be concentrated at the $C.G.$ and all the external forces are transferred parallel to themselves to act at the $C.G.$ of the body.

The motion about the centre of gravity : By this we mean that the sum of moment of the effective forces about the $C.G.$ is equal to sum of the moments of the external forces about $C.G.$

Classical method for finding the equations of motion : Let us consider the motion of a rigid body (to start we may take a lamina) in a fixed plane (say), xy plane. Let $(\bar{x}, \bar{y})$ be the coordinate of $C.G.$ of the body and M be the mass of the body concentrated at G, then we have

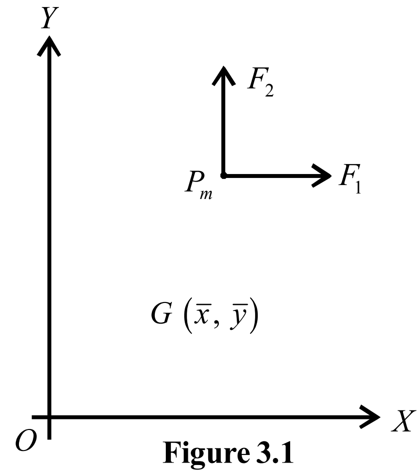


Figure 3.1

$$M\,\frac{d^2x}{dt^2} = \sum F_1 \qquad ...(1)$$

and $$M\,\frac{d^2y}{dt^2} = \sum F_2 \qquad ...(2)$$

where $\sum F_1$ = the algebraic sum of the components of the external forces parallel to x-axis, supposed to be acting at the $C.G.$

and $\sum F_2$ = The algebraic sum of the components of the external forces paralle to y-axis.

For rotational motion, Let P be any particle of body of mass m whose coordinates relative to $C.G.\ G$ as origin be (x^1, y^1), then from earlier knowledge else from unit-1, we have the following equation

$$\sum m\left(x^1\,\frac{d^2y^1}{dt^2} - y^1\,\frac{d^2x^1}{dt^2}\right) = \sum\,(x^1F_2 - y^1F_1)$$

or $$\sum m\,\frac{d}{dt}\left(x^1\,\frac{dy^1}{dt} - y^1\,\frac{dx^1}{dt}\right) = \sum\,(x^1F_2 - y^1F_1)$$

or $$\frac{d}{dt}\sum m\left(x^1\,\frac{dy^1}{dt} - y^1\,\frac{dx^1}{dt}\right) = \sum\,(x^1F_2 - y^1F_1) \qquad ...(3)$$

But $\left(x^1\,\frac{dy^1}{dt} - y^1\,\frac{dx^1}{dt}\right)$ is the moment about G of the velocity of particle of mass m relative to G.

Let GL be line fixed in space (in xy plane) and GA be line fixed in the body (fixed with lamina)

Let $GP = r$ and let GP makes an angle ϕ with line GL. Again Let θ an angle which the line GA makes with GL, then from fig. 3.2, it is clear that

$$\phi = \theta + \angle AGP \qquad ...(4)$$

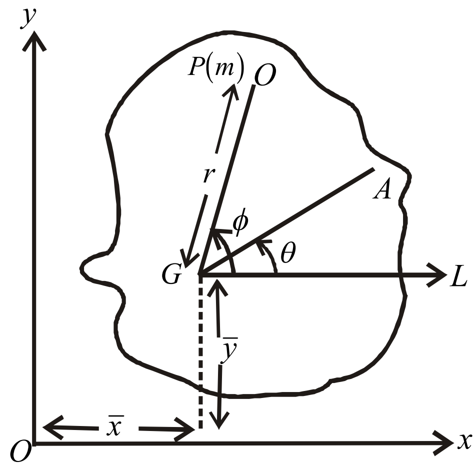


Figure 3.2

But $\angle AGP$ = constant, because it is the angle between two fixed lines GA and GP so on differentiating (4)

$$\frac{d\phi}{dt} = \frac{d\theta}{dt} \qquad ...(5)$$

Now the velocity of particle of mass m at P relative to G is $r\,\dfrac{d\phi}{dt}$ and hence its moment about G is

o

$$r\,.\left(r\,\frac{d\phi}{dt}\right) = r^2\,\frac{d\phi}{dt} = r^2\,\frac{d\theta}{dt} \qquad \text{[from (5)]} \qquad ..(6)$$

$$\therefore \left(x^1\,\frac{dy^1}{dt} - y^1\,\frac{dx^1}{dt}\right) = r^2\,\frac{d\theta}{dt} \qquad ...(7)$$

Therefore from eqns (3) and (7), we have

$$\frac{d}{dt}\left[\sum m\left(r^2\,\frac{d\theta}{dt}\right)\right] = \sum\,\left(x^1 F_2 - y^1 F_1\right)$$

or $$\frac{d}{dt}\left[\sum m\,r^2\,\frac{d\theta}{dt}\right] = \sum\,\left(x^1 F_2 - y^1 F_1\right)$$

or $$\frac{d}{dt}\left[\frac{d\theta}{dt}\,.\sum m\,r^2\right] = \sum\,\left(x^1 F_2 - y^1 F_1\right)$$

or $$\frac{d}{dt}\left[\frac{d\theta}{dt}\,.\,Mk^2\right] = \sum\,\left(x^1 F_2 - y^1 F_1\right) \qquad ...(8)$$

where k is the radius of gyration of the body about an axis through G perpendicular to the plane of motion i.e. xy plane.

or $\qquad M k^2 \dfrac{d^2 \theta}{d t^2} = \sum \left(x^1 F_2 - y^1 F_1\right) = (say)\ L$ ...(9)

where L is the moment of all external forces acting on the body about G.

Equation (1), (2) and (9) are taken as the equations of motion of rigid body moving in two dimensions under finite forces.

3.3 Kinetic Energy in terms of the Motion of the Centre of Inertia and motion relative to Centre of Inertia

Let (x, y) be the coordinates of a particle of mass m of the rigid body referred to the fixed axes ox and oy and let $(\bar{x}, \bar{y})$ be coordinates of G referred to O as origin and $\left(x^1, y^1\right)$ be coordinates of P referred to G as origin, then (see fig. 3.1)

$$x = \bar{x} + x^1,\ y = \bar{y} + y^1 \qquad ...(1)$$

clearly $\dfrac{\sum m x^1}{\sum m} = x$ - coordinate of $C.G. = 0$

$$\frac{\sum m y^1}{\sum m} = y \text{ - coordinates of } C.G. = 0$$

as $\left(x^1, y^1\right)$ are the coordinate of P referred to G as origin

Hence $\sum m x^1 = 0, \sum m y^1 = 0$ ...(2)

$$\therefore \qquad \sum m \frac{d x^1}{d t} = 0, \sum m \frac{d y^1}{d t} = 0 \qquad ...(3)$$

Thus, the K.E. of the particle P of mass m is

$$= \frac{1}{2} m \left\{\left(\frac{d x}{d t}\right)^2 + \left(\frac{d y}{d t}\right)^2\right\}$$

Hence the K.E. of the whole body is

$$= \frac{1}{2} \sum m \left\{\left(\frac{d x}{d t}\right)^2 + \left(\frac{d y}{d t}\right)^2\right\}$$

$$= \frac{1}{2} \sum m \left\{\left(\frac{d \bar{x}}{d t} + \frac{d x^1}{d t}\right)^2 + \left(\frac{d \bar{y}}{d t} + \frac{d y^1}{d t}\right)^2\right\}$$

$$= \frac{1}{2} \sum m \left\{\left(\frac{d \bar{x}}{d t}\right)^2 + \left(\frac{d \bar{y}}{d t}\right)^2\right\} + \frac{1}{2} \sum m \left\{\left(\frac{d x^1}{d t}\right)^2 + \left(\frac{d y^1}{d t}\right)^2\right\}$$

$$+ \sum m \frac{d \bar{x}}{d t} \cdot \frac{d x^1}{d t} + \sum m \frac{d \bar{y}}{d t} \cdot \frac{d y^1}{d t}$$

Last two terms will be zero by virtue of eqn (3),

Hence K.E. of whole body

$$= \frac{1}{2}\left[\left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2\right]\left(\sum m\right) + \frac{1}{2}\sum m\, v'^2$$

where v^1 is the velocity of any particle of mass m relative to G and it is equal to $r\,\frac{d\phi}{dt} = r\,\frac{d\theta}{dt}$

$\therefore$ K.E. of whole body

$$= \frac{1}{2}M\left[\left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2\right] + \frac{1}{2}\sum m\left(r\,\frac{d\theta}{dt}\right)^2$$

$$= \frac{1}{2}MV^2 + \frac{1}{2}\left(\sum m\, r^2\right)\left(\frac{d\theta}{dt}\right)^2$$

$$= \frac{1}{2}MV^2 + \frac{1}{2}Mk^2\left(\frac{d\theta}{dt}\right)^2$$

where k is radius of gyration of the body about a line through $C.G.$ perpendicular to the plane of the motion. V is velocity of $C.G.$, hence $\frac{1}{2}MV^2$ is the K.E. of a particle of mass M placed at $C.G.$, and

$\frac{1}{2}Mk^2\dot{\theta}^2 = \frac{1}{2}\sum m v^{1^2} = \frac{1}{2}\sum m\left[\left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dy^1}{dt}\right)^2\right]$ is the K.E. of the body relative to G.

$$\therefore \text{K.E.} = \frac{1}{2}MV^2 + \frac{1}{2}Mk^2\dot{\theta}^2$$

= (K.E. of a particle of a mass M placed at G and moving with it)

+ (K.E. of the body relative to $C.G.$)

= K.E. due to translation motion + K.E. due to rotation

3.4 Moment of Momentum of the body about the Fixed Origin

Let (x, y) be the coordinates of P referred to O as origin, $(\bar{x}, \bar{y})$ be coordinates of G referred to O as origin and $\left(x^1, y^1\right)$ are coordinates of P referred to G as origin, then the moment of momentum of a particle of mass m at P about origin O is

$= m\left(x\,\frac{dy}{dt} - y\,\frac{dx}{dt}\right)$, where $\frac{dx}{dt}, \frac{dy}{dt}$ are the velocities of particle of mass m parallel to coordinate axes.

Therefore moment of momentum of whole body is

$$= \sum m\left(x\,\frac{dy}{dt} - y\,\frac{dx}{dt}\right) \qquad ...(1)$$

Now $x = \bar{x} + x^1,\ y = \bar{y} + y^1$ [by eqn (1) and art. 3.3] ...(2)

Also $\left(\dfrac{\sum m x^1}{\sum m}, \dfrac{\sum m y^1}{\sum m}\right)$, are the coordinates of $C.G.$ referred to G as origin, so $\left(\dfrac{\sum m x^1}{\sum m}, \dfrac{\sum m y^1}{\sum m}\right) = (0, 0)$

$$\left.\begin{aligned}
&\therefore \quad \sum mx^1 = 0, \sum my^1 = 0\\
&\text{or} \quad \sum m\frac{dx^1}{dt} = 0, \sum m\frac{dy^1}{dt} = 0\\
&\text{Also} \quad \sum mx^1\frac{d\bar{y}}{dt} = \frac{d\bar{y}}{dt}\left(\sum mx^1\right) = 0\\
&\text{and} \quad \sum my^1\frac{d\bar{x}}{dt} = \frac{d\bar{x}}{dt}\left(\sum my^1\right) = 0
\end{aligned}\right\} \qquad ...(3)$$

Now, from eqn (1), after using (2), we have

$$\sum m\left[\left(\bar{x} + x^1\right)\left(\frac{d\bar{y}}{dt} + \frac{dy^1}{dt}\right) - \left(\bar{y} + y^1\right)\left(\frac{d\bar{x}}{dt} + \frac{dx^1}{dt}\right)\right]$$

$$= \sum m\left(\bar{x}\,\frac{d\bar{y}}{dt} - \bar{y}\,\frac{d\bar{x}}{dt}\right) - \sum m\left(x^1\,\frac{dy^1}{dt} - y^1\,\frac{dx^1}{dt}\right)$$

Rest of the terms vanish in view of (3)

$\therefore$ Moment of momentum of whole body

$$= \left(\bar{x}\,\frac{d\bar{y}}{dt} - \bar{y}\,\frac{d\bar{x}}{dt}\right)\left(\sum m\right) + \sum m\left(x^1\,\frac{dy^1}{dt} - y^1\,\frac{dx^1}{dt}\right)$$

$$= M\left(\bar{x}\,\frac{d\bar{y}}{dt} - \bar{y}\,\frac{d\bar{x}}{dt}\right) + \sum m\left(x^1\,\frac{dy^1}{dt} - y^1\,\frac{dx^1}{dt}\right) \qquad ..(4)$$

$$= M_G + M_R \ (say)$$

where $M_G = M\left(\bar{x}\,\dfrac{d\bar{y}}{dt} - \bar{y}\,\dfrac{d\bar{x}}{dt}\right)$ = moment of momentum about O of a particle of mass M placed at $G = M\,v\,p$ ...(5)

where v is the velocity of $C.G.$ of mass M placed at G and p is the perpendicular from O of upon the direction of velocity v of $C.G.\ G$,

and $$M_R = \sum m\left(x^1 \frac{d\,y^1}{d\,t} - y^1 \frac{d\,x^1}{d\,t}\right) = \text{Moment of momentum of body about } G$$

$$= \sum m\left(r\,\frac{d\,\phi}{d\,t}\right) r$$

where $r\,\dfrac{d\,\phi}{d\,t}$ is the velocity of particle of mass m relative to G which is equal to $r\,\dfrac{d\,\theta}{d\,t}$ (as shown in art. 3.2)

$$\therefore \quad M_R = \sum m\left(r\,\frac{d\,\theta}{d\,t}\right) r = \sum m\left(r^2\,\frac{d\,\theta}{d\,t}\right) = \frac{d\,\theta}{d\,t}\left(\sum m\,r^2\right) = \frac{d\,\theta}{d\,t}\,M\,k^2 \qquad ...(6)$$

where k is the radius of gyration of the body about a line through $G \perp$ to $x\,y$ plane

$$\therefore \quad \text{Moment of momentum} = M\,v\,p + M\,k^2\,\frac{d\theta}{d\,t}$$

= (Moment of momentum of $C.G.$ of the body) + (Moment of momentum relative to $C.G.$)

3.5 Rolling and Sliding Friction

Friction is a self adjusting force which tends to prevent the relative motion of the point at which it acts. But if μ is the coefficient of friction and R be normal reaction then it has a limiting value $\mu\,R$

During the motion of rough bodies in any direction, we assume a friction F opposite to the direction of relative motion and assume that the point of contact is at relative rest, which is expressed by a geometrical equation.

Further if

(i) $F < \mu\,R$, it is the case of pure rolling.

(ii) $F = \mu\,R$, body is on the commencement of slipping. It is the case of rolling with limiting friction.

(iii) $F > \mu\,R$, it is the case of rolling and sliding combined. Sliding or slipping follows and F is to be replaced by $\mu\,R$. Motion need to be discussed fresh without the geometrical relations.

(iv) $F = 0 \Rightarrow \mu = 0$, so motion is on smooth surface. It is the case of pure sliding.

3.6 Rolling of a sphere on a rough inclined plane

To discuss the motion of a uniform sphere which rolls down an inclined plane, rough enough to prevent any sliping.

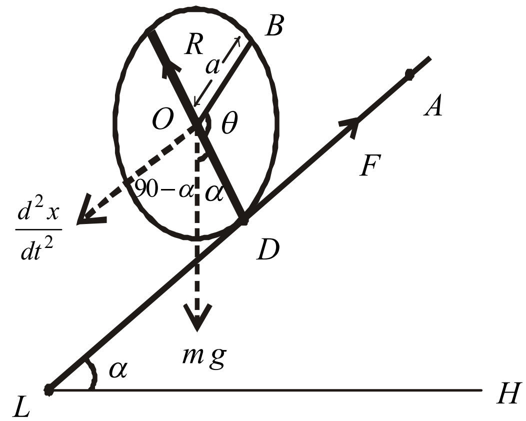


Figure 3.3

Let M be the mass, a be the radius and O be the centre of the sphere. LH is horizontal plane, LA is an inclined plane. Initially, let the sphere be at rest with its point B in contact with A.

$\therefore$ At A, $t = 0$, $x = 0$ and $\dot{x} = 0$ ...(1)

After time t, let the centre of sphere describe a distance x parallel to the inclined plane and radius of sphere OB have turned through an angle θ, i.e. $\angle BOD = \theta$

Since there is no sliding, therefore

$$\left.\begin{array}{l} \text{distance } AD = arc\,DB \text{ or } x = a\theta \\ \text{so that } \dot{x} = a\dot{\theta},\ \ddot{x} = a\ddot{\theta} \end{array}\right\} \quad ...(2)$$

Here OB is a line fixed in the body which makes an angle θ with the normal to the plane which is a line fixed in space.

Let R be the normal reaction and F be the frictional force then we have the following equations of motion of centre of gravity O of sphere.

$$M\frac{d^2x}{dt^2} = Mg\sin\alpha - F \qquad \text{(motion of } O \text{ in direction of } AL) \quad ...(3)$$

and $$O = Mg\cos\alpha - R \qquad \text{(motion of } O \text{ perpendicular to inclined } AL) \quad ...(4)$$

Also, for the motion about centre of inertia, taking the moments about O, we get

$$Mk^2\ddot{\theta} = F.a + o.R$$

or $$Mk^2\frac{\ddot{x}}{a} = F.a \qquad \text{[from (2)]}$$

or $$Mk^2\frac{d^2x}{dt^2} = a^2\left(Mg\sin\alpha - M\frac{d^2x}{dt^2}\right) \qquad \text{[from (3)]}$$

or $\quad M\left(a^2 + k^2\right)\dfrac{d^2x}{dt^2} = a^2\ Mg\sin\alpha$

or $\quad \dfrac{d^2x}{dt^2} = \dfrac{a^2 g\sin\alpha}{\left(a^2 + k^2\right)}$ ...(5)

R.H.S. is constant, so the sphere rolls down with constant acceleration. Integrating eqn (5), we get

$$\frac{dx}{dt} = \frac{a^2 g\sin\alpha}{\left(a^2 + k^2\right)}t + C_1 \quad ...(6)$$

but by eqn (1) $t = 0,\ \dot{x} = 0$

$\therefore \quad C_1 = 0$

$\therefore \quad \dfrac{dx}{dt} = \dfrac{a^2 g\sin\alpha}{\left(a^2 + k^2\right)}t$ ...(7)

Again integrating it, we get

$$x = \frac{a^2 g\sin\alpha}{\left(a^2 + k^2\right)}\cdot\frac{t^2}{2} + C_2 \quad ...(8)$$

but again $x = 0,\ t = 0 \Rightarrow C_2 = 0$

Also from (5)

$$2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} = \frac{2a^2 g\sin\alpha}{\left(a^2 + k^2\right)}\frac{dx}{dt} \rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g\sin\alpha}{\left(a^2 + k^2\right)}x + C_3$$

with above conditions $C_3 = 0$

$\therefore \quad (\dot{x})^2 = \dfrac{2a^2 g\sin\alpha}{\left(a^2 + k^2\right)}\cdot x$ ...(8 A)

$\therefore \quad x = \dfrac{1}{2}\dfrac{a^2 g\sin\alpha}{\left(a^2 + k^2\right)}t^2$, which is the required distance described by the sphere in time t

...(9)

For solid sphere $k^2 = \dfrac{2}{5}a^2$, using in eqn (5), we get

$$\ddot{x} = \frac{a^2 g\sin\alpha}{\left(a^2 + \frac{2}{5}a^2\right)} = \frac{5a^2 g\sin\alpha}{7a^2} = \frac{5}{7}g\sin\alpha \quad ...(10)$$

Using $\ddot{x}$ from (10) in (3), we get

$$M\cdot\frac{5}{7}g\sin\alpha = Mg\sin\alpha - F$$

$$\Rightarrow \qquad F = Mg\sin\alpha - \frac{5}{7}Mg\sin\alpha = \frac{2}{7}Mg\sin\alpha \qquad ...(11)$$

which is the frictional force and normal reaction, from eqn (4)

$$R = Mg\cos\alpha \qquad ...(12)$$

Condition of pure Rolling :

Here $\dfrac{F}{R} = \dfrac{\frac{2}{7}Mg\sin\alpha}{Mg\cos\alpha} = \dfrac{2}{7}\tan\alpha$ [by eqn (11) and (12)]

$\therefore$ for pure rolling, $\dfrac{F}{R} < \mu$ or $F < \mu R$

$$\therefore \qquad \mu > \frac{2}{7}\tan\alpha$$

which should be satisfied during the motion.

Kinetic energy of the body :

$$\text{K.E. of sphere (at any time } t) = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}Mk^2\dot{\theta}^2$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}Mk^2\frac{\dot{x}^2}{a^2} \qquad \left(\because \dot{\theta} = \frac{\dot{x}}{a}\right)$$

$$= \frac{1}{2}M\dot{x}^2\left(1 + \frac{k^2}{a^2}\right)$$

$$= Mg(x\sin\alpha) \qquad \text{from (8 A)}$$

which is the same as the work done by gravity and also follows from the principle of energy and work.

Particular Cases : From the symmetry of the body, we may derive corresponding results for acceleration by subtituting the value of k (radius of gyration) for different bodies.

For different bodies

Body	**radius of gyration** k^2	**Acceleration**
Uniform sphere of radius a	$\frac{2}{5}a^2$	$\frac{5}{7}g\sin\alpha$ (see 10)
Thin solid disc of radius a	$\frac{1}{2}a^2$	$\frac{2}{3}g\sin\alpha$
Thin spherical shell of radius a	$\frac{2}{3}a^2$	$\frac{3}{5}g\sin\alpha$
Uniform thin ring of radius a	a^2	$\frac{1}{2}g\sin\alpha$

3.7 Slipping of Rods

(i) A uniform rod is held in a vertical position with one end resting upon a rough table, and when released rotates about the end in contact with the table to discuss the motion.

Let ox be top of perfectly rough table and OA be a rod of mass M and length $2a\ (= OA)$. The rod is capable of rotating about end O end let it make an angle θ with vertical.

Choosing O as origin and horizontal and vertical lines through O as axes. Then coordinates of G ($C.G.$ of rod) will be (x, y) $OG = a$, $GB = a\sin\theta$, $OB = a\cos\theta$

$$\therefore \quad x = a\sin\theta, \quad y = a\cos\theta \qquad ...(1)$$

on differentiating $w.r.$ to t

$$\dot{x} = a\cos\theta\,\dot{\theta},\ \ddot{x} = -a\sin\theta\,\dot{\theta}^2 + a\cos\theta\,\ddot{\theta} \qquad ...(2)$$

and $$\dot{y} = -a\sin\theta\,\dot{\theta},\ \ddot{y} = -a\cos\theta\,\dot{\theta}^2 - a\sin\theta\,\ddot{\theta} \qquad ...(3)$$

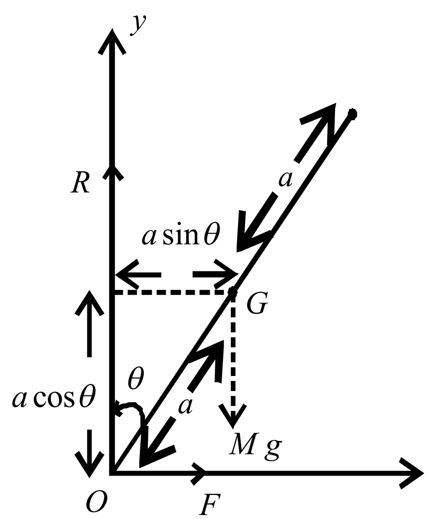


Figure 3.4

Let F be the frictional force on the table acting along ox and R be the normal reaction at O acting along oy (fig. 3.4) then equations of motion of $C.G.$ are

$$M\frac{d^2x}{dt^2} = M\left(-a\sin\theta\,\dot{\theta}^2 + a\cos\theta\,\ddot{\theta}\right) = F \quad \text{(In horizontal direction)} \qquad ...(4)$$

and $$M\ddot{y} = M\left(-a\cos\theta\,\dot{\theta}^2 - a\sin\theta\,\ddot{\theta}\right) = R - Mg \quad \text{(In vertical direction)} \qquad ...(5)$$

Also $$v^2 = \dot{x}^2 + \dot{y}^2 = a^2\dot{\theta}^2 \text{ and } k^2 = \frac{a^2}{3} \quad \text{(for rod } OA \text{ of length } 2a) \qquad ...(6)$$

The distance fallen by $C.G.$ in downward direction $= (a - a\cos\theta)$ ...(7)

Hence energy equation is

$$\frac{1}{2}Mv^2 + \frac{1}{2}Mk^2\dot{\theta}^2 = Mg(a - a\cos\theta)$$

or $\quad \dfrac{1}{2} M a^2 \dot{\theta}^2 + \dfrac{1}{2} M \dfrac{a^2}{3} \dot{\theta}^2 = M g a (1 - \cos\theta)$ [by eqn (6)]

or $\quad \dfrac{2}{3} M a^2 \dot{\theta}^2 = M g a (1 - \cos\theta)$

or $\quad \dot{\theta}^2 = \dfrac{3g}{2a} (1 - \cos\theta)$ ...(8)

Differenting eqn (8) with respect to t

$$2\dot{\theta}\ddot{\theta} = \frac{3g}{2a}\left(\sin\theta\,\dot{\theta}\right) \Rightarrow \ddot{\theta} = \frac{3g}{4a}\sin\theta \qquad ...(9)$$

Substituting values of $\dot{\theta}$ and $\ddot{\theta}$ from (8) and (9) in (4), we get

$$F = Ma\left[\cos\theta \,.\, \frac{3g}{4a}\sin\theta - \sin\theta\,\frac{3g}{2a}(1-\cos\theta)\right]$$

or $\quad F = \dfrac{3}{4} M g \sin\theta\,(3\cos\theta - 2)$ ...(10)

Also from eqn (5), on putting values of $\dot{\theta}$ and $\ddot{\theta}$, we get

$$R = \frac{1}{4} M g (1 - 3\cos\theta)^2 \qquad ...(11)$$

these relation (10) and (11) determine the motion.

Analysis :

(i) R does not change sign and it is always positive (eqn 11). It vanishes when $\cos\theta = \dfrac{1}{3}$, therefore end O does not leave the table.

(ii) F changes its sign as θ passes through the angle $\cos^{-1}\dfrac{2}{3}$ and hence its direction is then reversed (eqn 10)

(iii) The ratio $\dfrac{F}{R} \to \infty$ as $\cos\theta \to \dfrac{1}{3}$

Hence unless the plane is rough enough, there must be sliding then. (eqn (10) and (11))

(iv) The end O of rod will begin to slip for some value of θ less than $\cos^{-1}\left(\dfrac{1}{3}\right)$, and it will slip backwards or forward accroding as the slipping occurs before or after the inclination of rod is $\cos^{-1}\left(\dfrac{1}{3}\right)$.

(ii) A uniform straight rod slides down a vertical plane its end being in contact with two smooth planes; one horizontal and other vertical. If it starts from rest at an angle α with the horizontal, to discuss the motion.

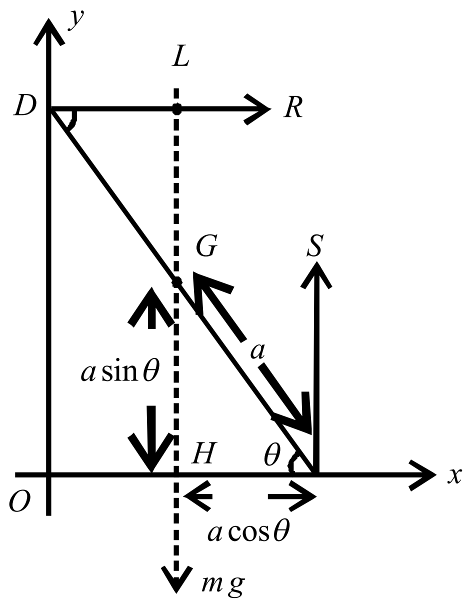


Figure 3.5

Let CD be the rod of length $2a$ and mass M, which is in contact with two smooth planes, one horizontal (OCx plane) and the other vertical (ODy plane) at which the normal reaction are S and R respectively. Let G be $C.G.$ of rod and θ be the inclination of the rod to the horizontal at any time t.

Initially $t = 0,\ \theta = \alpha,\ \dot{\theta} = 0$ ...(1)

Let coordinates of G about ox, oy as axes be (x, y) so that

$$\left.\begin{aligned} x &= OH = DL = a\cos\theta \\ y &= HG = a\sin\theta \end{aligned}\right\} \quad ...(2)$$

on differentiating with respect to t, we get

$$\left.\begin{aligned} \dot{x} &= -a\sin\theta\,\dot{\theta}\ ;\ \ddot{x} = -a\cos\theta\,\dot{\theta}^2 - a\sin\theta\,\ddot{\theta} \\ \dot{y} &= a\cos\theta\,\dot{\theta}\ ;\ \ddot{y} = -a\sin\theta\,\dot{\theta}^2 + a\cos\theta\,\ddot{\theta} \end{aligned}\right\} \quad ...(3)$$

Equations of motion of centre of gravity are

$$M\ddot{x} = R \Rightarrow M\left(-a\cos\theta\,\dot{\theta}^2 - a\sin\theta\,\ddot{\theta}\right) = R \quad \text{(In } ox \text{ direction)} \quad ...(4)$$

and $$M\ddot{y} = S - Mg \Rightarrow M\left(-a\sin\theta\,\dot{\theta}^2 + a\cos\theta\,\ddot{\theta}\right) = S - Mg \text{(in } oy \text{ direction)} \quad ...(5)$$

Initially, when the inclination of rod was α, the hieght of $C.G.$ was $a\sin\alpha$ and now it is $a\sin\theta$ so that the distance described by $C.G.$ is $a\,(\sin\alpha - \sin\theta)$. Also, if v be velocity of $C.G.$, then

$$v^2 = \left(\dot{x}^2 + \dot{y}^2\right) = \left(a^2\,\dot{\theta}^2\right)$$

Therefore from energy equation

$$\frac{1}{2}Mv^2 + \frac{1}{2}Mk^2\,\dot{\theta}^2 = \text{(work done by gravity)}$$

we get $\frac{M}{2}\left[a^2\dot{\theta}^2 + \frac{a^2}{3}\dot{\theta}^2\right] = M g a\,(\sin\alpha - \sin\theta)$ $\qquad \left[\because k^2 = \frac{a^2}{3}\right]$

or $\qquad \dot{\theta}^2 = \frac{3g}{2a}(\sin\alpha - \sin\theta)$ ...(6)

Differentiating with respect to t, and dividing by $2\dot{\theta}$

$$\ddot{\theta} = -\frac{3g}{4a}\cos\theta \qquad \text{...(7)}$$

Now putting values of $\dot{\theta}^2$ and $\ddot{\theta}$ from (6) and (7) in (4), we get

$$M\left\{-a\cos\theta\,.\,\frac{3g}{2a}(\sin\alpha - \sin\theta) - a\sin\theta\left(-\frac{3g}{4a}\cos\theta\right)\right\} = R$$

Simplifying

$$R = \frac{3M}{4}g\cos\theta\,(3\sin\theta - 2\sin\alpha) \qquad \text{...(8)}$$

Now putting values of $\dot{\theta}^2$ and $\ddot{\theta}$ from (6) and (7), in (5), we get

$$S = Mg + M\left\{-a\sin\theta\times\frac{3g}{2a}(\sin\alpha - \sin\theta) + a\cos\theta\times\frac{-3g}{4a}\cos\theta\right\}$$

$$= Mg + Mg\left\{-\frac{3}{2}\sin\theta\sin\alpha + \frac{3}{2}\sin^2\theta - \frac{3}{4}\cos^2\theta\right\}$$

$$= \frac{Mg}{4}\left\{4 - 6\sin\theta\sin\alpha + 6\sin^2\theta - 3\cos^2\theta\right\}$$

$$= \frac{Mg}{4}\left\{1 - 6\sin\theta\sin\alpha + 6\sin^2\theta + 3\left(1 - \cos^2\theta\right)\right\}$$

$$= \frac{Mg}{4}\left\{1 - 6\sin\theta\sin\alpha + 9\sin^2\theta\right\}$$

$$= \frac{Mg}{4}\left\{9\sin^2\theta - 6\sin\theta\sin\alpha + \sin^2\alpha + \left(1 - \sin^2\alpha\right)\right\}$$

or $\qquad S = \frac{Mg}{4}\left\{(3\sin\theta - \sin\alpha)^2 + \cos^2\alpha\right\}$ ...(9)

From equation (8), we find that reaction R at wall vanishes if $\sin\theta = \frac{2}{3}\sin\alpha$, and for smaller values of θ, R becomes negative. In this stuation the rod will leave the wall with angular velocity $\dot{\theta}^2$

i.e. $\qquad \dot{\theta} = \sqrt{\frac{3g}{2a}(\sin\alpha - \sin\theta)} = \sqrt{\frac{g\sin\alpha}{2a}}$ $\qquad \left(\text{on using } \sin\theta = \frac{2}{3}\sin\alpha\right)$ ...(10)

The centre of mass G will have the horizontal velocity

$$\dot{x} = -a \sin\theta \, \dot{\theta} = -a \frac{2}{3} \sin\alpha \sqrt{\frac{g \sin\alpha}{2a}}$$

$$= -\frac{1}{3} \sqrt{2ag \sin^3 \alpha} \qquad ...(11)$$

Second Part of the Motion : (When the end D leaves the plane)

When $\sin\theta = \frac{2}{3} \sin\alpha$, and $R = 0$

Let $\theta = \phi$ and let S_1 be the normal reaction at floor. Then the equations governing the motion of rod are

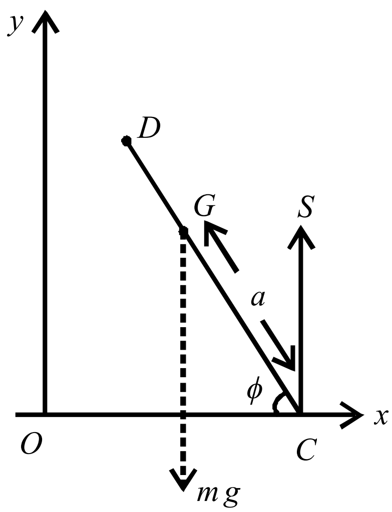


Figure 3.6

$$M\ddot{x} = 0 \qquad ...(12)$$

and $$M\ddot{y} = S_1 - Mg \qquad ...(13)$$

taking moments about G, we have

$$M \frac{a^2}{3} \ddot{\phi} = -S_1 (a \cos\phi) \qquad ...(14)$$

where $y = a \sin\phi, \therefore \dot{y} = a \cos\phi \, \dot{\phi}, \ddot{y} = -a \sin\phi \, \dot{\phi}^2 + a \cos\phi \, \ddot{\phi}$ ...(15)

using value of $\ddot{y}$ from (15) in (13), we have

$$M\left(-a \sin\phi \, \dot{\phi}^2 + a \cos\phi \, \ddot{\phi}\right) = S_1 - Mg$$

$$= \frac{Ma^2 \ddot{\phi}}{3a \cos\phi} - Mg \qquad \text{[from (14)]}$$

or $$\left(\frac{1}{3} + \cos^2\phi\right) \ddot{\phi} - \sin\phi \cos\phi \, \dot{\phi}^2 = -\frac{g}{a} \cos\phi$$

on integrating it, we get

$$\left(\frac{1}{3} + \cos^2\phi\right)\dot{\phi}^2 = -\frac{2g}{a}\sin\phi + C_1 \qquad ...(16)$$

Initially when $\phi = \theta$, $\sin\phi = \frac{2}{3}\sin\alpha$

and $\dot{\phi} = \dot{\theta} = \sqrt{\dfrac{g\sin\alpha}{2g}}$,

$$C_1 = \frac{2g\sin\alpha}{a}\left(1 - \frac{\sin^2\alpha}{9}\right) \qquad ...(17)$$

using this values of C_1 in (16), we get

$$\left(\frac{1}{3} + \cos^2\phi\right)\dot{\phi}^2 = \frac{2g\sin\alpha}{a}\left(1 - \frac{\sin^2\alpha}{9}\right) - \frac{2g}{a}\sin\phi \qquad ...(18)$$

when the rod becomes horiozontal i.e. when $\phi = 0$, Let w be the angular velocity i.e. $\dfrac{d\phi}{dt} = w$ where $\phi = 0$

$$w^2 = \frac{3g\sin\alpha}{2a}\left(1 - \frac{\sin^2\alpha}{9}\right) \qquad ...(19)$$

and by eqn (11) the rod in this elapsed time had its constant horizontal velocity $\dot{x} = -\frac{1}{3}\sqrt{2ag\sin^3\alpha}$

Self Learning Exercise - I

1. Define two deminsional motion under finite forces.
2. Write exprssion for K.E. of a rigid body in a two dimensional motion under finite forces.
3. What is friction?
4. What is condition for pure sliding?
5. What is condition for pure rolling?

Illustrative Examples :

Example 1 : Two equal cylinders each of mass m are bound together by an elastic string whose tension is T and roll with their axes horizontal down a rough plane of inclination α. Show that their acceleration is $\frac{2}{3}g\sin\alpha\left[1 - \frac{2\mu T}{mg\sin\alpha}\right]$, where μ is the coefficient of friction between the cylinders.

Solution : Let R_1, F_1 be the normal reaction and the frictional force on the upper cylinder respectively

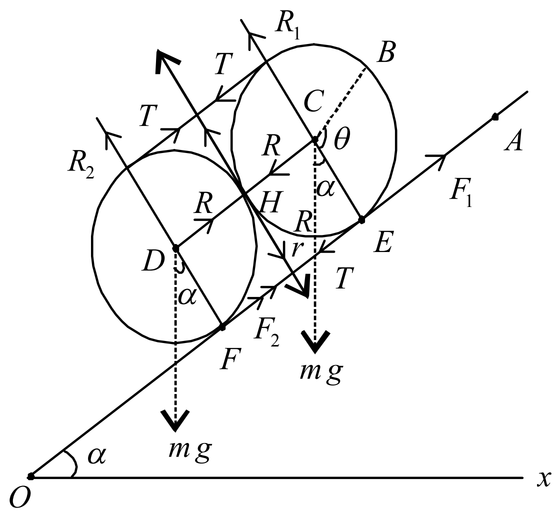


Figure 3.7

and R_2, F_2 be the normal reaction and the frictional force on the lower cylinder respectively, due to plane OA which is inclined at angle α with the horizon. Let R be normal reaction between the two cylinders at their point of contact H (fig. 3.7). Then at H the frinctional force μR acts away from the inclined plane for upper cylinder and towards the inclined plane for lower cylinder. Let T be the tension in the string. At any time t let the cylinders move through a distance x along the inclined plane and θ be the angle turned by radius CB of cylinders. Since there is no slipping

$$\therefore\ x = a\theta, \quad \text{on differentiating } \dot{x} = a\dot{\theta} \ \ \ddot{x} = a\ddot{\theta} \qquad ...(1)$$

Now the equations of motion of upper cylinder are

$$m\ddot{x} = mg\sin\alpha + 2T - F_1 - R \qquad \text{(parallel to inclined plane } AO\text{)} \qquad ...(2)$$

and $$m\ddot{y} = 0 = R_1 - mg\cos\alpha + \mu R \qquad \text{(perpendicular to inclined plane)} \qquad ...(3)$$

and moment equation is

$$m\,k^2\,\ddot{\theta} = F_1 - \mu R a \qquad (k \text{ is radius of gyration about axis through } C) \qquad ...(4)$$

where m is mass of cylinder.

Equations of motion for lower cylinder are

$$m\ddot{x} = mg\sin\alpha - 2T - F_2 + R \qquad \text{(parallel to inclined plane)} \qquad ...(5)$$

$$m\ddot{y} = 0 = R_2 - mg\cos\alpha - \mu R \qquad \text{(perpendicular to inclined plane)} \qquad ...(6)$$

and moment equation is

$$mk^2\,\ddot{\theta} = F_2\,a - \mu R a \qquad ...(7)$$

from eqn (4) and (7), we get

$$F_1 = F_2 \qquad ...(8)$$

Now, from equation (2) and (5), we get

$$mg\sin\alpha + 2T - F_1 - R = mg\sin\alpha - 2T - F_2 + R$$

or $$4T - 2R = 0 \Rightarrow R = 2T \qquad ...(9)$$

Also from eqn (4),

$$F_1 = \mu R + m\frac{k^2}{a}\ddot{\theta}$$

$$= \mu(2T) + m\,.\,\frac{a^2}{2a}\ddot{\theta} \qquad \left(\because k^2 = \frac{a^2}{2}\right)$$

or $$F_1 = 2\mu T + \frac{ma}{2}\frac{\ddot{x}}{2} = 2\mu T + \frac{1}{2}m\ddot{x}$$

Now using this value of F_1 and (9) in eqn (2), we get

$$m\ddot{x} = mg\sin\alpha + 2T - \left(2\mu T + \frac{m\ddot{x}}{2}\right) - 2T$$

or $$\frac{3m}{2}\ddot{x} = mg\sin\alpha - 2\mu T$$

or $$\ddot{x} = \frac{2}{3}g\sin\alpha\left(1 - \frac{2\mu T}{mg\sin\alpha}\right)$$

Example 2 : A uniform solid cylinder is placed with its axis horizontal on a plane, whose incilination to the horizon is α, show that the least coefficient of friction between it and the plane, so that is may roll and not slide, is $\frac{1}{3}\tan\alpha$.

If the cylinder be hollow, and of small thickness, the least value is $\frac{1}{2}\tan\alpha$.

Solution : At any time t let the axis of cylinder describe a distance x and θ be the angle turned (fig. 3.3.) than $x = a\theta$ (since there is no slinding) $\rightarrow$ $\dot{x} = a\dot{\theta}$ and $\ddot{x} = a\ddot{\theta}$

If F be the frictional force and R be normal reaction, then equations of motion of $C.G.$ are

$$M\ddot{x} = Mg\sin\alpha - F \qquad \text{(parallel to inclined plane)} \qquad ...(1)$$

$$O = Mg\cos\alpha - R \qquad \text{(perpendicular to inclined plane)} \qquad ...(2)$$

taking moments about the axis through the centre of gravity O, we have

$$mk^2\ddot{\theta} = F\,.\,a$$

or $$Mk^2\frac{\ddot{x}}{a} = F\,.\,a \qquad ...(3)$$

Eliminating $M\ddot{x}$ from (3) and (1), we have

$$\frac{k^2}{a}(Mg\sin\alpha - F) = F\,.\,a \qquad \text{or} \qquad F = \frac{k^2}{a^2 + k^2}Mg\sin\alpha \qquad ...(4)$$

and from eqn (2), $R = Mg\cos\alpha$

$\therefore \quad \frac{F}{R} = \left(\frac{k^2}{a^2 + k^2} \tan\alpha\right)$, but $k^2 = \frac{a^2}{2}$ (for solid cylinder)

$\therefore \quad \frac{F}{R} = \frac{1}{3} \tan\alpha$

For pure rolling $F < \mu R$ or $\frac{F}{R} < \mu$

or $\quad \mu > \frac{F}{R} \Rightarrow \mu > \left(\frac{1}{3} \tan\alpha\right)$ ...(5)

Hence for pure rolling least coefficient of friction is $\frac{1}{3} \tan\alpha$.

If cylinders be hollow, then $k^2 = a^2$, therefore from (4)

$$F = \frac{1}{2} Mg \sin\alpha \qquad \text{and} \quad \frac{F}{R} = \frac{1}{2} \tan\alpha$$

For pure rolling $\mu > \frac{F}{R} \Rightarrow \mu > \frac{1}{2} \tan\alpha$

$\therefore$ In case of hollow cylinders for pure rolling least coefficient of frictions is $\frac{1}{2} \tan\alpha$.

Example 3 : A uniform rod is placed with one end in contact with a horizontal table, and is then at an inclination α to the horizon and is allowed to fall. When it becomes horizontal, show that its angular velocity is $\sqrt{\left(\frac{3g \sin\alpha}{2a}\right)}$, whether the plane be perfectly smooth or perfectly rough. Show also that the end of the rod will not leave the plane in either case.

Solution : At any time t, a rod OA of length $2a$ and mass m makes angle θ with the horizontal G be $C.G.$ of the rod, then its coordinates are $(x = a\cos\theta,\ y = a\sin\theta)$ Initially the rod is inclined at an angle α to the horizontal so that height of $C.G.$ was $a\sin\alpha$ and now it is $a\sin\theta$. So in this time distance travelled by $C.G.$ in downward direction

$(a\sin\alpha - a\sin\theta)$ ...(1)

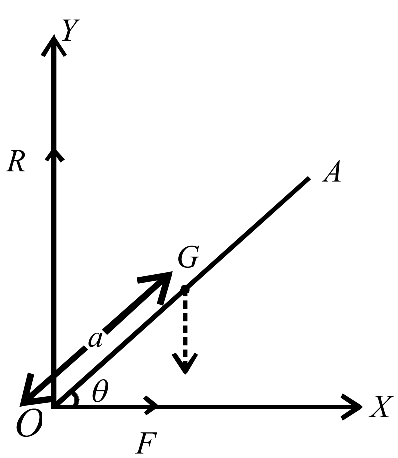


Figure 3.8

We have to find the angular velocity when the rod becomes horizontal i.e. when $\theta = 0$

Case - I : When the plane is perfectly rough : Let F be the friction and R be the normal reaction then the equation of energy gives

$$\frac{1}{2} M v^2 + \frac{1}{2} M k^2 \dot{\theta}^2 = \text{work done by gravity}$$

or $$\frac{1}{2} M \left(a^2 \dot{\theta}^2 + \frac{a^2}{3} \dot{\theta}^2 \right) = m g \,(a \sin\alpha - a \sin\theta) \qquad \text{[by eqn (1)]}$$

$\because$ $$v^2 = \dot{x} + \dot{y}^2 = \left(- a \sin\theta \dot{\theta}\right)^2 + \left(a \cos\theta \dot{\theta}\right)^2 = a^2 \dot{\theta}^2$$

and $k^2 = \dfrac{a^2}{3}$

$\therefore$ $$\frac{m}{2} \left(\frac{4}{3} a^2 \dot{\theta}^2 \right) = m g a \left(\sin\alpha - \sin\theta\right)$$

$\Rightarrow$ $$\dot{\theta}^2 = \frac{3g}{2a} (\sin\alpha - \sin\theta) \qquad \text{...(2)}$$

when $\theta = 0$ i.e. the rod becomes horizontal then if

$\dot{\theta} = w$, then from eqn (2)

$$w^2 = \frac{3g}{2a} \sin\alpha \Rightarrow w = \sqrt{\frac{3g \sin ga}{2a}} \qquad \text{...(3)}$$

Now we have to prove that the end O will not leave the table i.e. we have to show that R is always positive. Differentiating eqn (2), $w.\ r.$ to t

$$2\, \dot{\theta} \ddot{\theta} = \frac{3g}{2a} (- \cos\theta) \dot{\theta}$$, on dividing by $2\,\dot{\theta}$, we get

$$\ddot{\theta} = - \frac{3g}{4a} \cos\theta \qquad \text{...(4)}$$

The equation of motion of $C.G.$ in *oy* direction is

$$m \ddot{y} = R - m g \Rightarrow m \left(- a \sin\theta \dot{\theta}^2 + a \cos\theta \ddot{\theta}\right) = R - m g \qquad \text{...(5)}$$

Putting values of $\dot{\theta}^2$ and $\ddot{\theta}$ from equations (2) and (4), we get

$$R = m g + m a \left[- \sin\theta \ . \frac{3g}{2a} (\sin\alpha - \sin\theta) + \cos\theta \left(\frac{-3g}{4a} \cos\theta \right) \right]$$

$$= \frac{1}{4} m g \left[4 - 6 \sin\alpha \sin\theta + 6 \sin^2\theta - 3 \cos^2\theta\right]$$

$$= \frac{mg}{4}\left[1 + 3\left(1 - \cos^2\theta\right) - 6\sin\alpha\sin\theta + 6\sin^2\theta\right]$$

$$= \frac{mg}{4}\left[1 + 9\sin^2\theta - 6\sin\alpha\sin\theta + 9\sin^2\alpha\sin^2\theta - 9\sin^2\alpha\sin^2\theta\right]$$

$$= \frac{mg}{4}\left[\left(1 - 6\sin\alpha\sin\theta + 9\sin^2\alpha\sin^2\theta\right) + 9\sin^2\theta - 9\sin^2\alpha\sin^2\theta\right]$$

$$= \frac{mg}{4}\left[(1 - 3\sin\alpha\sin\theta)^2 + 9\sin^2\theta\left(1 - \sin^2\alpha\right)\right]$$

or $$R = \frac{mg}{4}\left[(1 - 3\sin\alpha\sin\theta)^2 + 9\sin^2\theta\cos^2\alpha\right] \qquad ...(6)$$

This shows that R is always positive for all values of θ and α, hence the end O of the rod never leaves the plane.

Case - II : When the plane is perfectly smooth. In this case there is no force of friction in the horizontal direction so that $C.G.$ moves in a vertical line, the only velocity of G being along the vertical.

$$y = a\sin\theta,\ \dot{y} = a\cos\theta\,\dot{\theta} \qquad \text{so} \qquad v^2 = \dot{y}^2 = \left(a\cos\theta\,\dot{\theta}\right)^2$$

then the enrgy equation is

$$\frac{m}{2}\left(v^2 + k^2\dot{\theta}^2\right) = \text{work done by gravity}$$

$$\therefore \quad \frac{m}{2}\left(a^2\cos^2\theta\dot{\theta}^2 + \frac{a^2}{2}\dot{\theta}^2\right) = mga(\sin\alpha - \sin\theta)$$

$$\frac{ma^2\dot{\theta}^2}{6}\left(3\cos^2\theta + 1\right) = mga(\sin\alpha - \sin\theta)$$

$$\therefore \quad \dot{\theta}^2 = \frac{6g}{a}\frac{(\sin\alpha - \sin\theta)}{\left(1 + 3\cos^2\theta\right)} \qquad ...(7)$$

when the rod becomes horizontal $\theta = 0$

$$\therefore \quad \dot{\theta}^2 = \frac{3g}{2a}\sin\alpha \Rightarrow \dot{\theta} = \sqrt{\left(\frac{3g}{2a}\sin\alpha\right)} \qquad ...(8)$$

Differentiating eqn (7), $w.r.$ to t, we get

$$2\dot{\theta}\ddot{\theta} = \frac{6g}{a}\left[\frac{\left(1 + 3\cos^2\theta\right)\left(-\cos\theta\,\dot{\theta}\right) - (\sin\alpha - \sin\theta)\left(-6\cos\theta\sin\theta\,\dot{\theta}\right)}{\left(1 + 3\cos^2\theta\right)^2}\right]$$

or $$2\dot{\theta}\ddot{\theta} = \frac{6g}{a}\cos\theta\,\dot{\theta}\left[\frac{-1 - 3\cos^2\theta + (\sin\alpha - \sin\theta)\times 6\sin\theta}{\left(1 + 3\cos^2\theta\right)^2}\right]$$

or $$\ddot{\theta} = -\frac{3g\cos\theta}{a}\left[\frac{1 + 3\left(1 - \sin^2\theta\right) - 6\sin\alpha\sin\theta + 6\sin^2\theta}{\left(1 + 3\cos^2\theta\right)^2}\right]$$

$$= -\frac{3g\cos\theta}{a}\left[\frac{4 + 3\left(\sin^2\theta - 2\sin\alpha\sin\theta\right) + 3\sin^2\alpha - 3\sin^2\alpha}{\left(1 + 3\cos^2\theta\right)^2}\right]$$

or $$\ddot{\theta} = -\frac{3g\cos\theta}{a}\left[\frac{1 + 3\cos^2\alpha + 3\left(\sin\alpha - \sin\theta\right)^2}{\left(1 + 3\cos^2\theta\right)^2}\right]$$

taking moment about G, we get

$$mk^2\ddot{\theta} = -Ra\cos\theta$$

or $$m\frac{a^2}{3}\ddot{\theta} = -Ra\cos\theta \qquad \left(\because k^2 = \frac{a^2}{3}\right)$$

$$\therefore \quad R = -m\frac{a^2}{3}\cdot\frac{1}{a\cos\theta}\ddot{\theta}$$

or $$R = \frac{-ma^2}{3}\cdot\frac{1}{a\cos\theta}\left[\frac{-3g\cos\theta}{a}\left\{\frac{1 + 3\cos^2\alpha + 3\left(\sin\alpha - \sin\theta\right)^2}{\left(1 + 3\cos^2\theta\right)^2}\right\}\right]$$

or $$R = mg\left[\frac{1 + 3\cos^2\alpha + 3\left(\sin\alpha - \sin\theta\right)^2}{\left(1 + 3\cos^2\theta\right)^2}\right] \qquad ...(9)$$

This also show that R is always positive. Therefore the end O does not leave the plane in this case also.

Example 4 : A uniform rod of mass m, is placed at right angles to a smooth plane of inclination α with one end in contact with it. The rod is then released. Show that when the inclination to the plane is ϕ, the reaction of the plane will be $\left\{\frac{3\left(1 - \sin\phi\right)^2 + 1}{\left(1 + 3\cos^2\phi\right)^2}\right\} mg\cos\alpha$.

Solution : The inclined plane AB is smooth and uniform rod BC of mass m and length $2a$ placed at right angles to it. So there is no force along the plane and there was no initial motion along the plane. Hence centre of mass G moves perpendicular to the plane. Initially its distance from the inclined plane AB was $BG = a$, so that its vertical height above the plane was $GH = a\cos\alpha$ (In ΔBGH)

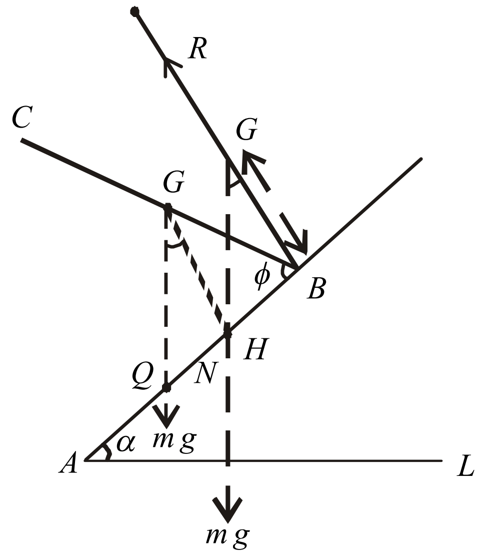


Figure 3.9

Now, when the rod is released then after time t let the rod make an angle ϕ with the inclined plane AB. Referred to BA and perpendicular to BA as axes let the coordinates of G be (x, y)

$$\therefore \quad \left.\begin{aligned} x &= BN = a\cos\phi \\ y &= GN = a\sin\phi \end{aligned}\right\} \qquad (\text{In } \Delta BGN) \qquad ...(1)$$

$\therefore$ Equation of motion G perpendicular to the plane is

$$m\ddot{y} = m\left(a\cos\phi\,\ddot{\phi} - a\sin\phi\,\dot{\phi}^2\right) = -R - mg\cos\alpha$$

Also taking moment about G

$$m\frac{a^2}{3}\ddot{\phi} = -R\,a\cos\phi \qquad ...(2)$$

Initial vertical height of $C.G.$ above the inclined plane was $GH = a\cos\alpha$, and now its vertical height above the plane is $GQ = GN\cos\alpha = (a\sin\phi)\cos\alpha$ [by eqn (1)]

Therefore distance fallen by G in downward direction

$$= GH - GQ = a\cos\alpha - a\sin\phi\cos\alpha$$

$$= a\cos\alpha\,(1 - \sin\phi) \qquad ...(3)$$

$\therefore$ Energy equation gives

$$\frac{m}{2}\left[v^2 + k^2\dot{\phi}^2\right] = \text{work done by gravity}$$

or $$\frac{m}{2}\left[a^2\cos^2\phi\,\dot{\phi}^2 + \frac{a^2}{3}\dot{\phi}^2\right] = mg\,a\cos\alpha\,(1 - \sin\phi) \qquad [\text{by eqn (3)}] \qquad ...(4)$$

$$\left(\because\ v^2 = \dot{y}^2 = \left(a\cos\phi\,\dot{\phi}\right)^2\right)$$

$$\therefore \quad \frac{ma^2}{6}\left[3\cos^2\phi + 1\right]\dot{\phi}^2 = mga\cos\alpha\,(1-\sin\phi)$$

or $$\dot{\phi}^2 = \frac{6g}{a}\frac{(1-\sin\phi)}{\left(1+3\cos^2\phi\right)}\cos\alpha \qquad ...(5)$$

Differentiating this $w.r.$ to t, we get

$$2\dot{\phi}\ddot{\phi} = \frac{6g}{a}\cos\alpha\left[\frac{\left(1+3\cos^2\phi\right)\left(-\cos\phi\dot{\phi}\right)-(1-\sin\phi)\left(-6\cos\phi\sin\phi\,\dot{\phi}\right)}{\left(1+3\cos^2\phi\right)^2}\right]$$

dividing $2\dot{\phi}$ and simplifying, we get

$$\ddot{\phi} = -\frac{3g\cos\alpha\cos\phi}{a\left(1+3\cos^2\phi\right)^2}\left[1+3(1-\sin\phi)^2\right] \qquad ...(6)$$

Substituting this value of $\ddot{\phi}$ in eqn (2), we get

$$R = mg\left\{\frac{3(1-\sin\phi)^2+1}{\left(1+3\cos^2\phi\right)^2}\right\}.\cos\alpha$$

Example 5 : A cylinder rolls down a smooth plane whose inclination to the horizon is α, unwrapping as it goes a fine string fixed to the highest point of the plane, find its acceleration and the tension of the string.

Solution : Let a be radius and m be mass of cylinder, BO is inclined plane which is smooth so that there will be no frictional force. But when cylinder rolls the inclined plane there will be a tension T in the string which is fixed to the highest point O of the inclined plane. When cylinder rolls along the plane, it unwraps a length (say) x of string. Since string remains tight geometrically. we have

$$x = a\theta, \text{ so that } \dot{x} = a\dot{\theta},\ \ddot{x} = a\ddot{\theta} \qquad ...(1)$$

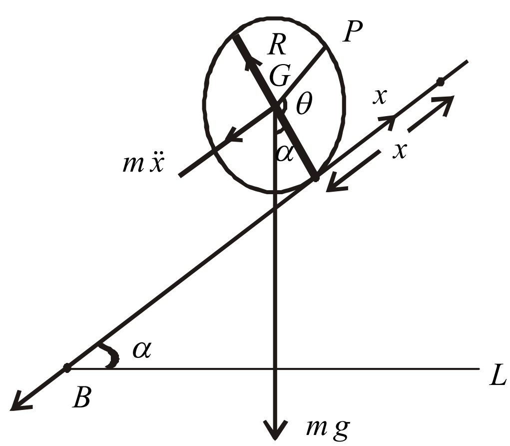


Figure 3.10

where θ is the angle which radius GP has turned in this time, initially P was at O.

Let R be normal reaction as shown in fig. 3.10, then the equations of motion of cylinder along and perpendicular to the inlcined plane are respectively

$$m\ddot{x} = mg\sin\alpha - T \qquad ...(2)$$

and $\quad O = m\ddot{y} = mg\cos\alpha - R \qquad ...(3)$

Now taking moment of forces about the centre G of cross section of cylinder, then

$$mk^2\ddot{\theta} = T\,.\,a$$

or $\quad m\dfrac{a^2}{2}\,.\,\dfrac{\ddot{x}}{a} = T\,.\,a \qquad \left(\because k^2 = \dfrac{a^2}{2}, \dfrac{\ddot{x}}{a} \equiv \dot{\theta}\right)$

or $\quad \dfrac{m\ddot{x}}{2} = T \qquad ...(4)$

Using this value of T in eqn (2), we get

$$m\ddot{x} = mg\sin\alpha - \frac{1}{2}m\ddot{x} \Rightarrow \frac{3}{2}\ddot{x} = g\sin\alpha$$

or $\quad \ddot{x} = \dfrac{2}{3}g\sin\alpha \qquad ...(5)$

which is acceleration of cylinder. To find tention T in the string, using this value of $\ddot{x}$ in (4), we get

$$T = \frac{1}{3}mg\sin\alpha \qquad ...(6)$$

3.8 Sliding and Rolling of a sphere on an inclined plane

An imperfactly rough sphere moves from rest down a plane inclined at an angle α to the horizon; To discuss its motion.

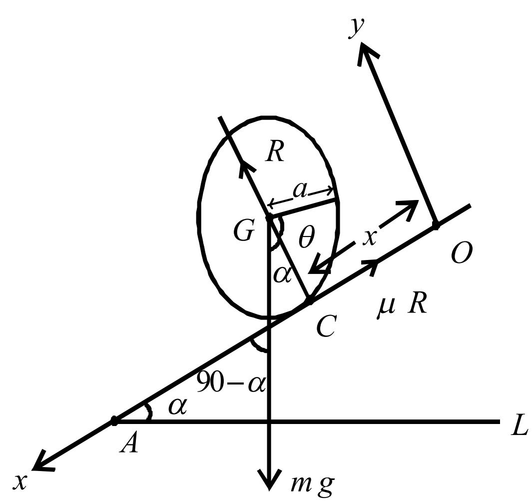


Figure 3.11

Let G be centre of sphere of mass m and radius a, and move along a rough inclined plane OCA of inclination α with horizon. Also let x be the distance describe by the centre G in time t and the sphere has rolled through an angle θ, i.e. $\angle CGB = \theta$. θ is the angle between radius GB, which was initially normal to the inclined plane and normal GC at time t. Initially point B was at O, so there

$$t = 0,\ x = 0,\ \dot{x} = 0,\ \theta = 0,\ \dot{\theta} = 0 \qquad ...(1)$$

Referred to ox and oy as axes, coordinates of G are (x, y). We take that the friction is not enough to produce pure rolling hence the sphere slides as well as turns, and the maximum friction μR acts up the inclined plane as shown in fig. 3.11, where μ is coefficient of friction.

There is no motion perpendicular to the inclined plane and as such the $C.G.\,G$ of the sphere always moves parallel to the plane

$\therefore$ y coordinate of $C.G.$ = constant, $\therefore \dot{y} = 0,\ \ddot{y} = 0$ ...(2)

Hence the equation of motionof centre of gravity (along and perpendicular to inclined plane) are

$$m\ddot{x} = mg\sin\alpha - \mu R \quad ...(3)$$

$$m\ddot{y} = 0 = R - mg\cos\alpha \quad ...(4)$$

and moment equation about G is

$$mk^2\ddot{\theta} = \mu Ra \quad ...(5)$$

where k is radius of gyration of sphere about axis through G, so $k^2 = \frac{2}{5}a^2$

$$\therefore \quad m\,.\,\frac{2}{5}a^2\ddot{\theta} = \mu Ra \quad ...(6)$$

Eliminating R between eqn (3) and (4)

$$\ddot{x} = (g\sin\alpha - \mu g\cos\alpha) \quad ...(7)$$

Integrating it with respect to t

$$\dot{x} = (g\sin\alpha - \mu g\cos\alpha)\,t + C_1 \quad ...(8)$$

but $t = 0,\ \dot{x} = 0,\ \therefore\ C_1 = 0$ [by eqn (1)]

$$\therefore \quad \dot{x} = (g\sin\alpha - \mu g\cos\alpha)\,t \quad ...(9)$$

again integrating

$$x = g\,(\sin\alpha - \mu\cos\alpha)\,\frac{t^2}{2} + C_2$$

again by eqn (1), $x = 0,\ t = 0$, we get $C_2 = 0$

$$\therefore \quad x = g\,(\sin\alpha - \mu\cos\alpha)\,\frac{t^2}{2} \quad ...(10)$$

Now from eqn (4), using value of R in eqn (6), we get

$$\frac{2}{5}ma^2\ddot{\theta} = \mu a\,(mg\cos\alpha)$$

or $$\ddot{\theta} = \frac{5\mu g}{2a}\cos\alpha \quad ...(11)$$

integrating it with respect to t

$$\dot{\theta} = \frac{5\mu g}{2a}\cos\alpha \,.\, t + C_3$$

Initially $t = 0, \dot{\theta} = 0$, hence $C_3 = 0$

$$\therefore \quad \dot{\theta} = \frac{5\mu g}{2a}\cos\alpha \; t \quad ...(12)$$

again integrating

$$\theta = \frac{5\mu g \cos\alpha}{2a}\frac{t^2}{2} + C_4$$

Again, $\theta = 0, t = 0$, hence $C_4 = 0$

$$\therefore \quad \theta = \frac{5\mu g \cos\alpha}{4a} t^2 \quad ...(13)$$

Now velocity of point of contact C down the plane

= velocity of centre G + velocity of C relative to G

$$= \dot{x} + \left(-a\dot{\theta}\right) = gt\,(\sin\alpha - \mu\cos\alpha) - a \times \frac{5\mu g\cos\alpha}{2a} t \quad \text{[by eqn (9) and (12)]}$$

velocity of C down the plane

$$= \frac{g}{2}\left[2\sin\alpha - 7\mu\cos\alpha\right] t \quad ...(14)$$

From eqn (8), we will have the following three cases for **discussion :**

First Case : Slding with rolling :

If $(2\sin\alpha - 7\mu\cos\alpha) > 0$ or $2\sin\alpha > 7\mu\cos\alpha$ or $\mu < \frac{2}{7}\tan\alpha$, then the velocity of the point of contact C does not vanish and is always positive for all values of t. So the point of contact C always slides down and the maximum friction μR always acts. In other words sliding of the sphere with turning takes place. Therefore the motion is entirely determined by equations (9), (10) and (12), (13).

Second Case : Rolling with limiting friction :

When $(2\sin\alpha - 7\mu\cos\alpha) = 0 \Rightarrow \mu = \frac{2}{7}\tan\alpha$.

In this case the velocity of C vanishes at the start and is always zero. Therefore it is the case of pure rolling, the maximum friction μR is always being exterted. Hence motion is determined by equations (12) and (13). We can also apply geometrical relation $x = a\theta$.

Third Case : Pure rolling : when $(2\sin\alpha - 7\mu\cos\alpha) < 0$ or $\mu > \frac{2}{7}\tan\alpha$, then velocity of the point of contact appears to be negative. In other words if maximum force of friction μR is allowed to e x e r t , the point of contact will slide upwards the inclined plane which is not possible as the friction acts only with a force which is sufficient to keep the point of contact at rest. Hence in this case pure rolling takes place from the begining but maximum force of friction is not exerted. So the equation of motion written above (eqn (9), (10), (11) and (12)) will not hold good. If F be the force of friction (not maximum) then corresponding equations of motion are

$$m\ddot{x} = mg\sin\alpha - F \qquad \text{(along the plane)} \qquad ...(15)$$

$$m\ddot{y} = 0 = R - mg\cos\alpha \qquad \text{(perpendicular to the plane)} \qquad ...(16)$$

and $$mk^2\ddot{\theta} = Fa$$

or $$m\frac{2}{5}a^2\ddot{\theta} = Fa \qquad \text{(moment eqn about } G\text{)} \qquad ...(17)$$

Since the point of contact is at rest

$$\therefore \quad \dot{x} - a\dot{\theta} = 0 \quad \text{or} \quad \dot{x} = a\dot{\theta} \qquad ...(18)$$

which may also be obtained by the geometrical equation $x = a\theta$.

Using value of F from (17) in (15), we get

$$m\ddot{x} = mg\sin\alpha - m\,.\,\frac{2}{5}a\ddot{\theta} \qquad \text{or} \qquad \ddot{x} + \frac{2}{5}\ddot{x} = g\sin\alpha$$

$$\ddot{x} = \frac{5}{7}g\sin\alpha = a\ddot{\theta} \qquad ...(19)$$

On integrating and using eqn (1), we get

$$\dot{x} = \frac{5}{7}g\sin\alpha\, t = a\dot{\theta} \qquad ...(20)$$

again integrating and using eqn (1), we get

$$x = a\frac{5}{14}g\sin\alpha\, t^2 = a\theta \qquad ...(21)$$

Remarks : (1) when $\mu = 0$ (pure sliding) i.e. motion on smooth plane, then from eqn (9), (10), (12) and (13), we get

$$\dot{x} = gt\sin\alpha, \quad x = \frac{1}{2}gt^2\sin\alpha$$

$$\dot{\theta} = 0, \ \theta = 0$$

this shows that the sphere will slide on the inclined plane as particle whose mass is equal to the mass of the sphere. There will no rolling at all.

(ii) Application of work and energy principle :

(a) where there is pure sliding (no friction), there is no loss of kinetic energy.

(b) where there is pure rolling, there is no loss of K.E.

(c) where, there is sliding and rolling combined, energy is lost.

3.9 Rolling and Sliding of a Sphere on a Fixed Sphere

A solid homogeneous sphere, resting on the top of another fixed sphere, is slightly displaced and begins to roll down it, then it will slip when the common normal makes with the vertical an angle θ given by the equation

$$2\sin(\theta - \alpha) = 5\sin\lambda\,(3\cos\theta - 2)$$

where λ is the angle of friction. Also the upper sphere would leave the lower when $\theta = \cos^{-1}\left(\frac{10}{17}\right)$.

Further, if both the sphere are taken as smooth, then the upper would leave the lower when $\theta = \cos^{-1}\left(\frac{2}{3}\right)$.

Let O be the centre of the fixed sphere of radius $OD = a$, whose highest point is D. The moving sphere of radius b was initially resting at the top of fixed sphere, so that QB was vertical and the point of contact Q was at D. After time t Let point of contact be A and the position BQ is as shown in the fig. 3.12.

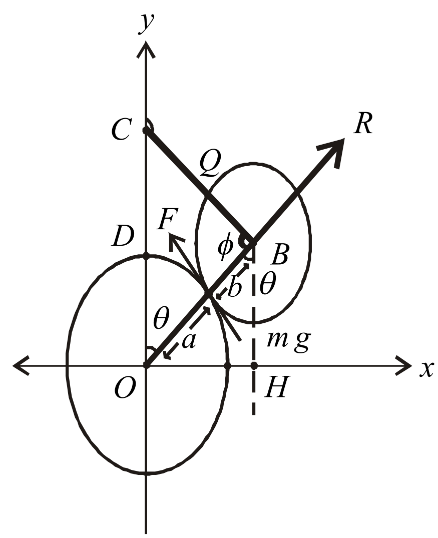


Figure 3.12

As the sphere rolls down,

$$arc\ DA = arc\ QA$$

$$\Rightarrow \qquad a\theta = b\phi, \qquad \text{then } a\dot{\theta} = b\dot{\phi} \qquad \text{...(1)}$$

where θ and ϕ are the angles which common normal OAB makes with the vertical and BQ, a line fixed in the moving sphere, respectively. Let R and F be the normal reactions and the friction acting on the upper sphere. Now point B describes a circle of radius $OB\ (= a + b)$ about O as centre, so that its accelerations are

$(a+b)\,\dot{\theta}^2$ along BO and

$(a+b)\,\ddot{\theta}$ perpendicular to BO

Hence equations of motion of B are

$$m(a+b)\dot{\theta}^2 = mg\cos\theta - R \qquad \text{(along } BO) \qquad ...(2)$$

and $$m(a+b)\ddot{\theta} = mg\sin\theta - F \qquad \text{(perpendicular to } BO) \qquad ...(3)$$

where m is mass of moving sphere.

Referred to O as the origin, the coordinates of centre B are

$$\left(x = (a+b)\sin\theta,\ y = (a+b)\cos\theta\right)$$

so that $$v^2 = \dot{x}^2 + \dot{y}^2 = \left((a+b)\cos\theta\,\dot{\theta}\right)^2 + \left(-(a+b)\sin\theta\,\dot{\theta}\right)^2 = (a+b)^2\dot{\theta}^2 \qquad ...(4)$$

Also initially (when Q was at B) B was at a height $(a+b)$ and now its height is $(a+b)\cos\theta = BH$, so that the distance fallen in downward direction is

$$= (a+b) - (a+b)\cos\theta = (a+b)(1-\cos\theta) \qquad ...(5)$$

The energy equation gives

$$\frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{m}{2}k^2\left(\dot{\phi} + \dot{\theta}\right)^2 = \text{work done by gravity}$$

$$= mg \times (\text{distance fallen}) = mg(a+b)(1-\cos\theta)$$

or $$(a+b)^2\dot{\theta}^2 + \frac{2b^2}{5}\left(\frac{a+b}{b}\right)^2\dot{\theta}^2 = 2g(a+b)(1-\cos\theta) \qquad ...(6)$$

$$\left[\because k^2 = \frac{2}{5}b^2 \text{ and } \dot{\phi} = \frac{a}{b}\dot{\theta} \text{ by eqn (1)}\right]$$

or $$\dot{\theta}^2 = \frac{10g}{7(a+b)}(1-\cos\theta) \qquad ...(7)$$

Differentiating (7), $w.r.$ to t and can celling $2\dot{\theta}$, we have

$$\ddot{\theta} = \frac{5g}{7(a+b)}\sin\theta \qquad ...(8)$$

Subtituting value of $\dot{\theta}^2$ from (7) in eqn (2), we get

$$R = mg\cos\theta - m\frac{10g}{7}(1-\cos\theta) = \frac{mg}{7}(17\cos\theta - 10) \qquad ...(9)$$

Now using value of $\ddot{\theta}$ from (8) in eqn (3), we get

$$F = mg\sin\theta - \frac{5}{7}mg\sin\theta = \frac{2}{7}mg\sin\theta \qquad ...(10)$$

Now, the sphere will slip, when the friction becomes limiting

if $F = \mu R$ or $F = R\tan\lambda$ $\qquad (\because \mu = \tan\lambda)$

or $\quad \frac{2}{7} mg \sin\theta = \tan\lambda \,.\, \frac{mg}{7} (17 \cos\theta - 10)$

[using value of F and R from eqn (10) and (9)]

or $\quad 2 \sin\theta \cos\lambda = (17 \cos\theta - 10) \sin\lambda$

or $\quad 2 (\sin\theta \cos\lambda - \cos\theta \sin\lambda) = 5 (3 \cos\theta - 2) \sin\lambda$

or $\quad 2 \sin(\theta - \lambda) = 5 \sin\lambda (3 \cos\theta - 2)$ ...(11)

which is required result.

The upper sphere will leave the lower one, when $R = 0$, so from eqn (9), we have,

$$(17 \cos\theta - 10) = 0 \Rightarrow \theta = \cos^{-1}\left(\frac{10}{17}\right)$$

which is another required result.

When both the spheres are smooth : In this case $F = 0$ so that energy equation becomes

$$\frac{1}{2} m v^2 = \frac{m}{2} (a + b) \dot{\theta}^2 = mg (a + b) (1 - \cos\theta)$$

or $\quad \dot{\theta}^2 = \frac{2g}{(a + b)} (1 - \cos\theta)$ ...(12)

Now, equation of motion of B, along BO will remains unchanged, so from eqn (2)

$$R = mg \cos\theta - m (a + b) \dot{\theta}^2$$

$$= mg \cos\theta - m (a + b) \frac{2g}{(a + b)} (1 - \cos\theta) \quad \text{(from (12)}$$

$$= mg (3 \cos\theta - 2) \quad ...(13)$$

The upper sphere will leave the lower if $R = 0$

i.e. $\quad 3 \cos\theta - 2 = 0 \Rightarrow \theta = \cos^{-1}\left(\frac{2}{3}\right)$

3.10 Unstable equilibrium between two smooth spheres

Two unequal smooth spheres are placed one on the top of the other in unstable equilibirum, the lower sphere resting on a smooth table. The system is slighty disturbed, then the sphere will separate when the line joining their centres makes an angle θ with the vertical given by the equation

$$\frac{m}{m + M} \cos^3\theta - 3 \cos\theta + 2 = 0,$$

Where M is the mass of the lower and m of the upper sphere.

Suppose A and B be the centres, a and b be radii and M, m the given masses of lower and upper spheres respectively. Let G be their common cetnre of gravity which lies on the line of centres AB.

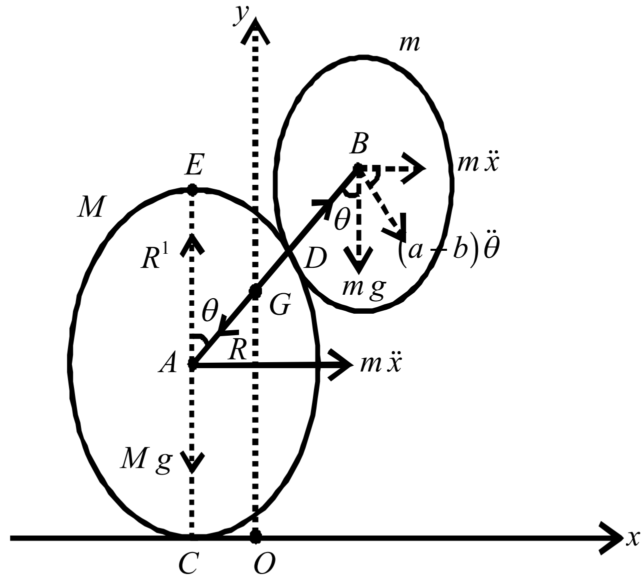


Figure 3.13

When the system is disturbed, the centre B of upper sphere describes a circle of radius $AB\ (= a + b)$ about A as centre.

The coordinates of $C.G.$ of lower and upper spheres with respect to O as centre and OX and OY as fixed horizontal and vertical axes are

$$A\ (-\ x,\ a) \qquad \text{and} \qquad B\left[(a + b)\sin\theta - x,\ a + (a + b)\cos\theta\right]$$

Since the table and sphere are smooth there will be no horizontal force on the combined system of two spheres. Thus the linear momentum equation in horizontal direction is

$$\frac{d}{dt}\left[-\ M\dot{x} + m\left\{-\ \dot{x} + (a + b)\cos\theta\ \dot{\theta}\right\}\right] = 0$$

or $\quad -\ (M + m)\ \dot{x} + m\ (a + b)\cos\theta\ \dot{\theta} = K$ (constant of integration) Initially, $\dot{x} = 0 = \dot{\theta}$, when $t = 0$, so that $K = 0$

$$\dot{x} = \frac{m}{(M + m)}\ (a + b)\cos\theta\ \dot{\theta} \qquad \text{...(1)}$$

Also, the kinetic energy of the system is

$$= \frac{1}{2}\ M\dot{x}^2 + \frac{1}{2}\ m\left[\left(-\ \dot{x} + (a + b)\cos\theta\ \dot{\theta}\right)^2 + \left(-\ (a + b)\sin\theta\ \dot{\theta}\right)^2\right]$$

$\because$ Velocity of $A = (-\ \dot{x})^2 + O^2$, velocity of $B = \left[\left((a + b)\cos\theta\ \dot{\theta}\right)^2 + \left(-\ (a + b)\sin\theta\ \dot{\theta}\right)^2\right]$

$$= \frac{1}{2} m (a + b)^2 \left[1 - \frac{m}{M + m} \cos^2 \theta\right] \dot{\theta}^2 \qquad ...(2)$$

Further the work energy equation gives

$$\frac{1}{2} m (a + b)^2 \left[1 - \frac{m}{M + m} \cos^2 \theta\right] \dot{\theta}^2 = m g \left[(a + b) - (a + b) \cos\theta\right]$$

Since B has fallen through a distance $\left[(a + b) - (a + b) \cos\theta\right]$

$$\left(M + m \sin^2 \theta\right) \dot{\theta}^2 = \frac{2g}{a + b} (M + m) (1 - \cos\theta) \qquad ...(3)$$

Differentiating and simplifying

$$\left(M + m \sin^2 \theta\right) \ddot{\theta} + m \sin\theta \cos\theta \, \dot{\theta}^2 = \frac{(m + M)}{a + b} g \sin\theta \qquad ...(4)$$

By the horizontal motion of lower sphere

$$M \ddot{x} = R \sin\theta$$

where R is the reaction between the spheres

Using this, with (1), we get

$$\frac{M m}{(m + M)} (a + b) \left(\cos\theta \, \ddot{\theta} - \sin\theta \, \dot{\theta}^2\right) = R \sin\theta \qquad ...(5)$$

The spheres will seperate when $R = 0$, so from (5), we get

$$\cos\theta \, \ddot{\theta} = \sin\theta \, \dot{\theta}^2 \qquad ...(6)$$

Eliminating $\dot{\theta}^2$ from (4) and (6), we get

$$\ddot{\theta} = \frac{g}{a + b} \sin\theta \qquad ...(7)$$

Again from (6) and (7)

$$\dot{\theta}^2 = \frac{g \cos\theta}{a + b} \qquad ...(8)$$

In the end, eliminating $\dot{\theta}^2$ from (3) and (8)

$$m \cos^3 \theta = (m + M) (3 \cos\theta - 2)$$

or $$\frac{m}{m + M} \cos^3 \theta - 3 \cos\theta + 2 = 0$$

Self Learning Exercise - 2

1. In case of pure sliding of a sphere on an inclined plane, is there loss of K.E.?
2. In case of sliding and rolling combined of a sphere on inclined plane, is there loss of K.E.?
3. In case of pure rolling of a sphere on an inclined plane, is there loss of K.E.?
4. Write a relation between μ (coefficient of friction) and λ (angle of friction).

Illustrative Examples :

Example 6 : A sphere is projected with an under twist down a rough inclined plane, show that it will turn back in the course of its motion if $2aw(\mu - \tan\alpha) > 5\mu u$, where u is the initial linear velocity and w the initial angulars velocity of the sphere, μ is the coefficient of friction and α is the inclination of the plane.

Solution : Let ABO be inclined plane and G be centre of sphere of mass m and radius a. The sphere has been projected with an under hand twist down the rough inclined plane OBA, so that its linear velocity is u and angular velocity w as shown in the fig. 3.14.

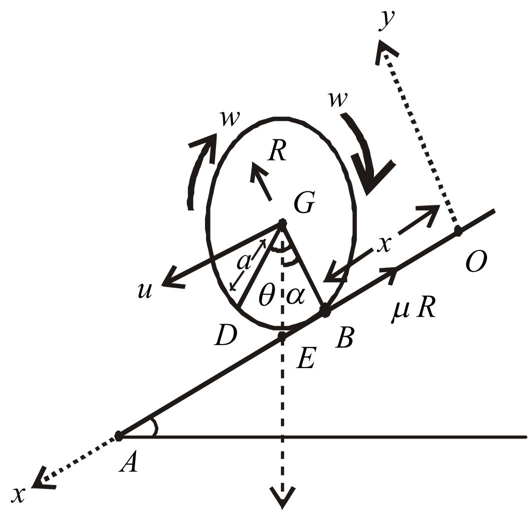


Figure 3.14

Initially $t = 0,\ \dot{x} = w,\ x = 0,\ \theta = 0,\ \dot{\theta} = w$ ...(1)

where θ is angle turned by diametre GD and x be distance moved by sphere.

Hence initial velocity of point of contact B, down the plane

$=$ velocity of G + velocity of B relative to G

$= (u + aw) > 0$ ($\because$ motion is sliding and rolling)

Therefore the force of friction μR acts up the plane. Then equations of motion of G are

$m\ddot{x} = mg\sin\alpha - \mu R$ (parallel to inclined OBA in direction OBA) ...(2)

and $m\ddot{y} = 0 = R - mg\cos\alpha$ (perpendicular to inclined plane) ...(3)

or $R = mg\cos\alpha$ ...(4)

and moment equation of G is

$$m k^2 \ddot{\theta} = - \mu R \, . \, a$$

or $$m \frac{2}{5} a^2 \ddot{\theta} = - \mu R a \qquad \left(\because \textit{ for sphere } k^2 = \frac{2}{5} a^2 \right) \qquad ...(5)$$

Now putting the value of R from (4) in (2), we have

$$m \ddot{x} = m g \sin \alpha - \mu \, m g \cos \alpha$$

or $$\ddot{x} = g \left(\sin \alpha - \mu \cos \alpha \right) \qquad ...(6)$$

Now using value of R from (4) in (5), we have

$$\frac{2}{5} m a^2 \ddot{\theta} = - \mu a \, . \, m g \cos \alpha$$

or $$a \ddot{\theta} = - \frac{5 \mu g \cos \alpha}{2} \qquad ...(7)$$

on integrating (6) $w.r.$ to t

$$\dot{x} = g \left(\sin \alpha - \mu \cos \alpha \right) t + C_1$$

but initially by eqn (1), $t = 0, \dot{x} = u$

$$\therefore C_1 = u$$

$\therefore$ $$\dot{x} = g \left(\sin \alpha - \mu \cos \alpha \right) t + u \qquad ...(8)$$

Now integrating eqn (7), we have

$$a \dot{\theta} = - \frac{5}{2} \mu g \cos \alpha + C_2$$

by eqn (1), $t = 0, \dot{\theta} = w$

$$\therefore C_2 = a w$$

$\therefore$ $$a \dot{\theta} = - \frac{5}{2} \mu g \cos \alpha \; t + a w \qquad ...(9)$$

Suppose that the sphere will cease to go down after time $t = t_1$

so $\dot{x} = 0$, when $t = t_1$

$\therefore$ from (8), we get

$$0 = g \left(\sin \alpha - \mu \cos \alpha \right) t_1 + u \Rightarrow t_1 = \frac{4}{g \left(\mu \cos \alpha - \sin \alpha \right)} \qquad ...(10)$$

Putting this value of t in eqn (9), we get value of $a \dot{\theta}$ at this time as

$$a \dot{\theta} = - \frac{5}{2} \mu g \cos \alpha \left[\frac{4}{g \left(\mu \cos \alpha - \sin \alpha \right)} \right] + a w$$

Now $a\dot{\theta}$ will be positive if

$$a\,w - \frac{5\,\mu\,\cos\alpha\;u}{2\,g\,(\mu\,\cos\alpha\,-\,\sin\alpha)} > 0$$

or $\quad 2\,a\,w\,(\mu\,\cos\alpha\,-\,\sin\alpha) > 5\,\mu\,u\,\cos\alpha$

or $\quad 2\,a\,w\,(\mu\,-\,\tan\alpha) > 5\,\mu\,u$

which is the desired result.

Example 7 : A rough cylinder, of mass M, is capable of motion about its axis, which is horizontal; a particle of mass m is palced on it vertically above the axis and the system is slightly disturbed. Show that the particle will slip on the cylinder when its has moved through an angle θ given by

$$\mu\,(M\,+\,6\,m)\,\cos\theta\,-\,M\,\sin\theta\,=\,4\,m\,\mu$$

where μ is the coefficient of friction.

Solution : Rough cylinder of mass M moves about its axis OZ which is horizontal with a particle of mass m placed on it vertically above the axis. After slightly disturbing after a time t, the radius $OA\,(=\,a)$ makes an angle θ with the vertical and F is the force of friction which keeps the particle at rest. Referred to O as origin and ox, oy as coordinate axes the coordinates of particle at A are $(x\,=\,a\,\cos\theta,\;y\,=\,a\,\sin\theta)$

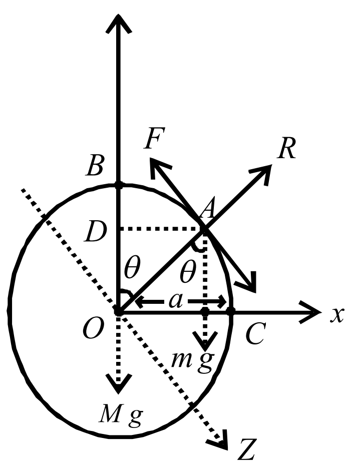


Figure 3.15

$\therefore \quad \dot{x}\,=\,-\,a\,\sin\theta\,\dot{\theta},\;\dot{y}\,=\,a\,\cos\theta\,\dot{\theta}$

$\therefore \quad v^2\,=\,\dot{x}^2\,+\,\dot{y}^2\,=\,a^2\,\dot{\theta}^2 \qquad ...(1)$

Now the energy equation gives

$$\frac{1}{2}\,M\,\frac{a^2}{2}\,\dot{\theta}^2 \;+\;\frac{1}{2}\,m\,a^2\,\dot{\theta}^2 \;=\; \text{work done by gravity} = m\,g\,a\,(1\,-\,\cos\theta)$$

$(K.E.\ due\ to\ rotation)\ (K.E.\ of\ \ particle)$

or $\quad a\,(M\,+\,2\,m)\,\dot{\theta}^2\,=\,4\,m\,g\,(1\,-\,\cos\theta) \qquad ...(2)$

Differentiating $w.r.$ to t and dividing by $2\dot{\theta}$, we get

$$a\,(M + 2m)\,\ddot{\theta} = 2\,mg\sin\theta \qquad ...(3)$$

The particle m describes a circle about O as centre therefore its equation of motion are

$$ma\dot{\theta}^2 = mg\cos\theta - R \qquad \text{(along } AO\text{)} \qquad ...(4)$$

and $$ma\ddot{\theta} = mg\sin\theta - F \qquad \text{(perpendicular to } AO\text{)} \qquad ...(5)$$

Now, we shall find values of F and R using equation (2), (3), (4) and (5).

Using value of $\dot{\theta}^2$ from (2) in (4), we get

$$m\left[\frac{4mg\,(1-\cos\theta)}{(M+2m)}\right] = mg\cos\theta - R$$

$$\therefore \qquad R = \frac{mg}{M+2m}\left[(M+2m)\cos\theta - 4m\,(1-\cos\theta)\right]$$

$$= \frac{mg}{M+2m}\left[(M+6m)\cos\theta - 4m\right] \qquad ...(6)$$

Now putting value of $\ddot{\theta}$ from eqn (3) in (5), we have

$$m\left[\frac{2mg\sin\theta}{M+2m}\right] = mg\sin\theta - F$$

or $$F = mg\sin\theta - \frac{2m^2 g\sin\theta}{M+2m} = \frac{mMg}{M+2m}\sin\theta \qquad ...(7)$$

on dividing eqn (7) by (6), we have

$$\frac{F}{R} = \frac{M\sin\theta}{(M+6m)\cos\theta - 4m} \qquad ...(8)$$

The particle will slips from the cylinder when $F = \mu R$

or $$\frac{F}{R} = \mu \Rightarrow \frac{M\sin\theta}{(M+6m)\cos\theta - 4m} = \mu$$

or $$(M+6m)\,\mu\cos\theta - 4m\mu = M\sin\theta$$

or $$(M+6m)\,\mu\cos\theta - M\sin\theta = 4m\mu$$

which is desired result.

3.11 Motion of a Hollow Cylinder Inside a Cylinder

A hollow cylinder of radius a, is fixed with its axis horizontal; inside it moves a solid cylinder, of radius b, whose angular velocity in its lowest position is given, if the friction between the cylinder be sufficient to prevent any sliding, discuss the motion.

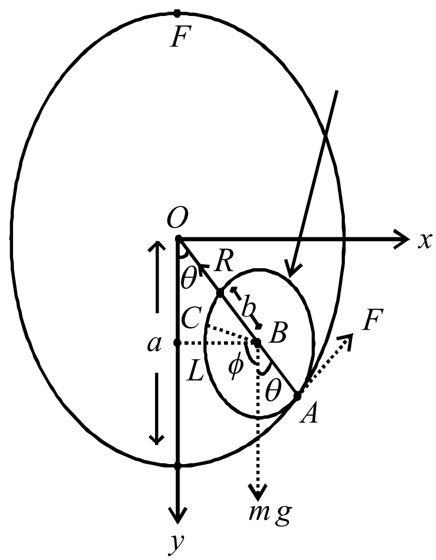


Figure 3.16

The Fig. 3.16 represents vertical section through the centre of gravity B of the rolling cylinder. Let the radius of fixed cylinder be $a\ (= OD)$ with O as centre. Let ox, oy be axes. Let m be mass and $b\ (= BC)$ be radius of moving cylinder with centre at B. Initially C was at its lowest position D and OD is vertical Let BC turn through an angle ϕ with vertical in time t.

Since there is no sliding between the cylinders, therefore

$$arc\ DA = arc\ CA$$

$$\Rightarrow a\theta = b\,(\theta + \phi) \Rightarrow (a-b)\,\theta = b\phi$$

$$\therefore \qquad \theta = \frac{b}{a-b}\phi \quad \text{and} \quad (a-b)\,\dot{\theta} = b\dot{\phi} \qquad ...(1)$$

where θ is the angle which the line of centres makes with vertical. Let R and F be the normal reaction and frictional force at A respectively. The centre B describe a circle of radius $OB\ (= a-b)$ about O as centre. Then, the radial acceleration of $B = (a-b)\,\dot{\theta}^2$ along BO and transverse acceleration of $B = (a-b)\,\ddot{\theta}$, perpendicular to BO. So equations of motion of B are

$$m\,(a-b)\,\dot{\theta}^2 = R - mg\cos\theta \qquad ...(2)$$

$$\text{and} \qquad m\,(a-b)\,\ddot{\theta} = F - mg\sin\theta \qquad ...(3)$$

The coordinates of B with respect to O as origin are

$$B\,(x = LB,\ y = OL)$$

$$\text{or} \qquad x = LB = OB\cos\theta = (a-b)\cos\theta$$

$$y = OL = OB\sin\theta = (a-b)\sin\theta$$

$$\text{or} \qquad \dot{x} = -\,(a-b)\sin\theta\,\dot{\theta},\ \ \dot{y} = (a-b)\cos\theta\,\dot{\theta} \qquad \text{(on differentiating)}$$

$$\therefore \qquad \dot{x}^2 + \dot{y}^2 = \left\{-(a-b)\sin\theta\,\dot{\theta}\right\}^2 + \left\{(a-b)\cos\theta\,\dot{\theta}\right\}^2 = (a-b)^2\,\dot{\theta}^2 \qquad ...(4)$$

So kinetic energy of moving cylinder at any time t is

$$= \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2} m k^2 \dot{\phi}^2$$

$$= \frac{m}{2}(a-b)^2\dot{\theta}^2 + \frac{1}{2} m \frac{b^2}{2}\dot{\phi}^2 \qquad \left(\because k^2 = \frac{b^2}{2}\right)$$

$$= \frac{m}{2}\left[(a-b)^2\dot{\theta}^2 + \frac{(a-b)^2}{2}\dot{\theta}^2\right] \qquad \text{[by eqn (1)]}$$

$$= \frac{m}{2}\left[\frac{3}{2}(a-b)^2\dot{\theta}^2\right] = \frac{3}{4} m (a-b)^2\dot{\theta}^2 \qquad ...(5)$$

If $\dot{\theta} = w$ be the angular velocity at the lowest position of the moving cylinder, then

K.E. at time of projection $= \frac{3}{4} m (a-b)^2 w^2$ [by eqn (5)] ...(6)

Hence by work energy equation, we have

Change in K.E. = Work done by gravity

$$\therefore \qquad \frac{3}{4} m (a-b)^2\dot{\theta}^2 - \frac{3}{4} m (a-b)^2 w^2 = -mg(a-b)(1-\cos\theta)$$

or
$$\dot{\theta}^2 = w^2 - \frac{4g}{3(a-b)}(1-\cos\theta) \qquad ...(7)$$

this equation cannot be integrated, in a compact form. So on differentiating eqn (7) with respect to t and dividing by $2\dot{\theta}$, we have

$$\ddot{\theta} = \frac{-2g}{3(a-b)}\sin\theta \qquad ...(8)$$

Now from eqn (2),

$$R = mg\cos\theta + m(a-b)\dot{\theta}^2$$

$$= mg\cos\theta + m(a-b)\left\{w^2 - \frac{4g}{3(a-b)}(1-\cos\theta)\right\} \qquad \text{[by eqn (7)]}$$

or
$$R = mg\cos\theta + m(a-b)w^2 - \frac{4mg}{3}(1-\cos\theta)$$

$$= m(a-b)w^2 + \frac{mg}{3}(7\cos\theta - 4) \qquad ...(9)$$

From eqn (3),

$$F = mg \sin\theta + m(a-b)\ddot{\theta}$$

$$= mg \sin\theta + m(a-b)\left\{\frac{-2g}{3(a-b)}\sin\theta\right\} \qquad \text{[by eqn (8)]}$$

or $$F = \frac{1}{3} mg \sin\theta \qquad ...(10)$$

Hence equations (7), (9) and (10) determine motion.

From (10), when $\theta = 0 \Rightarrow F = 0$, which shows that friction is zero at the lowest point and for any other position F is positive.

Following are other main points of discussion :

Case - I : When the cylinder makes complete revolution :

The cylinder may roll round completely when reaction R is zero at highest point E, i.e. when $\theta = \pi$ so from eqn (9)

$$0 = m(a-b)w^2 + \frac{mg}{3}(-7-4)$$

$$\Rightarrow \quad w^2 = \frac{11g}{3(a-b)} \qquad ...(11)$$

Hence velocity of centre B is

$$= (a-b)w = \sqrt{\frac{11g(a-b)}{3}} \quad \text{(on using value of } w) \qquad ...(12)$$

This gives least velocity of projection in order that moving cylinder may roll completely round the outer fixed cylinder.

Case - II : When the moving cylinder leaves the fixed cylinder :

The moving cylinder, if it does not roll round completely, will leave the lower body when $R = 0$, that position (value of angle θ) is given by putting $R = 0$ in equation (9), so

$$0 = m(a-b)w^2 + \frac{mg}{3}(7\cos\theta - 4)$$

or $$\cos\theta = \frac{4g - 3(a-b)w^2}{7g} \Rightarrow \theta = \cos^{-1}\frac{1}{7}\left\{4 - \frac{3(a-b)w^2}{g}\right\}$$

at this angle moving cylinder will leave the fixed cylinder.

Case - III : When the moving cylinder makes oscillations :

If the moving cylinder makes small oscillation about the lowest point D of the fixed cylinder. Then angle θ is always small $(\theta \to 0)$

Hence from eqn (8), taking θ for $\sin\theta$, we have

$$\ddot{\theta} = -\frac{2g}{3(a-b)}\theta$$

which is a S.H.M. of time period

$$T = 2\pi\sqrt{\frac{3(a-b)}{2g}}$$

Illustrative Example :

Example 8 : A homogeneous sphere of radius a rotating with angular velocity w about a horizontal diameter, is gently placed on a table whose coefficient of friction is μ. Show that there will be slipping at the point of contact for a time $\frac{2aw}{7\mu g}$, and then the sphere will roll with angular velocity $\frac{2w}{7}$.

Solution : POL is a table and G be centre, m be mass and a is the radius of sphere, which is placed on table.

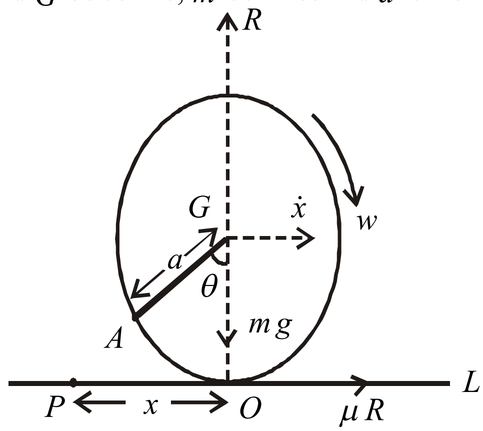


Figure 3.17

Initially the sphere is gently placed on the table, so velocity (linear) of the $C.G.$ G of the sphere is zero while initial angular velocity is w. Let at time t, x be the distance described by the $C.G.$ in the horizontal direction and θ the angle turned by the sphere.

So initially $\quad t = 0,\ x = 0,\ \dot{x} = 0,\ \dot{\theta} = w$ ...(1)

Now, the initial velocity of point of contact O

= (initial velocity of $C.G.G$) + (initial velocity of point of contact with respect to G) = $(O + aw)$

$\therefore$ Initial velocity of $O = aw$ (In the backward direction $\leftarrow$)

Since, initially sphere was rotating, the point of contact will slip in the direction ($\leftarrow$) (backward) and consequently the force of friction μR will act in forward ($\rightarrow$) direction. (As shown in the fig. 3.17) Therefore the equations of motion are

$m\ddot{x} = \mu R$ (In horizontal direction $\rightarrow$) ...(2)

and $\quad O = R - mg$ (In vertical direction $\uparrow$) ...(3)

and by moment equation

$$m k^2 \ddot{\theta} = -\mu R a \qquad ...(4)$$

from (3), $R = mg$ using in (2) and (4)

$$m\ddot{x} = \mu m g \Rightarrow \ddot{x} = \mu g \qquad ...(5)$$

and $\quad m \frac{2}{5} a^2 \ddot{\theta} = -\mu a \, . \, m g \qquad \left(\because k^2 = \frac{2}{5} a^2\right)$

or $\quad a\ddot{\theta} = -\frac{5}{2}\mu g \qquad ...(6)$

Integrating eqn (5), we have $\quad \dot{x} = \mu g t + C_1$

by eqn (1), $\dot{x} = 0, t = 0 \Rightarrow C_1 = 0$

$\therefore \quad \dot{x} = \mu g t \qquad ...(7)$

Now integrating eqn (6), we have

$$a\dot{\theta} = -\frac{5}{2}\mu g t + C_2$$

again by eqn (1), $t = 0, \dot{\theta} = w$

$\therefore \quad a w = O + C_2 \Rightarrow C_2 = a w$

$\therefore \quad a\dot{\theta} = -\frac{5}{2}\mu g t + a w \qquad ...(8)$

Now, the velocity of point of contact $= \left(\dot{x} - a\dot{\theta}\right)$

$$= \mu g t + \frac{5}{2}\mu g t - a w$$

$$= \left(\frac{7}{2}\mu g t - a w\right) \qquad ...(9)$$

The point of contact will come to rest, when $\dot{x} - a\dot{\theta} = 0$

$\therefore$ by eqn (9), $\frac{7}{2}\mu g t = a w \Rightarrow t = \frac{2}{7}\frac{a w}{\mu g} \qquad ...(10)$

Therefore after time $\frac{2}{7}\frac{aw}{\mu g}$, the slipping will stop and pure rolling will begin and at this time the angular velocity is obtained by putting value of t from (10) in (8)

$$\therefore \quad a\dot{\theta} = -\frac{5}{2}\mu g\left(\frac{2aw}{7\mu g}\right) + aw = \frac{2wa}{7}$$

$$\Rightarrow \quad \dot{\theta} = \frac{2w}{7} \qquad ...(11)$$

As rolling starts, suppose frictional force be F and then the equations of motion are

$$m\ddot{x} = F \qquad \text{(along Table)} \qquad ...(12)$$

and $$m\frac{2}{5}a^2\ddot{\theta} = -F \,.\, a \qquad \text{(Moment eqn)} \qquad ...(13)$$

Also $\dot{x} - a\dot{\theta} = 0 \Rightarrow \dot{x} = a\dot{\theta}, \quad \therefore \ddot{x} = a\ddot{\theta}$ ...(14)

using value of F from (12) in (13)

$$m\frac{2}{5}a^2\ddot{\theta} = -m\ddot{x} \,.\, a$$

$$= -m\left(a\ddot{\theta}\right)a \qquad \text{(from (14)]}$$

$$\Rightarrow \quad \frac{7}{5}ma\ddot{\theta} = 0, \therefore \ddot{\theta} = 0$$

on itegrating, $\dot{\theta} =$ (constant)

or $$\dot{\theta} = \frac{2w}{7} \qquad \text{[when the rolling commenced, by eqn (11)]}$$

which is the desired result.

Example 9 : A disc rolls on the inside of a fixed hollow circular cylinder whose axis is horizontal, the plane of the disc being vertical and perpendicular to the axis of the cylinder, if when in its lowest position, its centre is moving with a velocity $\sqrt{\frac{8}{3}g(a-b)}$ show that the centre of the disc will describe an angle ϕ about the centre of the cylinder in time $\left\{\frac{3(a-b)}{2g}\right\}^{1/2}\log\tan\left(\frac{\pi}{4}+\frac{\phi}{4}\right)$.

Solution : As shown in the fig. 3.18, let O be the centre of the fixed hollow cylinder of radius $a\ (= OB)$. Let D be centre of the disc of mass m and radius $DC = b$ (line fixed in the moving body) DC was initially vertical when C coincided with B and making an angle θ with the vretical (line fixed in space) at time t. The disc rolls on the inside of a fixed cylinder so that

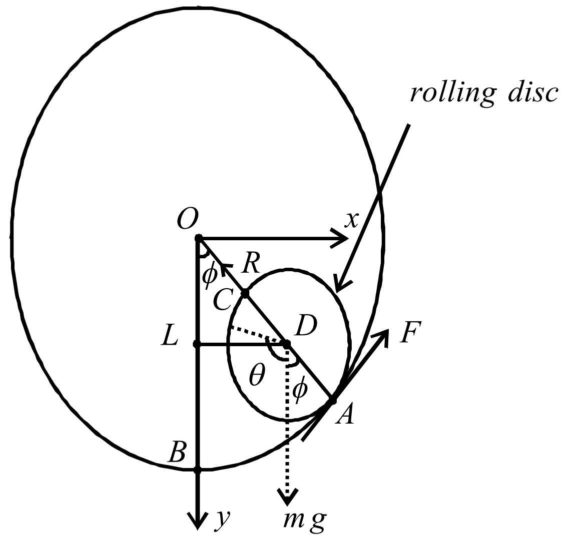


Figure 3.18

$$arc\ BA = arc\ CA$$

or $\quad a\phi = b\,(\theta + \phi)$

or $\quad b\theta = (a - b)\,\phi \quad$ ...(1)

on differentiating with respect to t

$$b\dot{\theta} = (a - b)\dot{\phi} \quad ...(2)$$

Now, coordinates of D referred to O as origin are

$$(x = LD,\ y = OL)$$

$$x = (a - b)\sin\phi,\ y = (a - b)\cos\phi \quad ...(3)$$

On differentiating

$$\dot{x} = (a - b)\cos\phi\ \dot{\phi},\ \dot{y} = -\,(a - b)\sin\phi\ \dot{\phi}$$

$\therefore \quad \left(\dot{x}^2 + \dot{y}^2\right) = \left\{(a - b)\cos\phi\ \dot{\phi}\right\}^2 + \left\{-\,(a - b)\sin\phi\ \dot{\phi}\right\}^2 = (a - b)^2\ \dot{\phi}^2$

since initial velocity of the centre = $\sqrt{\dfrac{8g}{3}(a - b)}$

i.e. $\quad (a - b)\,\dot{\phi} = \sqrt{\dfrac{8g}{3}(a - b)}$

$\Rightarrow \quad \dot{\phi}^2 = \dfrac{8g}{3\,(a - b)}$, at $t = 0 \quad$ ...(5)

Now kinetic energy of rolling disc at any time t is given by

$$= \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2} m k^2 \dot{\theta}^2$$

$$= \frac{1}{2} m (a-b)^2 \dot{\phi}^2 + \frac{m}{2} \cdot \frac{b^2}{2} \dot{\theta}^2 \qquad \left[\textit{by eqn } (4) \textit{ and } k^2 = \frac{b^2}{2} \textit{ for disc}\right]$$

or K.E. of disc at time $t = \frac{m}{2}(a-b)^2 \dot{\phi}^2 + \frac{m}{4}(a-b)^2 \dot{\phi}^2$ [by eqn (2)]

$$= \frac{3}{4}(a-b)^2 \dot{\phi}^2 \qquad \text{...(6)}$$

Initial K.E. of disc $= \frac{3}{4} m (a-b)^2 \times \frac{8g}{3(a-b)}$

$$= 2mg(a-b) \qquad \text{...(7)}$$

Initially distance of D below O was $(a-b)$ and after time t it is OL, so distance risen by D in upward direction $= (OD - OL) = (a-b) - (a-b)\cos\phi = (a-b)(1-\cos\phi)$

$\therefore$ Energy equation gives

$$\frac{3}{4} m (a-b)^2 \dot{\phi}^2 - 2mg(a-b) = \text{work done by gravity} = -mg(a-b)(1-\cos\phi)$$

or $$\frac{3}{4} m (a-b)^2 \dot{\phi}^2 = mg(a-b)\left[2 - 1 + \cos\phi\right]$$

$$= mg(a-b)(1+\cos\phi)$$

or $$\dot{\phi}^2 = \frac{4g}{3(a-b)}(1+\cos\phi) \Rightarrow \dot{\phi}^2 = \frac{8g\cos^2 \phi/2}{3(a-b)}$$

or $$\dot{\phi} = \sqrt{\frac{8g}{3(a-b)}} \cdot \cos\frac{\phi}{2}$$

Integrating it, we have

$$\int_0^t dt = \sqrt{\frac{3(a-b)}{8g}} \int_0^\phi \sec\frac{\phi}{2}\, d\phi$$

or $$t = \sqrt{\frac{3(a-b)}{8g}} \cdot 2 \log\tan\left(\frac{\pi}{4} + \frac{\phi}{4}\right) = \sqrt{\frac{3(a-b)}{2g}} \cdot \log\tan\left(\frac{\pi}{4} + \frac{\phi}{4}\right)$$

which is desired result.

3.12 Two Dimensional Motion of a Rigid Body Under Implusive Forces

Here, in this part we shall study instantaneous motion in two dimensions of a rigid body under impulsive forces. While studying the effect of impulsive forces on the motion of the body the effect of finite forces on the body is neglacted. First we shall define impuluse of a force and impulsive force.

When very large force act on a body for a very short time then their effects are measured by their impulses. Let u be the velocity of a particle of mass m at any time t and F be the force acting on it in the same direction, its acceleration being $\frac{dv}{dt}$.

Then equation of motion is, $m\,\frac{dv}{dt} = F$ ...(1)

Let T be the time for which the force acts and velocity changes from v to v^1, then on integrating eqn (1), we have

$$\int_v^{v^1} m\,dv = \int_0^T F\,dt$$

or $$m\left(v^1 - v\right) = \int_0^T F\,dt \qquad ...(2)$$

If force F increases infinitely and at the same time time T decreases without limit, then R.H.S. of eqn (2) may have a definite limit (say) I, and as such eqn can be written as

$$m\left(v^1 - v\right) = I \qquad ...(3)$$

Now, if V be the greatest value of velocity during the interval T then the space described by particle is less than VT. But as $T \to 0 \Rightarrow VT \to 0$, From which we conclude that the particle has not displaced during the action of the force F. In fact particle had no time to displace but the velocity has changed from v to v'.

So, we observe that when infinite forces act on a body for indefinitely short time, the displacement of body is zero and the force is measured by change in velocity.

Definition : A force measured by the change of velocity is called impulsive force. Alos I (asin eqn (3)) is the impulse of this impulsive force.

3.13 Equation of Motion in Two-Dimensions Under Impulsive Forces

The equations of motion in two dimensions when the forces acting on the body are finite, are [by art 3.2, equations (1), (2) and (9)]

$$M\,\frac{d^2\bar{x}}{dt^2} = \sum F_1 \quad \text{or} \quad \frac{d}{dt}\left(M\,\frac{d\bar{x}}{dt}\right) = \sum F_1 \qquad ...(1)$$

$$M\,\frac{d^2 y}{dt^2} = \sum F_2 \quad \text{or} \quad \frac{d}{dt}\left(M\,\frac{d\bar{y}}{dt}\right) = \sum F_2 \qquad ...(2)$$

and moment equation is

$$Mk^2\,\ddot{\theta} = \sum\left(x^1 F_2 - y^1 F_1\right)$$

or $$\frac{d}{dt}\left(Mk^2\,\dot{\theta}\right) = \sum\left(x^1 F_2 - y^1 F_1\right) \qquad ...(3)$$

Where $(\bar{x}, \bar{y})$ are the coordinates of centre of gravity. M be mass of body. When large forces act for a very short time than we say that it is an impulse and than we shall deduce from above equations the corresponding equations which give the initial motion of a body under the action of implusive forces.

If the forces be impulsive, let T be the short time during which the impulsive forces act on the body, then on integrating eqn (1), we have

$$\left\{M \frac{d\bar{x}}{dt}\right\}_{t=0}^{T=0} = \sum \left\{\int_0^T F_1\, dt\right\}$$

or $$\left\{M \frac{d\bar{x}}{dt}\right\}_0^T = \sum F_1, \quad \text{where } F_1^1 = \int_0^T F_1\, dt \qquad ...(4)$$

then F_1^1 is the impulse acting at the point (x, y) parallel to x-axis

If u and u^1 be the velocities of the centre of gravity parallel to x-axis, just before and just after the application of the impulsive forces, then integrating eqn (4), we have

$$M\left(u^1 - u\right) = \sum F_1^1 \qquad ...(5)$$

Similarly integrating equation (2), we get

$$\left\{M \frac{d\bar{y}}{dt}\right\}_{t=0}^{T=0} = \sum \left\{\int_0^T F_2\, dt\right\} = \sum F_2^1$$

where $F_2^1 = \int_0^T F_2\, dt$ is the impulse parallel to y-axis

or $$M\left(v^1 - v\right) = \sum F_2^1 \qquad ...(6)$$

where v and v^1 are the velocities of the centre of gravity parallel to y-axis, just before and just after the application of the impulsive forces.

Equation (5) and (6) state that change in the linear momentum of thes mass M, supposed collected at the centre of gravity, in any direction is equal to the sum of impulses in that direction.

Now integrating eqn (3), we have

$$\left\{M k^2 \dot{\theta}\right\}_{t=0}^{T=0} = \sum \left\{x^1 \int_0^T F_2\, dt - y^1 \int_0^T F_1\, dt\right\} = \sum \left(x^1 F_2^1 - y^1 F_1^1\right)$$

where $F_2^1 = \int_0^T F_2\, dt$ etc.

or $$M k^2 \left(w^1 - w\right) = \sum \left(x^1 F_2^1 - y^1 F_1^1\right) \qquad ...(7)$$

where w and w^1 be the angular velocities of the body just before and just after the application of impulsive forces.

Equation (7) states that change in moment of momentum (or angular momentum) about the centre of gravity is equal to the sum of moments of the impulses of the forces about the centre of gravity.

Hence equations (5), (6) and (7) give the equation of motion in two dimension under impulsive forces.

Illustrative Examples :

Example 10 : Two equal uniform rods AB and AC are freely hinged at A and rest in a straight line on a smooth table. A blow is struck at it perpendicular to the rods; show that the K.E. generated is $\frac{7}{4}$ times what it would be if the rods were rigidly fastened together at A.

Solution :

Case - I : **When rods are freely hinged at A.**

Let m be mass and length $2a$ of each rod AB and AC. Let P be the impulse applied at end B and let u_1, w_1 and u_2, w_2 be linear and angular velocities of the rods AB and AC respectively after the application of blow.

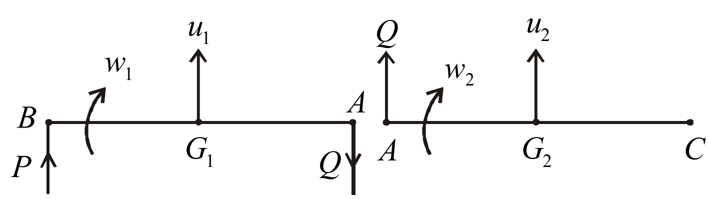


Figure 3.19

Let Q be the impulsive action at the joint A equal and opposite on the two rods when blow is struck (fig. 3.19).

The equation of motion (for impulsive forces) are (for rod AB)

$$m\,(u_1 - 0) = P - Q \qquad ...(1)$$

and $$m\,.\,\frac{a^2}{3}\,w_1 = (P + Q)\,a \qquad ...(2)$$

Rod AB started from rest.

Similarly for rod AC

$$m\,u_2 = Q \qquad ...(3)$$

and $$m\,\frac{a^2}{3}\,w_2 = Q\,.\,a \qquad ...(4)$$

Now both rods AB and AC are connected at A, therefore the velocity of A as found from each rod must be the same

$$\therefore \quad u_1 - a w_1 = u_2 + a w_2 \qquad ...(5)$$

From equations (1)-(4), we have

$$u_1 = \frac{P - Q}{m} \qquad \text{...(6)}$$

$$w_1 = \frac{3\,(P + Q)}{m\,a} \qquad \text{...(7)}$$

$$u_2 = \frac{Q}{m} \qquad \text{...(8)}$$

$$w_2 = \frac{3Q}{m\,a} \qquad \text{...(9)}$$

Using above values of u_1, w_1, u_2 and w_2 in eqn (5), we get

$$\frac{P - Q}{m} - \frac{3\,(P + Q)}{m\,a}\,.\,a = \frac{Q}{m} + a\,.\,\frac{3Q}{m\,a}$$

or $$-\,2P = 8Q \Rightarrow Q = -\,\frac{P}{4} \qquad \text{...(10)}$$

Now using value of Q from (10), in (6) and (7), we have

$$u_1 = \frac{5P}{4m},\; a\,w_1 = \frac{9P}{4m} \qquad \text{...(11)}$$

Similarly using value of Q from (10), in (8) and (9), we have

$$u_2 = -\,\frac{P}{4m},\; a\,w_2 = -\,\frac{3P}{4m} \qquad \text{...(12)}$$

therefore K.E. of the system = K.E. of rod AB + K.E. of rod AC

$$= \frac{m}{2}\left(u_1^2 + \frac{1}{3}\,a^2\,w_1^2\right) + \frac{m}{2}\left(u_2^2 + \frac{1}{3}\,a^2\,w_2^2\right) \quad \left[\because K.E. = \frac{1}{2}\,m\,v^2 + \frac{1}{2}\,m\,k^2\,\dot{\theta}^2\right]$$

K.E. of the system $= \dfrac{m}{2}\left[\dfrac{25P^2}{16m^2} + \dfrac{27P^2}{16m^2}\right] + \dfrac{m}{2}\left[\dfrac{P^2}{16m^2} + \dfrac{3P^2}{16m^2}\right]$

[on using value of u_1, u_2 and w_1, w_2 from (11) and (12)]

or K.E. of the system $= \left(\dfrac{7P^2}{4m}\right) = E_1\ (say)$ ...(13)

Case - II : When the rods are fastened together at A :

In this case rods AB and AC will be considered as forming one rod BC, the reaction Q at A vanishes.

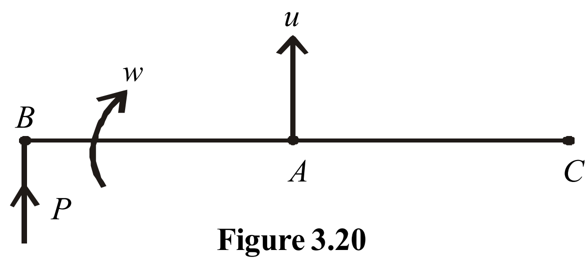


Figure 3.20

The length of complete rod BC is takes as $4a$ and mass $2m$ acting at the centre of gravity of the system at A.

Just after applying blow at B, let u be the linear velocity and w be angular velocity of rod BC. The equations of motion of rod BC for impulsive force are

$$2mu = P \qquad \text{(motion of } C.G.) \qquad ...(14)$$

and $$2m \cdot \frac{4}{3}a^2 w = P \cdot 2a \qquad \text{(motion about } C.G.) \qquad ...(15)$$

then from eqn (14), $$u = \frac{P}{2m} \qquad ...(16)$$

and $$aw = \frac{3P}{4m} \qquad \text{(from eqn (15)} \qquad ...(17)$$

therefore K.E. of the system $= \frac{1}{2}(2m)\left(u^2 + \frac{4}{3}a^2 w^2\right)$

$$= \frac{P^2}{m} \qquad \text{(by eqn (16) and (17))}$$

$\therefore$ K.E. of system in this case $= \frac{P^2}{m} = E_2 \ (say)$ ...(18)

Required relation between the K.E. in the two cases is given by

$$\frac{E_1}{E_2} = \left(\frac{7P^2}{4m}\right) \Big/ \left(\frac{P^2}{m}\right) = \frac{7}{4}$$

or $$E_1 = \frac{7}{4}E_2$$

$\therefore$ (K.E. when the rods are freely hinged at A) $= \left(\frac{7}{4}\right)$ (K.E. when they are rigidly fastened at A)

which is required result.

3.14 Change in K.E. due to action of impulse

A body of mass M is acted upon by of blow of impulse I at given point A. If V and V^1 are the velocities of A in the direction of I just before and just after the action of I. Show that the change in K.E. of the body is $\frac{1}{2} I \left(V + V^1\right)$.

Let M be mass and G be centre of gravity of the rigid body. Take x-axis parallel to the direction of I, where I is blow of impulse at A.

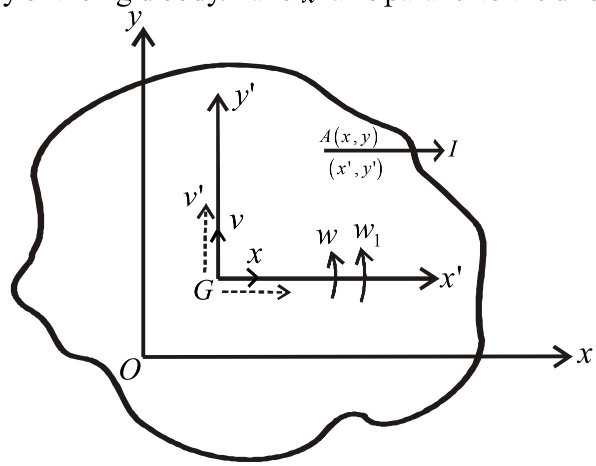


Figure 3.21

Let u, v be the velocities of $C.G.$ G in x and y direction before the action of I and let u_1, v_1 be the velocities of G in x and y direction respectively.

Let w and w_1 be angular velocities just before and just after the action of blow I about G.

Let coordinates of A about ox, oy be (x, y) and coordinates of A about Gx^1, Gy^1 be $\left(x^1, y^1\right)$.

Then the equations of motion are

$$M\left(u_1 - u\right) = I \qquad \text{(change in linear momentum is equal to sum of impulses in that direction)} \qquad ...(1)$$

$$M\left(v_1 - v\right) = O \qquad ...(2)$$

and $$M k^2 \left(w_1 - w\right) = -y^1 I \qquad ...(3)$$

(change in angular momentum is equal to moment of impulses)

Now, we are given that

$V =$ (velocity of A parallel to ox before impulse) = (velocity of G parallel to ox) + (velocity of A relative to G)

or $$V = \left(u - y^1 w\right) \qquad ...(4)$$

Similarly $$V^1 = \left(u_1 - y^1 w_1\right) \qquad ...(5)$$

Thus change in K.E. of body

$$= \frac{1}{2} M\left(u_1^2 + v_1^2 + k^2 w_1^2\right) - \frac{1}{2} M\left(u^2 + v^2 + k^2 w^2\right)$$

$$= \frac{M}{2}\left(u_1^2 - u^2\right) + \frac{M}{2}\left(v_1^2 - v^2\right) + \frac{1}{2} M k^2 \left(w_1^2 - w^2\right)$$

$$= M\left(u_1 - u\right) . \frac{1}{2}\left(u_1 + u\right) + \frac{1}{2} M\left(v_1^2 - v^2\right) + \frac{1}{2} M k^2 \left(w_1 - w\right)\left(w_1 + w\right)$$

$$= \frac{1}{2} I\left(u_1 + u\right) + 0 + \left\{-\frac{I}{2} y^1 \left(w_1 + w\right)\right\} \qquad \text{[by eqn (1), (2) and (3)]}$$

$$= \frac{1}{2} I\left\{\left(u_1 - y^1 w_1\right) + \left(u - y^1 w\right)\right\} = \frac{1}{2} I\left(V + V^1\right) \qquad \text{[by eqn (4) and (5)]}$$

$\therefore$ The change in K.E. $= \frac{I}{2}\left(V + V^1\right)$

3.15 Impact of Rotating elastic sphere on a fixed horiozontal rough plane

A uniform sphere, rotating with an angular velocity w about an axis perpendicular to the plane of motion of its centre, impinges on a rough horizontal plane, to find the resulting change in its motion.

First we assume that the plane is rough enough to prevent sliding.

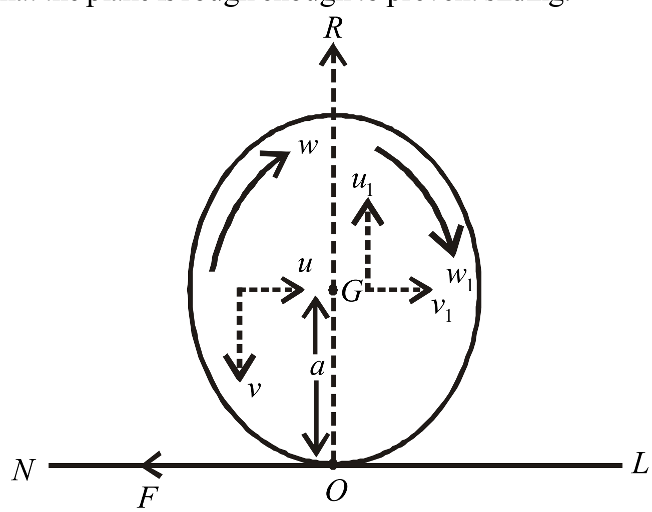


Figure 3.22

Let G be centre of the impnging sphere of mass m and radius a. Let u and v be the components of the linear velocities along and perpendicular to the plane (NL) and w be the angular velocity just before the impact whereas u^1, v^1 be the components of linear velocity and w^1 the angular velocity just after the impact. Let R be normal impulsive reaction and F be impulsive friction, as shown in the fig. 3.22. Then equation of motion are

$m\left(u^1 - u\right) = -F$ ...(1) (*change in momentum = sum of impulses along the plane*)

$m\left(v^1 + v\right) = R$ ...(2) (*change in momentum = sum of impulses perpendicular to the plane*)

and $m k^2\left(w^1 - w\right) = F.a$ ...(3)

(change in moment of momentum about G = moment of Impulses)

Where k is radius of gyration $\left(k^2 = \frac{2}{5} a^2 \;\; for\ solid\ sphere\right)$ about axis through G.

Since, the point of contact O is instantaneously reduced to rest, there being no sliding,

$$\therefore \quad u^1 - a w^1 = 0 \qquad ...(4)$$

Also, if e is the cofficient of restitution, then by Newton's law of elastic collision

Velocity of separation = e (velocity of approach)

$$\text{or} \quad v^1 = e v \qquad ...(5)$$

Now using value of F from eqn (1) in (3), we get

$$m k^2 \left(w^1 - w\right) = - a m \left(u^1 - u\right)$$

$$\text{or} \quad \frac{2}{5} a^2 \left(w^1 - w\right) = - a \left(u^1 - u\right)$$

$$\text{or} \quad 2 \left(a w^1 - a w\right) = - 5 u^1 + 5 u$$

From (4)

$$7 u^1 = (5u + 2 a w) \Rightarrow u^1 = \frac{1}{7} (5u + 2 a w) = a w^1 \qquad ...(6)$$

Also form (1),

$$F = m \left(u - u^1\right) = m \left[u - \frac{1}{7} (5u + 2 a w)\right]$$

$$= \frac{2m}{7} (u - a w) \qquad ...(7)$$

Now we consider following cases :

Case - I : When $u = a w$, then from eqn (7), $F = 0$. Hence there is no frictional force, then by eqn (1) and (3) $u^1 = u$ and $w^1 = w$, which shows that u and w do not change.

Case - II : When $u < a w$

Then by eqn (7), F is $-ve$, which show that direction of F will be opposite to what has been shown fig. 3.22. By eqn (1), $u^1 > u$ and by eqn (3) $w^1 < w$, therefore when the point of contact O is moving ($\leftarrow$ *direction*) before impact, the angular velocity is decreased by the impact, while the horizontal velocity is increased.

Case - III : When $u > a w$, then by eqn (7), F is $+ve$, i.e. F acts in ($\leftarrow$ *direction*) as shown in fig. 3.22. Then by eqn (1) and (3) $u^1 < u$ and $w^1 < w$. Therefore the point of contact O before impact is moving, in direction ($\rightarrow$), and the angular velocitiy is increased while horizontal velocity is diminished.

Case - IV : When the angular velocity w before impact is reversed. In this sign of w is changed and we have

$$u^1 = a w^1 = \frac{1}{7}(5u - 2aw) \qquad ...(8)$$

and $$F = \frac{2m}{7}(u + aw) \qquad \text{[by eqn (7)]} \qquad ...(9)$$

Again there arise two cases :

(a) If $u = \frac{2}{5} a w$, then by eqn (8), $u^1 = 0 = w^1$ therefore we conclude that after impact the sphere rebounds from the plane vertically with no spin as $w^1 = 0$.

(b) If $u < \frac{2}{5} a w$, then u^1 is negative and the sphere rebounds towards the direction from which it came. Alos vertical velocity after impact is $e v$ and $R = m(1 + e) v$.

Illustrative Examples :

Example 11 : Two equal uniform rods, AB and BC, are freely jointed at B and turn about a smooth joint at A. When the rods are in a straight line, w being the angular velocity of AB and x the velocity of the centre of mass of BC : BC impinges on a fixed inelastic obstale at a point D; Show that the rods are instantaneous brought to rest if $BD = 2a\left(\frac{2u - aw}{3u + 2aw}\right)$, where $2a$ is the length of either rod.

Solution : Let m be mass and $2a$ be the length of each rod AB and BC.

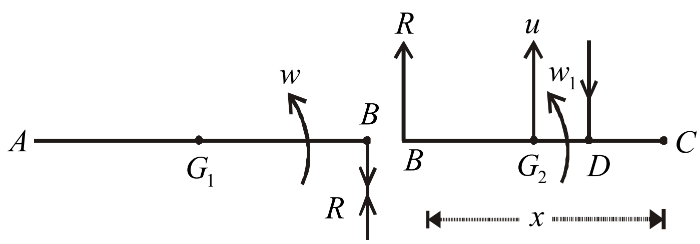


Figure 3.23

Let G_1 and G_2 be the centres of gravity of rods AB and BC respectively. Let R be an impulsive action between the rods at B, equal and opposite. When the rods BC impings on a fixed inelastic obstacle at a point D, such that $BD = x$.

$\therefore G_2 D = (x - a)$, Let P be the impulsive action due to the impingement at the fixed obstacle D.

If the rods are instantaneously brought to rest, then the equation of motion of rod AB is

$$m \cdot k^2 w = R \cdot 2a \qquad \text{(Taking moment about } A) \qquad \left(k^2 = \frac{4}{3}a^2\right) \quad ...(1)$$

and equation of motion of rod BC are

$$mu = P - R \qquad ...(2)$$

$$m\frac{a^2}{3} w_1 = P(x - a) + R \cdot a \qquad ...(3)$$

Since the rods are connected at B, the motion of B as deducted from each rod must be the same

$$\therefore \quad 2aw = \text{velocity of } G_2 + \text{velocity of } G_1 = u - aw_1 \qquad ...(4)$$

$$w_1 = \left(\frac{u - 2aw}{a}\right) \qquad ...(5)$$

Using values of P and w_1 from eqn (2) and eqn (5) in (3), we get

$$m \cdot \frac{a}{3}(u - 2aw) = (mu + R)(x - a) + Ra$$

or $$\frac{ma}{3}(u - 2aw) = mu(x - a) + Rx \qquad ....(6)$$

Now using value of R from (1) in above eqn (6), we have

$$\frac{ma}{3}(u - 2aw) = mu(x - a) + m \cdot \frac{2}{3} aw \cdot x$$

or $$\frac{a}{3}(u - 2aw + 3u) = \frac{1}{3}(3u + 2wa) \Rightarrow x = 2a \frac{2u - aw}{3u - 2aw} = BD$$

which is required result t.

3.16 Summary

In this unit you have studied about equations of motion in two dimensions under finite forces and under impulsive forces, some results of Kinetic Energy of a rigid body in two dimensions, Rolling and Sliding friction conditions, sliding of a rod, sliding and rolling of a sphere on an inclined plane, equation of motion of impulsive forces. Change in K.E. due to action of impulse will help the students to easily understand various results obtained in this unit.

3.17 Exercise

1. A rough uniform rod of length $2a$, is placed on a rough table at right angles to its edge; if its centre of gravity initially be at a distance b beyond the edge, show that the rod will begin to slide when it has turned through an angle $\tan^{-1}\left(\frac{\mu a^2}{a^2 + 9b^2}\right)$, where μ is the coefficient of friction.

2. A uniform rod is held at an inclination α to the horizon with one end in contact with a horizontal table whose coefficient of friction is μ. If it is then released, show that it will commence to slide if

$$\mu < \left(\frac{3 \sin\alpha \cos\alpha}{1 + 3\sin^2\alpha}\right)$$

if $\alpha = 45^\circ$, then $\mu < \frac{3}{5}$.

3. A cylinder rolls down a smooth plane whose inclination to the horizon is α, unwrapping as it goes, a fine string fixed to the highest point of the plane; find its acceleration and the tension of the string.

(Ans. : Acceleration $= \frac{a^2}{a^2 + k^2} g \sin\alpha$, tension $= \frac{k^2}{a^2 + k^2} mg \sin\alpha$)

4. If a sphere be projected up an inclined plane, for which $\mu = \frac{1}{7} \tan\alpha$, with velocity V and an initial angular velocity w (in the direction in which it would roll up), and if $V > aw$, Show that the friction acts downwards at first and upwards after wards, and prove that the whole time during which the sphere rises is $\frac{17V + 4aw}{18g \sin\alpha}$.

5. A solid uniform sphere resting on another fixed sphere is slightly displaced and begins to roll down. Show that it will slip when the common normal makes with the vertical an angle given by $2 \sin\theta = \mu (17 \cos\theta - 10 \cos\alpha)$, where α is the initial angle of the common normal with the vertical and μ is the coefficient of friction.

6. A circular cylinder of radius a and radius of gyration k rolls without slipping inside a fixed hollow cylinder of radius b. show that the plane through their axes moves like a circular pendulum of length $(a - b)\left(1 + \frac{k^2}{a^2}\right)$.

7. Four equal uniform, rods AB, BC, CD and DE are freely jointed at B, C and D and lie on a smooth table in the form of a square. The rod AB is struck by a blow at A at right angles to AB from the inside of the square; Show that the inital velocity of A is 79 times that of E.

8. Two equal uniform rods, AB and AC, are freely jointed at A and are placed on a smooth table so as to be at right angles. The rod AC is struck by a blow at C in a direction perpendicular to itself; Show that the resulting velocities of the middle point of AB and AC are in the ratio 2 : 7.

3.18 Answers of Self Learning Exercise

Self Learning Exercise - I

1. See art 3.1

2. K.E. of body = $\frac{1}{2} Mv^2 + \frac{1}{2} Mk^2 \dot{\theta}$

3. Friction is self adjusting force which tends to prevent the relative motion of the point at which it acts.

4. $\mu = 0$

5. $F < \mu R$

Self Learning Exercise - II

1. No 2. Yes 3. No 4. $\mu = \tan\lambda$

□□□□

UNIT - 4
Motion in Three Dimensions Under Finite Forces

Structure of the unit

4.0 Objective

When a rigid body performs three dimensional motion, its motion is translatory as well as rotational. In many situation a point of rigid body is fixed, in this case the body performs a rotational motion. Here in this unit our objective is to consider such types of motion of a rigid body.

4.1 Introduction

A rigid body, free to move has six degrees of freedom. Its position, is fully determined when three points of it are given. The nine coordinates of these three points are connected by three relations expressing the invariable lengths of the three lines joining them. Hence in all, the body has six degrees of freedom.

When a rigid body is moving about a fixed point, it has 6-3 = 3 degrees of freedom and therefore three constraints. In this unit we establish Euler's equations of motion to discuss such motions of a rigid body.

4.2 Moving axes and fixed axes

Suppose a rigid body is moving about a fixed point O of itself. We take OA, OB, OC the principal axes which are fixed in the body and moving with the body; and OX, OY, OZ be axes fixed in space. Let θ_1, θ_2, θ_3 be components of angular velocity of the body at time t about OX, OY, OZ; and

w_1, w_2, w_3 be components of angular velocity about OA, OB, OC respectively. Consider a vector V.

Let V_x, V_y, V_z be its components along the axes OX, OY, OZ and V_1, V_2, V_3 be its components along the principal axes OA, OB, OC respectively. If α, β, γ be the inclinations of OA, OB, and OC to OX respectively. Then we have

$$V_x = V_1 \cos\alpha + V_2 \cos\beta + V_3 \cos\gamma$$

Differentiating with respect to t, we get

$$\dot{V}_x = \dot{V}_1 \cos\alpha - V_1 \dot{\alpha} \sin\alpha + \dot{V}_2 \cos\beta - V_2 \dot{\beta} \sin\beta + \dot{V}_3 \cos\gamma - V_3 \dot{\gamma} \sin\gamma$$

Let OA, OB, OC momentarily coincide with OX, OY, OZ at time t, then

$$\alpha = 0, \qquad \beta = \frac{\pi}{2}, \qquad \gamma = \frac{\pi}{2}$$

then the above relation becomes

$$\dot{V}_x = \dot{V}_1 - V_2 \dot{\beta} - V_3 \dot{\gamma}$$

Since β is the angle which OB makes with OX, therefore $\dot{\beta}$ denotes the rate at which OB is receding from OX; also the same rate is denoted by θ_3 and in the same sense, therefore

$$\dot{\beta} = \theta_3$$

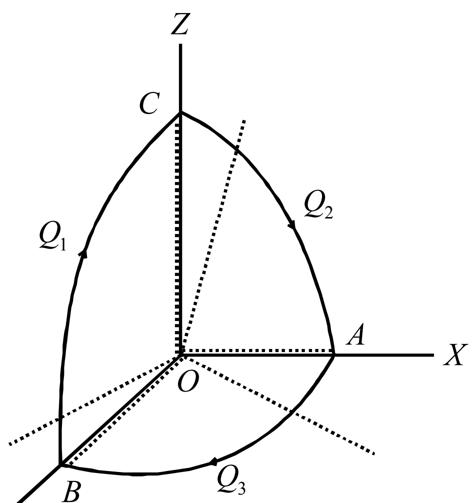


Figure 4.1

Similarly $\dot{\gamma}$ denotes the rate at which OC is receding from OX; also the same rate is denoted by θ_2 but in the opposite sense, therefore $\dot{\gamma} = \theta_2$. Hence the above relation becomes

$$\dot{V}_x = \dot{V}_1 - V_2 \theta_3 + V_3 \theta_2$$

similarly $\dot{V}_y = \dot{V}_2 - V_3 \theta_1 + V_1 \theta_3$

$$\dot{V}_z = \dot{V}_3 - V_1 \theta_2 + V_2 \theta_1$$

If OA, OB, OC coincide with OX, OY, OZ at time t, then $\theta_1 = w_1$, $\theta_2 = w_2$, $\theta_3 = w_3$;

amd above three relations can be written as

$$\left.\begin{aligned} \dot{V}_x &= \dot{V}_1 - V_2 w_3 + V_3 w_2 \\ \dot{V}_y &= \dot{V}_2 - V_3 w_1 + V_1 w_3 \\ \dot{V}_z &= \dot{V}_3 - V_1 w_2 + V_2 w_1 \end{aligned}\right\} \quad ...(1)$$

These relations establish a rule that how the components of a vector along the axes fixed in space can be expressed in terms of the components of the same vector along the principal axes and angular velocities w_1, w_2, w_3 about them.

As an example, if h_1, h_2, h_3 be components of angular momentum of the body about OA, OB, OC and h_x, h_y, h_z those about OX, OY, OZ, then

$$\left.\begin{aligned} \dot{h}_x &= \dot{h}_1 - h_2 w_3 + h_3 w_2 \\ \dot{h}_y &= \dot{h}_2 - h_3 w_1 + h_1 w_3 \\ \dot{h}_z &= \dot{h}_3 - h_1 w_2 + h_2 w_1 \end{aligned}\right\} \quad ...(2)$$

As another example, if V denotes the resultant angular velocity of a body about an instantaneous axis with components w_1, w_2, w_3 about $OA, OB, OC,$ and w_x, w_y, w_z about OX, OY, OZ, then

$$\left.\begin{aligned} \dot{w}_x &= \dot{w}_1 - w_2 \theta_3 + w_3 \theta_2 \\ \dot{w}_y &= \dot{w}_2 - w_3 \theta_1 + w_1 \theta_3 \\ \dot{w}_z &= \dot{w}_3 - w_1 \theta_2 + w_2 \theta_1 \end{aligned}\right\} \quad ...(3)$$

4.3 Euler's Dynamical Equations of Motion

Suppose a rigid body is moving about a fixed point O of itself under the action of external forces. Let OX, OY, OZ be axes fixed in space and OA, OB, OC be principal axes fixed in the body and moving with the body. Let L, M, N be components of external forces along OX, OY, OZ; w_1, w_2, w_3 be components of angular velocity and A, B, C be moments of inertia about OA, OB, OC respectively.

Let h_x, h_y, h_z be the components of angular momentum about OX, OY, OZ; and h_1, h_2, h_3 the corresponding quantities about OA, OB, OC, then we have from (form last article ...... eqn (2))

$$\dot{h}_x = \dot{h}_1 - h_2 w_3 + h_3 w_2 \quad ...(1)$$

with two more similar relations.

Let (x, y, z) be coordinates of a particle of mass m of rigid body relative to OA, OB, OC. Therefore position vector of this point $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and velocity $\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$, if $\vec{w}$ be resultant angular velocity, then $\vec{w} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k}$ and

$$\vec{v} = \vec{w} \times \vec{r}$$

$$\Rightarrow \quad \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} = \left(w_1\hat{i} + w_2\hat{j} + w_3\hat{k}\right) \times \left(\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}\right)$$

$$\Rightarrow \quad \left.\begin{aligned} \dot{x} &= z\,w_2 - y\,w_3 \\ \dot{y} &= x\,w_3 - z\,w_1 \\ \dot{z} &= y\,w_1 - x\,w_2 \end{aligned}\right\} \qquad \text{...(2)}$$

If $\vec{H}$ be resultant angular momentum, then

$$\vec{H} = \sum \vec{r} \times m\,\vec{v}$$

$$\Rightarrow \quad h_1\,\hat{i} + h_2\,\hat{j} + h_3\,\hat{k} = \sum \left(x\,\hat{i} + y\,\hat{j} + z\,\hat{k}\right) \times \left(\dot{x}\,\hat{i} + \dot{y}\,\hat{j} + \dot{z}\,\hat{k}\right)$$

$$\therefore \quad h_1 = \sum m\,(y\,\dot{z} - z\,\dot{y})$$

using (2) we get

$$h_1 = \sum m\left[y\left(y\,w_1 - x\,w_2\right) - z\left(x\,w_3 - z\,w_1\right)\right]$$

$$= w_1 \sum m\left(y^2 + z^2\right) - w_2 \sum m\,x\,y - w_3 \sum m\,z\,x$$

$$= A\,w_1 - F\,w_2 - E\,w_3$$

Similarly

$$h_2 = B\,w_2 - D\,w_3 - F\,w_1$$

$$h_3 = C\,w_3 - E\,w_1 - D\,w_2$$

where A, B, C are M.I. and D, E, F are P.I. of rigid body about principal axes OA, OB, OC.

But $D = E = F = 0$

so that $h_1 = A\,w_1, \qquad h_2 = B\,w_2, \qquad h_3 = C\,w_3$

$$\Rightarrow \quad \dot{h}_1 = A\,\dot{w}_1, \qquad \dot{h}_2 = B\,\dot{w}_2, \qquad \dot{h}_3 = C\,\dot{w}_3$$

Substituting these in (1)

$$\dot{h}_x = A\,\dot{w}_1 - \left(B - C\right) w_2\,w_3$$

Also we have $\dot{h}_x = L,\ \dot{h}_y = M \qquad \dot{h}_z = N$

$$\therefore \quad A\,\dot{w}_1 - (B - C)\,w_2\,w_3 = L \qquad \text{...(3)}$$

Similarly

$$B\,\dot{w}_2 - (C - A)\,w_3\,w_1 = M \qquad \text{...(4)}$$

$$C\,\dot{w}_3 - (A - B)\,w_1\,w_2 = N \qquad \text{...(5)}$$

Equation (3), (4) and (5) are known as Euler's Dynamical equations of motion.

4.4 Illustrative Example

Example 1 : A body having an axis of symmetry OC, moves about a fixed point O under no forces except a constant retarding couple $k\,C$ about the axis OC. If $A,\ A, C$ are the moments of Inertia and

w_1, w_2, w_3 the angular velocities about the principal axes OA, OB, OC. Show that at time t,

$$w_1 = \Omega \cos\left[\lambda t\left(\Omega - \frac{1}{2} k t\right)\right],$$

$$w_2 = -\Omega \sin\left[\lambda t\left(\Omega - \frac{1}{2} k t\right)\right]$$

$$w_3 = \Omega - k t$$

where $\lambda = \dfrac{A - C}{A}$, the initial values of w_1, w_2, w_3 being Ω, O, Ω respectively.

Solution : Here components of external forces are

$$L = 0,\ M = 0, N = k C$$

and M.I.'s are A, A, C

From Euler's equations of motion

$$A \dot{w}_1 - (A - C)\, w_2 w_3 = 0 \qquad ...(1)$$

$$A \dot{w}_2 - (C - A)\, w_3 w_1 = 0 \qquad ...(2)$$

$$C \dot{w}_3 = -k C \qquad ...(3)$$

From (3) $\quad \dot{w}_3 = -k \qquad \therefore\ w_3 = \Omega - k t$

when $\quad t = 0,\ w_3 = \Omega, \qquad \therefore\ C_1 = \Omega \qquad \therefore\ w_3 = \Omega - k t$

Hence from (1) and (2)

$$\dot{w}_1 = \frac{A - C}{A} w_2 w_3 = \lambda (\Omega - k t)\, w_2$$

$$\dot{w}_2 = -\frac{A - C}{A} w_3 w_1 = -\lambda (\Omega - k t)\, w_1$$

Let $\quad u = w_1 + i w_2$

$\therefore \quad \dot{u} = \lambda(\Omega - k t)(w_2 - i w_1) = -\lambda i (\Omega - k t)\, u$

$\Rightarrow \quad \dfrac{\dot{u}}{u} = -\lambda i (\Omega - k t)$

on integration and applying initial conditions

$$u = -\Omega e^{-i\lambda}\left(\Omega t - \frac{1}{2} k t^2\right)$$

Equating real and imaginary parts we get the required results.

Example 2 : If $2T = Aw_1^2 + Bw_2^2 + Cw_3^2$ and C be the moments of the impressed forces about the instantaneous axis of rotation and w be the resultant angular velocity, prove that

$$\frac{dT}{dt} = wG$$

Also, prove that $$\frac{d}{dt} = \left(\frac{1}{2} I w^2\right) = wG$$

where I is the moment of inertia about the instantaneous axis and A, B, C have their usual meaning

Solution : Given that $$2T = Aw_1^2 + Bw_2^2 + Cw_3^2$$

therefore $$\frac{dT}{dt} = Aw_1 \dot{w}_1 + Bw_2 \dot{w}_2 + Cw_3 \dot{w}_3 \qquad ...(1)$$

we know that the Euler's dynamical equations referred to principal axes are

$$\left.\begin{aligned} A\dot{w}_1 - (B - C)\, w_2 w_3 &= L \\ A\dot{w}_2 - (C - A)\, w_3 w_1 &= M \\ A\dot{w}_3 - (A - B)\, w_1 w_2 &= N \end{aligned}\right\} \qquad ...(2)$$

From (1) and (2), we have

$$\frac{dT}{dt} = w_1 \left[L + (B - C)\, w_2 w_3\right] + w_2 \left[M + (C - A)\, w_3 w_1\right]$$

$$+ w_3 \left[N + (A - B)\, w_1 w_2\right] = Lw_1 + Mw_2 + Nw_3 \qquad ...(3)$$

If l, m, n are the direction cosines of the instantaneous axis of rotation referred to the principal axis, fixed in body, then

$$w_1 = lw,\ w_2 = mw,\ w_3 = nw \qquad \text{and} \qquad G = Ll + Mm + Nn \qquad ...(4)$$

Therefore (3) becomes

$$\frac{dT}{dt} = w\,(Ll + Mm + Nn) = Gw \qquad ...(5)$$

Further, from (5)

$$2T = \left(Al^2 + Bm^2 + Cn^2\right) w^2 = Iw^2 \qquad ...(6)$$

or $$T = \frac{1}{2} Iw^2$$

so that $$\frac{d}{dt}\left(\frac{1}{2} Iw^2\right) = Gw^2$$

4.5 Intantaneous axis of rotation

When a body moves about a fixed point O and at time t if w_1, w_2, w_3 are the components of angular velocities about the axes OA, OB, OC fixed in the body then direction cosines of the resultant axis of rotation with these axes are $\frac{w_1}{w}, \frac{w_2}{w}, \frac{w_3}{w}$, where $w = \sqrt{w_1^2 + w_2^2 + w_3^2}$ since, w_1, w_2, w_3 are functions of time t, therefore resultant axis of rotation changes with time. Hence at different instant, there are different axes of rotation. On account of its temporary character, it is called instantaneous axis of rotation or simply instantaneous axis.

Example 3 : A rigid body, symmetrical about an axis, so that $B = A$, is supported at its centre of gravity which is fixed and the only forces that have moments about the $C.G.$ are equivalent to a retarding frictional couple, proportational to the angular velocity and acting in a plane at right angles to the instantaneous axis. Assuming $C > A$, prove that the equations of motion can be integrated in the form

$$w_1 = \alpha\, e^{-C\lambda t/A} \,.\, \sin\left(\frac{\sigma}{\lambda} e^{-\lambda t} + \in\right)$$

$$w_2 = \alpha\, e^{-C\lambda t/A} \,.\, \cos\left(\frac{\sigma}{\lambda} e^{-\lambda t} + \in\right)$$

$$w_3 = n\, e^{-\lambda t}$$

where n, α, σ are constants, $\sigma = \frac{n\,(C - A)}{A}$ and λ is a constant defined by a constant couple.

Solution : Here $L = -\lambda w_1, M = -\lambda w_2, N = -\lambda w_3, B = A$, so Euler's equations of motion are

$$A\,\dot{w}_1 - (A - C)\, w_2\, w_3 = -\lambda w_1 \qquad ...(1)$$

$$A\,\dot{w}_2 + (A - C)\, w_3\, w_1 = -\lambda w_2 \qquad ...(2)$$

$$C\,\dot{w}_2 = -\lambda w_3 \qquad ...(3)$$

Integrating (3) we get $w_3 = n\, e^{-\lambda t/C}$ ...(4)

Using (4) in (1) and (2), we have

$$A\dot{w}_1 - (A - C)\, n\, e^{-\lambda t/C}\, w_2 = -\lambda\, w_1$$

$$A\dot{w}_2 + (A - C)\, n\, e^{-\lambda t/C}\, w_1 = -\lambda\, w_2$$

Putting $u = w_1 + i\,w_2$, we get

$$\dot{u} = \left[\sigma\, i\, e^{-\lambda t/C} - \frac{\lambda}{A}\right] u$$

Integrating and separating real and imaginary parts, we get

$$w_1 = \alpha\, e^{-\lambda t/A} \cos\left\{\frac{\sigma}{\lambda}\, e^{-\lambda t/C}\right\}$$

$$w_2 = -\,\alpha\, e^{-\lambda t/A} \sin\left\{\frac{\sigma}{\lambda}\, e^{-\lambda t/C}\right\}$$

where α is a constant.

Putting $C\lambda$ for λ and $\in$ for $\frac{\pi}{2}$ in above equations we get the required results.

Also we note that

$$\frac{w_1^2 + w_2^2}{w_3^2} = \frac{\alpha^2}{m^2}\, e^{-2\lambda\left(\frac{1}{A}-\frac{1}{C}\right)t} \to o \qquad \text{as} \qquad t \to \infty \qquad (\because C > A)$$

Ultimately $w_1^2 + w_2^2 = 0 \Rightarrow w_1 = 0 = w_2$

Hence the axis of rotation is the axis of greatest moments i.e. the least axis of the figure.

Example 4 : An uniaxial body is supported at its centre of mass and is rotating initially with angular velocity w about an axis perpendicular to the axis of symmetry. Prove that if a couple of constant moment l is applied above the axis of symmetry, the instantaneous axis will describe a cone whose equation referred to the axis fixed in the body, of which that of z coincides with the axis of symmetry, is $2\,A\,l\,(x^2 + y^2)\tan^{-1}\frac{y}{x} = C(C - A)\,w^2 z^2$.

Solution : Initially we take $w_1 = w,\ w_2 = 0,\ w_3 = 0$

here $B = A,\ L = 0,\ M = 0,\ N = l$

Euler's equations of motion are

$$A\,\dot{w}_1 - (A - C)\,w_2\,w_3 = 0 \qquad ...(1)$$

$$A\,\dot{w}_2 + (A - C)\,w_3\,w_1 = 0 \qquad ...(2)$$

$$C\,\dot{w}_3 = l \qquad ...(3)$$

From (1) and (2) we get

$$\frac{\dot{w}_1}{\dot{w}_2} = -\frac{w_2}{w_1}$$

$\therefore$ on integration $w_1^2 + w_2^2 = \text{constant} = w^2$

eqn (3) gives $w_3 = \frac{l}{C}\,t + \text{constant} = \frac{l\,t}{C}$ as $t = 0,\ w_3 = 0$

Hence (1) and (2) imply

$$A\,\dot{w}_1 = (A - C)\,\frac{l}{C}\,w_2\,t$$

$$A\dot{w}_2 = -(A - C)\frac{l}{C} w_1 t$$

Putting $u = w_1 + i\, w_2$, we get

$$u = -\frac{A - C}{AC} l\, t\, i\, u$$

on integration $u = w\, e^{-i \frac{A-C}{AC} l t^2 / 2}$

Equating real and imaginary parts.

$$w_1 = w \cos\left\{\frac{C - A}{2AC} l\, t^2\right\}$$

$$w_2 = w \sin\left\{\frac{C - A}{2AC} l\, t^2\right\}$$

If w^1 be the resultant angular velocity, $dc's$ of instantaneous axis are $\frac{w_1}{w'}, \frac{w_2}{w'}, \frac{w_3}{w'}$, for which we will write x, y, z to find the equation of cone.

Now $\quad w'^2 = w_1^2 + w_2^2 + w_3^2 = w^2 + w_3^2$

$$\tan^{-1}\frac{w_2}{w_1} = \frac{C - A}{2AC} l t^2 = \frac{(C - A)C}{2Al} w_3^2 = \frac{(C - A)}{2Al}\frac{C w_3^2}{w'^2}\ w'^2$$

But $\quad x^2 + y^2 = \frac{w_1^2 + w_2^2}{w'^2}\quad$ i.e. $\quad w'^2 = \frac{w^2}{x^2 + y^2}$

Hence the locus of instantaneous axis is

$$\tan^{-1}\frac{y}{x} = \frac{(C - A)\, C}{2Al} z^2 \frac{w^2}{(x^2 + y^2)}$$

or $\quad 2Al(x^2 + y^2)\tan^{-1}\frac{y}{x} = C(C - A) w^2 z^2$

4.6.1 (a) Kinetic Energy of a body with one point fixed

If (x, y, z) be coordinates of a particle of mass m referred to OA, OB, OC with O as fixed point, we have

velocity components $\dot{x} = z w_2 - y w_3,\ \dot{y} = x w_3 - z w_1,\ \dot{z} = y w_1 - x w_2$

$$\therefore \quad K.E.\quad T = \frac{1}{2}\sum m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$= \frac{1}{2}\sum m\left[(z w_2 - y w_3)^2 + (x w_3 - z w_1)^2 + (y w_1 - x w_2)^2\right]$$

$$= \frac{1}{2} \left(A w_1^2 + B w_2^2 + C w_3^2 - 2 D w_2 w_3 - 2 E w_3 w_1 - 2 F w_1 w_2 \right)$$

If OA, OB, OC be principal axes at O, then

$$D = E = F = 0$$

$$T = \frac{1}{2} \left(A w_1^2 + B w_2^2 + C w_3^2 \right)$$

4.6.2 (b) Angular Momentum about a fixed point

If $\vec{H}$ is angular momentum about the fixed point O, then

$$\vec{H} = \sum m \vec{r} \times \vec{v} = \sum m \vec{r} \times \left(\vec{w} \times \vec{r} \right)$$

$$= \sum m \left[r^2 \vec{w} - \left(\vec{r} \cdot \vec{w} \right) \vec{r} \right]$$

$$= \sum m \left[\left(x^2 + y^2 + z^2 \right) \left(w_1 \hat{i} + w_2 \hat{j} + w_3 \hat{k} \right) \right.$$

$$\left. - \left(w_1 x + w_2 y + w_3 z \right) \left(x \hat{i} + y \hat{j} + z \hat{k} \right) \right]$$

$$= \left(A w_1 - F w_2 - E w_3 \right) \hat{i} + \left(B w_2 - D w_3 - F w_1 \right) \hat{j} + \left(C w_3 - E w_1 - D w_2 \right) \hat{k}$$

If OA, OB, OC be principal axes of the body at O, then

$$\vec{H} = A w_1 \hat{i} + B w_2 \hat{j} + C w_3 \hat{k}$$

i.e. components of the angular momentum about OA, OB, OC are

$$H_1 = A w_1, \; H_2 = B w_2, \; H_3 = C w_3$$

4.7 Vector form of Euler's equations of motion

If $\vec{H}$ be the angular momentum of the body relative to the fixed point O and $\vec{G}$ be the sum of the moments of the external forces about O then equation of motion of the body is

$$\frac{d\vec{H}}{dt} = \vec{G}$$

or

$$\frac{d\vec{H}}{dt} + \vec{W} \times \vec{H} = \vec{G}$$

4.8 Eulerian angles θ, ϕ, ψ

Suppose a rigid body turns about a fixed point O. In order to know the position of the body in space, at any time t, with reference to initial position, three angles θ, ϕ, ψ are choosen to define the position of the principal axes and therefore the body itself. The angles θ, ϕ, ψ are called Eulerian angles.

To understand the concepts of θ, ϕ, ψ construet a spherical surface of radius unity with centre at O. Let the axes OX, OY, OZ which are fixed in space, meet the spherical surface in the points x, y, z. At any time t, when the body is rotating about O, let the moving axes meet the surface in the points A, B, C, then θ is angle between OZ and OC, ϕ is the angle between the plane ZOC and the plane CDA and ψ in the angle between the plane ZOC and the plane ZOX.

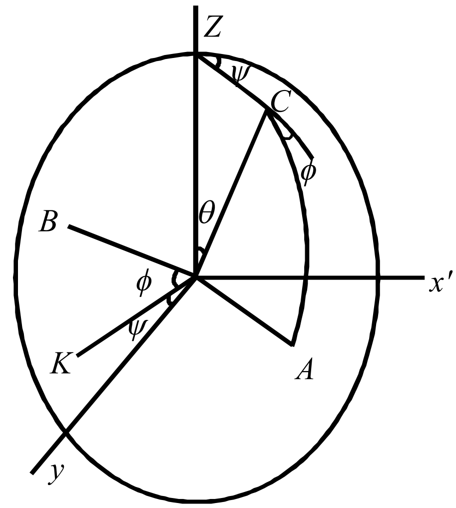


Figure 4.2

To express ϕ and ψ more precisely, draw a line OK perpendicular to the plane ZOC. Normals to the planes COA and ZOX are respectively OB and OY.

Since the angles between any two planes is the same as the angle between their normals, therefore we observe from the figure that, ϕ is the angle between OK and OB. ψ is the angle between OK and OY.

4.9 Euler's Geomatrical Equations of Motion

Let a rigid body be moving about a fixed point O on it. If w_1, w_2, w_3 be the component angular velocities about the principal axes OA, OB, OC, to determine w_1, w_2, w_3 in term of Eulerian angles θ, ϕ, ψ and $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$.

Let OX, OY, OZ be the axes fixed in the space. Suppose θ, ϕ, ψ be the Eulerian angles determining the position of the body at time t.

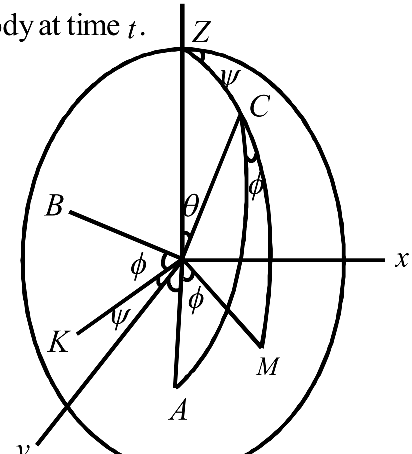


Figure 4.3

Angle θ is in ZOC plane therefore the angular velocity $\dot{\theta}$ is about the line OK

which is normal to this plane. The line OK being perpendicular the both OZ and OC lies in the XOY plane as well as in AOB plane.

Angle ϕ is in AOB plane, therefore angular velocity $\dot{\phi}$ is about OC which is normal to the plane AOB.

Angle ψ is in XOY plane, therefore angular velocity $\dot{\psi}$ is about OZ which is normal to the plane XOY.

Now we find the direction cosines of lines OK, OC and OZ with respect to the principal axes OA, OB, OC.

Since OK is perpendicular to OC and is in AOB plane making angle ϕ with OB therefore direction cosines of OK are $\sin\phi$, $\cos\phi$, O and $dc's$ of OC are clearly $0, 0, 1$. To find the direction OZ draw a line OM perpendicular to OC in the plane ZOC, The line OM, in virtue of its being in ZOC plane, is perpendicular to OK and by virtue of its being perpendicular to OC, lies in AOB plane. Thus both OM and OK lie in AOB plane and are mutually perpendicular, and as OK makes the angle ϕ with OB, therefore OM must make an angle ϕ with OA.

Now the direction cosines of OZ about the co-planer lines OC and OM are $\cos\theta, \sin\theta$.

But the direction cosines of OM about OA, OB are

$$\cos\phi, -\sin\phi$$

$\therefore$ direction cosines of OZ about OA, OB, OC are

$$-\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta$$

Hence components of $\dot{\theta}$ about OA, OB, OC are $\dot{\theta}\sin\phi, \dot{\theta}\cos\phi, O$ the components of $\dot{\phi}$ about OA, OB, OC are $O, O, \dot{\phi}$ and the components of $\dot{\psi}$ about OA, OB, OC are

$$-\dot{\psi}\sin\theta\cos\phi, \dot{\psi}\sin\theta\sin\phi, \dot{\psi}\cos\theta$$

Now, if w_1, w_2, w_3 denote the components of angular velocities about the principal axes OA, OB, OC respectively then from above results, we have

$$w_1 = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi$$

$$w_2 = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi$$

$$w_3 = \dot{\phi} + \dot{\psi}\cos\theta$$

These are called Euler's geometrical equations of motion.

4.10 Deduction of Euler's Equations from Lagrange's Equations

Suppose a rigid body is moving about a fixed point O under the action of external forces. Let OA, OB, OC be principal axes at O and at time t. θ, ϕ, ψ be Eulerian angles which are the generlised coordinates of the body. Let w_1, w_2, w_3 be angular velocities and A, B, C are moments of inertia of the body about OA, OB, OC respectively. The kinetic energy T is given by

$$T = \frac{1}{2}\left(A w_1^2 + B w_2^2 + C w_3^2\right)$$

The Euler's Geometrical equations are

$$w_1 = \dot{\theta} \sin\phi - \dot{\psi} \sin\theta \cos\phi$$

$$w_2 = \dot{\theta} \cos\phi + \dot{\psi} \sin\theta \sin\phi$$

$$w_3 = \dot{\phi} + \dot{\psi} \cos\theta$$

Hence kinetic energy T can be regarded as the function of the generalised coordinates θ, ϕ, ψ and generalised velocities $\dot{\theta}, \dot{\phi}, \dot{\psi}$.

$$\therefore \quad \frac{\partial T}{\partial \dot{\phi}} = \frac{\partial T}{\partial w_1}\frac{\partial w_1}{\partial \dot{\phi}} + \frac{\partial T}{\partial w_2}\frac{\partial w_2}{\partial \dot{\phi}} + \frac{\partial T}{\partial w_3}\frac{\partial w_3}{\partial \dot{\phi}}$$

$$= \frac{\partial T}{\partial w_1} . 0 + \frac{\partial T}{\partial w_2} . 0 + \frac{\partial T}{\partial w_3} . 1 = C w_3$$

$$\frac{\partial T}{\partial \phi} = \frac{\partial T}{\partial w_1}\frac{\partial w_1}{\partial \phi} + \frac{\partial T}{\partial w_2}\frac{\partial w_2}{\partial \phi} + \frac{\partial T}{\partial w_3}\frac{\partial w_3}{\partial \phi}$$

$$= A w_1 \left(\dot{\theta} \cos\phi + \dot{\psi} \sin\theta \sin\phi\right) + B w_2 \left(- \dot{\theta} \sin\phi + \dot{\psi} \sin\theta \cos\phi\right) + 0$$

$$= A w_1\, w_2 - B w_2\, w_1$$

$$= (A - B) w_1\, w_2$$

If $\cup$ be the work function of the system and N be the moment of external forces about axis OC, then

$$\frac{\partial \cup}{\partial \phi} = N$$

Lagrange's ϕ equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial \cup}{\partial \phi}$$

or $$\frac{d}{dt}(C w_3) - (A - B)\, w_1\, w_2 = N$$

or $$C \dot{w}_3 - (A - B)\, w_1\, w_2 = N$$

This is third Euler's Equation.

Similarly we can duduce other two Euler's equations.

4.11 Illustrative Examples

Example 5 : Show that in the free motion of body with an axis of symmetry (C) about its $C.G.$ If n denotes the spin about the axis C and ϕ denotes the Euler's third angle then

$$A\dot{\phi} = (A - C)\, n.$$

Solution : Here θ = constant and $\dot{\theta} = 0$

From Euler's first two geometrical equations of motion we have by squaring

$$\sin^2\theta\,\dot{\psi}^2 = w_1^2 + w_2^2$$

$$\Rightarrow \quad \sin\theta\ \dot{\psi} = \sqrt{w_1^2 + w_2^2} = n\ \tan\ i$$

From Euler's third equation

$$\cos\theta\ \dot{\psi} = n - \dot{\phi}$$

$$\tan\theta = n\tan i \big/ \left(n - \dot{\phi}\right)$$

but we have $\tan\theta = \dfrac{A}{C}\ \tan i$

$$\therefore \quad \frac{n\tan i}{n - \dot{\phi}} = \frac{A}{C}\tan\ i$$

or $\dfrac{C}{A}\, n = \left(n - \dot{\phi}\right)$ or $Cn = A\left(n - \dot{\phi}\right)$

$$\therefore \quad A\,\dot{\phi} = (A - C)\, n$$

Example 6 : Prove that for a rigid body moving about a fixed point $\dfrac{dT}{dt} = \vec{W}\cdot\vec{G}$ where G is the moment of external forces about fixed point and $T = \dfrac{1}{2}\ \vec{H}\cdot\vec{w}$ where H is the angular momentum about the fixed point.

Solution : Here $\vec{W} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k}$

$$\vec{G} = L\hat{i} + M\hat{j} + N\hat{k}$$

$$\vec{H} = \{Aw_1 - Fw_2 - Ew_3\}\,\hat{i} + \{Bw_2 - Dw_3 - Fw_1\}\,\hat{j} + \{Cw_3 - Ew_1 - Dw_2\}\,\hat{k}$$

$$T = \frac{1}{2}\left\{Aw_1^2 + Bw_2^2 + Cw_3^2 - 2Dw_2w_3 - 2Ew_3w_1 - 2Fw_1w_2\right\}$$

$$\therefore \qquad \vec{H} . \vec{W} = \{A w_2 - F w_2 - E w_3\} w_1 + \{B w_2 - D w_3 - F w_1\} w_2$$

$$+ \{C w_3 - E w_1 - D w_2\} w_3$$

$$= 2T$$

$$\frac{dT}{dt} = (A w_1 \dot{w}_1 + B w_2 \dot{w}_2 + ...)$$

The equation of motion is

$$\frac{d\vec{H}}{dt} = \vec{G} \qquad \text{or} \qquad \frac{d\vec{H}}{dt} + \vec{W} \times \vec{H} = \vec{G}$$

$$\therefore \qquad \vec{W} . \frac{d\vec{H}}{dt} = \vec{W} . \vec{G} \qquad ...(1)$$

$$\vec{W} . \frac{d\vec{H}}{dt} = w_1 (A\dot{w}_1 - F\dot{w}_2 - E\dot{w}_3) + w_2 (B\dot{w}_2 - D\dot{w}_3 - F\dot{w}_1)$$

$$+ w_3 (C\dot{w}_3 - E\dot{w}_1 - F\dot{w}_2)$$

$$= \frac{dT}{dt}$$

$\therefore$ using (1) $\qquad \frac{dT}{dt} = \vec{W} . \vec{G}$

Example 7 : A body turns about a fixed point and

$$2T = A w_1^2 + B w_2^2 + C w_3^2 - 2 D w_2 w_3 - 2 E w_3 w_1 - 2 F w_1 w_2 .$$

Show that, if the axes are fixed in the body, but are not necessarily principal axes, Euler's equations of motion may be written in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial w_1}\right) - \frac{\partial T}{\partial w_2} w_2 + \frac{\partial T}{\partial w_3} w_3 = L$$

with two similar expressions.

Solution : If the axes are not the principal axes the equations of motion are

$$\frac{dh_1}{dt} - h_2 w_2 + h_3 w_3 = L \text{ etc}$$

where $h_1 = A w_1 - F w_2 - E w_3 = \frac{\partial T}{\partial w_1}$

Similarly $h_2 = \frac{\partial T}{\partial w_2}, h_3 = \frac{\partial T}{\partial w_3}$

Substituting the values of h_1, h_2, h_3 we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial w_1}\right) - \frac{\partial T}{\partial w_2} w_2 + \frac{\partial T}{\partial w_3} w_3 = L \text{ etc}$$

similarly we can find other two equation of motion

4.12 Self Evaluation Exercise

1. What do you mean by moving axes?
2. Define isntantaneous axis of rotation?
3. Write Euler's Equations of motion?
4. What are the Euler's angles?
5. Write dwon the Euler's geometrical equation of motion?

4.13 Summary

In this unit we have discussed the motion of a rigid body about a fixed point O. The fixed point O lies in the body. We have derived Euler's dynamical equations of motion. To determine the position of rigid body in space three angles θ, ϕ, ψ are defined, these are called Eulerian angles. The relations between angular velocities $\dot{\theta}, \dot{\phi}, \dot{\psi}$ and w_1, w_2, w_3, called Euler's geometrical equation of motion are also stablished.

4.14 Exercise

1. A rigid body possesses an axis of symmetry OC and moves about O under a retarding couple $\lambda C w_3$ about OC ; λ being a constant, A, A, C be principal moments of inertia at O and w_1, w_2, w_3 being the angular velocity components about the principal axes, the third axis being OC. Initially the body is given an angular velocity w about a line inclined at γ to OC. Prove that ultimately the instantneous axis will be perpendicular to OC.

2. A body is moving about a fixed point O and has all its moment of inertia at O equal. If θ, ϕ, ψ be the Eulerian coordinates of the axes OA, OB, OC fixed in the body. Show that the angular moments about the axes fixed in space are respectively.

$$h_x = A\left(-\sin\psi\,\dot{\theta} + \sin\theta\,\cos\psi\,\dot{\phi}\right)$$

$$h_y = A\left(\cos\psi\,\dot{\theta} + \sin\theta\,\sin\psi\,\dot{\phi}\right)$$

$$h_z = A\left(\dot{\phi}\,\cos\theta + \dot{\psi}\right)$$

□□□□

UNIT - 5
Motion in Three Dimensions
(Under no Forces)

Structure of the unit

5.0 Objective

In this unit we will consider centain important situation arising in rigid body dynamics, where the Euler's equations are applicable and there is, no force acting on the rigid body. This type of motion of a rigid body is called force free motion or simply free motion. In this motion the centre of gravity of rigid body is either at reast or moving with a uniform velocity. Consequently we need to consider only the rotational motion of the body.

5.1 Introduction

When a rigid body turns about a fixed point O under no forces, the angular momentum about every axis through O fixed in space is constant. Thus the rigid body has a constant angular momentum which we denote by H. The axis of H is fixed in space and is known as **Invariable Line**. A plane perpendicular to it is called an invariable plane.

Under force free motion of rigid body the kinetic energy is also constant. In this unit we consider the problems on the motion of rigid body under no force condtion.

5.2 Invariable Line

Suppose a rigid body is moving about a fixed point O under no forces. Let OA, OB, OC be the principal axes at O; w_1, w_2, w_3 the angular velocities and A, B, C the moments of inertia of the body about these axes. Then components of angular momentum about these axes are $A\,w_1$, $B\,w_2$, $C\,w_3$ and the resultant axis of these components is called **Invariable Line.**

5.3 Integrals of Energy and Angular Momentum

Let us now consider above system regarding the rigid body then Euler's equations under not forces are

$$A\dot{w}_1 - (B - C)\,w_2\,w_3 = 0 \qquad ...(1)$$

$$B\dot{w}_2 - (C - A)\,w_3\,w_1 = 0 \qquad ...(2)$$

$$C\dot{w}_3 - (A - B)\,w_1\,w_2 = 0 \qquad ...(3)$$

Multiplying (1) by w_1, (2) by w_2 and (3) by w_3 then adding we get

$$A w_1 \dot{w}_1 + B w_2 \dot{w}_2 + C w_3 \dot{w}_3 = 0$$

on integration $\left(A w_1^2 + B w_2^2 + C w_3^2\right) = \text{constant}$ ...(4)

$$\Rightarrow 2T = \text{constant}$$

where T is the kinetic energy and is constant.

This is becuase there in no external force to do the work, so the Kinetic Energy of the body remains constant.

Equation (4) is called integral of energy.

Again multiplying (1) by Aw_1, (2) by Bw_2, (3) by Cw_3 then adding we get

$$A^2 w_1 \dot{w}_1 + B^2 w_2 \dot{w}_2 + C^2 w_3 \dot{w}_3 = 0$$

on integtration $A^2 w_1^2 + B^2 w_2^2 + C^2 w_3^2 = \text{constant}$ ...(5)

or $\quad H = \text{constant}$

where H is the angular momentum and is constant.

This is due to the fact that the moment of external forces is zero as there is no force, so that angular momentum remains constant.

5.3.1 Locus of Invariable Line :

On dividing (4) by (5), we have

$$\frac{A^2 w_1^2 + B^2 w_2^2 + C^2 w_3^2}{A w_1^2 + B w_2^2 + C w_3^2} = \frac{H^2}{2T}$$

$$\Rightarrow \qquad A^2 w_1^2 + B^2 w_2^2 + C^2 w_3^2 = \frac{H^2}{2T}\left(\frac{A^2 w_1^2}{A} + \frac{B^2 w_2^2}{B} + \frac{C^2 w_3^2}{C}\right)$$

or $$A^2 w_1^2 \left(1 - \frac{H^2}{2AT}\right) + B^2 w_2^2 \left(1 - \frac{H^2}{2BT}\right) + C^2 w_3^2 \left(1 - \frac{H^2}{2CT}\right) = 0$$

Let (x, y, z) be coordinates of any point on the invariable line. Since it passes through O and has direction cosines proportioanl to $Aw_1,\ Bw_2,\ Cw_3$

$$\frac{x}{Aw_1} = \frac{y}{Bw_2} = \frac{z}{Cw_3} = r$$

Substituing these in (6) we get

$$x^2\left(1-\frac{H^2}{2AT}\right)+y^2\left(1-\frac{H^2}{2BT}\right)+z^2\left(1-\frac{H^2}{2CT}\right)=0$$

which is the equation of a cone with vertex at O.

Since quantities A, B, C, H, T are constant and A, B, C are constant with respect to body, hence the above cone is fixed in the body and it is called invariable cone.

5.4 Illustrative Examples

Example 1 : Prove that, if a rectangular parallelopiped (edges $2a, 2a, 2b$) rotates about its centre of gravity, its angular velocity about one principal axis is constant and about the other principal axes is periodic, the period being to the period about the first mentioned principal axis as

$$\left(b^2+a^2\right):\left(b^2-a^2\right)$$

Solution : Here $A=\frac{1}{3}M\left(a^2+b^2\right)=B, \qquad C=\frac{2}{3}Ma^2$

$$L=M=N=0$$

Euler's equations of motion are

$$A\dot{w}_1-(A-C)\,w_2\,w_3=0 \qquad \text{...(1)}$$

$$B\dot{w}_2-(C-A)\,w_3\,w_1=0 \qquad \text{...(2)}$$

$$C\dot{w}_3-(A-A)\,w_1\,w_2=0 \qquad \text{...(3)}$$

From (3)

$$w_3=n\text{ (constant) (say)}$$

Again, from (1) and (2)

$$\left(a^2+b^2\right)\dot{w}_1=\left(b^2-a^2\right)n\,w_2 \qquad \text{...(4)}$$

$$\left(a^2+b^2\right)\dot{w}_2=-\left(b^2-a^2\right)n\,w_1 \qquad \text{...(5)}$$

$\therefore \qquad \left(a^2+b^2\right)^2\ddot{w}_1=-\left(b^2-a^2\right)^2n^2\,w_1$

or $\qquad \ddot{w}_1=-\left(\frac{b^2-a^2}{b^2+a^2}\right)^2n^2\,w_1$

which is charctristic of S.H.M.

The periodic time $T=\dfrac{2\pi}{n\left(b^2-a^2\right)/\left(a^2+b^2\right)}=\dfrac{2\pi\left(a^2+b^2\right)}{n\left(b^2-a^2\right)}$

The period of first $T_1 = \dfrac{2\pi}{n}$

Hence $T : T_1 = (a^2 + b^2) : (b^2 - a^2)$

Example 2 : A body moves under no forces about a point O, the principal moments of inertia at O being $6A$, $3A$ and A. Initially the angular vleocity of the body has components $w_1 = n, w_2 = 0, w_3 = 3n$ about the principal axes. Show that at any later time

$$w_2 = -\sqrt{5}\, n \tan h\, \sqrt{5}\, nt$$

and ultimately the body rotates about the mean axis.

Solution : Here $A = 6A,\ B = 3A,\ C = A\ ;\ L = M = N = 0$

Euler's equations of motion are

$$A\dot{w}_1 - (B - C)\, w_2\, w_3 = 0$$

$$B\dot{w}_2 - (C - A)\, w_3\, w_1 = 0$$

$$C\dot{w}_3 - (A - B)\, w_1\, w_2 = 0$$

Substituting for A, B, C, we get

$$3\dot{w}_1 = w_2\, w_3 \qquad ...(1)$$

$$3\dot{w}_2 = -5\, w_3\, w_1 \qquad ...(2)$$

$$\dot{w}_3 = 3\, w_1\, w_2 \qquad ...(3)$$

Dividing (1) by (2)

$$\frac{\dot{w}_1}{\dot{w}_2} = -\frac{w_2}{5w_1}$$

$\Rightarrow$ $5\, w_1\, \dot{w}_1 + w_2\, \dot{w}_2 = 0$

Integrating $5\, w_1^2 + w_2^2 = a$, where a is constant.

Initially $w_1 = n, w_2 = 0$ $\quad \therefore\ a = 5n^2$

$\therefore$ $5\, w_1^2 + w_2^2 = 5n^2 \qquad ...(4)$

Dividing (1) by (3), we have

$$\frac{3\dot{w}_1}{\dot{w}_3} = \frac{w_3}{3w_1} \text{ or } 9w_1\dot{w}_1 - w_3\dot{w}_3 = 0$$

Integrating $9w_1^2 - w_3^2 = b$, where b is constant

initially $w_1 = n, w_3 = 3n$ $\quad \therefore b = 0$

$\therefore$ $9w_1^2 - w_3^2 = 0$

$\Rightarrow \quad 3w_1 = w_3$ ...(5)

From (3) $3\dot{w}_2 = -5w_3 w_1 = -15w_1^2$ using (5)

or $\quad \dot{w}_2 = w_2^2 - 5n^2$

or $\quad dt = -\dfrac{dw_2}{5n^2 - w_2^2}$

on integration $t = -\dfrac{1}{n\sqrt{5}} \tanh^{-1}\left(\dfrac{w_2}{n\sqrt{5}}\right) + C$

initially $t = 0,\ w_2 = 0, \qquad \therefore C = 0$

Hence $w_2 = -n\sqrt{5} \tan h\left(nt\sqrt{5}\right)$

As $\quad t \to \infty, \tan h\left(nt\sqrt{5}\right) \to 1 \qquad \left(\because \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \to 1 \text{ as } x \to \infty\right)$

$\therefore \quad w_2 = -n\sqrt{5}$ as $\quad t \to \infty$

From (4) and (5) we get

$w_1 = 0,\ w_3 = 0$

Thus ultimately $w_1 = 0,\ w_3 = 0$ but $w_2 = -\sqrt{5}\,n$

i.e. the body rotates about mean axis.

Example 3 : A lamina rotating with uniform angular velocity n about an axis through its centre of gravity perpendicular to its plane has an additional angular velocity λn impressed upon it about the axis of least moments ; $(A < B < C)$ where $\lambda^2 = \dfrac{B + A}{B - A}$, Prove that at time t, its angular velocities are $\lambda n \operatorname{sech} nt$, $\lambda n \tanh nt$ and $n \operatorname{sech} nt$. Also show that it will ultimately revolve about the axis of mean moment.

Solution : Axis of least moment is OA, since $A < B < C$

$\therefore$ initial conditions are $w_1 = \lambda n,\ w_2 = 0,\ w_3 = n$

Euler's equations of motion under no force are

$A\dot{w}_1 - (B - C)\,w_2\,w_3 = 0$ ...(1)

$B\dot{w}_2 - (C - A)\,w_3\,w_1 = 0$ ...(2)

$C\dot{w}_3 - (A - B)\,w_1\,w_2 = 0$ ...(3)

For lamina $\quad C = A + B$, therefore above equations become

$\dot{w}_1 = -w_2\,w_3$ ...(4)

$\dot{w}_2 = w_3\,w_1$ ...(5)

$$\lambda^2 \dot{w}_3 = - w_1 \, w_2 \qquad ...(6)$$

Dividing (4) by (5), we have

$$\frac{\dot{w}_1}{\dot{w}_2} = \frac{w_2}{w_1} \qquad \text{or} \qquad w_1 \, \dot{w}_1 + w_2 \, \dot{w}_2 = 0$$

Integrating, $w_1^2 + w_2^2 = a$, where a is constant

initially $w_1 = \lambda n, \, w_2 = 0$, $\therefore a = \lambda^2 n^2$

$$\therefore \qquad w_1^2 + w_2^2 = \lambda^2 n^2 \qquad ...(7)$$

Dividing (4) by (6), we get

$$\frac{\dot{w}_1}{\lambda \dot{w}_3} = \frac{w_3}{w_1} \qquad \text{or} \qquad w_1 \, \dot{w}_1 - \lambda \, w_3 \, \dot{w}_3 = 0$$

Integrating, $w_1^2 - \lambda w_3^2 = b$ where b is consnntant

Initially $w_1 = \lambda n, \, w_3 = n$ $\therefore b = 0$

Hence $w_1^2 = \lambda^2 \, w_3^2$ or $w_1 = \lambda w_3$

Now from eqn (5) $\dot{w}_2 = w_3 \, w_1 = \dfrac{1}{\lambda} \, w_1^2$

$$= \frac{1}{\lambda} \left(\lambda^2 n^2 - w_2^2 \right), \text{ using (7)}$$

On integration, we get

$$t = \lambda \int \frac{d \, w_2}{\lambda^2 n^2 - w_2^2}$$

$$= \lambda \, . \, \frac{1}{\lambda n} \tan h^{-1} \frac{w_2}{\lambda n} + C$$

Initially $t = 0, \, w_2 = 0$ $\therefore e = 0$

$$\therefore \qquad t = \frac{1}{n} \tan h^{-1} \frac{w_2}{\lambda n}$$

$$\therefore \qquad w_2 = \lambda n \, \tan \, h \, nt$$

consequently from (4), we obtain

$$w_1^2 = \lambda n \operatorname{sec} h n t$$

$$\therefore \qquad w_3 = \frac{1}{\lambda} \, w_1 = n \operatorname{sec} h n t$$

Thus at any time t angular velocities are

$$w_1 = \lambda \, n \operatorname{sec} h n t, \; w_2 = \lambda \, n \, \tan h n t, \; w_3 = n \, \operatorname{sec} h n t$$

Now $\underset{t \to \infty}{Limit} \operatorname{sec} h\, n\, t = \underset{t \to \infty}{Limit} \dfrac{2}{e^{nt} + e^{-nt}} = 0$

and $\underset{t \to \infty}{Limit} \tan h\, n\, t = \underset{t \to \infty}{Limit} \dfrac{e^{nt} - e^{-nt}}{e^{nt} + e^{-nt}} = 0$

Thus ultimately the values of w_1, w_2, w_3 are $O, \lambda n, O$ i.e. the body will ultimately rotate about mean axis.

Example 4 : A rigid body rotates freely about its centre of mass under no forces. The body is rotating with angular velocity w about the axis of greatest moment C when a small angular momentum Ap is applied about the axis of least moment A. Show that the instantaneous axis moves in the body with period $\dfrac{2\pi}{\Omega}$, where $\Omega^2 = \dfrac{(C-B)(C-A)}{AB} w^2$.

Solution : Initially $w_3 = w, w_1 = p$ (since initially $A w_1 = Ap$) Euler's equations of motion, under no force, are

$$A\dot{w}_1 - (B - C) w_2 w_3 = 0 \quad ...(1)$$

$$B\dot{w}_2 - (C - A) w_3 w_1 = 0 \quad ...(2)$$

and $$C\dot{w}_3 - (A - B) w_1 w_2 = 0 \quad ...(3)$$

Dividing (1) by (2), we have

$$\frac{A\dot{w}_1}{B\dot{w}_2} = -\left(\frac{C-B}{C-A}\right)\frac{w_3}{w_1},$$

or $A(C-A) w_1 \dot{w}_1 + B(C-B) w_2 \dot{w}_2 = 0$

Intigrating $A(C-A) w_1^2 + B(C-B) w_2^2 = a$ (constant)

initally $w_1 = p, w_2 = 0$ $\quad\therefore a = A(C-A)p^2$

Hence $A(C-A)w_1^2 + B(C-B)w_2^2 = A(C-A)p^2$

both sides are positive as $C > B > A$

This relations shows that when p is small, w_1 and w_2 both must be small, so the product of w_1, w_2 may be neglacted, and then, Euler's third equation reduces to

$$\dot{w}_3 = 0 \qquad \text{or} \qquad w_3 = \text{constant} = w \text{ (initial)}$$

Differentaiting (1) and taking $w_3 = w$, we have

$$A\ddot{w}_1 = -(C-B) w\dot{w}_2 = -\frac{(C-B)(C-A)}{B} w^2 w_1 \qquad \text{by (2)}$$

or $$\ddot{w}_1 = -\frac{(C-B)(C-A)}{A\,.\,B} w^2 w_1$$

or $\quad \ddot{w}_1 = -\Omega^2 w_1 \quad$ where $\quad \Omega^2 = \dfrac{(C-B)(C-A)}{A \, . \, B} w^2$

Hence period $= \dfrac{2w}{\Omega}$

Example 5 : A rectangular parallelopiped whose edges are $a, 2a, 3a$ can turn freely about is centre and is set rotating about a line perpendicular to the mean axis and making an angle $\cos^{-1}\dfrac{5}{8}$ with the least axis. Prove that ultimately the body will rotate about mean axis.

Solution : Let w be the initial angular velocity then

Initially $w_1 = w\cos\theta = \dfrac{5}{8}w$, $w_2 = 0$, $w_3 = w\sin\theta = \dfrac{\sqrt{39}}{8}w$ as $\theta = \cos^{-1}\left(\dfrac{5}{8}\right)$

Here $\quad A = \dfrac{b^2 + c^2}{3} = \dfrac{(2a)^2 + (3a)^2}{3} = \dfrac{13}{3}a^2$

$$B = \frac{c^2 + a^2}{3} = \frac{(3a)^2 + a^2}{3} = \frac{10}{3}a^2$$

$$C = \frac{a^2 + b^2}{3} = \frac{a^2 + (2a)^2}{3} = \frac{5a^2}{3}$$

There is no external force, therefore Euler's equations of motion are

$$A\dot{w}_1 - (B-C)\, w_2\, w_3 = 0$$

$$B\dot{w}_2 - (C-A)\, w_3\, w_1 = 0$$

and $\quad C\dot{w}_3 - (A-B)\, w_1\, w_2 = 0$

Putting the values of A, B, C, we have

$$13\dot{w}_1 = 5\, w_2\, w_3 \qquad ...(1)$$

$$10\dot{w}_2 = -8\, w_3\, w_1 \qquad ...(2)$$

and $\quad 5\dot{w}_3 = 3\, w_1\, w_2 \qquad ...(3)$

Dividing (1) by (2), we have

$$\frac{13\dot{w}_1}{10\dot{w}_2} = -\frac{5w_2}{8w_1} \quad \text{or} \quad 52w_1\dot{w}_1 + 25w_2\dot{w}_2 = 0$$

Integrating, $52w_1^2 + 25w_2^2 = a$, where a is constant

Initially $w_1 = \dfrac{5}{8}w$, $w_2 = 0 \qquad \therefore\ a = \dfrac{325}{16}w^2$

Hence $\quad 52w_1^2 + 25w_2^2 = \dfrac{325}{16}w^2 \qquad ...(4)$

Dividing (1) by (3), we have

$$\frac{13\dot{w}_1}{5\dot{w}_3} = \frac{5w_3}{3w_1} \text{ or } 39w_1\ \dot{w}_1 - 25\ w_3\ \dot{w}_3 = 0$$

on integration $39w_1^2 - 25\ w_3^2 = b$ where b is constant

initially $w_1 = \frac{5}{8}w,\ w_3 = \frac{\sqrt{39}}{8}w, \therefore b = 0$

Hence $39w_1^2 - 25w_3^2 = 0$ ...(5)

using (5) in (2)

$$10\dot{w}_2 = -\frac{8\sqrt{39}}{5}\ w_1^2$$

$$= -\frac{8\sqrt{39}}{5\sqrt{52}}\left(\frac{325}{16}w^2 - 25w_2^2\right) \qquad \text{by (4)}$$

$$= -\frac{4\sqrt{3}}{5}\left(\frac{325}{16}w^2 - 25w_2^2\right)$$

on Integration $\frac{4\sqrt{3}\,t}{5} = -10\int\frac{dw_2}{\left(\frac{325}{16}w^2 - 25w_2^2\right)}$

$$= -10\ \frac{4}{5\sqrt{325}\,w}\ \tan h^{-1}\ \frac{5w_2}{\sqrt{\frac{325}{4}}\,w}$$

$$= -\frac{8}{5\sqrt{13}\,w}\ \tan h^{-1}\ \frac{20w_2}{\sqrt{325}\,w}$$

or $\frac{\sqrt{39}\,w}{2}\ t = -\tan h^{-1}\ \frac{20w_2}{\sqrt{325}\,w}$

or $w_2 = \frac{w\sqrt{325}}{20}\ \tan h\left[\frac{\sqrt{39}}{2}\ wt\right]$

$$= \frac{w\sqrt{13}}{4}\ \tan h\left[\frac{\sqrt{39}}{2}\ wt\right]$$

But when $t \to \infty$, $\tan h\left[\frac{\sqrt{39}}{2}\ wt\right] \to 1$ $\qquad \therefore w_2 = -\frac{w\sqrt{13}}{4}$

and from (4) and (5) $w_1 = 0,\quad w_3 = 0$

Hence the body ultimately rotates about the mean axis.

Example 6 : A rectangular lamina $ABCD$ in which $BC = \sqrt{2}\ AB$, can turn freely about the middle point O of AD. Initially it is set rotating with angular velocity Ω about a line through O perpendicular to AD and making a angles 30° with the plane of the rectangle. Show that after time t, the components of angular velocity of the rectangle about the principal axis at O are $\frac{1}{2}\Omega\sqrt{3}\ \sec h\ \frac{1}{2}\Omega t, \frac{1}{2}\Omega\sqrt{3}\ \tan h\ \frac{1}{2}\Omega t$ and $\frac{1}{2}\Omega\ \sec h\ \frac{1}{2}\Omega t$.

Solution : Let G be the centre of gravity of the lamina and ON be the normal to the plane of lamina.Then OG, OA, ON are the principal axes at O. Initially the lamina is rotating about a line OK perpendicular to AD and making an angle 30° with the plane of the lamina. Thus OK is in the plane NOG to which OA is normal. So direction cosines of OK are $\cos 30^\circ, \cos 90^\circ, \cos 60^\circ$

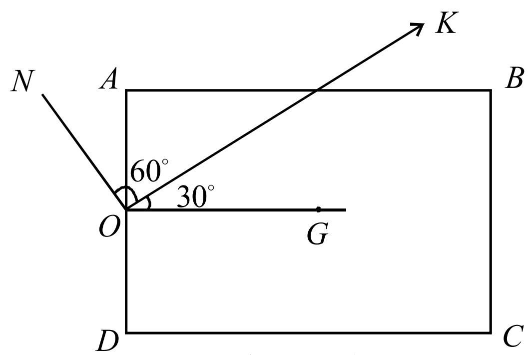


Figure 5.1

i.e. $\frac{\sqrt{3}}{2}, 0, \frac{1}{2}$

Initially the lamina is rotating about OK, so that initially

$$w_1 = \frac{\sqrt{3}}{2}\Omega,\ w_2 = 0,\ w_3 = \frac{1}{2}\Omega$$

Euler's equations of motion under no forces, are

$$A\dot{w}_1 - (B - C)\ w_2\ w_3 = 0$$

$$B\dot{w}_2 - (C - A)\ w_3\ w_1 = 0$$

$$C\dot{w}_3 - (A - B)\ w_1\ w_2 = 0$$

Let $AB = 2a \quad \therefore BC = 2\sqrt{2}\,a$

$$A = \text{M.I. of lamina about } OG = \frac{2}{3}Ma^2$$

$$B = \text{M.I. of lamina about } OA = \frac{1}{3}Ma^2 + Ma^2 = \frac{4}{3}Ma^2$$

$$C = \text{M.I. of lamina about } ON = A + B = 2\ Ma^2$$

Substituting the values of A, B, C in the Euler's equations we have

$$\dot{w}_1 = - w_2 \; w_3 \qquad ...(1)$$

$$\dot{w}_2 = w_3 \; w_1 \qquad ...(2)$$

$$3 \; \dot{w}_3 = - w_3 \; w_2 \qquad ...(3)$$

Dividing (1) by (2) and intigrating, we get

$$w_1^2 + w_2^2 = a \text{ (constant)}$$

Initially $w_1 = \frac{\sqrt{3}}{2} \Omega, w_2 = 0 \quad \therefore a = \frac{3}{4} \Omega^2$

$$\therefore \qquad w_1^2 + w_2^2 = \frac{3}{4} \Omega^2 \qquad ...(4)$$

Dividing (1) by (3) and integrating , we get

$$w_1^2 = 3 \; w_3^2 + b \text{ (constant)}$$

But initially $w_1 = \frac{\sqrt{3}}{2} \Omega, w_2 = \frac{1}{2} \Omega,$ so that $b = 0$

$$\therefore \qquad w_1^2 = 3 \; w_3^2 \quad \text{or} \quad w_1 = \sqrt{3} \; w_3 \qquad ...(5)$$

Equation (2) is $\dot{w}_2 = w_3 \; w_1 = \frac{1}{\sqrt{3}} w_1^2$ by (5)

$$= \frac{1}{\sqrt{3}} \left(\frac{3}{4} \Omega^2 - w_2^2 \right) \qquad \text{by (4)}$$

on integration, $t = \sqrt{3} \int \frac{d \, w_1}{\frac{3}{4} \Omega^2 - w_2^2}$

$$= \sqrt{3} \; . \; \frac{2}{\Omega \sqrt{3}} \tan \, h^{-1} \frac{w_2}{\left(\frac{\sqrt{3}}{2} \Omega \right)} + C$$

Initially, $t = 0, \; w_2 = 0 \qquad \therefore c = 0$

$$\therefore \qquad t = \frac{2}{\Omega} \tan^{-1} \left(\frac{2 \, w_2}{\sqrt{3} \; \Omega} \right)$$

or $\qquad w_2 = \frac{\sqrt{3}}{2} \Omega \tan \, h \, \left(\frac{1}{2} \Omega t \right)$

using (4) $\qquad w_1 = \frac{\sqrt{3}}{2} \Omega \sec \, h \, \left(\frac{1}{2} \Omega t \right)$

from (5) $\quad w_3 = \frac{1}{\sqrt{3}} w_1 = \frac{1}{2} \Omega \sec\left(\frac{1}{2}\Omega t\right)$

Example 7 : A uniform elliptic disc is free to move about a focus and in set rotating with initial angular velocity Ω about an axis perpendicular to the corresponding latus retum and making an angle θ with the plane of the disc. If $\cos 2\theta = \frac{A}{B}$, where A, B are moments of inertia of the disc about the major axis and latus rectum respectively. Prove that after time t the component angular velocity of the disc about the major axis will be

$$\Omega \sqrt{\left(\frac{B-A}{2B}\right)} \sec h \left[\Omega t \sqrt{\frac{B-A}{2B}}\right].$$

Solution : Let S be a focus and SL the semi latus rectum and SC normal to the plane of the disc.

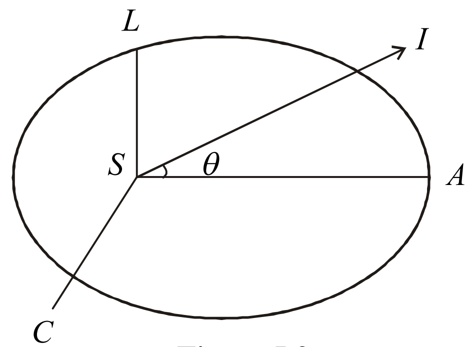


Figure 5.2

Thus SA, SL, SC are the principal axes at S. Suppose initial axis of rotation is SI which is given perpndicular to SL, therefore it lies in the plane CSA making angle θ with SA, so that direction cosines of SI are

$$\cos\theta,\ \cos 90^\circ,\ \cos\left(90^\circ - \theta\right), \text{ i.e } \cos\theta,\ 0,\ \sin\theta$$

$\therefore$ Initially $\quad w_1 = \Omega\cos\theta,\ w_2 = 0,\ w_3 = \Omega\sin\theta$

Euler's equations of motion under no forces are

$$A\dot{w}_1 - (B - C)\, w_2\, w_3 = 0$$

$$B\dot{w}_2 - (C - A)\, w_3\, w_1 = 0$$

$$C\dot{w}_3 - (A - B)\, w_1\, w_2 = 0$$

For lamina $C = A + B$

$\therefore$ Above equation becomes

$$\dot{w}_1 = -\, w_2\, w_3 \qquad ...(1)$$

$$\dot{w}_2 = w_3\, w_1 \qquad ...(2)$$

$$\frac{B+A}{B-A}\dot{w}_3 = -\, w_1 w_2 \quad \text{or} \quad \lambda^2 \dot{w}_3 = -\, w_1 w_2 \qquad ...(3)$$

where $\lambda^2 = \dfrac{B + A}{B - A}$

Dividing (1) by (2) and integrating we obtain

$$w_1^2 + w_2^2 = a \text{ (constant)}$$

Initially $w_1 = \Omega \cos\theta, w_2 = 0 \;\therefore\; a = \Omega^2 \cos^2\theta$

$$\therefore \quad w_1^2 + w_2^2 = \Omega^2 \cos^2\theta \qquad \text{...(4)}$$

Again dividing (1) by (3) and integrating

$$w_1^2 = \lambda^2 w_3^2 + b \text{ (constant)}$$

Initially $w_1 = \Omega \cos\theta,\; w_3 = \Omega \sin\theta$

$$\therefore \quad b = \Omega^2 \left(\cos^2\theta - \lambda^2 \sin^2\theta\right) = 0 \qquad \left\{\because \cos 2\theta = \frac{A}{B}\right\}$$

Hence $w_1^2 = \lambda^2 w_3^2$ ...(5)

Now equation (2) is

$$\dot{w}_2 = w_3\, w_1 = \frac{1}{\lambda} w_1^2 \qquad \text{using (5)}$$

$$= \frac{1}{\lambda}\left(\Omega^2 \cos^2\theta - w_2^2\right) \qquad \text{using (4)}$$

on integration $t = \lambda \displaystyle\int \frac{d w_2}{\Omega^2 \cos^2\theta - w_2^2} = \lambda \frac{1}{\Omega \cos\theta} \tan h^{-1} \frac{w_2}{\Omega \cos\theta} + C$

Initially $t = 0,\; w_2 = 0 \qquad \therefore c = 0$

Hence $t = \dfrac{\lambda}{\Omega \cos\theta} \tan h^{-1} \dfrac{w_2}{\Omega \cos\theta}$

or $\quad w_2 = \Omega \cos\theta \tan h \left(\dfrac{\Omega \cos\theta}{\lambda} t\right)$

Then from (4), we have

$$w_1^2 = \Omega^2 \cos^2\theta - w_2^2 = \Omega^2 \cos^2\theta - \Omega^2 \cos^2\theta \tan h^2 \left(\frac{\Omega \cos\theta}{\lambda} t\right)$$

$$\therefore \quad w_1 = \Omega \cos\theta \sec h \left(\frac{\Omega \cos\theta}{\lambda} t\right)$$

$$= \Omega \sqrt{\frac{A + B}{2B}} \sec h \left(\Omega\, t \sqrt{\frac{A + B}{2B}}\right) \qquad \text{as } \frac{1}{\lambda} \cos\theta = \sqrt{\frac{B - A}{2B}}$$

Hence from (5)

$$w_3 = \frac{1}{\lambda} w_1 = \sqrt{\frac{B-A}{B+A}}\ \Omega \sqrt{\frac{A+B}{2B}} \sec h\left(\Omega t \sqrt{\frac{B-A}{2B}}\right)$$

$$= \Omega \sqrt{\frac{B-A}{2B}} \sec h\left(\Omega t \sqrt{\frac{B-A}{2B}}\right).$$

Example 8 : The principal moments of inertia of a body at the centre of mass are $A, 3A, 6A$. The body is so started that its angular velocities about the axis are $3n, 2n, n$ respectively. If in the subsequent motion under no forces w_1, w_2, w_3 denote the angular velocities about the principal axis at time t, prove that

$$w_1 = 3w_3 = \frac{9n}{\sqrt{5}} \sec hu \qquad \text{and} \qquad w_2 = 3n \tan hu$$

where $u = 3nt + \frac{1}{2}\log 5$.

Solution : Euler's equations of motion under no forces, are

$$A\dot{w}_1 - (B-C)\, w_3\, w_2 = 0$$

$$B\dot{w}_2 - (C-A)\, w_3\, w_1 = 0$$

$$C\dot{w}_3 - (A-B)\, w_1\, w_2 = 0$$

Here $A = A,\ B = 3A,\ C = 6A$

Substituting in the above equations, we have

$$\dot{w}_1 = -3w_2 w_3 \qquad ...(1)$$

$$3\dot{w}_2 = 5w_3 w_1 \qquad ...(2)$$

$$3\dot{w}_3 = -w_1 w_2 \qquad ...(3)$$

Dividing (1) by (3)

$$\frac{\dot{w}_1}{3\dot{w}_3} = \frac{3w_3}{w_1} \qquad \text{or} \qquad w_1\dot{w}_1 = 9w_3\dot{w}_3$$

On integration

$$w_1^2 = 9w_3^2 + a, \qquad a \text{ is a constant}$$

Initially $w_1 = 3n,\ w_3 = n, \qquad \therefore a = 0$

$$\therefore \quad w_1^2 = 9w_3^2 \qquad \text{or} \qquad w_1 = 3w_3 \qquad ...(4)$$

Dividing (2) by (3)

$$\frac{\dot{w}_2}{\dot{w}_3} = -5\frac{w_3}{w_2} \qquad \text{or} \qquad w_2\dot{w}_2 = -5\, w_3\dot{w}_3$$

On integration

$$w_2^2 = -5w_3^2 + b, \qquad b \text{ is a constant}$$

Initially $w_2 = 2n,\ w_3 = n, \qquad \therefore\ b = 9n^2$

Hence $w_2^2 = -5w_3^2 + 9n^2$ ...(5)

Now equation (2) gives

$$3\dot{w}_2 = 5w_3\ w_1$$

$$= 15w_3^2 \qquad \text{by (4)}$$

$$\Rightarrow \qquad \dot{w}_2 = 5w_3^2 = 9n^2 - w_2^2 \qquad \text{by (5)}$$

Integrating, $t = \dfrac{1}{3n}\tan h^{-1}\left(\dfrac{w_2}{3n}\right) + c, \qquad c$ is a constant

Initially $t = 0,\ w_2 = 2n \qquad \therefore C = -\dfrac{1}{3n}\tan\ h^{-1}\dfrac{2}{3}$

$$= -\frac{1}{3n}\frac{1}{2}\log\frac{3+2}{3-2} = -\frac{1}{3n}\log\sqrt{5}$$

Hence $t = \dfrac{1}{3n}\tan\ h^{-1}\dfrac{w_2}{3n} - \dfrac{1}{3n}\log\sqrt{5}$

or $\qquad 3nt + \log\sqrt{5} = \tan h^{-1}\dfrac{w_3}{3n}$

or $\qquad w_2 = 3n\tan\ h\left(3nt + \log\sqrt{5}\right)$

$$= 3n\tan\ h\ u, \qquad \text{where } u = 3nt + \log\sqrt{5}$$

which is required result.

Example 9 : An elliptical lamina acted on by no external forces is set rotating about a line through its centre which makes equal angles with the major and minor axes and with the normal to its plane. Prove that, if e be the acceutricity of the ellipse, and $w_1,\ w_2,\ w_3$ the angular velocities about the major and minor axes and the normal to the plane. then

$$w_1 = F\cos x,\ w_2 = F\sin x$$

$$w_3 = \dot{x} = F\sqrt{\left\{\left(1 - e^2\sin^2 x\right)/\left(2 - e^2\right)\right\}},$$

where F is a constant, equal to $\lambda\sqrt{2}$ where λ is the initial angular velocity along major, minor axis and normal to its plane.

Solution : Here $A = \dfrac{1}{4}Mb^2,\ B = \dfrac{1}{4}Ma^2, C = A + B = \dfrac{1}{\phi}M\left(a^2 + b^2\right)$

Euler's equations of motion under no forces are

$$A\dot{w}_1 - (B - C)\, w_2\, w_3 = 0$$

$$B\dot{w}_2 - (C - A)\, w_3\, w_1 = 0$$

$$C\dot{w}_3 - (A - B)\, w_1\, w_2 = 0$$

Putting the values of $A,\ B,\ C$, we get

$$\dot{w}_1 = -\, w_2\, w_3 \qquad \text{...(1)}$$

$$\dot{w}_2 = w_3\, w_1 \qquad \text{...(2)}$$

$$\dot{w}_3 = \frac{b^2 - a^2}{b^2 + a^2}\, w_1\, w_2 \qquad \text{...(3)}$$

Dividing (1) by (2), we get

$$\frac{\dot{w}_1}{\dot{w}_2} = -\frac{w_2}{w_1} \;\Rightarrow\; w_1\,\dot{w}_1 + w_2\,\dot{w}_2 = 0$$

On integration

$$w_1^2 + w_2^2 = F^2 \;(\text{constant})$$

Let $w_1 = F\cos x,\ \ w_2 = F\sin x$

Initially $w_1 = \lambda,\ w_2 = \lambda,\ w_3 = \lambda \qquad \therefore F = \lambda\sqrt{2}$

Dividing (2) by (3) we have

$$\frac{\dot{w}_2}{\dot{w}_3} = \frac{b^2 + a^2}{b^2 - a^2}\,\frac{w_3}{w_2}$$

or $\quad w_3\,\dot{w}_3 = \dfrac{b^2 - a^2}{b^2 + a^2}\, w_2\,\dot{w}_2$

On integration

$$w_3^2 = \frac{b^2 - a^2}{b^2 + a^2}\, w_2^2 + C$$

Initially $w_2 = \lambda,\ w_3 = \lambda$

$$\therefore \quad C = \lambda^2\left(1 - \frac{b^2 - a^2}{b^2 + a^2}\right)$$

$$= \lambda^2\left(\frac{2\,a^2}{b^2 + a^2}\right)$$

Putting the values of C and w_2

$$w_3^2 = \frac{b^2 - a^2}{b^2 + a^2} (F \sin x)^2 + F^2 \frac{a^2}{b^2 + a^2} \qquad \{\because b^2 = a^2 (1 - e^2)\}$$

$$= \frac{-a^2 e^2}{a^2 (2 - e^2)} F^2 \sin^2 x + F^2 \frac{a^2}{a^2 (2 - e^2)}$$

$$= F^2 \frac{1 - e^2 \sin^2 x}{(2 - e^2)}$$

$$\therefore \quad w_3 = F \sqrt{\{(1 - e^2 \sin^2 x) / (2 - e^2)\}}$$

5.5 Motion of Symmetrical Bodies under no forces

A solid of revolution whose principal moments at the centre of inertia are A, A, C $(C > A)$ is set spinning with angular velocity w about an axis passing through the centre of inertia and making an angle i with the axis of figure. Prove that the instantaneous axis of rotation describes in the body a right cone of semivertical angle i in the period $\frac{2 \pi A \sec i}{(C - A) w}$ and that the axis of figure discribes a right cone in space of semi vertical angle θ where $\tan\theta = \frac{A}{C} \tan i$, in the period $\frac{2\pi}{\Omega}$, where $\Omega \sin\theta = w \sin i$ or $\Omega = \frac{w}{A} \sqrt{A^2 \sin^2 \hat{i} + C^2 \cos^2 \hat{i}}$.

Here $B = A$ and $L = M = N = 0$

Initial conditions are $w_3 = w \cos i$, $w_1 = w \sin i$, $w_2 = 0$ Euler's equations under no forces are

$$A \dot{w}_1 - (B - C) w_2 w_3 = 0 \quad \text{or} \quad \dot{w}_1 = -\frac{C - A}{A} w_2 w_3 \qquad ...(1)$$

$$B \dot{w}_2 - (C - A) w_3 w_1 = 0 \quad \text{or} \quad \dot{w}_2 = \frac{C - A}{A} w_3 w_1 \qquad ...(2)$$

$$C \dot{w}_3 - (A - B) w_1 w_2 = 0 \quad \text{or} \quad \dot{w}_3 = 0 \qquad ...(3)$$

From (3) $w_3 = \text{constant} = w \cos i$ (Initially)

Dividing (1) by (2)

$$\frac{\dot{w}_1}{\dot{w}_2} = -\frac{w_2}{w_1} \quad \text{or} \quad w_1 \dot{w}_1 + w_2 \dot{w}_2 = 0$$

Integrating $w_1^2 + w_2^2 = a$ (constant)

Initially $w_1 = w \sin i, \quad w_2 = 0, \qquad \therefore a = w^2 \sin^2 i$

$$\therefore \quad w_1^2 + w_2^2 = w^2 \sin^2 i \qquad ...(4)$$

$$\therefore \quad w_1^2 + w_2^2 + w_3^2 = w^2 \sin^2 i + w^2 \cos^2 i = w^2$$

which shows that the angular velocity is w throughout the motion.

Now (1) $\Rightarrow$ $\ddot{w}_1 = -\dfrac{C - A}{A} w \cos i \; \dot{w}_2$ as $w_3 = w \cos i$

$$= -\left(\frac{C - A}{A}\right)^2 w^2 \cos^2 i \; w_1 \qquad \text{using (2)} \qquad ...(5)$$

The direction cosines of instantaneous axis are proportional to w_1, w_2, w_3; therefore the angle with the instantaneous axis makes with the axis of solid is

$$\cos^{-1} \frac{w_3}{\sqrt{w_1^2 + w_2^2 + w_3^2}} = \cos^{-1}\left(\frac{w \cos i}{\sqrt{w^2}}\right) = \cos^{-1}(\cos i) = i$$

which shows that instantaneous axis makes a constant angle i with the axis of figure OC. Hence we can say that it describes a right circular cone of semi vertical angle i, the period is seen from (5) as

$$\frac{2\pi}{\sqrt{\left(\dfrac{C - A}{A} w \cos i\right)^2}} = \frac{2\pi A \sec i}{w (C - A)}$$

The direction cosines of invariable line (line fixed in space) are proportional to $A w_1, B w_2, C w_3$. Hence the angle θ which the axis of figure makes with the invariable line is given by

$$\cos \theta = \frac{C w_3}{\sqrt{A w_1^2 + B w_2^2 + C w_3^2}}$$

$$= \frac{C w_3}{\sqrt{A^2 \left(w_1^2 + w_2^2\right) + C w_3^2}} \qquad \text{as } B = A$$

$$= \frac{C w \cos i}{\sqrt{\left(A^2 w^2 \sin^2 i + C^2 w^2 \cos^2 i\right)}}$$

$$= \frac{C}{\sqrt{A^2 \tan^2 i + C^2}}$$

or $\tan \theta = \dfrac{A}{C} \tan i$, thus θ is constant which shows that the axis of figure OC describes about invariable lien (in space) a right circular cone of semivertical angle θ, where $\tan \theta = \dfrac{A}{C} \tan i$

Since OC describes a cone of semivertical angle θ, about the invariable line with angular velocity Ω, therefore

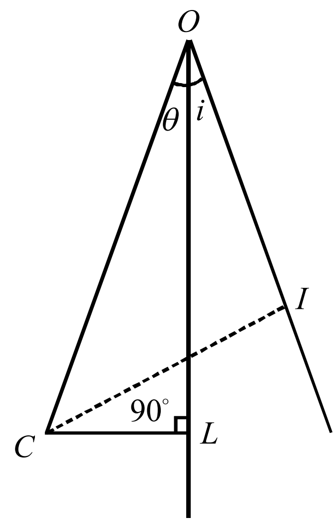


Figure 3

velocity of $C = \Omega \,.\, CK$

where K is the foot of perpendicular from C on invariable line.

Also, since throughout the motion angular velocity of the body is w about the instantaneous axis, therefore velocity of $C = w \,.\, CI$

where I is the foot of the perpendicular from C on the instantaneous axis.

Both these values of velocity of C must be identical, therefore

$$\Omega \,.\, CK = w.\, CI$$

$$\Rightarrow \qquad \Omega \,.\, OC \sin\theta = w.\, OC \sin i$$

$$\Rightarrow \qquad \Omega = w \frac{\sin i}{\sin \theta}$$

$$= \frac{w}{A}\sqrt{A^2 \sin^2 i + C^2 \cos^2 i} \qquad \text{as } \tan\theta = \frac{A}{C} \tan i$$

5.6 Illustrative Examples

Example 10 : A uniform thin circular disc is set rotating with an angular velocity w about an axis through the centre making an angle i with the normal. Prove that the semi vertical angle θ of the cone described by the axis of disc is given by

$$\tan\theta = \frac{1}{2} \tan i$$

If w be angular velocity, prove that above cone is discribed in the period $\dfrac{2\pi}{w\sqrt{1 + 3\cos^2 i}}$.

Solution : Here normal to the disc is the axis of figure. Initial conditions are $w_3 = w\cos i$, $w_1 = w\sin i$, $w_2 = 0$ since $B = A$, $C = A + B$

Euler's equation of motion under no forces are

$$A\dot{w}_1 - (B - C)\, w_2\, w_3 = 0 \quad \Rightarrow \quad \dot{w}_1 = -\frac{(C - A)}{A}\, w_2\, w_3 \qquad ...(1)$$

$$B\dot{w}_2 - (C - A)\, w_3\, w_1 = 0 \quad \Rightarrow \quad \dot{w}_2 = \frac{(C - A)}{A}\, w_3\, w_1 \qquad ...(2)$$

$$C\dot{w}_3 - (A - B)\, w_1\, w_2 = 0 \quad \Rightarrow \quad \dot{w}_3 = 0 \qquad ...(3)$$

Integrating (3), we get $w_3 = \text{constant} = w\cos i$ ...(4)

Dividing (1) by (2), we have

$$\frac{\dot{w}_1}{\dot{w}_2} = -\frac{w_2}{w_1} \qquad \text{or} \qquad w_1\,\dot{w}_1 + w_2\,\dot{w}_2 = 0$$

On integration $w_1^2 + w_2^2 = a$ (constant)

Initially $w_1 = w\sin i,\ w_2 = 0 \qquad \therefore\ a = w^2 \sin^2 i$

$$\therefore\ w_1^2 + w_2^2 = w^2 \sin^2 i \qquad ...(5)$$

The direction cosines of invariable line are proportional to $A\,w_1,\ A\,w_2,\ C\,w_3$ and if θ be the angle between the axis of figure and the invariable line, then

$$\cos\theta = \frac{C\,w_3}{\sqrt{A^2\left(w_1^2 + w_2^2\right) + C^2 w_3^2}}$$

$$= \frac{C}{\sqrt{A^2 \tan^2 i + C^2}} \qquad \text{using (4) and (5)}$$

or $\tan\theta = \dfrac{A}{C}\tan i$

Here $A = \dfrac{1}{4}Ma^2,\quad C = \dfrac{1}{2}Ma^2$

$\therefore \quad \tan\theta = \dfrac{1}{2}\tan i$

This proves the required result.

If Ω be the angular velocity of the axis of disc about the invariable line, than we have from art 5.5

$$\Omega = \frac{w}{A}\sqrt{\left(A^2 \sin^2 i + C^2 \cos^2 i\right)}$$

$$= w\sqrt{\left(\sin^2 i + \frac{C^2}{A^2}\cos^2 i\right)}$$

$$= w\sqrt{\sin^2 i + 4\cos^2 i}$$

$$= w\sqrt{1 + 3\cos^2 i}$$

The period $= \frac{2\pi}{\Omega}$

$$= \frac{2\pi}{w\sqrt{1 + 3\cos^2 i}}$$

Example 11 : If the earth be regarded as a solid of revolution, whose principal moments of inertia at its centre of gravity are A, A, C. Show that its axis of rotation describes a cone of very small angle about the axis of the figure in period $\frac{A}{C - A}$ siderial days.

Solution : Let the earth be set rotating with angular velocity w about an axis, passing through the centre of gravity and making an angle i with the axis of figure. Thus initially $w_3 = w \cos i$,

$w_1 = w \sin i$, $w_2 = 0$

Euler's equations under no forces are

$$A\dot{w}_1 = (B - C)\, w_2\, w_3 \Rightarrow \dot{w}_1 = -\left(\frac{C - A}{A}\right) w_3\, w_2 \qquad ...(1)$$

$$B\dot{w}_2 = (C - A)\, w_3\, w_1 \Rightarrow \dot{w}_2 = \left(\frac{C - A}{A}\right) w_3\, w_1 \qquad ...(2)$$

$$C\dot{w}_3 = (A - B)\, w_1\, w_2 \Rightarrow \dot{w}_3 = 0 \qquad ...(3)$$

from (3) $\qquad w_3 = \text{constant} = w\cos i \qquad ...(4)$

Dividing (1) by (2) $\frac{\dot{w}_1}{\dot{w}_2} = -\frac{w_2}{w_1}$ $\qquad$ or $\qquad w_1\dot{w}_1 + w_2\dot{w}_2 = 0$

on integration $\quad w_1^2 + w_2^2 = C \text{ (constant)}$

Initially $w_1 = w \sin i, \quad w_2 = 0 \qquad \therefore C = w^2 \sin^2 i$

$\therefore \qquad w_1^2 + w_2^2 = w^2 \sin^2 i \qquad ...(5)$

Differentiating (1), we get

$$\ddot{w}_1 = -\left(\frac{C - A}{A}\right) w \cos i\, \dot{w}_2$$

$$= -\left(\frac{C - A}{A}\right) w^2 \cos^2 i\; w_1$$

The direction cosines of instantaneous axis are proportional to w_1, w_2, w_3

$\therefore$ Instantaneous axis makes with the axis of figure an angle

$$= \cos^{-1}\left(\frac{w_3}{\sqrt{w_1^2 + w_2^2 + w_3^2}}\right) = \cos^{-1}\frac{w\cos i}{\sqrt{w^2\left(\cos^2 i + \sin^2 i\right)}} = \cos^{-1}(\cos i) = i$$

Hence instantaneous axis describes about the axis of figure a right cone of semivertical angle i in the period

$$= \frac{2\pi}{w\left(\frac{C-A}{A}\right)\cos i} = \frac{2\pi A \sec i}{w(C-A)}$$

$$= \frac{2\pi A}{w(C-A)} \qquad \text{(since } i \text{ is small so } \sec i = 1\text{)}$$

$$= \frac{A}{C-A} \quad \text{siderial days.}$$

as one siderial day $= \frac{2\pi}{w}$.

5.7 Motion under impulsive forces

Suppose a rigid body is moving about a fixed point O. At O let OA, OB, OC be the principal axes which are fixed in the body but move along with the body.

Let the rigid body be given an impulse such that the moments of the impulsive force about OA, OB, OC are respectively L, M, N. Suppose that w_1, w_2, w_3 and w_1', w_2', w_3' are the components of angular velocities just before and just after the application of impulsive force. Hence angular momentum about OA just before and just after the impulse are $A w_1$ and $A w_1'$.

Similarly angular momentum just before and just after the impulse are

$B w_2, B w_2'$ about OB

and $C w_3, C w_3'$ about OC

Since change in angular momentum about any line is equal to the moment of the impulse about the same line, therefore taking moments about OA, OB, OC, we have respectively

$$A(w_1' - w_1) = L$$

$$B(w_2' - w_2) = M$$

$$C(w_3' - w_3) = N$$

These are Euler's equations of motion for impulsive forces.

5.8 Illustrative Examples

Example 12 : Show that for a body of revolution the maximum value of the angle between the axis of the impulsive couple acting on it and the instantaneous axis of initial motion set up by the couple in the body is $\sin^{-1}\left(\frac{C-A}{C+A}\right)$.

Solution : Let L, M, N be the components of impulsive couple about the axes of moments of inertia (A, A, C), equations of the motion are

$$A(w_1 - 0) = L$$

$$B(w_2 - 0) = M$$

$$C\left(w_3 - 0\right) = N$$

These equation show that the axis of couple is same as the invariable line, the angle between the instantaneous axis and the invariable line is $(i - \theta)$. For max $(i - \theta)$

$$\frac{d}{dt}(i - \theta) = 0 \qquad \Rightarrow \frac{di}{dt} = \frac{d\theta}{dt} \qquad ...(1)$$

Now $\tan\theta = \dfrac{A}{C}\tan i$

$$\therefore \quad \sec^2\theta\,\frac{d\theta}{dt} = \frac{A}{C}\sec^2 i\,\frac{di}{dt}$$

$$\Rightarrow \quad \sec^2\theta = \frac{A}{C}\sec^2 i \qquad \text{Using (1)}$$

$$\Rightarrow \quad 1 + \tan^2\theta = \frac{A}{C}\left(1 + \tan^2 i\right)$$

$$\Rightarrow \quad 1 + \frac{A^2}{C^2}\tan^2 i = \frac{A}{C}\left(1 + \tan^2 i\right)$$

$$\therefore \quad \tan^2 i = \frac{C}{A} \qquad \therefore\ \tan^2\theta = \frac{A}{C}$$

$$\therefore \quad \tan i = \sqrt{\frac{C}{A}},\ \tan\theta = \sqrt{\frac{A}{C}}$$

Hence $\sin(i - \theta) = \sin i\cos\theta - \cos i\sin\theta = \dfrac{C - A}{C + A}$

$$\therefore \quad (i - \theta) = \sin^{-1}\left(\frac{C - A}{C + A}\right)$$

which is required angle.

Example 13 : A disc, in the form of a portion of parabola bounded by its latus rectum and its axis, has its vertex A fixed, and is stuck by a blow through the end of its latus rectum perpendicular to its plane. Show that the disc starts revolving about a line through A inclined at an angle $\tan^{-1}\left(\dfrac{14}{25}\right)$ to the axis.

Solution : Taking parabolic portion in ZX plane. equation of parabola is

$$z^2 = 4ax$$

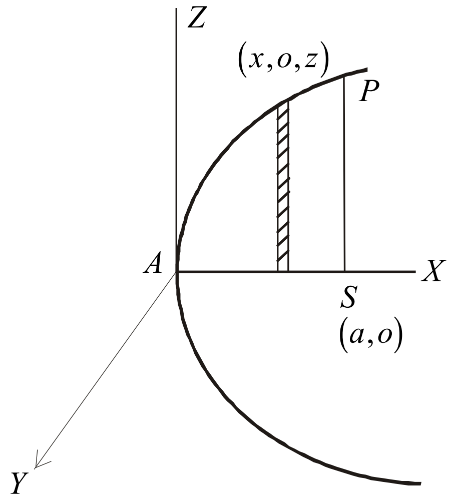


Figure 5.4

Let M.I. of portion about AX, AY, AZ are A, B, C respectively, then

$$A = \int_0^a z\,dx\,\rho \,.\, \frac{1}{3} z^2 = \frac{16}{15} a^4 \rho$$

$$C = \int_0^a \rho z\,dx \,.\, x^2 = \frac{4}{7} a^4 \rho$$

$$B = A + C = \frac{172}{105} \rho\, a^4$$

If D, E, F be products of inertia about these axes then $D = O = F$,

$$E = \int_0^a \rho\, z\,dx \,.\, x\, \frac{1}{2} z = \frac{2}{3} a^4 \rho$$

Using Euler's equations for impulsive forces

$$\frac{16}{15} \rho\, a^4 w'_x - \frac{2}{3} a^4 \rho w'_z = -P \,.\, 2a \qquad ...(1)$$

$$\frac{172}{105} \rho\, a^4 w'_y = 0 \qquad ...(2)$$

$$\frac{4}{7} \rho\, a^4 w'_z - \frac{2}{3} \rho\, a^4 w'_x = P \,.\, a \qquad ...(3)$$

Eliminating P between (1) and (3)

$$25\, w'_z = 14\, w'_x$$

If axis of rotation make an angle ϕ with the axis of parabola then

$$\tan \phi = \frac{w'_z}{w'_x} = \frac{14}{25}$$

$$\therefore \qquad \phi = \tan^{-1}\left(\frac{14}{25}\right)$$

5.9 Self Evaluation Questions

1. Define invariable line.
2. What is the locus of Invariable line?
3. Write down the equations of motion for impulsive forces.

5.10 Summary

This unit is devoted to the study of motion of rigid body when there is no force acting on the body. It has been shown that under no forces situation the Kinetic Energy and Angular Momentum remain constant. In these conditions the body will rotate about an axis, which does not change its position in space, this axis is known as invariable line. It has also been shown that the instantaneous axis describes a cone about axis if figure V (in the body) and the axis of figure describes a cone about invariable line in space. So many examples are given on the conditions. In the last Euler's equations for impulsive forces have been derived and some solved examples are given on these.

5.11 Exercise

1. A solid cube is in motion about an angular point which is fixed. If there are no external forces and w_1, w_2, w_3 are the angular velocities about the edges through the fixed point, prove that $w_1 + w_2 + w_3$ and $w_1^2 + w_2^2 + w_3^2$ are each constant.

2. A uniform circular disc free to turn about its centre which is fixed, is set rotating with angular velocity w about an axis which makes an angle 45° with the axis of disc. Prove that in the subsequent motion the axis of the disc describes a right cone about an axis making an angle $\tan^{-1}\frac{1}{2}$ with the initial axis of rotation with constant angular velocity $\frac{1}{2}w^2\sqrt{10}$.

3. A right circular cone whose altitude h is double of the radius of the base, can turn about its centre of gravity as a fixed point and is set rotating about an axis inclined at an angle i to the axis of figure. Prove that the vertex of the cone will describe a circle of radius $\frac{3h}{4}\sin i$.

4. A uniform right ciruclar cone of vertical angle 2α moves under no force except at its vertex which is fixed. It is set rotating about a generater. Show that its axis describes in space a right cone of angle 2β where

$$\tan\beta = \frac{1}{2}\tan\alpha + 2\cot\alpha$$

□□□□

UNIT - 6
Conservation of Momentum and Energy

Structure of the unit

6.0 Objective

The linear momentum of a particle is defined as the product of its mass m and velocity v i.e. mv. The rigid body is considered as the collection of particles, so the linear momentum of rigid body is given by the vector sum of the linear momentum of the particles of the body moving in parallel straight lines with equal velocity v and therefore the linear momentum of body is equal to the total mass of body and its velocity v.

Generally, the motion of body is not always translatory, there arise some situations in which motion of body is rotational also. Thus, we use another term called Angular Momentum which relates to the rotational motion. The energy of a body is defined as its capacity of doing work. In dynamics we consider only mechanical energy. It is of two types, the kinetic and potential. The Kinetic Energy of a body is by virtue of its motion and measured by the amount of work which it does in coming to rest. The potential energy of a body is the work it can do in moving from its actual position to some standard position.

Our objective in this unit is to study the principales of conservation of momentum and energy.

6.1 Introduction

In mechanics the mass and energy are considered as two distinct physical quantities and there are separate conservation laws. In this unit we are interested in the formulation of laws in which linear as well as angular momentum of a body is considered in cases when the forces are finite as well as impulsive. The law of conservation of energy has also been studied in this unit.

6.2 Conservation of Momentum under finite forces

In this section we study the principles of conservation of linear momentum and angular momentum of rigid body under the action of finite forces.

6.2.1 Principle of Conservation of Linear Momentum :

"If a rigid body is moving under the action of some external forces whose sum of resolved parts parallel to a line is zero, throughout the motion, the momentum of the body parallel to that line remains constant throughout the motion."

Suppose a rigid body is moving under the action of some external forces and (x, y, z) be coordinates of its any particle, of mass m, at any time t refered to fixed axes. Let X, Y, Z be the resolved parts, parallel to the axes, of external forces acting on the particle. By D'Alembert Principle, the general equations of motion of rigid body are

$$\sum m\ddot{x} = \sum X \qquad ...(1)$$

$$\sum m\ddot{y} = \sum Y \qquad ...(2)$$

$$\sum m\ddot{z} = \sum Z \qquad ...(3)$$

Let X-axis be the fixed striaght line, and it is so choosen that sum of resolved parts of the external forces parallel to it is zero throughout the motion, i.e. $\sum X = 0$.

Hence from (1) $\sum m\ddot{x} = 0$ ...(4)

We know that

$$\bar{x} = \frac{\sum mx}{\sum m}, \sum m = M, \bar{x} \text{ is the } x- \text{ coordinates of centre of gravity of body.}$$

Then from (4)

$$M\ddot{\bar{x}} = 0 \qquad \Rightarrow \quad M\dot{\bar{x}} = \text{constant} \qquad ...(5)$$

Eqn (5) states that total linear momentum of the rigid body parallel to x-axis remains constant throughout the motion.

Hence it is shown that if a line is such that sum of resolved parts of the forces parallel to it is zero throughout the motion, the total linear momentum parallel to it remains unchanged during the motion.

6.2.2 Principle of Conservation of Angular Momentum :

"If a rigid body is moving under the action of external forces the sum of whose moments about a given line is zero throughout the motion, the moment of momentum (or angular momentum) of the body about that line remains unchanged during the motion."

Let X, Y, Z be resolved parts of the external forces parallel to the axes acting on the particle of mass m, whose coordinates are (x, y, z) at time t.

Taking the given line as x-axis, then sum of the moments of external forces about the given line $(x - axis)$ is $\sum (yZ - zY)$

By D'Alembert principle, we have

$$\sum m(y\ddot{z} - z\ddot{y}) = \sum (yZ - zY) \qquad ...(6)$$

If $\quad \sum m(yZ - zY) = 0$, then (6) gives

$$\frac{d}{dt}\sum m(y\dot{z} - z\dot{y}) = 0$$

$$\Rightarrow \quad \sum m(y\dot{z} - z\dot{y}) = \text{constant} \qquad ...(7)$$

Now $\sum m(y\dot{z} - z\dot{y})$ is the angular momentum of the rigid body. Hence eqn (7) shows that the total moment of momentum of the body about x-axis is constant throughout the motion.

Hence it is shown that if a line is such that moment of external forces about it is zero throughout the motion, the angular momentum (or moment of momentum) about it remains unchanged during the motion.

6.2.3 Illustrative Examples :

Example 1 : A small insect moves along a uniform bar, of mass equal to itself and of length $2a$, the ends of which are constrained to remain on the circumference of a fixed circle whose radius is $\frac{2a}{\sqrt{3}}$, If the insect starts from the middle point of the bar and move along the bar with relative velocity V, show that the bar in time t will turn through angular

$$\frac{1}{\sqrt{3}}\tan^{-1}\frac{Vt}{a}.$$

Solution : Let θ be the angle through which the bar AB turns in time t Let P be the position of the insect after time t after starting from C.

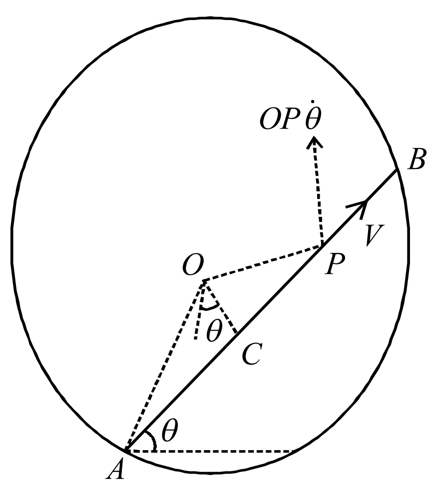


Figure 6.1

$$\therefore\ CP = Vt$$

and $$OC = \sqrt{OA^2 - AC^2} = \sqrt{\left[\left(\frac{2a}{\sqrt{3}}\right)^2 - a^2\right]} = \frac{a}{\sqrt{3}}$$

$$\therefore \quad OP = \sqrt{\left(OC^2 + CP^2\right)}$$

$$= \sqrt{\left(\frac{1}{3}a^2 + V^2t^2\right)}$$

Since the moment of external forces about vertical axis through O vanishes throughout the motion, therefore the moment of momentum about this line remains unchanged

$$\therefore \quad m\left(OC\dot{\theta}\,.\,OC + \frac{1}{3}a^2\dot{\theta}\right) + m\left(V\,.\,OC + OP\dot{\theta}\,.\,OP\right) = 0$$

$$\Rightarrow \quad \frac{2}{3}a^2\dot{\theta} + V\,.\,\frac{a}{\sqrt{3}} + \left(\frac{1}{3}a^2 + V^2t^2\right)\dot{\theta} = 0$$

$$\Rightarrow \quad \dot{\theta} = \frac{d\theta}{dt} = -\frac{Va}{\sqrt{3}\left(a^2 + V^2t^2\right)}$$

Hence the angle turned by the bar in time t is given by

$$\theta = -\frac{1}{\sqrt{3}}\int_0^t \frac{Va}{a^2 + V^2t^2} = -\frac{1}{\sqrt{3}}\tan^{-1}\frac{Vt}{a}$$

negative sign signifies that the rod turns in a dirrection opposite to the motion of insect.

Example 2 : A uniform straight rod, of length $2a$, has two small rings at its ends which can respectively slide on thin smooth horizontal and vertical wires OX and OY. The rod starts at an angle α to the horizon with angular velocity $\sqrt{\{3g\,(1 - \sin\alpha)/2a\}}$ and moves down wards. Show that it will strike the horizontal wire at the end of time $2\sqrt{\left(a/3g\right)}\ \log\left\{\cot\left(\frac{\pi}{8} - \frac{\alpha}{4}\right)\tan\frac{\pi}{8}\right\}$.

Solution : Suppose G be the centre of gravity of rod $AB\ (= 2a)$ and θ the angle made by it with the horizontal at any time t. The coordinates of $C.G.\,G$ of the rod are $(a\cos\theta,\ a\sin\theta)$ as shown. Therefore the velocity of the $C.G. = \sqrt{\left\{\left(-a\sin\theta\,\dot{\theta}\right)^2 + \left(a\cos\theta\,\dot{\theta}\right)^2\right\}} = a\dot{\theta}$

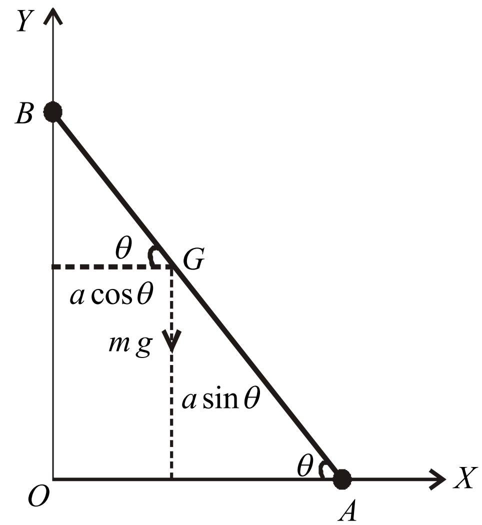


Figure 6.2

Hence the Kinetic Energy of rod at any time t

$$= \frac{1}{2}\, m \left\{ \frac{1}{3}\, a^2\, \dot{\theta}^2 + a^2\, \dot{\theta}^2 \right\}$$

$$= \frac{2}{3}\, m\, a^2\, \dot{\theta}^2 \qquad ...(1)$$

Initial $K.E. = \frac{2}{3}\, m\, a^2 \cdot \frac{3g}{2a}\, (1 - \sin\alpha) = m\, g\, a\, (1 - \sin\alpha)$ ...(2)

and Initial angular velocity

$$\dot{\theta} = \sqrt{\left\{ \frac{3g}{2a}\, (1 - \sin\alpha) \right\}} \qquad \text{(given)}$$

Now the work energy equation gives

$$\frac{2}{3}\, m\, a^2\, \dot{\theta}^2 - m\, g\, a\, (1 - \sin\alpha) = m\, g\, a\, \sin\alpha - m\, g\, (a \sin\theta)$$

or $\dot{\theta}^2 = \frac{3g}{2a}\, (1 - \sin\theta)$

or $\frac{d\theta}{dt} = -\sqrt{\frac{3g}{2a}}\, \sqrt{(1 - \sin\theta)}$ ($-ive$ sign shows that θ decreases with time t) ...(3)

Therefore, the required time from $\theta = \alpha$ to $\theta = 0$ is

$$t = -\sqrt{\left(\frac{2a}{3g} \right)} \int_{\alpha}^{0} \frac{d\theta}{\sqrt{1 - \sin\theta}}$$

$$= \sqrt{\left(\frac{2a}{3g} \right)} \int_{0}^{\alpha} \frac{d\theta}{\sqrt{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}}$$

$$= \sqrt{\left(\frac{a}{3g}\right)} \int_0^{\alpha} \operatorname{cosec}\left(\frac{\pi}{4} - \frac{\theta}{2}\right) d\theta$$

$$= -2\sqrt{\left(\frac{a}{3g}\right)} \left\{\log \tan\left(\frac{\pi}{8} - \frac{\theta}{4}\right)\right\}_0^{\alpha}$$

$$= 2\sqrt{\left(\frac{a}{3g}\right)} \log \left\{\frac{\tan\frac{\pi}{8}}{\tan\left(\frac{\pi}{8} - \frac{\alpha}{4}\right)}\right\}_0^{\alpha}$$

$$\therefore \qquad t = 2\sqrt{\left(\frac{a}{3g}\right)} \log\left\{\cot\left(\frac{\pi}{8} - \frac{\alpha}{4}\right)\tan\frac{\pi}{8}\right\}.$$

Example 3 : A uniform circular board, of mass M and radius a, is placed on a perfectly smooth horizontal plane and free to rotate about a vertical axis through its centre; a man of mass M', walks round the edge of the board whose upper surface is rough enough to privent his slipping : when he has walked completely round the board to his starting point, show that board has turned through an angle

$$\frac{M'}{M + 2M'} 4\pi.$$

Solution : Let the board has turned through an angle θ at time t and radius to the man makes an angle ϕ with the fixed line OA_0.

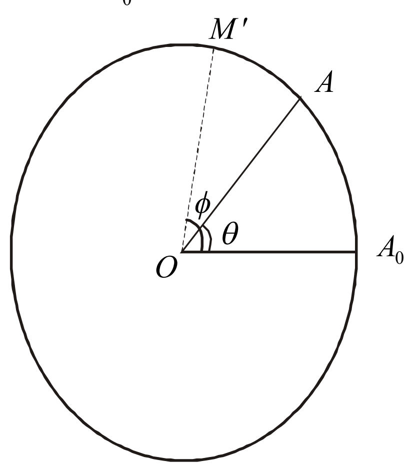


Figure 6.3

The forces acting on the system are all vertical; therefore there moments about the vertical through O are zero throughout the motion; hence moment of momentum about this axis through O remains unaltered throughout the motion.

i.e. $$M\frac{a^2}{2}\dot{\theta} + M'a^2\dot{\phi} = 0$$

integrating $\frac{1}{2}M\theta + M'\phi = 0$ ...(1)

[The constnt of integration vanishes becuase intially θ and ϕ are zero]

When the man returnes to his starting point A,

$$\therefore \quad \phi - \theta = 2\pi \qquad \text{...(2)}$$

Eliminating ϕ between (1) and (2), we have

$$\frac{1}{2} M\theta + M' (2\pi + \theta) = 0$$

or $$\theta = -\frac{M'}{(M + 2M')} 4\pi$$

The negative sign shows that the board turns in a direction opposite to that of the man.

Example 4 : A particle of mass m within a rough circular tube, of mass M lying on a horizontal plane and initially the tube is at rest while particle has an angular vleocity round the tube. Show that by the time relative motion ceases the fraction $\frac{M}{M + 2m}$ of the initial kinetic energy has been dissipated by friction.

Solution : Let G be the common $C.G.$ of M and m, then

$$\frac{x}{m} = \frac{a - x}{M} = \frac{a}{M + m} \qquad \text{...(1)}$$

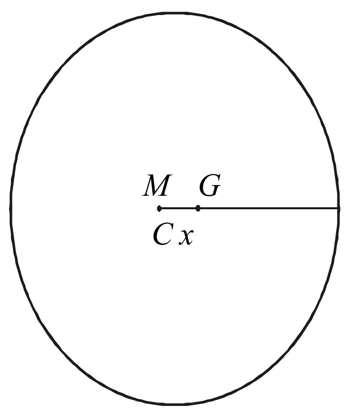


Figure 6.4

Let u be the velocity of the particle P when motion started, the tube being at rest.

Since there is no force on the system in the horizontal plane throughout the motion, therefore horizontal momentum remains unaltered throughout the motion.

i.e. $(M + m)\, v = mu$ or $v = \frac{mu}{M + m}$ ...(2)

This determines the motion of common centre of gravity G.

When the system begins to move, let w and w' be the angular velocities of the tube and the particle about G.

The forces acting on the system are all vertical; therefore their moments vanish about the vertical axis through G throughout the motion, hence moment of momentum about G remains unaltered throughout the motion

i.e. $M\left(a^2 w + x^2 w\right) + m\,(a - x)^2\, w' = m u\,(a - x)$

The relative motion ceases when $w' = w$ and then

$$w = \frac{mu\,(a - x)}{M\left(a^2 + x^2\right) + m\,(a - x)^2} = \frac{mu\,(a - x)}{M\left\{(a - x)^2 + 2ax\right\} + m\,(a - x)^2}$$

$$= \frac{mu}{M\left\{(a - x) + \dfrac{2ax}{a - x}\right\} + m\,(a - x)}$$

$$= \frac{mu}{(M + m)\,(a - x) + 2aM\,\dfrac{x}{a - x}}$$

From (1) we have

$$w = \frac{mu}{aM + 2aM} = \frac{mu}{a\,(M + 2m)} \qquad ...(3)$$

At this time kinetic energy of the system

$$= \frac{1}{2}\,(M + m)\,v^2 + \frac{1}{2}\,M\left(a^2 + x^2\right) w^2 + \frac{1}{2}\,m\,(a - x)^2\, w^2$$

now putting the values of v^2 and w from (2) and (3)

$$K.E. = \frac{1}{2}\,\frac{m^2 u^2}{M + m}\left[1 + \frac{M}{M + m}\right] = \frac{1}{2}\,m^2 u^2\,\frac{2}{M + 2m}$$

$$= \frac{1}{2}\,mu^2\left[1 - \frac{M}{M + 2m}\right] = \frac{1}{2}\,mu^2 - \frac{1}{2}\,mu^2\,\frac{M}{M + 2m}$$

$$\therefore \text{ loss of } K.E. = \frac{1}{2}\,mu^2\,\frac{M}{M + 2m}$$

$$= \frac{M}{M + 2m} \quad \text{of intial kinetic energy.}$$

Example 5 : A uniform square plate $ABCD$ of mass M and side $2a$, lies on a smooth horizontal plane; it is struck at A by a particle of mass m moving with velocity V in the direction of AB, the particle remaining attached to the plate. Determine the subsiquent motion of the system and show that its angular velocity is

$$\frac{m}{M + 4m}\cdot\frac{3V}{2a}$$

Solution : Let O be the centre of square and G the centre of gravity of the system (particle and the square), then

$$\frac{OG}{m} = \frac{AG}{M} = \frac{AO}{M+m} = \frac{a\sqrt{2}}{M+m} \qquad ...(1)$$

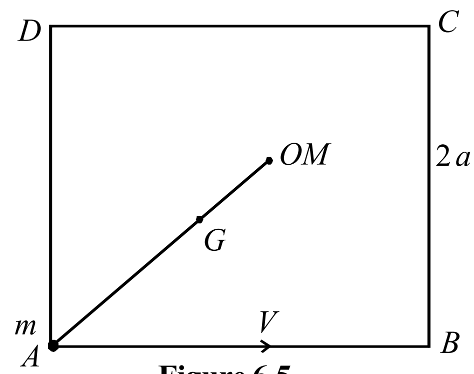


Figure 6.5

Let V^1 be the velocity of the system after impact, then by principle of linear momentum we have

$$(M+m)\,V^1 = mV \quad \text{or} \quad V^1 = \frac{mV}{M+m}$$

This gives velocity of G after impact.

After impact let w be the angular velocity about G. Since moments of foces (weights and reaction) about vertical through G is zero, therefore angular momentum about it remains unchanged, hence

$$M\left(\frac{2a^2}{3} + OG^2\right)w + M\,AG\,w\,.\,AG = m\left(V\sin 45^\circ\right).\,AG$$

or $$M\,\frac{2a^2}{3}\,w + M(OG)^2\,w + m\left(AG^2\right)w = \frac{mV}{\sqrt{2}}\,AG$$

or $$M\,.\,\frac{2a^2}{3}\,w + M\frac{2a^2m^2}{(M+m)^2}\,w + m\,\frac{2a^2m^2}{(M+m)^2} = \frac{mV}{\sqrt{2}}\,\frac{a\sqrt{2}\,M}{M+m} \qquad \text{from (1)}$$

or $$M\,.\,\frac{2a^2}{3}\,w + 2a^2\,Mm\,\frac{(m+M)}{(m+M)^2}\,w = \frac{mMVa}{(m+M)}$$

or $$M\,.\,\frac{2a^2}{3}\,w + \frac{2a^2\,Mmw}{M+m} = \frac{mMVa}{M+m}$$

or $$2a^2\left(\frac{1}{3} + \frac{m}{M+m}\right)w = \frac{mVa}{M+m}$$

or $$\frac{2a}{3}\,(M+4\,m)\,w = mV$$

or $$w = \frac{m}{M + 4\,m} \cdot \frac{3\,V}{2\,a}$$

this gives angular velocity after impact.

6.3 Conservation of Momentum Under Impulsive Forces

This section is denoted to the study of motion of rigid bodies under the action of impulsive forces. Here principles of conservation of linear momentum and angular momentum have been discussed.

6.3.1 Principle of Conservation of Linear Momentum :

"If the sum of impulses of the forces parallel to a certain fixed line vanishes, the momentum in that directions remains the same just before and after the applications of impulses."

Let the fixed line be taken as x-axis then by D'Alembert's principle, the equation of motion is

$$\sum m \frac{d^2x}{dt^2} = \sum X$$

or $$\frac{d}{dt}\left(\sum m \frac{dx}{dt}\right) = \sum X$$

Now when the forces are impulsive and duration of the impluse be a small time τ, then integrating the last equation, we have

$$\left[\sum m \frac{dx}{dt}\right]_0^{\tau} = \int_0^{\tau} \sum X\, dt = \sum X'$$

where X' be the impulse of the forces parallel to the x-axis. If u, v are the velocities of the particle before and after the impulse, then the equation reduces to

$$\sum m\,(u - v) = \sum X'$$

The equation shows, that the change in total momentum parallel to the x-axis is equal to the sum of the impulses in that directions. If $\sum X' = 0$ then $\sum m\,(u - v) = 0 \Rightarrow \sum m u = \sum m v$.

i.e. total momentum after impulse = total momentum before impulse.

Hence it is shown that if the sum of impulses parallel to a line is zero, the total momentum in that direction remains unchanged.

6.3.2 Principle of Conservation of Angular Momentum :

"If the sum of the moments of the impulses about a certain line vanishes, the angular momentum about that line remains the same before and after the application of the impulses."

Let the fixed line be supposed as x-axis. The moment equation of motion using D'Alembert's principle

is $$\sum m\left(y \frac{d^2z}{dt^2} - z \frac{d^2y}{dt^2}\right) = \sum (yZ - zY)$$

or $$\sum m \frac{d}{dt}\left(y \frac{dz}{dt} - z \frac{dy}{dt}\right) = \sum (yZ - zY)$$

Now, when the forces are impulses and duration of impulse be a small time τ, then integrating the last equation, we have

$$\left[\sum m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right)\right]_0^\tau = \sum \int_0^\tau (yZ - zY)\, dt$$

$$= \sum (yZ' - zY')$$

i.e. the angular momentum after impulse - angular momentum before impulse = sum of the moments of the impulses.

If the sum of the moments of impulses about x-axis is zero, then

angular momentum after impulse = angular momentum before impulse.

Hence, it has been shown that if the sum of moments of impulses about a line is zero, the angular momentum or moment of momentum about it remains unchanged.

6.2.3 Illustrative Examples :

Example 6 : A uniform square lamina, of mass M and side $2a$, is moving freely about a diagonal with uniform angular velocity w when one of the corners not in the diagonal becomes fixed, show that the new angular velocity is $\frac{1}{7}w$ and that the impulse of the force on the fixed point is $\frac{\sqrt{2}}{7}Maw$.

Solution : It is given that the square lamina $ABCD$ of mass M is rotating about a diagonal BD with angular velocity w. Let initial direction of rotation be such that C was moving upward from the paper. Since the diagonal BD is set free and a corner A is suddenly fixed, then the square begins to turn about a line AL parallel to BD, through A. Let w^1 be the angular velocity of the square about AL.

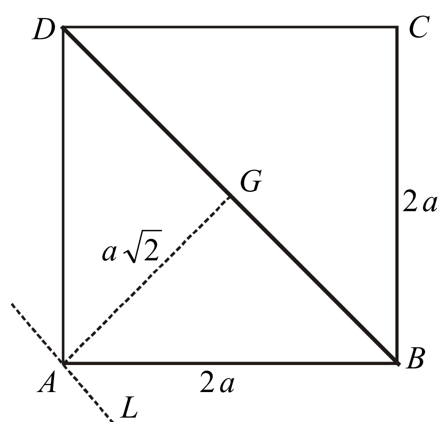


Figure 6.6

Since the impulse is at A, therefore moment of the impulse about AL is zero. Hence the moment of momentum about AL remains same before and after the fixing, therefore,

$$M\left\{k^2 + (AG)^2\right\} w^1 = Mk^2 w$$

$$\left\{\text{Here } k^2 \text{ about } BD = \frac{1}{3}a^2\cos^2\frac{\pi}{4} + \frac{1}{3}a^2\sin^2\frac{\pi}{4} = \frac{1}{3}a^2\right\}$$

or $\quad M\left\{\frac{a^2}{3} + 2a^2\right\} w^1 - M \cdot \frac{a^2}{3} w$

$\Rightarrow \quad w^1 = \frac{w}{7}$

After fixing the velocity of centre of gravity of the square right angle to AG is $a\sqrt{2}\, w^1$. Hence, the impulse in given by

Impulse = change of momentum

$$= M a \sqrt{2}\, w^1 - 0$$

$$= \frac{\sqrt{2}}{7} M a w.$$

Example 7 : A rod of length $2a$, is moving about one end with uniform angular velocity upon a smooth horizontal plane. Suddenly this end is set free and a point, distant b from this end, is fixed; find the motion, considering the cases when $b <, =, > \frac{4a}{3}$.

Solution : Let AB be the rod of length $2a$ and mass m and G be its centre of gravity. Suppose the rod is moving with uniform angular velocity w about the end A. Let w^1 be angular velocity of rod when the end A is set free and a point O of rod is suddenly fixed, when $AO = b$. As the impulses is at O, then the moment of impulse about O will be zero, therefore the moment of momentum about the axis through O before and after the fixing i.e. impulse remains unchanged. Hence

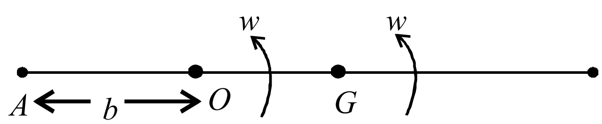


Figure 6.7

$$m\left\{\frac{1}{3}a^2 + (a-b)^2\right\} w^1 = m\left\{\frac{1}{3}a^2 + a(a-b)\right\} w$$

or $\quad w^1 = \dfrac{3a\left(\frac{4}{3}a - b\right)}{a^2 + 3(a-b)^2} w.$

If (i) $b < \frac{4a}{3}$, then w^1 is positive, which shows that the rod will rotate in the same direction after the impulse as before.

(ii) $\quad b = \frac{4a}{3}$, $w^1 = 0$, which shows that the rod will reduce to rest after the impulse and

(iii) $\quad b > \frac{4a}{3}$, then w^1 is negative, which depicts that the rod will rotate in opposite such after the impulse.

Example 8 : A circular plate is turning in its own plane about a point A on its circumference. Suddenly A is freed and point B, also on the circumference, fixed show that the plate will be reduced to rest if the arc AB is one third of the cirumference.

Solution : A is set free and B is suddenly fixed. Let w^1 be the angular velocity just after fixture.

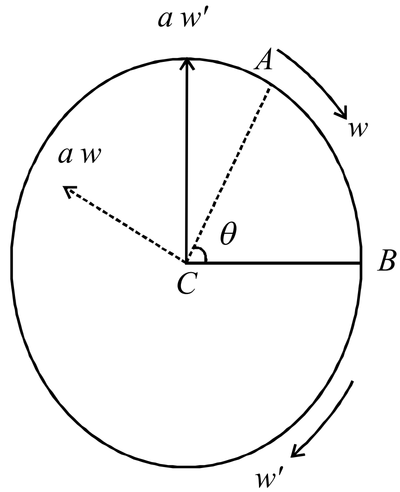


Figure 6.8

Let C be the centre and $ACB = \theta$. On account of sudden fixture at B the impulse is at B, therfore moment of impulse about B is zero; hence moment of momentum about B remains the same before and after fixture.

i.e. $$m\left[\frac{a^2}{2}\, w^1 + a^2\, w^1\right] = m\left[\frac{a^2}{2}\, w + a\,w\,.\,a\,\cos\theta\right]$$

i.e. $$w^1 = \frac{1}{3}(1 + 2\cos\theta)\,w$$

The plate will be reduced to rest of $w^1 = 0$

i.e. if $1 + 2\cos\theta = 0$ or $\cos\theta = -\frac{1}{2}$

i.e. $\theta = \frac{2\pi}{3}$ $\Rightarrow$ $a\theta = \frac{2\pi a}{3}$

$\Rightarrow$ Arc AB is one-third of the circumference.

Example 9 : A uniform cube is spinning freely with angular velocity w about a diagonal through a corner O, when suddenly the diagonal through O of one of the faces through O becomes fixed. Show that the new angular velocity is $\frac{2\sqrt{2}}{5\sqrt{3}}\, w$.

Solution : Let $2a$ be the side of cube; therefore length of its diagonal

$$= \sqrt{(2a)^2 + (2a)^2 + (2a)^2} = 2a\sqrt{3}$$

and length of a diagonal of a face $= \sqrt{4a^2 + 4a^2} = 2\sqrt{2}\,a$

So inclination of the diagonal of the cube to the diagonal of the face

$$= \cos^{-1} \frac{2\sqrt{2}\,a}{2\sqrt{3}\,a} = \cos^{-1}\left(\sqrt{\frac{2}{3}}\right)$$

Before sudden fixture cube was revolving about its diagonal with angular velocity w^1, resolved parts of this angular velocity about the diagonal of the face $= w\sqrt{\frac{2}{3}}$

The distance of the centre of gravity of the cube from the diagonal of the face is equal to a. For the cube about any axis through its centre,

$$k^2 = \frac{2a^2}{3}$$

Due to the radius fixing of the diagonal of the face, the impulse is at this diagonal, so moment of the impulse about this diagonal is zero; hence moment of momentum about this diagonal of the face must remain the same before and after the fixture.

i.e. $$m\left(\frac{2a^2}{3}w^1 + a^2 w^1\right) = m\left(\frac{2a^2}{3} w \sqrt{\frac{2}{3}}\right)$$

i.e. $$5w^1 = 2\sqrt{\frac{2}{3}}\,w \quad \text{or} \quad w^1 = \frac{2}{5}\sqrt{\frac{2}{3}}\,w$$

Example 10 : Three equal uniform rods placed in a straight line are freely joined and move with velocity v perpendicular their lengths. If the middle point of the middle rod be suddenly fixed, show that the ends of the two rods will meet in time $\frac{4\pi a}{9v}$, where a is the length of each rod.

Solution : Let AB, BC, CD be three rods each of length a and mass m. Consider the motion of AB impulsive force acts at B; therefore moments of impulsive force about B vanishes. Hence the moment of momentum of AB about B before and after fixture remains unaltered.

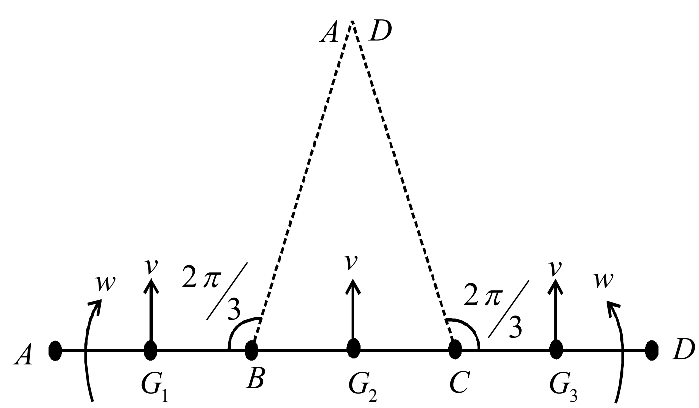


Figure 6.9

If w be angular velocity of AB just after the fixture we have $m \,.\, v \,.\, \frac{1}{2} a = m \frac{1}{3} a^2 w$

This gives $w = \frac{3v}{2a}$

Similarly for the motion of CD, we get its angular velocity also equal to $\frac{3v}{2a}$.

Now when the ends of rods will meet, that is when A and D will meet, then ABC and DBC will be equilateral triangle, that is when AB and DC each has turned through an angle $\frac{2\pi}{3}$ as shown in figure.

$$\therefore \qquad \text{Required time} = \frac{2\pi/3}{3v/2a} = = \frac{4\pi a}{9v}.$$

6.4 Conservation of Energy

6.4.1 Conservative Forces :

The forces acting in a system are called conservative when work done by the forces *viz* $\int (X\,dx + Y\,dy + Z\,dz)$ is independent of the path followed from initial to the final position of the body and depends only on the cofiguration, of the body at times t_1 and t_2.

If the forces acting on the system are conservatives, then $\int (X\,dx + Y\,dy + Z\,dz)$ is complete differential of some quantity V, then forces are said to be conservative

$$\int (X\,dx + Y\,dy + Z\,dz) = dV$$

$$\therefore \qquad \sum \int (X\,dx + Y\,dy + Z\,dz) = \int dV$$

or in other words the forces have the potential V. If A and B are configurations of the system at times t_1 and t_2, then we have

$$\int_A^B dV = V_B - V_A$$

6.4.2 Principle of Conservation of Energy :

"If a system moves under the action of finite forces and if the geometric relations of the system do not contain time explicitly, the change in the Kinetic Energy of the system in passing from one configuration to another is equal to the corresponding work done by the forces."

Let X, Y, Z be the components of external forces parallel to the axes acting on the particle of mass m whose coordinates are (x, y, z) at time t.

We have by D'Alembert's principle, the forces $X - m\frac{d^2x}{dt^2}, Y - m\frac{d^2y}{dt^2}, Z - m\frac{d^2z}{dt^2}$ acting on the particle and similar force acting on the other particles of the system, form a system of forces in equilibrium.

Therefore by principle of virtual work.

$$\sum \left[\left(X - m\frac{d^2x}{dt^2}\right)\delta x + \left(Y - m\frac{d^2y}{dt^2}\right)\delta y + \left(Z - m\frac{d^2z}{dt^2}\right)\delta z\right] = 0$$

where δx, δy, δz are very small arbitrary displacements of the particlem consistent with the geometrical conditions at time t.

If the geometrical relations do not contain time explicitly, then the geometrical relations which hold at time t will hold throughout the time δt, therefore we can take the arbitrary displacements $\frac{dx}{dt}\delta t$, $\frac{dy}{dt}\delta t, \frac{dz}{dt}\delta t$ of the particle in time δt,

Sustituting these in the last equations, we have

$$\sum m\left[\frac{d^2x}{dt^2}\frac{dx}{dt}+\frac{d^2y}{dt^2}\frac{dy}{dt}+\frac{d^2z}{dt^2}\frac{dz}{dt}\right]=\sum\left(X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt}\right)$$

Integrating with respect to t

$$\sum\frac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]_{t_1}^{t_2}=\sum\int(X\,dx+Y\,dy+Z\,dz)$$

i.e. the changes in the kinetic energy of the system from time t_1 to t_2 is equal to the work done by the external forces on the body from one configuration of the body at time t_1 to the configuration at time t_2.

6.4.3 Theorem : When a body moves under the action of a system of conservative forces, the sum of its Kinetic and Potential energies is constant throughout the motion.

Proof : We know that if A and B are configurations of the body at times t_1 and t_2, then

K.E. at time t_2 $-$ $K.E.$ at time $t_1 = \int_A^B dV = V_B - V_A$ ...(1)

The potential energy of the body in any position A is the work, which the forces do in moving the body from this position A to a standard position, say C.

$\therefore$ P.E. at time $t_1 = \int_A^C (X\,dx + Y\,dy + Z\,dz) = \int_A^C dV = V_C - V_A$ ...(2)

P.E. at time $t_2 = \int_B^C (X\,dx + Y\,dy + Z\,dz) = \int_B^C dV = V_C - V_B$ ...(3)

subtracting (3) from (2), we have

P.E. at time t_1 - P.E. at time $t_2 = (V_C - V_A) - (V_C - V_B) = V_B - V_A$ ...(4)

comparing (1) and (4), we get

K.E. at time t_2 - K.E. at time t_1 = P.E. at time t_1 - P.E. at time t_2 $\Rightarrow$ K.E. at time t_2 + P.E. at time t_2 = K.E. at time t_1 + P.E. at time t_1

$\therefore$ sum of K.E & P.E time t_2 = sum of K.E. & P.E at time t_1

Hence sum of K.E. & P.E. of a body is same at all times.

6.4.4 Theorem : The K.E. of rigid body, moving in any manner is at any instant equal to the Kinetic Energy of the whole mass, supposed to be collected at its centere of inertia and moving with it together with the kinetic energy of the whole mass relative to its centre of inertia.

Proof : Let (x, y, z) be the coordinates of any elementary particle of mass m in the body at time t. $(\bar{x}, \bar{y}, \bar{z})$ be the coordinate of centre of gravity G of the body. Let (x^1, y^1, z^1) be coordinates of the particle referred to $C.G.$ as origin, therefore we have

$$x = \bar{x} + x^1, \; y = \bar{y} + y^1, \; z = \bar{z} + z^1$$

Since $\sum m x' = 0, \quad \sum m y' = 0, \quad \sum m z' = 0$

$$\Rightarrow \quad \sum m \dot{x}' = 0, \quad \sum m \dot{y}' = 0, \quad \sum m \dot{z}' = 0 \qquad \text{...(1)}$$

Now, total K.E. of the body

$$= \frac{1}{2} \sum m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$= \frac{1}{2} \sum m \left[\left(\dot{\bar{x}} + \dot{x}'\right)^2 + \left(\dot{\bar{y}} + \dot{y}'\right)^2 + \left(\dot{\bar{z}} + \dot{z}'\right)^2\right]$$

$$= \frac{1}{2} \sum m \left[\left(\dot{\bar{x}}^2 + \dot{\bar{y}}^2 + \dot{\bar{z}}^2\right) + \frac{1}{2} \sum m \left(\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2\right)\right.$$

$$+ \dot{\bar{x}} \sum m \dot{x}' + \dot{\bar{y}} \sum m \dot{y}' + \dot{\bar{z}} \sum m \dot{z}'$$

$$= \frac{1}{2} \sum m \left(\dot{\bar{x}}^2 + \dot{\bar{y}}^2 + \dot{\bar{z}}^2\right) + \frac{1}{2} \sum m \left(\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2\right) \qquad \text{from (1)}$$

$= \frac{1}{2}$ (square of velocity $C.G.$) $\sum m \; + \frac{1}{2} \sum m$ (square of velocity of $m \, w.r$ to G)

$= \frac{1}{2} M v^2$ + K.E. of body relative to G

where v is velocity of $C.G.$

Hence

The total K.E. of the body

= K.E. of mass M supposed to be collected at $C.G.$ of body

+ K.E. of body relative to G.

6.4.5 Illustrative Exmaples :

Example 11 : A uniform rod, of length $2a$, is placed with one end in contact with a smooth horizontal table and is then allowed to fall, if α be the initial inclination to the vertical, show that its angular velocity when it is inclined at angle θ is $\left\{\frac{6g}{a} \frac{\cos\alpha - \cos\theta}{1 + 3\sin^2\theta}\right\}^{1/2}$. Find also the reaction of the table.

Solution : Let AB be the rod of length $2a$ and mass m. The end A is in contact with the table. Since there is no horizontal force on the rod, therefore its $C.G.$ moves downword in a vertical line, through G, let this vertical through G cut horizontal plane at D, then D is a fixed point.

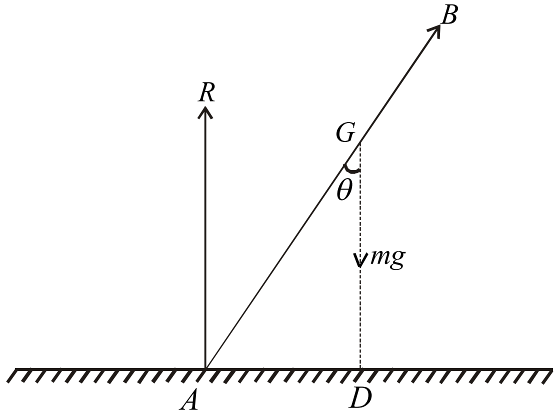


Figure 6.10

Let the horizontal and the vertical lines through D are coordinate axes, then coordinates of G are $(0,\ a\cos\theta)$

We have from energy equation

$$\frac{1}{2}\,m\left\{\frac{a^2}{3} + a^2\sin^2\theta\,\dot{\theta}^2\right\} = mg\left(a\cos\alpha - a\cos\theta\right)$$

$$\Rightarrow \qquad \dot{\theta}^2 = \frac{6g}{a}\cdot\frac{\cos\alpha - \cos\theta}{1 + 3\sin^2\theta} \qquad (1)$$

$$\Rightarrow \qquad \dot{\theta} = \left[\frac{6g}{a}\cdot\frac{\cos\alpha - \cos\theta}{1 + 3\sin^2\theta}\right]^{1/2}$$

it proves first result.

Differenting (1) and dividing by $2\dot{\theta}$, we get

$$\ddot{\theta} = \frac{3g}{a}\,\frac{\sin\theta\left(1 + 3\sin^2\theta\right) - 6\sin\theta\cos\theta\,(\cos\alpha - \cos\theta)}{\left(1 + 3\sin^2\theta\right)^2}$$

$$= \frac{3g\sin\theta\left[4 + 3\left(\cos^2\theta - 2\cos\alpha\cos\theta\right)\right]}{a\left(1 + 3\sin^2\theta\right)^2}$$

$$= \frac{3g\sin\theta\left[1 + 3\sin^2\alpha + 3\,(\cos\theta - \cos\alpha)^2\right]}{a\left(1 + 3\sin^2\theta\right)^2} \qquad ..(2)$$

Taking moments of all forces about G, we have

$$m\,\frac{a^2}{3}\,\ddot{\theta} = Ra\sin\theta \qquad ...(3)$$

Using (2) and (3) we get

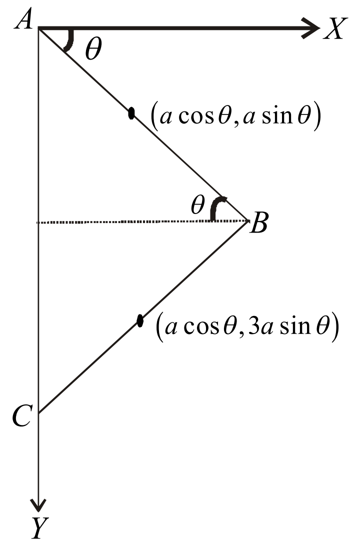
$$R = mg\,\frac{1 + 3\sin^2\alpha + 3(\cos\theta - \cos\alpha)^2}{(1 + 3\sin^2\theta)}$$

This gives reaction of the table.

Example 12 : Two link rods AB and BC, each of length $2a$ are freely joined at B, AB can turn round the rod A and C can move freely on a vertical straight line through A. Initially the rods are held in a horizontal line, C being in coincidence with A and they are then released. Show that when the rods are inclined at an angle θ to the horizontal, the angular velocity of either is

$$\sqrt{\left(\frac{3g}{a}\,\frac{\sin\theta}{1 + 3\cos^2\theta}\right)}.$$

Solution : It is given that $2a$ be the length and m be the mass of each rod AB and BC, which are freely joined at B. Taking fixed point A as origin and horizontal and vertical through A as axes shown in the figure. The coordinates of the centre of gravity of the rod AB are $(a\cos\theta, a\sin\theta)$ and the rod BC are $(a\cos\theta, 3a\sin\theta)$



Figue 6.11

Hence the work energy equation gives

$$\frac{1}{2}m\left(\frac{a^2}{3}\dot{\theta}^2 + a^2\dot{\theta}^2\right) + \frac{1}{2}m\left\{\frac{a^2}{3}\dot{\theta}^2 + \left(-\sin\theta\dot{\theta}\right)^2 + \left(3a\cos\theta\dot{\theta}\right)^2\right\}$$

$$= mga\sin\theta + mg\,.\,3a\sin\theta$$

or $\quad a\dot{\theta}^2\left(1 + 3\cos^2\theta\right) = 3g\sin\theta$

or $\quad \dot{\theta} = \sqrt{\left(\frac{3g}{a}\,\frac{\sin\theta}{1 + 3\cos^2\theta}\right)}$

It is the required angular velocity.

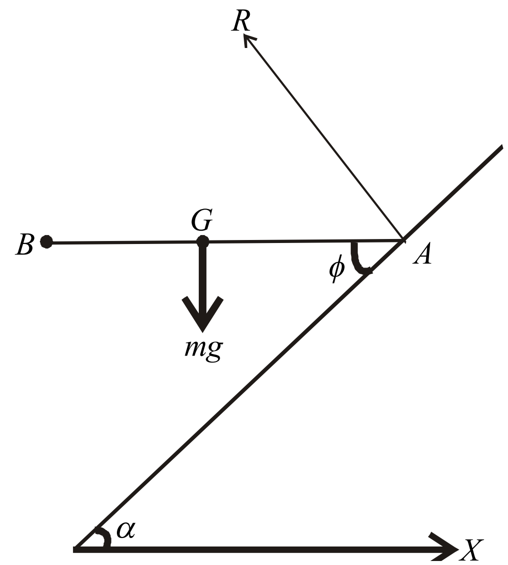
Example 13 : A straight uniform rod of mass m, is placed at right angles to a smooth plane of inclination α with one end in contact with it, the rod is then released. Show that when its inclination to the plane is ϕ, the reaction of the plane will be

$$mg\,\frac{3\,(1-\sin\phi)^2+1}{\left(3\cos^2\phi+1\right)^2}\cos\alpha$$

Solution : We have equation of motion of rod

$$R - mg\cos\alpha = m\frac{d^2}{dt^2}(a\sin\phi) = ma\left(\cos\phi\,\ddot{\phi} - \sin\phi\,\dot{\phi}^2\right) \qquad ...(1)$$

and $$m\frac{a^2}{3}\ddot{\phi} = -Ra\cos\phi \qquad ...(2)$$



Figue 6.12

Eliminating R between (1) & (2), we get

$$m\frac{a^2}{3}\ddot{\phi} = -a\cos\phi\left[mg\cos\alpha + ma\cos\phi\ddot{\phi} - ma\sin\phi\dot{\phi}^2\right]$$

i.e. $$a\left(\frac{1}{3}+\cos^2\phi\right)\ddot{\phi} - a\sin\phi\cos\phi\dot{\phi}^2 = -g\cos\alpha\cos\phi \qquad ...(3)$$

Integrating $a\left(\frac{1}{3}+\cos^2\phi\right)\dot{\phi}^2 = -2g\cos\alpha\sin\phi + C$

when $\phi = \frac{\pi}{2}, \dot{\phi} = 0 \qquad \therefore C = 2g\cos\alpha$

Hence $a\left(\frac{1}{3}+\cos^2\phi\right)\dot{\phi}^2 = 2g\cos\alpha\,(1-\sin\phi)$

or $$a\dot{\phi}^2 = \frac{6g\cos\alpha\,(1-\sin\phi)}{1+3\cos^2\phi} \qquad ...(4)$$

Now from (2), (3) and (4), we have

$$R = m g \cos\alpha \; \frac{3(1-\sin\phi)^2 + 1}{(1+3\cos^2\phi)^2}, \text{ which is positive.}$$

Hence lower end of the rod will never leave the plane.

Example 14 : A sphere of radius b, rolls without slipping down the cycloid. $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$. It starts from rest with its centre on the horizontal line $y = 2a$. Show that the velocity V of its centre, when at its lowest point is given by

$$V^2 = \frac{10g}{7}(2a - b).$$

Solution : Let C be the centre of the sphere of radius b and the radius CB is fixed. Initially, the point B was in contact with A and let at time t the radius CB makes angle ϕ with the vertical. The point of contact is P at which the tangent makes an angle ψ with the horizontal and length $OP = c - s$ length $AP = c$

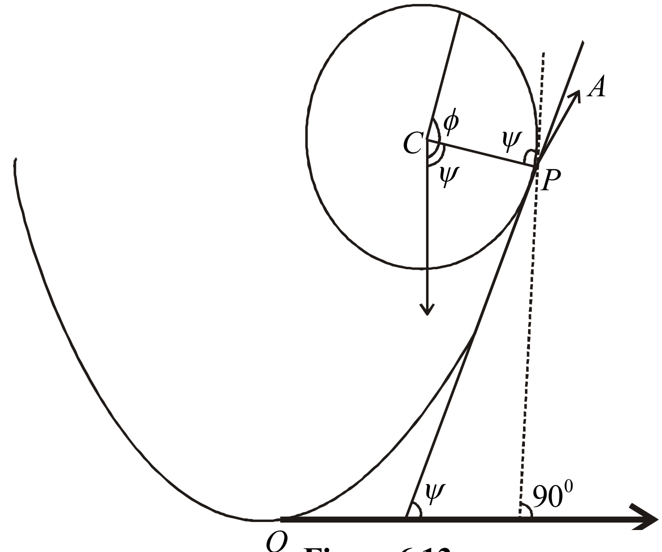


Figure 6.13

Since there is no sliding between the cycloid and the sphere, therefore

length $AP =$ arc BP

i.e. $\quad c - s = b(\phi - \psi)$ ...(1)

Let V be the velocity of centre C

$\therefore \quad V =$ velocity of point P + velocity of C relative to P

$= \dot{s} - b\,\dot{\psi}$

$= -b(\dot{\phi} - \dot{\psi}) - b\dot{\psi}$ [from (1)]

$= -b\,\dot{\phi}$

i.e. $\quad V = -b\,\dot{\phi}$ ...(2)

Now energy equation gives

$$\frac{1}{2} m \left[\frac{2b^2 \dot{\phi}^2}{5} + V^2 \right] = m g \left[2a - b \right]$$

i.e. $$\frac{1}{2} m \left[\frac{2}{5} V^2 + V^2 \right] = m g \left[2a - b \right] \qquad \text{[from (2)]}$$

i.e. $$\frac{7V^2}{5} = 2g\,(2a - b)$$

or $$V^2 = \frac{10g}{7} (2a - b).$$

6.5 Self Evaluation Questions

1. Write Principle of Consevation of linear momentum under finite forces.
2. State the Principle of Conservation of angular momentum under finite force.
3. What is the Principle of work and energy?
4. What do you mean by Conservation forces?

6.6 Summary

The unit is devoted to the study of Conservation of momentum and energy in case of finite forces and also for impulsive forces. Principle of Conservation of linear and angular momentum for finite forces resepectively have been discussed. Also the cases of impulsive forces are considered. At the end principle of conservation of energy in described.

6.7 Exercise

1. A circular disc of radius a lies on a smooth horizontal table when a point A on the cirucmference is compelled to move in the direction of the tengent at the point with velocity u. Show that the disc begins to the turn with angular velocity $\left(\frac{2u}{3a}\right)$.

2. A circular plate rotates about an axis through its centre perpendicular to its plane with angular velocity w. This axis set free and a point on the cirucmference of the plate fixed. Show that the resulting angular velocity is $\frac{1}{3} w$.

3. An equilateral triangle, formed of uniform rods freely hinged at their ends is falling freely with one side horizontal and uppermost. If the middle of this side be suddenly stopped, Show that the impulsive actions of the upper and lower hinges are in the ratio $\sqrt{13} : 1$.

4. A rectangular lamina, whose sides are of length $2a$ and $2b$ is at rest when one corner is caught and suddenly made to move with prescribed speed V in the plane of the lamina. Show that the greatest angular velocity which can be imparted to the lamina is $\frac{3V}{4\sqrt{a^2 + b^2}}$.

5. A circular disc is moving with angular velocity Ω about an axis through its centre perpendicular to its plane. An insect alights on its edge and crawls along a curve drawn on the disc in the form of the laminscate with uniform relative angular velocity $\frac{1}{4}\Omega$, the curve touching the edge of the disc. The mass of the insect being $\frac{1}{16}$th of that of the disc, Show that the angle turned through by the disc when the insects gets to the centre is $\left(\frac{24}{\sqrt{7}}\right)\tan^{-1}\left(\frac{\sqrt{7}}{3}\right)-\frac{\pi}{4}$.

6. If the earth, supposed to be uniform sphere, had in a certain period contracted slightly, so that its radius was less $\left(\frac{1}{n}\right)$th then before, show that the length of the day would have shortened by $\frac{48}{n}$ hours.

7. An elleptic lamina is rotating about its centre on a smooth horizontal table. If w_1, w_2, w_3 be its angular velocities when the extremity of major axis, its focus and the extremity of minor axis, respectively becomes fixed, prove that $\frac{7}{w_1}=\frac{6}{w_2}+\frac{5}{w_3}$.

8. An elliptic area, of eccentricity e, is rotating with angular velocity w about one latus rectum: suddenly this latus rectum is loosed and the other fixed. Show that the new angular velocity is $\frac{1-4e^2}{1+4e^2}w$.

9. A cube is rotating with angular velocity w about a diagonal when suddenly the diagonal is let go and one of the edges which does not meet the diagonal is fixed, show that the resulting angular velocity about this edges is $\frac{1}{12}\sqrt{3}\,w$.

10. A uniform cricular disc of radius a is rolling without slipping along a smooth horizontal plane with velocity V when the highest point becomes suddenly fixed. Prove that the disc will make a com plete revolution round the point if $V^2 > 24\,ag$.

□□□□

UNIT - 7
Generalised Co-ordinates

Structure of the unit

7.0 Objective

In this unit we will learn about the generalised coordintes degrees of freedom. Also we will understand about solving of dynamical problems (i) with the help of generalised coordinates components of a given dynamical system and (ii) Kinetic Energy. We will also learn about Lagrange equations of Holonomic and non - Holonomic systems. Also we will learn about Lagrange's equation for motion due to impulsive forces.

7.1 Introduction

In order to attack a class of dynamical problems for conservative system there were generally two approaches which were considered viz (i) D'Alemberts equations of motion, and (ii) principle of conservation of energy However, since impulsive forces are non-conservative, these methods present a good amount of difficulty to the extent that the second approach is not applicable.

For such problems, Lagrange's equations are found to be useful for solving all dynamical problems of conservative or non-conservative system. This unit deals with these equation in solving certain such problems through examples.

7.2 Generalised Co-ordinates

All those independent variables which determine the position of dynamical system or a material system on a body at any given instant are called its **generalised coordinates**. These generalised coordinates may be distances, angles or other quantities relating to them.

7.2.1 Degrees of Freedom

The number of independent motions which a dynamical system can have (admits) are called its **degrees of freedom**. Also, the number of independent motion is the same as the number of generalised coordinates. Hence the number of degrees of freedom of the system is equal to the number of generalised coordinates.

The degree of freedom of a particle moving in space is three, because three co-ordinates, say $x, y\ and\ z$ are required to specify its position in space. A free rigid body possesses six degrees of freedom, because its position is determined by the three coordiantes x, y, z of an assigned point of the body together with three angle $i.e.$ the Eulerian angles $\theta,\ \phi,\ \psi$, which determine its orientation. The degrees of freedom of a system containing n particles moving in space is $3n$ as it requires $3n$ coordinates to specify its position.

7.2.2 Holonomous System

Let $\theta,\ \phi,\ \psi, \ldots.$ be the generalised coordinates of a system, then the cartesian coordinates (x, y, z) of any point of it at any time t can be expressed as functions of generalised coordinates and time t as

$$x = x\ (t, \theta, \phi, \psi, \ldots.),$$

$$y = y\ (t, \theta, \phi, \psi, \ldots.)$$

$$z = z\ (t, \theta, \phi, \psi, \ldots.)$$

If these functions do not contain derivatives of generalised coordinates with respect to time $i.e.$ they do not involve quantities like $\dot{\theta},\ \dot{\phi},\ \dot{\psi}, \ldots$ or higher derivatives with respect to t then such a system is called a holonomous or holonomic system. Alternatively the independent variables in terms of which the motion is to be found may be any we please, with the restriction, that the coordinates of every particle of the body can, if required, be expressed in terms of them by means of equations which do not contain any differential coefficients with regard to the time. When the systems admits of such a choice of independent coordinates, it is said to be holonomous.

Note : In holonomous system all generalised coordinates are independent to each other.

7.2.3 Conservative system

If all the forces acting on a system and doing work ar derivable from a potential function (or potential enegy) V (say), then the system is called conservative, otherwise it is non-conservative and the forces are known as conservative forces.

7.2.4 Self learning Exercise

1. The degree of freedom for a single particle moving in space at any time t is

   (a) 2 (b) 3 (c) 4 (d) 6

2. A system in which functions which do not contain derivatives of generalised coordinates with respect to time are called ....

3. Conservative forces are derivable form ....

7.3 Lagrange's Equations

To derive Lagrange's equations of motion in generalised coordinates for a holonomic dynamical system under finite forces.

Let P be a typical particle of mass m of a rigid body at a time t and let (x, y, z) be its

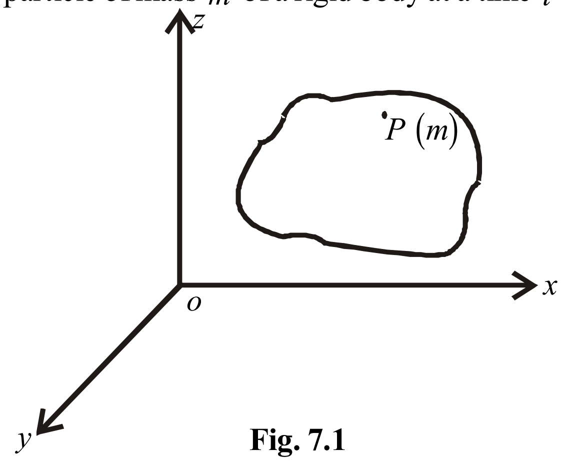


Fig. 7.1

coordinates referred to rectangular axes. Also let θ, ϕ, ψ ,.... be the generalised coordinates of the system. Since the cartesian coordinates (x, y, z) be expressed in terms of generalised coordinates and time t, we have

$$x = x\left(t, \theta, \phi, \psi, \ldots\right),\ y = y\left(t, \theta, \phi, \psi, \ldots\right),\ z = z\left(t, \theta, \phi, \psi, \ldots\right) \qquad \ldots(1)$$

Assume that the system is holonomous, therefore, the equation (1) do not contain $\dot{\theta}, \dot{\phi}, \dot{\psi}, \ldots$ or any other derivative with respect to time. Differentiating (1) with respect to t, we have

$$\frac{dx}{dt} = \dot{x} = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial x}{\partial \phi}\frac{d\phi}{dt} + \frac{\partial x}{\partial \psi}\frac{d\psi}{dt} + \ldots..$$

or $$\dot{x} = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial \theta}\dot{\theta} + \frac{\partial x}{\partial \phi}\dot{\phi} + \frac{\partial x}{\partial \psi}\dot{\psi} + \ldots.. \qquad \ldots(2)$$

with similar expressions for $\dot{y}$ and $\dot{z}$.

On differentiating (2) partially with regard to $\dot{\theta}$, we have

$$\frac{\partial \dot{x}}{\partial \dot{\theta}} = \frac{\partial x}{\partial \theta} \qquad \ldots(3)$$

Similarly $\dfrac{\partial \dot{y}}{\partial \dot{\theta}} = \dfrac{\partial y}{\partial \theta}$, $\dfrac{\partial \dot{z}}{\partial \dot{\theta}} = \dfrac{\partial z}{\partial \theta}$ etc.

Now, differentiating (2) again partially with regard to θ, we have

$$\frac{\partial \dot{x}}{\partial \theta} = \frac{\partial^2 x}{\partial \theta\, \partial t} + \frac{\partial^2 x}{\partial \theta^2}\dot{\theta} + \frac{\partial^2 x}{\partial \theta\, \partial \phi}\dot{\phi} + \frac{\partial^2 x}{\partial \theta\, \partial \psi}\dot{\psi} + \ldots$$

$$= \frac{\partial}{\partial t}\left(\frac{\partial x}{\partial \theta}\right) + \frac{\partial}{\partial \theta}\left(\frac{\partial x}{\partial \theta}\right)\dot{\theta} + \frac{\partial}{\partial \phi}\left(\frac{\partial x}{\partial \theta}\right)\dot{\phi} + \frac{\partial}{\partial \psi}\left(\frac{\partial x}{\partial \theta}\right)\dot{\psi} + \ldots$$

If we compare R.H.S. of (4) with R.H.S. of (2) we observe that (4) can be obtained from (2) if we replace x by $\frac{\partial x}{\partial \theta}$ and hence L.H.S. of (4) takes the form $\frac{d}{dt}\left(\frac{\partial x}{\partial \theta}\right)$ in the L.H.S. of (2)

$$\therefore \qquad \frac{\partial \dot{x}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial x}{\partial \theta}\right) \qquad \ldots(5)$$

similarly $\frac{\partial \dot{y}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial y}{\partial \theta}\right), \frac{\partial \dot{z}}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial z}{\partial \theta}\right)$ etc.

Let T be the kinetic energy of the system, then

$$T = \frac{1}{2}\sum m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \qquad \ldots(6)$$

Differentiating it partially with regard to $\dot{\theta}$ we have

$$\frac{\partial T}{\partial \dot{\theta}} = \sum m\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{\theta}} + \dot{y}\frac{\partial \dot{y}}{\partial \dot{\theta}} + \dot{z}\frac{\partial \dot{z}}{\partial \dot{\theta}} + \ldots\right) \qquad \ldots(7)$$

Again differentiating (6) partially with respect to θ, we have

$$\frac{\partial T}{\partial \theta} = \sum m\left(\dot{x}\frac{\partial \dot{x}}{\partial \theta} + \dot{y}\frac{\partial \dot{y}}{\partial \theta} + \dot{z}\frac{\partial \dot{z}}{\partial \theta} + \ldots\right) \qquad \ldots(8)$$

Now, by D'Alembert's principle (which states that the reversed effective forces acting at each point of the system and the impressed (external) forces form a system of forces in equilibrium) therefore, by giving the systems a small virtual displacement consistent with the geometrical conditions at time t, the total virtual work done by the system is zero, in other words,

the virtual work done by effective forces = the virtual work done by the impressed forces.

Now, the virtual work done by effective forces for a variation of θ alone

$$= \sum m\left(\ddot{x}\frac{\partial x}{\partial \theta} + \ddot{y}\frac{\partial y}{\partial \theta} + \ddot{z}\frac{\partial z}{\partial \theta}\right)\delta\theta \qquad \ldots(9)$$

$$= \frac{d}{dt}\left\{\Sigma\, m\left(\dot{x}\frac{\partial x}{\partial \theta} + \dot{y}\frac{\partial y}{\partial \theta} + \dot{z}\frac{\partial z}{\partial \theta}\right)\delta\theta\right\}$$

$$- \Sigma\, m\left\{\dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial \theta}\right) + \dot{y}\frac{d}{dt}\left(\frac{\partial y}{\partial \theta}\right) + \dot{z}\frac{d}{dt}\left(\frac{\partial z}{\partial \theta}\right)\right\}\delta\theta$$

$$= \frac{d}{dt}\left\{\sum m\left(\dot{x}\,\frac{\partial \dot{x}}{\partial \dot{\theta}} + \dot{y}\,\frac{\partial \dot{y}}{\partial \dot{\theta}} + \dot{z}\,\frac{\partial \dot{z}}{\partial \dot{\theta}}\right)\delta\theta\right\}$$

$$- \sum m\left\{\dot{x}\,\frac{\partial \dot{x}}{\partial \theta} + \dot{y}\,\frac{\partial \dot{y}}{\partial \theta} + \dot{z}\,\frac{\partial \dot{z}}{\partial \theta}\right\}\delta\theta \quad \text{(Using (3) and (5)} \qquad ...(10)$$

Again using (7) and (8), we have

the virtual work done by effective forces for a variation of θ alone

$$= \left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta}\right\}\delta\,\theta \qquad ...(11)$$

Now, virtual work done by impressed forces for a variation of θ alone,

$$= \sum\left(X\,\frac{\partial x}{\partial \theta} + Y\,\frac{\partial y}{\partial \theta} + Z\,\frac{\partial z}{\partial \theta}\right)\delta\theta \qquad ...(12)$$

where X, Y, Z are the impressed forces on the system. Again, if W be the work function of the system, then we have

$$X = \frac{\partial W}{\partial x} \text{ etc.}$$

Therefore, the virtual work done by impressed forces for a variation of θ alone

$$= \sum\left(\frac{\partial W}{\partial x}\,\frac{\partial x}{\partial \theta} + \frac{\partial W}{\partial y}\,\frac{\partial y}{\partial \theta} + \frac{\partial W}{\partial z}\,\frac{\partial z}{\partial \theta}\right)\delta\theta$$

$$= \frac{\partial W}{\partial \theta}\,\delta\theta \qquad ...(13)$$

Equating (11) and (13), we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$$

Similarly, we have the equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi} = \frac{\partial W}{\partial \psi} \qquad ...(14)$$

and so on, there being one equation corresponding to each generalised coordinate of the system.

These equations are the Lagrange's equations in generalised coordinates for finite forces under a holonomous dynamical system.

If V be the potential function and W be the work function, then we know that

$$V + W = \text{constant}$$

$$\Rightarrow W = \text{constant} - V \qquad \text{...(15)}$$

Then the Lagrange's equations (14) become

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = -\frac{\partial V}{\partial \theta}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = -\frac{\partial V}{\partial \phi}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi} = -\frac{\partial V}{\partial \psi} \quad \text{etc.} \qquad \text{...(16)}$$

Equations (16) are the Lagrange's equations in generalised coordinates for finite forces under a holonomous conservative dynamical system.

7.3.1 Lagrange's function

When the forces are conservative and a potential function V exists the Lagrange's θ-equation from (16) is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = -\frac{\partial V}{\partial \theta}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial}{\partial \theta}(T - V) = 0$$

$$\Rightarrow \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{\theta}}(T - V)\right\} - \frac{\partial}{\partial \theta}(T - V) = 0 \qquad (\because V \text{ does not contain } \dot{\theta}, \dot{\phi}, \text{etc.})$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Similarly

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \psi} = 0 \quad \text{etc.} \qquad \text{...(17)}$$

where $L = T - V$, is called the Lagrange's function or Lagrangian function or kinetic potential.

7.4 Principle of Energy

To deduce the principle of energy from the Lagrange's equations.

The principle of energy can be deduced from Lagrange's equations when the geometrical relation do not contain the time explicitly.

If $\theta, \phi, \psi, ...$ are the generalised coordinates of a holonomous conservative dynamical system with potential function V, then the Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = -\frac{\partial V}{\partial \theta},$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = -\frac{\partial V}{\partial \phi},$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi} = -\frac{\partial V}{\partial \psi} \quad \text{etc} \qquad ...(18)$$

If x, y, z do not contain time t explicitly, we have

$$\dot{x} = \frac{\partial x}{\partial \theta}\dot{\theta} + \frac{\partial x}{\partial \phi}\dot{\phi} + \frac{\partial x}{\partial \psi}\dot{\psi} + \underset{G}{...} \quad \text{etc.} \qquad ...(19)$$

so that the kinetic energy T gives

$$T = \frac{1}{2}\sum m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$= \frac{1}{2}\sum m\left\{\left(\frac{\partial x}{\partial \theta}\dot{\theta} + \frac{\partial x}{\partial \phi}\dot{\phi} + \frac{\partial x}{\partial \psi}\dot{\psi} + ...\right)^2 + \left(\frac{\partial y}{\partial \theta}\dot{\theta} + \frac{\partial y}{\partial \phi}\dot{\phi} + \frac{\partial y}{\partial \psi}\dot{\psi} + ...\right)^2\right.$$

$$\left. + \left(\frac{\partial z}{\partial \theta}\dot{\theta} + \frac{\partial z}{\partial \phi}\dot{\phi} + \frac{\partial z}{\partial \psi}\dot{\psi} + ...\right)\right\}$$

$$= A_{11}\dot{\theta}^2 + A_{22}\dot{\phi}^2 + A_{33}\dot{\psi}^2 + ... + 2A_{12}\dot{\theta}\dot{\phi} + 2A_{13}\dot{\theta}\dot{\psi} + 2A_{23}\dot{\phi}\dot{\psi} + ... \qquad ...(20)$$

where the coefficients $A_{11}, A_{22}, A_{33}, ... A_{12}, A_{13}, A_{23}, ...$ are function of θ, ϕ, ψ etc.

Equation or relation (20) shows that, the kinetic energy T is a homogeneous quadratic function of $\dot{\theta}, \dot{\phi}, \dot{\psi}, ...$

Hence by Euler's theorem, we have

$$\dot{\theta}\frac{\partial T}{\partial \dot{\theta}} + \dot{\phi}\frac{\partial T}{\partial \dot{\phi}} + \dot{\psi}\frac{\partial T}{\partial \dot{\psi}} + ... = 2T \qquad ...(21)$$

Also T is a function of θ, ϕ, ψ .... and $\dot{\theta}, \dot{\phi}, \dot{\psi}, ...$ etc, hence

$$\frac{dT}{dt} = \frac{\partial T}{\partial \theta}\dot{\theta} + \frac{\partial T}{\partial \phi}\dot{\phi} + \frac{\partial T}{\partial \psi}\dot{\psi} + ... + \frac{\partial T}{\partial \dot{\theta}}\ddot{\theta} + \frac{\partial T}{\partial \dot{\phi}}\ddot{\phi} + \frac{\partial T}{\partial \dot{\psi}}\ddot{\psi} + ... \qquad ...(22)$$

Now multiplying the Lagrange's equations (18) by $\dot{\theta}, \dot{\phi}, \dot{\psi}, ...$ respectively and then adding, we have

$$\left[\dot{\theta}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) + \dot{\phi}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) + \dot{\psi}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) + ...\right]$$

$$-\left[\dot{\theta}\frac{\partial T}{\partial \theta} + \dot{\phi}\frac{\partial T}{\partial \phi} + \dot{\psi}\frac{\partial T}{\partial \psi} + ...\right] = -\left(\dot{\theta}\frac{\partial V}{\partial \theta} + \dot{\phi}\frac{\partial V}{\partial \phi} + \dot{\psi}\frac{\partial V}{\partial \psi} + ...\right)$$

or $$\left[\frac{d}{dt}\left(\dot{\theta}\frac{\partial T}{\partial \dot{\theta}} + \dot{\phi}\frac{\partial T}{\partial \dot{\phi}} + \dot{\psi}\frac{\partial T}{\partial \dot{\psi}} + ...\right) - \left(\ddot{\theta}\frac{\partial T}{\partial \dot{\theta}} + \ddot{\phi}\frac{\partial T}{\partial \dot{\phi}} + \ddot{\psi}\frac{\partial T}{\partial \dot{\psi}} + ...\right)\right]$$

$$-\left(\dot{\theta}\frac{\partial T}{\partial \theta} + \dot{\phi}\frac{\partial T}{\partial \phi} + \dot{\psi}\frac{\partial T}{\partial \psi} + ...\right) = -\left(\dot{\theta}\frac{\partial V}{\partial \theta} + \dot{\phi}\frac{\partial V}{\partial \phi} + \dot{\psi}\frac{\partial V}{\partial \psi} + ...\right)$$

or $$\frac{d}{dt}(2T) - \frac{dT}{dt} = -\frac{dV}{dt} \qquad \{\text{using (21) and (22)}\}$$

or $$\frac{d}{dt}(T+V) = 0$$

$$\therefore T+V = \text{constant} \qquad ...(23)$$

i.e. the sum of the kinetic and potential enegies of a system of conservative forces is constant throughout the motion.

Illustrative examples :

Example 1 : Use Lagrange's equations to find the equation of motion of a simple pendulum.

Solution : Let l be the length of the simple pendulum, m be the mass and θ the angle made by

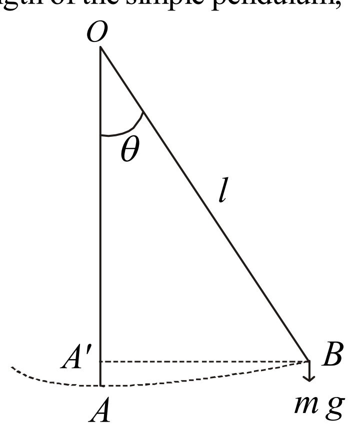


Fig. 7.2

the string OB with the vertical OA after time t. Here θ is only the generalised coordinate. Now the velocity V_B of the bob at B will be equal to $l\dot{\theta}$.

Therefore, if T be the $K.E.$ of the system, then

$$T = \frac{1}{2} m \left(l\dot{\theta}\right)^2 = \frac{m}{2} l^2 \dot{\theta}^2$$

Again, if W be the work function of the system, then

$$W = m g OA' + C = m g . l \cos\theta + C$$

where C, is any constant, which adjusts the initial position of the pendulum (bob).

Now, applying Lagrange's θ - equation, we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$$

or $$\frac{d}{dt}\left(2 . \frac{m}{2} l^2 \dot{\theta}\right) - 0 = - m g l \sin\theta$$

or $$m l^2 \ddot{\theta} = - m g l \sin\theta$$

or $$\ddot{\theta} = -\left(\frac{g}{l}\right) \sin\theta$$

or $$\ddot{\theta} = -\left(\frac{g}{l}\right) \theta \qquad \{\text{Since } \theta \text{ is very small } \therefore \sin\theta \approx \theta\}$$

which is the required equation of the motion.

Example 2 : A bead of mass M, slides on a smooth fixed wire, whose inclination to the vertical is α, and has hinged to it a rod of mass m and length $2l$, which can move freely in the vertical plane throgh the wire. If the system starts from rest with the rod hanging vertically, show that

$$\left\{4 M + m \left(1 + 3\cos^2\theta\right)\right\} l\dot{\theta}^2 = 6 \left(M + m\right) g \sin\alpha \left(\sin\theta - \sin\alpha\right)$$

where θ is the angle between the rod and the lower part of the wire.

Solution : Let $A_0\, Q_0$ be the fixed wire whose inclination to the vertical $A_0\, B_0$ is α, where $A_0\, B_0$ be the initial position of the rod. At time t, let the bead of mass M be at A and the rod AB

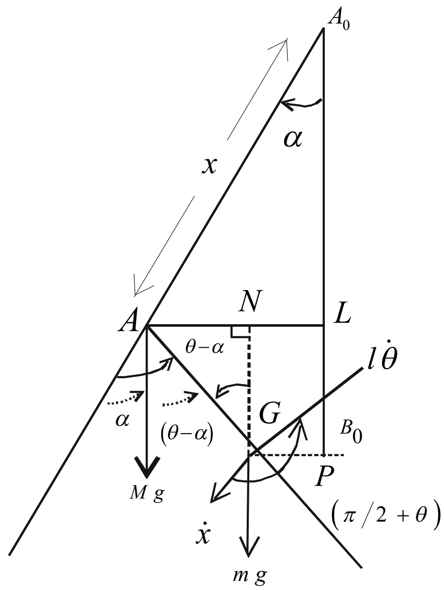


Fig. 7.3

of mass m and length $2l$ be inclined at an angle θ to the lower part of the fixed wire $A_0\,Q_0$. Let $A_0\,A = x$ be the distance moved by bead on the fixed wire in time t. Here x and θ give the position of the system at any time t, therefore, x and θ are taken as generalised coordinates of the system.

The weights of the bead (of mass M) and the rod (of mass m) are acting vertically downwards at A and G ($C.G.$ fo the rod) respectively. Thus, if T and W are the $K.E.$ and the work function of the system respectively then with the help of figure, we get

$T = K.E.$ of the system $=$ $K.E.$ of the bead $+$ $K.E.$ of the rod

$$= \frac{1}{2}\,M\dot{x}^2 + \frac{m}{2}\left\{\left(\dot{x}^2 + l^2\,\dot{\theta}^2 + 2\,\dot{x}\,l\,\dot{\theta}\,\cos\left(\pi/2 + \theta\right)\right) + \frac{l^2}{3}\,\dot{\theta}^2\right\}$$

$$= \frac{1}{2}\,(M+m)\dot{x}^2 + \frac{2}{3}\,m\,l^2\,\dot{\theta}^2 - m\,l\,\dot{x}\,\dot{\theta}\,sin\,\theta \qquad ...(1)$$

and

$W =$ work function $=$ work done by weight of the bead $+$ work done by weight of the rod

$$= M\,g\,.\,A_0L + m\,g\,.\,A_0P + C\text{, where } C \text{ is any constant}$$

$$= M\,g\,.\,x\,cos\,\alpha + m\,g\,(A_0L + NG) + C$$

$$= M\,g\,.\,x\,cos\,\alpha + m\,g\,\{x\,cos\,\alpha + l\,cos(\theta - \alpha)\} + C$$

or $$W = (M+m)\,g\,x\,cos\,\alpha + m\,g\,l\,cos(\theta-\alpha) + C \qquad ...(2)$$

Now, applying Lagrange's equations

Lagrange's x - equation is $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{x}}\right) - \dfrac{\partial T}{\partial x} = \dfrac{\partial W}{\partial x}$

Here

$$\frac{d}{dt}\left\{(M+m)\,\dot{x} - ml\dot{\theta}\sin\theta\right\} - 0 = (M+m)g\cos\alpha$$

or $$(M+m)\,\ddot{x} - ml\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) = (M+m)\,g\cos\alpha \qquad ...(3)$$

Similarly, Lagrange's θ - equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

Here

or $$\frac{d}{dt}\left\{\frac{4}{3}ml^2\dot{\theta} - ml\dot{x}\sin\theta\right\} + ml\dot{x}\dot{\theta}\cos\theta = -mgl\sin(\theta-\alpha)$$

or $$\frac{4}{3}ml^2\ddot{\theta} - ml\left\{\dot{x}\cos\theta\,\dot{\theta} + \ddot{x}\sin\theta\right\} + ml\dot{x}\dot{\theta}\cos\theta = -mgl\sin(\theta-\alpha)$$

or $$\frac{4}{3}ml^2\ddot{\theta} - ml\,\ddot{x}\sin\theta = -mgl\sin(\theta-\alpha) \qquad ...(4)$$

Eliminating $\ddot{x}$ between equations (3) and (4), we get

$$\frac{4}{3}ml^2\ddot{\theta} - ml\sin\theta\left\{\frac{(M+m)g\cos\alpha + ml\dot{\theta}^2\cos\theta + ml\ddot{\theta}\sin\theta}{(M+m)}\right\} = -mgl\sin(\theta-\alpha)$$

or $$\left\{\frac{4}{3}(M+m) - m\sin^2\theta\right\}l^2\ddot{\theta} - ml^2\dot{\theta}^2\sin\theta\cos\theta$$

$$= (M+m)gl\left\{\cos\alpha\sin\theta - \sin(\theta-\alpha)\right\} = (M+m)gl\sin\alpha\cos\theta \qquad ...(5)$$

Multiplying it by $2\dot{\theta}$, we get

$$\left\{\frac{4}{3}(M+m) - m\sin^2\theta\right\}2l^2\dot{\theta}\ddot{\theta} - ml^2\dot{\theta}^2\sin\theta\cos\theta\,2\dot{\theta}$$

$$= (M+m)\,gl\sin\alpha\cos\theta\,2\dot{\theta}$$

or $$l^2\frac{d}{dt}\left[\left\{\frac{4}{3}(M+m) - m\sin^2\theta\right\}\dot{\theta}^2\right] = 2(M+m)\,gl\sin\alpha\cos\theta\,\dot{\theta}$$

Integrating it, we get

$$l^2\left\{\frac{4}{3}(M+m) - m\sin^2\theta\right\}\dot{\theta}^2 = 2(M+m)\,gl\sin\alpha\,\sin\theta + C_1 \qquad ...(6)$$

Initially when the rod was vertical at A_0 i.e.

when $\theta = \alpha$, $\dot{\theta} = 0$ from (6) we have

$$C_1 = -\,2(M+m)\,g\,l\,sin^2\alpha \qquad ...(7)$$

Hence, the equation (6) gives

$$l^2\left\{\frac{4}{3}(M+m) - m\,sin^2\theta\right\}\dot{\theta}^2 = 2(M+m)\,g\,l\left\{sin\alpha\ sin\theta - sin^2\alpha\right\}$$

or $$l\left\{4(M+m) - 3m\,sin^2\theta\right\}\dot{\theta}^2 = 6(M+m)\,g\left\{sin\alpha\ sin\theta - sin^2\alpha\right\}$$

or $$l\left\{4M + 4m - 3m\left(1-\cos^2\theta\right)\right\}\dot{\theta}^2 = 6\,(M+m)\,g\sin\alpha\,(\sin\theta - \sin\alpha)$$

or $$l\left\{4M + m\left(1+3\cos^2\theta\right)\right\}\dot{\theta}^2 = 6\,(M+m)\,g\sin\alpha\,(\sin\theta - \sin\alpha)$$

Example 3 : A uniform rod, of mass $3m$ and length $2l$ has its middle point fixed and a mass m attached at one extremity. The rod when in a horizontal position is set rotating about a vertical axis through its centre with an angular velocity equal to $\sqrt{\frac{2ng}{l}}$. Show that the heavy end of the rod will fall till the inclination of the rod to the vertical is $cos^{-1}\left\{\sqrt{n^2+1} - n\right\}$, and will then rise again.

Solution : Let AB be the rod of mass $3m$ and length $2l$. The middle point O of the rod be fixed and a mass m be attached at the extrimity A of the rod. Initially let the rod be at rest along

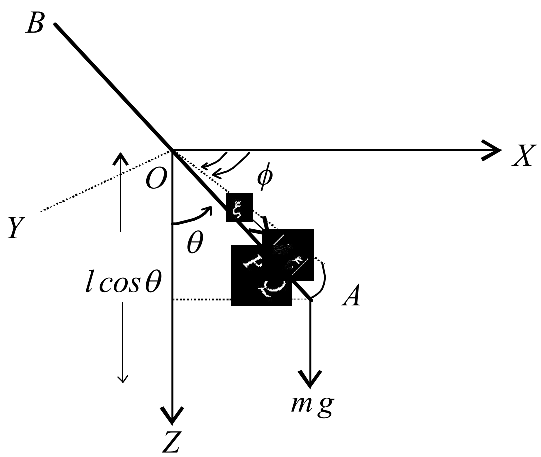


Fig. 7.4

OX taken as axis of x. Let a line OY perpendicular to the plane of paper and a line OZ perpendicular to OX be taken as axes of y and z respectively. O is taken as origin.

Let at any time t, the rod turn through an angle ϕ from its initial position OX with an angular velocity $\dot{\phi} = \sqrt{\frac{2ng}{l}}$. Also let θ be the inclination of the rod with OZ at this time t. Take an element

$PQ = d\,\xi$ fo the rod such that $OP = \xi$, then the coordinates of the point P on the rod are (x, y, z), where

$$x = \xi \sin\theta \cos\phi\,,\ \ y = \xi \sin\theta \sin\phi\,,\ \ z = \xi \cos\theta \qquad ...(1)$$

Then, $\dot{x} = \xi \cos\theta \, \cos\phi \, \dot{\theta} - \xi \sin\theta \, \sin\phi \, \dot{\phi}$

$$\dot{y} = \xi \cos\theta \, \sin\phi \, \dot{\theta} + \xi \sin\theta \, \cos\phi \, \dot{\phi} \qquad ...(2)$$

$$\dot{z} = -\, \xi \sin\theta \, \dot{\theta}$$

Therefore, the velocity of P, V_p is

$$V_P^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

$$= \xi^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) \qquad ...(3)$$

If V_A be the velocity of mass m at A, then we have on putting $\xi = l$

$$V_A^2 = l^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) \qquad ...(4)$$

Now, the mass of the element $PQ = \dfrac{3m}{2l}\, d\,\xi$.

therefore, the $K.E.$ of the element $PQ = \dfrac{1}{2} \cdot \left(\dfrac{3m}{2l}\, d\,\xi\right) . V_P^2$

$$= \frac{1}{2}\left(\frac{3m}{2l}\, d\,\xi\right) \xi^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$$

Thus, the $K.E.$ of the rod $AB = \displaystyle\int_{-l}^{l} \frac{1}{2}\left(\frac{3m}{2l}\, d\,\xi\right) \xi^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$

$$= \frac{3m}{4l} \int_{-l}^{l} \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) \xi^2 \, d\,\xi$$

$$= \frac{3m}{4l} \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) . \, 2 \int_{0}^{l} \xi^2 \, d\,\xi$$

or, $K.E.$ of the rod $AB = \dfrac{m}{2} \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) l^2 \qquad ...(4)$

Also, the $K.E.$ of the mass m at $A = \dfrac{1}{2}\, m l^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) \qquad ...(5)$

Thus, if T be the $K.E.$ of the system, then

$= K.E.$ of the rod $AB + K.E.$ of the mass m at A

$$= \frac{ml^2}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{ml^2}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$$

$$T = ml^2\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) \qquad ...(6)$$

Again, if W be the work function of the system, then

$W =$ work done by weight of the mass m at $A = mg \,.\, OR$

$$= mgl\cos\theta \qquad ...(7)$$

Now, applying Lagrange's equations, we get

Lagrange's θ - equation as

$$\frac{d}{dt}\left\{2ml^2\dot{\theta}\right\} - 2ml^2\,\dot{\phi}^2 \sin\theta\cos\theta = -\,mgl\sin\theta$$

or $$l\,\ddot{\theta} - l\,\dot{\phi}^2\,\sin\theta\,\cos\theta = -\,\frac{g}{2}\,\sin\theta \qquad ...(8)$$

Lagrange's ϕ - equation gives

$$\frac{d}{dt}\left(2ml^2\dot{\phi}\sin^2\theta\right) - 0 = 0$$

or $$\frac{d}{dt}\left(\dot{\phi}\sin^2\theta\right) = 0$$

on integration, we get

$$\dot{\phi}\sin^2\theta = C \quad \text{(Say)} \qquad ...(9)$$

But Initially when $\theta = \frac{\pi}{2}$, $\dot{\phi} = \sqrt{\frac{2ng}{l}}$ (given)

which gives $$C = \sqrt{\frac{2ng}{l}} \qquad ...(10)$$

Therefore, the equation (9) gives

$$\dot{\phi}\sin^2\theta = \sqrt{\frac{2ng}{l}} \qquad ...(11)$$

Sublituting the value of $\dot{\phi}$ in equation (8), we get

$$l\,\ddot{\theta} - l\left(\frac{2ng}{l\sin^4\theta}\right)\sin\theta\cos\theta = -\frac{g}{2}\sin\theta$$

or $$2l\,\ddot{\theta} - 4ng\cot\theta\,\mathrm{cosec}^2\theta = -g\sin\theta \qquad ...(12)$$

Multiplying it by $2\,\dot{\theta}$ and then integrating, we get

$$l\,\dot{\theta}^2 + 2ng\cot^2\theta = g\cos\theta + C_1 \qquad ...(13)$$

But initially, when $\theta = \pi/2$, $\dot{\theta} = 0$,

which gives $C_1 = 0$

Thus, we get

$$l\,\dot{\theta}^2 + 2ng\cot^2\theta = g\cos\theta \qquad ...(14)$$

Now, the rod will fall till $\dot{\theta} = 0$, i.e.

$$2ng\cot^2\theta = g\cos\theta$$

or $$\cos\theta\left(2n\cos\theta - \sin^2\theta\right) = 0 \qquad ...(15)$$

which gives

either $\cos\theta = 0$ or $2n\cos\theta - \sin^2\theta = 0$

$\therefore$ when $\cos\theta = 0$ then $\theta = \pi/2$, which gives initial position of the system.

When $2n\cos\theta - \sin^2\theta = 0$

then $2n\cos\theta - \left(1 - \cos^2\theta\right) = 0$

or $$\cos\theta = \frac{-2n \pm \sqrt{4n^2 + 4}}{2}$$

or $\cos\theta = \sqrt{n^2 + 1} - n$ (leaving - sign because $\theta \le \pi/2$)

or $$\theta = \cos^{-1}\left\{\sqrt{n^2 + 1} - n\right\} \qquad ...(16)$$

which gives the required result.

Again, from equation (12), we get

$$\ddot{\theta} = -\left(\frac{g}{2l}\right)\left\{\frac{\sin^4\theta - 4n\cos\theta}{\sin^3\theta}\right\} \qquad ...(17)$$

If we substitute the value of θ from (16) in the R.H.S. of (17), then $\ddot{\theta}$ comes out to be positive, thus at that time the rod begins to rise again.

7.5 Small Oscillations

To explain how Lagrange's equations are used in case of small oscillations.

or

Use of Lagrange's equations in finding the small oscillations of a conservative system about a position of equlibrium.

In the case of small oscillations the generalised coordinates $\theta, \phi, \psi, \ldots$ should be so chosen that they vanish in the position of equlibrium.

Since the system makes small oscillations about the position of equilibrium the generalised coordinates $\theta, \phi, \psi, \ldots$ as also their derivatives will remain small during the whole motion.

If x, y, z do not contain time t explicity then the $K.E.$ is a homogeneous quadratic function of $\dot{\theta}, \dot{\phi}, \dot{\psi} \ldots$ etc. Therefore, $K.E.\ T$ and the work function W of the system are given by

$$T = A_{11}\dot{\theta}^2 + A_{22}\dot{\phi}^2 + A_{33}\dot{\psi}^2 + \ldots + 2A_{12}\dot{\theta}\dot{\phi} + 2A_{13}\dot{\theta}\dot{\psi} + 2A_{23}\dot{\phi}\dot{\psi} \ldots \qquad \ldots(1)$$

and $$W = C + B_1\theta + B_2\phi + B_3\psi + \ldots + B_{11}\theta^2 + B_{22}\phi^2 + B_{33}\dot{\psi}^2 \ldots \qquad \ldots(2)$$

Now, choosing X, Y, Z such that $\theta,\ \phi,\ \psi, \ldots$ be expressed in terms of X, Y, Z by the following equations

$$\left.\begin{aligned} \theta &= \lambda_1 X + \lambda_2 Y + \lambda_3 Z \\ \phi &= \mu_1 X + \mu_2 Y + \mu_3 Z \\ \psi &= v_1 X + v_2 Y + v_3 Z \end{aligned}\right\} \qquad \ldots(3)$$

Again choose $\lambda_i's,\ \mu_i's,\ v_i's$ in such a way that when the above values of $\theta,\ \phi,\ \psi, \ldots$ and their derivatives are substituted in equation (1) and (2), then there is no term containing $\dot{X}\dot{Y},\ \dot{Y}\dot{Z},\ \dot{Z}\dot{X}$ in T and there is no term containing $XY,\ YZ,\ ZX$ in W, then $X,\ Y,\ Z$ are called the **Principal** or **Normal coordinates.**

Thus when $X,\ Y,\ Z$ are the principal coordinates then from (1) and (2), we have

$$T = A'_{11}\dot{X}^2 + A'_{22}\dot{Y}^2 + A'_{33}\dot{Z}^2 + \ldots \qquad \ldots(4)$$

and $$W = C' + B'_1 X + B'_2 Y + B'_3 Z + \ldots + B'_{11}X^2 + B'_{22}Y^2 + B'_{33}Z^2 + \ldots \qquad \ldots(5)$$

Then the Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}}\right) - \frac{\partial T}{\partial X} = \frac{\partial W}{\partial X}, \text{ etc}$$

or $$\frac{d}{dt}\left(2A'_{11}\dot{X}\right) - 0 = \left(B'_1 + 2B'_{11}X\right)$$

or $\quad 2\, A'_{11}\ddot{X} = B'_1 + 2\, B'_{11} X$ etc.

which can be put in the forms

$$\ddot{X} = -\, W_1^2 X,\ \ \ddot{Y} = -\, W_2^2\, Y,\ \ \ddot{Z} = -\, W_3^2\, Z \ \text{ etc.}$$

which represent S.H.M. giving the small oscillations about the position of equilibrium. Thus we can conclude that, when the equation of motion of a system can be reduced by the variables X, Y, Z to equations of S.H.M., then the coordinates X, Y, Z, are called the principal or normal coordinates.

Example 4 : Two equal rods AB and BC, each of length l, smoothly jointed at B, are suspended from A and oscillates in a vertical plane through A. Show that the periods of normal oscillations are $\dfrac{2\pi}{n}$, where,

$$n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}.$$

Solution : Let m be the mass of the rod AB and BC which are smoothly jointed at B. Also Let G_1, and G_2 be the centres of gravity of the rods AB and BC respectively and θ and ϕ their inclinations to the vertical at any time t.

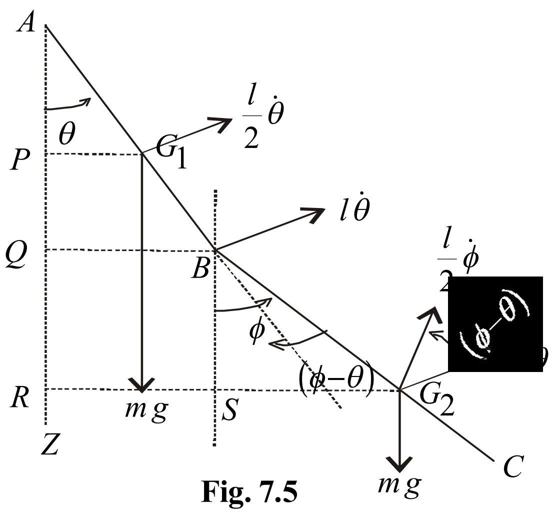


Fig. 7.5

G_2 is turning round B with velocity $\frac{l}{2}\dot{\phi}$, while B is turning round A with velocity $l\dot{\theta}$. Now if T and W are the $K.E.$ and work function of the system, then, we have

$$T = K.E. \text{ of the system}$$

$$= K.E. \text{ of the rod } AB + K.E. \text{ of the rod } BC$$

$$= \frac{m}{2}\left\{V_{G_1}^2 + K_1^2\,\dot{\theta}^2\right\} + \frac{m}{2}\left\{V_{G_2}^2 + K_2^2\,\dot{\phi}^2\right\} \qquad ...(1)$$

Where V_{G_1} and V_{G_2} are the velocities of the G_1 and G_2, $C.G.$ of the rods AB and BC and K_1 and K_2 are their radii of gyration of these rods respectively.

$$\therefore \quad T = \frac{m}{2}\left\{\left(\frac{l}{2}\dot{\theta}\right)^2 + \frac{1}{3}\left(\frac{l}{2}\right)^2 \dot{\theta}^2\right\}$$

$$+ \frac{m}{2}\left\{\left(l\,\dot{\theta}\right)^2 + \left(\frac{l}{2}\dot{\phi}\right)^2 + 2\left(l\dot{\theta}\right)\left(\frac{l}{2}\dot{\phi}\right)\cos\left(\phi - \theta\right) + \frac{1}{3}\left(\frac{l}{2}\right)^2 \dot{\phi}\right\}$$

$$= \frac{m}{2}\left\{\left(\frac{l^2\dot{\theta}^2}{3}\right) + l^2\dot{\theta}^2 + \frac{l^2\dot{\phi}^2}{3} + l^2\dot{\theta}\,\dot{\phi}\,.\,1\right\} \quad \text{(for small oscillation } \cos(\phi-\theta) \approx 1\text{)}$$

$$= \frac{ml^2}{6}\left\{4\dot{\theta}^2 + \dot{\phi}^2 + 3\dot{\theta}\dot{\phi}\right\} \qquad \text{...(2)}$$

and

$$W = m\,g\,.\,AP + m\,g\,.\,AR + C, \text{ where } C \text{ is any constant}$$

$$= m g\,.\,\frac{l}{2}\cos\theta + m g\left(AQ + BS\right) + C \qquad \text{...(3)}$$

$$= m g\,.\,\frac{l}{2}\cos\theta + m g\left(l\cos\theta + \frac{l}{2}\cos\phi\right) + C$$

$$= \frac{m g l}{2}\left(3\cos\theta + \cos\phi\right) + C \qquad \text{...(4)}$$

Now, here θ and ϕ are taken as generalised coordinates. Therefore, Lagrange's θ - equation gives

$$\frac{d}{dt}\left\{\frac{ml^2}{6}\left(8\dot{\theta} + 3\dot{\phi}\right)\right\} - 0 = -\frac{m g l}{2}\,.\,3\sin\theta$$

or $$8\,\ddot{\theta} + 3\,\ddot{\phi} = -\,9\left(\frac{g}{l}\right)\sin\theta \qquad \{\text{For small oscillation } \sin\theta \approx \theta\}$$

or $$8\,\ddot{\theta} + 3\,\ddot{\phi} = -\,9\left(\frac{g}{l}\right)\theta \qquad \text{...(5)}$$

and Lagrange's ϕ - equation gives

$$\frac{d}{dt}\left\{\frac{nl^2}{6}\left(2\dot{\phi} + 3\dot{\theta}\right)\right\} - 0 = -\frac{m g l}{2}\,.\,\sin\phi$$

or $$2\ddot{\phi} + 3\ddot{\theta} = -\,3\left(\frac{g}{l}\right)\sin\phi$$

$$= -3\left(\frac{g}{l}\right)\phi \qquad \{\text{For small oscillation } sin\,\phi \approx \phi)$$

or $$2\ddot{\phi} + 3\ddot{\theta} = -3\left(\frac{g}{l}\right)\phi \qquad ...(6)$$

Now, the equation (5) and (6) can be written as

$$\left\{8D^2 + 9\left(\frac{g}{l}\right)\right\}\theta + 3D^2\phi = 0,$$

and $$3D^2\theta + \left\{2D^2 + 3\left(\frac{g}{l}\right)\right\}\phi = 0 \qquad ...(7)$$

where $D = \dfrac{d}{dt}$

Eliminating ϕ in the equation (7), we get

$$\left[\left\{8D^2 + 9\left(\frac{g}{l}\right)\right\}\left\{2D^2 + 3\left(\frac{g}{l}\right)\right\} - 9D^4\right]\theta = 0$$

on $$\left[7D^4 + 4z\left(\frac{g}{l}\right)D^2 + 27\left(\frac{g}{l}\right)^2\right]\theta = 0 \qquad ...(8)$$

If the periods of small oscillation are $\dfrac{2\pi}{n}$, then

we put $\theta = A\,cos(nt + B)$, where A and B are constants

then $\quad D\theta = -An\sin(nt + B)$, and

$$D^2\theta = -n^2 A\cos(nt + B)$$

or $\quad D^2\theta = -n^2\theta$, similarly $D^4\theta = n^4\theta$ $\qquad ...(9)$

$\therefore$ with the help of (9), equation (8) becomes

$$\left\{7n^4 - 4z\left(\frac{g}{l}\right)n^2 + 27\left(\frac{g}{l}\right)^2\right\}.\ \theta = 0, \text{ which gives}$$

$$7n^4 - 4z\left(\frac{g}{l}\right)n^2 + 27\left(\frac{g}{l}\right)^2 = 0 \qquad (\because \theta \neq 0)$$

Therefore, we get

$$n^2 = \frac{4z\left(\frac{g}{l}\right) \pm \sqrt{\left\{4z\left(\frac{g}{l}\right)\right\}^2 - 4.7.27\left(\frac{g}{l}\right)^2}}{2.7}$$

or $$n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\left(\frac{g}{l}\right).$$

Example 5 : A unifrom rod, of mass $5m$ and length $2a$, turns freely about one end which is fixed, to its other extermity is attached one end of a light string of length $2a$, which carries at its other end a particle of mass m, show that the periods of small oscillations in a vertical plane are the same as those of simple perdulums of lengths $\frac{2a}{3}$ and $\frac{20}{7}a$.

Solution : Let OA be a rod of length $2a$ and mass $5m$, whose one end is fixed at O and G be its centre of gravity. Let a partical of mass m be attached at the end B of a string AB of length $2a$ which is attached to the rod at the end A.

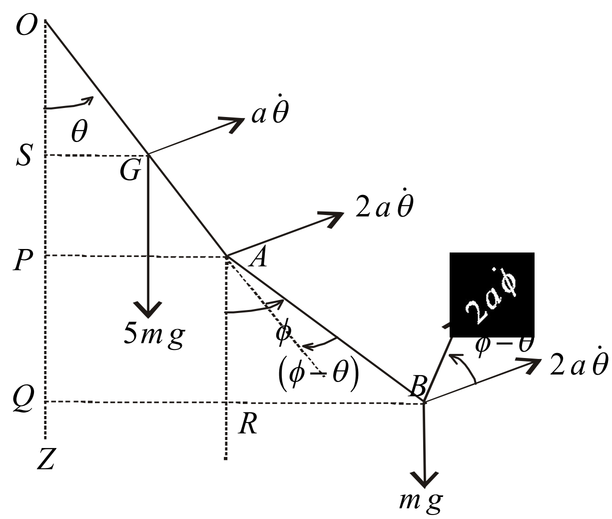


Fig. 7.6

Let after time t, the inclinations of the rod and string to the vertical be θ and ϕ respectively Now, if T be the $K.E.$ of the system, then

$$T = K.E. \text{ of the rod} + K.E. \text{ of the particle of mass } m \text{ at } B$$

$$= \frac{5m}{2}\left\{V_G^2 + K^2\dot{\theta}^2\right\} + \frac{m}{2}V_B^2, \text{ where } V_G \text{ be the velocity}$$

of the $C.G.$ of the rod, k be its radius of gyration and V_B be the velocity of mass m at B.

$$T = \frac{5m}{2}\left\{\left(a\dot{\theta}\right)^2 + \frac{a^2}{3}\dot{\theta}^2\right\} + \frac{m}{2}\left\{\left(2a\dot{\theta}\right)^2 + \left(2a\dot{\phi}\right)^2 + 2.\left(2a\dot{\theta}\right)\left(2a\dot{\phi}\right)\cos(\phi-\theta)\right\}$$

$$= \frac{5m}{2}\left\{a^2\dot{\theta}^2 + \frac{a^2}{3}\dot{\theta}^2\right\} + \frac{m}{2}\left\{4a^2\dot{\theta}^2 + 4a^2\dot{\phi}^2 + 8a^2\dot{\theta}\dot{\phi}\right\}$$

{For small oscillation $cos\,(\phi-\theta)\approx 1$}

or $$T = \frac{ma^2}{3}\left\{16\dot{\theta}^2 + 6\dot{\phi}^2 + 12\dot{\theta}\dot{\phi}\right\} \qquad ...(1)$$

Again, if W be the work function of the system, then

$W =$ work done by weight of the rod at G, $C.G.$ of the rod

$+$ work done by weight of the particle at B

$$= 5mg\,.\,OS + mg.OQ + C, \text{ where } C \text{ is any constant}$$

$$= 5mg\,.\,a\cos\theta + mg\,.\,(OP + AR) + C$$

$$= 5mg\;a\cos\theta + mg\,\{2a\cos\theta + 2a\cos\phi\} + C$$

or $$W = 7mga\,cos\,\theta + 2mga\,cos\,\phi + C \qquad ...(2)$$

Here θ and ϕ are taken an generalised coordinates of the system, therefore, by Lagrange's equations, Lagrange's θ - equation gives.

$$\frac{d}{dt}\left\{\frac{ma^2}{3}\left(32\dot{\theta} + 12\dot{\phi}\right)\right\} - 0 = -\,7mga\,sin\,\theta$$

or $$32\ddot{\theta} + 12\ddot{\phi} = -\,21\left(\frac{g}{a}\right)sin\,\theta$$

or $$32\ddot{\theta} + 12\ddot{\phi} = -\,21\left(g/a\right)\theta \qquad \{\text{For small oscillation } sin\,\theta \approx \theta\} \qquad ...(3)$$

Lagrange's ϕ-equation gives

$$\frac{d}{dt}\left\{\frac{ma^2}{3}\left(12\dot{\phi} + 12\dot{\theta}\right)\right\} - 0 = -\,2mga\sin\phi$$

or $$4\ddot{\phi} + 4\ddot{\theta} = -\,2\left(\frac{g}{a}\right)sin\,\phi$$

or $$2\ddot{\phi} + 2\ddot{\theta} = -\left(\frac{g}{a}\right)\phi \qquad (\text{for small oscillation } sin\,\phi \approx \phi)...(4)$$

Equation (3) and (4) can be written as

$$\left\{32D^2 + 21\left(\frac{g}{l}\right)\right\}\theta + 12D^2\phi = 0 \qquad ...(5)$$

and $$2D^2\theta + \left\{2D^2 + \frac{g}{a}\right\}\phi = 0 \qquad ...(6)$$

Eliminating ϕ between equation (5) and (6),

$$\left[\left\{32D^2 + 21\left(\frac{g}{a}\right)\right\}\left\{2D^2 + \left(\frac{g}{a}\right)\right\} - 24D^4\right]\theta = 0$$

or $$\left[40D^4 + 74\left(\frac{g}{a}\right)D^2 + 21\left(\frac{g}{a}\right)^2\right]\theta = 0 \qquad ...(7)$$

Now, put $\theta = A\,cos(wt + B)$, then

$$D\theta = -\,w\,A\,sin(wt + B),\ D^2\theta = -\,w^2 A\cos(wt + B)$$

or $D^2\theta = -\,w^2\theta$ and similarly $D^4\theta = w^4\theta$

Then the equation (7) becomes

$$\left\{40w^4 - 74w^2\left(\frac{g}{a}\right) + 21\left(\frac{g}{a}\right)^2\right\}\theta = 0$$

or $$40w^4 - 74w^2\left(\frac{g}{a}\right) + 21\left(\frac{g}{a}\right)^2 = 0 \qquad (\because \theta \neq 0)$$

or $$\left\{20w^2 - 7\left(\frac{g}{a}\right)\right\}.\left\{2w^2 - 3\left(\frac{g}{a}\right)\right\} = 0$$

or $$w_1^2 = \frac{7}{20}\left(\frac{g}{a}\right) \quad \text{or} \quad w_2^2 = \frac{3}{2}\left(\frac{g}{a}\right)$$

Thus, if t be the time period of small oscillation then $t = \frac{2\pi}{w}$ and if l be the length of simple equivalent pendulum, then

$$t = 2\pi\sqrt{\frac{l}{g}} = \frac{2\pi}{w}\text{, which gives } l = \frac{g}{w^2}$$

Thus, in the present case, the lengths of equivalent simple pendulum are $\frac{g}{w_1^{\ 2}}$ and $\frac{g}{w_2^{\ 2}}$

or $$\frac{20a}{7} \text{ and } \frac{2a}{3}$$

Example 6 : A perfectly rough sphere lying inside a hollow cylinder, which rests on a perfectly rough plane, is slightly displaced from its position of equilibrium. Show that the time of a small oscillation is

$$2\pi\sqrt{\left(\frac{a-b}{g}\right)\left(\frac{14M}{10M+7m}\right)}$$

where a is the radius of the cylinder, b that of the sphere, and M, m are the masses of the cylinder and sphere respectively.

Solution :

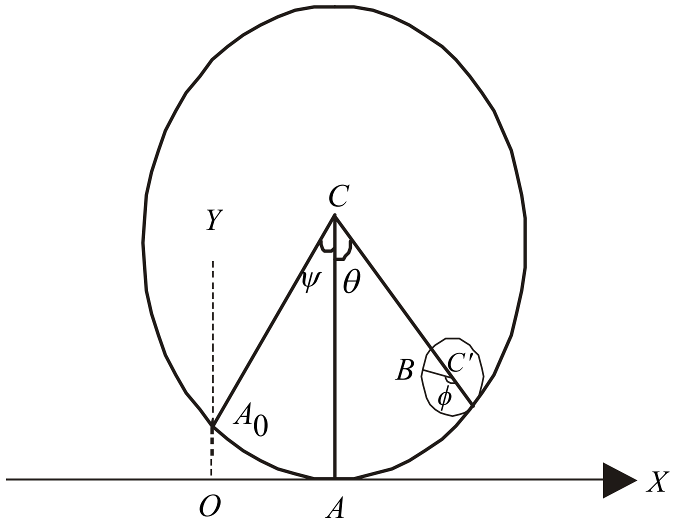


Fig. 7.7

The figure in the vertical cross - section through the centres of the cylinders and sphere.

Let C and C' be the centres of the cylinder and the sphere. Initially the point A_0 of the cylinder and sphere be coincident, therefore, let CA_0 and $C'A_0$ are lines fixed in the cylinder and sphere respectively These lines are initially vertical and after a time t, make angles ψ and ϕ with the vertical. Let θ be the angle which the line joining centres C and C' makes with the vertical. There is no slipping between the cylinder and horizontal plane and also between the cylinder and sphere. Let the cylinder ralls a distance x on the plane, Therefore, with the help of figure, we have

$$x = A_0A = Arc\ A_0A = a\,\psi$$

and $$Arc\ A_0P = Arc\ B_0P$$

$$i.e.\ a\,(\psi+\theta) = b\,(\theta+\phi)$$

or $$b\,\phi = (a-b)\,\theta + a\,\psi$$

so that $b\,\dot{\phi} = C\,\dot{\theta} + a\,\dot{\psi}$ ...(1)

where $C = a - b$

Referred to the horizontal and vertical through

A_0 as coordinates axes, the coordinates (x_c, y_c) of C and (x_c', y_c') of C' are given by

$$\left.\begin{aligned} x_c &= OA\ (or\ A_0A) = a\psi\ ,\ y_c = a \\ x'_c &= OK\ (or\ A_0K) = OA + AK = a\psi + C\sin\theta \end{aligned}\right\} \qquad ...(2)$$

$$y'_c = C'K = a - C\cos\theta$$

If V_C and $V_{C'}$ be the velocitites of C and C' respectively then we get

$$V_C{}^2 = \dot{x}_c{}^2 + \dot{y}_c{}^2 = a^2\,\dot{\psi}^2 \qquad ...(3)$$

and $$V_{C'}{}^2 = \dot{x}'^2_c + \dot{y}'^2_c = \left(a\dot{\psi} + C\cos\theta\,\dot{\theta}\right)^2 + \left(C\sin\theta\,\dot{\theta}\right)^2$$

$$= a^2\,\dot{\psi}^2 + C^2\,\dot{\theta}^2 + 2\,aC\dot{\psi}\,\dot{\theta} \qquad \begin{pmatrix} \because\ \theta \text{ is very small} \\ \therefore\ \cos\theta \approx 1,\ sin\theta \approx \theta \end{pmatrix} \qquad ...(4)$$

Now, if T be the $K.E.$ of the system, then we have

$$T = K.E. \text{ of the cylinder} + K.E. \text{ of the sphere}$$

$$= \frac{M}{2}\left\{\dot{x}^2 + K_1^2\,\dot{\psi}^2\right\} + \frac{m}{2}\left\{V_{C'}{}^2 + K_2^2\,\dot{\phi}^2\right\}$$

where K_1 and K_2 are the radii of gyration of the cylinder and sphere respectively.

or $$T = \frac{M}{2}\left\{\dot{x}^2 + a^2\,\dot{\psi}^2\right\} + \frac{m}{2}\left\{a^2\,\dot{\psi}^2 + C^2\,\dot{\theta}^2 + 2\,ac\,\dot{\psi}\,\dot{\theta} + \frac{2}{5}b^2\dot{\phi}^2\right\}$$

$$= \frac{M}{2}\left\{a^2\,\dot{\psi}^2 + a^2\,\dot{\psi}^2\right\} + \frac{m}{2}\left\{a^2\,\dot{\psi}^2 + C^2\,\dot{\theta}^2 + 2\,aC\,\dot{\psi}\,\dot{\theta} + \frac{2}{5}\left(C\dot{\theta} + a\,\dot{\psi}\right)^2\right\}$$

$$= \frac{(10\,M + 7\,m)}{10}\,a^2\,\dot{\psi}^2 + \frac{7\,m}{10}\left(C^2\,\dot{\theta}^2 + 2\,aC\,\dot{\psi}\,\dot{\theta}\right) \qquad ...(5)$$

Again, if W be the work function of the system, then

$W =$ work done by weight of the sphere

$$= -\,mg(a - C\cos\theta) \quad \{\text{Distance is measured in upward direction}\} \qquad ...(6)$$

Here ψ and θ are taken as generalised coordinates of the system. We have from Lagrange's equations :

Lagrange's ψ equation gives.

$$\frac{d}{dt}\left\{\frac{(10\,M + 7\,m)\,a^2\,\dot{\psi}^2}{5} + \frac{7}{5}\,maC\dot{\theta}\right\} - 0 = 0$$

or $$7\,mC\ddot{\theta} + (10\,M + 7\,m)\,a\,\ddot{\psi} = 0 \qquad ...(7)$$

and Lagrange's θ - equation gives

$$\frac{d}{dt}\left\{\frac{7m}{10}\left(2C^2\dot{\theta}+2aC\dot{\psi}\right)\right\} - 0 = -mgC\theta \qquad (\because \theta \text{ is small, } \therefore \sin\theta \approx \theta)$$

or $$7C\ddot{\theta} + 7a\ddot{\psi} = -5g\theta \qquad ...(8)$$

Eliminating $\ddot{\psi}$ between equations (7) and (8), we get

$$\{7aC(10M+7m) - 49maC\}\ddot{\theta} = -5ag(10M+7m)\theta$$

or $$\ddot{\theta} = -\left\{\left(\frac{10M+7m}{14M}\right)\cdot\left(\frac{g}{a-b}\right)\right\}\theta \qquad ...(9)$$

which represents an equation of simple harmonic motion (S.H.M.) with the time of small oscillation

$$= 2\pi\sqrt{\left(\frac{a-b}{g}\right)\left(\frac{14M}{10M+7m}\right)}.$$

Example 7 : A uniform straight rod, of length $2a$ is freely movable about its centre and a particle of mass one third that of the rod is attached by a light inextensible sting, of length a, to one end of the rod; show that one period of principal oscillation is $(\sqrt{5}+1)\,\pi\sqrt{\frac{a}{g}}$.

Solution : Let AB be a rod of length $2a$ of mass m and having its centre of gravity at G, Also let a particle of mass $\left(\frac{m}{3}\right)$ attached at C and is connected by a light string BC of length a at the lower end B of the rod.

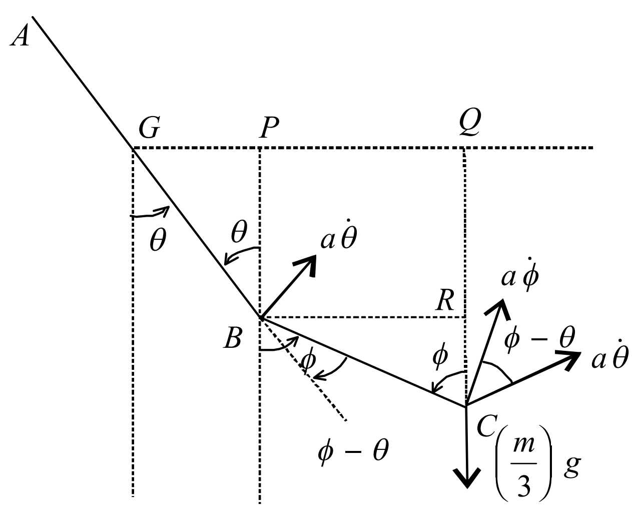


Fig. 7.8

At any time t, let θ and ϕ be the inclinations of the rod and the string to the vertical respectively.

Here θ and ϕ are taken as generalised coordinats of the system.

Now, if T be the $K.E.$ of the system, then we get

$$T = \frac{m}{2}\left\{V_G^2 + K^2\,\dot{\theta}^2\right\} + \frac{1}{2}\left(\frac{m}{3}\right)\left\{\left(a\,\dot{\theta}\right)^2 + \left(a\,\dot{\phi}\right)^2 + 2\left(a\,\dot{\theta}\right)\left(a\,\dot{\phi}\right)\,cos\left(\phi-\theta\right)\right\}$$

where V_G be the velocity of $C.G.$ of the rod (which is zero) and K be its radius of gyration.

or $$T = \frac{m}{2}\left\{\frac{a^2}{3}\;\dot{\theta}^2\right\} + \frac{m}{6}\left(a^2\,\dot{\theta}^2 + a^2\,\dot{\phi}^2 + 2\,a^2\,\dot{\theta}\,\dot{\phi}\right)$$

(for small oscillation $cos\left(\phi-\theta\right)\approx 1$)

$$= \frac{m\,a^2}{6}\left\{2\,\dot{\theta}^2 + \dot{\phi}^2 + 2\,\dot{\theta}\,\dot{\phi}\right\} \qquad ...(1)$$

Again, if W be the work function, then

$W =$ work done by the particle of mass $\left(\frac{m}{3}\right)$ at C

$= \frac{m}{3}g.\,\{QC\} + C'$, where C' is any constant.

$$= \frac{m\,g}{3}\left\{a\cos\theta + a\cos\phi\right\} + C' \qquad ...(2)$$

$\left(as\;QC = QR + RC = PB + RC = a\cos\theta + a\cos\phi\right)$

Now, Lagrange's θ equation gives

$$\frac{d}{dt}\left\{\frac{m\,a^2}{6}\left(4\,\dot{\theta} + 2\,\dot{\phi}\right)\right\} - 0 = \frac{m\,g}{3}\left(-a\,sin\,\theta\right)$$

or $$4\,\ddot{\theta} + 2\,\ddot{\phi} = -2\left(\frac{g}{a}\right)sin\,\theta$$

or $$2\,\ddot{\theta} + \ddot{\phi} = -2\left(\frac{g}{a}\right)\theta$$ (for small oscillation $sin\,\theta \approx \theta$)

and Lagrange's ϕ equation gives

$$\frac{d}{dt}\left\{\frac{m\,a^2}{6}\left(2\,\dot{\phi} + 2\,\dot{\theta}\right)\right\} - 0 = -\frac{m\,g}{3}\,a\,sin\,\phi$$

or $$\ddot{\phi} + \ddot{\theta} = -\left(\frac{g}{a}\right)\phi$$ (For small oscillation, $sin\,\phi \wp \phi$) ...(4)

If we put $D = \frac{d}{dt}$, then the equation (3) and (4) become

$$\left\{2D^2 + \left(\frac{g}{a}\right)\right\}\theta + D^2\phi = 0 \qquad ...(5)$$

and $$D^2\theta + \left\{D^2 + \left(\frac{g}{a}\right)\right\}\phi = 0 \qquad ...(6)$$

Eliminating ϕ between equation (5) and (6) on multiplying equation (5) by $\left\{D^2 + \left(\frac{g}{a}\right)\right\}$ and equation (6) by D^2 and then on subtacting, we get

$$\left[\left\{2D^2 + \left(\frac{g}{a}\right)\right\}\left\{D^2 + \left(\frac{g}{a}\right)\right\} - D^4\right]\theta = 0$$

or $$\left[D^4 + 3\left(\frac{g}{a}\right)D^2 + \left(\frac{g}{a}\right)^2\right] \qquad ...(7)$$

Now, put $\theta = A\cos(wt + B)$, then

$$D\theta = -wA\sin(wt + B), \qquad D^2\theta = -w^2 A\cos(wt + B) = -w^2\theta$$

and similarly

$$\therefore D^4\theta = w^4\theta$$

The equation (7) thus becomes

$$\left\{w^4 - 3\left(\frac{g}{a}\right)w^2 + \left(\frac{g}{a}\right)^2\right\}\theta = 0$$

or $$w^4 - 3\left(\frac{g}{a}\right)w^2 + \frac{g^2}{a^2} = 0, \qquad \because \theta \neq 0$$

which is a quedratic in w^2 let the two roots be w_1 and w_2

so that $w_1^2 = \frac{g}{2a}\left(3 - \sqrt{5}\right)$ and $w_2^2 = \frac{g}{2a}\left(3 + \sqrt{5}\right)$

$\therefore$ one period of principal oscillation corespending to w_1 is given by

$$\frac{2\pi}{w_1} \Rightarrow 2\pi\sqrt{\frac{2a}{g\left(3 - \sqrt{5}\right)}}$$

rationalize

$$\frac{2\pi}{w_1} = 2\pi\sqrt{\left(\frac{g}{a}\right)\left(\frac{6+2\sqrt{5}}{4}\right)} = \pi\sqrt{\frac{a}{g}}\ \sqrt{\left(1+\sqrt{5}\right)^2}$$

$$= \pi\left(\sqrt{5}+1\right)\sqrt{\frac{a}{g}}$$

7.6 Lagrange's Equation for Impulsive Forces

To establish Lagrange's equations for impulsive forces

Let $\theta, \phi, \psi, ...$ are generalised coordinates of the system and (x, y, z) be the cartesion coordinates of any particle of mass m of the system referred to the rectangular axes. Since the cartesian coordinates are functions of generalised coordinates and time t as

$$x = x(t,\theta,\phi,\psi,...),\ y = y(t,\theta,\phi,\psi,...),\ z = z(t,\theta,\phi,\psi,...) \quad ...(1)$$

Now differentiating (1) with respect to t, we get

$$\frac{dx}{dt} = \dot{x} = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial \theta}\dot{\theta} + \frac{\partial x}{\partial \phi}\dot{\phi} + \frac{\partial x}{\partial \psi}\dot{\psi} + \cdots \text{etc} \quad ...(2)$$

Again, differentiating (2) partially with respect to $\dot{\theta}$ we get

$$\frac{\partial \dot{x}}{\partial \dot{\theta}} = \frac{\partial x}{\partial \theta}$$

Similarly, we have $\frac{\partial \dot{y}}{\partial \dot{\theta}} = \frac{\partial y}{\partial \theta}, \frac{\partial \dot{z}}{\partial \dot{\theta}} = \frac{\partial z}{\partial \theta}$, etc. ...(3)

Since we are here dealing with non-conservative systems, the geometrical equations may contain the time explicitly, but they do not contain $\dot{\theta}, \dot{\phi}, \dot{\psi}, ...$ Now, consider a number of simultaneously applied impulses acting on a dynamical system, we also have D'Alemberts principle for impulsive forces, which states that "the change of momentum in a certain direction is equal to the impulsive of the forces in that direction."

Therefore, the sum of the virtual moments of the components of the change of momentum is equal to the sum of the virtual moments of the applied impulses.

Let $(\dot{x}_0, \dot{y}_0, \dot{z}_0)$ and $(\dot{x}_1, \dot{y}_1, \dot{z}_1)$ be the velocities of the particle of mass m just before and just after the application of the impulses respectively. Then the virtual work of the system the changes of moments is

$$\sum m\left\{(\dot{x}_1 - \dot{x}_0)\,\delta x + (\dot{y}_1 - \dot{y}_0)\,\delta y + (\dot{z}_1 - \dot{z}_0)\,\delta z\right\} \quad ...(4)$$

and if X, Y, Z be the components of the applied impulses at (x, y, z) then the virtual work of the system of impulses is

$$\sum \left\{X\,\delta x + Y\,\delta y + Z\,\delta z\right\} \quad ...(5)$$

Since the virtual works of the two systems are equal, we get

$$\sum m\left\{(\dot{x}_1-\dot{x}_0)\delta x+(\dot{y}_1-\dot{y}_0)\delta y+(\dot{z}_1-\dot{z}_0)\delta z\right\}=\sum\left\{X\delta x+Y\delta y+Z\delta z\right\}$$

Now, if T be the $K.E.$ of the system, then

$$T=\frac{1}{2}\sum m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)$$

then $$\frac{\partial T}{\partial\dot{\theta}}=\sum m\left(\dot{x}\frac{\partial\dot{x}}{\partial\dot{\theta}}+\dot{y}\frac{\partial\dot{y}}{\partial\dot{\theta}}+\dot{z}\frac{\partial\dot{z}}{\partial\dot{\theta}}\right) \qquad ...(7)$$

Now, let the syffix $'0'$ to values refer just before, and the suffix $'1'$ to values just after the application of the impulse. Also, we have

$$\left.\begin{aligned}\delta x&=\frac{\partial x}{\partial\theta}\delta\theta+\frac{\partial x}{\partial\phi}\delta\phi+\frac{\partial x}{\partial\psi}\delta\psi,....\\ \delta y&=\frac{\partial y}{\partial\theta}\delta\theta+\frac{\partial y}{\partial\phi}\delta\phi+\frac{\partial y}{\partial\psi}\delta\psi,.... \quad \text{etc.}\end{aligned}\right\} \qquad ...(8)$$

Now,

$$\sum m\left(\dot{x}_0\delta x_0+\dot{y}_0\delta y_0+\dot{z}_0\delta z_0\right)=\sum m\left\{\left(\dot{x}_0\frac{\partial x_0}{\partial\theta_0}+\dot{y}_0\frac{\partial y_0}{\partial\theta_0}+\dot{z}_0\frac{\partial z_0}{\partial\theta_0}\right)\delta\theta\right.$$

$$\left.+\left(\dot{x}_0\frac{\partial x_0}{\partial\phi_0}+\dot{y}_0\frac{\partial y_0}{\partial\phi_0}+\dot{z}_0\frac{\partial z_0}{\partial\phi_0}\right)\delta\phi+....\right\}$$

$$=\sum m\left\{\left(\dot{x}_0\frac{\partial\dot{x}_0}{\partial\dot{\theta}_0}+\dot{y}_0\frac{\partial\dot{y}_0}{\partial\dot{\theta}_0}+\dot{z}_0\frac{\partial\dot{z}_0}{\partial\dot{\theta}_0}\right)\delta\theta\right.$$

$$\left.+\left(\dot{x}_0\frac{\partial\dot{x}_0}{\partial\dot{\phi}_0}+\dot{y}_0\frac{\partial\dot{y}_0}{\partial\dot{\phi}_0}+\dot{z}_0\frac{\partial\dot{z}_0}{\partial\dot{\phi}_0}\right)\delta\phi+....\right\}$$

Therefore, using (7)

$$=\left(\frac{\partial T}{\partial\dot{\theta}}\right)_0\delta\theta+\left(\frac{\partial T}{\partial\dot{\phi}}\right)_0\delta\phi+........ \qquad ...(9)$$

Similarly

$$\sum m\left(\dot{x}_1\delta x_1+\dot{y}_1\delta y_1+\dot{z}_1\delta z_1\right)=\left(\frac{\partial T}{\partial\dot{\theta}}\right)_1\delta\theta+\left(\frac{\partial T}{\partial\dot{\phi}}\right)_1\delta\phi+........ \qquad ...(10)$$

Also,

$$\sum \left(X\delta x + Y\delta y + Z\delta z\right)$$

$$= \sum \left\{\left(X\frac{\partial x}{\partial\theta} + Y\frac{\partial y}{\partial\theta} + Z\frac{\partial z}{\partial\theta}\right)\delta\theta + \left(X\frac{\partial x}{\partial\phi} + Y\frac{\partial y}{\partial\phi} + Z\frac{\partial z}{\partial\phi}\right)\delta\phi + \ldots\ldots\right\}$$

$$= P\delta\theta + Q\delta\phi + \ldots \qquad \ldots(11)$$

where $P, Q, \ldots\ldots$ are the generalised components of the impulse.

Hence, with the help of (9), (10), and (11), the equation (6) takes the form

$$\left\{\left(\frac{\partial T}{\partial\dot{\theta}}\right)_1 - \left(\frac{\partial T}{\partial\dot{\theta}}\right)_0\right\}\delta\theta + \left\{\left(\frac{\partial T}{\partial\dot{\phi}}\right)_1 - \left(\frac{\partial T}{\partial\dot{\phi}}\right)_0\right\}\delta\phi + \ldots = P\delta\theta + Q\delta\phi + \ldots\ldots(12)$$

Since $\delta\theta, \delta\phi, \ldots\ldots$ are independent, therefore, we get

$$\left(\frac{\partial T}{\partial\dot{\theta}}\right)_1 - \left(\frac{\partial T}{\partial\dot{\theta}}\right)_0 = P$$

$$\left(\frac{\partial T}{\partial\dot{\phi}}\right)_1 - \left(\frac{\partial T}{\partial\dot{\phi}}\right)_0 = Q \qquad \text{etc.} \qquad \ldots(13)$$

there being one equation corresponding to each of the generalised coordinates of the system. These equations are the Lagrange's equations for impulsive forces.

Example 8 : Three equal uniform rods AB, BC, CD, each of the mass m and length $2a$, are at rest in a straight line smoothly jointed at B and C. A blow I is given to the middle rod at distance c from its centre O in a direction perpendicular to it; show that the initial velocity of O is $\frac{2I}{3m}$, and that the initial angular velocities of the rods are

$$\frac{(5a+9c)I}{10ma^2}, \frac{6cI}{5ma^2} \text{ and } \frac{(5a-9c)I}{10ma^2}.$$

Solution :

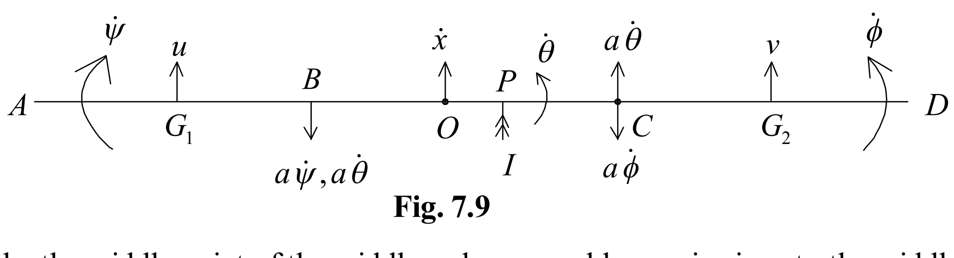


Fig. 7.9

Let O be the middle point of the middle rod BC, A blow I is given to the middle rod in a direction perpendicular to it at the point P, such that $OP = c$. Let $\dot{x}$ and $\dot{\theta}$ be the linear and angular velocities of the rod BC just after the impulse. Also let $\dot{\psi}$ and $\dot{\phi}$ be the angular velocities of the rods AB and CD respectively at time t. Let u and v be the linear velocities of $C.G.$ G_1 of rod AB and that of

G_2 of rod CD respectively.

Now,

Velocity of end B of the rod AB in a direction $\perp$ to it

= velocity of end B of the rod BC in a direction $\perp$ to it.

or $\quad u - a\,\dot{\psi} = \dot{x} - a\dot{\theta}$

or $\quad u = \dot{x} - a\dot{\theta} + a\,\dot{\psi}$ ...(1)

Similar

velocity of end C of the rod BC in a direction $\perp$ to it

= velocity of end B of the rod CD in a direction $\perp$ to it.

or $\quad \dot{x} + a\dot{\theta} = v - a\,\dot{\phi}$

or $\quad v = \dot{x} + a\dot{\theta} + a\,\dot{\phi}$ ...(2)

If T be the $K.E.$ of the system, then we have

$$T = K.E.\text{ of the rod } AB + K.E.\text{ of the rod } BC + K.E.\text{ of the rod } CD$$

$$= \frac{m}{2}\left\{u^2 + \frac{a^2}{3}\dot{\psi}^2\right\} + \frac{m}{2}\left\{\dot{x}^2 + \frac{a^2}{3}\dot{\theta}^2\right\} + \frac{m}{2}\left\{v^2 + \frac{a^2}{3}\dot{\phi}^2\right\}$$

$$= \frac{m}{2}\left[\left(\dot{x} - a\dot{\theta} + a\dot{\psi}\right)^2 + \frac{a^2}{3}\dot{\psi}^2 + \dot{x}^2 + \frac{a^2}{3}\dot{\theta}^2 + \left(\dot{x} + a\dot{\theta} + a\dot{\psi}\right)^2 + \frac{a^2}{3}\dot{\phi}^2\right]$$

or

$$T = \frac{m}{2}\left[3\dot{x}^2 + \frac{7}{3}a^2\dot{\theta}^2 + \frac{4}{3}a^2\dot{\phi}^2 + \frac{4}{3}a^2\dot{\psi}^2 + 2a\dot{x}\dot{\phi} + 2a\dot{x}\dot{\psi} + 2a^2\dot{\theta}\dot{\phi} - 2a^2\dot{\theta}\dot{\psi}\right]$$

...(3)

Since before impulse the system was at rest.

$\therefore$ $K.E.$ of the system just before impulse $(T)_0 = 0$

Also If $\delta U = I\,(\delta x + c\,\delta\theta)$ ...(3)

δU is the virtual work of the impulse, then on applying Lagrange's equations, we get

Lagrange's x equation gives

$$\left(\frac{\partial T}{\partial \dot{x}}\right)_1 - \left(\frac{\partial T}{\partial \dot{x}}\right)_0 = \text{coefficient of } \delta x \text{ in } \delta U$$

or $\quad \dfrac{m}{2}\left(6\dot{x} + 2a\dot{\phi} + 2a\dot{\psi}\right) - 0 = I$

or $$3\dot{x} + a\dot{\phi} + a\dot{\psi} = \frac{I}{m}$$ ...(4)

Similarly, Lagrange's θ - equation gives

$$\frac{m}{2}\left(\frac{14}{3}a^2\dot{\theta}^2 + 2a^2\dot{\phi}^2 - 2a^2\dot{\psi}^2\right) - 0 = Ic$$

or $$\frac{7}{3}a\dot{\theta} + a\dot{\phi} - a\dot{\psi} = \frac{Ic}{am}$$ ...(5)

Lagrange's ϕ-equation gives

$$\frac{m}{2}\left(\frac{8}{3}a^2\dot{\phi} + 2a\dot{x} + 2a^2\dot{\theta}^2\right) - 0 = 0$$

or $$\dot{x} + a\dot{\theta} + \frac{4}{3}a\dot{\phi} = 0$$ ...(6)

and Lagrange's ψ -equation gives

$$\frac{m}{2}\left(\frac{8}{3}a^2\dot{\psi} + 2a\dot{x} - 2a^2\dot{\theta}\right) - 0 = 0$$

or $$\dot{x} - a\dot{\theta} + \frac{4}{3}a\dot{\psi} = 0$$ ...(7)

Now, adding equation (6) and (7), we get

$$2\dot{x} + \frac{4}{3}a\dot{\phi} + \frac{4}{3}a\dot{\psi} = 0$$

or $$\frac{3}{2}\dot{x} + a\dot{\phi} + a\dot{\psi} = 0$$ ...(8)

Again, subtacting equation (8) from equation (4), we get

$$\frac{3}{2}\dot{x} = \frac{I}{m}$$

or $$\dot{x} = \frac{2I}{3m}$$ ...(9)

which gives the initial velocity of 0.

Similarly, subtacting equation (7) from equation (6), we get

$$2a\dot{\theta} + \frac{4}{3}a\dot{\phi} - \frac{4}{3}a\dot{\psi} = 0$$

or $$\frac{3}{2}a\dot{\theta} + a\dot{\phi} - a\dot{\psi} = 0$$ ...(10)

Subtracting this from the equation (5), we get

$$\left(\frac{7}{3}-\frac{3}{2}\right)a\dot{\theta}=\frac{Ic}{am}$$

or $$\dot{\theta}=\frac{6cI}{5ma^2}, \qquad ...(11)$$

which gives the angular velocity of the middle rod BC.

Substituting the values of $\dot{x}$ from (9) and $\dot{\theta}$ from (11) in equation (7), we get

$$\frac{2I}{3m}-\frac{6cI}{5ma}+\frac{4}{3}a\dot{\psi}=0$$

or $$\dot{\psi}=-\frac{(5a-9c)\,I}{10ma^2}, \qquad ...(12)$$

which gives the angular velocity of the rod CD.

Similarly, on substituting the values of $\dot{x}$ and $\dot{\theta}$ in equation (6), we get

$$\dot{\phi}=-\frac{(5a+9c)\,I}{10ma^2}, \qquad ...(13)$$

which gives the angular velocity of the rod AB.

Example 9 : Three equal uniform rods AB, BC, CD are smoothly jointed at B and C and the ends A and D are fastened to smooth fixed points whose distance apart is equal to the length of either rod. The frame being at rest in the form of a square. A blow I is given perpendicular to AB at its middle point and in the plane of the square. Show that the energy set up is $\frac{3I^2}{4om}$, where m is the mass of each rod. Find also the blows at the joints A and C.

Solution :

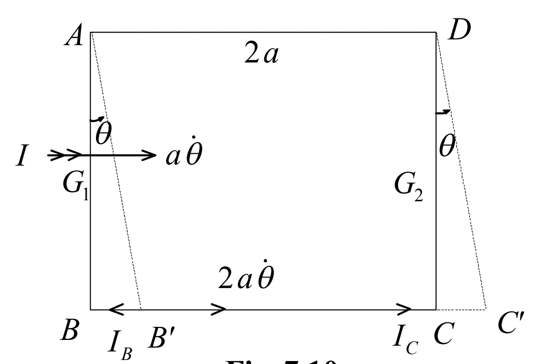


Fig. 7.10

Here m is the mass of each of the rods AB, BC, CD each of length $2a$. Also let A and D are fixed points such that $AD = 2a$ so that $ABCD$ form a square. The blow I is given at the middle point G_1 of the rod AB in a direction perpendicular to it. After the blow, the rods AB and CD will turn through the same angle say θ and the rod BC will remain parallel to AD. Therefore, just after the application of flow, Velocity of rod $AB =$ velocity of rod $DC = a\dot{\theta}$ and velocity of the rod $BC = 2a\dot{\theta}$

Now, if T be the $K.E.$ of the system just after the blow then

$$T = K.E. \text{ of the rod } AB + K.E. \text{ of the rod } DC + K.E. \text{ of the rod } BC$$

$$= 2\,(K.E. \text{ of the rod } AB) + K.E. \text{ of the rod } BC$$

$$= 2\left\{\frac{m}{2}\left(a^2\dot{\theta}^2 + \frac{a^2}{3}\dot{\theta}^2\right)\right\} + \frac{m}{2}\left(2a\dot{\theta}\right)^2$$

or $$T = \frac{10}{3}ma^2\dot{\theta}^2 \qquad ...(1)$$

and if δu be the virtual work done by the impulse, then

$$\delta u = I.a\,\delta\theta \qquad ...(2)$$

Since, before the impulse the system was at rest, therefore $K.E.$ just before the impulse was zero.

Now, Lagrange's θ-equation gives

$$\left(\frac{\partial T}{\partial\dot{\theta}}\right)_1 - \left(\frac{\partial T}{\partial\dot{\theta}}\right)_0 = \text{coefficient of } \delta\theta \text{ in } \delta u$$

or $$\frac{20}{3}ma^2\dot{\theta} - 0 = Ia$$

or $$\dot{\theta} = \frac{3I}{20ma} \qquad ...(3)$$

Substituting this value of $\dot{\theta}$ in (1), we get the required energy setup by blow is

$$T = \frac{10}{3}ma^2\left(\frac{3I}{20ma}\right)^2$$

or $$T = \frac{3I^2}{40m} \qquad ...(4)$$

Again let I_B and I_C are the impulses at the ends B and C as shown in the figure.

Now, considering the motion of the rod AB and taking moments about A, we have

Change in angular momentum about the axis through A

= moments of the impulses about this axis

or $$m\,.\,\frac{4}{3}a^2\dot{\theta} = I.a - I_B\,.\,2a$$

or $$I_B = \frac{I}{2} - \frac{2}{3}\, m\, a\, \dot{\theta} = \frac{I}{2} - \frac{2}{3}\, m\, a . \left(\frac{3I}{20\, m\, a} \right) \qquad \text{(form (3))}$$

or $$I_B = \frac{2}{5} I$$

Again considering the motion of the rod CD and taking moment about D, we have

$$m \, . \, \frac{4}{3} a^2 \dot{\theta} = I_C \, . \, 2a \;\; \text{or} \;\; I_C = \frac{I}{10} \qquad \text{(form (3))}$$

7.7 Summary

To solve dynamical poblems of conservative or non-conservative system kinetic energy forces and couples and the generalised coordinates of the components of this system have been found to be very useful. In this unit the degrees of freedom, generalized coordinates have been explained. The Lagrange's equations have been established for the finite and impulsive froces. Principle of energy has been derived using Lagranges equations moreover the Lagranges equations are used in case of small oscillation. For every type of above a good number of examples have been solved.

7.8 Answers to self learning exercise

Exercise

1. (B) (2) Holonomous (3) Potential function

7.9 Exercise

1. A uniform bar, of length $2a$, is hung from a fixed point by a string of length b fastened to one end of the bar. Show that when the system makes small oscillation in a vertical plane, the length l of the simple equivalent pendulum is a root of the quadratic

$$l^2 - \left(\frac{4a}{3} + b \right) l + \frac{ab}{3} = 0$$

2. At the lowest point of a smooth circular tube, of mass M and radius a, is placed a particle of mass m; the tube hangs in a vertical plane from its highest point, which is fixed and can turn freely in its own plane about this point. If the system be slightly displaced, show that the periods of the two independent oscillations of the system are

$$2\pi \sqrt{\left(\frac{2a}{g} \right)} \text{ and } 2\pi \sqrt{\left(\frac{M}{M+m} \right) \frac{a}{g}}.$$

3. A uniform rod, of length $2a$, which has one end attached to a fixed point by a light inextensible string, of length $\left(\frac{5}{12} \right) a$, in performing small oscillations in a vertical plane about its position of equilibrium. Find its position at any time and show that the periods of its principal oscillations are

$$2\pi \sqrt{\frac{5a}{3g}} \text{ and } \pi \sqrt{\frac{a}{3g}}$$

4. A particle P moves on a smooth horizontal circular wire of radius a, which is free to rotate about a vertical axis through a point O, distance c from the centre C. If the $\angle PCO = \theta$, show that

$$a\ddot{\theta} + \dot{w}(a - c\cos\theta) = c\,w^2 \sin\theta,$$

where w is the angular velocity of the wire.

5. Two heavy particles m, m' are attached to the points A, B of a light inextensible string, of which the upper extermity O is fixed. Prove that the periods of small oscillations are $\frac{2\pi}{n}$, where n is given by

$$n^4 - n^2\left(\frac{m+m'}{m}\right)\left(\frac{1}{l} + \frac{1}{l'}\right)g + \left(\frac{m+m'}{m\,l\,l'}\right)g^2 = 0,$$

where $OA = l,\ AB = l'$.

6. Six equal uniform rods form a regular hexagon, loosely jointed at the angular points, and rest on a smooth table; a blow is given perpendicularly to one of them at its middle point; find the resulting motion and show that the opposite rod begins to move with one tenth of the velocity of the rod that is struck.

□□□□

UNIT - 8
Motion of a top

Structure of the unit

8.0 Objective

The objective of this unit is to understand a top and its motion about the axis of symmerty, The stability of this motion in the cases when the axis of the top can either be vertical or non-vertical is to be understood.

8.1 Indtroduction

In this unit we shall study the motion of a top, defined here in, about its axis of symmetry on a plane which is rough enough to prevent slipping.

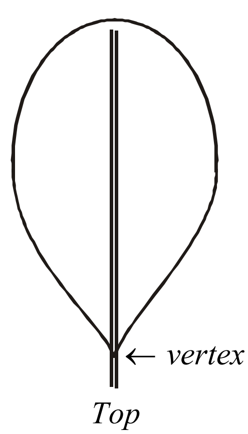


Fig. 8.1

Top : A rigid body which is sysmmetrical about an axis and teminates at one end of the axis in a sharp point is known as a top or a gyro-state. The sharp point is called the vertex of the top. The top thus is a sysmmertical solid of revolution whose centre of gravity lies on its axis.

8.2 Equation of Motion of a Top

A top, two of whose principal moments about the centre of inertia are equal due to its symmetry about its axis, moves under the action of gravity about a fixed point O in the axis of unequal moment the top be initially set spinning about its axis which was initially at rest.

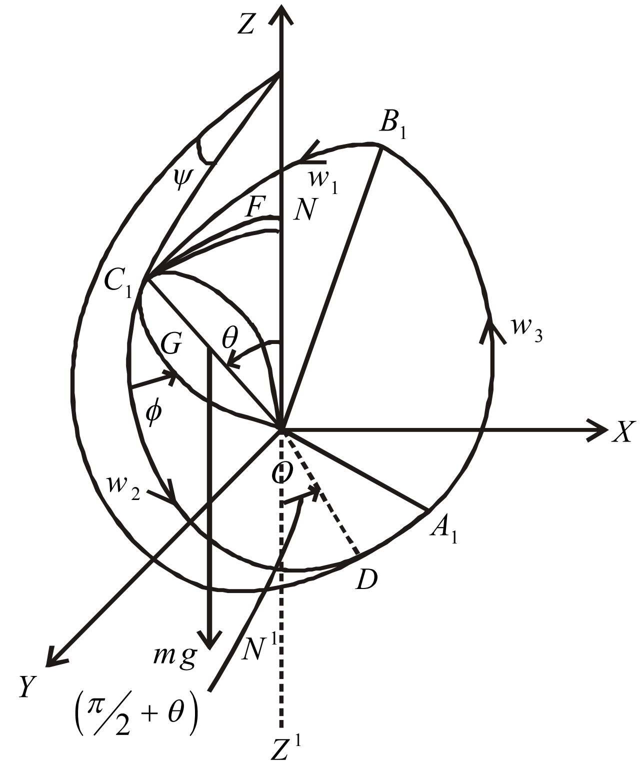


Fig. 8.2

Here we shall apply Lagrange's equations to determine the equation of motion of the top

Let a top spin with its vertex O fixed in a contact with a floor rough enough to prevent slipping. Let OC_1 be the axis of the top and G its centre of gravity, OZ the vertical is such that in which the axis OZ was at zero time, OY and OX are horizontal and are at right angles.

Let OA_1, OB_1 be two perpendicular lines, each perpendicular to OC_1, (OA_1, OB_1, OC_1 are principal axes at O). Let A be the moment of inertia about OA_1 or OB_1, and C that about OC_1.

At time t, let OC_1 be inclined at an angle θ to the vertical OZ and let the plane ZOC_1 be turned through an angle ψ from its initial position. Also let ϕ be the angle between the plane ZOC_1 and the plane C_1OA_1. Thus θ, ϕ, ψ are the Eulerian angles at O. Again let w_1, w_2, w_3 are the angular velocities of the top about OA_1, OB_1 and OC_1 respectively, then the Euler's geometrical relations i.e. relations between θ, ϕ, ψ and w_1, w_2, w_3 are given by

$$w_1 = \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi$$

$$w_2 = \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi$$

$$w_3 = \dot{\phi} + \dot{\psi}\cos\theta \qquad ...(1)$$

Now, if T be the K.E. of the system. then

$$T = \frac{1}{2}\left(A w_1^2 + A w_2^2 + C w_3^2\right)\{\because A, B, C \text{ are principal moments of inertia and } A = B\}$$

$$= \frac{1}{2}\left\{A\left(w_1^2 + w_2^2\right) + C w_3^2\right\}$$

$$T = \frac{1}{2}\left\{A\left(\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta\right) + C\left(\dot{\phi} + \dot{\psi}\cos\theta\right)^2\right\} \qquad ...(2)$$

Also, if W be the work function of the system, then

$$W = \text{constant - } mgh\cos\theta$$

or $$W = mgh(\cos i - \cos\theta) \qquad ...(3)$$

where m is the mass of the top, $h = OG$ and i be the initial value of θ.

Now, here θ, ϕ and ψ are taken as generalised coordinates of the system, then applying Lagrange's equations,

θ - equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial w}{\partial \theta}$$

i.e. $$\frac{d}{dt}\left(A\dot{\theta}\right) - A\dot{\psi}^2 \sin\theta\cos\theta - C\left(\dot{\phi} + \dot{\psi}\cos\theta\right)(-\dot{\psi}\sin\theta) = mgh\sin\theta$$

or $$A\ddot{\theta} - A\dot{\psi}^2\sin\theta\cos\theta + C\dot{\psi}\left(\dot{\phi} + \dot{\psi}\cos\theta\right)\sin\theta = mgh\sin\theta \qquad ...(4)$$

similarly, Lagrange's ϕ-equation gives

$$\frac{d}{dt}\left\{C\left(\dot{\phi} + \dot{\psi}\cos\theta\right)\right\} = 0$$

or $$\dot{\phi} + \dot{\psi}\cos\theta = \text{constant} = \text{n (say)} \qquad ...(5)$$

and Lagrange's ψ -equation gives

$$\frac{d}{dt}\left\{A\dot{\psi}\sin^2\theta + C\cos\theta\left(\dot{\phi} + \dot{\psi}\cos\theta\right)\right\} = 0$$

or $$A\dot{\psi}\sin^2\theta + C\cos\theta\left(\dot{\phi} + \dot{\psi}\cos\theta\right) = \text{constant} = \text{D (say)} \qquad ...(6)$$

with the help of equation (5), the equations (4) and (6) become

$$A\ddot{\theta} - A\dot{\psi}^2\sin\theta\, C\cos\theta + Cn\dot{\psi}\sin\theta = mgh\sin\theta \qquad ...(7)$$

and $$A\dot{\psi}\sin^2\theta + Cn\cos\theta = D \qquad ...(8)$$

Also, $T = \frac{1}{2}\left\{A\left(\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta\right) + cn^2\right\}$ since the top is rotating under gravity which is a conservative system of forces, the law of conservation of energy gives

$T + V =$ constant, where $V =$ constant - w

thus $T =$ constant + w

or $A\left(\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta\right) + cn^2 = \text{constant} - 2\,m\,g\,h\cos\theta$

or $A\left(\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta\right) + 2\,m\,g\,h\cos\theta = \text{constant} = \text{E}$ (say) ...(9)

where E is any constant which includes cn^2.

Thus the equations (7), (8) and (9) are the equations of motion of a top. The constant D and E in the equations of motion of a top are usually determined from the initial values given for the motion.

Note : (1) Generally the equation (8) which is also known as angular momentum equation and equation (9), which is energy equation are considered as equations of motion of a top.

(2) The constant D and E are not the products of inertia but simple constants.

(3) The motion of the axis of a top due to change in 'θ' is called **nutation** and due to change in ψ is is called **precession.** The general motion of the top about its fixed vertex O is a combination of these two when it is given a constant angular velocity $w_3(= n)$, called **spin** about its axis.

8.3 Steady Motion of a Top

The motion of a top is said to be steady if it spins about its vertex O in such a way that its axis OC_1 makes the same angle with the vertical OZ throughout the motion. In other words, when the axis of the top describes a cone around the vertical with constant angular velocity at a constant angle to the vertical, then the motion of the top is said to be steady motion.

Now for steady motion of the top

$\theta = \text{constant} = \alpha$ (say), then

$\dot{\theta} = 0 = \ddot{\theta}$, Also $\dot{\psi}\cos\theta = \text{constant}$ $\left(\because \dot{\phi} = 0\right)$

or $\dot{\psi} = \text{constant} = w$ (say)

Therefore, from the θ-equation of the motion of the top i.e. from $A\ddot{\theta} - A\dot{\psi}^2 \sin\theta\cos\theta + Cn\dot{\psi}\sin\theta = m\,g\,h\sin\theta$ we get

$$-A\cos\alpha\sin\alpha\; w^2 + Cn\,w\sin\alpha = m\,g\,h\sin\alpha$$

or $A\cos\alpha\, w^2 - Cn\,w + m\,g\,h = 0$ ...(10)

Thus there are two possible real values of w, if

$C^2n^2 > 4\,A\,m\,g\,h\cos\alpha$ ...(11)

This is the necessary condition for steady motion of a top and the angular velocities are called precessional angular velocities.

Particular Case : If $\alpha = \pi/2$, then from equation (10), we have

$$w = \left(\frac{mgh}{Cn}\right) \quad ...(12)$$

Then, if the top is given an angular velocity n about OC_1, when OC_1 is horizontal and OC_1 be given an angular velocity $\left(\frac{mgh}{Cn}\right)$ about OZ, then OC_1 will continue to revolve uniformaly in a horizontal plane round the vertical OZ Again, if the top is set in motion in the usual manner and n is very large, then from equation (10), we get

$$w = \frac{Cn \pm \sqrt{C^2n^2 - 4Amgh\cos\alpha}}{2A\cos\alpha}$$

$$= \frac{1}{2A\cos\alpha}\left\{Cn \pm Cn\sqrt{1 - \frac{4Amgh\cos\alpha}{C^2n^2}}\right\}$$

$$= \frac{1}{2A\cos\alpha}\left\{Cn \pm Cn\left(1 - \frac{1}{2}\frac{4Amgh\cos\alpha}{C^2n^2}\right)\right\}$$

or $$w = \frac{Cn}{2A\cos\alpha}\left\{1 \pm \left(1 - \frac{2Amgh\cos\alpha}{C^2n^2}\right)\right\} \text{ (nearly)}$$

or $$w = \frac{Cn}{A\cos\alpha} \text{ and } \frac{mgh}{Cn} \text{ (nearly)} \quad ...(13)$$

obviously the first value of w is large and second is very small, since n is considered very large.

8.4 Stability Conditions for the Motion of a Top

8.4.1 When the axis of the top is vertical

A top is spinning with an angular velocity n about an axis which is vertical, to find the condition of stability, if the axis be given a slight nutation. (or a top is executing steady motion with angular velocity n about its axis which is vertical, to show that the motion is stable).

Let the motion of a top be disturbed in such a way that θ is very small i.e. there is slight nutation initially, so the general equation of motion of the top are

$$A\ddot{\theta} - A\dot{\psi}^2\sin\theta\cos\theta + Cn\,\dot{\psi}\sin\theta = mgh\sin\theta \quad ...(1)$$

and $$A\dot{\psi}\sin^2\theta + Cn\cos\theta = D \quad ...(2)$$

For steady motion with axis vertical (initially)

we have $\theta = 0$, $\dot{\psi} = n$

then the equation (2) gives $D = Cn$

But for steady motion, the necessary condition is

$$C^2 n^2 > 4\,A\,m\,g\,h\cos\alpha$$

therefore, when $\alpha = 0$, this reduces to

$$C^2 n^2 > 4\,A\,m\,g\,h$$

Now, let the motion be slightly disturbed. The disturbed motion is a general motion of the top therefore, its equations are

$$A\ddot{\theta} - A\dot{\psi}^2 \sin\theta\cos\theta + C\,n\,\dot{\psi}\sin\theta = m\,g\,h\sin\theta \qquad ...(3)$$

and $$A\dot{\psi}\sin^2\theta + C\,n\cos\theta = C\,n \qquad ...(4)$$

From equation (4), we get

$$A\dot{\psi} = \frac{C\,n\left(1-\cos\theta\right)}{\sin^2\theta} = \frac{C\,n}{1+\cos\theta} \qquad ...(5)$$

with the help of this, the equation (1) becomes

$$A^2\ddot{\theta} - \frac{C^2 n^2}{\left(1+\cos\theta\right)^2}\sin\theta\cos\theta + \frac{C^2 n^2}{\left(1+\cos\theta\right)}\sin\theta = A\,m\,g\,h\sin\theta$$

Since θ is small, therefore writing θ for $\sin\theta$ and 1 for $\cos\theta$, the above equation reduces to

$$A^2\ddot{\theta} - \frac{C^2 n^2}{4}\,\theta + \frac{C^2 n^2}{2}\,\theta = A\,m\,g\,h\,\theta$$

or $$\ddot{\theta} = -\left(\frac{C^2 n^2 - 4\,A\,m\,g\,h}{4\,A^2}\right)\theta \qquad ...(6)$$

Since $C^2 n^2 > 4\,A\,m\,g\,h$, therefore, the coefficient of θ is negative. Thus, the above equation (6) is characteristic of harmonic motion (S.H.M.). Thus the motion of the axis of the top is S.H.M. about the vertical OZ from which θ is measured. In other words, axis of the top, if disturbed slightly from its vertical position is in steady motion, will tend to come back to its original postion. Hence the steady motion, in which axis of the top is vertical, is stable,

The period of nutation is

$$= 2\pi\sqrt{\frac{4\,A^2}{C^2 n^2 - 4\,A\,m\,g\,h}}$$

$$= \frac{4\,A\,\pi}{\sqrt{C^2 n^2 - 4\,A\,m\,g\,h}} \qquad ...(7)$$

8.4.2 When the Axis of top is not Vertical

A top is executing steady motion, with its axis, inclined at a constant angle α to the vertical, and precessional velocity w; to show that motion is stable.

a top is executing steady motion with a constant spin n about its axis which is inclimed at a constant angle α to the vertical, then initially $\theta = \alpha$, $\dot{\theta} = 0$, $\dot{\psi}$ =constant = w (say), then the equation of motion of the top (1) and (2) give

$$A w^2 \cos\alpha - C n w + m g h = 0 \qquad ...(8)$$

and $$D = A w \sin^2\alpha + C n \cos\alpha \qquad ...(9)$$

Let the steady motion of the top be slightly disturbed so that motion of the top then are general and so its equations are

$$A\ddot{\theta} - A\dot{\psi}^2 \sin\theta\cos\theta + C n\dot{\psi}\sin\theta = m g h \sin\theta \qquad ...(10)$$

and $$A\dot{\psi}\sin^2\theta + C n\cos\theta = A w \sin^2\alpha + C n\cos\alpha \qquad ...(11)$$

eliminating $\dot{\psi}$ between (10) and (11), we see

$$A^2 \sin^3\theta\ddot{\theta} - \left\{A w \sin^2\alpha + C n(\cos\alpha - \cos\theta)\right\}^2 \cos\theta + C n \sin^2\theta \times$$

$$\left\{A w \sin^2\alpha + C n(\cos\alpha - \cos\theta)\right\} = A m g h \sin^4\theta \qquad ...(12)$$

Now, put $\theta = \alpha + \in$, where $\in$ is small (i.e. $\in << 1$) then neglecting quantities of second and higher order of $\in$ and on simplification the abov equation (12) becomes

$$\ddot{\in} = -\left\{w^2 - \frac{2mgh}{A}\cos\alpha + \left(\frac{mgh}{Aw}\right)^2\right\}\in$$

or $$\ddot{\in} = -\left\{\frac{\left(A w^2 - m g h \cos\alpha\right)^2 + m^2 g^2 h^2 \sin^2\alpha}{A^2 w^2}\right\}\in \qquad ...(13)$$

which shows that the coefficient of $\in$ is always negative. Therefore, the equation (13) represents a S.H.M. which shows that the axis of the top, if disturbed slightly from its position in steady motion, will tend to come back to the original position i.e. the steady motion, with axis inclined at an angle α to the vertical, is stable. The period of this small oscillation is

$$= 2\pi\sqrt{\frac{A^2 w^2}{\left\{A^2 w^4 - 2 A m g h w^2 \cos\alpha + (m g h)^2\right\}}} \qquad ...(14)$$

In case the top is set in motion such that n is very large and the two values of the precessional velocity are appronimately given as

$$w = \frac{Cn}{A\cos\alpha} \text{ or } \frac{mgh}{Cn}$$

therefore, when n is large, the first value is large and the second is small as shown earlier. For a small oscillation w is considered small then the period of oscillation is given by

$$\frac{2\pi A w}{m g h} \qquad \text{(neglecting } w^2 \text{ and } w^4 \text{ in the denominater of (14))}$$

which when considered with (8), wherein on neglecting w^2, $w = \dfrac{mgh}{Cn}$ we

get the form of this period as $\dfrac{2\pi A}{Cn}$

8.5 Limits of θ

A top is set spinning with angular velocity n about its axis. Initially the axis being instantaneously at rest and inclined at an angle i to the vertical; discuss the range of angle θ, between the axis of top and vertical, in which its subsequent motion is included.

Initially given that $\theta = i,\ \dot{\theta} = 0,\ \dot{\psi} = 0,\ w_3 = n$ then the equation of motion of the top

$$A\dot{\psi}\, sin^2\theta + Cn\, cos\theta = Cn\, cos i \qquad ...(1)$$

and $$A\left(\dot{\theta}^2 + \dot{\psi}^2 sin^2\theta\right) = 2mgh\left(cos i - cos\theta\right) \qquad ...(2)$$

eliminating $\dot{\psi}$ from (1) and (2), we get

$$A^2 \sin^2\theta\dot{\theta}^2 = A\sin^2\theta\ 2mgh\ (\cos i - \cos\theta) - n^2C^2\ (\cos i - \cos\theta)^2$$

$$= 2Amgh\left(cos i - cos\theta\right)\left\{sin^2\theta - 2p\left(cos i - cos\theta\right)\right\}$$

$$= 2Amgh\ (\cos\theta - \cos i)\left\{(\cos\theta - p)^2 - \left(p^2 - 2p\cos i + 1\right)\right\}$$

where $$p = \frac{n^2C^2}{4Amgh} \qquad ...(3)$$

Now, $\dot{\theta}$ will vanish when $\theta = i$ or θ_1 or θ_2, where

$$cos\theta_1 = p - \sqrt{p^2 - 2p\,cos i + 1} \qquad ...(4)$$

or $$cos\theta_2 = p + \sqrt{p^2 - 2p\,cos i + 1} \qquad ...(5)$$

since p is always positive and if $o < i \le \pi/2$,

$$p^2 - 2p\,cos i + 1 \le 1 + p^2$$

therefore, $cos\theta_2 > 1$, which means θ_2 is imaginary and is not admissible

Also $$p - cos i < \sqrt{\left(p - cos i\right)^2 + \left(1 - cos^2 i\right)}$$

$$< \sqrt{p^2 - 2p\,cos i + 1}$$

therefore $p - cos\, i < p - cos\,\theta_1$ (From (4))

i.e. $cos\,\theta_1 < cos\, i$

or $\theta_1 > i$

Again, it may be noted that if $cos\,\theta > cos\, i$, i.e. $\theta < i$, the equation (3) implies that $\dot{\theta}^2$ is negative which will give an imaginary value of $\dot{\theta}$, and is therefore, inadmissible. Thus θ cannot be less than i.

Hence the top is never at an inclination less than i or at a greater inclination than θ_1 i.e. the motion is included between these limits. Thus

$$i \leq \theta \leq cos^{-1}\left\{p - \sqrt{p^2 - 2\,p\,cos\,i + 1}\right\}$$

Illustrative Examples :

Example 1 : A circular disc, of radius a, has a thin rod pushed through its centre perpendicular to its plane, the length of the rod being equal to the radius of the disc. Show that the system can not spin with the rod vertical unless the angular velocity is greater than

$$\sqrt{\frac{2og}{a}}$$

Solution : Here the body formed by pushing a rod into a circular disc at its centre and perpendicular to the plane, be a top with one free end of the rod be a vertex. At any time t, let the rod i.e. axis of the top makes an angle θ with the vertical OZ.

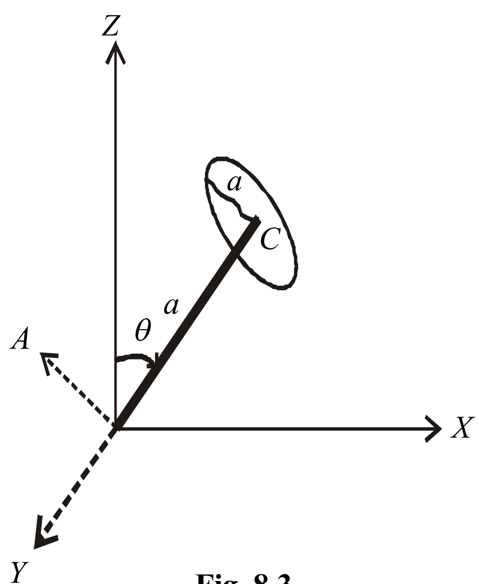


Fig. 8.3

Now, we know that for a steady motion of a top,

$$C^2 n^2 > 4\,A\,m\,g\,h\cos\alpha \qquad ...(1)$$

where A is the moment of inertia of the top about an axis through its vertex and perpendicular to its axis, C be the M.I. about its axis OC, h be the distance of the CG of the top from the vertex, m be the mass of top and α be the inclination of axis of the top with vertical .

Here, we have

A = M.I. of the top about an axis perpendicular to its axis OC and passing through vertex O.

$$= \frac{1}{4} ma^2 + ma^2 \qquad \text{(by parallel theorem)}$$

$$= \frac{5}{4} ma^2$$

C = M.I. of the top about its axis OC

$$= \frac{ma^2}{2}$$

$h = a$ $\qquad$ ($\because$ CG of the top is at C)

$\alpha = 0$ $\qquad$ (rod is vertical)

$\therefore$ the top will execute steady motion with axis vertical, if

$$C^2n^2 > 4\,A\,m\,g\,h$$

or $$\left(\frac{1}{2} ma^2\right)^2 . n^2 > 4 . \frac{5}{4} ma^2 . mg . a$$

or $$n^2 a > 2o\,g$$

or $$n > \sqrt{\frac{2o\,g}{a}}$$

Example 2 : If initially the axis of the top is horizontal and it is set spinning with angular velocity w in a horizontal plane, prove that the axis will start to rise if $nCw > mgh$ and that, when $nCw = 2mgh$, the axis will rise to an angular distance $cos^{-1}\left(\frac{Aw}{nc}\right)$, provided that $Aw < nC$, and will there be at instantaneous rest. A, C and n have their usual meanings.

Solution : The equations of motion of a top are

$$A\ddot{\theta} - A\dot{\psi}^2 \sin\theta \cos\theta + Cn\dot{\psi}\sin\theta = mgh\sin\theta \qquad ...(1)$$

$$A\dot{\psi}\sin^2\theta + Cn\cos\theta = D \qquad ...(2)$$

and $$A\left(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta\right) + 2mgh\cos\theta = E \qquad ...(3)$$

Initially

$$\theta = \pi/2,\ \dot{\theta} = 0,\ \dot{\psi} = w \qquad \text{(given)}$$

Now, the axis of the top will start rising from the initial position if the initial value of $\ddot{\theta}$ is negative, therefore from equation (1), we get

$$A\left(\ddot{\theta}\right)_{\text{initial}} = -\left(Cnw - mgh\right) \qquad \text{...(4)}$$

thus, $\left(\ddot{\theta}\right)_{\text{initial}}$ will be negative only when $Cnw > mgh$. Further, applying initial conditions to equations (2) and (3), we get

$D = Aw$ and $E = A\dot{\psi}^2$

Thus, the equation of motion of the top (2) and (3) become

$$A\,\dot{\psi}\,sin^2\theta + Cn\,cos\theta = Aw \qquad \text{...(5)}$$

and $$A\dot{\theta}^2 + A\dot{\psi}^2\,sin^2\theta + 2mgh\,cos\theta = Aw^2 \qquad \text{...(6)}$$

Eliminating $\dot{\psi}$ between equations (5) and (6), we get

$$A\dot{\theta}^2 + A\,sin^2\theta\left\{\frac{Aw - Cn\,cos\theta}{A\,sin^2\theta}\right\}^2 + 2mgh\,cos\theta = Aw^2$$

or $$A^2\,sin^2\theta\,\dot{\theta}^2 + \left(Aw - Cn\,cos\theta\right)^2 + 2Amgh\,cos\theta\,sin^2\theta = A^2w^2\,sin^2\theta$$

If $2mgh = Cnw$, then we have

$$A^2\,sin^2\theta\,\dot{\theta}^2 + \left(Aw - Cn\,cos\theta\right)^2 = A^2w^2\,sin^2\theta - Cnw\,A\,sin^2\theta\,cos\theta$$

or $$A^2\,sin^2\theta\,\dot{\theta}^2 = Aw\,sin^2\theta\left(Aw - Cn\,cos\theta\right) - \left(Aw - Cn\,cos\theta\right)^2$$

$$= \left(Aw - Cn\,cos\theta\right)\left\{Aw\left(1 - cos^2\theta\right) - \left(Aw - Cn\,cos\theta\right)\right\}$$

$$= \left(Aw - Cn\,cos\theta\right)\left\{Cn\,cos\theta - Aw\,cos^2\theta\right\}$$

or $$A^2\sin^2\theta\,\dot{\theta}^2 = \cos\theta\left(Aw - Cn\cos\theta\right)\left(Cn - Aw\cos\theta\right) \qquad \text{...(7)}$$

Now, for instantaneous rest, $\dot{\theta} = 0$, then from equation (7), the rod will be at instantaneously rest, when

$$cos\theta\left(Aw - Cn\,cos\theta\right)\left(Cn - Aw\,cos\theta\right) = 0 \qquad \text{...(8)}$$

$\Rightarrow$ either $cos\theta = 0$ or $Aw - Cn\,cos\theta = 0$ or $Cn - Aw\,cos\theta = 0$, but when $cos\theta = 0$ or $\theta = \pi/2$, this gives the initial position, and when

$Aw - Cn\,cos\theta = 0$, which gives $\theta = \cos^{-1}\left(\frac{Aw}{Cn}\right)$, provided $Aw < Cn$ and when $Cn - Aw\cos\theta = 0$ which gives

$$\theta = cos^{-1}\left(\frac{Cn}{Aw}\right)$$

which is impossible when $Aw < Cn$. Thus the rod will rise to an angular distance $cos^{-1}\left(\frac{Aw}{Cn}\right)$, provided $Aw < Cn$.

Example 3 : A symmetrical top is set in motion on a rough horizontal plane with an angular motion about its axis of figure, the axis being inclined at an angle i to the vertical. Show that between the greatest approach to and recess from the vertical, the centre of gravity describes an arc

$$h\ tan^{-1}\left(\frac{sin\,i}{p - cos\,i}\right)$$

where p and h have their usual meanings.

Solution : The equations of motion of the top are

$$A\dot{\psi}\sin^2\theta + Cn\cos\theta = D \qquad ...(1)$$

and $$A\left(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta\right) + 2mgh\cos\theta = E \qquad ...(2)$$

where D and E are constants, which can be determined by initial conditions. Here, the initial condition are when $\theta = i$, $\dot{\theta} = 0$, $\dot{\psi} = 0$, therefore, equations (1) and (2) give $D = Cn\,cos\,i$ and $E = 2mgh\,cos\,i$ thus, the equations (1) and (2) become

$$A\dot{\psi}\,sin^2\theta = Cn(cos\,i - cos\,\theta) \qquad ...(3)$$

and $$A\left(\dot{\theta}^2 + \dot{\psi}^2\,sin^2\theta\right) = 2mgh(cos\,i - cos\,\theta) \qquad ...(4)$$

eliminating $\dot{\psi}$ from equations (3) and (4), we get

$$A\dot{\theta}^2 + A\frac{c^2n^2(cos\,i - cos\,\theta)^2}{A^2\,sin^4\theta} = 2mgh(cos\,i - cos\,\theta)$$

or $$A^2\,sin^2\theta\,\dot{\theta}^2 = A\,sin^2\theta\;2mgh(cos\,i - cos\,\theta) - c^2n^2(cos\,i - cos\,\theta)^2$$

if $c^2n^2 = 4Amgh.p$, then, we have

$$A^2\,sin^2\theta\,\dot{\theta}^2 = 2Amgh(cos\,i - cos\,\theta)\left\{sin^2\theta - 2p\,(cos\,i - cos\,\theta)\right\}$$

$$= 2Amgh(\cos\theta - \cos i)\left\{2p\cos i - 2p\cos\theta - 1 + \cos^2\theta\right\}$$

or $$A\,sin^2\theta\,\dot{\theta}^2 = 2mgh(cos\,\theta - cos\,i)\left\{(p - cos\,\theta)^2 - (1 - 2p\,cos\,i + p^2)\right\}$$

or $$A\sin^2\theta\,\dot{\theta}^2 = 2mgh(\cos\theta - \cos i)\left\{(p - \cos\theta)^2 - \lambda^2\right\}$$

where $\lambda^2 = 1 - 2\,p\,cos\,i + p^2$

or $$A\,sin^2\,\theta\,\dot{\theta}^2 = 2\,m\,g\,h\left(cos\,i - cos\,\theta\right)\left\{\lambda^2 - \left(p - cos\,\theta\right)^2\right\}$$

or $$2\,m\,g\,h\left(cos\,i - cos\,\theta\right) = \frac{A\,sin^2\,\theta\,\dot{\theta}^2}{\lambda^2 - \left(p - cos\,\theta\right)^2} \qquad ...(5)$$

with the help of equation (5), the equation (4) becomes

$$A\left(\dot{\theta}^2 + \dot{\psi}^2\,sin^2\,\theta\right) = \frac{A\,sin^2\,\theta\,\dot{\theta}^2}{\lambda^2 - \left(p - cos\,\theta\right)^2}$$

or $$\dot{\theta}^2 + \dot{\psi}^2\,sin^2\,\theta = \frac{sin^2\,\theta\,\dot{\theta}^2}{\lambda^2 - \left(p - cos\,\theta\right)^2}$$

Dividing both the sides by $\dot{\theta}^2$, we get

$$1 + \left(\frac{d\psi}{d\theta}\right)^2 sin^2\,\theta = \frac{sin^2\,\theta}{\lambda^2 - \left(p - cos\,\theta\right)^2} \qquad ...(6)$$

But from geometry, we know that the length of an elementry are ds is given by

$$ds^2 = h^2\left(d\theta^2 + sin^2\,\theta\,d\dot{\psi}^2\right)$$

or $$\frac{1}{h^2}\left(\frac{ds}{d\theta}\right)^2 = 1 + \left(\frac{d\psi}{d\theta}\right)^2 sin^2\,\theta \qquad ...(7)$$

Therefore with the help of (7), the equation (6) becomes

$$\frac{1}{h^2}\left(\frac{ds}{d\theta}\right)^2 = \frac{sin^2\,\theta}{\lambda^2 - \left(p - cos\,\theta\right)^2}$$

or $$\frac{1}{h}\left(\frac{ds}{d\theta}\right) = \frac{sin\,\theta}{\sqrt{\lambda^2 - \left(p - cos\,\theta\right)^2}}$$

or $$d\,s = \frac{h\,sin\,\theta\,d\theta}{\sqrt{\lambda^2 - \left(p - cos\,\theta\right)^2}} \qquad ...(8)$$

since, we know that $\dot{\theta}$ vanish for $\theta = i$ and $\theta = \theta_1$, where $cos\,\theta_1 = p - \lambda$ or $p - \sqrt{1 - 2\,p\,cos\,i + p^2}$. Therefore, on integrating the equation (8), we get

$$\int_{s=0}^{s} ds = h \int_{\theta=i}^{\theta=\theta_1} \frac{sin\,\theta\,d\theta}{\sqrt{\lambda^2 - \left(p - cos\,\theta\right)^2}}$$

or $$s = h \int_{i}^{\theta_1} \frac{\sin\theta\, d\theta}{\sqrt{\lambda^2 - (p - \cos\theta)^2}}$$

or $$s = h\left[\sin^{-1}\left(\frac{p-\cos\theta}{\lambda}\right)\right]_i^{\theta_1}$$

$$= h\left\{\sin^{-1}\left(\frac{p-\cos\theta_1}{\lambda}\right) - \sin^{-1}\left(\frac{p-\cos i}{\lambda}\right)\right\}$$

$$= h\left\{\sin^{-1}\left(\frac{p-\overline{p-\lambda}}{\lambda}\right) - \sin^{-1}\left(\frac{p-\cos i}{\lambda}\right)\right\} \qquad (\because \cos\theta_1 = p-\lambda)$$

$$= h\left\{\pi/2 - \sin^{-1}\left(\frac{p-\cos i}{\lambda}\right)\right\}$$

or $$s = h\cos^{-1}\left(\frac{p-\cos i}{\lambda}\right)$$

$$= h\tan^{-1}\left\{\frac{\sqrt{\lambda^2-(p-\cos i)^2}}{p-\cos i}\right\}$$

$$= h\tan^{-1}\left\{\frac{\sqrt{1-2p\cos i+p^2-p^2-\cos^2 i+2p\cos i}}{p-\cos i}\right\}$$

$$= h\tan^{-1}\left(\frac{\sin i}{p-\cos i}\right)$$

Example 4 : When the axis of a symmetrical top is stationary and then spin is large and equal to n, a blow J is applied perpendiucalar to the axis at a distance d from the fixed point. Prove that the maximum angular deflection of the axis is approximately $2\tan^{-1}\left(\frac{Jd}{Cn}\right)$, C being the moment of inertia of the top about its axis of symmetry.

Solution : The equations of motion of the top are

$$A\dot{\psi}\sin^2\theta + Cn\cos\theta = D \qquad ...(1)$$

and $$A\left(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta\right) + 2mgh\cos\theta = E \qquad ...(2)$$

Let OK be a line perpendicular to vertical OZ and axis of symmetry of the top, OC both. Thus the angular velocity $\dot{\theta}$ is about OK. Initially, OC is stationary $i.e$. coincident with vertical OZ. Thus OK coincides with OA and $\dot{\theta}$ is about OA. Thus the value of $\dot{\theta}$ before the action of impulse is zero.

When the blow J is applied to a point on OC at a distance d from vertex (fixed point) O, perpendicular to its axis.

Since, the change in angular momentum about any line

= sum of moments of the impulses about the same line.

Therefore, by taking moment about OA, we have

$$A\dot{\theta} = Jd \qquad \text{or} \qquad \dot{\theta} = \frac{Jd}{A}$$

Thus initially $\theta = 0$, $\dot{\theta} = \dfrac{Jd}{A}$ ...(3)

Therefore, applying initial conditions to equations (1) and (2), we get

$$D = Cn \text{ and } \quad E = 2mgh + \frac{J^2 d^2}{A} \qquad ...(4)$$

with the help of (4), the equations (1) and (2) beocome

$$A\dot{\psi}\, sin^2\theta + Cn\,cos\theta = Cn \qquad ...(5)$$

and $$A\left(\dot{\theta}^2 + \dot{\psi}\, sin^2\theta\right) + 2mgh\,cos\theta = 2mgh + J^2d^2/A \qquad ...(6)$$

Eliminating $\dot{\psi}$ between equations (5) and (6), we get

$$A\dot{\theta}^2 + A\left\{\frac{Cn(1-cos\theta)}{A\,sin^2\theta}\right\}^2 sin^2\theta = 2mgh\,(1-cos\theta) + \frac{J^2d^2}{A} \qquad ...(7)$$

But for maximum deviation $\dot{\theta} = 0$, therefore, the equation (7) gives

$$\frac{C^2n^2(1-cos\theta)^2}{A\,sin^2\theta} = 2mgh\,(1-cos\theta) + \frac{J^2d^2}{A}$$

or $$\frac{J^2d^2}{A} = \frac{C^2n^2\left(2\sin^2\theta/2\right)^2}{A\left(2\sin\frac{\theta}{2}\cos\theta/2\right)^2} - 2mg\left(2\sin^2\frac{\theta}{2}\right)$$

or $$\frac{J^2d^2}{A} = \frac{C^2n^2}{A}\tan^2\theta/2 - 4mgh\,\sin^2\frac{\theta}{2}$$

or $$J^2d^2 = C^2n^2\tan^2\frac{\theta}{2}\left\{1 - 4Amgh\,.\,\cos^2\frac{\theta}{2}\right\} \qquad ...(8)$$

Since the spin n is large, therefore, the second term in the bracket on the R.H.S of equation (8) can be neglected, thus, we get

$$J^2 d^2 = C^2 n^2 \tan^2 \frac{\theta}{2}$$

or $$C n \tan \frac{\theta}{2} = J d$$

or $$\theta = 2 \tan^{-1}\left(\frac{J d}{C n}\right)$$

which is the required result.

8.6 Summary

In this unit we have learnt about the motion of a top which has also been defined. The equation of this motion have been derived along with the equation of energy. The steady motion of this top too has been considered for the case when the axis of the top continues to make the same angle with origin. Condition of stability have been discussed when during the motion of the axis of top is (i) vertical (ii) not vertical.

8.7 Exercise

1. Show that in order that it may be possible for a uniform cone of height h and vertical angle 2α, to spin as a top on its vertex on a rough horizontal plane, it is necessary that

$$n^2 > \frac{5g\left(4 + \tan^2 \alpha\right)\sin\alpha}{h \tan^4 \alpha}$$

where $w_3 = n$.

2. If the top be started when its axis makes an angle $\frac{\pi}{3}$ with the upward drawn vertical, so that the initial spin about its axis is $\frac{A}{C}\sqrt{\frac{3Mgh}{A}}$, and the angular velocity of its axis in azimuth is $2\sqrt{\frac{Mgh}{3A}}$, its angular velocity in the meridian plane being initially zero, show that the inclination θ of its axis to the vertical at any time t is given by the equation

$$\sec\theta = 1 + \sec h\left\{t\sqrt{\frac{Mgh}{A}}\right\}$$

so that the axis continually approaches to the vertical without ever reaching it.

3. If a top is made by running a thin pin through the centre of a circular dise of radius 3 inches, so that the length of the pin below the disc is 2 inches, prove that, for steady motion in which the rim does not touch the ground, the number of revolutins per second about the axis must exceed

$$\frac{80}{3\pi\,(13)^{1/4}}\ (=4.5\ approx).$$

4. A top consists of a thin uniform spherical shell of radius a and centre C and is free to move about a fixed point O of its surface. The radius OC makes an angle θ with the upward vertical OZ and ψ is the angle between the plane ZOC and a fixed vertical plane. Initially, $\theta=\frac{\pi}{2},\ \dot{\theta}=0,\ \dot{\psi}=\frac{2}{5}n$, where $n=\sqrt{\frac{15g}{2n}}$ is the spin of the top about its axis. Show that subsequently

$$\cos\theta=\tan h^2\left\{t\sqrt{\frac{3g}{10a}}\right\}.$$

5. Show tha the vertical pressure of the top on the point of support is equal to its weight when the inclination of its axis to the vertical in given by

$$3Amgh\cos^2\theta-\left(n^2c^2+2Ambgh\right)\cos\theta+n^2c^2a-Amgh=0$$

where a and b are constants depending on the initial circumstances of motion.

□□□□

UNIT - 9
Hamilton's Principle and Principle of Least Action

Structure of the unit

9.1 Introduction

In this unit two important principles of **dynamics viz Hamilton's Principle** and the principle of least action are disscused. A clear distinction between these two principles which is also discussed in the unit. The Principle due to Hamiltion has been found to be useful in deducting Lagrange's equations of motion.

9.2 Hamilton's Principle and Principle of Least Action

Consider a holonomous dynamical conservative system. Suppose the system to have definite motion from terminus A to terminus B, which may be called its actual motion. This may be visualised by particle of mass m at any point p, whose coordinates at time t are (x, y, z) to move continuously along the arc of a curve AB so that successive positions of the particle respresent successive configurations of the system. Let t_0 and t_1, be the times at which the configurations are represented by A and B respectively. Again consider a slighty different motion in which the path is contiguous to the actual path but having the same terminii, the times at the terminii being the same as for the actual path.

Now, by D'Alembert's principle, "the reversed effective forces acting at each point of the system and there external (impressed) forces form a system in equilibium." Therefore, giving the system a virtual displacement consistant with the geometrical conditions at time t, we have

$$\sum \left\{(X - m\ddot{x})\,\delta x + (Y - m\ddot{y})\,\delta y + (Z - m\ddot{z})\,\delta z\right\} = 0 \qquad ...(1)$$

where X, Y, Z are the impressed forces on the system and $m\ddot{x}, m\ddot{y}, m\ddot{z}$ are the effective forces.

But $$\sum \left\{X\,\delta x + Y\,\delta y + Z\,\delta z\right\} = -\left(\frac{\partial V}{\partial x}\delta x + \frac{\partial V}{\partial y}\delta y + \frac{\partial V}{\partial z}\delta z\right) = -\,\delta V \qquad ...(2)$$

where V is the potential function (energy) of the system.

Let (x, y, z) be the coordinates of the particle of mass m in the actual motion and (x', y', z') are the corresponding coordinates in the displaced path, then we have

$$\delta x = x' - x,\ \delta y = y' - y,\ \delta z = z' - z \qquad ...(3)$$

Similarly, if (u, v, w) be the components of velocity of the particle for the actual motion and (u', v', w') be the corresponding velocity components for the displaced path, then

$$\delta u = u' - u,\ \delta v = v' - v,\ \delta w = w' - w \qquad ...(4)$$

Now

$$\ddot{x}\,\delta x = \frac{d}{dt}(\dot{x}\,\delta x) - \dot{x}\,\frac{d}{dt}(\delta x)$$

$$= \frac{d}{dt}(\dot{x}\,\delta x) - \dot{x}\,\frac{d}{dt}(x' - x)$$

$$= \frac{d}{dt}(\dot{x}\,\delta x) - \dot{x}\left\{\frac{dx'}{dt} - \frac{dx}{dt}\right\}$$

$$= \frac{d}{dt}(\dot{x}\,\delta x) - \dot{x}\,\{u' - u\}$$

$$= \frac{d}{dt}(\dot{x}\,\delta x) - \dot{x}\,\delta u$$

$$\ddot{x}\,\delta x = \frac{d}{dt}(\dot{x}\,\delta x) - \dot{x}\,\delta\dot{x}$$

similarly,

$$\ddot{y}\,\delta y = \frac{d}{dt}(\dot{y}\,\delta y) - \dot{y}\,\delta\dot{y} \text{ and } \ddot{z}\,\delta z = \frac{d}{dt}(\dot{z}\,\delta z) - \dot{z}\,\delta\dot{z} \qquad ...(5)$$

Then

$$-m(\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z) = -m\left[\left\{\frac{d}{dt}(\dot{x}\delta x) + \frac{d}{dt}(\dot{y}\delta y) + \frac{d}{dt}(\dot{z}\delta z)\right\} - \{\dot{x}\delta\dot{x} + \dot{y}\delta\dot{y} + \dot{z}\delta\dot{z}\}\right]$$

$$= -m\left[\frac{d}{dt}(\dot{x}\delta x + \dot{y}\delta y + \dot{z}\delta z) - (\dot{x}\delta\dot{x} + \dot{y}\delta\dot{y} + \dot{z}\delta\dot{z})\right] \qquad ...(6)$$

Therefore, with the help of (2) and (6), the equation (1) becomes

$$-\delta V - \sum m\left\{\frac{d}{dt}(\dot{x}\delta x + \dot{y}\delta y + \dot{z}\delta z) - (\dot{x}\delta\dot{x} + \dot{y}\delta\dot{y} + \dot{z}\delta\dot{z})\right\} = 0$$

Integrating this form t_0 to t_1, we

$$\int_{t_0}^{t_1} \delta V\, dt + \int_{t_0}^{t_1} \sum m \left\{ \frac{d}{dt} (\dot{x}\,\delta x + \dot{y}\,\delta y + \dot{z}\,\delta z) - (\dot{x}\,\delta \dot{x} + \dot{y}\,\delta \dot{y} + \dot{z}\,\delta \dot{z}) \right\} dt = 0$$

or $$\int_{t_0}^{t_1} \delta V\, dt + \left[\sum m\, (\dot{x}\,\delta x + \dot{y}\,\delta y + \dot{z}\,\delta z)\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \sum m\, (\dot{x}\,\delta \dot{x} + \dot{y}\,\delta \dot{y} + \dot{z}\,\delta \dot{z})\, dt = 0 \quad (6)$$

But at the terminii, which are fixed

$$\delta x = 0 = \delta y = \delta z$$

therefore, $$\sum m \left(\dot{x}\,\delta x + \dot{y}\,\delta y + \dot{z}\,\delta z\right)\Big]_{t_0}^{t_1} = 0$$

Thus, the equation (6) becomes

$$\int_{t_0}^{t_1} \delta V\, dt - \int_{t_0}^{t_1} \sum m \left(\dot{x}\,\delta \dot{x} + \dot{y}\,\delta \dot{y} + \dot{z}\,\delta \dot{z}\right) dt = 0 \quad ...(7)$$

Now, if T be the kinetic energy of the system,

then,

$$T = \frac{1}{2} \sum m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$\therefore \quad \delta T = \sum m \left(\dot{x}\,\delta \dot{x} + \dot{y}\,\delta \dot{y} + \dot{z}\,\delta \dot{z}\right) \quad ...(8)$$

with the help of this, the equation (7) become

$$\int_{t_0}^{t_1} \delta V\, dt - \int_{t_0}^{t_1} \delta T\, dt = 0$$

or $$\int_{t_0}^{t_1} \delta (V - T)\, dt = 0$$ or $$\int_{t_0}^{t_1} \delta (T - V)\, dt = 0$$

or $$\int_{t_0}^{t_1} \delta L\, dt = 0 \quad ...(9)$$

where $L = T - V$, is the Lagrange's function.

But $$\int_{t_0}^{t_1} \delta L\, dt = \int_{t_0}^{t_1} (L' - L)\, dt = \int_{t_0}^{t_1} L'\, dt - \int_{t_0}^{t_1} L\, dt$$

or $$\int_{t_0}^{t_1} \delta L\, dt = \delta \int_{t_0}^{t_1} L\, dt \quad ...(10)$$

Hence, (9) becomes.

$$\delta \int_{t_0}^{t_1} L\,d\,t = 0 \qquad ...(11)$$

We define $S = \int_{t_0}^{t_1} L\,d\,t$, as the Hamilton's Principal function, then (11) gives

$$\delta S = 0 \qquad ...(12)$$

This is known as Hamilton's principle. It states that "if the time from one configuration to another is prescribed, the principal function S has a stationary value in the actual path as compared to a contiguous path i.e. the system will choose that path for which the principal function is stationary."

Again, If E be the total energy of the system,

then, $E = T + V$

Therefore, Lagranges function

$$L = T - V = T - (E - T) = 2\,T - E$$

therefore, if E is constant, then equation (11) gives

$$\delta \int_{t_0}^{t_1} (2\,T - E)\,d\,t = 0$$

or $$\delta \int_{t_0}^{t_1} 2\,T\,d\,t = 0 \qquad ...(13)$$

We define $A = \int_{t_0}^{t_1} 2\,T\,d\,t$, as the Hamilton's characteristic function, then (13) gives

$$\delta A = 0 \qquad ...(14)$$

This is known as the **Principle of Least Action.** It states that "if the total energy of a system is prescribed as it passes from one configuration to another, the action in the actual path is a minimum when compared with a contiguous path i.e. the system will choose that path for which the action is the least."

9.3 Distinction between Hamilton's Principle and Principle of Least Action

In **Hamilton's Principle** i.e. $\delta S = 0$, the time of description $t_1 - t_0$ is prescribed (fixed) as the system moves from one configuration A to the another configuration B, whereas in the **Principle of Least Action** i.e. $\delta A = 0$, there is no restriction of the time of description $t_1 - t_0$ but the total energy (sum of kinetic and potential energies) between the end point, A and B is prescribed (fixed).

9.4 Deduction of Lagrange's Equations from Hamilton's Principle

Consider a holonomous, conservative dynamical system with n generalised coordinates $\theta_1, \theta_2, \theta_3, ...\theta_n$. Let T and V are the kinetic and potential energies of the system and L be the Lagrangian function defined as $L = T - V$. Since the Lagrangian function $L = T - V$, is a function of $\theta_1, \theta_2, \theta_3, ...\theta_n$ and $\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3, ...\dot{\theta}_n$ at time t, then we have

$$\delta L = \sum_{r=1}^{n} \frac{\partial L}{\partial \theta_r} \delta \theta_r + \sum_{r=1}^{n} \frac{\partial L}{\partial \dot{\theta}_r} \delta \dot{\theta}_r \qquad ...(1)$$

Now, the Hamilton's Principle gives,

$$\delta S = \delta \int_{t_0}^{t_1} L \, dt = 0$$

or $$\int_{t_0}^{t_1} \delta L \, dt = 0$$

or $$\int_{t_0}^{t_1} \left\{ \sum \frac{\partial L}{\partial \theta_r} \delta \theta_r + \sum \frac{\partial L}{\partial \dot{\theta}_r} \delta \dot{\theta}_r \right\} dt = 0 \qquad \text{(from (1))}$$

or $$\int_{t_0}^{t_1} \left\{ \sum \frac{\partial L}{\partial \theta_r} \delta \theta_r \right\} dt + \int_{t_0}^{t_1} \left\{ \sum \frac{\partial L}{\partial \dot{\theta}_r} \delta \dot{\theta}_r \right\} dt = 0$$

or $$\int_{t_0}^{t_1} \left\{ \sum \frac{\partial L}{\partial \theta_r} \delta \theta_r \right\} dt + \int_{t_0}^{t_1} \left\{ \sum \frac{\partial L}{\partial \dot{\theta}_r} \frac{d}{dt} (\delta \theta_r) \right\} dt = 0 \qquad ...(2)$$

Integrating the second integral by parts taking $\frac{d}{dt}(\delta \theta_r)$ as the second function, we get

$$\int_{t_0}^{t_1} \left\{ \sum \frac{\partial L}{\partial \theta_r} \delta \theta_r \right\} dt + \left[\sum \frac{\partial L}{\partial \dot{\theta}_r} . \delta \theta_r \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \left\{ \sum \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_r} \right) . \delta \theta_r \right\} dt = 0$$

or $$\int_{t_0}^{t_1} \left\{ \sum \frac{\partial L}{\partial \theta_r} \delta \theta_r \right\} dt - \int_{t_0}^{t_1} \left\{ \sum \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_r} \right) . \delta \theta_r \right\} dt = 0$$

(the middle term vanishes as all $\delta \theta_r = 0$ at t_0 and t_1)

or $$\int_{t_0}^{t_1} \left[\sum \frac{\partial L}{\partial \theta_r} \delta \theta_r - \sum \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_r} \right) . \delta \theta_r \right] dt = 0$$

or $$\int_{t_0}^{t_1}\left[\sum\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_r}\right)-\frac{\partial L}{\partial\theta_r}\right\}\delta\theta_r\right]dt=0 \qquad ...(3)$$

Since the system is a holonomous system, so, $\delta\theta_1, \delta\theta_2, \delta\theta_3 \; .... \; \delta\theta_n$ are arbitrary and independent to each other, therefore, the above integral (3) will vanish, if the coefficient of $\delta\theta_r$ separately vanishes, that is

$$\frac{d}{dt}\left(\frac{\partial L}{\delta\dot{\theta}_r}\right)-\frac{\partial L}{\delta\theta_r}=0,\; r=1,2,3,\ldots.n$$

which are the Lagrange's equations of motion for conservative system in terms of Lagrange's function.

Illustrative Exmaples :

Example 1 : Show that the Hamilton's principle function S for simple harmonic motion in a straight line is

$$\frac{\sqrt{\mu}\,\left(x^2+x_0^2\right)\cos\left(t-t_0\right)\sqrt{\mu}-2\,x\,x_0}{2\sin\left(t-t_0\right)\sqrt{\mu}}$$

where x, x_0 are the displacements from the centre of force at times t, t_0 respectively.

or

A particle moves in a straight line with central acceleration μx between two points x_0 and x_1 in the prescribed time $t_1 - t_0$. Show that Hamilton's principle function S is

$$\frac{\sqrt{\mu}\,\left\{\left(x_1^2+x_0^2\right)\cos\left(t_1-t_0\right)\sqrt{\mu}-2\,x_1\,x_0\right\}}{2\sin\left(t_1-t_0\right)\sqrt{\mu}}$$

Solution : The equation of simple harmonic motion is

$$\ddot{x}=-\mu x \qquad ...(1)$$

Solution of this equation is

$$x=A\cos t\sqrt{\mu}+B\sin t\sqrt{\mu} \qquad ...(2)$$

so that $$x_0=A\cos t_0\sqrt{\mu}+B\sin t_0\sqrt{\mu} \qquad ...(3)$$

where A and B are constants

If T be the kinetic energy per unit mass of the system, then

$$T=\frac{1}{2}\dot{x}^2=\frac{1}{2}\left\{-\sqrt{\mu}\;A\sin t\sqrt{\mu}+\sqrt{\mu}\;B\cos t\sqrt{\mu}\right\}^2$$

or $$T=\frac{\mu}{2}\left(-A\sin t\sqrt{\mu}+B\cos t\sqrt{\mu}\right)^2 \qquad ...(4)$$

For the simple harmonic motion, the force per unit mass is $-\mu x$, therefore, if V be the potential function, then

$$-\frac{\partial V}{\partial x} = -\mu x$$

or
$$V = \frac{1}{2}\mu x^2$$

$$= \frac{1}{2}\mu\left(A\cos t\sqrt{\mu} + B\sin t\sqrt{\mu}\right)^2 \qquad ....(5)$$

Now, the Hamilton's principle function S is defined as

$$S = \int_{t_0}^{t} L\,dt = \int_{t_0}^{t}(T-V)\,dt$$

$$= \frac{\mu}{2}\int_{t_0}^{t}\left\{\left(-A\sin t\sqrt{\mu} + B\cos t\sqrt{\mu}\right)^2 - \left(A\cos t\sqrt{\mu} + B\sin t\sqrt{\mu}\right)^2\right\}dt$$

$$= \frac{\mu}{2}\int_{t_0}^{t}\left\{\left(B^2 - A^2\right)\cos 2t\sqrt{\mu} - 2AB\sin 2t\sqrt{\mu}\right\}dt$$

$$= \frac{\sqrt{\mu}}{4}\left[\left(B^2 - A^2\right)\sin 2t\sqrt{\mu} + 2AB\cos 2t\sqrt{\mu}\right]_{t_0}^{t}$$

$$= \frac{\sqrt{\mu}}{4}\left[\left(B^2 - A^2\right)\left(\sin 2t\sqrt{\mu} - \sin 2t_0\sqrt{\mu}\right) + 2AB\left(\cos 2t\sqrt{\mu} - \cos 2t_0\sqrt{\mu}\right)\right]$$

$$= \frac{\sqrt{\mu}}{2}\left[\left(B^2 - A^2\right)\sin(t-t_0)\sqrt{\mu}\cos(t+t_0)\sqrt{\mu} + 2AB\sin(t-t_0)\sqrt{\mu}\sin(t+t_0)\sqrt{\mu}\right]$$

$$= \frac{\sqrt{\mu}}{2}\sin(t-t_0)\sqrt{\mu}\left\{\left(B^2 - A^2\right)\cos(t+t_0)\sqrt{\mu} - 2AB\sin(t-t_0)\sqrt{\mu}\right\} \qquad ...(6)$$

Now, solving equation (2) and (3) for A and B, we get

$$\frac{A}{x\sin t_0\sqrt{\mu} - x_0\sin t\sqrt{\mu}} = \frac{B}{x_0\cos t\sqrt{\mu} - x\cos t_0\sqrt{\mu}}$$

$$= \frac{1}{\sin t_0\sqrt{\mu}\cos t\sqrt{\mu} - \sin t\sqrt{\mu}\cos t_0\sqrt{\mu}}$$

$$\therefore \quad A = \frac{x\sin t_0\sqrt{\mu} - x_0\sin t\sqrt{\mu}}{\sin(t-t_0)\sqrt{\mu}}, \quad B = \frac{x_0\cos t\sqrt{\mu} - x\cos t_0\sqrt{\mu}}{\sin(t_0-t)\sqrt{\mu}}$$

$$B^2 - A^2 = \frac{1}{\sin^2(t-t_0)\sqrt{\mu}}\left\{x_0^2 \cos 2t\sqrt{\mu} + x^2\cos 2t_0\sqrt{\mu} - 2xx_0\cos(t+t_0)\sqrt{\mu}\right\}$$

and

$$2AB = \frac{1}{\sin^2(t-t_0)\sqrt{\mu}}\left\{-x_0^2\sin 2t\sqrt{\mu} - x^2\sin 2t_0\sqrt{\mu} + 2x_0 x\sin(t+t_0)\sqrt{\mu}\right\}$$

substituting the values of $(B^2 - A^2)\sin(t-t_0)\sqrt{\mu}$ and $2AB\sin(t-t_0)\sqrt{\mu}$ in (6), we get

$$S = \frac{\sqrt{\mu}}{2\sin(t-t_0)\sqrt{\mu}}\left[\left\{x_0^2\cos 2t\sqrt{\mu} + x^2\cos 2t_0\sqrt{\mu} - 2xx_0\cos(t+t_0)\sqrt{\mu}\right\}.\right.$$

$$\left.\cos(t+t_0)\sqrt{\mu} + \left\{x_0^2\sin 2t\sqrt{\mu} + x^2\sin 2t_0\sqrt{\mu} - 2x_0 x\sin(t+t_0)\right\}\sin(t+t_0)\sqrt{\mu}\right]$$

$$= \frac{\sqrt{\mu}}{2\sin(t-t_0)\sqrt{\mu}}\left[x_0^2\left\{\cos 2t\sqrt{\mu}\cos(t+t_0)\sqrt{\mu} + \sin 2t\sqrt{\mu}\sin(t+t_0)\sqrt{\mu}\right\}\right.$$

$$+x^2\left\{\cos 2t_0\sqrt{\mu}\cos(t+t_0)\sqrt{\mu} + \sin 2t_0\sqrt{\mu}\sin(t+t_0)\right\}$$

$$\left.-2xx_0\left\{\cos^2(t+t_0)\sqrt{\mu} + \sin^2(t+t_0)\sqrt{\mu}\right\}\right]$$

or $$S = \frac{\sqrt{\mu}}{2\sin(t-t_0)\sqrt{\mu}}\left[x_0^2\cos(t-t_0)\sqrt{\mu} + x^2\cos(t_0-t)\sqrt{\mu} - 2xx_0\right]$$

or $$S = \frac{\sqrt{\mu}}{2\sin(t-t_0)\sqrt{\mu}}\left\{(x^2 + x_0^2)\cos(t-t_0)\sqrt{\mu} - 2xx_0\right\}$$

which is the required result.

Example 2 : A heavy bead of mass m is freely movable on a smooth circular wire of radius a, which is made to rotate about the vertical diameter with spin w, θ being the angle made by the radius through the bead at any time with the downwards vertical, prove that the action A is

$$A = \int_{\theta_1}^{\theta_2} ma^2\left\{\frac{2H}{ma^2} + \frac{2g}{a}\cos\theta + w^2\sin^2\theta\right\}^{1/2} d\theta$$

where H is the Hamiltonian of the system.

Solution : Let O be the centre of the smooth circular wire of radius a which is made to rotate about a vertical diameter AB with spin w. Let P be the position of the bead of mass m, at time t, such that $\angle AOP = \theta$ and distance of P from the vertical diameter AB be x. Thus at time t, the particle is moving on the fixed circular were under the following forces :

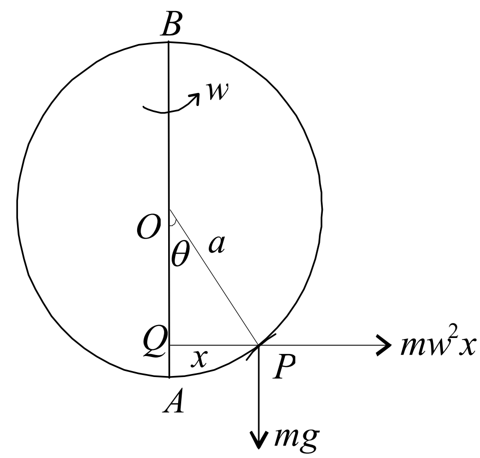


Fig. 9.1

(i) Weight mg acting vertically downward

(ii) the force $m w^2 x \left(= m w^2 a \sin\theta\right)$ in the horizontal direction.

Hence if w be the work function of the system, then

$$w = mg\ a\cos\theta + \int_0^x m w^2 x\, d\,x$$

$$= mg\ a\cos\theta + \frac{1}{2}\ m w^2\ x^2$$

$$= mg\ a\cos\theta + \frac{1}{2}\ m w^2\ a^2 \sin^2\theta \qquad ...(1)$$

Again, if velocity of the bead at P is $a\dot{\theta}$ therefore, if T be the K.E. of the system, then

$$T = \frac{1}{2}\ m\, a^2\, \dot{\theta}^2 \qquad ...(2)$$

Now, the action $A = \int_{t_0}^{t_1} 2\, T\, d\, t$

$$= \int_{t_0}^{t_1} 2\, T \frac{d\, t}{d\,\theta} d\,\theta$$

$$= \int_{\theta_0}^{\theta_1} 2\ .\ \frac{1}{2}\ m^2\ a^2\ \dot{\theta}^2\ \frac{1}{\dot{\theta}} d\,\theta$$

$$= m\, a^2 \int_{\theta_0}^{\theta_1} \dot{\theta}\, d\,\theta \qquad ...(3)$$

Since the potential exists, the Hamiltonian H is equal to sum of the kinetic and the potential energies, that is

$$H = T + V = \frac{1}{2}\, m\, a^2 \dot{\theta} - \left(mg\ a\ \cos\theta + \frac{1}{2}\, m\, w^2\, a^2 \sin^2\theta \right)$$

or $$\dot{\theta}^2 = \frac{2H}{m\,a^2} + \frac{2g}{a}\cos\theta + w^2 \sin^2\theta \qquad ...(4)$$

Hence, from (3) the action A is given

$$A = m a^2 \int_{\theta_0}^{\theta_1} \left\{ \frac{2H}{ma^2} + \frac{2g}{a}\cos\theta + w^2 \sin^2\theta \right\}^{1/2} d\theta .$$

Example 3 : A particle of unit mass moves along OX under a constant force f starting from rest at the origin at time $t = 0$. If T and V are the kinetic and potential energies of the particle, calculate

$$\int_0^{t_0} (T - V)\, dt$$

Evaluate this for the varied motion in which the position of particle is given by $x = \frac{1}{2} f\, t^2 + \in f\, t\,(t - t_0)$ where $\in$ is a constant; and show that the result is in agreement with Hamilton's principle. What are the essential features of the varried motion that ensure this agreement?

Solution : The particle moves along OX with acceleration f starting from rest, therefore its actual path is

$$x = \frac{1}{2} f\, t^2 \text{ so that } \dot{x} = f\, t$$

Thus, $$T = \frac{1}{2}\dot{x}^2 = \frac{1}{2} f^2 t^2 \qquad ...(1)$$

Since particle moves from $t = 0$ to $t = t_0$, therefore end points of the actual motion are O and $\frac{1}{2} f\, t_0^2$.

The potential V at a distance x, at time t is

$$V = -\, f\, x = -\, f\left(\frac{1}{2} f\, t^2\right) = -\frac{1}{2} f^2 t^2 \qquad ...(2)$$

Therefore,

$$T - V = \frac{1}{2} f^2 t^2 - \left(-\frac{1}{2} f^2 t^2\right) = f^2 t^2 \qquad ...(3)$$

$\therefore$ In actual path,

$$\int_0^{t_0} (T-V)\,dt = \int_0^{t_0} f^2 t^2\,dt = \frac{1}{3} f^2 t_0^3 \qquad ...(4)$$

and the varied path is given by

$$x = \frac{1}{2} f t^2 + \in f t (t - t_0) = \frac{1}{2}\left\{(1+2\in) f t^2 - 2\in f t t_0\right\} \qquad ...(5)$$

Putting $t=0$ and $t=t_0$, we see that end points of the varied motion are also O and $\frac{1}{2} f t_0^2$. Hence end points, of the actual path and the varied path, coincide.

Now, velocity in varied path $= (1+2\in) f t - \in f t_0$

Therefore, in varied path

$$T = \frac{1}{2}\left\{(1+2\in) f t - \in f t_0\right\}^2 = \frac{1}{2}\left\{(1+4\in+4\in^2) f^2 t^2 - (2\in+4\in_0^2) f^2 t_0 t + \in^2 f^2 t_0^2\right\} \qquad ...(6)$$

and potential

$$V = -f\left[\frac{1}{2}\left\{(1+2\in) f t^2 - 2\in f t_0 t\right\}\right] = -\frac{1}{2}\left[(1+2\in) f^2 t^2 - 2\in f^2 t_0\right] \qquad ...(7)$$

therefore, in the varied path

$$T - V = \frac{1}{2}\left[\left\{(1+4\in+4\in^2) + (1+2\in)\right\} f^2 t^2 - (4\in+4\in^2) f^2 t_0 t + \in^2 f^2 t_0^2\right]$$

$$= \frac{1}{2}\left[(2+6\in+4\in^2) f^2 t^2 - (4\in+4\in^2) f^2 t_0 t + \in^2 f^2 t_0^2\right] \qquad ...(8)$$

Hence in the varied path

$$\int_0^{t_0} (T-V)\,dt = \int_0^{t_0} \frac{1}{2}\left[(2+6\in+4\in^2) f^2 t^2 - (4\in+4\in^2) f^2 t_0 t + \in^2 f^2 t_0^3\right] dt$$

$$= \frac{1}{2}\left[(2+6\in+4\in^2)\frac{f^2 t_0^3}{3} - (4\in+4\in^2)\frac{1}{2} f^2 t_0^3 + \in^2 f^2 t_0^3\right]$$

$$= \frac{1}{2}\left[\frac{1}{3}(2+6\in+4\in^2) - (2\in+2\in^2) + \in^2\right] f^2 t_0^3$$

$$= \frac{1}{2}\left[\frac{2}{3} + \frac{1}{3}\in^2\right] f^2 t_0^3$$

$$= \frac{1}{3}\left(1+\frac{1}{2}\in^2\right) f^2 t_0^3 \qquad ...(9)$$

which is minimum if $\in = 0$ and then the minimum value is $\frac{1}{3} f^2 t_0^3$, which is the same as in the actual path. Also we see that when $\in = 0$, the varied path coincides with the actual path. Hence the result is in agreement with Hamilton's principle.

Example 4 : A particle of unit mass is projected so that its total energy is h in a field of force of which the potential energy is $\phi(r)$ at a distance r from the origin. Deduce from the principle of energy and least action that the differential equation of the path is

$$c^2\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right] = r^4\left[h-\phi(r)\right].$$

Solution : Let T and V be the kinetic and potential energies of the system and h be the total energy then

$$h = T + V$$

or $$T = h - V = h - \phi(r) \qquad \text{(since } V = \phi(r) \text{ is given)}$$

or $$\frac{1}{2}V^2 = h - \phi(r)\text{, where } V \text{ is the velocity}$$

or $$V = \sqrt{2}\,\{h-\phi(r)\}^{1/2} \qquad ...(1)$$

Now, the action $A = \int_{t_0}^{t_1} 2T\,dt$

$$= \int_{t_0}^{t_1} 2.\frac{1}{2}V^2\,dt$$

$$= \int_{t_0}^{t_1} V\,ds \qquad \left\{\because V = \frac{ds}{dt}\right\}$$

$$= \sqrt{2}\int_{t_0}^{t_1}\{h-\phi(r)\}^{1/2}\,ds \qquad \left\{\because \frac{ds}{dr} = \left\{1+r^2\left(\frac{d\theta}{dr}\right)^2\right\}^{1/2}\right\}$$

or $$A = \sqrt{2}\int_{t_0}^{t_1}\left[\{h-\phi(r)\}^{1/2}\left\{1+r^2\left(\frac{d\theta}{dr}\right)^2\right\}^{1/2} dr\right] \qquad ...(2)$$

or $$A = \sqrt{2} \int_{t_0}^{t_1} f\left(r, \theta^1\right) d r \qquad ...(3)$$

where $f\left(r,\theta^1\right) = \left\{h - \phi(r)\right\}^{1/2} \left\{1 + r^2 \left(\frac{d\theta}{dr}\right)^2\right\}^{1/2}$

Since θ is absent from f, therefore, we have

$$\frac{\partial f}{\partial \theta^1} = c \text{ (constant)}, \text{ where } \theta^1 = \frac{d\theta}{dr}$$

or $$\frac{\partial}{\partial \theta^1}\left[\left\{h - \phi(r)\right\}^{1/2} \left\{1 + r^2 \theta^1\right\}^{1/2}\right] = c$$

or $$\left\{h - \phi(r)\right\}^{1/2} \frac{1}{2} \left\{1 + r^2 \theta'^2\right\}^{-1/2} . \left(2 r^2 \theta^1\right) = c$$

or $$\left\{h - \phi(r)\right\}^{1/2} \frac{r^2}{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{1/2}} = c$$

or $$c^2 \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\} = r^4 \left\{h - \phi(r)\right\}$$

which is the required equation.

9.5 Summary

In this unit we have derived two important principles of dynamics viz Hamilton's principle and principle of least action. We have also discussed the important distinction between the two principles. The usefulness of the Hamilton's principle has been emphasized by deriving Lagrange's equations using Hamiltin's principle.

9.6 Exercise

1. State and prove the principle of least action for a conservation holonomic system.

2. A particle moves in a plane curve, under the central acceleration $w^2 r$, between two fixed points $\left(x_0, y_0\right)$ and $\left(x_1, y_1\right)$ in the prescribed time $t_1 - t_0$; prove that Hamilton's principle function S is

$$\frac{w}{2 \sin w \left(t_1 - t_0\right)} \left[\left(x_1^2 + y_1^2 + x_0^2 + y_0^2\right) \cos w \left(t_1 - t_0\right) - 2 \left(x_1 x_0 + y_1 y_0\right)\right]$$

3. A projectile is launched in a vertical plane with a velocity whose horizontal and vertical components are V_x and V_y respectively. Calculate the value of the integral,

$$\int_0^{t_0} L\,dt \text{ , where } t_0 = \frac{n\pi}{w}$$

Evaluate this integral for the varied path given by the equations

$$x = V_x t \text{ , } y = V_y t - \frac{1}{2} g t^2 + \in \sin wt \text{ ,}$$

where $\in$ is a small constant quantity.

Show that the integral $\int_0^{t_0} L\,dt$ is greater for the varied path than, that for the actual path, but the result is in agreement with Hamilton's principle.

4. A mass m attached to a coil spring having a constant k, oscillates along a smooth horizonatal line with a motion given by

$$x = A\sin wt \text{ , where } w = \sqrt{\frac{k}{m}} \text{ .}$$

Assuming a varied path represented by

$$x = A\sin wt + \in \sin 2wt \text{ .}$$

Where $\in$ is a small constant quantity, show that for the actual path taken over the integral

$$\int_{t=0}^{t=\frac{\pi}{2w}} L\,dt = 0$$

and that for the varied path this integral is equal to $\frac{3}{8} m\pi w \in^2$.

5. A particle of unit mass is projected so that its total energy is h, in a field of a force is a which the potential energy in $\phi(r)$ at a distance r from the origin. Find the differential equation of the path.

□□□□

UNIT - 10
Kinematics

Structure of the unit

10.0 Objective

In this unit, our object is to be aware about the fundamental propeties of a fluid and approches to solve hydrodynamical problems. We also study about path line, stream line and function in it.

10.1 Introduction

In general, the matter is classified in two categories viz solids and fluids. A solid may be rigid or elastic. Here we will not go into further classfication of solids but keep ourselves concerned with the fluids. The fluid differs from solid in that it yields to a shearing stress and cannot be in equilibrium under such a stress. This stress leads to the deformation of the fluid and culminates into a flow. The properties of the fluids are directly related to the molecular structure and to the nature of the forces between molecules.

Hydrodynamics or Fluid dynamics is that branch of science which is concerned about the study of the motion of fluids or that of bodies in contact with the fluid.

10.2 Characteristics of a fluid

The fluids are divided in two categories viz liquids and gases. this classification is based on the binding intermolicular force existing in the fluids. In case of liquids, this is such that the volume of the liquids remains unchanged to a great extent, while the shape alters to take the shape of the container. In case of the gases, this intermolecular force is too small resulting in change in volume and shape depending on the capacity and shape of the container. The property by virtue of which the volume of the fluid changes is called the compressibility. In general the liquids are regarded as incompressible fluids while gases are regarded as compressible fluids. Real fluids have five physical characteristics; These are pressure, density, volume, termperature and viscosity.

10.2.1 Denstiy :

ρ at a point may be defined as

$$\rho = \lim_{\delta v \to 0} \frac{\delta m}{\delta v}$$

where δv is the volume element around a point in the fluid and δm is the mass of the fluid contained within δv. The specific volume of a fluid is defind as the volume per unit mass and it is clearly the reciprocal of the density.

10.2.2 Pressure :

When a fluid is contained in a vessel, it exerts a force at each point of the inner side of the vessel. Such a force per unit area is called pressure. Mathematically, the pressure P at any point may be defined as

$$p = \lim_{\delta A \to 0} \frac{\delta F}{\delta A}$$

where δA is an elementry area around the point and δF is the normal force due to fluid on δA.

10.2.3 Compressibility :

The compressibility of a fluid is defined as the variation of the density, with the variation of pressure. Mathematically

$$dp \, \alpha \frac{\delta \rho}{\rho} \Rightarrow dp = k \frac{d\rho}{\rho}$$ where k is called the Bulk Modulus of the fluid.

10.2.4 Viscosity :

It is common experiance that when two solids are in contact and one solid moves over another, there is a property which prevents the slipping. This property is known as friction. A corresponding property of the fluid is called viscosity. Viscosity is that property of fluids as a result of which they present some resistance to sharing i.e. sliding movement of one particle past or near another. It was observed that viscous forces vary directly with the relative velocity of sliding between adjacent particles or parts of a fluid. Hance the shear stress is proportional to the velocity gradient, that is

$$\tau \, \alpha \frac{du}{dy} \rightarrow \tau = \mu \frac{du}{dy}$$

Where τ is the shear stress and μ, the constant of proportionality is called the coefficient of viscosity of the fluid. This equation is called Newton's law of viscosity. The ratio of the coefficient of viscosity μ to the density ρ of the fluid is known as the coefficient of kinematic viscosity and represented as $\upsilon = \mu / \rho$. The viscosity of a fluid is practically independent of pressure and observed to be dependent on temperature only.

10.3 Kinds of Fluid

The fluids are classified in two forms viz Ideal (perfect) fluid and Real (Actual) fluid.

10.3.1 Ideal Fluid :

The fluids which have no viscosity, surface tension and incompressibility are known as Ideal fluids. In ideal fluid, there are no tangential forces between the adjoining layers of the fluid but only normal stresses are present. The pressure at every point of an ideal fluid is equal in all directions, whether the fluid be at rest or in motion. This theory defines some concepts of the flow such as wave motion, the lift and the induced drag of an airfoil etc, but it fails to define the phenomena such as skinfriction, drag of a body etc Ideals fluids also known as perfect fluids or Inviscid fluids.

10.3.2 Real Fluid :

The fluid which actually exist in nature are considered real fluid. These fluids possess all the properties of fluids i.e. viscosity, pressure, density, volume and temperature are present. Real fluids are further classified as follows.

(i) Incompressible fluid :

If the density of the fluid is constant then it is called incompressible fluid. In real liquid (e.g. oil, water, mercury, etc.) the density is not exactly constant, but considered almost constant. Generally water is considered as an incompressible fluid.

(ii) Compressible Fluid :

A fluid in which density is not constant but it is assumed that density is a function of hydrostatic pressure is considered as a compressible fluid. Generally air is considered as compressible fluid.

(iii) Newtonian Fluid :

Fluid which obeys Newton's law of viscosity is classified as Newtonian fluid.

(iv) Non-Newtonian Fluid :

Fluids which donot obey Newton's law of viscosity (relation between stress and rate of strain) are known as Non-Newtonian fluid. Some Non-Newtonian fluids are power law fluid, prandtl fluid, oldroyd fluid, walters fluid depending on the relation between stress and rate of strain.

(v) Isotropic Fluid :

Fluid in which the relation between the components of stress and rate of strain remains unchanged by a rotation of the coordinate system are called Isotropic fluid.

10.4 Kinds of Fluid Flow

Fluid flow may be characterized in the following kinds :

10.4.1 Steady Flow :

The fluid flow in which the fluid characteristics, i.e. velocity, density, pressure, temperature are independent of time is called steady flow.

In the steady flow, fluid particles move along the stream line and the position of stream line does not change with the time.

(About the stream lines we will know more later.)

10.4.2 Unsteady flow :

The fluid flow in which all physical properties of the fluid in the motion vary with time is known as unsteady flow.

10.4.3 Laminar flow :

A flow, in which each fluid particle traces out a definite curve and the curves traced out by any two different fluid particles do not intersect, is said to be laminar flow. So laminar flow is a well ordered flow in which layers are assumed to slide over one another in similar fashion.

10.4.4 Turbulent flow :

A flow, in which each fluid particle does not trace out a definite curve and the curves traced out by fluid particles intersects, is said to be turbulent. So turbulent flow is an irregular nature of flow in which various layers move in disorderly manner.

10.4.5 Uniform Flow :

A flow, in which the fluid particles possess equal velocities at each section of the channel or pipe is called uniform.

10.4.6 Non-Uniform Flows :

A flow, in which the fluid particles possess different velocities at each section of the channel or pipe is called non-uniform. These terms are usually used in connection with flow in channels.

10.4.7 Rotational Flow :

A flow, in which the fluid particles go on rotating about their own axes, while flowing, is said to be rotational.

10.4.8 Irrotational Flow :

A flow, in which the fluid particles do not rotate about their own axes, while flowing, is said to be irrotational.

10.4.9 Barotropic Flow :

A flow is said to be barotropic, when the pressure is a function of the density.

10.5 Description of fluid motion

A flow field is the description of the flow of a fluid by representing all fundamental flow properties as a function of position in space and time. Here we shall study only the kinematics of the flow field. The fluid motion can be studied through the following two methods

1. Lagrange's Method
2. Euler's Method

Both these methods are due to Euler.

10.5.1 Lagrange's Method

In this motion, any fluid particle is selected and its motion is studied. In other words we try to study the history of each fluid particle to deal with the position, velocity and acceleration of the individual particles.

Let a particle be initially at a point P_o with cartesian coordinates (a, b, c) move to another point P with coordinates (x, y, z) after the lapse of time t. Clearly these coordinates at P will depend on the initial position and the lapsed time, so that

$$x = f(a,b,c,t) \ ; \ y = g(a,b,c,t) \ ; \ z = h(a,b,c,t) \ .$$

The function f , g and h are continous function of a,b,c and t in most problems and they possess partial differential coefficients with respect to a,b,c and t . This method enables us to have the components of velocity and acceleration as

$$\dot{x} = \frac{\partial f}{\partial t} \ ; \qquad \dot{y} = \frac{\partial g}{\partial t} \ ; \qquad \dot{z} = \frac{\partial h}{\partial t}$$

$$\ddot{x} = \frac{\partial^2 f}{\partial t^2} \ ; \qquad \ddot{y} = \frac{\partial^2 g}{\partial t^2} \ ; \qquad \ddot{z} = \frac{\partial^2 h}{\partial t^2}$$

The fundamental equation of motion in Lagrangian form are non-linear and it leads to many difficulties while solving a problem. This method is useful in some special refrences such as certain one-dimensional problems. Generally it is not as conveniant and fruitful as the Euler's method.

10.5.2 Euler's Method

In this method, we fix a point in space and study the motion of fluid particles as they pass through this point. In other words, the consideration is given to velocities and acceleration of different particles at a particular point rather than the variation in velocities and accerlerations of particle along their various paths.

In this method. if the velocity components are u, v and w in the directions of axes at the point (x, y, z) at time t then u, v, w are the functions of the position and time.

10.6 Relationship between the Lagrangian and Eulerian method

To establish relationship between these two methods, we investigate a relation between the particle parameters and space parameters.

10.6.1 Lagrangian to Eulerian :

Let φ be some physical quantity involving Lagrangian description

$$\varphi = \varphi\,(a, b, c, t) \qquad \text{... (1)}$$

Here we want to express a, b, c in terms of the coordinates x, y, z of a point in the space. In Lagrangian method, it is defind as

$$x = f_1\,(a, b, c, t)\ ;\ y = f_2\,(a, b, c, t)\ ;\ z = f_3\,(a, b, c, t) \qquad \text{...(2)}$$

On solving these relations, we obtain

$$a = g_1\,(x, y, z, t)\ ;\ b = g_2\,(x, y, z, t)\ ;\ c = g_3\,(x, y, z, t) \qquad \text{...(3)}$$

Using (3) in (1), we get

$$\varphi = \varphi\left[g_1(x, y, z, t), g_2(x, y, z, t), g_3(x, y, z, t)\right] \qquad \text{...(4)}$$

Which express φ in terms of Euler's description

10.6.2 Eulerian to Lagrangian :

Let ψ be some physical quantity involving Euler's description

$$\psi = \psi\,(x, y, z, t) \qquad \text{...(5)}$$

Let u, v, w are the velocity components at the point (x, y, z) at any time t, which is defined as

$$u = F_1(x, y, z, t)\ ;\ v = F_2(x, y, z, t)\ ;\ w = F_3(x, y, z, t) \qquad \text{...(6)}$$

We know that in Lagrangian method,

$$u = \frac{\partial x}{\partial t},\ v = \frac{\partial y}{\partial t},\ w = \frac{\partial z}{\partial t} \qquad \text{...(7)}$$

Where x, y, z are function of the variables a, b, c and t.

Using (7) in (6), the velocity components of a fluid particle is given by

$$\frac{\partial x}{\partial t} = F_1(x, y, z, t) \; ; \; \frac{\partial y}{\partial t} = F_2(x, y, z, t) \; ; \; \frac{\partial z}{\partial t} = F_3(x, y, z, t) \qquad ...(8)$$

On integrating, we obtain

$$x = f_1(x_0, y_0, z_0, t); \; y = f_2(x_0, y_0, z_0, t); \; z = f_3(x_0, y_0, z_0, t) \qquad ...(9)$$

Where x_0, y_0, z_0 are taken as initial coordinates of the fluid particle. Choosing the particle parameters a, b, c equal to x_0, y_0, z_0 respectively, then

$$\psi = \psi\left[f_1(a,b,c,t), f_2(a,b,c,t); f_3(a,b,c,t)\right]$$

Which express ψ in terms of Lagrangian description.

10.7 Velocity of a fluid Particle

Le P and Q be the positions of the fluid particle at any time t and $t + \delta t$ with respect to the fixed point O such that

$$\overrightarrow{OP} = \vec{r} \text{ and } \overrightarrow{OQ} = \vec{r} + \delta \vec{r}$$

Then the velocity $\vec{q}$ of the fluid particle at P is defind as

$$\vec{q} = \lim_{\delta t \to 0} \frac{\left(\vec{r} + \delta \vec{r}\right) - \vec{r}}{\delta t} = \frac{d\vec{r}}{dt}$$

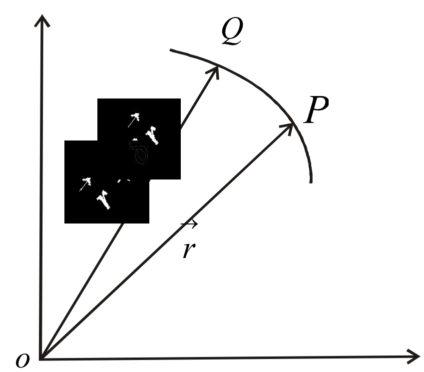


Fig. 10.1

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then

$$\vec{q} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = u\hat{i} + v\hat{j} + w\hat{k}$$

Hence $u = \frac{dx}{dt}$, $v = \frac{dy}{dt}$ and $w = \frac{dz}{dt}$.

10.8 The Material Derivative

Let $f(r, t)$ represent a flow parameter i.e. velocity, density etc at any instant t. If in a small time δt, the particle moves to a new position and flow parameter is represented by $f(r + \delta r, t + \delta t)$, then

$$\delta r = f(r + \delta r, t + \delta t) - f(r, t)$$

This may be expressed in two parts, one as a change due to local variation with time of the fluid property at a given position and other due to change of position at a given time. Hence

$$\delta f = f(r + \delta r, t + \delta t) - f(r, t + \delta t) + f(r, t + \delta t) - f(r, t)$$

$$\delta f = \delta r \ \nabla \ f(r, t+\delta t) + \delta t \ \frac{\partial}{\partial t} f(r, t)$$

$$\because \frac{\delta f}{\delta t} = \frac{\delta r}{\delta t} \ \nabla \ f(r, t+\delta t) + \frac{\partial}{\partial t} f(r, t)$$

$$\underset{\delta t \to 0}{Lt} \frac{\delta f}{\delta t} = \underset{\delta t \to 0}{Lt} \left\{ \frac{\delta r}{\delta t} . \nabla f(r, t+\delta t) \right\} + \underset{\delta t \to 0}{Lt} \frac{\partial}{\partial t} f(r, t)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{d\vec{r}}{dt} . \nabla f$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \left(\vec{q} . \nabla\right) f$$

Hence $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \left(\vec{q} . \nabla\right)$. This is known as material derivative or differential following the motion of the fluid. This implies that we are calculating the rate of change of some quantities associated with the same fluid particle as it moves about. In cartesian coordinate system, this material derivative is expressed as

$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$ where u, v, w are the velocity components of a fluid paritcle at (x, y, z) at time t in the direction of the axes.

If a_x, a_y and a_z are the acceleration components of the fluid particle, then acceleration in vector notation is $\vec{a} = \frac{d\vec{q}}{dt} = \frac{\partial \vec{q}}{\partial t} + \left(\vec{q} . \nabla\right) \vec{q}$

and hence $\vec{a}_x = \frac{du}{dx} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}$,

$$\vec{a}_y = \frac{dv}{dy} = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z},$$

$$\vec{a}_z = \frac{dw}{dz} = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}.$$

10.9 Stream Lines

A stream line is a curve drawn in the fluid such that at any instant of time the tangent at point is in the direction of motion of the fluid at that point. In other words, the stream lines are imaginary lines in the fluid such that the tangent at each point represents the direction of the velocity. Stream lines are also called lines of flow.

If $d\vec{r}$ be the element of arc length along a stream line and $\vec{q}$ be the fluid velocity then the directions of tangent and the velocity are parallel. So the equation of stream line is given by the relation $\vec{q} \times d\vec{r} = 0$

$$\Rightarrow \left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right) \times \left(u\hat{i} + v\hat{j} + w\hat{k}\right) = 0$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ dx & dy & dz \\ u & v & w \end{vmatrix} = 0$$

$$\Rightarrow (w\,dy - v\,dz)\hat{i} - (w\,dx - u\,dz)\hat{j} + (v\,dx - u\,dy)\hat{k} = 0$$

Hence $w\,dy - v\,dz = 0$

$$w\,dx - u\,dz = 0$$

$$v\,dx - u\,dy = 0$$

From these, we arrive at

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dw}{z}$$

which is the differential equation of stream lines in cartesian, coordinates, where u, v, w are the velocity components in the direction of three axes at any point (x, y, z).

Thus the velocity of the fluid at any point on the stream line is along the tangent to it at the point.

10.9.1 Stagnation Point

The two stream lines cannot intersect except at a point where the velocity is zero and that point is called the stagnation point.

10.9.2 Stream filament

If stream lines are drawn through every point of a closed curve then we get a stream tube. A stream tube whose cross section is a curve of infinitesimal dimension, is called a stream filament.

In the steady motion, the product of the speed and cross-section is constant along stream filament of a liquid. The stream filament is widest at places where the speed is least and is narrowest at place where the speed is greatest.

10.10 Path lines

A path line is the curve or trajectory along which a particular fluid particle travels during its motion. The differential equation of path lines are

$$\frac{d\vec{r}}{dt} = \vec{q}$$

and $\frac{dx}{dt} = u \; ; \; \frac{dy}{dt} = v \; ; \; \frac{dz}{dt} = w$

where $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

10.11 Difference between the Stream line and Path line

The stream lines indicate how each particle moves at a given instant while the path lines indicate how a given particle moves at each instant. In case of steady motion, the stream lines and the path line become identical.

10.12 Surfaces Orthogonal to Stream Lines

We know that the curve given by differential equations

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad ...(1)$$

orthogonally cut the surfaces whose differential equation is

$$u\,dx + v\,dy + w\,dz = 0 \qquad ...(2)$$

If the curve (1) represents the stream line, then the surface (2) is cut orthogonally by the stream line if $(u\,dx + v\,dy + w\,dz)$ is integrable. To find the integrability condition for equation (2), let

$$d\varphi = \frac{1}{\lambda}\,(u\,dx + v\,dy + w\,dz)$$

or $$\frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz = \frac{1}{\lambda}\,(u\,dx + v\,dy + w\,dz)$$

$$\Rightarrow \quad u = \lambda\,\frac{\partial \varphi}{\partial x}\ ;\ v = \lambda\,\frac{\partial \varphi}{\partial y}\ ;\ w = \lambda\,\frac{\partial \varphi}{\partial z}$$

Hence $$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \lambda\,\frac{\partial^2 \varphi}{\partial y \partial z} + \frac{\partial \lambda}{\partial y}\,\frac{\partial \varphi}{\partial z} - \lambda\,\frac{\partial^2 \varphi}{\partial y \partial z} - \frac{\partial \lambda}{\partial z}\,\frac{\partial \varphi}{\partial y}$$

or $$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial \lambda}{\partial y}\cdot\frac{\partial \varphi}{\partial z} - \frac{\partial \lambda}{\partial z}\,\frac{\partial \varphi}{\partial y} \qquad ...(3)$$

Similarly $$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{\partial \lambda}{\partial z}\,\frac{\partial \varphi}{\partial x} - \frac{\partial \lambda}{\partial x}\cdot\frac{\partial \varphi}{\partial z} \qquad ...(4)$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial \lambda}{\partial x}\cdot\frac{\partial \varphi}{\partial y} - \frac{\partial \lambda}{\partial y}\cdot\frac{\partial \varphi}{\partial x} \qquad ...(5)$$

On multiplying (3), (4) and (5) by u, v and w respectively then adding, we obtain

$$u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

Which is the condition that such orthogonal surfaces exist.

10.13 Two dimensional motion

When the fluid motion is such that it is same in all planes parallel to a fixed plane and also there is no velocity parallel to the fixed line normal to fixed plane, it is called two dimensional motion. Generally the fixed plane is taken as xy-plane and the fixed line is taken as z-axis. Hence in two diamensional motion, the velocity components are only u and v with $w=0$, where u and v are functions of x, y and t only.

10.14 Stream function in two diamensions

Let u and v be the velocity components in two diamensional motion, then the differential equation of the stream lines is

$$\frac{dx}{u} = \frac{dy}{v} \Rightarrow v\,dx - u\,dy = 0 \qquad ...(1)$$

Which is the differential equation of the form

$$M\,dx + N\,dy = 0$$

It is an exact differential equation if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$$\text{ie. } \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad ...(2)$$

which is the equation of continuity for the incompressible fluid in two dimensions. Hence, eqn (1) is exact, say $d\psi$ then, $v\,dx - u\,dy = d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0$ ...(3)

So that $u = \frac{-\partial \psi}{\partial y}$ and $v = \frac{\partial \psi}{\partial x}$ ...(4)

which leads to $d\,\psi = 0$

or ψ = constant

This function ψ is known as the stream function or current function. The stream function is constant along a stream line and consequantly equation of stream lines are obtained from $\psi = c$, by giving arbitrary values to the constant c.

10.15 Physical significance of stream function

Let A be any fixed point and P is any orbitrary point in the plane. ABP and ACP be two of possible curves joining A and P. These curves bound a region R between them and we assume that no fluid can either be created nor distroyed in R. If the fluid motion is from right to left then

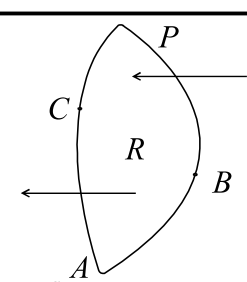


fig. 10.2

the rate at which the fluid flows in R across ABP is equal to the rate at which the fluid flows out of R from right to left across ACP. Thus the flux across ACP is equal to the flux across any curve joining A and P. For any fixed point A, the flux solely depends on the position of P and time t. Thus flux is called the stream function and is denoted by ψ. The stream function ψ is a function of the position and time. The difference of two values of the current function at two points in the plane represents the flow across any line joining the two point.

10.16 Velocity in terms of stream function

Let δs be an element of an arc AB Only the velocity perpendicular to δs will contribute to two flux across δs. The velocity along δs contributes nothing

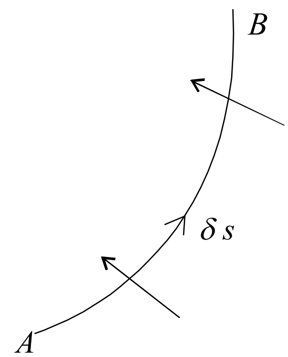


fig. 10.3

$\because$ Fluid across the element of arc δs

$$(\psi + \delta\psi) - \psi = \text{normal velocity} \times \delta s$$

$$\Rightarrow \text{normal velocity} = \frac{(\psi + \delta\psi) - \psi}{\delta s} = \frac{\delta\psi}{\delta s}$$

$$\text{Hence normal velocity} = \underset{\delta s \to 0}{Lt} \frac{\delta\psi}{\delta s} = \frac{\partial\psi}{\partial s}.$$

Where ψ and $\psi + \delta\psi$ are the values of stream function at A and B respectively.

Velocity in terms of stream function in cartesian coordinate

If u and v are components of velocity at the point $P(x, y)$

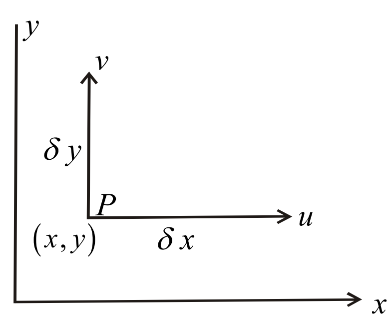


fig. 10.4

Velocity from right to left across δy

$$= -u = \lim_{\delta y \to 0} \frac{\delta \psi}{\delta y} = \frac{\partial \psi}{\partial y} \Rightarrow u = -\frac{\partial \psi}{\partial y}$$

and velocity from right to left across δx

$$= v = \lim_{\delta x \to 0} \frac{\delta \psi}{\delta x} = \frac{\partial \psi}{\partial x} \Rightarrow v = \frac{\partial \psi}{\partial x}$$

Hence $u = \frac{-\partial \psi}{\partial y}$; $v = \frac{\partial \psi}{\partial x}$; are the velocity components in terms of stream function in cartesian coordiate as seen earlier.

If q_r and q_θ be the velocity components in the directions of r and θ respectively then velocity form right to left acorss $r\delta\theta = -q_r$

$$= \lim_{\delta\theta \to 0} \frac{\delta \psi}{r\delta\theta} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

and velocity from right to left across δr

$$= q_\theta$$

$$= \lim_{\delta r \to 0} \frac{\delta \psi}{\delta r} = \frac{\partial \psi}{\partial r}$$

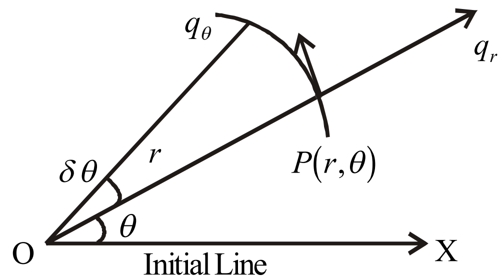


fig. 10.5

Thus,. $q_r = -\frac{1}{r}\frac{\delta \psi}{\delta \theta}$ and $q_\theta = \frac{\partial \psi}{\partial r}$ are the velocity components in terms of stream function in polar coordinate.

Self Learning exercise :

1. Fill in the blanks in following

(a) Fluid which obeys Newton's law of viscosity in known as ....

(b) Fluid in which density is not constant is known as ....

(c) The fluid flow in which all physical properties of the fluid in the motion vary with the time is known as ....

(d) The ratio of the coefficient of viscosity to the density of the fluid is known as the coefficient of ....

(e) Generally air is considered as .... fluid.

2. A flow, in which the fluid particle possess different velocity at each section of the channel is known as

(a) uniform flow (b) non-uniform flow

(c) unsteady flow (d) barotropic flow

3. In which approach the fluid particle is selected and studied its motion?

(a) Lagrangian approch (b) Eulerian approch

(c) Both (d) None of these

4. In which method, a point is fixed in space and study the motion of fluid particle as they pass through the point?

(a) Lagrange method (b) Euler's method

(c) In both methods (d) None of these

5. In the stream line the direction of tangent and velocity are

(a) perpendicullar (b) parallel

(c) opposite (d) not defined

6. The stream line and path lines are same if the flow motion is

(a) unsteady (b) steady

(c) turbulent (d) of any type

7. Write down the velocity components in terms of stream function in polar coordinates of any fluid motion.

8. Find the equation of the stream line for the flow where $u = -x$ and $v = y$.

9. Define stream function and its physical significance.

10. Determine the acceleration of a fluid particle of the flow field

$$\vec{q} = xy^2 t\,\hat{i} + x^2 y t\,\hat{j} + x y z\,\hat{k}$$

Example 1 : For a two-dimensional flow the velocities at a point in the fluid may be expressed in the Eulerian coordinates by

$$u = x + y + 2t \text{ and } v = 2y + t$$

Determine the Lagrange coordinate as function of the initial position x_0 and y_0 and the time t.

Solution : Given $u = x + y + 2t$ and $v = 2y + t$ ...(1)

then $u = \dfrac{dx}{dt} = x + y + 2t$ and $v = \dfrac{dy}{dt} = 2y + t$ ...(2)

or $Dx - x - y = 2t \quad \Rightarrow (D-1)x - y = 2t$ ...(3)

and $Dy - 2y = t \quad \Rightarrow (D-2)y = t$ ...(4)

where $D = \dfrac{d}{dt}$.

From eqn (4) we have

$$y = \frac{1}{(D-2)} t$$

$$y = C_1 e^{2t} - \frac{1}{4}(2t+1) \qquad \text{...(5)}$$

and using (5) in (3) we have

$$(D-1)x - C_1 e^{2t} + \frac{1}{4}(2t+1) = 2t$$

or $$(D-1)x = C_1 e^{2t} + \frac{1}{4}(6t-1)$$

or $$x = C_1 e^{2t} + C_2 e^{t} - \frac{1}{4}(6t+5) \qquad \text{...(6)}$$

$$\because t = 0;\ x = x_0;\ y = y_0$$

From eqn (5) $$y_0 = C_1 - \frac{1}{4} \Rightarrow C_1 = y_0 + \frac{1}{4}$$

and from eqn (6) $$x_0 = y_0 + \frac{1}{4} + C_2 - \frac{5}{4} \Rightarrow C_2 = x_0 - y_0 + 1$$

Hence $$x = (x_0 - y_0 + 1)e^{t} + \left(y_0 + \frac{1}{4}\right)e^{2t} - \frac{1}{4}(6t+5)$$

$$y = \left(y_0 + \frac{1}{4}\right)e^{2t} - \frac{1}{4}(2t+1)$$

Where, the Lagrangian system is the function of the initial position x_0, y_0 and the time t.

Example 2 : The velocity components for a two dimensional flow system can be given in the Eulerian system by

$$u = 2x + 2y + 3t \ ;\ v = x + y + \frac{1}{2}t$$

Find the displacement of a fluid particle in the Lagrangian system.

Solution : The velocities may be expressed in terms of the displacement as

$$u = \frac{dx}{dt} = 2x + 2y + 3t \Rightarrow (D-2)x - 2y = 3t \qquad \text{...(1)}$$

$$v = \frac{dy}{dt} = x + y + \frac{1}{2} \Rightarrow (D-1)y - x = \frac{t}{2} \qquad \text{...(2)}$$

On eliminating x from (1) and (2), we have

$$D(D-3)y = 2t + \tfrac{1}{2}$$

$$\Rightarrow \; y = C_1 + C_2 e^{3t} - \frac{7}{18}t - \frac{1}{3}t^2 \qquad ...(3)$$

Substituting the value of y in the equation (2), we have

$$x = -C_1 + 2C_2 e^{3t} + \frac{1}{3}t^2 - \frac{7}{9}t - \frac{7}{18} \qquad ...(4)$$

The arbitrary constants C_1 and C_2 are determined by using the initial condtions :

$x = x_0$, $y = y_0$ at $t = 0$, we obtain

$$y_0 = C_1 + C_2 \,;\; x_0 = -C_1 + 2C_2 - \frac{7}{18}$$

Hence

$$C_1 = -\frac{1}{3}\left(x_0 - 2y_0 + \frac{7}{18}\right)$$

$$C_2 = \frac{1}{3}\left(x_0 + y_0 + \frac{7}{18}\right) \qquad ...(5)$$

Using (5) in (3) and (4), we have

$$x = \frac{1}{3}\left[x_0 - 2y_0 + \frac{7}{18}\right] + \frac{2}{3}\left[x_0 + y_0 + \frac{7}{18}\right]e^{3t} + \frac{1}{3}t^2 - \frac{7}{9}t - \frac{7}{18}$$

and

$$y = -\frac{1}{3}\left[x_0 - 2y_0 + \frac{7}{18}\right] + \frac{1}{3}\left[x_0 + y_0 + \frac{7}{18}\right]e^{3t} - \frac{7}{18}t + \frac{1}{3}t^2$$

which are the required displacements x and y in the Lagrangian system where $x_0,\, y_0$ are initial position and the time t.

Example 3 : Find the equation of the stream lines for the flow $\vec{q} = x\hat{i} - y\hat{j}$

Solution : By the definition of the stream lines, we have

$$\vec{q} \times d\vec{r} = 0$$

or

$$\left(x\hat{i} - y\hat{j}\right) \times \left(dx\,\hat{i} + dy\,\hat{j}\right) = 0$$

$$\Rightarrow \quad \left(x\,dy + y\,dx\right)\hat{k} = 0$$

$$\Rightarrow \quad x\,dy + y\,dx = 0$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{-dy}{y}$$

on solving

$$xy = c$$

which represents the equation of stream line.

Example 4 : Find the equation of the stream lines passing through the point (1, 1, 1) for an incompressible flow $\vec{q} = 2x\hat{i} - y\hat{j} - z\hat{k}$.

Solution : Given $\vec{q} = 2x\hat{i} - y\hat{j} - z\hat{k} = u\hat{i} - v\hat{j} - w\hat{k}$

then $u = 2x$; $v = -y$ and $w = -z$

The differential equation of the stream lines are

$$\frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{-z} \qquad ...(1)$$

Taking the first two members of (1), we obtain

$$xy^2 = C_1 \qquad ...(2)$$

Similary on taking first and third member of (1), we have

$$\frac{dx}{x} = -2\frac{dz}{z}$$

$$xz^2 = C_2 \qquad ...(3)$$

Where C_1 and C_2 are integration constants.

The stream lines are passing through point (1, 1, 1) then

$$C_1 = 1 = C_2$$

Hence the required stream lines are

$$xy^2 = 1 \text{ and } xz^2 = 1$$

Example 5 : Given $u = -Wy$, $v = Wx$ and $w = 0$: show that the surfaces intersecting the stream line orthogonally exist and are the planes through z-axis.

Solution : The diffrential equation of stream line

are $$\frac{dx}{-Wy} = \frac{dy}{Wx} = \frac{dz}{0} \qquad ...(1)$$

$\Rightarrow$ $Wx\,dx + Wy\,dy = 0$ and $dz = 0$

or $x\,dx + y\,dy = 0$ and $dz = 0$

On integrating

$$x^2 + y^2 = C_1 \text{ and } z = C_2 \qquad ...(2)$$

Hence the stream line are circles given by the intersection of surfaces (2).

Now, the surfaces which cut the stream lines orthogonally are

$$u\,dx + v\,dy + w\,dz = 0$$

i.e. $$-Wy\,dx + Wx\,dy + 0.\,dz = 0$$

or $$\frac{dx}{x} - \frac{dy}{y} = 0$$

On integrating

$$\frac{x}{y} = c \quad \Rightarrow x = cy$$

which represents a plane through z-axis and cuts the stream line (2) orthogonally.

Example 6 : The velocity components of fluid are given by $u = -\frac{c^2 y}{r^2}$, $v = \frac{c^2 x}{r^2}$ and $w = 0$ where r distance from z-axis, find the surfaces which are orthogonal to stream line, the liquid being homogenous.

Solution : The differential equation of stream lines

are $$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

i.e. $$\frac{dx}{-c^2 y / r^2} = \frac{dy}{c^2 x / r^2} = \frac{dz}{0}$$

or $$\frac{-dx}{y} = \frac{dy}{x} = \frac{dz}{0}$$

which gives $x^2 + y^2 = C_1$ and $z = C_2$

The sufraces which cut the stream line orthogonally, are given by

$$u\,dx + v\,dy + w\,dz = 0$$

i.e. $$\frac{-c^2 y}{x^2 + y^2} dx + \frac{c^2 x}{x^2 + y^2} dy = 0$$

$$\Rightarrow \quad -\frac{dx}{x} + \frac{dy}{y} = 0$$

$$\Rightarrow \quad \frac{y}{x} = c_3 \quad \Rightarrow y = c_3 x$$

The curve $y = c_3 x$ gives the surfaces which are othogonal to the stream lines $x^2 + y^2 = c_1$ and $z = c_2$.

Example 7 : The velocity field at a point in fluid is given as $\vec{q} = \frac{x}{t}\hat{i} + y\hat{j} + o.\hat{k}$, obtain path lines.

Solution : The differential equation of path lines are given by

$$\vec{q} = \frac{d\vec{r}}{dt} \Rightarrow \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = \frac{x}{t}\hat{i} + y\hat{j} + o.\hat{k}$$

or $$\frac{dx}{dt} = \frac{x}{t} \Rightarrow \frac{dx}{x} = \frac{dt}{t}$$

$$\Rightarrow \frac{x}{t} = c_1 \Rightarrow x = c_1 t$$

$$\frac{dy}{dt} = y \Rightarrow \frac{dy}{y} = dt \Rightarrow y = c_2 e^t \text{ and } dz = 0 \Rightarrow z = c_3.$$

Hence the path lines are

$$x = c_1 t \; ; \; y = c_2 e^t \text{ and } z = c_3$$

Example 8 : Determine the stream line if the velocity of an incompressible fluid at the point (x, y, z) is given by

$$\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2 - r^2}{r^5}\right)$$

where $r^2 = x^2 + y^2 + z^2$

Solution : Given that $u = \frac{3xz}{r^5}$, $v = \frac{3yz}{r^5}$ and $w = \frac{3z^2 - r^2}{r^5}$ then the differential equation of the stream line are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

i.e. $$\frac{dx}{\frac{3xz}{r^5}} = \frac{dy}{\frac{3yz}{r^5}} = \frac{dz}{\frac{(3z^2 - r^2)}{r^5}}$$

or $$\frac{dx}{3xz} = \frac{dy}{3yz} = \frac{dz}{(3z^2 - r^2)} \qquad ...(1)$$

Taking the first two member of (1), we have

$$\frac{y\,dx}{3xyz} = \frac{x\,dy}{3xyz} \Rightarrow \frac{dx}{x} = \frac{dy}{y}$$

$$\Rightarrow x = c_1 y \qquad ...(2)$$

Now from (1), we have

$$\frac{dx}{3xz} = \frac{dy}{3yz} = \frac{dz}{\left(3z^2 - r^2\right)} = \frac{x\,dx + y\,dy + z\,dz}{3x^2 z + 3y^2 z + 3z^2 - zr^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{3z\left(x^2 + y^2 + z^2\right) - r^2 z}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{2z\left(x^2 + y^2 + z^2\right)} \qquad ...(3)$$

Now from (3), we have

$$\frac{dx}{3xz} = \frac{x\,dx + y\,dy + z\,dz}{2z\left(x^2 + y^2 + z^2\right)}$$

$$\Rightarrow \quad \frac{2}{3}\frac{dx}{x} = \frac{1}{2}.\frac{2\,(x\,dx + y\,dy + z\,dz)}{\left(x^2 + y^2 + z^2\right)}$$

On integrating

$$\frac{2}{3}\log x = \frac{1}{2}\log\left(x^2 + y^2 + z^2\right) + \log C_2$$

$$\Rightarrow \quad x^{2/3} = c_2 \left(x^2 + y^2 + z^2\right)^{1/2} \qquad ...(4)$$

Hence the required stream lines are the curves of intersection of curves. $x = c_1 y$ and $x^{2/3} = c_2\left(x^2 + y^2 + z^2\right)^{1/2}$.

Example 9 : Find the stream lines and paths of the particles when

$$u = \frac{x}{(1+t)},\ v = \frac{y}{(1+t)},\ w = \frac{z}{(1+t)}$$

Solution : Stream lines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\Rightarrow \quad \frac{dx}{x/(1+t)} = \frac{dy}{y/(1+t)} = \frac{dz}{z/(1+t)}$$

or $\quad \Rightarrow \quad \dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dz}{z} \qquad ...(1)$

Taking the first two members of (1), $\frac{dx}{x}=\frac{dy}{y}\Rightarrow\frac{x}{y}=c_1$ and on taking the first and third members of (1), $\frac{dx}{x}=\frac{dz}{z}\Rightarrow\frac{x}{z}=c_2$ Hence the required stream lines are given by the intersection of curves $x=c_1y$ and $x=c_2z$.

The p ath of the particles (Path line) are given by

$$\therefore\ u=\frac{dx}{dt}=\frac{x}{(1+t)}\Rightarrow\frac{dx}{x}=\frac{dt}{(1+t)}\Rightarrow\log x=\log(1+t)+\log c_3$$

$$\Rightarrow x=c_3(1+t)$$

similarly $$v=\frac{dy}{dt}=\frac{y}{(1+t)}\Rightarrow\frac{dy}{y}=\frac{dt}{(1+t)}\Rightarrow y=c_4(1+t)$$

and $$z=c_5(1+t)$$

Hence the required path of the fluid particles are given by $x=c_3(1+t)$; $y=c_4(1+t)$; $z=c_5(1+t)$.

Example 10 : Find the stream lines and path lines of the particles of the velocity field.

$$u=\frac{x}{(1+t)},\ v=y \text{ and } w=0$$

Solution : Given that

$$u=\frac{x}{1+t}\ ;\ v=y\ ;\ w=0 \qquad ...(1)$$

Stream line :

The differential equation of the stream lines are

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

i.e. $$\frac{dx}{x/(1+t)}=\frac{dy}{y}=\frac{dz}{0} \qquad ...(2)$$

On integrating

$$(1+t).\frac{dx}{x}=\frac{dy}{y} \text{ and } dz=0$$

$$\Rightarrow \quad (1+t)\log x=\log y-\log c_1 \text{ and } z=c_2$$

or $$y=c_1x^{(1+t)} \text{ and } z=c_2$$

Hence in the plane $z = c_2$, the stream lines are $y = c_1 x^{(1+t)}$.

Path Line :

The path lines are given by

$$u = \frac{dx}{dt} = \frac{x}{(1+t)} \Rightarrow \frac{dx}{x} = \frac{dt}{(1+t)} \Rightarrow x = c_3(1+t)$$

$$v = \frac{dx}{dt} = y \Rightarrow \frac{dy}{y} = dt \Rightarrow y = c_4 e^t$$

$$w = \frac{dz}{dt} = 0 \Rightarrow dz = 0 \Rightarrow z = c_5$$

Hence the path lines are

$$x = c_3(1+t);\ y = c_4 e^t \text{ and } z = c_5.$$

Example 11 : A velocity field is given by $\vec{q} = -x\hat{i} + (y+t)\hat{j}$. Find the stream function and the stream line for this field at $t = 2$.

Solution : Given that the velocity field $\vec{q} = -x\hat{i} + (y+t)\hat{j}$ then $u = -x$ and $v = y + t$.

We know that

$$u = -\frac{\partial \psi}{\partial y} = -x \qquad ...(1)$$

and $$v = \frac{\partial \psi}{\partial x} = y + t \qquad ...(2)$$

By integrating (1) with respect to y, we have

$$\psi = xy + f(x,t) \qquad ...(3)$$

or $$\frac{\partial \psi}{\partial x} = y + \frac{\partial f}{\partial x} = y + t$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = t$$

or $$f(x,t) = xt + g(t) \qquad ...(4)$$

Using (4) in (3) we have

$$\psi = xy + xt + g(t)$$

If $t = 2$; $\psi = xy + 2x + g(2)$ $\quad \because g(2) =$ constant

The stream lines are given by ψ = constant

therefore $\quad x(y+2) =$ constant

which are rectangular hyperbolas.

Example 12 : Find the stream function $\psi(x, y, t)$ for the given velocity field $u = \bigcup t$ and $v = x$.

Solution : We know that

$$u = -\frac{\partial \psi}{\partial y} \text{ and } v = \frac{\partial \psi}{\partial x}$$

Then here given that $u = \bigcup t$ and $v = x$

$$\therefore \frac{\partial \psi}{\partial y} = -\bigcup t \Rightarrow \psi = -\bigcup y t + f(x,t) \quad ...(1)$$

$$\Rightarrow \frac{\partial \psi}{\partial x} = \frac{\partial f}{\partial x} \quad ...(2)$$

Also given that

$$v = \frac{\partial \psi}{\partial x} = x \quad ...(3)$$

From (2) and (3) we have

$$\frac{\partial f}{\partial x} = x \Rightarrow f = \frac{1}{2}x^2 + g(t) \quad ...(4)$$

Using (4) in (1) we have

$$\psi = -\bigcup y t + \frac{1}{2}x^2 + g(t)$$

Which in the required stream function.

Example 13 : If $u = 2xy$ and $v = (a^2 + x^2 - y^2)$ are the velocity components of a fluid motion, then find the stream function.

Solution : By the definition of stream function, we know that

$$u = -\frac{\partial \psi}{\partial y} \text{ and } v = \frac{\partial \psi}{\partial x}$$

Here given that $u = 2xy$ and $v = (a^2 + x^2 - y^2)$

so $$\frac{\partial \psi}{\partial y} = -2xy \quad ...(1)$$

and $$\frac{\partial \psi}{\partial x} = a^2 + x^2 - y^2 \qquad ...(2)$$

Integrating (1) with respect to y, we have

$$\psi = -xy^2 + f(x,t) \qquad ...(3)$$

Deffrentiating (3), w.r. to x, we have

$$\frac{\partial \psi}{\partial x} = -y^2 + \frac{\partial f}{\partial x} \qquad ...(4)$$

Using (2) in (4) we obtain

$$\frac{\partial f}{\partial x} = a^2 + x^2$$

On integrating w.r. to x, we have

$$f(x,t) = \left(a^2 x + \frac{x^3}{3}\right) + g(t)$$

which in the required stream function.

10.17 Summary

In this unit, we studied about charateristics of a fluid, kinds of fluids, kinds of fluid flow approches to solve hydrodynamical problems. We also studied about stream line, path line and stream function in two dimensional flow. Now we are in position to study the equation of continuity in various coordinate systems.

10.18 Answers to self Learning exercise

(1) (a) Newtonian Fluid

(b) Compressible Fluid

(c) Unsteady Flow

(d) Kinematic Viscosity

(e) Compressible

(2) (b) Non Uniform Flow

(3) (a) Lagrangian Approch

(4) (b) Euler's Method

(5) (b) parallel

(6) (b) steady

(7) $q_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}$ and $q_\theta = \frac{\partial \psi}{\partial r}$

(8) $xy =$ constant

(9) See text

(10) $\vec{a} = \left(xy^2 + xy^4t + 2x^3y^2t^2\right)\hat{i} + \left(x^2y + 2x^2y^3t + x^4yt^2\right)\hat{j} + \left(xy^3z + x^3yz\,t + x^2y^2z\right)\hat{k}$.

10.19 Exercise

1. The velocity components in two dimensional flow system in Eulerian system are given by

$$u = (x+y) + 2t \ ; \ v = (x+y) + t$$

The find the displacment of a fluid particle in Lagrangian system

Ans.

$$\left[\begin{array}{l} x = c_1 + c_2e^{2t} - \frac{3}{4}t + \frac{1}{4}t^2 - \frac{3}{4} \\ y = c_1 + c_2e^{2t} - \frac{3}{4}t - \frac{1}{4}t^2 \\ \text{where } c_1 = -\frac{1}{2}\left[x_0 - y_0 + \frac{3}{4}\right] \\ \qquad c_2 = \frac{1}{2}\left[x_0 + y_0 + \frac{3}{4}\right] \\ \text{where } x = x_0,\ y = y_0 \text{ when } t = 0 \end{array}\right]$$

2. The velocity components in a two dimensional flow field for an incompressible fluid are given by $u = -3y^2$ and $v = -6x$ then find the equation of stream line at the point (1, 1)

$\left[3x^2 = y^3 + 2\right]$

3. Find the equation of the stream line for the flow $\vec{q} = e^x \cosh y\,\hat{i} - e^x \sinh y\,\hat{j}\cdot$

$\left[e^x \sinh y = c\right]$

4. If the velocity components of a flow field are given by

$$u = \frac{3x^2 - r^2}{r^5} \ ; \ v = \frac{3xy}{r^5} \text{ and } w = \frac{3xz}{r^5}$$

where $r^2 = x^2 + y^2 + z^2$ then prove that the stream lines are intersection of the surfaces

$$\left(x^2 + y^2 + z^2\right)^3 = c\left(y^2 + z^2\right)$$

by the plane passing through OX.

5. Show that the velocity vector $\vec{q}$ is everywhere tangent to lines in the xy-plane along which ψ = constant.

6. What are stream lines? Are stream lines and the path lines of a fluid always the same?

7. Determine the stream line if the motion is specified by

$$\vec{q} = \frac{k\left(x\hat{i} - y\hat{j}\right)}{\left(x^2 + y^2\right)} ; \text{ where } k \text{ be the constant.}$$

$[x = cy]$

□□□□

UNIT - 11
Equation of Continuity-I

Structure of the unit

11.0 Objective

In this unit our objective is to study about the velocity potential, rotational and irrotational motion in fluid flow. We will also study about the concept of conservation of mass in mathematical form as equation of continuity.

11.1 Introduction

The equation of continuity simply expresses the law of conservation of mass in a mathematical expression. The law of conservation of mass states that the fluid mass can neither be created nor dstroyed. The fluid motion is possible only if the equation of continuity is satisfies for this fluid motion.

11.2 Velocity Potential (Velocity function)

Suppose that the fluid velocity of a particle at a point (x, y, z) at any instant is $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$.

If it is assumed that the expression $u\,dx + v\,dy + w\,dz$ is an exact differential, it is conventionally convenient to express

$$-d\phi = u\,dx + v\,dy + w\,dz$$

or $$-\left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz\right) = u\,dx + v\,dy + w\,dz$$

then $u = -\frac{\partial \phi}{\partial x}$, $v = -\frac{\partial \phi}{\partial y}$ and $w = -\frac{\partial \phi}{\partial z}$

which clearly show that ϕ is a function of x, y and z only. Hence

$$\phi = \phi(x, y, z)$$

Then the velocity of fluid at the point is given by

$$\vec{q} = -\frac{\partial \phi}{\partial x}\hat{i} - \frac{\partial \phi}{\partial y}\hat{j} - \frac{\partial \phi}{\partial z}\hat{k}$$

$$= -\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$\vec{q} = -\vec{\nabla}\phi = -grad\,\phi$$

Here, ϕ is called the velocity potential or velocity function. The negative sign is conventional but it ensures that the flow is from higher potential to lower potential.

11.3 Rotational and Irrotational motion

Rotation of a fluid particle is defined as the average of the angular velocities of two mutually perpendicular linear sides of elementary rectangular element of the fluid particle. In presence of this average the motion is said to be rotational and if this average is zero then the motion is said to be irrotational.

Consider a rectangular element ABCD in two dimensional flow such that AB = δx and AD = δy as shown in the figure.

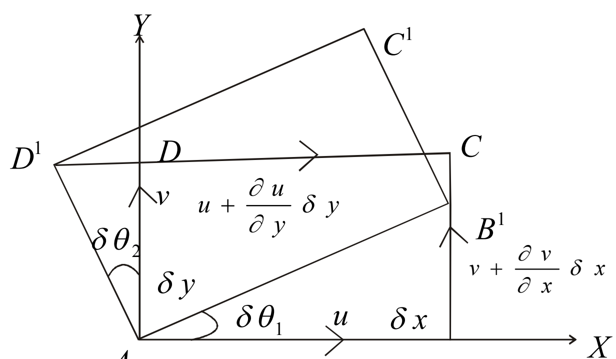


Figure 11.1

Upon rotating the element about A during a small interval δt, the sides of element become AB' and AD'. B' and D' approximately lie on BC and extended CD.

Let u and v be the components of velocity at A then the components of velocity along BC and DC are $v + \frac{\partial v}{\partial x}\delta x$ and $u + \frac{\partial u}{\partial y}\delta y$ respectively.

$\therefore$ velocity of B relative to A along $BC = \frac{\partial v}{\partial x}\delta x$

and velocity of D relative to A along $DC = \frac{\partial u}{\partial y}\delta x$

Hence is small interval δt, we have

$$BB' = \frac{\partial v}{\partial x}\delta x \delta t \text{ and } DD' = -\frac{\partial u}{\partial y}\delta y \delta t$$

If $\delta\theta_1$ and $\delta\theta_2$ are the angles through which AB and AD respectively have turned in time δt, then the angular velocity of AB about the z-axis i.e. perpendicular to the plane through A is

$$\lim_{\delta t \to 0}\frac{\delta\theta_1}{\delta t} = \lim_{\delta t \to 0}\frac{\tan\delta\theta_1}{\delta t} \qquad [\delta\theta_1 \; in\, small \Rightarrow \delta\theta_1 = \tan\delta\theta_1]$$

$$= \lim_{\delta t \to 0}\frac{BB'/AB}{\delta t} = \lim_{\delta t \to 0}\frac{BB'}{AB.\delta t}$$

$$= \lim_{\delta t \to 0}\frac{\frac{\partial v}{\partial x}\delta x.\delta t}{\delta x.\delta t} = \frac{\partial v}{\partial x}$$

$$\lim_{\delta t \to 0}\frac{\delta\theta_1}{\delta t} = \frac{\partial v}{\partial x}$$

Again the angular velocity of AD about z-axis is

$$\lim_{\delta t \to 0}\frac{\delta\theta_2}{\delta t} = \lim_{\delta t \to 0}\frac{\tan\delta\theta_2}{\delta t}$$

$$= \lim_{\delta t \to 0}\frac{DD'/AD}{\delta t} = \lim_{\delta t \to 0}\frac{DD'}{AD.\delta t}$$

$$= \lim_{\delta t \to 0}\frac{-\frac{\partial u}{\partial y}\delta y\,\delta t}{\delta y.\delta t}$$

$$= -\frac{\partial u}{\partial y}$$

The average of the angular velocities of AB and AD about the z-axis, considered perpendicular to $x - y$ plane, be w_z and is given as

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad ...(1)$$

In the similar manner the average angular velocity components ω_x, ω_y and ω_z in three dimensional flow may be obtained as follows :

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) ; \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

and $$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad ...(2)$$

Hence the angular velocity $\vec{\omega}$ of a fluid element is given by

$$\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k} \qquad ...(3)$$

This $\vec{\omega}$ is known as rotation and ω_x, ω_y and ω_z are known as the camponents of rotation.

If this average angular velocity is zero then the particle is said to have zero rotation and the flow is said to be irrotational. Therefore, in two dimensional irrotational flow, we have

$$\omega_z = 0 \Rightarrow \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

or $$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

In three dimensional flow, the conditions for irrotational are given by

$$\vec{\omega} = 0 \Rightarrow \omega_x = 0 ; \omega_y = 0; \omega_z = 0$$

or $$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} ; \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} ; \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

If the fluid motion is rotational then the spin components ξ, η, ζ are given by

$$\xi = \frac{1}{2}\omega_x \qquad \Rightarrow \qquad \xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\eta = \frac{1}{2}\omega_y \qquad \Rightarrow \qquad \eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\zeta = \frac{1}{2}\omega_z \qquad \Rightarrow \qquad \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

We know that the velocity components u and v are functions of x, y, t and $w = 0$ in two dimensional flow. Hence the spin components are given by

$$\xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = 0$$

$$\eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = 0$$

and $$\zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right)\right]$$

or $$2\zeta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi$$

If the motion be irrotational so that $\zeta = 0$ then we obtain

$$\nabla^2 \psi = 0$$

Showing that for irrotational motion of a fluid stream function satisfies Laplace's equation. It is also evident that when the velocity potential ϕ exists, all the above conditions are satisfied, then the motion is irrotational.

11.4 Concept of Equation of Continuity

The equation of continuity simply expresses the law of conservation of mass in a mathematical form. As is known that the law of conservation of mass states that the fluid mass can neither be created nor destroyed. Thus if we consider any given volume of fluid in space bounded by a closed surface, then at any instant, the flow of fluid across the boundary surface from without inwards minus that from within outward must be equal to the increase in the mass of the fluid within the surface.

If V be the volume and ρ be the density of a small element of the fluid, then the equation of continuity states that the mass of the fluid remains unchanged in the time inteval δt following the motion. Therefore

$$\frac{D}{Dt}(\rho V) = 0$$

or $$\rho \frac{DV}{Dt} + V\frac{D\rho}{Dt} = 0$$

11.5 Equation of Continuity (Vector Form) by Euler's Method

Consider a volume V of moving fluid enclosed within a surface S. Let $P(x, y, z)$ be any

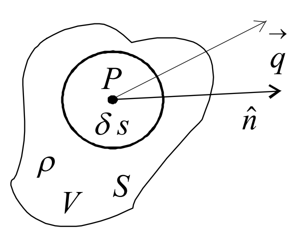


Figure 11.2

point of S and ρ the density of the fluid which can be considered a function of time and space coordinates.

Let δs denote an element of the surface S enclosing P. Let $\hat{n}$ be the unit vector in the direction of the outward drawn normal at any point of the surface δS and $\vec{q}$ be the velocity of the fluid. Then the velocity in the outward drawn normal direction will be $\hat{n}.\hat{q}$. Thus,

rate of mass flow across $\delta s = \rho\left(\hat{n}.\vec{q}\right)\delta S$

The mass of the fluid entering the whole surface S in unit time will be

$$= -\int_S \rho\left(\hat{n}.\vec{q}\right) dS$$

$$= -\int_V \nabla.\left(\rho\vec{q}\right) dV \qquad \text{(By Gauss divergance theorem)} \quad ...(1)$$

Also the mass of the fluid within the volume V bounded by the surface S is.

$$= \int_V \rho\, dV$$

Hence rate of increase of mass within S

$$= \frac{\partial}{\partial t}\int_V \rho\, dV$$

$$= \int_V \frac{\partial \rho}{\partial t}\, dV \qquad ...(2)$$

Then by the law of conservation of the fluid mass the rate of increase of the mass of fluid within V in unit time must be equal to the total rate of mass flowing into V.

Hence from (1) and (2), we have

$$= \int_V \frac{\partial \rho}{\partial t}\, dV = -\int_V \nabla.\left(\rho\vec{q}\right) dV$$

or
$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla.\left(\rho\vec{q}\right)\right] dV = 0$$

Since this result is true for every value of the volume, we get the above expression in the form

$$\frac{\partial \rho}{\partial t} + \nabla.\left(\rho\vec{q}\right) = 0 \qquad ...(3)$$

or
$$\frac{\partial \rho}{\partial t} + div\left(\rho\vec{q}\right) = 0$$

which is the equation of continuity.

Equation (3) may be written as

$$\frac{\partial \rho}{\partial t} + \rho \, div \, \vec{q} + \vec{q} \, . \, grad \, \rho = 0$$

or $$\frac{D\rho}{Dt} + \rho \, div \, \vec{q} = 0 \qquad ...(4)$$

where $\frac{D}{Dt}$ is the known differential following the motion. If the fluid is incompressible and uniform density ρ is constant so that $\frac{\partial \rho}{\partial t} = 0$. Thus for incompressible fluid the equation of continuity becomes

$$div \, \vec{q} = 0 \qquad ...(5)$$

Cor. Thus equation of continuity can be obtained in Cartesian coordinates by taking $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k} \cdot$

Hence from (3)

$$\frac{\partial \rho}{\partial t} + \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) . \left(\hat{i} \rho u + \hat{j} \rho v + \hat{k} \rho w \right) = 0$$

or $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \qquad ...(6)$$

Where u, v and w are components of fluid velocity in x, y and z directions

Eqn (6) can also be derived directly and we will deal with it in next unit. It must also be clear that this approach is due to Eulerian approach of study.

11.6 Equation of Continuity by the Lagrangian Method

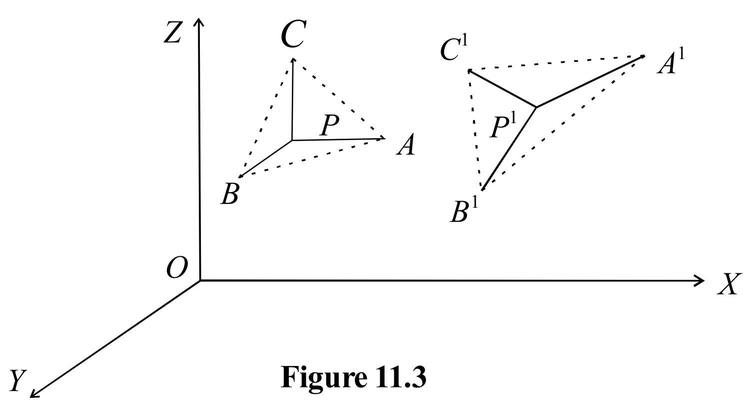


Figure 11.3

At any instant, let the cartesian coordinate of a fluid particle P be (a,b,c). After a time t, let the coordinate of this particle be (x,y,z), then these coordinates will depend on the initial position and the time elapsed.

Hence $x = f(a,b,c,t)$; $y = g(a,b,c,t)$; $z = h(a,b,c,t)$

Consider a fluid tetrahedron such that one of the vertices is at P and the edges are of lengths δa, δb and δc parallel to three axes respectively. Hence the coordinate of the other three vertices are $A(a+\delta a,b,c)$, $B(a,b+\delta b,c)$ and $C(a,b,c+\delta c)$ where δa, δb and δc are taken to be small.

Now if ρ_0 is the initial density of the fluid, then the mass of the fluid within the tetrahedron $PABC$

$$=\frac{1}{6}\rho_0\,\delta a\,\delta b\,\delta c \qquad ...(1)$$

After a time t, let the new positions of P, A, B and C be P', A', B' and C' respectively. If the coordinates of A', B' and C' are (x_1,y_1,z_1), (x_2,y_2,z_2) and (x_3,y_3,z_3) respectively, then

$$x_1 = f(a+\delta a,b,c,t)$$

$$= f(a,b,c,t)+\frac{\partial f}{\partial a}\delta a + \text{(Higher power of } \delta a\text{)}$$

$$x_1 = x+\frac{\partial x}{\partial a}\delta a. \qquad \text{(Neglecting higher power of } \delta a\text{)}$$

Similarly

$$y_1 = g(a+\delta a,b,c) = y+\frac{\partial y}{\partial a}\delta a$$

and $$z_1 = h(a+\delta a,b,c) = z+\frac{\partial z}{\partial a}\delta a$$

Threfor the relative coordinate of A' with respect to P' will be

$$\left(\frac{\partial x}{\partial a}\delta a,\frac{\partial y}{\partial a}\delta a,\frac{\partial z}{\partial a}\delta a\right)$$

Proceding in similar manner, we obtain the relative coordinate of B' and C' with respect to P' as.

$$\left(\frac{\partial x}{\partial b}\delta b,\frac{\partial y}{\partial b}\delta b,\frac{\partial z}{\partial b}\delta b\right) \text{ and } \left(\frac{\partial x}{\partial c}\delta c,\frac{\partial y}{\partial c}\delta c,\frac{\partial z}{\partial c}\delta c\right)$$

Hence the volume of the tetrahedron $P'A'B'C'$ are

$$= \frac{1}{6}\begin{vmatrix} \frac{\partial x}{\partial a}\,\delta a & \frac{\partial y}{\partial a}\,\delta a & \frac{\partial z}{\partial a}\,\delta a \\ \frac{\partial x}{\partial b}\,\delta b & \frac{\partial y}{\partial b}\,\delta b & \frac{\partial z}{\partial b}\,\delta b \\ \frac{\partial x}{\partial c}\,\delta c & \frac{\partial y}{\partial c}\,\delta c & \frac{\partial z}{\partial c}\,\delta c \end{vmatrix}$$

$$= \frac{1}{6}\;\frac{\partial(x\ y\ z)}{\partial(a\ b\ c)}\;\delta a\ \delta b\ \delta c$$

$$= \frac{1}{6}\;J\ \ \delta a\ \delta b\ \delta c \text{ where } J = \frac{\partial(x\ y\ z)}{\partial(a\ b\ c)}$$

If ρ is the density of the fluid after a time t the mass of the fluid within $P'A'B'C'$ will be

$$= \frac{1}{6}\ J\rho\ \delta a\ \delta b\ \delta c \qquad ...(2)$$

Since the mass contained within the tetrahedron does not change due to law of conservation of mass, we have from (1) and (2)

$$\frac{1}{6}\ \rho J\ \delta a\ \delta b\ \delta c = \frac{1}{6}\ \rho_0\,\delta a\ \delta b\ \delta c$$

or $\quad \rho J = \rho_0$

which is the required equation of continuity in Lagrangian form.

11.7 Equivalence of two forms of equation of continuity

It can be shown that the two forms of equation of continuity obtained through Eulerian approach and Lagrangian approach are equivalent.

The equation of continuity in Lagrangian form is

$$\rho J = \rho_0 \qquad \text{so that } \frac{d}{dt}(\rho J) = 0$$

or
$$\rho\frac{dJ}{dt} + J\frac{d\rho}{dt} = 0 \qquad ...(1)$$

where $\frac{d}{dt}$ is the differential following the motion.

The velocity components in the two systems are connected by the equation

$$u = \frac{dx}{dt}\ ,\ v = \frac{dy}{dt} \text{ and } w = \frac{dz}{dt} \qquad ...(2)$$

Also $x = x(a,b,c,t)$; $y = y(a,b,c,t)$; $z = z(a,b,c,t)$ ...(3)

$$\therefore \frac{\partial u}{\partial a} = \frac{\partial}{\partial a}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(\frac{\partial x}{\partial a}\right) \text{ etc.} \quad ...(4)$$

we can demonstrate the following two dimensional differential coefficients to extend to the case of three dimensional coefficient

$$\frac{d}{dt}\left[\frac{\partial(x, y)}{\partial(a, b)}\right] = \frac{d}{dt}\left[\frac{\partial x}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial x}{\partial b}\frac{\partial y}{\partial a}\right]$$

$$= \frac{d}{dt}\left(\frac{\partial x}{\partial a}\right)\frac{\partial y}{\partial b} + \frac{\partial x}{\partial a} \cdot \frac{d}{dt}\left(\frac{\partial y}{\partial b}\right)$$

$$- \frac{d}{dt}\left(\frac{\partial x}{\partial b}\right)\frac{\partial y}{\partial a} - \frac{\partial x}{\partial b}\frac{d}{dt}\left(\frac{\partial y}{\partial a}\right)$$

$$= \frac{\partial u}{\partial a} \cdot \frac{\partial y}{\partial b} + \frac{\partial x}{\partial a} \cdot \frac{\partial v}{\partial b} - \frac{\partial u}{\partial b}\frac{\partial y}{\partial a} - \frac{\partial x}{\partial b}\frac{\partial v}{\partial a}$$

$$= \left(\frac{\partial u}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial u}{\partial b}\frac{\partial y}{\partial a}\right) + \left(\frac{\partial x}{\partial a} \cdot \frac{\partial v}{\partial b} - \frac{\partial x}{\partial b}\frac{\partial v}{\partial a}\right)$$

$$\therefore \frac{d}{dt}\left[\frac{\partial(x, y)}{\partial(a, b)}\right] = \frac{\partial(u, y)}{\partial(a, b)} + \frac{\partial(x, v)}{\partial(a, b)} \quad ...(5)$$

Similarly

$$\frac{d}{dt}\left[\frac{\partial(x, y, z)}{\partial(a, b, c)}\right] = \frac{\partial(u, y, z)}{\partial(a, b, c)} + \frac{\partial(x, v, z)}{\partial(a, b, c)} + \frac{\partial(x, y, w)}{\partial(a, b, c)}$$

$$= \frac{dJ}{dt} \quad ...(6)$$

Also, we know that

$$\frac{\partial u}{\partial a} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial a} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial a} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial a}$$

$$\frac{\partial u}{\partial b} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial b} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial b} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial b}$$

$$\frac{\partial u}{\partial c} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial c} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial c} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial c} \quad ...(7)$$

On Eliminating $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$, we obtain

$$\begin{vmatrix} \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial a}-\frac{\partial u}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial b}-\frac{\partial u}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial c}-\frac{\partial u}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix} = 0 \qquad ...(8)$$

or $$\frac{\partial u}{\partial x}\cdot\frac{\partial(x,y,z)}{\partial(a,b,c)}-\frac{\partial(u,y,z)}{\partial(a,b,c)}=0$$

or $$J\ \frac{\partial u}{\partial x}=\frac{\partial(u,y,z)}{\partial(a,b,c)} \qquad ...(9)$$

similarly $$J\ \frac{\partial u}{\partial y}=\frac{\partial(x,v,z)}{\partial(a,b,c)} \qquad ...(10)$$

and $$J\ \frac{\partial w}{\partial z}=\frac{\partial(x,y,w)}{\partial(a,b,c)} \qquad ...(11)$$

On adding (9) to (11) we obtain

$$\frac{\partial(u,y,z)}{\partial(a,b,c)}+\frac{\partial(x,v,z)}{\partial(a,b,c)}+\frac{\partial(x,y,w)}{\partial(a,b,c)}=J\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

Using (6), we have

$$\frac{dJ}{dt}=J\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \qquad ...(12)$$

Using (1), we have

$$\rho J\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+J\ \frac{\partial \rho}{\partial t}=0$$

or $$\frac{\partial \rho}{\partial t}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$$

or $$\frac{\partial \rho}{\partial t}+\nabla\ \left(\rho\vec{q}\right)=0$$

which is Eulerian form of equation of continuity as shown earlier.

11.8 Some Symmetrical forms of equation of continuity

The equation of continuity takes a simplified form in cases when the motion of the fluid possesses certain symmetrical properties as shown below.

11.8.1 Cylindrical Symmetry :

Let $P(r,\theta,z)$ be a fluid particle. Let $\rho\,(r,t)$ be the density and $q_r(r,t)$ be the velocity at P perpendicular to axis OZ.

Consider an element of fluid consisting of two cylinders of radii r and $r+\delta r$ with axis of it be z-axis and bounded by planes at unit distance apart.

Then rate of flow across the inner surface

$$= \rho q_r . 2\pi r = f(r,t) \qquad ...(1)$$

Rate of flow across the outer surface

$$= f(r+\delta r, t) \qquad ...(2)$$

Rate of change of mass within the element

$$= \frac{\partial}{\partial t}(\rho . 2\pi r . \delta r)$$

$$= 2\pi r\ \delta r\ \frac{\partial \rho}{\partial t} \qquad ...(3)$$

Suppose that the element of the fluid contains neither sources nor sinks. Then by the law of conservation of mass, the rate of increase of the mass within the element must be equal to the rate of mass flowing into the element.

Hence

$$2\pi r\ \delta r \frac{\partial \rho}{\partial t} = f(r,t) - f(r+\delta r, t)$$

$$= f(r,t) - \left[f(r,t) + \delta r\ \frac{\partial}{\partial r} f(r,t) + ... \right]$$

$$2\pi r\ \delta r\ \frac{\partial \rho}{\partial t} = -\ \delta r\ \frac{\partial}{\partial r} f(r,t) \qquad \text{(on neglecting higher powers)}$$

Now using (1)

or $$2\pi r\ \frac{\partial \rho}{\partial t} = -\ \frac{\partial}{\partial r}[\rho q_r 2\pi r]$$

or $$2\pi r\ \frac{\partial \rho}{\partial t} = -\ 2\pi \frac{\partial}{\partial r}(\rho q_r r)$$

or $$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho q_r r)=0 \qquad ...(4)$$

If $\rho =$ constant then $\frac{\partial \rho}{\partial t}=0$, we have

$$\rho \frac{\partial \rho}{\partial r}(q_r r)=0$$

On integrating with respect to r, we have

$$r\, q_r = g(t) \qquad ...(5)$$

If the flow is steady, $g(t)$ reduces to an absolute constant. Hence for the steady flow.

$$r\, q_r = \text{constant} \qquad ...(6)$$

Which is the equation of continuity due to cylindrical symmetry.

11.8.2 Spherical Symmetry :

Let $P(r,\theta,\phi)$ be a fluid particle. Also let $\rho(r,t)$ be the density and $q_r(r,t)$ the velocity at P is the radial direction .

Consider an element of the fluid consisting of two concentric spheres of radii r and $r+\delta r$ with O as centre as Plying on the inner sphere.

Hence rate of flow across the inner surface

$$=\rho q_r 4\pi r^2 = f(r,t) \qquad ...(1)$$

Thus rate of flow across the outer surface

$$= f(r+\delta r,t) \qquad ...(2)$$

Rate of change of mass within the element

$$=\frac{\partial}{\partial t}(\rho . 4\pi r^2 . \delta r)$$

$$=4\pi r^2\, \delta r \frac{\partial \rho}{\partial t} \qquad ...(3)$$

Using concept of law of conservation of mass, the rate of increase of the mass within the element must be equal to the rate of mass flowing into the element minus the rate of mass flowing out of the element

Hence

$$4\pi r^2\, \delta r \frac{\partial \rho}{\partial t} = f(r,t) - f(r+\delta r,t)$$

or $$= f(r,t) - \left[f(r,t) + \delta r \, . \, \frac{\partial}{\partial t} f(r,t) + ... \right]$$

$$= - \, \delta r \, \frac{\partial}{\partial t} f(r,t)$$ (neglecting higher powers of small δr)

or $$4\pi r^2 \, \delta r \, \frac{\partial \rho}{\partial t} = - \, \delta r \, . \, \frac{\partial}{\partial r} \left[\rho q_r \, 4\pi r^2 \right]$$ by (1)

or $$4\pi r^2 \, \delta r \, \frac{\partial \rho}{\partial t} = - \, \delta r \; 4\pi \frac{\partial}{\partial r} \left(r^2 \, \rho q_r \right)$$

or $$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left(\rho r^2 q_r \right) = 0 \qquad ...(4)$$

If ρ be constant then $\frac{\partial \rho}{\partial t} = 0$ then we have

$$\frac{\partial}{\partial r} \left(\rho r^2 q_r \right) = 0 \Rightarrow \frac{\partial}{\partial r} \left(r^2 q_r \right) = 0 \qquad ...(5)$$

On integrating with respect to r, we have

$$r^2 q_r = g(t)$$

If the motion is steady, $g(t)$ reduces to an absolute constant. Thus for a steady flow

$$r^2 q_r = \text{constant} \qquad ...(6)$$

which is the equation of continuity due to spherical symmetry.

Self learning Exercise :

1. The fluid motion is said to be irrotational if

   (a) curl $\vec{q} = 0$ (b) div $\vec{q} = 0$

   (c) grad $\vec{q} = 0$ (d) None of these

2. In rotational motion, the components of the rotation are called ---------.

3. What is the condition for a possible liquid motion?

4. Whether the motion specified by $\vec{q} = \frac{k^2(xj - yi)}{x^2 + y^2}$ is a possible motion for an incompressible fluid motion?

5. Test the motion specified by

$$u = \frac{3xz}{r^5}\ ,\ v = \frac{3yz}{r^5}\ \text{and}\ w = \frac{3z^2 - r^2}{r^5}$$

for possible incompressible fluid motion.

6. Find the velocity vector $\vec{q}$ for the velocity potential $\phi = c(x^2 - y^2)$.

11.9 Summary

In this unit, we studied about the concept of law of conservation of mass in terms of equation of continuity for the fluid motions. We also studied the equation of continuity in Euler's and Lagrangian approch and their equivalance. In this unit, we also familearized with the velocity potential, rotational and irrotational motions in two dimensional fluid motion.

11.10 Answers to self learning exercise

1. (a) curl $\vec{q} = 0$
2. Spin components
3. Statisfies the equation of continuity
4. Yes, possible fluid motion
5. Possible fluid motion
6. $\vec{q} = 2c\,(y\hat{j} - x\hat{i})$

Example 1 : Show that

$$u = \frac{-2xyz}{(x^2 + y^2)^2}\ ;\ v = \frac{(x^2 - y^2)z}{(x^2 + y^2)^2}\ \text{and}\ w = \frac{y}{(x^2 + y^2)}$$

are the velocity components of a possible fluid motion. Is this motion irrotational?

Solution : A fluid motion is only possible if the velocity components satisfy the equation of continuity. Here

$$u = \frac{-2xyz}{(x^2 + y^2)^2} \Rightarrow \frac{\partial u}{\partial x} = \frac{-2yz(y^2 - 3x^2)}{(x^2 + y^2)^3}$$

$$v = \frac{(x^2 - y^2)z}{(x^2 + y^2)^2} \Rightarrow \frac{\partial v}{\partial y} = \frac{2yz(y^2 - 3x^2)}{(x^2 + y^2)^3}$$

$$w = \frac{y}{(x^2 + y^2)} \Rightarrow \frac{\partial w}{\partial z} = 0$$

so that $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$

Since this equation of continuity is satisfied the flow is possible.

For the motion to be irrotational, we must show that spin components are zero.

That is we have show that

$$\xi=0 \Rightarrow \frac{\partial w}{\partial y}=\frac{\partial v}{\partial z} \ ; \ \eta=0 \ \Rightarrow \frac{\partial u}{\partial z}=\frac{\partial w}{\partial x}$$

and $\zeta=0 \Rightarrow \frac{\partial v}{\partial x}=\frac{\partial u}{\partial y}$

Here $\frac{\partial w}{\partial y}=\frac{x^2-y^2}{\left(x^2+y^2\right)^2}=\frac{\partial v}{\partial z}$

$$\frac{\partial u}{\partial z}=\frac{-2xy}{\left(x^2+y^2\right)^2}=\frac{\partial w}{\partial x}$$

and $\frac{\partial v}{\partial x}=\frac{-2xz\left(x^2-3y^2\right)^2}{\left(x^2+y^2\right)^3}=\frac{\partial u}{\partial y}$

Hence the motion is irrotational.

Example 2 : If the velocity of an incompressible fluid at the point (x,y,z) is given by $\left(\frac{3xz}{r^5},\frac{3yz}{r^5},\frac{3z^2-r^2}{r^5}\right)$ prove that the liquid motion is possible and the velocity potential is $\frac{\cos\theta}{r^2}$.

Solution : Given that

$$u=\frac{3xz}{r^5} \ , \ v=\frac{3yz}{r^5} \text{ and } w=\frac{3z^2-r^2}{r^5} \text{ where } r^2=x^2+y^2+z^2 .$$

For the motion to be possible, we must show that the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \text{...(1)}$$

is satisfied. Here

$$\frac{\partial u}{\partial x}=\frac{3z}{r^5}-\frac{15x^2z}{r^7}$$

$$\frac{\partial v}{\partial y}=\frac{3z}{r^5}-\frac{15y^2z}{r^7}$$

and $\quad \dfrac{\partial w}{\partial z} = \dfrac{9z}{r^5} - \dfrac{15z^3}{r^7}$

Thus $\quad \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0$

Hence the fluid motion is possible.

Further, velocity potential ϕ is given by

$$d\phi = -\left(\frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy + \frac{\partial \phi}{\partial z}\,dz\right)$$

$$= -(u\,dx + v\,dy + w\,dz)$$

or $$d\phi = -\frac{1}{r^5}\left[3z\,(x\,dx + y\,dy + z\,dz) - r^2 d\,z\right]$$

$$= -\frac{1}{r^5}\left[3z\;r\,dr - r^2 d\,z\right]$$

$$= \frac{r^2 d\,z - 3z\;r\,d\,r}{r^5}$$

$$= \frac{r^3 d\,z - z.3r^2.d\,r}{r^6}$$

$$d\phi = d\left(\frac{z}{r^3}\right)$$

$$\Rightarrow \phi = \frac{z}{r^3} = \frac{r\cos\theta}{r^3} = \frac{\cos\theta}{r^2}$$

Hence. Velocity potential $\phi = \dfrac{\cos\theta}{r^2}$

Example 3 : Show that if the velocity potential of an irrotational fluid motion is equal to $A\left(x^2+y^2+z^2\right)^{-3/2}\, z\tan^{-1}\left(y/x\right)$

The lines of flow will be on the series of the surfaces

$$\left(x^2+y^2+z^2\right) = c^{2/3}\left(x^2+y^2\right)^{2/3}$$

Solution : The velocity potential ϕ is given

by $\phi = A\left(x^2+y^2+z^2\right)^{-\frac{3}{2}} z\tan^{-1}\left(y/x\right)$

$$\phi = A\,r^{-3}\,z\tan^{-1}\left(y/x\right) \qquad ...(1)$$

So that

$$u = -\frac{\partial\phi}{\partial x} = 3Azxr^{-5}\tan^{-1}\frac{y}{x} + \frac{Azyr^{-3}}{\left(x^2+y^2\right)}$$

$$v = -\frac{\partial\phi}{\partial y} = 3Azyr^{-5}\tan^{-1}\frac{y}{x} - \frac{Azxr^{-3}}{\left(x^2+y^2\right)}$$

$$w = -\frac{\partial\phi}{\partial z} = 3Az^2r^{-5}\tan^{-1}\frac{y}{x} - Ar^{-3}\tan^{-1}\left(y/x\right)$$

Where $r^2 = x^2+y^2+z^2$ and $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$.

The equation of lines of flow are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

i.e. $$\frac{dx}{\left(3Azxr^{-5}\tan^{-1}y/x + \frac{Azyr^{-3}}{\left(x^2+y^2\right)}\right)} = \frac{dy}{\left[3Azyr^{-5}\tan^{-1}\frac{y}{x} - \frac{Azxr^{-3}}{\left(x^2+y^2\right)}\right]}$$

$$= \frac{dz}{A\left(3z^2r^{-5} - r^{-3}\right)\tan^{-1}y/x} \qquad ...(2)$$

$$\Rightarrow \quad \frac{x\,dx + y\,dy + z\,dz}{\left(3x^2+3y^2+3z^2\right)r^{-2} - 1} = \frac{x\,dx + y\,dy}{\left(3x^2+3y^2\right)r^{-2}}$$

$$\frac{x\,dx + y\,dy + z\,dz}{2} = \frac{r^2(x\,dx + y\,dy)}{3\left(x^2+y^2\right)}$$

or $$\frac{2\left(x\,dx + y\,dy + z\,dz\right)}{\left(x^2+y^2+z^2\right)} = \frac{2}{3}\,\frac{\left(2x\,dx + 2y\,dy\right)}{\left(x^2+y^2\right)}$$

On integrating

$$\log\left(x^2+y^2+z^2\right)=\frac{2}{3}\left[\log\left(x^2+y^2\right)+\log c\right]$$

$$x^2+y^2+z^2=\left[c\left(x^2+y^2\right)\right]^{2/3}$$

which gives the required series of the surfaces on which the lines of flow will lie.

Example 4 : Show that $\phi=(x-t)(y-t)$ represents the velocity potential of an incompressible two diamensional fluid. Show that the stream line at time "t" are the curves $(x-t)^2-(y-t)^2$ = constant and the path of the fluid particles have the equation

$$\log\left(x-y\right)=\frac{1}{2}\left[\left(x+y\right)-a\left(x-y\right)^{-1}\right]+b$$

where a and b are constants.

Solution : Given that velocity potential

$$\phi=\left(x-t\right)\left(y-t\right) \qquad ...(1)$$

$$\Rightarrow \quad u=-\frac{\partial\phi}{\partial x}=-\left(y-t\right)$$

$$v=-\frac{\partial\phi}{\partial y}=-\left(x-t\right)$$

and $\frac{\partial u}{\partial x}=0$ ‘ $\frac{\partial v}{\partial y}=0$. Thus the equation of continuity $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ is satisfied. Hence given (1) represents the velocity potential of an incompressible two dimensional flow.

Also the equations of stream line are

$$\frac{dx}{u}=\frac{dy}{v}\Rightarrow\frac{dx}{-(y-t)}=\frac{dy}{-(x-t)}$$

or $\quad (x-t)dx\ -\left(y-t\right)dy=0$

On integrating

$$\left(x-t\right)^2-\left(y-t\right)^2=\text{constant} \qquad ...(2)$$

which are the required stream lines.

And the paths of particles, are given by

$$u=\frac{dx}{dt}=-\left(y-t\right) \qquad \Rightarrow \quad \frac{dx}{dt}=t-y \qquad ...(3)$$

$$v = \frac{dy}{dt} = -(x-t) \qquad \Rightarrow \frac{dy}{dt} = t - x \qquad ...(4)$$

From (3) and (4) we have

$$\frac{dx}{dt} + \frac{dy}{dt} = 2t - (x+y) \qquad ...(5)$$

Now suppose that $z = x + y$, $\therefore \frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$

Then (5) gives

$$\frac{dz}{dt} = 2t - z$$

or $$\frac{dz}{dt} + z = 2t$$

which is a linear different equation whose solution is

$$z\,e^t = \int 2\,t\,e^t dt + C_1$$

or $$z = 2t - 2 + C_1 e^{-t}$$

or $$x + y = 2t - 2 + C_1 e^{-t} \qquad ...(6)$$

Again from (3) and (4) we have

$$\frac{dx}{dt} - \frac{dy}{dt} = x - y \Rightarrow \frac{dx - dy}{x - y} = dt$$

On integrating $\log(x-y) - \log c_2 = t$

$$\Rightarrow x - y = c_2 e^t \qquad ...(7)$$

From (6) and (7), we have

$$(x+y) - a(x-y)^{-1} = 2t - 2 + c_1 e^{-t} - \frac{a}{c_2}\, e^{-t}$$

$$= 2t - 2 \quad \text{taking } c_1 = \frac{a}{c_2}$$

Hence $\frac{1}{2}\left[(x+y) - a(x-y)^{-1}\right] = (t-1)$ ...(8)

From (7) we have

$$t = \log(x-y) - \log c_2$$

$$\Rightarrow \quad (t-1) = \log(x-y) - b \qquad ...(9)$$

Using (9) in (8), we have

$$\frac{1}{2}\left[(x+y) - a(x-y)^{-1}\right] = \log(x-y) + b$$

11.11 Exercise

1. Prove that the liquid motion is possible when velocity at point (x, y, z) is given by

$u = \dfrac{3x^2 - r^2}{r^5}$, $v = \dfrac{3xy}{r^5}$, $w = \dfrac{3xz}{r^5}$ where $r^2 = x^2 + y^2 + z^2$ and the stream linea are the intersection of the surfaces $\left(x^2 + y^2 + z^2\right)^3 = c\left(y^2 + z^2\right)^2$ by the plane passing through OX. State if the motion is irrotational, with reasons.

2. Show that the following velocity field is a possible case of irrotational flow of an incompressible fluid :

$$u = yzt\,;\ v = zxt \text{ and } w = xyt$$

3. Show that the velocity potential $\phi = \dfrac{1}{2}\, a\left(x^2 + y^2 - 2z^2\right)$ satisfies the Laplace equation and represents the flow against a fixed plane wall. Also find the stream line. $\left[y^2 z = c\right]$

4. Give the physical significance implied in the equation of continuity in fluid motion.

5. Show that for an incompressible fluid, the equation of continuity becomes $div\,\vec{q} = 0$.

6. Find the equation of continuity in Lagrange's method. Show that it is equivalant to

$$\frac{\partial \rho}{\partial t} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0.$$

□□□□

UNIT - 12
Equation of Continuity-II

Structure of the unit

12.0 Objective

In this unit, we will learn about the equation of conitnuity in various coordinate systems i.e. cartesian, cylindrical, spherical polar coordinate etc. We will also study about the concept of boundary surface.

12.1 Introduction

The equation of continuity is always found by the fact that the mass contained inside a given volume of fluid remains unaltered throughout the motion. Using this conecpt, we will obtain equation of continuity in cartesian coordinate system, cylindrical polar coordinate system and spherical polar coordinate system. These forms help in dealing problems of fluid motion through various geometric forms of the flow. We will also obtain the condition that the surface represent the boundary surface.

12.2 Equation of continuity in cartesian coordinates system

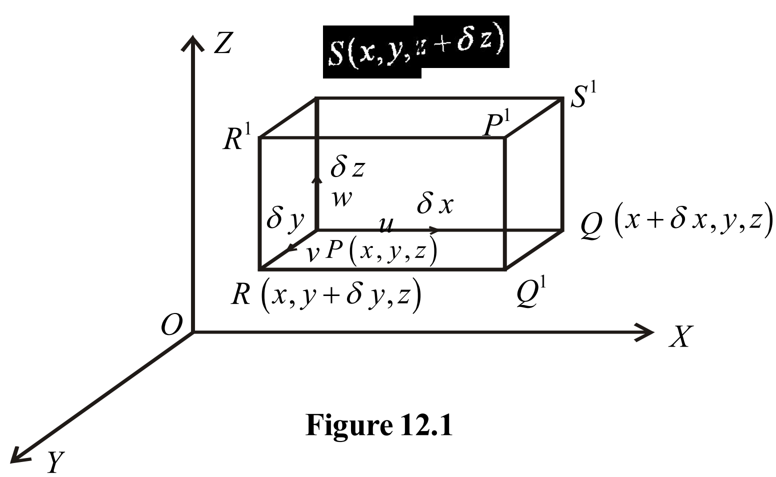


Figure 12.1

Let there be a fluid particle at $P(x,y,z)$ and $\rho(x,y,z,t)$ be the density of the fluid at P at any time t. Let u,v,w be the velocity components at P parallel to the rectangular coordinate axes.

Now construct a small parallelopiped with edges δx, δy and δz parallel to coordinate axes having point P at one of the angular point as shown in figure.

Now mass of fluid that passes through the plane face $PRR'S$ per unit time parallel to OX

$$= \rho(\delta y\ \delta z)u = f(x,y,z) \qquad ...(1)$$

Where $\delta y\ \delta z$ is the area of the cross-section and u is the velocity with which the fluid crosses the face $PRR'S$.

Now mass of fluid that passes out through the plane face $P'Q'QS$ per unit time

$$= f(x+\delta x, y, z)$$

$$= f(x,y,z) + \delta x \,.\, \frac{\partial}{\partial x} f(x,y,z) + \text{higher powers of } \delta x$$

$$= f(x,y,z) + \delta x \,.\, \frac{\partial}{\partial x} f(x,y,z) \text{ to the first order approximation.}$$

Therefore the excess (increase) fluid in the parallelopiped in a unit time in x-direction

= Fluid mass enters in through $PRR'S$ - Fluid mass leaves through $PQ'QS'$ per unit time

$$= f(x,y,z) - f(x+\delta x, y, z)$$

$$= -\delta x \,.\, \frac{\partial f}{\partial x}(x,y,z)$$

$$= -\delta x \,.\, \frac{\partial}{\partial x}(\rho u\ \delta y\ \delta z) \qquad \text{using (1)}$$

$$= -\frac{\partial(\rho u)}{\partial x}\delta x\ \delta y\ \delta z \qquad ...(2)$$

Similarly increase in the fluid mass in parallelopiped in y-direction through faces $PQS'S$ and $RQ'P'R'$ is

$$= -\frac{\partial(\rho v)}{\partial y}\delta x\ \delta y\ \delta z \qquad ...(3)$$

and increase in the fluid mass in parallelopiped in z-direction through faces. $PQQ'R$ and $SS'P'R'$ is

$$= -\frac{\partial(\rho w)}{\partial y}\delta x\,\delta y\,\delta z \qquad ...(4)$$

The total excess flow in parallelopiped, on using (2), (3), (4)

$$= -\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]\delta x\,\delta y\,\delta z \qquad ...(5)$$

Again the total mass in the parallelo piped $= \rho.\delta x\,\delta y\,\delta z$

Hence increase in mass of the parallelopiped in unit time $= \frac{\partial \rho}{\partial t}.\delta x\,\delta y\,\delta z$ ...(6)

Now, by the law of conservation of mass, the rate of increase of the mass of the fluid within the parallelopiped must be equal to the total excess flow in the parallelopiped in unit time. Hence from (5) and (6), we have

$$\delta x\,\delta y\,\delta z\frac{\partial \rho}{\partial t} = -\,\delta x\,\delta y\,\delta z\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]$$

or $$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

which is the equation of continuity in cartesian coordinates.

If the fluid is homogenous and incompressible, density ρ is constant the equation of continuity reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

12.3 Equation of continuity in cylindrical polar coordinates

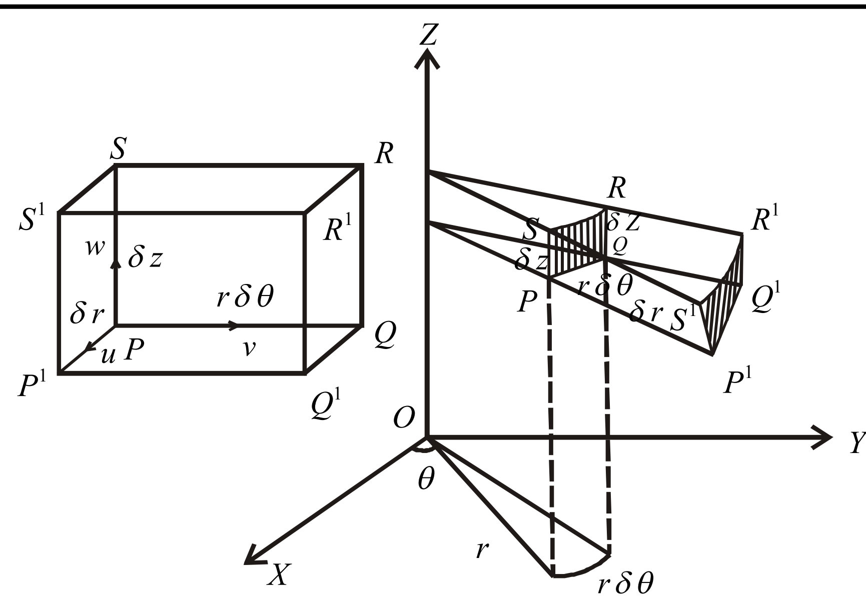


Figure 12.2

Let P be a point whose cylindrical polar coordinate are (r,θ,z), with P as one corner. construct a parallelopiped with edges

$PQ = r\delta\theta$, $PS = \delta z$ and $PP' = \delta r$. Let $u, v,$ and w are the velocity components along PP', PQ and PS respectively.

Thus mass flow entering through $PQRS$ per unit time along r direction is

$$= \rho u(r\ \delta\theta\ \delta z) = \rho u r\ \delta\theta\ \delta z = f(r,\theta,z)$$

and fluid mass getting out through the face $P'Q'R'S'$ per unit time is

$$= f(r + \delta r\ \theta z)$$

$$= f(r,\theta,z) + \delta r.\frac{\partial f(r,\theta,z)}{\partial r} + \text{Higher powers of } \delta r$$

Therefore excess of flow along r - direction is

= man that enters through $PQRS$ - mass that flows out through $P'Q'R'S'$

$$= -\ \delta r.\frac{\partial}{\partial r} f\ (r,\theta,\phi)$$

$$= -\ \delta r\ \frac{\partial}{\partial r}(\rho u r\ \delta\theta\ \delta z)$$

$$= -\ \frac{\partial}{\partial r}(\rho u r)\ \delta r\ \delta\theta\ \delta z \qquad ...(1)$$

Similarly the excess of flow along θ direction when fluid mass entering through $PP'S'S$ and out through $QQ'R'R$ per unit time $= -\ \dfrac{\partial}{\partial\theta}(\rho v)\ \delta r\ \delta\theta\ \delta z$ ...(2)

and the excess of flow along z -direction when fluid mass entering through $PQQ'P'$ and out flow through $SRR'S'$ per unit time $= -\ \dfrac{\partial}{\partial z}(\rho w r)\ \delta r\ \delta\theta\ \delta z$ ...(3)

Hence the total excess flow in parallelopiped in unit time is

$$= -\left[\frac{\partial}{\partial r}(\rho u r) + \frac{\partial}{\partial\theta}(\rho v) + \frac{\partial}{\partial z}(\rho w r)\right]\delta r\ \delta\theta\ \delta z \qquad ...(4)$$

The vloume of the parallelopiped is

$$= r\delta\theta.\delta r.\delta z$$

The fluid mass in the parallelopiped at any time

$$= \rho r \delta r\ \delta\theta\ \delta z$$

The rate of change in fluid mass in parallelopiped

$$= \frac{\partial}{\partial t}(\rho r)\, \delta r\, \delta \theta\, \delta z \qquad ...(5)$$

Hence, by the law of conservation of the fluid mass, the rate of change of the mass of the fluid within the parallelopiped must be equal to the total excess flow in parallelopiped in unit time (4) and (5), we have

$$\frac{\partial(\rho r)}{\partial t} + \frac{\partial(\rho u r)}{\partial r} + \frac{\partial(\rho v)}{\partial \theta} + \frac{\partial}{\partial z}(\rho w r) = 0$$

or $$r\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho u r) + \frac{\partial}{\partial \theta}(\rho v) + \frac{\partial}{\partial z}(\rho w r) = 0$$

or $$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho u r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \qquad ...(6)$$

which is the equation of continuity in cylindrical polar coordinates.

If the fluid in homigenous and incompressible, density ρ is constant then equation of conitnuity (6) reduces to

$$\frac{1}{r}\frac{\partial(u r)}{\partial r} + \frac{1}{r}\frac{\partial(v)}{\partial \theta} + \frac{\partial w}{\partial z} = 0$$

12.4 Equation of continuity in spherical polar coordinates

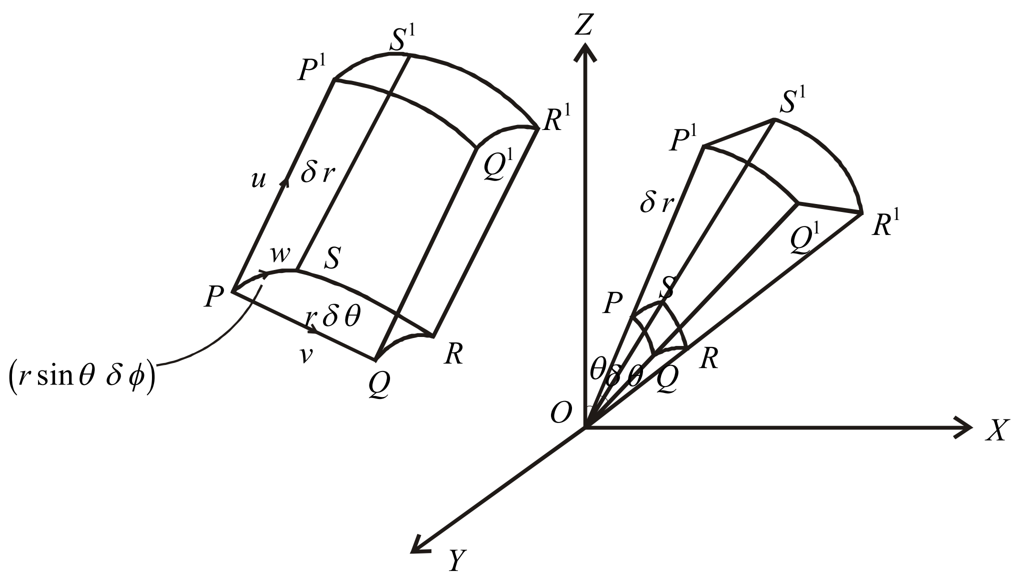


Figure 12.3

Let $P\,(r,\theta,\phi)$ be a point in the fluid. On constructing a curvilinear parallelopiped with three ajdacent edges as $PP' = \delta r$, $PQ = r\,\delta\theta$ and $PS = r\sin\theta\,\delta\phi$ where P is one corner of it. Let the velocity components be u, v, w in the directions of r, θ, ϕ respectively.

The mass of the fluid that flows in through the face $PQRS$ due to velocity component u

per unit time is

$$= \rho \left(r\ \delta\theta\ .\ r \sin\theta\ \delta\theta \right) u = f\ \left(r, \theta, \phi \right) \qquad ...(1)$$

The mass of the fluid that flows out through the face $P'Q'R'S'$ due to flow along PP' (r-direction) per unit time

$$= f\left(r + \delta r, \theta, \phi\right)$$

$$= f\left(r, \theta, \phi\right) + \delta r . \frac{\partial}{\partial r} f\ \left(r, \theta, \phi\right) + \text{Higher powers } \delta r$$

Therefore excess of flow along PP'

= Fluid mass enters through $PQRS$ - Fluid mass flows out through $P'Q'R'S'$

$$= -\ \delta r . \frac{\partial}{\partial r} f\ \left(r, \theta, \phi\right) \qquad \text{(neglecting higher powers of } \delta r \text{)}$$

$$= -\ \delta r . \frac{\partial}{\partial r} \left(\rho r^2 \sin\theta\right) \delta\theta\ \delta\phi$$

$$= -\ \frac{\partial}{\partial r} \left(\rho r^2 \sin\theta\right) \delta r\ \delta\theta\ \delta\phi$$

$$= -\ \frac{\partial}{\partial r} \left(\rho r^2\right) \sin\theta\ \delta r\ \delta\theta\ \delta\phi \qquad ...(2)$$

Similarly excess of flow in over out through faces $PP'S'S$ and $QQ'R'R$ due to velocity v along PQ $\left(\theta - \text{direction}\right)$ per unit time is

$$= -\ r\ \delta\theta \frac{\partial}{\partial\theta} \left(\rho \sin\theta\, v\right) \delta r\ \delta\phi$$

$$= -\ r\ \delta r\ \delta\theta\ \delta\phi\ \frac{\partial}{\partial\theta} \left(\rho v\ \sin\theta\right) \qquad ...(3)$$

and the excess of flow from faces $PQQ'P'$ and $SRR'S'$ due to flow along PS $\left(\phi - direction\right)$ per unit time is

$$= -\ r\ \frac{\partial}{\partial\phi} \left(\rho w\right) \delta r\ \delta\theta\ \delta\phi$$

$$= -\ r\ \delta r\ \delta\theta\ \delta\phi\ \frac{\partial}{\partial\phi} \left(\rho w\right) \qquad ...(4)$$

Therefore the total excess flow in the parallelopiped per unit time from all faces

$$= -\,\delta r\,\delta\theta\,\delta\phi\left[\sin\theta\,\frac{\partial}{\partial r}\left(\rho u r^2\right) + r\,\frac{\partial}{\partial\theta}\left(\rho v\sin\theta\right) + r\,\frac{\partial}{\partial\phi}\left(\rho w\right)\right] \qquad ...(5)$$

Also, we know that the total mass of the fluid in parallelopiped

$$= \rho r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\phi$$

Hence the rate of change in the fluid mass in the parallelopiped

$$= \frac{\partial}{\partial t}\left(\rho r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\phi\right)$$

$$= \delta r\,\delta\theta\,\delta\phi\,\frac{\partial}{\partial t}\left(\rho r^2 \sin\theta\right) \qquad ...(6)$$

Using the concept of law of mass conservation, the rate of change in fluid mass

= total excess flow per unit time

then from (5) and (6), we have

$$\left[\frac{\partial}{\partial t}\left(\rho r^2\sin\theta\right) + \sin\theta\,\frac{\partial}{\partial r}\left(\rho u r^2\right) + r\frac{\partial}{\partial\theta}\left(\rho v\sin\theta\right) + r\,\frac{\partial}{\partial\theta}\left(\rho w\right)\right] = 0$$

or $$r^2\sin\theta\,\frac{\partial\rho}{\partial t} + \sin\theta\,\frac{\partial}{\partial r}\left(\rho u r^2\right) + r\frac{\partial}{\partial\theta}\left(\rho v\sin\theta\right) + r\,\frac{\partial}{\partial\theta}\left(\rho w\right) = 0$$

or $$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(\rho u r^2\right) + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial\theta}\left(\rho v\sin\theta\right) + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial\phi}\left(\rho w\right) = 0$$

which is the equation of continuity in spherical polar coordinates.

The equation for homogeneous incompressible fluid reduces to

$$\frac{1}{r}\,\frac{\partial}{\partial r}\left(u r^2\right) + \frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(v\sin\theta\right) + \frac{1}{\sin\theta}\,\frac{\partial w}{\partial\phi} = 0$$

12.5 Equation of continuity of a liquid flow through a channel or a pipe

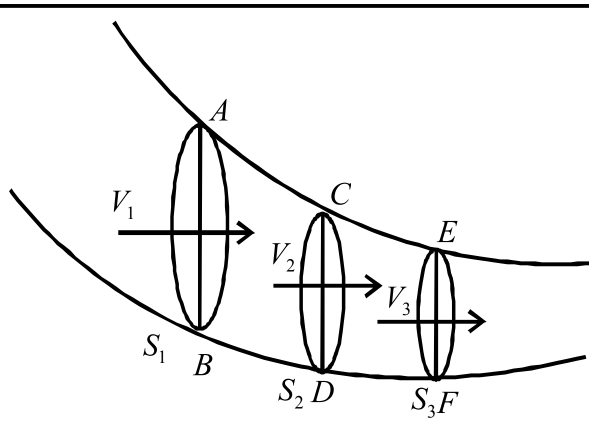


Figure 12.4

Let an incompressible fluid continuously flow through a channel or a pipe whose crossection area may or may not be fixed. Then the quantity of liquid passing through all section is the same per second due to law of conservation of mass.

Let us suppose that some liquid is flowing through a tapering pipe as shown in figure. Let S_1, S_2, S_3 be the areas of the pipe and V_1, V_2, V_3 are the velocities of liquid at the section AB, CD, EF respectively. If Q_1, Q_2, Q_3 be the total quantity of liquid flowing across the section AB, CD and EF respectively then

$$Q_1 = S_1V_1,\ Q_2 = S_2V_2 \text{ and } Q_3 = S_3V_3.$$

From the law of conservation of mass, the total quantity of liquid flowing across the section AB, CD and EF must be same. Hence

$$Q_1 = Q_2 = Q_3$$

Therefore $\quad S_1V_1 = S_2V_2 = S_3V_3$

which is the equation of continuity of a liquid flowing through a channel or pipe.

Self Learning Exercies - I

1. The equation of continuity for the homogenous steady incompressible fluid is

   (a) curl $\vec{q} = 0$ (b) div $\vec{q} = 0$ (c) grad $\vec{q} = 0$ (d) None of these

2. Write down the equation of continuity of a liquid flow through a pipe.
3. What is the physical significance of equation of continuity?
4. Is the velocity field $u = 0$; $v = r + \frac{1}{r}$, $w = 0$ satisfy the equation of conitnuity

$$\frac{d^2v}{dr^2} + \frac{d}{dr}\left(\frac{v}{r}\right) = 0\ ?$$

Example 1 : A mass of fluid moves in such a way that each particle describes a circle in one plane about a fixed axis. Show that the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho w)}{\partial \theta} = 0$$

Where w be the angular velocity of a particle whose azimuthal angle is θ at time t.

Solution : Given that the motion is in a plane and a fluid particle describes a circle of radius r. At any instant with fluid particle at P consider an element $PQRS$ such that $PS = \delta r$ and $PQ = r\delta\theta$. There is no motion along radial direction PS. Hence

the excess of flow in over flow out along PQ per unit time

$$= - r\delta\theta \,.\, \frac{\partial}{r\partial\theta}(\rho w \delta r)$$

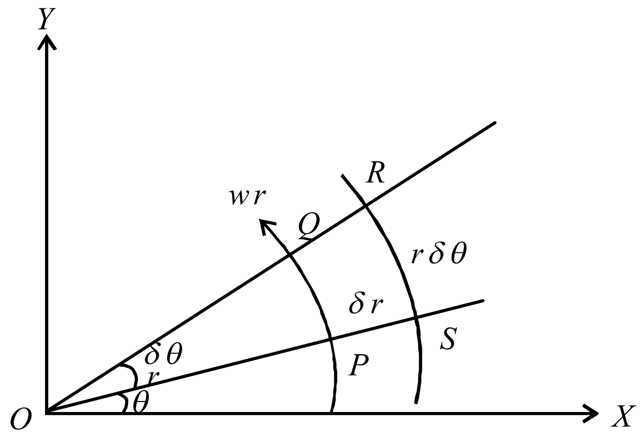


Figure 12.5

$$= -\frac{\partial}{\partial \theta}(\rho w r)\delta r\ \delta\theta$$

Also mass of the fluid inside the element

$$= \rho \delta r\ r\ \delta\theta$$

$\therefore$ change in the mass of the element

$$= \frac{\partial}{\partial t}(\rho \delta r\ r\ \delta\theta)$$

Hence the equation of continuity is

$$\frac{\partial}{\partial t}(\rho \delta r\ r\ \delta\theta) + \frac{\partial}{\partial \theta}(\rho w r)\ \delta r \delta\theta = 0$$

or $$= \frac{\partial \rho}{\partial t} + \frac{\partial(\rho w)}{\partial \theta} = 0$$

which is the required equation of continuity.

Example 2 : If σ is the cross sectional area of a stream filament, establish the equation of continuity in the form $\frac{\partial}{\partial t}(\rho\sigma) + \frac{\partial}{\partial s}(\rho\sigma q) = 0$ where s is measured along the filament in the direction of flow and q is the spead.

Solution : Let P be the point. Consider a volume bounded by the cross -sections through P and another at a distance ds from P. Hence $PQQ'P'$ be the stream filament whose cross sectional area is σ and arc $PQ = \delta s$.

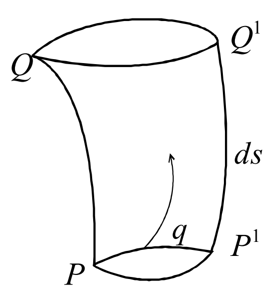


Figure 12.6

The rate of the excess of the flow in over the flow out along PQ per unit time

$$= -\delta s \frac{\partial}{\partial s}(\rho \sigma q)$$

Again, the total mass of the fluid within the stream filament is $= \rho \sigma \delta s$

$\therefore$ the rate of change in mass of the stream filament $= \frac{\partial}{\partial t}(\rho \sigma \delta s)$

Hence equation of continuity is

$$\frac{\partial}{\partial t}(\rho \sigma \delta s) = -\delta s \frac{\partial}{\partial s}(\rho \sigma q)$$

or $$\frac{\partial}{\partial t}(\rho \sigma) + \frac{\partial}{\partial s}(\rho \sigma q) = 0$$

which is the required equation of continuity.

Example 3 : If the lines of motion are curves on the surfaces of cones having their vertices at the origin and the axis of z for common axis, Prove that the equation of continuity is

$$\frac{\partial \rho}{\partial r} + \frac{\partial}{\partial r}(\rho w) + \frac{2\rho u}{r} + \frac{\operatorname{cosec}\theta}{r}\frac{\partial}{\partial \theta}(\rho w) = 0$$

where u and w are the velocity components in the directions in which r and ϕ increase.

Solution : Let origin O be the common vertex and oz the axis, be the axis of z. Consider a cone OAB of semi vertical angle θ. Let $P\,(r, \theta, \phi)$ be a point on the surface of the cone and $PP = \delta r$, $PS = r\,\delta\theta$, $PQ = r\sin\theta\;\delta\phi$ being edges of the curvilinear parallelopiped as shown in the figure.

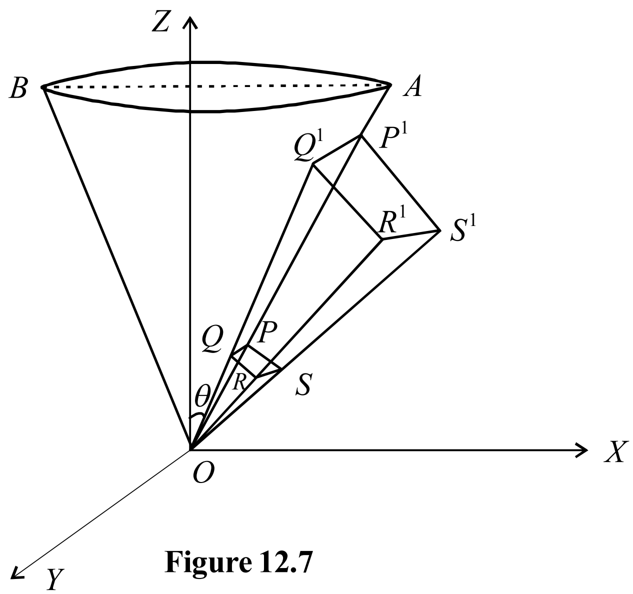


Figure 12.7

Since the lines of motion are curves on the surface of cone, there will be no motion perpendicular to the surface of the cone, hence there is no velocity along the edge PS (along θ-direction). Since velocity along PS is zero, the excess of flow in over flow out along PS vanishes. u and w are the velocity componates along r and ϕ directions respectively.

Again

rate of excess of flow-in over flow out along PP'

$$= -\,\delta r\,.\,\frac{\partial}{\partial r}\left(\rho u.r\,\delta\theta.r\sin\theta\;\delta\phi\right)$$

$$= -\,\sin\theta\,\frac{\partial}{\partial r}\left(r^2\rho u\right).\delta r\;\delta\theta\;\delta\phi$$

and rate of excess of flow in over flow out along PQ

$$= -\,r\sin\theta\,\frac{\partial}{r\sin\theta\,\partial\phi}\left(\rho w\;\delta r.r\;\delta\theta\right)$$

$$= -\,r\,\delta r\;\delta\theta\;\delta\phi\,\frac{\partial}{\partial\phi}\left(\rho w\right)$$

Also, the rate of increase in mass of the parallelopiped

$$= \frac{\partial}{\partial t}\left(\rho\,\delta r.r\;\delta\theta.r\sin\theta\;\delta\phi\right)$$

$$= r^2\sin\theta\,\frac{\partial\rho}{\partial t}\,.\,\delta r\;\delta\theta\;\delta\phi$$

Hence the equation of continuity is given by

$$r^2\sin\theta\,\frac{\partial\rho}{\partial t}\,.\,\delta r\;\delta\theta\;\delta\phi = -\,\delta r\;\delta\theta\;\delta\phi\left[\sin\theta\frac{\partial}{\partial r}\left(\rho u r^2\right)+r\,\frac{\partial}{\partial\phi}\left(\rho w\right)\right]$$

or $$r^2\sin\theta\,\frac{\partial\rho}{\partial t} + \sin\theta\frac{\partial}{\partial r}\left(\rho u r^2\right)+r\,\frac{\partial}{\partial\phi}\left(\rho w\right)=0$$

or $$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(\rho u r^2\right)+\frac{1}{r\sin\theta}\,\frac{\partial}{\partial\phi}\left(\rho w\right)=0$$

or $$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\left[r^2\,\frac{\partial(\rho u)}{\partial r}+2\,r\,\rho u\right]+\frac{1}{r\sin\theta}\,\frac{\partial}{\partial\phi}\left(\rho w\right)=0$$

or $$\frac{\partial\rho}{\partial t} + \frac{\partial\left(\rho u\right)}{\partial r}+\frac{2\,\rho u}{r}+\frac{\operatorname{cosec}\theta}{r}\,\frac{\partial}{\partial\phi}\left(\rho w\right)=0$$

which is the required equation of continuity.

Example 4 : A mass of fluid is in motion so that the liens of motion lie on the surface of coaxial cylinders, show that the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho u)}{\partial \theta} + \frac{\partial(\rho v)}{\partial z} = 0$$

where u, v are the velocity perpendicular and parallel to z.

Solution : Consider a fluid particle P, whose cylindrical coordinates are (r, θ, z). Construct a curilinear parallelopiped taking P as one corner of it and the edges be $PQ = \delta r$, $PS = r\,\delta\theta$ and $PP' = \delta z$.

The fluid motion lies on the surface of co-axial cylinders, so there is no motion along PQ then the excess flow along PQ is zero.

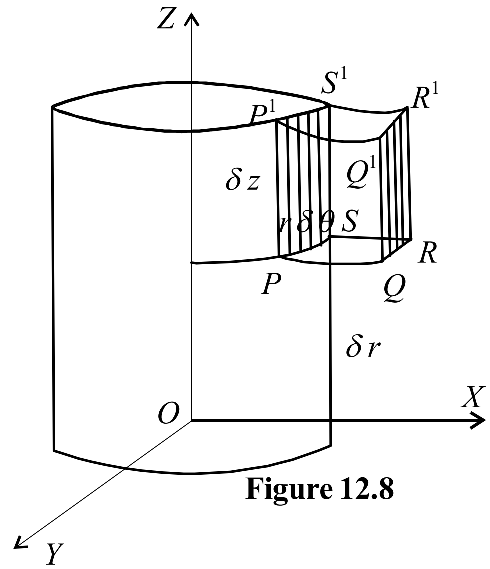


Figure 12.8

The excess flow in over flow out along PS

$$= -r\,\delta\theta \frac{\partial}{r\,\delta\theta}(\rho u\,\delta r\,\delta z)$$

$$= -\delta r\,\delta\theta\,\delta z \frac{\partial}{\partial\theta}(\rho u)$$

and the excess flow in over flow out along PP'

$$= -\delta z \frac{\partial}{\partial z}(\rho v r\,\delta\theta\;\delta r)$$

$$= -\delta r\;\delta\theta\;\delta z \frac{\partial}{\partial z}(\rho v r)$$

Also the mass of the fluid in the parallelopiped is

$$= \rho\; r\;\delta\theta\;\delta r\;\delta z$$

$\therefore$ The rate of change in mass of the parallelopiped

$$= r\;\delta\theta\;\delta r\;\delta z \frac{\partial\rho}{\partial t}$$

Hence the equation of continuity is

$$r\;\delta\theta\;\delta r\;\delta z \frac{\partial\rho}{\partial t} = -\;\delta\theta\;\delta r\;\delta z \left[\frac{\partial}{\partial\theta}(\rho u) + \frac{\partial}{\partial z}(\rho v r)\right]$$

or $$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho u) + \frac{\partial}{\partial z}(\rho v) = 0$$

which is the required equation of continuity.

Example 5 : If every particel moves on the surface of a sphere prove that the equation of continuity is

$$\frac{\partial \rho}{\partial t}\cos\theta+\frac{\partial}{\partial\theta}(\rho w\cos\theta)+\frac{\partial}{\partial\phi}(\rho w'\cos\theta)=0$$

ρ being the density, θ, ϕ the latitude and longitude of any element, and w, w' the angular velocities of the elelment in latitude and longitude respectively.

Solution : Consider a fluid P on the semi-circle APB making an angle ϕ with semi-circle ACB. Suppose that OP makes an angle θ with OC. Constructing an elementry parallelopiped on the surface of a sphere whose edges are $PQ=\delta r$, $PP'=r\,\delta\theta$ and $PS=r\cos\theta\,\delta\phi$ and as P one corner of it.

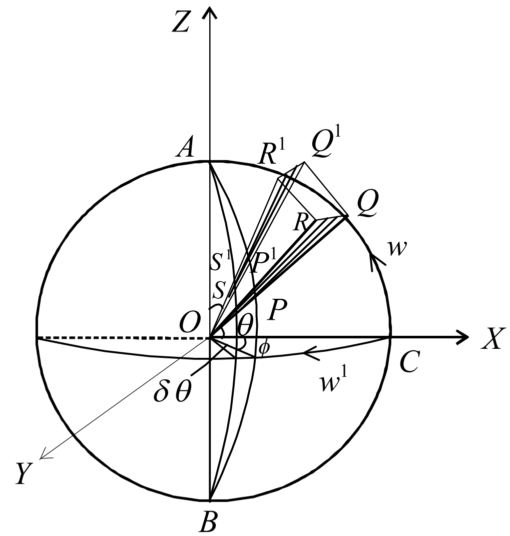


Figure 12.9

Let ρ be the density of the fluid at P. Since every particle moves on the surface of the sphere, there will be no velocity along PQ ie. in the radial direction. Here the velocity along PP' and PS are wr and $w'r\cos\theta$ because w and w' are the angular velocity in latitude and longitude directions respectively. Since velocity along PQ is zero then the rate of excess of law in over flow out along PQ is zero

Now, the excess of flow in over flow out along PP' per unit time

$$=-r\,\delta\theta\,\frac{\partial}{r\partial\theta}(\rho wr.\delta r.r\cos\theta\delta\phi)$$

$$=-r^2\,\delta r\,\delta\theta\,\delta\phi\frac{\partial}{\partial\theta}(\rho w\cos\theta)$$

and the excess of flow in over flow out along PS per unit time

$$=-r\,\cos\theta\,\delta\phi\,\frac{1}{r\,\cos\theta}\,\frac{\partial}{\partial\phi}\,(\rho\, r\cos\theta\, w'\;\;\delta r\, r\,\delta\theta)$$

$$=-r^2\,\delta r\,\delta\theta\,\delta\phi\frac{\partial}{\partial\phi}(\rho w'\cos\theta)$$

Hence the total excess flow through parallelopiped per unit time is

$$= -r^2\ \delta r\ \delta\theta\ \delta\phi\left[\frac{\partial}{\partial\theta}(\rho w\cos\theta) + \frac{\partial}{\partial\phi}(\rho w^1 \cos\theta)\right]$$

Again, the rate of increase in mass of the element

$$= \frac{\partial}{\partial t}(\rho.\delta r\ r\ \delta\theta . r\cos\theta\,\delta\phi)$$

$$= r^2 \cos\theta\,\delta r\delta\theta\delta\phi\frac{\partial\rho}{\partial t}$$

Hence the equation of continuity in given by

$$r^2\cos\theta\ \delta r\ \delta\theta\ \delta\phi\ \frac{\partial\rho}{\partial t} = -\ r^2\ \delta r\ \delta\theta\ \delta\phi\left[\frac{\partial}{\partial\theta}(\rho w\cos\theta) + \frac{\partial}{\partial\phi}(\rho w'\cos\theta)\right]$$

or $$\frac{\partial\rho}{\partial t}\cos\theta + \frac{\partial}{\partial\theta}(\rho w\cos\theta) + \frac{\partial}{\partial\phi}(\rho w'\cos\theta) = 0$$

Example 6 : If the lines of motion are curves on the surface of spheres all touching the plane of xy at the origin O, the equation of continuity is

$$r\sin\theta\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v)}{\partial\phi} + \sin\theta\frac{\partial(\rho u)}{\partial\theta} + (\rho u)(1 + 2\cos\theta) = 0$$

Where r in the radius CP of one sphere, D the angle PCO, u the velocity in the plane PCO, v the perpendicular velocity and ϕ the inclination of the plane PCO to a fixed plane through the axis of z.

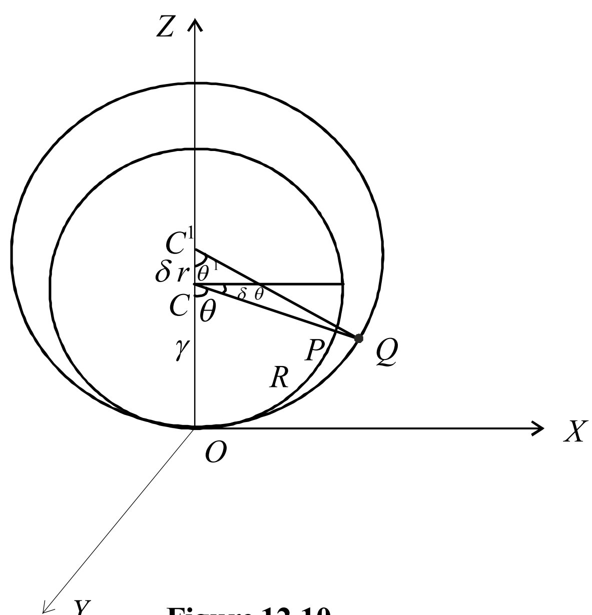


Figure 12.10

Solution : Let C and C^1 be the centres of two spheres of radii r and $r + \delta r$ respectively. P be the point on the smaller sphere.

Let PQ, PR and PS be the edges of the elementry parallelopiped. Here $PR = r\,\delta\,\theta$; $PS = r\sin\theta\;\delta\,\phi$ where ϕ is the angle that the plane PCO makes with a fixed plane through z-axis.

To Find the length PQ, we have

$$CP = r;\; C'Q = r + \delta\, r,\; CC' = \delta\, r \text{ and } \angle PCO = \theta$$

In $\Delta CC'Q$, we have

$$C'Q^2 = CC'^2 + CQ^2 - 2CC'.CQ\,\cos QCC'$$

or $$(r+\delta\, r)^2 = \delta\, r^2 + (r+PQ)^2 - 2\delta\, r.(r+PQ)\cos(\pi-\theta)$$

or $$2r\,\delta\, r = 2r.PQ + PQ^2 + 2r\,\delta\, r\,\cos\theta + 2\,\delta\, r\,PQ\,\cos\theta$$

or $$2r\,\delta\, r(1-\cos\theta) = 2r.PQ + PQ^2 + 2\,\delta\, r.PQ.\cos\theta$$

Since PQ and $\delta\, r$ are small quantitites, so neglacting PQ^2, $\delta\, r^2$ and $\delta\, r.PQ$. Thus we have

$$2r\,\delta\, r\,(1-\cos\theta) = 2\,r\,PQ$$

$$\Rightarrow PQ = (1-\cos\theta)\,\delta\, r$$

Thus we have determined the three edges of the elementary parallelopiped as $PQ = (1-\cos\theta)\,\delta\, r$, $PS = r\;\delta\theta$ and $PR = r\sin\theta\,\delta\,\phi$. Since the lines of motion are curves on the surface of spheres touching the plane of xy, there would be no motion along PQ, then the excess flow in over flow out along PQ is zero.

Now, the excess flow-in over flow out along PS per unit time

$$= -\,r\,\delta\,\theta\,\frac{\partial}{r\,\partial\theta}\left[\rho u(1-\cos\theta)\delta\, r.r\sin\theta\,\delta\,\phi\right]$$

$$= -\,r\,\delta r\;\delta\theta\;\delta\phi\left[\sin\theta\;(1-\cos\theta)\frac{\partial}{\partial\theta}(\rho u) + \rho u\left\{\cos\theta\;(1-\cos\theta) + \sin^2\theta\right\}\right]$$

$$= -\,r\,\delta r\;\delta\theta\;\delta\phi\left[\sin\theta\;(1-\cos\theta)\frac{\partial}{\partial\theta}(\rho u) + \rho u\left\{\cos\theta\;(1-\cos\theta) + (1-\cos^2\theta)\right\}\right]$$

$$= -\,r(1-\cos\theta)\;\delta r\;\delta\theta\;\delta\phi\left[\sin\theta\;\frac{\partial}{\partial\theta}(\rho u) + \rho u\;(1+2\cos\theta)\right]$$

and the excess of flow in over flow out along PR per unit time

$$= -\,r\sin\theta\;\delta\phi\;\frac{\partial}{r\sin\theta\,\partial\theta}\left[\rho v(1-\cos\theta)\;\delta\, r\;.\;r\;\delta\theta\right]$$

$$= -r(1-\cos\theta)\,\delta r\,\delta\theta\,\delta\phi\,\frac{\partial}{\partial\phi}(\rho v)$$

Also, the rate of increase in mass of the element

$$= \frac{\partial}{\partial t}\left[\rho(1-\cos\theta)\,\delta r.r\,\delta\theta.r\sin\theta\,\delta\phi\right]$$

$$= r^2\sin\theta\,(1-\cos\theta)\,\delta r\,\delta\theta\,\delta\phi\,\frac{\partial\rho}{\partial t}$$

Hence the equation of continuity is given by

$$r^2\sin\theta\,(1-\cos\theta)\,\delta r\,\delta\theta\,\delta\phi\,\frac{\partial\rho}{\partial t} =$$

$$-r(1-\cos\theta)\,\delta r\,\delta\theta\,\delta\phi\left[\sin\theta\,\frac{\partial(\rho u)}{\partial\theta}+\rho u\,(1+2\cos\theta)\right] - r(1-\cos\theta)\,\delta r\,\delta\theta\,\delta\phi\,\frac{\partial}{\partial\phi}(\rho v)$$

or $$r\sin\theta\,\frac{\partial\rho}{\partial t}+\sin\theta\,\frac{\partial}{\partial\theta}(\rho u)+\frac{\partial}{\partial\phi}(\rho v)+\rho u\,(1+2\cos\theta)=0$$

which is the required equation of continuity.

12.6 Boundary Surface

Physical conditions that should be satisfied on given boundaries of the fluid are called as boundary conditions. At the boundary of the fluid the equation of continuity is replaced by a special surface condition. When the fluid is in contact with an impermeable bounding surface, the velocity of a fluid portete at any point of the boundary relative to the surface must be tangential to the boundary.

A surface is a boundary surface if at any point of this surface the normal component of the fluid velocity is equal to the normal component of the velocity of the surface. If it is not so, then the contact between the fluid and the surface will break.

12.7 Condition for a surface may be boundary surface

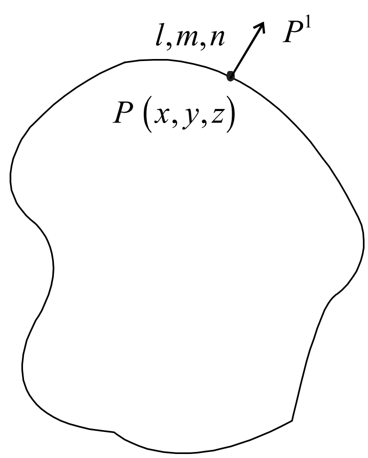


Figure 12.11

Let the surface be $F(x,y,z,t)=0$. Take a point $P(x,y,z)$ on the boundary of this surface such that the direction cosine of the normal be $l_1 m$ and n. Also the normal velocity of the boundary at P be V. After a small time δt, P moves to P' such that $PP' = V\,\delta t$. The projections of PP' on the axes x, y and z will be $l\,V\,\delta t,\ m\,V\,\delta t$ and $n\,V\,\delta t$. But after time $t+\delta t$, point P' continues to be on the surface hence

$$F\,(x + l\,V\,\delta t,\ y + m\,V\,\delta t,\ z + n\,V\,\delta t,\ t + \delta t) = 0$$

Expanding by Taylor's theorem, we obtain

$$F\,(x,y,z,t) + \left(\frac{\partial F}{\partial x}\,l\,V + \frac{\partial F}{\partial y}\,m\,V + \frac{\partial F}{\partial z}\,n\,V + \frac{\partial F}{\partial t}\right)\delta t + \text{Higher powers of } \delta t = 0$$

On neglecting higher powers of δt and using $F(x,y,z,t)=0$ we have

$$V = -\frac{\partial F/\partial t}{l\,\dfrac{\partial F}{\partial x} + m\,\dfrac{\partial F}{\partial y} + n\,\dfrac{\partial F}{\partial z}} \qquad \text{...(1)}$$

But l,m,n are direction cosine of PP' to the surface $F(x\ y\ z\ t)=0$; so

$$\frac{l}{\partial F/\partial x} = \frac{m}{\partial F/\partial y} = \frac{n}{\partial F/\partial z} = \frac{1}{\sqrt{\left(\dfrac{\partial F}{\partial x}\right)^2 + \left(\dfrac{\partial F}{\partial y}\right)^2 + \left(\dfrac{\partial F}{\partial z}\right)^2}}$$

$$\Rightarrow l\,\frac{\partial F}{\partial x} + m\,\frac{\partial F}{\partial y} + n\,\frac{\partial F}{\partial z} = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2} \qquad \text{...(2)}$$

Using (2) in (1), we have

$$V = -\frac{\partial F/\partial t}{\sqrt{\left(\dfrac{\partial F}{\partial x}\right)^2 + \left(\dfrac{\partial F}{\partial y}\right)^2 + \left(\dfrac{\partial F}{\partial z}\right)^2}} \qquad \text{...(3)}$$

and we know that the normal velocity in the direction PP' at P be $V = ul + vm + wn$ where u, v, w are velocity components and must satisfy the equation of continuity.

$$V = ul + vm + wn = \frac{u\,\dfrac{\partial F}{\partial x} + v\,\dfrac{\partial F}{\partial y} + w\,\dfrac{\partial F}{\partial z}}{\sqrt{\left(\dfrac{\partial F}{\partial x}\right)^2 + \left(\dfrac{\partial F}{\partial y}\right)^2 + \left(\dfrac{\partial F}{\partial z}\right)^2}} \qquad \text{...(4)}$$

From (3) and (4), we have

$$-\frac{\partial F}{\partial t} = u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z}$$

or $$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0$$

or $$\frac{DF}{Dt} = 0$$

Hence if $F(x,y,z,t) = 0$ be boundary surface, then at every point on it, $\frac{DF}{Dt} = 0$ where $\frac{D}{Dt}$ is the differential following the motion. Hence the expression for normal velocity of the boundary surface is given by

$$V = \frac{-\partial F/\partial t}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}$$

Self Learning Exerecise II

1. At the boundary surface the normal velocity component of the fluid is .... to the normal component of velocity of the surface.
2. Write down the condition for a surface representing a boundary surface.
3. Write down the condition for a surface to represent the boundary surface if the boundary surface is at rest.

Example 7 : Show that the ellipsoid

$$\frac{x^2}{a^2k^2t^{2n}} + kt^n\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 1$$

is a possible form of the boundary surface of a liquid.

Solution : We know that the surface $F(x,y,z,t) = 0$ represents the boundary surface if

$$\frac{DF}{Dt} = 0 \Rightarrow \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0 \qquad ...(1)$$

where u, v and w satisfy the equation of continuity Here

$$F = \frac{x^2}{a^2k^2t^{2n}} + kt^n\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right) - 1 = 0$$

$$\frac{\partial F}{\partial t} = -\frac{x^2}{a^2k^2} \cdot \frac{2n}{t^{2n+1}} + nkt^{n-1}\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$$

$$\frac{\partial F}{\partial x} = \frac{2x}{a^2 k^2 t^{2n}}$$

$$\frac{\partial F}{\partial y} = \frac{2k t^n y}{b^2}$$

and $$\frac{\partial F}{\partial z} = \frac{2k t^n z}{c^2}$$

Using there in (1), we have

$$-\frac{x^2}{a^2\ k^2}\frac{2n}{t^{2n+1}} + nk t^{n-1}\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)+\frac{2xu}{a^2k^2t^{2n}}+\frac{2kt^n yv}{b^2}+\frac{2kt^n zw}{c^2}=0$$

or $$\left(u-\frac{nx}{t}\right)\frac{2x}{a^2\ k^2t^{2n}}+\left(v+\frac{ny}{2t}\right)\frac{2kyt^n}{b^2}+\left(w+\frac{nz}{2t}\right)\frac{2kzt^n}{c^2}=0$$

which will hold if

$$u-\frac{nx}{t}=0\ ;\ v+\frac{ny}{2t}=0\ ;\text{ and } w+\frac{nz}{2t}=0$$

$$\Rightarrow\quad u=\frac{nx}{t}\ ;\ v=\frac{-ny}{2t}\ ;\text{ and } w=\frac{-nz}{2t}$$

These value of u, v, w can be seen to satisfy the equation of continuity. Hence the given surface is a boundary surface.

Example 8 : Show that

$$\frac{x^2}{a^2}\ f(t) + \frac{y^2}{b^2}\ \phi\ (t) + \frac{z^2}{c^2}\ \psi\ (t)=1$$, where $f(t).\phi\ (t).\psi\ (t)=1$ is a possible form of the boundary surface.

Solution : Given that

$$F\ (x,y,z,t)=\frac{x^2}{a^2}\ f\ (t) + \frac{y^2}{b^2}\ \phi\ (t) + \frac{z^2}{c^2}\ \psi\ (t)-1=0 \qquad ...(1)$$

will be a possible form of the boundary surface if it satisfies

$$\frac{\partial F}{\partial t} + u\ \frac{\partial F}{\partial x} + v\ \frac{\partial F}{\partial y} + w\ \frac{\partial F}{\partial z}=0 \qquad ...(2)$$

Using (1) is (2), we have

$$\frac{x^2}{a^2}\ f'(t)+\frac{y^2}{b^2}\ \phi'(t)+\frac{z^2}{c^2}\ \psi'(t)+u.\frac{2x}{a^2}f(t)+\frac{2vy}{b^2}\phi(t)+\frac{2wz}{c^2}\psi(t)=0$$

or $$\frac{2x}{a^2} f(t)\left[u+\frac{x f'(t)}{2 f(t)}\right]+\frac{2y}{b^2}\phi(t)\left[v+\frac{y\phi'(t)}{\phi(t)}\right]+\frac{2z}{c^2}\psi(t)\left[w+\frac{z\psi'(t)}{\psi(t)}\right]=0$$

which satisfies only when

$$u=-\frac{1}{2}x\frac{f'(t)}{f(t)}, \quad v=-\frac{y\phi'(t)}{2\phi(t)}\ ; \quad w=-\frac{z\psi'(t)}{\psi(t)}$$

But there value of u, v and w must satisfy the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\Rightarrow \quad -\frac{1}{2}\frac{f'(t)}{f(t)}-\frac{1}{2}\frac{\phi'(t)}{\phi(t)}-\frac{1}{2}\frac{z\psi'(t)}{\psi(t)}=0$$

or $$\frac{f'(t)}{f(t)}+\frac{\phi'(t)}{\phi(t)}+\frac{z\psi'(t)}{\psi(t)}=0$$

or $$\frac{d}{dt}\left[\log f(t)+\log\phi(t)+\log\psi(t)\right]=0$$

$$\Rightarrow \quad \frac{d}{dt}\left[\log f(t)\,.\,\phi(t)\,.\,\psi(t)\right]=0$$

which is true for given condition $f(t)\,.\,\phi(t)\,.\,\psi(t)=1$.

Example 9 : Show that ellipsoid

$$\frac{x^2}{a^2k^2t^4}+kt^2\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=1$$

is a possible form of the boundary surface of a liquid at time t.

Solution : Here $F(x,y,z,t)=\frac{x^2t^{-4}}{a^2k^2}+\frac{ky^2}{b^2}t^2+\frac{kz^2}{c^2}t^2-1=0$ ...(1)

Now (1) will represent a boundary surface if it satisfies

$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0 \quad ...(2)$$

Now $$\frac{\partial F}{\partial t}=\frac{-4x^2}{a^2k^2t^5}+\frac{2ky^2}{b^2}t+\frac{2z^2k}{c^2}t$$

$$\frac{\partial F}{\partial x}=\frac{2x}{a^2k^2t^4}$$

$$\frac{\partial F}{\partial y} = \frac{2ky}{b^2} t^2$$

and $$\frac{\partial F}{\partial z} = \frac{2kz}{c^2} t^2$$

Putting these in (2), we have

$$-\frac{4x^2}{a^2 k^2 t^5} + 2kt\frac{y^2}{b^2} + \frac{2kz^2}{c^2} t + \frac{2xu}{a^2 k^2 t^4} + \frac{2kyvt^2}{b^2} + \frac{2kzwt^2}{c^2} = 0$$

or $$\frac{2x}{a^2 k^2 t^4}\left[u - \frac{2x}{t}\right] + \frac{2ykt^2}{b^2}\left[v + \frac{y}{t}\right] + \frac{2zkt^2}{c^2}\left[w + \frac{z}{t}\right] = 0$$

It is satisfied if

$$u = \frac{2x}{t} \; ; \; v = -\frac{y}{t} \text{ and } w = -\frac{z}{t}$$

Which satisfies the equation of conitinuity therefore the given surface is a possible boundary surface of the fluid.

12.8 Summary

In this unit, we studied the forms of equation of continuity in different coordinate systems. The concept of boundary surface is introduced and the conditions required for a given surface to be a boundry surface are discused.

12.9 Answer to self learning exercise

Exercise I

1. b $\quad div\ \vec{q} = 0$
2. $SV =$ constant, where S is sectional area and V is the velocity
3. The mathematical representation of Law of mass conservation.
4. Yes : $\frac{dv}{dr} = 1 - \frac{1}{r^2}$; $\frac{d^2 v}{dr^2} = \frac{2}{r^3}$

$$\because \frac{d^2 v}{dr^2} + \frac{d}{dr}\left(\frac{v}{r}\right) = \frac{2}{r^3} - \frac{2}{r^3} = 0$$

Exercise II

1. Equal
2. $$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0$$
3. $$u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0$$

12.10 Exercise

1. Derive the equation of continuity for a constant density fluid.

2. Show that in a two-dimensional incompressible steady flow fleld the equation of continuity is satisfied with the velocity components in rectangular coordinates given by

$$u(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2} \text{ and } v(x,y) = \frac{2xy}{(x^2+y^2)^2}$$

3. The particle of a fluid moves symmetrically in space with regard to a fixed centre. Prove that the equation of continuity is

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) = 0$$

Where u is the velocity at distance r.

4. Each particle of a mass of liquid moves in a plane through the axis of z. Find the equation of continuity.

Ans. $$\left[\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho u r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho v \sin\theta) = 0\right]$$

5. Homogenous liquid moves so that the path of any particle P lies in the plane POX, where OX is fixed axis. Prove that if $OP = r$ and the $\angle XOP = \theta$, the equation of continuity may be written as

$$\frac{\partial}{\partial r}(ur^2) - \frac{\partial}{\partial \mu}(vr\sin\theta) = 0$$

Where u, v are the component velocities along and perpendicular to OP in the plane POX and $\mu = \cos\theta$.

6. Show that $\frac{x^2}{a^2} f(t) + \frac{y^2}{b^2} \cdot \frac{1}{f(t)} = 1$ is a possible form of the boundary surface of a liquid.

7. Show that $\frac{x^2}{a^2} \sin^2 t + \frac{y^2}{b^2} \operatorname{cosec}^2 t = 1$ is a possible form for the boundary surface of a liquid.

8. Show that $\frac{x^2}{a^2} e^t + \frac{y^2}{b^2} \cos t + \frac{z^2}{c^2} e^{-t} \sec t = 1$ is a possible form for the boundary surface of a liquid.

□□□□

UNIT - 13
Equation of Motion-I

Structure of the unit

13.0 Objective

In this unit, our aim is to study Euler's equations of motion in various forms and integrate the equation of motion to get the Bernoulli's equation and Bernoulli's theorem for the fluid motion. The Helmholtz equation is also obtained through various approches.

13.1 Introduction

The Euler's equations of motion are based on the law of conservation of mass and Newton's second law of motion. The momentum of a body is defined as the product of the mass of the body and its velocity. In the fluid motion, the momentum per unit volume is defined by $M = \rho q$, where ρ be the density of the fluid and q be the velocity of the fluid particle. The Newton's second law of motion states that the rate of change in momentum is directly proportional to the external force applied on it and this changes takes always in the direction of the fluid flow. In other words, the net force acting on the fluid particle is equal to the product of mass of the body and acceleration of the fluid particle.

13.2 Euler's dynamical equations of motion in vector notation

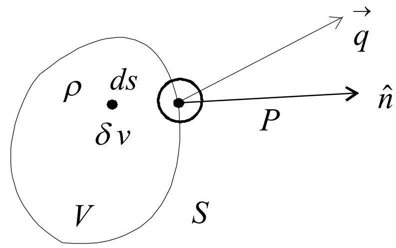


Figure 13.1

Consider a closed surface S such that it encloses a non-viscous fluid and is in motion with the fluid. This motion ensures that at any time S wll contain same fluid particles.

Now take any point P within the surface S, ρ be the density of the fluid at P and δV be an elementary volume which encloses P and $\vec{q}$ be the fluid velocity at P.

Since the mass $\rho \delta V$ of the element remains unchanged during the motion, the momentum $\vec{M}$ of the volume V in S is given by

$$\vec{M} = \int_V \vec{q} \rho d V \qquad \text{...(1)}$$

The rate of change of momentum is

$$\frac{d\vec{M}}{dt} = \int_V \frac{d\vec{q}}{dt} \rho d V + \int_V \vec{q} \frac{d}{dt}(\rho d V)$$

But here $\frac{d}{dt}(\rho d V) = 0$ since $\rho d V$ is always constant, so

$$\frac{d\vec{M}}{dt} = \int_V \frac{d\vec{q}}{dt} \rho d V \qquad \text{...(2)}$$

Again, let $\vec{F}$ be the external force per unit mass acting on fluid and p be the pressure at a point of the surface element ds then total force on the fluid in volume V

$$= \int_V \vec{F} \rho d V \qquad \text{...(3)}$$

And the force due to pressure in the outward normal direction

$$= - \int_S p.\hat{n}\, d s$$

$$= - \int_V \nabla p\, d V \qquad \text{(By Gauss throrem)} \qquad \text{...(4)}$$

Hence from the Newton's second law of motion, rate of change of momentum = total force acting on the mass in the direction of the momentum, we have

$$\int_V \frac{d\vec{q}}{dt} \rho d V = \int_V \vec{F} \rho d V - \int_V \nabla p\, d V$$

$$= \int_V \left(\rho \vec{F} - \nabla p\right) d V$$

or $$\int_V \left(\rho \frac{d\vec{q}}{dt} - \rho \vec{F} + \nabla p \right) dV = 0$$

But volume V enclosed in S is arbitrary, so we have

$$\rho \frac{d\vec{q}}{dt} = \rho \vec{F} - \nabla p$$

or $$\frac{d\vec{q}}{dt} = \vec{F} - \frac{1}{\rho} \nabla p \quad ...(5)$$

where $\frac{d}{dt}$ is the differential following the motion. Therefore equation (5) should be written as

$$\frac{\partial \vec{q}}{\partial t} + \left(\vec{q} . \nabla \right) \vec{q} = \vec{F} - \frac{1}{\rho} \nabla p \qquad \because \frac{d}{dt} = \frac{\partial}{\partial t} + \left(\vec{q} . \nabla \right)$$

Which is the Euler's dynamical equation in vector notation.

13.3 Euler's dynamical equations of motion in cartesian coordinate

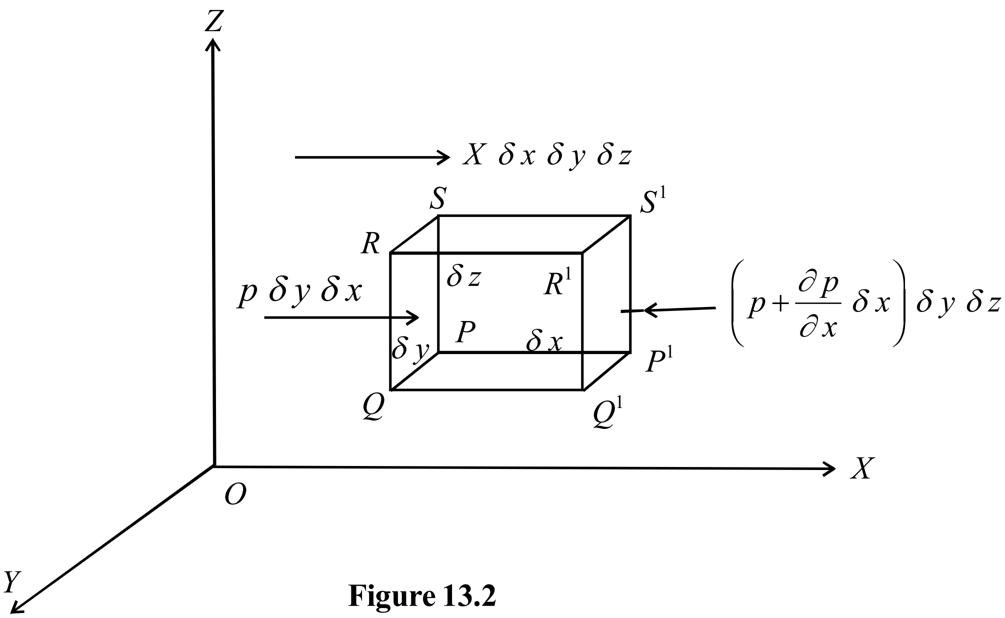


Figure 13.2

Consider a point $P(x, y, z)$ of a fluid. Let p be the pressure per unit area, ρ be the density, u, v, w be the components of velocity and X, Y, Z the components of external force per unit mass in the direction of axes respectively.

Construct a rectangular parallelopiped having faces parallel to coordinate planes and edges

$$PP' = \delta x \;,\;\; PQ = \delta y \;\text{ and }\; PS = \delta z$$

The pressure on the face $PQRS = p \, \delta y \, \delta z = f(x, y, z)$ (say)

Then the pressure on the face $P'Q'R'S'$

$$= f(x+\delta x, y, z)$$

$$= f(x,y,z) + \frac{\partial f}{\partial x}\delta x + \ldots$$

$$= p\,\delta y\,\delta z + \frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z$$ on neglecting second order terms

$$= \left(p + \frac{\partial p}{\partial x}\,\delta x\right)\delta y\,\delta z$$

Also since X is the force per unit mass parallel to x-axis, the force on the rectangular parallelopiped in x-direction $= X\,\rho\,\delta x\,\delta y\,\delta z$

Again, the acceleration along x-axis is $\frac{du}{dt}$ then the rate of change of momentum along x-axis is

$$= \frac{du}{dt}\,\rho\,\delta x\,\delta y\,\delta z$$

Hence, the equation of motion along x-axis is

$$\rho\,\delta x\,\delta y\,\delta z\,\frac{du}{dt} = X\,\rho\,\delta x\,\delta y\,\delta z + p\,\delta y\,\delta z - \left(p + \frac{\partial p}{\partial x}\,\delta x\right)\delta y\,\delta z$$

i.e. $$\rho\,\frac{du}{dt} = X\,\rho - \frac{\partial p}{\partial x}$$

or $$\frac{du}{dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

or $$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} \quad \ldots(1)$$

similarly, the equations of motion in y and z directions will be

$$\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + w\,\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} \quad \ldots(2)$$

and $$\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + w\,\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} \quad \ldots(3)$$

respectively.

Equation (1) to (3) are known as Euler's dynamical equations of motion in cartesian coordinates.

13.4 Conservative Field of Force

A field of force having components X, Y and Z parallel to axes is called conservative if the work done by the force per unit mass from one point to the other is independent of the path of the motion.

Hence $X\,dx + Y\,dy + Z\,dz = -dV$

or
$$X\,dx + Y\,dy + Z\,dz = -\frac{\partial V}{\partial x}\,dx - \frac{\partial V}{\partial y}\,dy - \frac{\partial V}{\partial z}\,dz$$

$$\Rightarrow \qquad X = -\frac{\partial V}{\partial x}\;;\; Y = -\frac{\partial V}{\partial y} \text{ and } Z = -\frac{\partial V}{\partial z}$$

where V is called potential function and depends upon the initial and final positions of the moving mass.

13.5 Integration of the equations of motion

Suppose the motion be irrotational, so that velocity potential ϕ exists and the system of force be conservative so that the external forces are derivable from a potential function V, such that

$$u = -\frac{\partial \phi}{\partial x},\; v = -\frac{\partial \phi}{\partial y},\; w = -\frac{\partial \phi}{\partial z} \qquad \text{...(1)}$$

and
$$X = -\frac{\partial V}{\partial x},\; Y = -\frac{\partial V}{\partial y},\; Z = -\frac{\partial V}{\partial z} \qquad \text{...(2)}$$

Since the motion is irrotational, so that spin components should vanish

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z},\; \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} \text{ and } \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \qquad \text{...(3)}$$

The equation of motion are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{...(4)}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{...(5)}$$

and
$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad \text{...(6)}$$

Using results from (1) to (3) in (4) to (6), we have

$$\frac{\partial}{\partial t}\left(-\frac{\partial \phi}{\partial x}\right) + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial}{\partial t}\left(-\frac{\partial \phi}{\partial y}\right) + u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial y} = -\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

and $$\frac{\partial}{\partial t}\left(-\frac{\partial \phi}{\partial z}\right)+u\frac{\partial u}{\partial z}+v\frac{\partial v}{\partial z}+w\frac{\partial w}{\partial z}=-\frac{\partial V}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z}$$

These equation may be written as

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial x}\left(u^2+v^2+w^2\right)=-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad ...(7)$$

$$\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial y}\left(u^2+v^2+w^2\right)=-\frac{\partial V}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad ...(8)$$

and $$\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}\frac{\partial}{\partial z}\left(u^2+v^2+w^2\right)=-\frac{\partial V}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad ...(9)$$

Multiplying (7) by dx, (8) by dy and (9) by dz then adding, we get

$$\left[\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial t}\right)dx+\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial t}\right)dy+\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial t}\right)dz\right]$$

$$+\frac{1}{2}\left[\frac{\partial}{\partial x}\left(u^2+v^2+w^2\right)dx+\frac{\partial}{\partial y}\left(u^2+v^2+w^2\right)dy+\frac{\partial}{\partial z}\left(u^2+v^2+w^2\right)dz\right]$$

$$=-\left[\frac{\partial V}{\partial x}dx+\frac{\partial V}{\partial y}dy+\frac{\partial V}{\partial z}dz\right]-\frac{1}{\rho}\left[\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy+\frac{\partial p}{\partial z}dz\right]$$

which may be written as

$$d\left(-\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}d\left(u^2+v^2+w^2\right)=-dV-\frac{1}{\rho}dp$$

or $$d\left(-\frac{\partial \phi}{\partial t}\right)+\frac{1}{2}dq^2+dV+\frac{1}{\rho}dp=0$$ where $q^2=u^2+v^2+w^2$

Assuming a functional relatioship between p and ρ and integrating we obtain

$$-\frac{\partial \phi}{\partial t}+\frac{1}{2}q^2+V+\int\frac{dp}{\rho}=\text{constant}$$

or $$\int\frac{dp}{\rho}-\frac{\partial \phi}{\partial t}+\frac{1}{2}q^2+V=\text{constant}=C \qquad ...(10)$$

Where C is an arbitrary function of the time.

Case I :

If ρ is constant, then

$$\frac{p}{\rho} - \frac{\partial \phi}{\partial t} + \frac{1}{2} q^2 + V = C$$

which is the Bernoulli's equation for the unsteady, irrotational motion of an incompressible fluid.

Case II :

If motion is steady, then the Bernoulli's equation takes the form

$$\frac{p}{\rho} + V + \frac{1}{2} q^2 = C \qquad \text{as} \qquad \frac{\partial \phi}{\partial t} = 0$$

Where C is an absolute constant.

13.6 Bernoulli's theorem

The theorem states that "in a steady fluid motion, which is not irrotational, if potential function exists such that the external forces are derivable from this, then

$$\int \frac{dp}{\rho} + \frac{1}{2} q^2 + V = C \text{ (OR)}$$

Show that

$$\int \frac{dp}{\rho} + \frac{1}{2} q^2 + V = C$$

Where the motion is steady and the velocity potential does not exists, V being the potential function from which the external forces are derivable.

Proof :

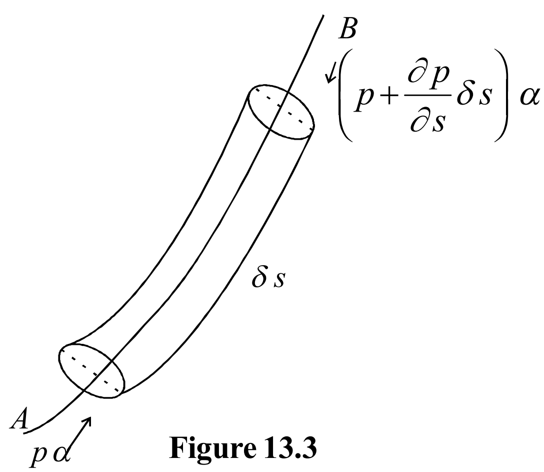


Figure 13.3

Let AB be a stream line in the fluid. Consider an element δs of this stream line and construct a small cylinder of cross-section α and δs as axis.

If $\vec{q}$ be the velocity and S the component of external force per unit mass along the stream line, then the equation of motion is

$$\rho\alpha\,\delta s.\frac{d\vec{q}}{dt} = \rho\alpha\,\delta s\ S + p\alpha - \left(p + \frac{\partial p}{\partial s}\,\delta s\right)\alpha$$

or $$\frac{d\vec{q}}{dt} = S - \frac{1}{\rho}\frac{\partial p}{\partial s}$$

or $$\frac{\partial \vec{q}}{\partial t} + q\frac{\partial \vec{q}}{\partial s} = S - \frac{1}{\rho}\frac{\partial p}{\partial s} \qquad \because \frac{d}{dt} \equiv \frac{\partial}{\partial t} + q\frac{\partial}{\partial s}$$

Here motion is steady therefore $\frac{\partial \vec{q}}{\partial t} = 0$ and since the component of external force S is derivable from potential function then $S = -\frac{\partial V}{\partial s}$

Hence

$$\vec{q}\ .\frac{\partial \vec{q}}{\partial s} = -\frac{\partial V}{\partial s} - \frac{1}{\rho}\frac{\partial p}{\partial s}$$

Integrating it along the stream line, we obtain

$$\frac{1}{2}q^2 = -V - \int\frac{dp}{\rho} + C$$

or $$\int\frac{dp}{\rho} + \frac{1}{2}q^2 + V = C$$

Where constant C depends upon the stream line choosen. This is the Bernoulli's theorem.

13.7 Helmholtz Equation

The Euler's dynamical equation of motion in the direction of x-axis in cartesian coordinates is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad ...(1)$$

where u, v, w are velocity components along axes respectively and

$X = -\frac{\partial V}{\partial x}$, $Y = -\frac{\partial V}{\partial y}$, $Z = -\frac{\partial V}{\partial z}$ are the components of external forces which are conservative with potential function V.

Now

$$\frac{\partial u}{\partial t} + \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial x}\right) + v\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + w\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

or $$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left(u^2 + v^2 + w^2\right) + v\,(-2\zeta) + 2\,w\,\eta = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

where ξ, η, ζ are the spin components and $q^2 = u^2 + v^2 + w^2$.

or $$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial q^2}{\partial x} - 2\,v\,\zeta + 2\,w\,\eta = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

or $$\frac{\partial u}{\partial t} - 2\,v\,\zeta + 2\,w\,\eta = -\frac{\partial}{\partial x}\left[V + \int\frac{d\,p}{\rho} + \frac{1}{2}q^2\right]$$

or $$\frac{\partial u}{\partial t} - 2\,v\,\zeta + 2\,w\,\eta = -\frac{\partial Q}{\partial x} \qquad ...(2)$$

where $Q = \int\frac{d\,p}{\rho} + \frac{1}{2}q^2 + V$ and ρ is a function of p.

Similarly we obtain

$$\frac{\partial v}{\partial t} - 2\,w\,\xi + 2\,u\,\zeta = -\frac{\partial Q}{\partial y} \qquad ...(3)$$

$$\frac{\partial w}{\partial t} - 2\,u\,\eta + 2\,v\,\xi = -\frac{\partial Q}{\partial z} \qquad ...(4)$$

Deffrantating (3) partially with respect to z and differentiating (4) partially with respect to y then eliminative Q, we have

$$\frac{\partial^2 v}{\partial z\,\partial t} - 2\,\xi\,\frac{\partial w}{\partial z} - 2\,w\,\frac{\partial \xi}{\partial z} + 2\,\zeta\,\frac{\partial u}{\partial z} + 2\,u\,\frac{\partial \zeta}{\partial z}$$

$$= \frac{\partial^2 w}{\partial y\,\partial t} - 2\,u\,\frac{\partial n}{\partial y} - 2\,\eta\,\frac{\partial u}{\partial y} + 2\,\xi\,\frac{\partial v}{\partial y} + 2\,v\,\frac{\partial \xi}{\partial y}$$

or $$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - 2\,u\left(\frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}\right) + 2\,v\,\frac{\partial \xi}{\partial y} + 2\,w\,\frac{\partial \xi}{\partial z}$$

$$+ 2\,\xi\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - 2\,\eta\,\frac{\partial u}{\partial y} - 2\,\zeta\,\frac{\partial u}{\partial z} = 0 \qquad ...(5)$$

On adding and subtracting $2\,u\,\frac{\partial \xi}{\partial x}$, we have

$$2\,\frac{\partial \xi}{\partial t} + 2u\,\frac{\partial \xi}{\partial x} + 2\,v\,\frac{\partial \xi}{\partial y} + 2\,w\,\frac{\partial \xi}{\partial z} + 2\,\xi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$- 2\xi \frac{\partial u}{\partial x} - 2\eta \frac{\partial u}{\partial y} - 2\zeta \frac{\partial u}{\partial z} = 0$$

or $$\left(\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y} + w\frac{\partial \xi}{\partial z}\right) + \xi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z}$$

or $$\frac{d\xi}{dt} + \xi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z} \qquad ...(6)$$

where $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$. From the equation of continuity

$$\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

we get $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{d\rho}{dt} \qquad ...(7)$$

Using (7) in (6), we obtain

$$\frac{d\xi}{dt} - \frac{\xi}{\rho}\frac{d\rho}{dt} = \xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z}$$

or $$\frac{1}{\rho}\frac{d\xi}{dt} - \frac{\xi}{\rho^2}\frac{d\rho}{dt} = \frac{\xi}{\rho}\frac{\partial u}{\partial x} + \frac{\eta}{\rho}\frac{\partial u}{\partial y} + \frac{\zeta}{\rho}\frac{\partial u}{\partial z}$$

or $$\frac{d}{dt}\left(\frac{\xi}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial u}{\partial x} + \frac{\eta}{\rho}\frac{\partial u}{\partial y} + \frac{\zeta}{\rho}\frac{\partial u}{\partial z} \qquad ...(8)$$

which is the first Helmholtz equation.

Similarly

$$\frac{d}{dt}\left(\frac{\eta}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial v}{\partial x} + \frac{\eta}{\rho}\frac{\partial v}{\partial y} + \frac{\zeta}{\rho}\frac{\partial v}{\partial z} \qquad ...(9)$$

and $$\frac{d}{dt}\left(\frac{\zeta}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial w}{\partial x} + \frac{\eta}{\rho}\frac{\partial w}{\partial y} + \frac{\zeta}{\rho}\frac{\partial w}{\partial z} \qquad ...(10)$$

These equation (8) to (10) are the Helmholtz equations. To obtain another form of Helmholtz equations, we take Helmholtz equation in x-direction (8),

$$\frac{d}{dt}\left(\frac{\xi}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial u}{\partial x} + \frac{\eta}{\rho}\frac{\partial u}{\partial y} + \frac{\zeta}{\rho}\frac{\partial u}{\partial z}$$

$$= \frac{\xi}{\rho}\frac{\partial u}{\partial x} + \frac{\eta}{\rho}\frac{\partial v}{\partial x} + \frac{\zeta}{\rho}\frac{\partial w}{\partial x} + \frac{\eta}{\rho}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \frac{\zeta}{\rho}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$= \frac{\xi}{\rho}\frac{\partial u}{\partial x} + \frac{\eta}{\rho}\frac{\partial v}{\partial x} + \frac{\zeta}{\rho}\frac{\partial w}{\partial x} + \frac{\eta}{\rho}(-2\zeta) + \frac{\zeta}{\rho}(2\eta)$$

Hence $\frac{d}{dt}\left(\frac{\xi}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial u}{\partial x} + \frac{\eta}{\rho}\frac{\partial v}{\partial x} + \frac{\zeta}{\rho}\frac{\partial w}{\partial x}$

Similarly other two equations (9) and (10) gives

$$\frac{d}{dt}\left(\frac{\eta}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial u}{\partial y} + \frac{\eta}{\rho}\frac{\partial v}{\partial y} + \frac{\zeta}{\rho}\frac{\partial w}{\partial y}$$

and $\frac{d}{dt}\left(\frac{\zeta}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial u}{\partial z} + \frac{\eta}{\rho}\frac{\partial v}{\partial z} + \frac{\zeta}{\rho}\frac{\partial w}{\partial z}$

which give the other form of Helmholtz equations.

13.8 Permanence of irrotational motion

If ξ, η, and ζ be zero at any instant of time t then

$$\frac{D}{Dt}\left(\frac{\xi}{\rho}\right) = \frac{D}{Dt}\left(\frac{\eta}{\rho}\right) = \frac{D}{Dt}\left(\frac{\zeta}{\rho}\right) = 0$$

$\Rightarrow \quad \frac{D\xi}{Dt} = \frac{D\eta}{Dt} = \frac{D\zeta}{Dt} = 0 \qquad$ if ρ is constant

$\Rightarrow \quad \xi$, η and ζ are constants.

Let us assume that $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \cdots\cdots,$ are all finite and less than a quantity l, then $\frac{\xi}{\rho}, \frac{\eta}{\rho}$ and $\frac{\zeta}{\rho}$ cannot increase if they satisfy the equations

$$\frac{D}{Dt}\left(\frac{\xi}{\rho}\right) = \frac{D}{Dt}\left(\frac{\eta}{\rho}\right) = \frac{D}{Dt}\left(\frac{\zeta}{\rho}\right) = \frac{l}{\rho}(\xi + \eta + \zeta)$$

Let $\xi + \eta + \zeta = \rho w$, we get from the above equation

$$\frac{D}{Dt}(w) = \frac{D}{Dt}\left(\frac{\xi}{\rho} + \frac{\eta}{\rho} + \frac{\zeta}{\rho}\right)$$

$$= \frac{3l}{\rho}(\xi + \eta + \zeta)$$

$$\frac{D}{Dt}(w) = 3l\,w$$

On integratip $w = Ce^{3lt}$, $w \neq 0$.

If when $t = w = 0$ then $C = 0$ Hence if $w = 0$ at time $t = 0$ it must be so for all times. Since w is the sum of three quantities ξ, η, ζ which cannot be negative. Hence $w = 0$, it follows that each of these three quantities must be zero,

$$\xi = 0 = \eta = \zeta$$

Hence, if the motion is irrotational at any instant, it must be so for all times. In other words, if once, the velocity potential exists, it exists for all time. This is known as the principle of permanance of irrotational motion.

13.9 Working rule to solve problems

1. Read the question and obtained the equations of motion as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

and $$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

The general problem of hydro-dynamics is to find values of u, v, w, p and ρ as functions of x, y, z and t.

2. The equation of motion are not sufficient for the purpose and we need other relation. In all problems we make use of the equation of continuity.

3. With the help of initial and boundary conditions, the problem is solved particularly.

Self Learninig Exercise

1. If the velocity potential does not exist then motion is

(a) rotational (b) irrotational

(c) translational (d) rotational as well as translational

2. Write down the Bernoulli's equation for the unsteady, irrotational motion of an incompressible fluid.

3. Define the conservative field of force.

Example 1 : Air, obeying Boyle's law, is in motion in a uniform tube of small section, Prove that if ρ be the density and v be the velocity at a distance x from a fixed point at time t,

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left\{\left(v^2 + k\right)\rho\right\}$$

Solution : Air, obeying Boyle's law

$$\therefore\ p = k\ \rho \qquad ...(1)$$

The equation of continuity in this case is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0 \qquad ...(2)$$

and the equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Using (1), $$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{k}{\rho}\frac{\partial \rho}{\partial x} \qquad ...(3)$$

On differentiating (2) partially with respect to t., we obtain

$$\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial t}\cdot\frac{\partial}{\partial x}(\rho v) = 0$$

i.e. $$\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial x}\left[\rho\frac{\partial v}{\partial t} + v\frac{\partial \rho}{\partial t}\right] = 0$$

$\therefore$ From (2) and (3)

or $$\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial x}\left[\rho\left\{-v\frac{\partial v}{\partial x} - \frac{k}{\rho}\frac{\partial \rho}{\partial x}\right\} + v\left\{-\frac{\partial(\rho v)}{\partial x}\right\}\right] = 0$$

or $$\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial}{\partial x}\left[\rho v\frac{\partial v}{\partial x} + v\frac{\partial(\rho v)}{\partial x} + k\frac{\partial \rho}{\partial x}\right] = 0$$

or $$\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(\rho v.v) + k\frac{\partial \rho}{\partial x}\right] = 0$$

or $$\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial^2}{\partial x^2}\left(\rho v^2 + k\rho\right) = 0 \quad \Rightarrow \frac{\partial^2 \rho}{\partial t^2} - \frac{\partial^2}{\partial x^2}\left[\left(v^2 + k\right)\rho\right] = 0$$

which is the required result.

Example 2 : A pulse travelling along a fine straight uniform tube filled with gas causes the density at time t and distance x from the origin where the velocity is u_0 to become $\rho_0\ \phi\ (vt - x)$. Prove that the velocity u is given by

$$v + \frac{(u_0 - v)\ \phi\ (vt)}{\phi\ (vt - x)}$$

Solution : If ρ be the density of the gas at a distance x and u the velocity there then according to given condition

$$\rho = \rho_0 \, \phi \, (v\,t - x) \qquad ...(1)$$

and the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

or $$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} = 0 \qquad ...(2)$$

From (1), we have $\dfrac{\partial \rho}{\partial t} = \rho_0 v \, \phi' \, (v\,t - x)$

and $$\frac{\partial \rho}{\partial x} = - \, \rho_0 \, \phi' \, (v\,t - x) \qquad ...(3)$$

Using (3) in (2), it becomes

$$\rho_0 \, v \phi' \, (v\,t - x) + \rho_0 \, \phi \, (v\,t - x) \frac{\partial u}{\partial x} - u \, \rho_0 \, \phi' \, (v\,t - x) = 0$$

or $$(v - u) \, \phi' \, (v\,t - x) + \phi \, (v\,t - x) \frac{\partial u}{\partial x} = 0$$

or $$\frac{du}{(v-u)} + \frac{\phi' \, (v\,t - x)}{\phi \, (v\,t - x)} \, dx = 0$$

On integrating, we get

$$-\log(v - u) - \log \phi(v\,t - x) = \text{constant}$$

or $$(v - u) \, \phi \, (v\,t - x) = \text{C}$$

which C is the constant of integration. But initially when $x = 0$; $u = u_0 \Rightarrow C = (v - u_0) \, \phi \, (v\,t)$

Hence $(v - u) \, \phi \, (v\,t - x) = (v - u_0) \, \phi \, (v\,t)$

or $$v - u = \frac{(v - u_0) \, \phi \, (v\,t)}{\phi \, (v\,t - x)}$$

or $$u = v + \frac{(u_0 - v) \, \phi \, (v\,t)}{\phi \, (v\,t - x)}$$

Which is the required result.

Example 3 : A sphere is at rest in an infinate mass of homogeneous liquid of density ρ, the pressure at infinity being π; show that, if the radius R of the sphere varies in any manner, the pressure at the surface of the sphere at any time is

$$\pi + \frac{1}{2}\rho\left\{\frac{d^2}{dt^2}\left(R^2\right) + \left(\frac{dR}{dt}\right)^2\right\}$$

Solution : Let v be the velocity of the fluid at a distance r from the centre of the sphere at any time t and p the pressure, then the equation of continuity is

$$r^2\, v = F(t)$$

$$\Rightarrow \qquad \frac{\partial v}{\partial t} = \frac{F'(t)}{r^2} \qquad \text{...(1)}$$

The equation of motion is

$$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial r} = -\,\frac{1}{\rho}\,\frac{\partial p}{\partial r}$$

Using (1), we have

$$\frac{F'(t)}{r^2} + v\,\frac{\partial v}{\partial r} = -\,\frac{1}{\rho}\,\frac{\partial p}{\partial r} \qquad \text{...(2)}$$

On integrating $w.r.t.r$, we get

$$-\,\frac{F'(t)}{r} + \frac{1}{2}\,v^2 = C - \frac{p}{\rho}$$

The initial conditions are

$$r = \infty\ ,\ v = 0\ ;\ p = \pi \qquad \Rightarrow\ C = \pi/\rho$$

Hence

$$-\,\frac{F'(t)}{r} + \frac{1}{2}v^2 = \frac{\pi - p}{\rho}$$

or $$p = \pi + \frac{\rho}{2}\left[2\,\frac{F'(t)}{r} - v^2\right] \qquad \text{...(3)}$$

If P be the pressure on the surface of the sphere of radius R and V be the velocity at $r = R$ (on the surface of the sphere) then

$$P = \pi + \frac{\rho}{2}\left[2\,\frac{F'(t)}{R} - V^2\right] \qquad \text{...(4)}$$

Again from (1), $R^2 V = F(t)$

or $\quad F(t) = R^2 . \dfrac{dR}{dt}$

$$F'(t) = R^2 . \frac{d^2 R}{dt^2} + 2R\left(\frac{dR}{dt}\right)^2 \qquad ...(5)$$

Using (5) in (4), we obtain

$$P = \pi + \frac{\rho}{2}\left[2\left\{R.\frac{d^2 R}{dt^2} + 2\left(\frac{dR}{dt}\right)^2\right\} - \left(\frac{dR}{dt}\right)^2\right]$$

$$P = \pi + \frac{\rho}{2}\left[2\,R.\frac{d^2 R}{dt^2} + 3\left(\frac{dR}{dt}\right)^2\right]$$

or $\quad P = \pi + \dfrac{\rho}{2}\left[\dfrac{d^2(R^2)}{dt^2} + \left(\dfrac{dR}{dt}\right)^2\right]$

which is the required result.

Example 4 : Steam is rushing from a boiler through a conical pipe the diameter of the ends of which are D and d . If V and v be the corresponding velocities of the steam and if the motion be suppossed to be that of divergance from the vertex of the cone,

Prove that $\quad \dfrac{v}{V} = \dfrac{D^2}{d^2}\, e^{(v^2 - V^2)/2k}$

where k is the pressure divided by the density and supposed constant.

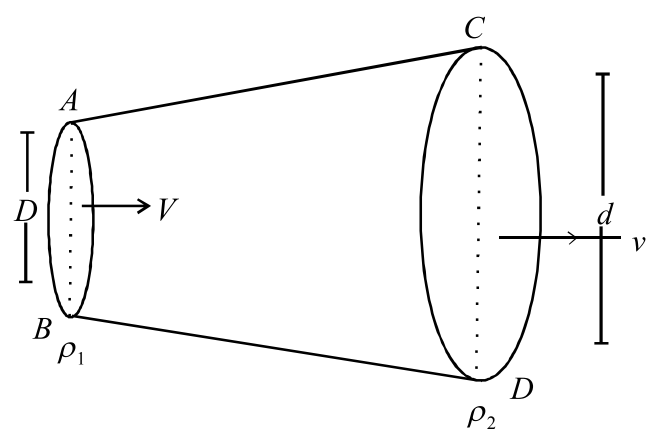


Figure 13.4

Solution : Let AB and CD be the ends of the conical pipe. V and v are the velocities at the ends where the diameters are D and d . Also let ρ_1 and ρ_2 be the densitites of steam at these ends. Then the equation of continuity can be written directly by the fact that mass of the steam that crosses AB and CD is the same. Thus equation of continuity is,

$$\pi\left(\frac{1}{2}D\right)^2 V\,\rho_1 = \pi\left(\frac{1}{2}d\right)^2 v.\rho_2$$

or $$\frac{\rho_1}{\rho_2} = \frac{v\,d^2}{VD^2} \qquad ...(1)$$

Now, if at any distance r from AB, the velocity is u, pressure is p and density is ρ then the equation of motion gives

$$u\,\frac{\partial u}{\partial r} = -\,\frac{1}{\rho}\,\frac{\partial p}{\partial r} \qquad ...(2)$$

Now as $p/\rho = k \;\Rightarrow\; p = \rho\,k$

then $$u\,\frac{\partial u}{\partial r} = -\,\frac{k}{\rho}\,\frac{\partial p}{\partial r}$$

On integrating, we have

$$\frac{u^2}{2} = -\,k\left[\log\rho - \log A\right] \qquad \Rightarrow\; \rho = A\,\bar{e}^{\,u^2/2k}$$

But here

$$\rho = \rho_1 \text{ when } u = V \Rightarrow \rho_1 = A\,\bar{e}^{\,V^2/2k}$$

and $$\rho = \rho_2 \text{ when } u = v \Rightarrow \rho_2 = A\,\bar{e}^{\,v^2/2k}$$

so that $$\frac{\rho_1}{\rho_2} = \frac{\bar{e}^{\,V^2/2k}}{\bar{e}^{\,v^2/2k}} \qquad ...(2)$$

From (1) and (2), we obtain

$$\frac{v\,d^2}{V\,D^2} = \frac{\bar{e}^{\,V^2/2k}}{\bar{e}^{\,v^2/2k}}$$

or $$\frac{v}{V} = \frac{D^2}{d^2}\,e^{\left(u^2 - V^2\right)/2k}$$

which is the required result.

Example 5 : An infinite mass of fluid is acted on by a force $\mu\,v^{-3/2}$ per unit mass directed to the origin. If initially the fluid is at rest and there is a cavity in the form of the surface $r = c$ in it, show that the cavity will be filled up after an interval of time $\left(2/5\,\mu\right)^{1/2} c^{5/4}$.

Solution : Let v' be the velocity at a distance r' from the origin and p be the pressure there at any time t. Then equation of continuity is $r'^2 v' = F(t)$ ...(1)

and the equation of motion is

$$\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial r'} = -\mu r'^{-3/2} - \frac{1}{\rho} \frac{\partial p}{\partial r'} \qquad ...(2)$$

Using (1) $\frac{\partial v'}{\partial t} = F'(t)/r'^2$, so that

$$\frac{F'(t)}{r'^2} + v' \frac{\partial v'}{\partial r'} = -\frac{\mu}{r'^{3/2}} - \frac{1}{\rho} \frac{\partial p}{\partial r'}.$$

Integrating $w.r.$ to r', we get

$$-\frac{F'(t)}{r'} + \frac{1}{2} v'^2 = \frac{2\mu}{r'^{1/2}} - \frac{p}{\rho} + C$$

When $r' \to \infty$; $v' = 0$, $p = 0$ $\Rightarrow C = 0$.

Hence $$\frac{-F'(t)}{r'} + \frac{1}{2} v'^2 = \frac{2\mu}{r'^{1/2}} - \frac{p}{\rho} \qquad ...(3)$$

Let r be the radius of the cavity at time t and v be the velocity there, then putting $p = 0$ (on the surface of cavity), the motion of the cavity is given by

$$\frac{-F'(t)}{r} + \frac{1}{2} v^2 = \frac{2\mu}{r^{1/2}} \qquad ...(4)$$

Now from equation (1), we have

$$r^2 v = F(t) \Rightarrow F'(t) = 2 r v^2 + r^2 v \frac{dv}{dr}$$

then equation (4) reduces to

$$-2v^2 - rv\frac{dv}{dr} + \frac{1}{2} v^2 = \frac{2\mu}{r^{1/2}}$$

or $$-r v \frac{dv}{dr} - \frac{3}{2} v^2 = \frac{2\mu}{r^{1/2}}$$

or $$v\frac{dv}{dr} + \frac{3}{2}\frac{v^2}{r} = -2\mu r^{-3/2}$$

or $$\frac{dv^2}{dr} + \frac{3}{r} v^2 = -4\mu r^{-3/2} \qquad ...(5)$$

which is a linear differential equation whose solution is given by

$$v^2 r^3 = -4\mu \int r^{3/2} dr + C_1 \qquad \because I.F = e^{\int \frac{3}{r}dr} = r^3$$

$$= -\frac{8\mu}{5} r^{5/2} + C_1$$

Initially $r = c \; ; v = 0 \Rightarrow C_1 = \frac{8\mu}{5} c^{5/2}$

then $\quad v^2 r^3 = \frac{8\mu}{5}\left(c^{5/2} - r^{5/2}\right)$

or $\quad v = \frac{dr}{dt} = -\sqrt{\frac{8\mu}{5}} \cdot \frac{\left(c^{5/2} - r^{5/2}\right)^{1/2}}{r^{3/2}}$

Here we take negative sign since r decreases with the increase in t.

$$t = -\sqrt{\frac{5}{8\mu}} \cdot \int_c^0 \frac{r^{3/2}\, dr}{\left(c^{5/2} - r^{5/2}\right)^{1/2}}$$

On taking $\quad r^{5/2} = c^{5/2} \sin^2\theta$

$$t = -\sqrt{\frac{5}{8\mu}} \cdot \int_0^{\pi/2} \frac{c^{3/2} \sin^3\theta . c . 2\sin\theta\cos\theta\, d\theta}{\left[c^{5/2}\left(1 - \sin^2\theta\right)\right]^{1/2}}$$

$$t = \sqrt{\frac{2}{5\mu}} \cdot c^{5/4}$$

which is the required result.

Example 6 : An infinite fluid in which is a spherical hollow shell of radius a is initially at rest under the action of no forces. If a constant pressure P is applied at infinity, show that the time of filling up the cavity is

$$\pi^2 a\left(\frac{\rho}{P}\right)^{1/2} . 2^{5/6}\left(\sqrt{1/3}\right)^{-3}$$

Solution : Let v' be the velocity at a distance r' and pressure p at any time t, also let v the velocity and r the radius of cavity. From the equation of continuity, we get

$$r'^2 v = F(t) = r^2 v \qquad ...(1)$$

which gives us that $\quad \frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2} \qquad ...(2)$

and the equation of motion is

$$\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

On using (2), $\dfrac{F'(t)}{r'^2} + v' \dfrac{\partial v'}{\partial r'} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r'}$

On integrating with respect to r', we get

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = C_1 - \frac{p}{\rho}$$

where C is the constant of integration and when

$$r' \to \infty \;;\; v' = 0 \;,\; p = P \;\to\; C_1 = P/\rho$$

Hence $-\dfrac{F'(t)}{r'} + \dfrac{v'^2}{2} = -\dfrac{P-p}{\rho}$

Now, on the surface of cavity $r' = r$, $v' = v$, $p = 0$

then $-\dfrac{F'(t)}{r} + \dfrac{v^2}{2} = \dfrac{P}{\rho}$ ...(3)

To calculate $F'(t)$, we differentiate (1), we have

$$F'(t) = 2rv^2 + r^2 v\frac{dv}{dr} \qquad ...(4)$$

From (3) and (4), we have

$$-2v^2 - vr\frac{dv}{dr} + \frac{1}{2}v^2 = P/\rho$$

or $\dfrac{dv^2}{dr} + \dfrac{3}{r}v^2 = -\dfrac{2P}{\rho}$

which is a linear differatial equation, solution of it is given by

$$v^2r^3 = -\frac{2P}{\rho}\int r^2 dr + C_2$$

$$v^2r^3 = -\frac{2Pr^3}{3\rho} + C_2$$

when $r = a$, $v = 0 \;\Rightarrow\; C_2 = \dfrac{2Pa^3}{3\rho} \;\Rightarrow\; v^2r^3 = \dfrac{2P}{3\rho}\left(a^3 - r^3\right)$

then $\quad v = \dfrac{dr}{dt} = -\sqrt{\dfrac{2P}{3\rho}}\,\dfrac{\left(a^3 - r^3\right)^{1/2}}{r^{3/2}}$

where negative sign before the radical is taken since r decreases as t increases. If t is the time of filling up the cavity

$$t = -\sqrt{\frac{3\rho}{2P}}\int_a^0 \frac{r^{3/2}\,dr}{\left(a^3 - r^3\right)^{1/2}} \qquad \text{on putting } r = a\sin^{2/3}\theta$$

$$= \sqrt{\frac{3\rho}{2P}}\int_0^{\pi/2} \frac{a^{3/2}\sin\theta\,\frac{2}{3}\,a\sin^{-1/3}\theta\cos\theta\,d\theta}{a^{3/2}\cos\theta}$$

$$t = \frac{2}{3}a\sqrt{\frac{3\rho}{2P}}\int_0^{\pi/2}\sin^{2/3}\theta\,d\theta$$

$$= \frac{2}{3}a\sqrt{\frac{3\rho}{2P}}\cdot\frac{\Gamma(5/6)\cdot\Gamma(1/2)}{2.\Gamma\left(5/6 + 1/2\right)}$$

$$t = \frac{2a}{3}\sqrt{\frac{3\rho}{2P}}\,\frac{\Gamma(5/6)\cdot\Gamma(1/2)}{2.\Gamma(4/3)} = \frac{2a}{3}.\sqrt{\frac{3\rho}{2P}}\cdot\frac{\Gamma(5/6)\cdot\Gamma(1/2)}{2.\frac{1}{3}\Gamma(1/3)} \qquad \text{...(5)}$$

we know that

$$\Gamma(n)\,.\,\Gamma\left(n + 1/2\right) = \frac{\sqrt{\pi}\,.\,\Gamma(2n)}{2^{2n-1}}$$

and $\quad \Gamma(n)\,.\,\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

putting $n = 1/3$, we have

$$\Gamma(1/3)\,.\,\Gamma(5/6) = \frac{\sqrt{\pi}\,\Gamma(2/3)}{2^{2/3-1}} \Rightarrow \Gamma(5/6) = \frac{\sqrt{\pi}\,.\,\Gamma\left(\frac{2}{3}\right).\,\Gamma(1/3)}{2^{-2/3}\left(\Gamma(1/3)\right)^2} \qquad \text{...(6)}$$

and $\quad \Gamma(1/3)\,.\,\Gamma(2/3) = \dfrac{\pi}{\sin \pi/3} = \dfrac{\pi}{\sqrt{3}/2} = \dfrac{2\pi}{\sqrt{3}} \qquad \text{...(7)}$

From (6) and (7), we have

$$\Gamma\left(5/6\right) = \frac{\sqrt{\pi}\;.\;2\pi/\sqrt{3}}{2^{-1/3}\;.\;\left(\Gamma\left(1/3\right)\right)^2} = \frac{2^{4/3}\;.\;\pi\;.\;\sqrt{\pi}}{\sqrt{3}\;.\;\left(\Gamma\left(1/3\right)\right)^2}$$

putting these values in (5), we obtain the time of filling up the cavity

$$t = \pi^2 a \left(2^{5/6}\right)\sqrt{\frac{\rho}{P}}\;.\;\left(\Gamma\left(1/3\right)\right)^{-3}$$

which is the required result.

Example 7 : An infinite mass of homogenous. incompressible fluid is at rest subject to a uniform pressure P and contains a spherical cavity of radius a filled with a gas at a pressure $m\,P$. Prove that if the inertia of the gas be neglected and Boyle's law be supposed to hold throughout the ensuing motion, the radius of the sphere will oscillate between the value a and na, where n is determined by the equation

$$1 + 3m\log n - n^3 = 0$$

If m be nearly equal to 1, the time of an oscillation will be $2\pi\sqrt{\frac{a^2\rho}{3P}}$, ρ being the density of the fluid.

Solution : Let v' be the velocity at a distance r' from the centre of cavity and p be the pressure there, then the equation of continuity is

$$r'^2 v' = F(t) \qquad ...(1)$$

and equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

on using (1), we have

$$\frac{F'(t)}{r'^2} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

On integrating with respect to r', we get

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = C_1 - \frac{p}{\rho}$$

when $r' \to \infty\,,\ v' = 0\,,\ p = P \Rightarrow C_1 = P/\rho$.

Hence $$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{P-p}{\rho} \qquad ...(2)$$

Now, let r be the radius of the cavity at time t, v the velocity and p' be the pressure there, then since air inside obeys Boyle's law,

$$\frac{4}{3}\pi r^3 p' = \frac{4}{3}\pi a^3 \,.\, mP$$

$$\Rightarrow \quad p' = \frac{a^3 mP}{r^3}$$

when $r' = r$, $p = p'$, $v' = v$ (motion of the surface of the cavity)

From (2), we have

$$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = \frac{P}{\rho} - \frac{a^3 m\, P}{r^3 . \rho} \qquad ...(3)$$

Now putting $v = \dfrac{F(t)}{r^2} \Rightarrow F'(t) = 2rv^2 + r^2 v\dfrac{dv}{dr}$

in equation (3), we have

$$\frac{dv^2}{dr} + \frac{3}{r}v^2 = -\frac{2P}{\rho r^4}\left(r^3 - ma^3\right)$$

which is linear differential equation whose solution is given as

$$r^3 v^2 = -\frac{2P}{\rho}\int \frac{1}{r}\left(r^3 - ma^3\right)dr + C_2$$

$$= -\frac{2P}{\rho}\left[\frac{r^3}{3} - ma^3 \log r\right] + C_2$$

when $r = a$; $v = 0$ $\Rightarrow C_2 = \dfrac{2P}{\rho}\left[\dfrac{a^3}{3} - ma^3 \log a\right]$

Hence $v^2 r^3 = \dfrac{2P}{\rho}\left[\dfrac{a^3 - r^3}{3} + ma^3 \log\dfrac{r}{a}\right]$ ...(4)

If $v = 0$; $r = na$ then eqn (4) gives

$$a^3 - n^3 a^3 + 3ma^3 \log n = 0$$

or $\quad 1 - n^3 + 3m\log n = 0$

which is the required relation for n.

2nd Part :

When m is nearly equal to 1. Let $r = a - x$ where x being small, so that $v = \dfrac{dr}{dt} = \dfrac{dx}{dt} = \dot{x}$ then equation (4) gives.

$$\dot{x}^2 (a+x)^3 = \frac{2P}{3\rho} a^3 \left[1 - \left(1+\frac{x}{a}\right)^3 + 3.1.\log\left(1+\frac{x}{a}\right)\right]$$

or $$\dot{x}^2 a^3 \left(1+\frac{x}{a}\right)^3 = \frac{2P}{3\rho} a^3 \left[1 - \left\{1+\frac{3x}{a}+\frac{3x^2}{a^2}+\ldots\right\} - 3\left\{\frac{x}{a}-\frac{1}{2}\frac{x^2}{a^2}+\ldots\right\}\right]$$

or $$\dot{x}^2 = \frac{2P}{3\rho}\left\{1+\frac{x}{a}\right\}^{-3}\left[-\frac{9}{2}\frac{x^2}{a^2}+\ldots..\right]$$

$$= \frac{2P}{3\rho}\left\{1-\frac{3x}{a}+\frac{6x^2}{a^2}+\ldots.\right\}\left[-\frac{9x^2}{2a^2}\right]$$

$$= \frac{2P}{3\rho}\left[-\frac{9x^2}{2a^2}\right]$$ (Neglecting higher power of x)

$$\dot{x}^2 = -\frac{3P}{\rho a^2} x^2$$

Now differentiating $w.r.to.t$, we have

$$2\dot{x}\ddot{x} = -\frac{3P}{\rho a^2} 2x\dot{x}$$

or $$\ddot{x} = -\frac{3P}{\rho a^2} x$$

which is the SHM and the time period T is given by

$$T = \frac{2\pi}{\sqrt{\frac{3P}{\rho a^2}}} \Rightarrow T = 2\pi\sqrt{\frac{\rho a^2}{3P}}.$$

Example 8 : A spherical hollow of radius a initially exists in an infinite fluid subject to constant pressure at infinity. Show that the pressure at distance r from the centre when the radius of the cavity is x is to the pressure at infinity as

$$3x^2r^4 + \left(a^3 - 4x^3\right)r^3 - \left(a^3 - x^3\right)x^3 : 3x^2r^4$$

Solution : Let at time t, v' be the velocity at a distance r' and p the pressure there, then equation of continuity is

$$r'^2 v' = F(t) \qquad ...(1)$$

and equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

on using (1)

$$\frac{F'(t)}{r'^2} + v' \frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

On integrating, with respect to r, we have

$$-\frac{F'(t)}{r'} + \frac{1}{2} v'^2 = C_1 - \frac{p}{\rho}$$

Let π be the pressure at infinity, so when $r' \to \infty$, $p = \pi$, $v' = 0 \Rightarrow C_1 = \frac{\pi}{\rho}$

Hence $-\frac{F'(t)}{r'} + \frac{1}{2} v'^2 = \frac{\pi - p}{\rho}$ ...(2)

Now let v be the velocity of the inner surface of radius x and pressure $p = 0$ then for the motion of inner surface, we have

$$-\frac{F'(t)}{x} + \frac{1}{2} v^2 = \frac{\pi}{\rho} \qquad ...(3)$$

or $$-\frac{F'(t)}{x} + \frac{1}{2}\frac{F^2(t)}{x^4} = \frac{\pi}{\rho} \qquad ...(4)$$

as $F(t) = x^2 v \Rightarrow v = \frac{F(t)}{x^2}$ or $\frac{dx}{dt} = \frac{F(t)}{x^2} \Rightarrow 2x^2 dx = 2F(t)\,dt$,

so multiplying (4) by $2x^2 dx = 2F(t)dt$

we get

$$-\frac{F'(t)}{x} 2F(t)\,dt + \frac{1}{2}\frac{F^2(t)}{x^4} . 2x^2\,dx = \frac{\pi}{\rho} . 2x^2\,dx$$

or $$\frac{-2F(t)\,F'(t)}{x} dt + \frac{1}{x^2} F^2(t) dx = 2\frac{\pi}{\rho} x^2\,dx$$

or $$d\left(-\frac{F^2(t)}{x}\right) = 2\frac{\pi}{\rho} x^2\,dx$$

On integrating, we have

$$-\frac{F^2(t)}{x} = \frac{2\pi}{3\rho} x^3 + C_2$$

Now, when $\quad x = a \; ; \; v = 0 \Rightarrow C_2 = -\dfrac{2\pi}{3\rho} a^3$

then $$-\frac{F^2(t)}{x} = \frac{2\pi}{3\rho}\left(x^3 - a^3\right)$$

On using $F(t) = x^2 v$,

$$v^2 x^3 = \frac{2\pi}{3\rho}\left(a^3 - x^3\right) \qquad \text{...(5)}$$

which is the velocity of the inner surface when it is a hollow of radius x. On using the value of v^2 in equation (2), we have

$$F'(t) = x\left[\frac{v^2}{2} - \frac{\pi}{\rho}\right] = x\left[\frac{\pi}{3\rho} \cdot \frac{a^3 - x^3}{x^3} - \frac{\pi}{\rho}\right]$$

$$F'(t) = \frac{\pi}{3\rho} \cdot \frac{\left(a^3 - 4x^3\right)}{x^2} \qquad \text{...(6)}$$

Now putting the value of $F'(t)$ from (6) in equation (2), we obtain the motion of any point in the fluid at the distance r' on using $r'^2 v' = x^2 v$

$$-\frac{1}{r'}\left[\frac{\pi}{3\rho} \cdot \frac{\left(a^3 - 4x^3\right)}{x^2}\right] + \frac{1}{2} v^2 \frac{x^4}{r'^4} = \frac{\pi - p}{\rho}$$

or $$\frac{\pi - p}{\rho} = -\frac{\pi}{3\rho} \cdot \frac{\left(a^3 - 4x^3\right)}{x^2 r} + \frac{\pi}{3\rho} \frac{x\left(a^3 - x^3\right)}{r^4} \qquad \text{at distance } r' = r$$

or $$\frac{p}{\pi} = 1 + \frac{\left(a^3 - 4x^3\right)}{3x^2 r} - \frac{x\left(a^3 - x^3\right)}{x^4}$$

or $$\frac{p}{\pi} = \frac{3x^2 r^4 + \left(a^3 - 4x^3\right) r^3 - \left(a^3 - x^3\right) x^3}{3x^2 r^4}$$

which is the required result.

Example 9 : A sphere is at rest in an infinite mass of homogeneous liquid of density ρ, the pres sure at infinity being P. If the radius R of the sphere varies in such a way that $R = a + b\cos nt$ where $b < a$. Show that pressure at the surface of the sphere at any time is

$$P + \frac{b n^2 \rho}{4}\left(b - 4a\cos nt - 5b\cos 2nt\right).$$

Solution : Let v' be the velocity at a distance r' at any time t and p be the pressure there. Again, let v be the velocity on the surface of sphere of radius R, where $R = a + b\cos nt$. The equation of continuity is

$$r'^2 v' = F(t) = R^2 v \qquad ...(1)$$

Therefore $$\frac{\partial v'}{\partial t} = \frac{F'(t)}{r^2} \qquad ...(2)$$

The equation of motion is

$$\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

On using (2)

$$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

On integrating

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{p}{\rho} + C$$

when $r \to \infty$, $v' = 0$, $p = P \Rightarrow C = P/\rho$

Thus $$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{P-p}{\rho} \qquad ...(3)$$

Now the pressure p at the surface of the sphere of radius R given by on putting

$r = R$, $v' = v$

$$-\frac{F'(t)}{R} + \frac{1}{2}v^2 = \frac{P-p}{\rho}$$

where $R = a + b\cos nt$.

or $$p = P + \rho\left[\frac{F'(t)}{R} - \frac{1}{2}v^2\right] \qquad ...(4)$$

From (1), we have

$$F'(t) = \frac{d}{dr}\left(vR^2\right) = 2Rv \, . \frac{dR}{dt} + R^2\frac{dv}{dt} = 2R\left(\frac{dR}{dt}\right)^2 + R^2\frac{d^2R}{dt^2} \text{ as } V = \frac{dR}{dt}$$

Using these values in (4), we have

$$p = P + \rho\left[2\left(\frac{dR}{dt}\right)^2 + R\,\frac{d^2R}{dt^2} - \frac{1}{2}\left(\frac{dR}{dt}\right)^2\right]$$

$$= P + \rho\left[\frac{3}{2}\left(\frac{dR}{dt}\right)^2 + R\,\frac{d^2R}{dt^2}\right]$$

$$= P + \rho\left[\frac{3}{2}(-b\,n\sin nt)^2 + (a + b\cos nt)\,\left(-b\,n^2\cos nt\right)\right]$$

$$= P + \frac{\rho b n^2}{2}\left[3\,b\sin^2 nt - 2\,b\cos^2 nt - 2\,a\cos nt\right]$$

$$= P + \frac{\rho b n^2}{4}\left[3\,b\,(1-\cos 2nt) - 2\,b\,(1+\cos 2nt) - 4\,a\cos nt\right]$$

$$p = P + \frac{\rho b n^2}{4}\left[b - 5\,b\,\cos 2nt - 4\,a\cos nt\right]$$

which is the required pressure.

Example 10 : Liquid is contained between two parallel planes, the surface is a circular cylinder of radius a whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius b is suddenly annihilated. Prove that if π be the pressure at the outher surface, the initial pressure at any point on the liquid distance r from the centre is

$$\pi\,\frac{\log r - \log b}{\log a - \log b}$$

Solution : Here the motion of the liquid will take place in such a manner so that each element of the liquid moves towards the axis of the cylinder $|z| = b$. Hence the free surface would be cylinderical. Thus the fluid velocity v' will be radial and it is a function of r' and time t only. Let p be the pressure at a distance r'. Then the equation of continuity is

$$r'\,v' = F(t) \qquad ...(1)$$

which gives $$\frac{\partial v'}{\partial t'} = \frac{F'(t)}{r'} \qquad ...(2)$$

The equation of motion is

$$\frac{\partial v'}{\partial t} + v'\,\frac{\partial v'}{\partial r'} = -\,\frac{1}{\rho}\,\frac{\partial p}{\partial r'}$$

Using (2), we have

$$\frac{F'(t)}{r'} + \frac{\partial}{\partial r'}\left(\frac{1}{2}\,v'^2\right) = -\,\frac{1}{\rho}\,\frac{\partial p}{\partial r'}$$

On integrating

$$F'(t)\,\log r' + \frac{1}{2}\,v'^2 = -\frac{p}{\rho} + C \qquad \text{...(3)}$$

Initially when $t = 0,\ v' = 0,\ p = P$

then $$F'(0)\,\log r' = -\frac{P}{\rho} + C \qquad \text{...(4)}$$

Again $P = \pi$ when $r' = a$ and $P = 0$ when $r' = b$

$$\therefore\ F'(0)\,\log a = -\frac{\pi}{\rho} + C$$

and $$F'(0)\,\log b = C$$

which gives that $C = -\log b \dfrac{\pi}{\rho \log\left(a/b\right)}$ and

$F'(0) = -\dfrac{\pi}{\rho \log\left(a/b\right)}$ then using these values in (4), we get

$$\frac{P}{\rho} = \frac{\pi}{\rho \log\left(a/b\right)} \log r' - \frac{\pi\ .\log b}{\rho \log\left(a/b\right)}$$

or $$P = \pi\,\frac{\log r' - \log b}{\log\left(a/b\right)}$$

$$= \pi\ .\ \frac{\log r' - \log b}{\log a - \log b}$$

On replacing $r' = r$, we obtain the required result

$$P = \pi\ .\ \frac{\log r - \log b}{\log a - \log b}$$

Example 11 : A stream in a horizontal pipe, after passing a contraction in the pipe at which its sectional area is A is delivered at atmospheric pressure at a place, where the sectional area is B. Show that if a side tube is connected with the pipe at the former place, water will be sucked up through it into the pipe from a reservior at a depth

$$\frac{s^2}{2g}\left(\frac{1}{A^2} - \frac{1}{B^2}\right)$$

below the pipe, s being the delivery per second.

Solution :

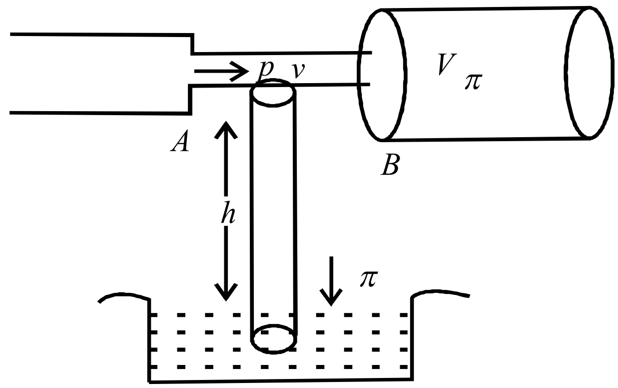


Figure 13.5

Let v be the velocity in the tube of smaller section A and p the pressure at that section. Further let V and π be the corresponding quantities at the bigger section B of the figure. Then by Bernoulli's theorem, we have $\frac{p}{\rho}+\frac{1}{2}q^2=$ constant then we obtain that

$$\frac{p}{\rho}+\frac{1}{2}v^2=\frac{\pi}{\rho}+\frac{1}{2}V^2$$

or $$\frac{1}{\rho}(p-\pi)=\frac{1}{2}\left(V^2-v^2\right) \qquad ...(1)$$

Let h be the height through which water is sucked up, then

$$\rho g h=\pi-p \qquad ...(2)$$

The equation of continuity is

$$Av=BV=s \qquad \text{(delivery per second)}$$

so that $v=s/A$ and $V=s/B$ ...(3)

On using (2) and (3) in equation (1), we have

$$\frac{1}{\rho}\rho g h=\frac{1}{2}\left[\frac{s^2}{A^2}-\frac{s^2}{B^2}\right]$$

or $$h=\frac{s^2}{2g}\left(\frac{1}{A^2}-\frac{1}{B^2}\right)$$

which is the required result.

13.10 Summary

In this unit, we studied the equations of motion in vector and cartesian forms. we integrated the equation of motion and obtained Bernoulli's theorem. We also derived the Helmholtz equation. Now we are capable solving the hydrodynamical problems for fluid motion by using equation of continuity, equations of motion with the help of given boundary conditions.

13.11 Answers to Self Learning Exercise

1. (d) rotational as well as translational.

2. $\frac{p}{\rho} - \frac{\partial \phi}{\partial t} + \frac{1}{2} q^2 + V =$ constant.

3. The work done by the force is independent of path of motion,

13.12 Exercise

1. An elastic fluid, the weight of which is neglected, obeying Boyle's law, is in motion in a uniform straight tube. Show that on the hypothesis of parallel section of the velocity at any time at a distance r from a fixed point in the tube is define by the equation

$$\frac{\partial v^2}{\partial t^2} + \frac{\partial}{\partial r}\left(2v \frac{\partial v}{\partial t} + v^2 \frac{\partial v}{\partial r}\right) = k \frac{\partial^2 v}{\partial r^2}.$$

2. A gas is moving in a uniform straight tube. Prove that if the density be $f(at - x)$ at a point where t is the time and x is the distance of the point from an end of the tube, its velocity is

$$\frac{a\, f(at - x) + (v - a) f(at)}{f(at - x)}$$

where v is the velocity at that end of the tube and a is a constant.

3. A mass of liquid of density ρ and volume $\frac{4}{3}\pi c^3$, is in the form of a spherical shell, a constant pressure π is exerted on the external surface of the shall, there is no pressure on the internal surface and no other forces act on the liquid. Initially the liquid is at rest and the internal radius of the shell is 2c. Prove that the velocity of the internal surface, when its radius is c is

$$\sqrt{\frac{14\,\pi}{3\rho} \cdot \frac{2^{1/3}}{2^{1/3} - 1}}$$

4. A volume $\frac{4}{3}\pi c^3$ of gravitating liquid of density ρ is initially in the form of a spherical shell of infinitely great radius. If the liquid shell contracts under the influence of its own attraction there being no external or internal pressure, show that when the radius of the inner spherical surface is x, its velocity will by given by

$$V^2 = \frac{4\pi\gamma\rho z}{15x^2}\left[2z^4 + 2z^3x + 2z^2x^2 - 3zx^3 - 3x^4\right]$$

where γ is the constant of gravitation and $z^3 = x^3 + c^3$.

5. A mass of uniform liquid is in the form of a thick spherical shall bounded by concentric spheres of radii a and b $(a < b)$. The cavity is filled with gas the pressure of which varies according to Boyle's law and is initially equal to the atmospheric pressure π and the mass of which may be

neglected. The other surface of the shell is exposed to atmospheric pressure. Prove that if the system is symmetrically distrubuted, so that each particle along a line joining it to the centre, the time of a small oscilation is

$$2\pi a\left\{\rho\,.\,\frac{(b-a)}{3\pi b}\right\}^{1/2}$$

where ρ is density of the fluid.

6. A mass of liquid of density ρ whose external surface is a long circular cylinder of radius a, which is subjcet to a constant pressure π, surrounds a co-axial long circular cylinder of radius b. The internal cylinder is suddenly destroyed. Show that if v is the velocity at the internal surface when the radius is r, then

$$v^2 = \frac{2\pi\left(b^2-r^2\right)}{\rho r^2\left[\log\dfrac{\left(r^2+a^2+b^2\right)}{r^2}\right]}$$

7. A sphere whose radius at time t is $b+a\cos nt$ is surrounded by liquid extending to infinity under no force. Prove that the pressure at distance r from the centre is less than the pressure at an ifinite distance by

$$\rho\,\frac{n^2 a}{r}\,(b+a\cos nt)\left[a\left(1-3\sin^2 nt\right)+b\cos nt+\frac{1}{2}\,\frac{a}{r^3}\sin^2 nt\,(b+a\cos nt)^3\right]$$

□□□□

UNIT - 14
Equations of Motion-II

Structure of the unit

14.0 Objective

In this unit, our aim is to study the Lagranges equations of motion and Cauchy's integral. The equation of motion under impulsive forces in vector and cartesian forms are also obtained and then solving the hydrodynamical problems based on it.

14.1 Introduction

In the previous unit, we studied the equations of motion and their applications to hydrodynamical problems under various condition. In present unit, we deal the equations of motion under impulsive forces and their applications. If sudden changes in velocity are produced at the boundaries of a perfect fluid or if impulsive forces are applied to it, then the disturbances produced are instantly transmitted to every part of the fluid. An impulsive force does not remain constant but changes first from zero to maximum and then maximum to zero Impulse of a force is a measure of total effect of the force.

14.2 Lagrange's Equation of Motion

Let (a,b,c) be initial coordinates of a fluid particle and after a any time t, (x,y,z) be the coordinate of the same particle. In this case a,b,c and t are independent variables. In order to get the required equations, we have to obtain x,y,z in terms of a,b,c and t. If we assume that the external forces are conservative and there exists a potential function V for them. We have the equations of motion as follows,

$$\frac{\partial^2 x}{\partial t^2} = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad ...(1)$$

$$\frac{\partial^2 y}{\partial t^2} = -\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad ...(2)$$

and $$\frac{\partial^2 z}{\partial t^2} = -\frac{\partial V}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad ....(3)$$

To get derivatives only with respect to a, b, c and t, we multiply (1) by $\frac{\partial x}{\partial a}$, (2) by $\frac{\partial y}{\partial a}$ and (3) by $\frac{\partial z}{\partial a}$ then adding, we get

$$\frac{\partial^2 x}{\partial t^2}\cdot\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\cdot\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\cdot\frac{\partial z}{\partial a} = -\frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial a} - \frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial a}$$

$$-\frac{\partial V}{\partial z}\cdot\frac{\partial z}{\partial a} - \frac{1}{\rho}\frac{\partial p}{\partial x}\cdot\frac{\partial x}{\partial a} - \frac{1}{\rho}\frac{\partial p}{\partial y}\cdot\frac{\partial y}{\partial a} - \frac{1}{\rho}\frac{\partial p}{\partial z}\cdot\frac{\partial z}{\partial a}$$

or $$\frac{\partial^2 x}{\partial t^2}\cdot\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\cdot\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\cdot\frac{\partial z}{\partial a} = -\frac{\partial V}{\partial a} - \frac{1}{\rho}\frac{\partial p}{\partial a} \qquad ...(4)$$

similarly, we obtain

$$\frac{\partial^2 x}{\partial t^2}\cdot\frac{\partial x}{\partial b} + \frac{\partial^2 y}{\partial t^2}\cdot\frac{\partial y}{\partial b} + \frac{\partial^2 z}{\partial t^2}\cdot\frac{\partial z}{\partial b} = -\frac{\partial V}{\partial b} - \frac{1}{\rho}\frac{\partial p}{\partial b} \qquad ...(5)$$

and $$\frac{\partial^2 x}{\partial t^2}\cdot\frac{\partial x}{\partial c} + \frac{\partial^2 y}{\partial t^2}\cdot\frac{\partial y}{\partial c} + \frac{\partial^2 z}{\partial t^2}\cdot\frac{\partial z}{\partial c} = -\frac{\partial V}{\partial c} - \frac{1}{\rho}\frac{\partial p}{\partial c} \qquad ...(6)$$

Equation (4), (5) and (6) are the Lagranges equations of motion. These equation (4) to (6), together with the equation of continuity $\rho\,\frac{\partial(x, y, z)}{\partial(a, b, c)} = \rho_0$ are known as Lagrange's hydrodynamical equations.

14.3 Cauchy's Integrals

Let a, b, c be the initial coordinates of a particle and x, y, z the coordinates of the same particle at time t. We know that a, b, c, t are the independent variable due to Lagrangian method. Now assuming the existence of a potential function V for the external forces, equations of motion in Lagrange's method are

$$\frac{\partial^2 x}{\partial t^2} = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad ...(1)$$

$$\frac{\partial^2 y}{\partial t^2} = -\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad ...(2)$$

$$\frac{\partial^2 z}{\partial t^2} = -\frac{\partial V}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad ....(3)$$

Taking ρ as a function of p, we take

$$Q = V + \int \frac{dp}{\rho} \qquad ...(4)$$

Then from (4), we have

$$\frac{\partial Q}{\partial a} = \frac{\partial V}{\partial a} + \frac{1}{\rho}\frac{\partial p}{\partial a} \qquad ...(5)$$

$$\frac{\partial Q}{\partial b} = \frac{\partial V}{\partial b} + \frac{1}{\rho}\frac{\partial p}{\partial b} \qquad ...(6)$$

and $$\frac{\partial Q}{\partial c} = \frac{\partial V}{\partial c} + \frac{1}{\rho}\frac{\partial p}{\partial c} \qquad ...(7)$$

On using (5) in equation (4) of Lagrange's form in preceding article, we have

$$\frac{\partial^2 x}{\partial t^2}\cdot\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\cdot\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\cdot\frac{\partial z}{\partial a} = -\frac{\partial Q}{\partial a} \qquad ...(8)$$

Similarly using (6) and (7) along with equations (5) and (6) of preceding article, we have

$$\frac{\partial^2 x}{\partial t^2}\cdot\frac{\partial x}{\partial b} + \frac{\partial^2 y}{\partial t^2}\cdot\frac{\partial y}{\partial b} + \frac{\partial^2 z}{\partial t^2}\cdot\frac{\partial z}{\partial b} = -\frac{\partial Q}{\partial b} \qquad ...(9)$$

and $$\frac{\partial^2 x}{\partial t^2}\cdot\frac{\partial x}{\partial c} + \frac{\partial^2 y}{\partial t^2}\cdot\frac{\partial y}{\partial c} + \frac{\partial^2 z}{\partial t^2}\cdot\frac{\partial z}{\partial c} = -\frac{\partial Q}{\partial c} \qquad ...(10)$$

Writing $\frac{\partial x}{\partial t} = u$, $\frac{\partial y}{\partial t} = v$ and $\frac{\partial z}{\partial t} = w$, equation (8), (9) and (10) may be written as

$$\frac{\partial u}{\partial t}\cdot\frac{\partial x}{\partial a} + \frac{\partial v}{\partial t}\cdot\frac{\partial y}{\partial a} + \frac{\partial w}{\partial t}\cdot\frac{\partial z}{\partial a} = -\frac{\partial Q}{\partial a} \qquad ...(11)$$

$$\frac{\partial u}{\partial t}\cdot\frac{\partial x}{\partial b} + \frac{\partial v}{\partial t}\cdot\frac{\partial y}{\partial b} + \frac{\partial w}{\partial t}\cdot\frac{\partial z}{\partial b} = -\frac{\partial Q}{\partial b} \qquad ...(12)$$

and $$\frac{\partial u}{\partial t}\cdot\frac{\partial x}{\partial c} + \frac{\partial v}{\partial t}\cdot\frac{\partial y}{\partial c} + \frac{\partial w}{\partial t}\cdot\frac{\partial z}{\partial c} = -\frac{\partial Q}{\partial c} \qquad ...(13)$$

Differentiating (12) and (13) partially with respect to c and b respectively for eliminating Q, we get

$$\frac{\partial}{\partial c}\left(\frac{\partial Q}{\partial b}\right) = \frac{\partial}{\partial b}\left(\frac{\partial Q}{\partial c}\right)$$

or $$\left(\frac{\partial^2 u}{\partial b \partial t}\frac{\partial x}{\partial c}-\frac{\partial^2 u}{\partial c \partial t}\frac{\partial x}{\partial b}\right)+\left(\frac{\partial^2 v}{\partial b \partial t}\frac{\partial y}{\partial c}-\frac{\partial^2 v}{\partial c \partial t}\frac{\partial y}{\partial b}\right)$$

$$+\left(\frac{\partial^2 w}{\partial b \partial t}\cdot\frac{\partial z}{\partial c}-\frac{\partial^2 w}{\partial c \partial t}\frac{\partial z}{\partial b}\right)=0$$

or $$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial b}\frac{\partial x}{\partial c}-\frac{\partial u}{\partial c}\frac{\partial x}{\partial b}\right)-\frac{\partial u}{\partial b}\frac{\partial^2 x}{\partial t \partial c}+\frac{\partial u}{\partial c}\frac{\partial^2 x}{\partial t \partial b}$$

$$+\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial b}\frac{\partial y}{\partial c}-\frac{\partial v}{\partial c}\frac{\partial z}{\partial b}\right)-\frac{\partial v}{\partial b}\cdot\frac{\partial^2 y}{\partial t \partial c}+\frac{\partial v}{\partial c}\frac{\partial^2 y}{\partial t \partial b}$$

$$+\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial b}\frac{\partial z}{\partial c}-\frac{\partial w}{\partial c}\frac{\partial z}{\partial b}\right)-\frac{\partial w}{\partial b}\cdot\frac{\partial^2 z}{\partial t \partial c}+\frac{\partial w}{\partial c}\frac{\partial^2 z}{\partial t \partial b}=0$$

But $\frac{\partial^2 x}{\partial t \partial b}=\frac{\partial u}{\partial b}$ and $\frac{\partial^2 x}{\partial t \partial c}=\frac{\partial u}{\partial c}$ etc, so that last two terms out of bracket are identical then we get

or $$\frac{\partial}{\partial t}\left[\left(\frac{\partial u}{\partial b}\cdot\frac{\partial x}{\partial c}-\frac{\partial u}{\partial c}\cdot\frac{\partial x}{\partial b}\right)+\left(\frac{\partial v}{\partial b}\cdot\frac{\partial y}{\partial c}-\frac{\partial v}{\partial c}\cdot\frac{\partial y}{\partial b}\right)+\left(\frac{\partial w}{\partial b}\cdot\frac{\partial z}{\partial c}-\frac{\partial w}{\partial c}\cdot\frac{\partial z}{\partial b}\right)\right]=0$$

On integrating $w.r.$ to t, we have

$$\left(\frac{\partial u}{\partial b}\cdot\frac{\partial x}{\partial c}-\frac{\partial u}{\partial c}\cdot\frac{\partial x}{\partial b}\right)+\left(\frac{\partial v}{\partial b}\cdot\frac{\partial y}{\partial c}-\frac{\partial v}{\partial c}\cdot\frac{\partial y}{\partial b}\right)+\left(\frac{\partial w}{\partial b}\cdot\frac{\partial z}{\partial c}-\frac{\partial w}{\partial c}\cdot\frac{\partial z}{\partial b}\right)=A$$

Initially

$x=a,\ y=b,\ z=c$, $u=u_0,\ v=v_0,\ w=w_0$ and initial spin components are

$\xi=\xi_0$, $\eta=\eta_0$, $\zeta=\zeta_0$; $\rho=\rho_0$ (initial density)

so $$\frac{\partial x}{\partial a}=\frac{\partial y}{\partial b}=\frac{\partial z}{\partial c}=1 \quad \text{and} \quad \frac{\partial x}{\partial b}=\frac{\partial x}{\partial c}=\frac{\partial y}{\partial a}=\frac{\partial y}{\partial c}=\frac{\partial z}{\partial a}=\frac{\partial z}{\partial b}=0$$

then $$A=\frac{\partial w_0}{\partial b}-\frac{\partial v_0}{\partial c}=2\,\xi_0$$

Hence $$\left(\frac{\partial u}{\partial b}\cdot\frac{\partial x}{\partial c}-\frac{\partial u}{\partial c}\cdot\frac{\partial x}{\partial b}\right)+\left(\frac{\partial v}{\partial b}\cdot\frac{\partial y}{\partial c}-\frac{\partial v}{\partial c}\cdot\frac{\partial y}{\partial b}\right)$$

$$+\left(\frac{\partial w}{\partial b}\cdot\frac{\partial z}{\partial c}-\frac{\partial w}{\partial c}\cdot\frac{\partial z}{\partial b}\right)=2\,\xi_0 \qquad ...(14)$$

Now using $\dfrac{\partial u}{\partial a} = \dfrac{\partial u}{\partial x} \cdot \dfrac{\partial x}{\partial a} + \dfrac{\partial y}{\partial y} \cdot \dfrac{\partial y}{\partial a} + \dfrac{\partial u}{\partial z} \cdot \dfrac{\partial z}{\partial a}$ etc ...(15)

in (14) we have

$$\frac{\partial x}{\partial c}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial b}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial b}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial b}\right)-\frac{\partial x}{\partial b}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial c}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial c}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial c}\right)$$

$$+\frac{\partial u}{\partial y}\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial b}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial b}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial b}\right)-\frac{\partial y}{\partial b}\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial c}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial c}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial c}\right)$$

$$+\frac{\partial z}{\partial c}\left(\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial b}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial b}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial b}\right)-\frac{\partial z}{\partial b}\left(\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial c}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial c}+\frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial c}\right)=2\xi_0$$

or
$$\left(\frac{\partial y}{\partial b}\frac{\partial z}{\partial c}-\frac{\partial z}{\partial b}\cdot\frac{\partial y}{\partial c}\right)\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)+\left(\frac{\partial z}{\partial b}\cdot\frac{\partial x}{\partial c}-\frac{\partial z}{\partial c}\cdot\frac{\partial x}{\partial b}\right)\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$

$$+\left(\frac{\partial x}{\partial b}\cdot\frac{\partial y}{\partial c}-\frac{\partial x}{\partial c}\cdot\frac{\partial y}{\partial b}\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=2\xi_0$$

or
$$2\xi\cdot\frac{\partial(y, z)}{\partial(b, c)}+2\eta\cdot\frac{\partial(z, x)}{\partial(b, c)}+2\zeta\,\frac{\partial(x, y)}{\partial(b, c)}=2\xi_0$$

or
$$\xi\,\frac{\partial(y, z)}{\partial(b, c)}+\eta\,\frac{\partial(z, x)}{\partial(b, c)}+\zeta\,\frac{\partial(x, y)}{\partial(b, c)}=\xi_0 \qquad ...(16)$$

where $2\xi = \dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}$, $2\eta = \dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x}$ and $2\zeta = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$ are the spin components.

Similarly

$$\xi\,\frac{\partial(y,z)}{\partial(c,a)}+\eta\,\frac{\partial(z,x)}{\partial(c,a)}+\zeta\,\frac{\partial(x,y)}{\partial(c,a)}=\eta_0 \qquad ...(17)$$

and
$$\xi\,\frac{\partial(y,z)}{\partial(a,b)}+\eta\,\frac{\partial(z,x)}{\partial(a,b)}+\zeta\,\frac{\partial(x,y)}{\partial(a,b)}=\zeta_0 \qquad ...(18)$$

On multiplying (16), (17), (18) by $\dfrac{\partial x}{\partial a}$, $\dfrac{\partial x}{\partial b}$, $\dfrac{\partial x}{\partial c}$ respectively and then adding, we obtain

$$\xi \frac{\partial(x,y,z)}{\partial(a,b,c)} = \xi_0 \frac{\partial x}{\partial a} + \eta_0 \frac{\partial x}{\partial b} + \zeta_0 \frac{\partial x}{\partial c} \qquad ...(19)$$

similarly

$$\eta \frac{\partial(x,y,z)}{\partial(a,b,c)} = \xi_0 \frac{\partial y}{\partial a} + \eta_0 \frac{\partial y}{\partial b} + \zeta_0 \frac{\partial y}{\partial c} \qquad ...(20)$$

and $$\zeta \frac{\partial(x,y,z)}{\partial(a,b,c)} = \xi_0 \frac{\partial z}{\partial a} + \eta_0 \frac{\partial z}{\partial b} + \zeta_0 \frac{\partial z}{\partial c} \qquad ...(21)$$

where $$\frac{\partial(x,y,z)}{\partial(a,b,c)} = \frac{\partial x}{\partial a} \cdot \frac{\partial(y,z)}{\partial(b,c)} + \frac{\partial x}{\partial b} \cdot \frac{\partial(y,z)}{\partial(c,a)} + \frac{\partial x}{\partial c} \cdot \frac{\partial(y,z)}{\partial(a,b)}$$

The equation of continuity in Lagrangian form is

$$\rho \frac{\partial(x,y,z)}{\partial(a,b,c)} = \rho_0$$

$$\Rightarrow \qquad \frac{\partial(x,y,z)}{\partial(a,b,c)} = \frac{\rho_0}{\rho} \qquad ...(22)$$

Hence

$$\frac{\xi}{\rho} = \frac{\xi_0}{\rho_0} \frac{\partial x}{\partial a} + \frac{\eta_0}{\rho_0} \frac{\partial x}{\partial b} + \frac{\zeta_0}{\rho_0} \frac{\partial x}{\partial c} \qquad ...(23)$$

$$\frac{\eta}{\rho} = \frac{\xi_0}{\rho_0} \frac{\partial y}{\partial a} + \frac{\eta_0}{\rho_0} \frac{\partial y}{\partial b} + \frac{\zeta_0}{\rho_0} \frac{\partial y}{\partial c} \qquad ...(24)$$

and $$\frac{\zeta}{\rho} = \frac{\xi_0}{\rho_0} \frac{\partial z}{\partial a} + \frac{\eta_0}{\rho_0} \frac{\partial z}{\partial b} + \frac{\zeta_0}{\rho_0} \frac{\partial z}{\partial c} \qquad ...(25)$$

These equations are called Cauchy's integrals.

Now we prove that the Cauchy's integrals are the integrals of the Helmholtz equation. To prove this we differentiate equation (23) with respect to t, we get

$$\frac{d}{dt}\left(\frac{\xi}{\rho}\right) = \frac{\xi_0}{\rho_0} \frac{\partial}{\partial a}\left(\frac{\partial x}{\partial t}\right) + \frac{\eta_0}{\rho_0} \frac{\partial}{\partial b}\left(\frac{\partial x}{\partial t}\right) + \frac{\zeta_0}{\rho_0} \frac{\partial}{\partial c}\left(\frac{\partial x}{\partial t}\right)$$

or $$\frac{d}{dt}\left(\frac{\xi}{\rho}\right) = \frac{\xi_0}{\rho_0} \frac{\partial u}{\partial a} + \frac{\eta_0}{\rho_0} \frac{\partial u}{\partial b} + \frac{\zeta_0}{\rho_0} \frac{\partial u}{\partial c} \qquad ...(26)$$

Again multiplying equation (23), (24) and (25) by $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ respectively then adding, we get

$$\frac{\xi}{\rho}\frac{\partial u}{\partial x}+\frac{\eta}{\rho}\frac{\partial u}{\partial y}+\frac{\zeta}{\rho}\frac{\partial u}{\partial z}=\frac{\xi_0}{\rho_0}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial a}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial a}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial a}\right)$$

$$+\frac{\eta_0}{\rho_0}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial b}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial b}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial b}\right)+\frac{\zeta_0}{\rho_0}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial c}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial c}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial c}\right)$$

or $$\frac{\xi}{\rho}\frac{\partial u}{\partial x}+\frac{\eta}{\rho}\frac{\partial u}{\partial y}+\frac{\zeta}{\rho}\frac{\partial u}{\partial z}=\frac{\xi_0}{\rho_0}\frac{\partial u}{\partial a}+\frac{\eta_0}{\rho_0}\frac{\partial u}{\partial b}+\frac{\zeta_0}{\rho_0}\frac{\partial u}{\partial c} \quad ...(27)$$

From (26) and (27), we get

$$\frac{d}{dt}\left(\frac{\xi}{\rho}\right)=\frac{\xi}{\rho}\frac{\partial u}{\partial x}+\frac{\eta}{\rho}\frac{\partial u}{\partial y}+\frac{\zeta}{\rho}\frac{\partial u}{\partial z}$$

Similarly

$$\frac{d}{dt}\left(\frac{\eta}{\rho}\right)=\frac{\xi}{\rho}\frac{\partial v}{\partial x}+\frac{\eta}{\rho}\frac{\partial v}{\partial y}+\frac{\zeta}{\rho}\frac{\partial v}{\partial z}$$

and $$\frac{d}{dt}\left(\frac{\zeta}{\rho}\right)=\frac{\xi}{\rho}\frac{\partial w}{\partial x}+\frac{\eta}{\rho}\frac{\partial w}{\partial y}+\frac{\zeta}{\rho}\frac{\partial w}{\partial z}$$

These are the Helmholtz equations. Hence Cauchy's integrals are the integrals of the Hemholtz equations.

14.4 Impulsive force

Let sudden velocity change be produces at the boundaries of an incompressible fluid or that impulsive forces be made to act to its interior. Then it is known that the impulsive pressure at any point is the same in every direction. Moreover the disturbances produced in both cases are propagated (trans mitted) instantaneously throughout the fluid. An impulsive force does not remain constant, but changes first from zero to maximum and then maximum to zero. Thus it is not possible to measure easily the value of impulsive force because it changes with time. In such cases, we measures the total effect of the force, called impulse.

14.5 Equation of motion under Impulsive forces in vector form

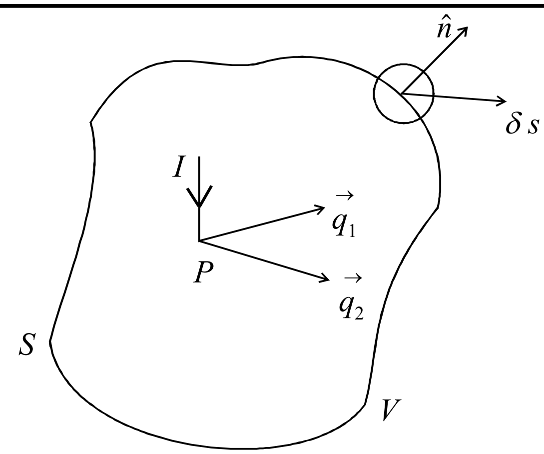


Figure 14. 1

Let S be an arbitrary small closed surface drawn in the incompressible fluid enclosing a volume V. Let I be the impulsive body force per unit mass. Let this impulse change the velocity v at $P(r,t)$ instantaneously from q_1 to q_2 and let it produce impulsive pressure on the boundary S.

Let $\overline{w}$ denote the impulsive pressure on the element δs of S. Let $\hat{n}$ be the unit outward drawn normal at δs. Let ρ be the density of the fluid.

On applying Newton's second law for impulsive motion to the fluid enclosed by S.

Total impulse applied = change of momentum

$$\therefore \int_V I\rho\, dV - \int_S \hat{n}\,\overline{w}\, ds = \int_V \rho\left(\vec{q_2} - \vec{q_1}\right) dV \qquad ...(1)$$

By Gauss Divergence theorem we know that

$$\int_S \hat{n}\ \overline{w}\, ds = \int_V \nabla\ \overline{w}\ dV \qquad ...(2)$$

From (1) and (2), we have

$$\int_S \left[I\ \rho - \nabla\ \overline{w} - \rho\left(\vec{q_2} - \vec{q_1}\right)\right] dV = 0$$

Since is an arbitrary small volume then

$$I\ \rho - \nabla\ \overline{w} - \rho\left(\vec{q_2} - \vec{q_1}\right) = 0$$

or $$\vec{q_2} - \vec{q_1} = I - \frac{1}{\rho}\nabla\ \overline{w} \qquad ...(3)$$

which is the equation of motion under impulsive forces in vector form.

Case I :- When external impulsive body forces are absent and impulsive pressures are present then eqn of motion reduces to

$$\vec{q_2} - \vec{q_1} = -\frac{1}{\rho}\nabla\ \overline{w} \qquad ...(4)$$

Taking divergence and using equation of continuity

$\nabla . \vec{q} = 0$ we have $\nabla^2\ \overline{w} = 0$ $\qquad \because \rho$ is constant

Case II :- If the external impulsive body force is absent and impulsive pressure be present then motion is started from rest then $\vec{q_1} = 0$; $I = 0$. The equation of motion reduces to

$$\vec{q} = -\nabla\left(\frac{\overline{w}}{\rho}\right) \qquad \because \vec{q_2} = \vec{q}$$

showing that there exists a velocity potential $\phi = \dfrac{\overline{w}}{\rho}$ and the motion is irrotational.

Case III :- Let there be no extraneous impulses. If ϕ_1 and ϕ_2 denote the velocity potential just before and just after the impulsive action. Then

$$\vec{q_1} = -\nabla \phi_1 \text{ and } \vec{q_2} = -\nabla \phi_2 \; ; \; I = 0$$

The equation of motion reduces to

$$-\nabla \phi_2 + \nabla \phi_1 = -\frac{1}{\rho} \nabla \overline{w}$$

or $$\nabla \overline{w} = \rho \nabla (\phi_2 - \phi_1)$$

Integrating taking density ρ as constant, we have

$$\overline{w} = \rho (\phi_2 - \phi_1) + C$$

where C be the constant and may be omitted by regarding it as an extra pressure and constant throughout the fluid then equation of motion is.

$$\overline{w} = \rho \phi_2 - \rho \phi_1$$

Case IV :- If $\phi_1 = 0$ and $\rho = 1$ then we find the actual motion, for which a single valued velocity potential exists, could be produced instantaneously from rest by applying appropriate impulses. Hence we can say that velocity potential is the impulsive pressure at any point. It is also seen that when a rotational motion exists in a fluid, the motion could neither be created nor destroyed by impulsive pressure.

14.6 Equations of motion under impulsive force in Cartesian form

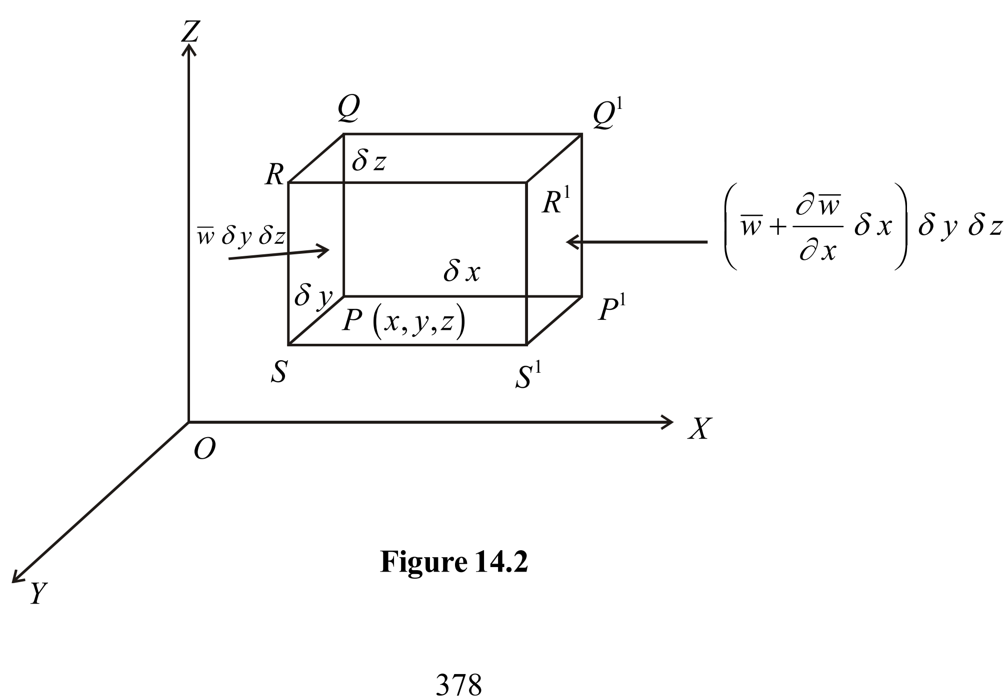


Figure 14.2

Let there be a fluid particle at $P(x,y,z)$ and ρ be the density of the incompressible fluid. Let u_1, v_1, w_1 and u_2, v_2, w_2 be the velocity components at the point P just before and just after the impulsive action. I_x, I_y and I_z be the components of the external impulsive forces per unit mass of the fluid. Construct a small parallelopiped whose edges $PP' = \delta x$, $PQ = \delta z$ and $PS = \delta y$ parallel to their respective coordinate axes having P as one of the angular corner as shown in figure. Let $\overline{w}$ be the impulsive pressure at P. Then we have

Inpulsive pressure on the face $PQRS = \overline{w}\,\delta y\,\delta z = f(x,y,z)$ ...(1)

Again Impulsive pressure on the face $P'Q'R'S'$

$$= f\left(x+\delta x, y, z\right)$$

$$= f\left(x,y,z\right)+\delta x \cdot \frac{\partial}{\partial x} f\left(x,y,z\right) + \text{.... higher power of } \delta x$$

The net impulsive pressure on the opposite faces $PQRS$ and $P'Q'R'S'$ along the x-axis.

$$= f\left(x,y,z\right)-\left[f\left(x,y,z\right)+\delta x \cdot \frac{\partial}{\partial x} f\left(x,y,z\right)+....\right]$$

$$= -\,\delta x \cdot \frac{\partial f\left(x,y,z\right)}{\partial x} \qquad \text{(neglecting higher powers of } \delta x\text{)}$$

$$= -\,\delta x \cdot \frac{\partial}{\partial x}\left(\overline{w}\,\delta y\,\delta z\right) \qquad \text{(by 1)}$$

$$= -\frac{\partial \overline{w}}{\partial x}\,\delta x\,\delta y\,\delta z \qquad ...(3)$$

Again, the impulse on the elementary parallelopiped along the x-axis due to external impulsive body force I_x

$$= \rho\,\delta x\,\delta y\,\delta z\,I_x \qquad ...(4)$$

The change in momentum along x-axis

$$= \rho\,\delta x\,\delta y\,\delta z\left(u_2-u_1\right) \qquad ...(5)$$

On using Newton's second law for impulsive motion to the fluid enclosed by the parallelopied in x-direction

Total impulse applied along x-axis = change of momentum along x-axis

$$\therefore -\,\delta x\,\delta y\,\delta z\left(u_2-u_1\right)\frac{\partial \overline{w}}{\partial x} + \rho I_x \delta x\,\delta y\,\delta z = \rho\,\delta x\,\delta y\,\delta z\left(u_2-u_1\right)$$

or $$\rho\left(u_2-u_1\right)=\rho I_x-\frac{\partial \bar{w}}{\partial x} \quad \text{...(6)}$$

Similarly

$$\rho\left(v_2-v_1\right)=\rho I_y-\frac{\partial \bar{w}}{\partial y} \quad \text{...(7)}$$

and $$\rho\left(w_2-w_1\right)=\rho I_z-\frac{\partial \bar{w}}{\partial z} \quad \text{...(8)}$$

These are the motion of an incompressible fluid under impulsive forces.

Now multiplying equation (6) by dx, (7) by dy and (8) by dz and then adding, we have

$$\rho\left(u_2-u_1\right) dx+\rho\left(v_2-v_1\right)+\rho\left(w_2-w_1\right) dz$$

$$=\rho\left[I_x\, dx+I_y\, dy+I_z\, dz\right]-\left[\frac{\partial \bar{w}}{\partial x} dx+\frac{\partial \bar{w}}{\partial y} dy+\frac{\partial \bar{w}}{\partial z} dz\right]$$

or $$d\bar{w}=\rho\left(I_x\, dx+I_y\, dy+I_z\, dz\right)-\rho\left[\left(u_2-u_1\right) dx+\left(v_2-v_1\right)+\left(w_2-w_1\right) dz\right] \quad \text{...(9)}$$

If ϕ_1 and ϕ_2 be the velocity potentials just before and just after, so that

$$d\phi_1=-\left(u_1\, dx+v_1\, dy+w_1\, dz\right)$$

and $$d\phi_2=-\left(u_2\, dx+v_2\, dy+w_2\, dz\right)$$

$$\because d\phi_1-d\phi_2=\left[\left(u_2-u_1\right)dx+\left(v_2-v_1\right)dy+\left(w_2-w_1\right)dz\right] \quad \text{...(10)}$$

and if external impulses are derived from a potential function V then

$$-dV=-\frac{\partial V}{\partial x} dx-\frac{\partial V}{\partial y} dy-\frac{\partial V}{\partial z} dz$$

$$-dV=I_x\, dx+I_y\, dy+I_z\, dz \quad \text{...(11)}$$

Now using (10) and (11) is equation (9), we have

$$d\bar{w}=-\rho\, dV+\rho\left(d\phi_2-d\phi_1\right)$$

If ρ is constant, then integrating, we have

$$\bar{w}=\rho\left(\phi_2-\phi_1\right)-\rho V+C \quad \text{...(12)}$$

C be the constant of integration and may be omitted as an extra pressure, which being constant throughout would not effect the motion and if $V=0$ then we get

$$\bar{w}=\rho\left(\phi_2-\phi_1\right) \quad \text{...(13)}$$

If initially the fluid is at rest so that $\phi = 0$ then $\overline{w} = \rho\, \phi_2$; which clearly implies that by applying a suitable impulse motion can be obtained whose velocity potential is same as the impulsive pressure.

Cor I :- Now we show that the impulsive pressure satisfies Laplace's equation. The equation of continuity, just before and just after the impulsive action are

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 = \frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial w_2}{\partial z}$$

these equation in terms of velocity potential ϕ_1 and ϕ_2 are

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} + \frac{\partial^2 \phi_1}{\partial z^2} = 0 = \frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} + \frac{\partial^2 \phi_2}{\partial z^2}$$

or $$\frac{\partial^2 (\phi_2 - \phi_1)}{\partial x^2} + \frac{\partial^2 (\phi_2 - \phi_1)}{\partial y^2} + \frac{\partial^2 (\phi_2 - \phi_1)}{\partial z^2} = 0$$

or $$\frac{\partial^2}{\partial x^2}\left(\frac{\overline{w}}{\rho}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\overline{w}}{\rho}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{\overline{w}}{\rho}\right) = 0$$

If ρ is constant then

$$\frac{\partial^2 \overline{w}}{\partial x^2} + \frac{\partial^2 \overline{w}}{\partial y^2} + \frac{\partial^2 \overline{w}}{\partial z^2} = 0$$

This shows that the impulsive pressure $\overline{w}$ satisfies the Laplace's equation.

Self Learning Exercise

1. The Cauchy's integrals are the integral of

   (a) Euler's dynamical equation of motion,

   (b) Lagrange's hydrodynamical equation,

   (c) Equation of continuity

   (d) Helmholtz equation.

2. In the absence of extraneous impulses the impulsive pressure at any point of a liquid satisfies

   (a) The equation of continuity

   (b) Laplace's equation

   (c) Newton's second law of motion

   (d) None of these.

3. How is impulse related to momentum?

Example 1 : A mass of liquid surrounds a solid sphere of radius a and its outer surface, which is a concentric spheres of radius b, is subject to a given constant pressure P, no other force being in action on the liquid. The solid sphere suddenly shrinks into a concentric sphere. It is required to determine the subsequent motion and the impulsive action on the sphere.

Solution : At any time t, let v' be the velocity at a distance r' from the centre, p the pressure and ρ the density there.

The equation of continuity is

$$r'^2 v' = F(t) \Rightarrow \frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2} \qquad ...(1)$$

and the equation of motion is

$$\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad ...(2)$$

Using (1) in (2), we get

$$\frac{F'(t)}{r'^2} + v' \frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

On integrating with respect to r', we get

$$-\frac{F'(t)}{r'} + \frac{1}{2} v'^2 = -\frac{p}{\rho} + C_1 \qquad ...(3)$$

Now let r and R be the internal and external radii of the fluid at any time t, v and V be the velocities there, so that when

$$r' = r \,,\ v' = v \,,\ p = 0 \qquad \Rightarrow -\frac{F'(t)}{r} + \frac{1}{2} V^2 = C_1 \qquad ...(4)$$

and $$r' = R,\ v' = V,\ p = P \qquad \Rightarrow -\frac{F'(t)}{R} + \frac{1}{2} V^2 = C_1 - \frac{P}{\rho} \qquad ...(5)$$

From (4) and (5), we have

$$-F'(t)\left(\frac{1}{r} - \frac{1}{R}\right) + \frac{1}{2}\left(v^2 - V^2\right) = \frac{P}{\rho} \qquad ....(6)$$

From eqn (1), we have

$$r^2 v = R^2 V = F(t)$$

i.e. $r^2\, dr = R^2\, dR = F(t)$ on taking $v = \dfrac{dr}{dt}$ and $V = \dfrac{dR}{dt}$.

Now, multiplying (6) by $2F(t)dt = 2R^2\, d\,R = 2r^2\, d\,r$ and putting $v = \dfrac{F(t)}{r^2}$ and $V = \dfrac{F(t)}{R^2}$, we have

$$-2F(t)\,F'(t)\left(\frac{1}{r}-\frac{1}{R}\right)dt + \frac{1}{2}F^2(t)\left[\frac{1}{r^2}dr - \frac{1}{R^2}dr\right] = \frac{2P}{\rho}r^2\,dr$$

On integrating, we obtain

$$-F^2(t)\left[\frac{1}{r}-\frac{1}{R}\right] = C_1 + \frac{2P}{3\rho}r^3 \qquad \text{...(7)}$$

Initially when $r = a$, $v = 0$, $F(t) = 0 \Rightarrow C_1 = -\dfrac{2P}{3\rho}a^3$

Thus (7) becomes

$$F^2(t)\left(\frac{1}{r}-\frac{1}{R}\right) = \frac{2P}{3\rho}\left(a^3 - r^3\right)$$

or
$$v^2 r^4\left(\frac{1}{r}-\frac{1}{R}\right) = \frac{2P}{3\rho}\left(a^3 - r^3\right) \qquad \text{...(8)}$$

where $R^3 - r^3 = b^3 - a^3$.

Equation (8) represent the required subsequent motion and now we find the impulsive pressure on the sphere. Let r be the radius of the solid sphere and $\overline{w}$ the impulsive pressure at a distance r', then

$$d\overline{w} = -\rho v'\,dr' = -\rho\frac{r^2 v}{r'^2}\,dr'$$

On integrating $\overline{w} = \dfrac{\rho r^2 v}{r'} + C_3$

But when $r' = R$; $\overline{w} = 0 \Rightarrow C_3 = -\dfrac{\rho r^2 v}{R}$

Hence the impulsive pressure when $r' = r$ is given by

$$\overline{w} = \rho r^2 v\left(\frac{1}{r}-\frac{1}{R}\right)$$

Also the whole impulse on the sphere

$$= 4\pi r^2\,\overline{w} = 4\pi r^4\,v\,\rho\left(\frac{1}{r}-\frac{1}{R}\right)$$

$$= 4\pi r^3\,v\,\rho\left(\frac{R-r}{R}\right)$$

Example 2 : A portion of homogenous fluid is confined between two concentric spheres of radii A and a, and is attracted towards their centre by a force varying inversely as the square of the distance. The inner spherical surface is suddenly annihilated and when the radii of the inner and outer surface of the fluid are r and R the fluid impinges on a solid ball concentric with these surfaces, prove that the impulsive pressure at any point of the ball for different values of R and r varies as

$$\left[\left(a^2 - r^2 - A^2 + R^2\right)\left(\frac{1}{r} - \frac{1}{R}\right)\right]^{1/2}$$

Solution : Let v' be the velocity at a distance r' from the centre of the sphere at any time t and p the pressure there. Then the equation of continuity is

$$r'^2 v' = F(t) \Rightarrow \frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2} \qquad ...(1)$$

Taking μ/r'^2 as the force towards the centre of the sphere, the equation of motion is

$$\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial r'} = -\frac{\mu}{r'^2} - \frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or
$$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2} v'^2\right) = -\frac{\mu}{r'^2} - \frac{1}{\rho}\frac{\partial p}{\partial r'} \qquad \text{(using (1))}$$

On integrating

$$-\frac{F'(t)}{r'} + \frac{1}{2} v'^2 = \frac{\mu}{r'} - \frac{p}{\rho} + C_1 \qquad ...(2)$$

Now r and R be the internal and external radii of the fluid at any time t, and v, V be the velocities there, thus we have

when $r' = R$, $v' = V$, $p = 0 \Rightarrow -\dfrac{F'(t)}{R} + \dfrac{1}{2}V^2 = C_1 + \dfrac{\mu}{R}$ ...(3)

and

when $r' = 0$; $v' = v$; $p = 0 \Rightarrow -\dfrac{F'(t)}{r} + \dfrac{1}{2}v^2 = C_1 + \dfrac{\mu}{r}$ ...(4)

On subtracting (4) from (5), we get

$$-F'(t)\left[\frac{1}{r} - \frac{1}{R}\right] + \frac{1}{2}\left(v^2 - V^2\right) = \mu\left(\frac{1}{r} - \frac{1}{R}\right) \qquad ...(5)$$

But from equation (1), we have

$$r^2 v = R^2 V = F(t)$$

therefore $r^2\,dr = R^2\,dR = F(t)\,dt$ ...(6)

On using (6) in (5), we get on multiplying by $2F(t)\,dt$

$$-2F(t)\,F'(t)\,dt\left(\frac{1}{r}-\frac{1}{R}\right)+\frac{1}{2}F^2(t)\left[\frac{2F(t)}{r^4}-\frac{2F(t)}{R^4}\right]dt=\mu\left[\frac{2F(t)}{r}-\frac{2F(t)}{R}\right]dt$$

or $$-2F(t)\,F'(t)\,dt\left(\frac{1}{r}-\frac{1}{R}\right)+\frac{1}{2}F^2(t)\left[\frac{2\,dr}{r^2}-\frac{2\,dR}{R^2}\right]=\mu\left[2r\,dr-2R\,dR\right]$$

On integrating

$$-F^2(t)\left(\frac{1}{r}-\frac{1}{R}\right)=\mu\left(r^2-R^2\right)+C_2$$

Since $r=a$; $R=A$; $v=0 \Rightarrow F(t)=0$ then $C_2=-\mu\left(a^2-A^2\right)$

Then $$-F^2(t)\left[\frac{1}{r}-\frac{1}{R}\right]=\mu\left(r^2-R^2-a^2+A^2\right) \qquad ...(7)$$

If $\overline{w}$ be the impulsive pressure at a distance r', then

$$d\overline{w}=-\rho v'\,dr'$$

$$=-\rho\frac{F(t)}{r'^2}dr$$

On integrating

$$\overline{w}=\rho\frac{F(t)}{r'}+C_3$$

But when $r'=R$, $\overline{w}=0$ then $C_3=-\dfrac{\rho F(t)}{R}$

Hence $$\overline{w}=\rho F(t)\left[\frac{1}{r'}-\frac{1}{R}\right]$$

Hence the impulsive pressure at any point of the ball where $r'=r$ is given by

$$\overline{w}=\rho F(t)\left[\frac{1}{r}-\frac{1}{R}\right] \qquad ...(8)$$

On substituting the value of $F(t)$ from (7), we have

$$\overline{w}=\rho\sqrt{\frac{\mu\left(r^2-R^2-a^2+A^2\right)}{\left(\frac{1}{R}-\frac{1}{r}\right)}}\cdot\left[\frac{1}{r}-\frac{1}{R}\right]$$

or $\quad \overline{w} = \rho \sqrt{\mu \left(a^2 - r^2 - A^2 + R^2\right) . \left(\frac{1}{r} - \frac{1}{R}\right)}$

Hence which shows that the required pressure varies as

$$\overline{w} \ \alpha \left[\left(a^2 - r^2 - A^2 + R^2\right)\left(\frac{1}{r} - \frac{1}{R}\right)\right]^{1/2}$$

Example 3 : A sphere of radius a is surrounded by infinite liquid of density ρ, the pressure at infinity being π. The sphere is suddenly annihilated. Show that the pressure at a distance r from the centre immediately falls to $\pi\left(1 - \frac{a}{r}\right)$. Show further that if the liquid is brought to rest by impinging on a concentric sphere of radius $a/2$, the impulsive pressure sustained by the surface of this sphere is

$$\left(7 \ \pi \ \rho^2 / 6\right)^{1/2}$$

Solution : Let v' be the velocity at a distance r' from the centre of the sphere at any time t and p the pressure there. Then the equation of continuity is

$$r'^2 \, v' = F(t) \ \Rightarrow \ \frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2} \qquad ...(1)$$

The equation of motion is

$$\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

On using (1), it reduces to

$$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2} v'^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

On integrating

$$-\frac{F'(t)}{r'} + \frac{1}{2} v'^2 = -\frac{p}{\rho} + C_1$$

When $\quad r' = \infty \ ; \ p = \pi \ , \ v' = 0 \ \Rightarrow C_1 = \frac{\pi}{\rho}$

Hence $\quad -\frac{F'(t)}{r'} + \frac{1}{2} v'^2 = \frac{\pi - p}{\rho} \qquad ...(2)$

When the sphere is suddenly annihilated, we have

$$t = 0 \, , \ r' = a \, , \ v' = 0 \ \text{and} \ p = 0$$

Then from equation (2), we have

$$-\frac{F'(0)}{a}=\frac{\pi}{\rho}$$

or $$F'(0)=-\frac{a\,\pi}{\rho}$$

So that immediately after annihilation $(t=0)$ the equation (2) becomes

$$\frac{a\,\pi}{\rho r'}+0=\frac{\pi-p}{\rho}$$

or $$\frac{a\,\pi}{r'}=\pi-p$$

Thus the pressure at the time of annihilation, when $r'=r$, is given by

$$p=\pi\left(1-\frac{a}{r}\right) \qquad ...(3)$$

which is the required first result.

Now if $\overline{w}$ be the impulsive pressure at distance r', then we have

$$d\,\overline{w}=-\rho v' \,d\,r'$$

$$d\,\overline{w}=-\rho v\,\frac{r^2}{r'^2}\,d\,r' \qquad as\ r^2\,v=r'^2\,v'$$

where r is the radius and v the velocity of the inner surface

On integrating $\overline{w}=\rho v\dfrac{r^2}{r'}+C_2$

when $r'=\infty$, $\overline{w}=0$ $\Rightarrow C_2=0$

thus $$\overline{w}=\rho v\frac{r^2}{r'} \qquad ...(4)$$

which gives the impulsive pressure $\overline{w}$ at a distance r'. Since $r=a/2$ then equation (4) reduces to

$$\overline{w}=\frac{1}{4}\,\rho v a^2\,.\frac{1}{r'} \qquad ...(5)$$

Now to find the velocity v at the inner surface, we have $r'=r$; $v'=v$ and $p=0$, then equation (2) reduces to

$$-\frac{F'(t)}{r}+\frac{1}{2}v^2=\frac{\pi}{\rho}$$

or $$-\frac{1}{r}\left[2rv^2 + r^2 v\frac{dv}{dr}\right] + \frac{1}{2}v^2 = \frac{\pi}{\rho} \qquad ...(6)$$

as $$F(t) = r^2 v \Rightarrow F'(t) = 2rv\frac{dr}{dt} + r^2\frac{dv}{dr}\cdot\frac{dr}{dt} = 2rv^2 + r^2 v\frac{dv}{dr}$$

On multiplying equation (6) by $2r^2 dr$ and then integrating we get

$$-v^2 r^3 = \frac{2\pi}{3\rho}r^3 + C_3$$

when $r = a$; $v = 0$; $\Rightarrow C_3 = -\frac{2\pi a^3}{3\rho}$

or $$v^2 r^3 = \frac{2\pi}{3\rho}\left(a^3 - r^3\right)$$

the velocity v on the surface of the sphere of radius $a/2$ is given by replacing r by $a/2$ then

$$v^2 = \frac{2\pi}{3\rho}\cdot\frac{a^3 - a^3/8}{a^3/8}$$

$$v^2 = \frac{14\pi}{3\rho} \qquad ...(7)$$

Using these value in (5), the impulsive pressure at a distance r' is given by

$$\overline{w} = \frac{\rho}{4}\left(\frac{14\pi}{3\rho}\right)^{1/2}\frac{a^2}{r'} \qquad ...(8)$$

On putting $r' = a/2$ in (8) we obtain the impulsive pressure at the surface of the sphere of radius $a/2$

i.e. $$\overline{w} = \frac{\rho}{4}\left(\frac{14\pi}{3\rho}\right)^{1/2}\cdot\frac{a^2}{a/2}$$

or $$\overline{w} = \left(\frac{7\pi\rho a^2}{6}\right)^{1/2}$$

which is the required result.

14.7 Summary

In this unit, we studied the Cauchy's integral and equations of motion under impulsive force. Now we are capable of solving hydrodynamical problems based on impulsive force and pressure with the use of given various condtions. We also learnt that the Cauchy's integrals are the integrals of the Helmholtz equations.

14.8 Answers to self learning exercise

1. (b) Helmhotz equation
2. (b) Laplace's equation
3. Impulse = change in momentum

14.9 Exercise

1. Prove that if $\overline{w}$ be the impulsive pressure, ϕ_1 and ϕ_2 be the velocity potentials immediately before and after an impulse acts, V the potential of the impulse,

$$\overline{w} + \rho v + \rho(\phi_2 - \phi_1) = \text{constant}$$

2. Prove that in the absence of external impuleses the impulsive pressure at any point of a liquid satisfies the Laplace's equation.
3. Prove that the Cauchy's integrals are the integrals of the Helmholtz equations.
4. If a bomb shell explodes at a great depth beneath the surface of the sea, prove that the impulsive pressure at any point varies inversly as the distance from the centre of the shell.

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UNIT - 15
Motion in Two Dimensions

Structure of the unit

15.0 Objective

In previous units, we studied the equation of continuity and equations of motion for the fluid motions. Now in this unit, we will discuss two dimensional motion and study the complex potential, Cauchy-Riemann equations, source, sink, Doublets and their images in two dimensional motion.

15.1 Introduction

When the fluid motion is such that it is same in all planes parallel to a fixed plane and also there is no velocity parallel to the fixed line and normal to fixed plane then it is called two dimensional motion. Generally the fixed plane is taken as xy -plane and the fixed line is taken as z -axis.

In two dimensional motion, the velocity components are only u and v with $w=0$ and also that u and v are the functions of x, y and t only. Moreover, when we speak of the flow across a curve in this plane, we means the flow across unit length of a cylinder on that curve with generators parallel to z -axis. When we speak of points in that planes, we mean straight line parallel to z -axis through that point.

15.2 Lagrange's stream function

Let u and v be the components of velocity in two dimensional motion, then the differential equation of the stream line is

$$\frac{dx}{u} = \frac{dy}{v} \quad \text{or} \quad v\,dx - u\,dy = 0 \qquad ...(1)$$

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad ...(2)$$

But (2) is the condition that (1) is an exact differential equations. Thus $v\,dx - u\,dy$ is a complete differential say $\partial\psi$, so that

$$v\,dx - u\,dy = \partial\psi = \frac{\partial\psi}{\partial x}\,dx + \frac{\partial\psi}{\partial y}\,dy$$

so that $u = -\frac{\partial\psi}{\partial y}$ and $v = \frac{\partial\psi}{\partial x}$

This function ψ is known as the stream function or the current function. In previous unit, we studied that the stream function is constant along a stream line. The stream function exists in all types of two-dimensional motion whether rotational or irrotational.

15.3 Irrotational motion in two-dimensions

We know that the velocity potential ϕ may exist only when the motion is irrotational. Thus if ϕ exists, we have

$$u = -\frac{\partial\phi}{\partial x} \text{ and } v = -\frac{\partial\phi}{\partial y} \qquad ...(1)$$

Also if ψ is stream function, then we have

$$u = -\frac{\partial\psi}{\partial y} \text{ and } v = \frac{\partial\psi}{\partial x} \qquad ...(2)$$

From (1) and (2), we have

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \qquad ...(3)$$

which are called Cauchy Riemann equation in Cartesian coordinates.

Hence $\phi + i\psi$ is an analytic function of $z = x + iy$. Moreover ϕ and ψ are known as conjugate functions. On multiplying and re - writing, (3) gives

$$\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} = 0 \qquad ...(4)$$

showing that the famalies of curves given by ϕ = constant and ψ = constant intersect orthogonally.

Differenting the equation given in (3) with respect to x and y respectively, we get

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \psi}{\partial x\, \partial y} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial y^2} = -\frac{\partial^2 \psi}{\partial y\, \partial x}$$

which gives $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad ...(5)$$

Again, deifferentiating the equation given in (3) with respect to y and x respectively, we get

$$\frac{\partial^2 \phi}{\partial y\, \partial x} = \frac{\partial^2 \psi}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial x\, \partial y} = -\frac{\partial^2 \psi}{\partial x^2}$$

which give $$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad ...(6)$$

Equations (5) and (6) show that ϕ and ψ satisfy Lapalce's equation when a two dimensional irrotational motion is considered. Functions ϕ and ψ are said to be conjugate functions if they satisfy (4), (5) and (6). In the light of the above discussions, the following points should be observed

(1) The stream function ψ exists whether the motion is irrotational or not.

(2) The velocity potential ϕ can exist only when the motion is irrotational.

(3) When the motion is not irrotational, the velocity potential does not exist.

15.4 Complex Potential

In an irrotational motion if ϕ represents the velocity potential and ψ the stream function, then expression $w = \phi + i\psi$ is defined as complex potential of the fluid motion since ϕ and ψ both are functions of x and y, the complex potential w can be expressed as the funtion of z

i.e. $w = f(z) = \phi + i\psi$ where $z = x + iy$

Thus if $w = \phi + i\psi = f(x + iy) = f(z)$

$$\Rightarrow \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = f'(x + iy)$$

and $\dfrac{\partial \phi}{\partial y} + i\,\dfrac{\partial \psi}{\partial y} = i\,f'(x+iy)$

which gives

$$\frac{\partial \phi}{\partial y} + i\,\frac{\partial \psi}{\partial y} = i\left(\frac{\partial \phi}{\partial x} + i\,\frac{\partial \psi}{\partial x}\right)$$

On equating real and imaginary parts, we have

$$\frac{\partial \psi}{\partial x} = -\,\frac{\partial \phi}{\partial y} \text{ and } \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x}$$

which are the same results we have obtained in the previous article. These are called Cauchy-Riemann equations to be satisfy by an analytic function $w = \phi + i\psi$. Such functions ϕ and ψ are called conjugate funtions.

15.5 Magnitude of velocity

Let $w = f(z)$ be the complex potential, then

$$w = \phi + i\psi \text{ and } z = x + iy \qquad ...(1)$$

Also we know that $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y}$ and $\dfrac{\partial \phi}{\partial y} = -\,\dfrac{\partial \psi}{\partial x}$ ...(2)

From (1), we have

$$\frac{dw}{dz}\cdot\frac{\partial z}{\partial x} = \frac{\partial \phi}{\partial x} + i\,\frac{\partial \psi}{\partial x} \text{ and } \frac{\partial z}{\partial x} = 1 \qquad \text{as } z = x + iy$$

$$\therefore\ \frac{dw}{dz} = \frac{\partial \phi}{\partial x} - i\,\frac{\partial \phi}{\partial y} \qquad \text{(On using (2))} \qquad ...(3)$$

or $\dfrac{dw}{dz} = -\,u + iv \qquad \because\ u = -\,\dfrac{\partial \phi}{\partial x}\ ;\ v = -\,\dfrac{\partial \phi}{\partial y}$ ...(4)

From (3) and (4), we see that the magnitude of velocity q at any point in a two dimensional irrotational motion is given by $\left|\dfrac{dw}{dz}\right|$, where

$$\left|\frac{dw}{dz}\right| = \left\{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(-\,\frac{\partial \phi}{\partial y}\right)^2\right\}^{1/2} = \left(u^2 + v^2\right)^{1/2} = q$$

Therefore the magnitude of the velocity $= \left|\dfrac{dw}{dz}\right| = q = \sqrt{u^2 + v^2}$. The points where velocity is zero are known as stagnation points.

15.6 Cauchy-Riemann equations in polar coordinates

Let $\phi + i\psi = f(z) = f\left(r e^{i\theta}\right)$ ...(1)

Differentating (1) with respect to r and θ, we get

$$\frac{\partial \phi}{\partial r} + i\,\frac{\partial \psi}{\partial r} = f'\left(r e^{i\theta}\right).\,e^{i\theta} \quad ...(2)$$

and $$\frac{\partial \phi}{\partial \theta} + i\,\frac{\partial \psi}{\partial \theta} = f'\left(r e^{i\theta}\right).\,r\,i\,e^{i\theta} \quad ...(3)$$

From (2) and (3), we easily obtain

$$\frac{\partial \phi}{\partial \theta} + i\,\frac{\partial \psi}{\partial \theta} = i\,r\left(\frac{\partial \phi}{\partial r} + i\,\frac{\partial \psi}{\partial r}\right)$$

On equating real and imaginary parts, we get

$$\frac{\partial \phi}{\partial \theta} = -\,r\,\frac{\partial \psi}{\partial r} \text{ and } \frac{\partial \psi}{\partial \theta} = r\,\frac{\partial \phi}{\partial r}$$

or $$\frac{\partial \phi}{\partial r} = \frac{1}{r}\,\frac{\partial \psi}{\partial \theta} \text{ and } \frac{1}{r}\,\frac{\partial \phi}{\partial \theta} = -\,\frac{\partial \psi}{\partial r} \quad ...(4)$$

which are the Cauchy-Riemann equations in polar coordinates.

15.7 Sources and Sinks

If the motion of a fluid consists of symmetrical radial flow in all directions proceeding from a point then the point is called a simple source. If the flow is such that the fluid is directed radially inwards to the point from all directions in a symmetrical manner, then the point is a simple sink. The sources involve continuous creation of fluid at the point and the sink involves continuous annihilation and the velocities in the region of such points approach infinite values.

15.7.1 Strength of a source :

The strength of a source is defined in terms of the amount of liquid flowing out from it. If $2m\pi$ is the volume of the fluid flowing out per unit time then m is called the strength of the source. Sink is a source of negative strength $(-m)$.

Let q_r be the radial velocity at a distance r from the source, then amount of liquid flowing out of circle of radius r is $2\pi m$

Therefore $$2\pi r\, q_r = 2\pi m$$

or $$q_r = \frac{m}{r} \quad ...(1)$$

Now, we try to get the expressions for the velocity potential ϕ and the stream function ψ due to a source. At any radial distance r from the source, the transverse velocity be zero and the entire velocity is radial which is given by

$$q_r = -\frac{1}{r}\frac{\partial\psi}{\partial\theta} = -\frac{\partial\phi}{\partial r} \qquad (\because \text{ Cauchy-Riemann equation}) \qquad ...(2)$$

From (1) and (2) we have

$$\frac{\partial\phi}{\partial r} = -\frac{m}{r} \text{ and } -\frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{m}{r}$$

On integrating, we obtain

$$\phi = -\log r \text{ and } \psi = -m\theta \qquad ...(3)$$

where the constant of integration will be merged in ϕ and ψ.

Thus curves of equi-velocity potential curves are $r =$ constant i.e. concentric circles centred on the source and stream line are $\theta =$ constant, i.e. straight lines radiating from the source.

15.8 Complex potential of sources

Let there be a source of strength m at origin. Then from the expressions for the velocity potential and the stream function for a souce of strength m at the origin, we have

$$w = \phi + i\psi = -m\log r - i\,m\theta$$

$$= -m\log r e^{i\theta}$$

or $$w = -m\log z \qquad ...(1)$$

which is the complex potential of the sources at the origin. If the source be at the point A whose coordinates are (α, β) so that $a = \alpha + i\beta$, then transferring the origin to A, we obtain

$$w = -\log z' = -m\log(x' + i\,y')$$

where (x', y') be the coordinates of any point referred to A. If P be the point (x, y) referred to origin O, we have

$$x' = x - \alpha \text{ and } y' = y - \beta$$

Hence $$w = -m\log[(x-\alpha) + i(y-\beta)]$$

$$w = -m\log[(x+iy) - (\alpha + i\beta)]$$

$$w = -m\log(z-a) \qquad ...(2)$$

The relation between w and z for sources of strengths m_1, m_2, m_3 ..... situated at the points z_1, z_2, z_3 ..... is

$$w = -m_1\log(z-z_1) - m_2\log(z-z_2) - m_3\log(z-z_3)..... \qquad ...(3)$$

or $$w = -m_1\log(r_1 . e^{i\theta_1}) - m_2\log(r_2 . e^{i\theta_2}) - m_3\log(r_3 . e^{i\theta_3}).....$$

where $r_1, r_2, r_3 .....$ are the distance of a point P of the plane from the points $z = a_1, a_2, a_3 ....$ and $\theta_1, \theta_2, \theta_3 ....$ are the angles these distances make with a fixed direction.

Therefore

$$w = -m_1(\log r_1 + i\,\theta_1) - m_2(\log r_2 + i\,\theta_2) - m_3(\log r_3 + i\,\theta_3).....$$

or $$\phi + i\psi = -m_1 \log r_1 - i\, m_1\,\theta_1 - m_2 \log r_2 - i\, m_2\,\theta_2 .....$$

On equating real and imaginary parts, we have

$$\phi = -m_1 \log r_1 - m_2 \log r_2 - m_3 \log r_3 .....$$

and $$\psi = -m_1\,\theta_1 - m_2\,\theta_2 - m_3\,\theta_3 - ....$$

15.9 Doublets

A combinations of a source of strength m and a sink of strength $-m$ at a small distance δs apart, where in the limit m is taken infinitely large and δs infinitely small such that the product $m\,\delta s$ remains finite and equal to μ, then it is called a doublet of strength μ. The line δs taken in the sense from sink to source is called the axis of the doublet.

15.10 Complex potential for a doublet

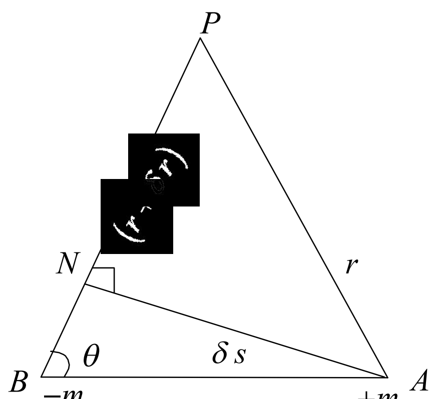

Figure 15.1

Let A and B be the positions of source and sinks of strengths $+m$ and $-m$ respectively at distance δs apart. Let P be any point a distance r from A at which we shall calculate the complex potential. AN is the perpendicular from A to PB. Then

$$BN = PB - PN$$

$$= r + \delta r - r = \delta r \text{ and } \cos\theta = \frac{\delta r}{\delta s}. \qquad ...(1)$$

Now the velocity potential at P due to the doublet

$$\phi = -m\log r - (-m)\log(r + \delta r)$$

$$= m\log\left(\frac{r+\delta r}{r}\right) = m\log\left(1 + \frac{\delta r}{r}\right)$$

$$= m\left[\frac{\delta r}{r} - \left(\frac{\delta r}{r}\right)^2 . \frac{1}{2} + \ldots\ldots\right]$$

On neglecting higher powers of δr, we have

$$\phi = m\,\frac{\delta r}{r}$$

or $$\phi = \frac{m\;.\;\delta s\cos\theta}{r} = \frac{\mu.\cos\theta}{r} \qquad \text{[from (1)]} \qquad \ldots(2)$$

where $\mu = m\,\delta\, s$ is the strength of the doublet when $m \to \infty$, $\delta s \to 0$.

we know $$\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad \ldots(3)$$

From (2), we have $$\frac{\partial\phi}{\partial r} = -\frac{\mu\cos\theta}{r^2} \qquad \ldots(4)$$

Using (4) in (3) we obtain

$$\frac{\partial\psi}{\partial\theta} = -\frac{\mu\cos\theta}{r}$$

On integrating, we get

$$\psi = -\frac{\mu\sin\theta}{r} + F(r) \qquad \ldots(5)$$

Again since

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r}$$

we have

$$-\frac{\mu\sin\theta}{r^2} = -\left[\frac{\mu\sin\theta}{r^2} + F'(r)\right]$$

or $$F'(r) = 0$$

or $$F(r) = \text{ constant} \qquad \ldots(6)$$

Therefore $$\psi = -\frac{\mu\sin\theta}{r} \qquad \ldots(7)$$

Hence the complex potential w at point P

$$w = \phi + i\psi = \frac{\mu\cos\theta}{r} - i\,\frac{\mu\sin\theta}{r} = \frac{\mu}{r}\,(\cos\theta - i\sin\theta)$$

$$w = \frac{\mu}{r} e^{-i\theta} = \frac{\mu}{r\, e^{i\theta}} = \frac{\mu}{z}$$

Note 1: Equi-potential curves are given by $\phi =$ constant

i.e. $$\frac{\mu \cos\theta}{r} = \text{constant}$$

or $$\frac{\cos\theta}{r} = C_1$$

or $$r\cos\theta = C_1 r^2$$

or $$x = C_1\left(x^2 + y^2\right)$$

which represent circles touching the y-axis at the origin.

Note 2 : Stream lines are given by $\psi =$ constant

i.e. $$\frac{-\mu \sin\theta}{r} = \text{constant} = C_2$$

or $$\sin\theta = C_2\, r$$

or $$r\sin\theta = C_2\ r^2$$

or $$y = C_2\left(x^2 + y^2\right)$$

which represent circle touching the x-axis at the origin.

Note 3 : If the doublet makes an angle α with x-axis, then we write $\theta - \alpha$ for θ, so that

$$w = \frac{\mu}{r\, e^{i(\theta-\alpha)}} = \frac{\mu\, e^{i\alpha}}{r\, e^{i\theta}} = \frac{\mu\, e^{i\alpha}}{z}$$

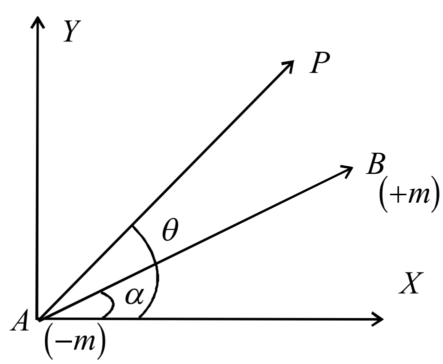

Figure 15.2

If the doublet be at point $A(x_1, y_1)$ where $z_1 = x_1 + i\, y_1$ then complex potential at the point is given by

$$w = \frac{\mu\, e^{i\alpha}}{z - z_1}$$

If there are a number of doublets of strength $\mu_1, \mu_2, ...$ at points $z_1, z_2, ...$ and whose axes are inclined at angles $\alpha_1, \alpha_2, ...$ with the x-axis then the complex potential of these will be given by

$$w = \frac{\mu_1 e^{i\alpha_1}}{z - z_1} + \frac{\mu_1 e^{i\alpha_2}}{z - z_2} + ... = \sum \frac{\mu_r e^{i\alpha_r}}{z - z_r}$$

15.11 Applications

(a) Let us take a source m at the point $(a, 0)$ and a sink $(-m)$ at the point $(-a, 0)$. The complex potential

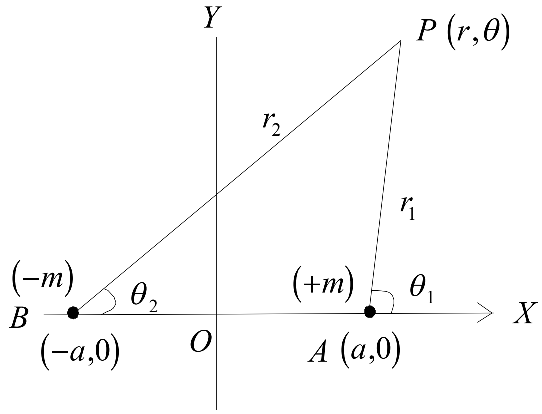

Figure 15.3

$$w = -m \log(z - a) + m \log(z + a)$$

$$w = -m \log \frac{(z - a)}{(z + a)}$$

and velocity potential

$$\phi = -m \log r_1 + m \log r_2$$

or $$\phi = -m \log \frac{r_1}{r_2}$$

where r_1 and r_2 are the distances of point P from $(a, 0)$ and $(-a, 0)$. Also, the stream function

$$\psi = -m\,\theta_1 + m\,\theta_2$$

or $$\psi = -m(\theta_1 - \theta_2)$$

where θ_1 and θ_2 are the angles as shown in figure.

Curves of equi-velocity potential (ϕ = constant) and stream line (ψ = constant) are such that the stream lines are co-axial circles passing through A and B. The flow is directed from source to sink. The equi-velocity potential lines are also circles with centre along x-axis such that the points A and B are the inverse points of these circles. The two families intersect orthogonally.

Now to calculate the magnitude of the velocity q, we have

$$q = \left|\frac{dw}{dz}\right| = \left|-\frac{m}{z-a} + \frac{m}{z+a}\right| = \left|\frac{-2am}{(z-a)\,(z+a)}\right|$$

thus $$q = \frac{2am}{|z-a|.|z+a|} = \frac{2am}{r_1\, r_2}$$

or $$q = \frac{2am}{\left|\left[r\,(\cos\theta + i\sin\theta) - a\right]\right|\,\left|\left[r\,(\cos\theta + i\sin\theta) + a\right]\right|}$$

or $$q = \frac{2am}{\sqrt{\left(r^2+a^2\right)^2 - 4a^2r^2\cos^2\theta}}$$

$$= \frac{2am}{\sqrt{r^4 - 2a^2r^2\cos 2\theta + a^4}}$$

Now to obtain an expression for ψ, we have

$$w = -\,m\,.\,\log\frac{z-a}{z+a}$$

or $$e^{-w/m} = \frac{z-a}{z+a}$$

or $$\exp\left[-\left(\frac{\phi + i\psi}{m}\right)\right] = \frac{r(\cos\theta + i\sin\theta) - a}{r(\cos\theta + i\sin\theta) + a}$$

or $$\bar{e}^{\,\phi/m}\left(\cos\frac{\psi}{m} - i\sin\frac{\psi}{m}\right) = \frac{r^2 - a^2 + i\,2ar\sin\theta}{(r\cos\theta + a)^2 + r^2\sin^2\theta}$$

Equating real and imaginary parts, we have

$$\bar{e}^{\,\phi/m}\,\cos\frac{\psi}{m} = \frac{\left(r^2 - a^2\right)}{r^2 + a^2 + 2ar\cos\theta}$$

and $$\bar{e}^{\,\phi/m}\,\sin\frac{\psi}{m} = \frac{-\,2ar\sin\theta}{r^2 + a^2 + 2ar\cos\theta}$$

From these, we get

$$\tan\frac{\psi}{m} = \frac{-\,2ar\sin\theta}{\left(r^2 - a^2\right)} \Rightarrow \psi = -\,m\tan^{-1}\left(\frac{2ar\sin\theta}{r^2 - a^2}\right)$$

$\Rightarrow \quad \psi =$ constant means stream lines are family of circles

$$x^2 + y^2 - 2ay - a^2 = 0$$

(b) Let us take source m at the point $(a, 0)$ and a sink $-m$ at $(0, a)$.

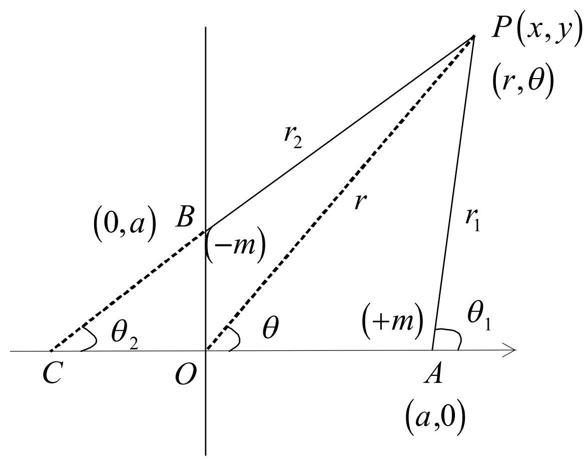

Figure 15.4

The complex potential w at P

$$w = -m \log(z-a) + m \log(z-ai)$$

$$= -m \log\left(\frac{z-a}{z-ai}\right) \qquad ...(1)$$

Therefore velocity potential

$$\phi = -m \log r_1 + m \log r_2 \qquad ...(2)$$

and the stream function

$$\psi = -m\,\theta_1 + m\,\theta_2 = -m(\theta_1 - \theta_2) \qquad ...(3)$$

where r_1, r_2 θ_1 and θ_2 are the distances AP, BP and their inclinations with the x-axis.

Then using (1)

$$w = \phi + i\psi = -m \log\left[(r\cos\theta - a) + i r\sin\theta\right] + m \log\left[r\cos\theta + i(r\sin\theta - a)\right]$$

or $\quad \phi + i\psi = -m.\log Re^{i\beta} + m \log R_1 e^{i\beta_1}$

where $\quad R\cos\beta = r\cos\theta - a \quad$ and $\quad R\sin\beta = r\sin\theta$

$$R_1\cos\beta_1 = r\cos\theta \quad \text{and} \quad R_1\sin\beta_1 = r\sin\theta - a \qquad ...(4)$$

or $\quad \phi + i\psi = -m \log\dfrac{R}{R_1} - im(\beta - \beta_1)$

Now equating real and imaginary parts using (4), we get

$$\phi = -m \log \left[\frac{\sqrt{(r\cos\theta - a)^2 + (r\sin\theta)^2}}{\sqrt{(r\cos\theta)^2 + (r\sin\theta - a)^2}} \right]$$

or $$\phi = -m \log \left[\frac{r^2 - 2ar\cos\theta + a^2}{r^2 - 2ar\sin\theta + a^2} \right]^{1/2}$$

and $$\psi = -m(\beta - \beta_1) = -m \left[\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta - a} \right) - \tan^{-1}\left(\frac{r\sin\theta - a}{r\cos\theta} \right) \right]$$

$$\psi = -m \tan^{-1} \left[\frac{ar(\sin\theta + \cos\theta) - a^2}{r^2 + ar(\sin\theta + \cos\theta)} \right]$$

Also $$\frac{dw}{dz} = -m \left[\frac{1}{z-a} + \frac{1}{z-ai} \right] = -m \frac{a(1-i)}{(z-a)(z-ai)}$$

$$q = \left| \frac{dw}{dz} \right| = \frac{am \,.\, |(1-i)|}{|z-a| \,.\, |z-ai|} = \frac{\sqrt{2} \,.\, am}{r_1 r_2}$$

$$= \frac{\sqrt{2} \,.\, a.m}{\left| \{(r\cos\theta - a) + ir\sin\theta\} \right| \,.\, \left| \{r\cos\theta + i(r\sin\theta - a)\} \right|}$$

$$q = \frac{\sqrt{2} \,.\, a.m}{\sqrt{(r^2 - 2ar\cos\theta + a^2)} \,.\, \sqrt{(r^2 - 2ar\sin\theta + a^2)}}$$

which is magnitude of the velocity.

15.12 Images in two dimensions

If there are a number of sources, sinks, doublets etc. in a liquid and if a surface S which is of purely geometrical character is drawn in the liquid so that

(1) a number of sources, sinks, doublets etc lie within the surface and others lie outside,

(2) there is no flux of liquid across this surface i.e. velocity normal to surface in zero.

then the system of sources, sinks, doublets etc. on one side of S is said to be the image of the system of sources, sinks, doublets etc on the other side with respect to S. The method of images is used to determine the complex potential due to source, sinks and doblets in the presence of rigid boundaries.

15.13 Image of a source with respect to a straight line

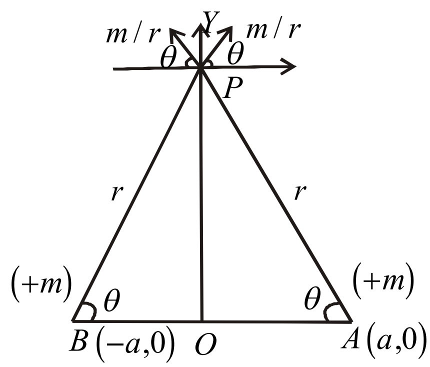

Figure 15.5

Suppose that image of the source m at $A\ (a,0)$ on x-axis is required with respect to OY. Take an equal source at B such that $OA = OB$. Let P be any point on OY such that $AP = BP = r$. Then the velocity at P due to source a A is $\frac{m}{r}$ along AP and velocity at P due to source at B is $\frac{m}{r}$ along BP.

The normal components of both of these radial velocitites at the point P will be $\frac{m}{r}\cos\theta$, but they will be is opposite directions. Therefore they will nullify each other. Hence at P, there will be no velocity across the straight line OY.

Therefore the image of a simple source with respect to a straight line in two dimension is an equal source at equi-distant from the straight line opposite to the source.

Alternate method :

Consider two equal sources of strength m at points $A\ (a,\ 0)$ and $B\ (-a,\ 0)$, then complex potential at

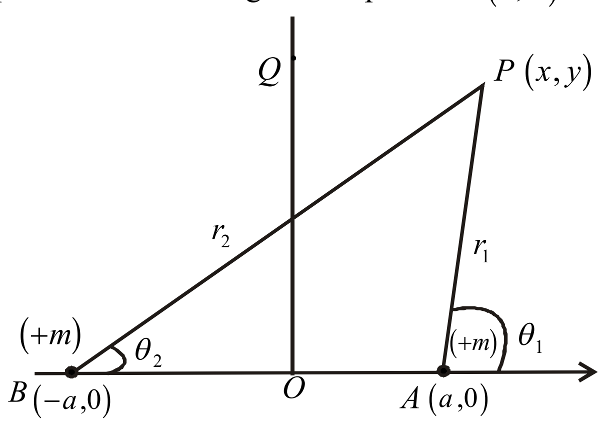

Figure 15.6

P is given by

$$w = -\ m\log(z-a) - m\log(z+a)$$

$$= -\ m\log\ (z-a)\ (z+a)$$

$$= -\ m\log\ \left(r_1\, e^{i\theta_1}\ .\ r_2\, e^{i\theta_2}\right)$$

$$w = -m \log r_1 r_2 - i\, m\left(\theta_1 + \theta_2\right) = \phi + i\psi$$

on equating real parts, we have

$$\phi = -m \log r_1 r_2$$

$$= -m \,.\, \log\left[\sqrt{(x-a)^2 + y^2}\,.\,\sqrt{(x+a)^2 + y^2}\right]$$

$$\phi = -\frac{m}{2} \,.\, \log\left[\left[(x-a)^2 + y^2\right].\left[(x+a)^2 + y^2\right]\right]$$

Hence the velocity components in x-direction will be

$$\frac{\partial \phi}{\partial x} = m\left[\frac{(x-a)}{(x-a)^2 + y^2} + \frac{(x+a)}{(x+a)^2 + y^2}\right]$$

If P takes a position on y-axis at Q so that velocity components at Q in x-direction will be obtained by putting $x = 0$ in above, and we will have

$$-\frac{\partial \phi}{\partial x} = 0$$

Hence there is no flow perpendicular to the line OY and thus the line OY is a stream line due to equal sources at A and B. The same is true for a sink.

15.14 Image of a doublet with respect to a straight line

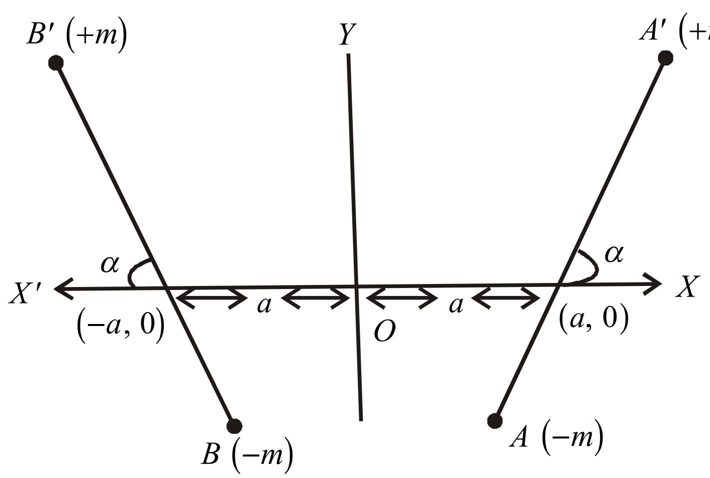

Figure 15.7

Take the y-axis along the straight line. Let the given doublet AA' be placed such that its axis is inclimed at angle α with the x-axis. Clearly the image of constitutent source A' and sink A of the given doublet will be equal source B' and sink B repectively. It is obvious that the location of B' will be such that it is at such a distance from the line which is equal to the distance of A'. Similarly A and B will be situated with respect to the given line. From geometry it is clear that AA' and BB' are antiparallel and both are inclined at an angle α with x-axis but in opposite sense. Hence the image of a doublet is a equal doublet but the axes are antiparallel.

15.15 Image of a source with respect to a circle :

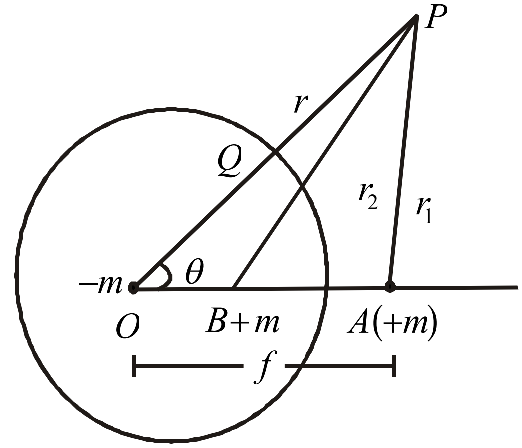

Figure 15.8

Let us determine the image of a source of strength m at a point A with respect to the circle with O as its centre. Let $OA = f$ and B be the inverse point of A with respect to the circle. If a be the radius of the circle, then $OA \,.\, OB = a^2$ so that $OB = a^2/f$.

Let there be a source of strength m at B. If w be the complex potential due to sources at A and B, then we get

$$w = -\, m \log\left(z - f\right) - m \log\left(z - a^2/f\right)$$

$$w = -\, m \left[\log\left\{(r\cos\theta - f) + i\,r\sin\theta\right\} + \log\left\{\left(r\cos\theta - \frac{a^2}{f}\right) + i\,r\sin\theta\right\}\right]$$

writing $\phi + i\,\psi = w$ and equating real part we get

$$\phi = -\frac{m}{2}\left[\log\left(r^2 + f^2 - 2\,f\,r\cos\theta\right) + \log\left(r^2 + \frac{a^4}{f^2} - \frac{2\,r\,a^2}{f}\cos\theta\right)\right]$$

Differentiating ϕ with respect to r, we get

$$\frac{\partial\phi}{\partial r} = -\frac{m}{2}\left[\frac{2\left(r - f\cos\theta\right)}{r^2 + f^2 - 2\,f\,r\cos\theta} + \frac{2\left(r - \dfrac{a^2}{f}\cos\theta\right)}{r^2 + a^4/f^2 - 2\,r\left(\dfrac{a^2}{f}\right)\cos\theta}\right]$$

Hence normal velocity at any point Q on the circle

$$= -\left(\frac{\partial\phi}{\partial r}\right)_{r=a}$$

$$= m\left[\frac{a - f\cos\theta}{a^2 + f^2 - 2fa\cos\theta} + \frac{\frac{a}{f}(f - a\cos\theta)}{\left(\frac{a^2}{f^2}\right)\left(a^2 + f^2 - 2fa\cos\theta\right)}\right]$$

$$= m\left[\frac{a - f\cos\theta + {f^2}/{a} - f\cos\theta}{a^2 + f^2 - 2fa\cos\theta}\right]$$

$$= \frac{m}{a}$$

Now if we place a sources of strength $(-m)$ (i.e. a sink) at O, the normal velocity due to it at Q will be $-{m}/{a}$ and hence the normal velocity of the system will be reduced to zero.

Hence the image system for a source outside a circle consists of an equal source at the inverse point and an equal sink at the centre of the circle.

15.16 Image of a doublet with respect to a circle

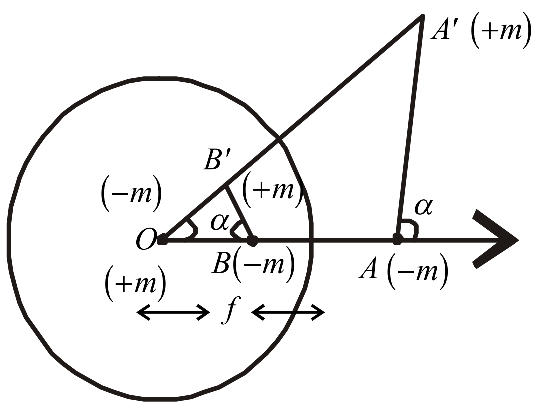

Figure 15.9

Let us determine the image of a doublet AA' outside the circle with its axis inclimed at angle α with OX such that, there is a source $+m$ at A and a sink $(-m)$ at A' and AA' small. Let B and B' be the inverse points of A and A' respectively with respect to circle whose centre is O.

so that $OA\,.\,OB = OA'\,.\,OB' = a^2$ ...(1)

Where a is the radius of the circle.

Now the image of source m at A' consists of a source m at B' and a sink $(-m)$ at O. Similarly, the image of sink $(-m)$ at A consists of a sink at B and a source m at O. Compounding these, we see that source m and sink $(-m)$ at O nullify each other then the image of the given doublet AA' is another doublet BB'.

Let the strength of the given doublet AA' be μ

then $\mu = \lim_{A \to A'} m \,.\, AA'$ ...(2)

From (1), we have $\frac{OA}{OA'} = \frac{OB'}{OB}$ ...(3)

which show that trinagle OAA' and $OB'B$ are similar.

From these similar triangles,

$$\frac{BB'}{AA'} = \frac{OB'}{OA} = \frac{OB' \,.\, OA'}{OA \,.\, OA'} = \frac{a^2}{OA \,.\, OA'} \qquad ...(4)$$

Now if strength of doublets $B'B$ is μ' then

$$\mu' = \lim_{B' \to B} \left(m \,.\, BB'\right)$$

$$= \lim_{A' \to A} \frac{a^2}{OA \,.\, OA'} \,.\, \left(m \; AA'\right) \qquad \text{On using (4)}$$

$$= \frac{\mu a^2}{f^2} \qquad OA = OA' = f$$

Also since $\angle OBB' = \alpha$, BB' is antiparallel to AA'. Thus the image of a doublet of strength μ at a distance $f(> a)$ from the centre of a circle of radius a is a doublet of strength $\frac{\mu a^2}{f^2}$ at the inverse point with its axis antiparallel to that of AA'.

15.17 Significance of Images

Whenever a two dimensional irrotational motion is confined to given rigid boundaries, we can consider this motion to have been caused by the presence of sources and sinks. By imagining a suitable set of sources and sinks, on either sides of the rigid boundary, called the image system, we can find such stream lines as will give the given boundaries as the stream lines, because velocity normal to the rigid boundary as well as to the stream lines, is zero. Thus a motion constrained by the boundaries is no longer so, with the advantage that we can predict the nature of the velocity and pressure at every point of the fluid.

Self Learning :

1. The curves of constant velocity potential cut the stream line ....
2. The points where velocity is .... are called as stagnation point.
3. When the motion is not irrotational, the velocity potential .... exists.
4. Write the complex potential when a source of strength m and a sink of strength $(-m)$ at a point $(a, 0)$ and $(0, a)$ respectively.

5. What arrangement of sources and sinks will give rise to the function $w = m \log \dfrac{(z^2 - a)}{z}$.

6. Write the complex potential due to a doublet which makes an angle α with x-axis.

7. Find the lines of flow in the two dimensional fluid motion given by

$$\phi + i\psi = -\frac{1}{2} n\,(x+iy)^2\ e^{2int}$$

Example 1 : Show that the velocity potential.

$$\phi = \frac{1}{2} \log \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}$$

gives a possible motion. Determine the stream lines and show also that the curves of equal speed (q = constant) are ovals of Cassinni given by $r r' =$ constant

Solution : Given that

$$\phi = \frac{1}{2} \log\left[(x+a)^2 + y^2\right] - \frac{1}{2} \log\left[(x-a)^2 + y^2\right]$$

$$\therefore\quad u = -\frac{\partial \phi}{\partial x} = -\frac{(x+a)}{(x+a)^2 + y^2} + \frac{(x-a)}{(x-a)^2 + y^2} \qquad ...(1)$$

and $$v = -\frac{\partial \phi}{\partial y} = -\frac{y}{(x+a)^2 + y^2} + \frac{y}{(x-a)^2 + y^2} \qquad ...(2)$$

From (1) $$\frac{\partial u}{\partial x} = -\frac{y^2 - (x+a)^2}{\left[(x+a)^2 + y^2\right]^2} + \frac{y^2 - (x-a)^2}{\left[(x-a)^2 + y^2\right]^2} \qquad ...(3)$$

From (2) $$\frac{\partial v}{\partial y} = -\frac{(x+a)^2 - y^2}{\left[(x+a)^2 + y^2\right]^2} + \frac{(x-a)^2 + y^2}{\left[(x-a)^2 + y^2\right]^2} \qquad ...(4)$$

Adding (3) and (4) we see that the equation of continuity

$\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$ is satisfied. Hence the given motion is possible.

To find the stream line, we have from Cauchy-Riemann eqns

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and } \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Now

$$\frac{\partial \psi}{\partial y} = \frac{(x+a)}{(x+a)^2 + y^2} - \frac{(x-a)}{(x-a)^2 + y^2}$$

On integrating $w.r.t.y$, we get

$$\psi = \tan^{-1} \frac{y}{(x+a)} - \tan^{-1} \frac{y}{(x-a)} + f(x) \qquad ...(5)$$

Therefore

$$\frac{\partial \psi}{\partial x} = -\frac{y}{(x+a)^2 + y^2} + \frac{y}{(x-a)^2 + y^2} + f'(x) \qquad ...(6)$$

As $\quad \dfrac{\partial \psi}{\partial x} = -\dfrac{\partial \phi}{\partial y}$

Thus by (2) and (6) we get $\quad f'(x) = 0$

or $\quad f(x) = \text{constant}$

Hence omitting the additive constant in ψ, we get (5) as

$$\psi(x, y) = \tan^{-1} \frac{y}{(x+a)} - \tan^{-1} \frac{y}{(x-a)}$$

$$= \tan^{-1} \frac{-2ay}{(x^2 + y^2 - a^2)}$$

Hence the stream line are given by $\psi = \text{constant}$

Therefore $\quad \tan^{-1}\left(\dfrac{-2ay}{x^2 + y^2 - a^2}\right) = \text{constant} = \text{c}$

or $\quad y = \dfrac{1}{c}(x^2 + y^2 - a^2)$

or $\quad x^2 + y^2 - cy = a^2$

which are circles. Now if $c = 0$, the stream line is the circle passing through $(a, 0)$ and $(-a, 0)$ and if c is infinite then stream line is $y = 0$.

Now $\quad w = \phi + i\psi$

$$w = \frac{1}{2} \log\left[(x+a)^2 + y^2\right] - \frac{1}{2} \log\left[(x-a)^2 + y^2\right] + i \tan^{-1}\left(\frac{y}{x+a}\right) - i \tan^{-1}\left(\frac{y}{x-a}\right)$$

$$= \log\left[(x+a) + iy\right] - \log\left[(x-a) + iy\right]$$

$$w = \log(z+a) - \log(z-a) \text{ where } z = x + iy$$

Hence $\quad q = \left|\dfrac{dw}{dz}\right| = \left|\dfrac{1}{z+a} - \dfrac{1}{z-a}\right| = \dfrac{2a}{|z+a| \,.\, |z-a|} = \dfrac{2a}{rr'}$

where r and r' are the distances of the point (x, y) from $(a, 0)$ and $(-a, 0)$. The curves of equal speed are given by

$$q = \text{constant} \qquad \text{or} \qquad \frac{2a}{rr'} = \text{constant}$$

$$\Rightarrow \; rr' = \text{constant}$$

These curves are called Cassinni's ovals.

Example 2 : What arrangement of sources and sinks will give rise to the function

$$w = \log\left(z - \frac{a^2}{z}\right)?.$$

Draw a rough sketch of a stream line. Prove that two of the stream lines sub divide into the circle $r = a$ and the axis of y.

Solution : Here $w = \log\left(z - \frac{a^2}{z}\right) = \log\left(\frac{z^2 - a^2}{z}\right)$

$$= \log\,(z-a) + \log\,(z+a) - \log z$$

which shows that there are two sinks of unit strength at the point $z = a$ and $z = -a$ and a source of unit strength at origin. Further

$$w = \phi + i\psi = \log(x - a + iy) + \log(x + a + iy) - \log(x + iy)$$

$$w = \phi + i\psi = \left[\frac{1}{2}\log\left[(x-a)^2 + y^2\right] + i\tan\frac{y}{(x-a)}\right]$$

$$+\left[\frac{1}{2}\log\left[(x+a)^2 + y^2\right] + i\tan^{-1}\frac{y}{(x+a)}\right] - \left[\frac{1}{2}\log\left(x^2 + y^2\right) + i\tan^{-1}\left(y/x\right)\right]$$

Equating imaginary parts on both sides, we have

$$\psi = \tan^{-1}\frac{y}{(x-a)} + \tan^{-1}\frac{y}{(x+a)} - \tan^{-1}\left(y/x\right)$$

$$= \tan^{-1}\frac{\frac{y}{x-a} + \frac{y}{x+a}}{1 - \frac{y}{x-a}\cdot\frac{y}{x+a}} - \tan^{-1} y/x$$

$$= \tan^{-1}\frac{2xy}{x^2 - y^2 - a^2} - \tan^{-1}\left(y/x\right)$$

$$= \tan^{-1}\frac{y\left(x^2 + y^2 + a^2\right)}{x\left(x^2 + y^2 - a^2\right)}$$

The stream line are given by ψ = constant

$$\frac{y\left(x^2+y^2+a^2\right)}{x\left(x^2+y^2-a^2\right)} = \text{constant}$$

If the constant tends to infinity, the stream lines are given by

$$x\left(x^2+y^2-a^2\right)=0$$

i.e. $x = 0$ and $x^2+y^2=a^2$ i.e. $r=a$

Hence the rough sketch of the stream lines is as shown in the following figure. In this figure there is a source of unit strength at origin O and there are two sinks each of unit strength at $A(a,0)$ and $B(-a,0)$.

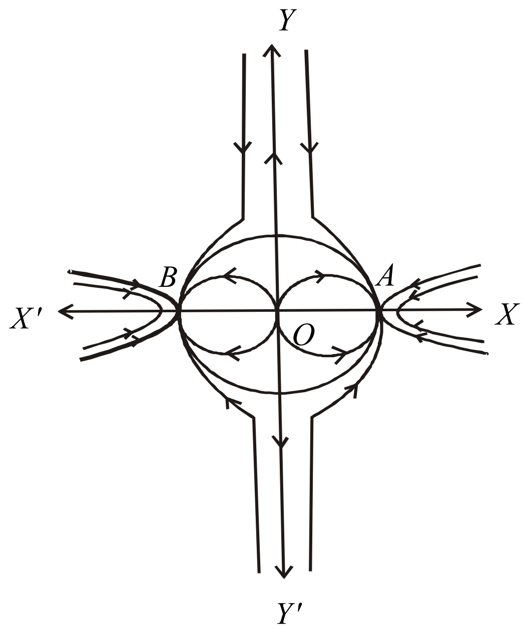

Figure 15.10

Example 3 : An area A is bounded by that part of the x-axis for which $x>a$ and by that branch of $x^2-y^2=a^2$ which is in the postive quadrant. There is a two dimensional unit source at $(a,0)$ which sends out liquid uniformaly in all directions. Show by means of the transformation $w=\log\left(z^2-a^2\right)$ that in steady motion the stream lines of the liquid within the area A are portions of rectangular hyperbola. Find the stream lines corresponding to $\psi=0$; $\pi/4$, $\pi/2$. If ρ_1 and ρ_2 are the distances of a point P within the fluid from the points $(\pm a,0)$ show that the velocity of the fluid at P is measured by $\frac{2\ OP}{\rho_1\cdot\rho_2}$, O being the origin.

Solution : Given $w=\log\left(z^2-a^2\right)$

or $w=\log\left[(x+iy)^2-a^2\right]$

or $\quad \phi + i\psi = \log\left[\left(x^2 - y^2 - a^2\right) + 2i\,xy\right]$

or $\quad = \frac{1}{2}\log\left[\left(x^2 - y^2 - a^2\right)^2 + 4x^2y^2\right] + i\tan^{-1}\frac{2xy}{x^2 - y^2 - a^2}$

Equating the imaginary parts, we have

$$\psi = \tan^{-1}\frac{2xy}{x^2 - y^2 - a^2} \qquad ...(1)$$

The stream lines are given by $\psi =$ constant $= \tan^{-1} C$

$$\frac{2xy}{x^2 - y^2 - a^2} = C \qquad ...(2)$$

When $C = 0$, stream line (2) reduce to $xy = 0$ i.e. $x = 0$ and $y = 0$. Again if $C \to \infty$, equation (2) reduces $x^2 - y^2 - a^2 = 0$ or $x^2 - y^2 = a^2$.

Hence the liquid flows in the area A bounded by $x = 0,\ y = 0$ and $x^2 - y^2 = a^2$ in the positive quadrant.

Now $\quad w = \log(z - a) + \log(z + a)$

which shows that there is a source of unit strength at $(a,0)$ and equal source at $(-a,0)$. Here the source at $(-a,0)$ is the image of the source at $(a,0)$ $w.r.$ to y-axis.

$$\frac{dw}{dz} = \frac{2z}{z^2 - a^2} = \frac{2z}{(z-a)(z+a)}$$

$$\because \quad q = \left|\frac{dw}{dz}\right| = \frac{2|z|}{|z-a|\,|z+a|} = \frac{2.OP}{\rho_1\ .\ \rho_2}$$

From (1), the stream line corresponding to $\psi = 0$ is

$$\frac{2xy}{x^2 + y^2 - a^2} = 0$$

or $\quad x = 0$ and $y = 0$

The stream lines corresponding to $\psi = \pi/4$ is

$$\frac{2xy}{x^2 + y^2 - a^2} = \tan\pi/4 = 1$$

or $\quad x^2 + y^2 - a^2 = 2xy$

or $\quad x^2 + y^2 - 2xy - a^2 = 0$

which are parallel lines $x - y = \pm a$

Stream line corresponding to $\psi = \pi/2$ is

$$\frac{2xy}{x^2 + y^2 - a^2} = \tan \pi/2 = \infty$$

or $\quad x^2 + y^2 - a^2 = 0$

these required stream lines are circles with centre at $(0, 0)$ and radius a.

Example 4 : Between the fixed boundaries $\theta = \frac{\pi}{6}$ and $\theta = -\frac{\pi}{6}$, there is a two-dimensional liquid motion due to a source of strength m at the point $(r = c, \theta = \alpha)$ and a sink at the origin, absorbing water at the same rate as the source produces it. Find the stream function and show that one of the stream lines is a part of the curve

$$\gamma^3 \sin 3\alpha = c^3 \sin 3\theta .$$

Solution : Let us transform the z-plane to ζ-plane by the transformation

$$\zeta = z^3$$

where $z = re^{i\theta}$ and $\zeta = Re^{i\phi}$ so that $Re^{i\phi} = r^3 e^{3i\theta}$

i.e. $\quad R = r^3$ and $\phi = 3\theta$

Therefore the boundaries $\theta = \pm \pi/6$ in z-plane transform to $\phi = \pm \pi/2$ in ζ-plane. The point (c, α) transforms to $(c^3, 3\alpha)$ and the point $(0, 0)$ remains there itself.

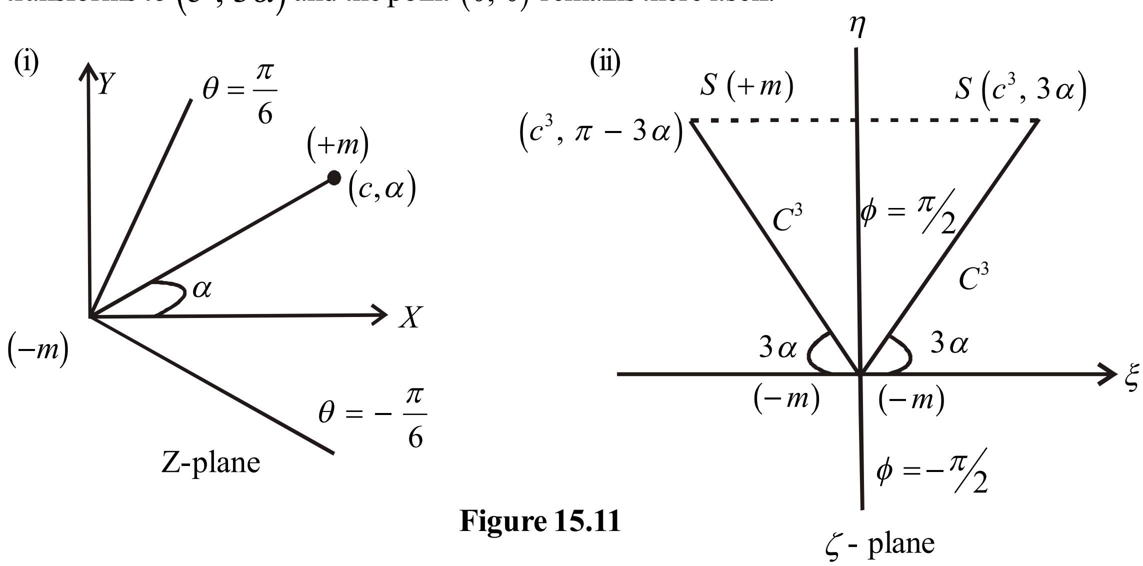

Figure 15.11

The image system with regards to imaginary axis $\phi = \pm \pi/2$ in ζ-plane consists of

(i) A source of strength $(+m)$ at $(c^3, 3\alpha)$ and its image an equal source $(+m)$ at $(c^3, \pi - 3\alpha)$

(ii) A sink of strength $(-m)$ at $(0, 0)$ and its image an equal sink $(-m)$ at $(0, 0)$.

And hence, the complex potential in (ζ-plane) will be given by

$$w = -m \log\left(\zeta - c^3 e^{3i\alpha}\right) + m\log\left(\zeta - 0\right) - m\log\left(\zeta - c^3 e^{i(\pi - 3\alpha)}\right) + m\log\left(\zeta - 0\right)$$

$$= -m \log\left(\zeta - c^3 e^{3i\alpha}\right) + m\log\zeta - m\log\left(\zeta - c^3 e^{-3i\alpha}\right) + m\log\zeta$$

On putting $\zeta = z^3$ we have

$$w = -m\log\left(z^3 - c^3 e^{3i\alpha}\right)\left(z^3 + c^3 e^{-3i\alpha}\right) + 2m\log z^3$$

$$= -m\log\left[z^6 - c^6 + z^3 c^3 \left(\cos 3\alpha - i\sin 3\alpha - \cos 3\alpha - i\sin 3\alpha\right)\right] + 6m\log z$$

$$= -m\log\left(r^6 e^{6i\theta} - c^6 - 2ic^3 r^3 e^{3i\theta}\sin 3\alpha\right) + 6m\log r e^{i\theta} \qquad \text{as } z = re^{i\theta}$$

$$w = -m\log\left[\left(r^6\cos 6\theta + 2c^3 r^3 \sin 3\alpha \sin 3\theta - c^6\right)\right.$$

$$\left. + i\left(r^6 \sin 6\theta - 2c^3 r^3 \sin 3\alpha \cos 3\theta\right)\right] + 6m\log\left(r\cos\theta + ir\sin\theta\right)$$

On equating imaginary parts to obatain stream line, we have

$$\psi = -m\tan^{-1}\left[\frac{r^6\sin 6\theta - 2c^3 r^3 \sin 3\alpha \cos 3\theta}{r^6\cos 6\theta + 2c^3 r^3 \sin 3\alpha \sin 3\theta - c^6}\right] + 6m\theta$$

on using formula $\quad \log(x + iy) = \frac{1}{2}\log\left(x^2 + y^2\right) + i\tan^{-1} y/x$

Now putting $\psi = 0$; the stream line is given by

$$\tan 6\theta = \frac{r^6\sin 6\theta - 2c^3 r^3 \sin 3\alpha \cos 3\theta}{r^6\cos 6\theta + 2c^3 r^3 \sin 3\alpha \sin 3\theta - c^6}$$

or $\sin 6\theta\left(r^6\cos 6\theta + 2c^3 r^3 \sin 3\alpha \sin 3\theta - c^6\right) = \cos 6\theta\left(r^6\sin 6\theta - 2c^3 r^3 \sin 3\alpha \cos 3\theta\right)$

or $\quad 2r^3 c^3 \sin 3\alpha\left[\sin 3\theta \sin 6\theta + \cos 3\theta \cos 6\theta\right] - c^6 \sin 6\theta = 0$

or $\quad 2r^3 c^3 \sin 3\alpha \cos 3\theta - c^6 \,.\, 2\sin 3\theta \cos 3\theta = 0$

or $\quad \cos 3\theta\left(r^3 \sin 3\alpha - c^3 \sin 3\theta\right) = 0$

when $\cos 3\theta = 0 \quad \Rightarrow \theta = \pm \pi/6$ which are the given boundaries.

$\therefore$ other stream line for $\psi = 0$ is part of the curve

$$r^3 \sin 3\alpha = c^3 \sin 3\theta.$$

Example 5 : In the case of the motion of liquid in a part of a plane bounded by a straight line due to a source in the plane, prove that if $m\rho$ is the mass of fluid of density ρ generated at the source per unit of time, the pressure on the length $2l$ of the boundary immediatly opposite to the source is less than that on an equal length at a great distance by

$$\frac{1}{2}\frac{m^2\rho}{\pi^2}\left\{\frac{1}{c}\tan^{-1}\frac{1}{c}-\frac{l}{l^2+c^2}\right\}$$

where c is the distance of the source from the boundary.

Solution :

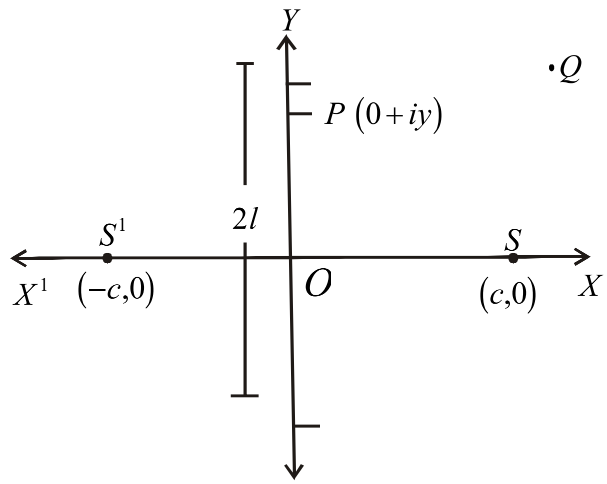

Figure 15.12

Let the bounding line OY be taken as y-axis and $\frac{m}{2\pi}$ be the source at $A\,(c,\,0)$. The equivalent image system consists of

a source $\frac{m}{2\pi}$ at $A\,(c,\,0)$ and its image S' a source $\frac{m}{2\pi}$ at $(-c,\,0)$. Hence the complex potential is given by

$$w = -\frac{m}{2\pi}\log(z-c) - \frac{m}{2\pi}\log(z+c)$$

$$w = -\frac{m}{2\pi}\log\left(z^2-c^2\right)$$

Hence the velocity $q = \left|\frac{dw}{dz}\right| = \frac{m}{2\pi}\left|\frac{2z}{z^2-c^2}\right|$

If $P\,(z = o+iy)$ is a point on y-axis, then the speed at this point is given by

$$q = \frac{m}{2\pi}\left|\frac{2iy}{-y^2-c^2}\right|$$

$$= \frac{m}{2\pi}\frac{y}{y^2+c^2}$$

Now for pressure, we use Bernoulli's theorem. Let $q = 0$ at infinite distance from y-axis and let p_0 be the pressure there, then

$$\frac{p}{\rho} = \frac{p_0}{\rho} - \frac{q^2}{2}$$

$$\therefore \quad \frac{p_0 - p}{\rho} = \frac{q^2}{2} = \frac{1}{2}\frac{m^2}{\pi^2}\frac{y^2}{\left(y^2+c^2\right)^2}$$

Therefore the required difference in pressure

$$= \int_{-l}^{l}(p_0 - p)\,dy = \frac{1}{2}\frac{m^2\rho}{\pi^2}\int_{-l}^{l}\frac{y^2}{\left(y^2+c^2\right)^2}\,dy$$

$$= \frac{m^2}{\pi^2}\rho\int_{0}^{l}\frac{y^2}{\left(y^2+c^2\right)^2}\,dy$$

On putting $y = c\tan\theta \Rightarrow dy = c\sec^2\theta\,d\theta$ and keeping the limits of integration for y as such, we get

$$= \frac{m^2\rho}{\pi^2}\int_{0}^{l}\frac{c^2\tan^2\theta}{c^4\sec^4\theta}\,c\,.\sec^2\theta\,d\theta$$

$$= \frac{m^2\rho}{\pi^2 c}\int_{0}^{l}\sin^2\theta\,d\theta = \frac{m^2\rho}{2\pi^2 c}\int_{0}^{l}(1-\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\frac{m^2\rho}{\pi^2}\left[\frac{1}{c}\theta - \frac{1}{c}\sin\theta\cos\theta\right]_0^l$$

$$= \frac{1}{2}\frac{m^2\rho}{\pi^2}\left[\frac{1}{c}\tan^{-1}\frac{y}{c} - \frac{y}{y^2+c^2}\right]_0^l = \frac{1}{2}\frac{m^2\rho}{\pi^2}\left[\frac{1}{c}\tan^{-1}\frac{l}{c} - \frac{l}{l^2+c^2}\right]$$

which is required result.

Example 6 : In the case of the two-dimensional fluid motion produce by a source of strength m placed at a point S outside a rigid circular disc of radius a whose centre is O. Show that the velocity of slip of the fluid in contact with the disc is greatest at the points where the line joining S to the ends diameter at right angles to OS cut the circle and prove that its magnitude at these points is

$$\frac{2m\,.\,OS}{\left(OS^2 - a^2\right)}$$

Solution :

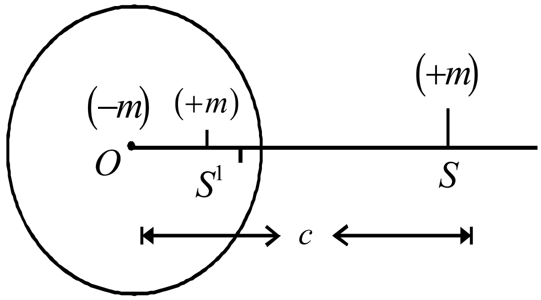

Figure 15.13

Let S' be the inverse point of S with regard to the circular disc, so that if $OS = c$ and $OS' = a^2/c$ where S' is in the inverse point of S and $OS \, . \, OS' = a^2$ where a be the radius of the circle. The equivalent image system consists of

(i) a source of m at S

(ii) a source of strength m at S', and

(iii) a sink of strength $(-m)$ at O.

The complex potential w at any point is

$$w = -\, m \log (z - c) - m \log \left(z - a^2/c\right) + m \log z$$

or $$\frac{dw}{dz} = \frac{-m}{(z-c)} + \frac{-m}{\left(z - a^2/c\right)} + \frac{m}{z}$$

or $$q = \left|\frac{dw}{dz}\right| = \left|\frac{-m}{(z-c)} + \frac{-m}{\left(z - a^2/c\right)} + \frac{m}{z}\right|$$

or $$q = m \left|\frac{(z-a)(z+a)}{(z-c)\, z \left(z - a^2/c\right)}\right| \qquad ...(1)$$

Now to find the velocity at any point on the boundary of the disc, we put $z = a e^{i\theta}$

$\therefore$ $$q = m \left|\frac{\left(a e^{i\theta} - a\right)\left(a e^{i\theta} + a\right)}{\left(a e^{i\theta} - c\right)\left(a e^{i\theta} - \dfrac{a^2}{c}\right) a e^{i\theta}}\right|$$

or $$q = m c \left|\frac{\left(1 - \bar{e}^{i\theta}\right)\left(1 + \bar{e}^{i\theta}\right)}{\left(a - c\bar{e}^{i\theta}\right)\left(c e^{i\theta} - a\right)}\right|$$

or $$q = \frac{2mc\sin\theta}{a^2 + c^2 - 2ac\cos\theta} \qquad ...(2)$$

For maximum/minimum value of q

$$\frac{dq}{d\theta} = 2mc\left[\frac{\cos\theta\left(a^2 + c^2 - 2ac\cos\theta\right) - \sin\theta\left(2ac\sin\theta\right)}{\left(a^2 + c^2 - 2ac\cos\theta\right)^2}\right] = 0$$

or $$\left(a^2 + c^2\right)\cos\theta - 2ac = 0$$

or $$\cos\theta = \frac{2ac}{a^2 + c^2} \Rightarrow \sin\theta = \frac{a^2 - c^2}{a^2 + c^2} \qquad ...(3)$$

As $\theta = 0$ gives minimum zero velocity, value of θ given by (3) corresponds to the maximum value of q. On using (2) in (1), we obtain

$$q_{max} = \frac{2mc \, . \left[\frac{a^2 - c^2}{a^2 + c^2}\right]}{a^2 + c^2 - 2ac\left(\frac{2ac}{a^2 + c^2}\right)}$$

$$= \frac{2mc}{c^2 - a^2}$$

$$q_{max} = \frac{2m \, . \, OS}{OS^2 - a^2}$$

The boundary being stream line, the velocity on the boundary is the velocity of the slip.

Example 7 : A source S and a sink T of equal strength m are situated within the space bounded by a circle whose centre is O. If S and T are at equal distances from O an opposite side of it and on the same diameter AOA'. Show that the velocity of the liquid at any point P is

$$2m \, . \, \frac{OS^2 + OA^2}{OS} \, . \, \frac{PA \, . \, PA'}{PS \, . \, PS' \, . \, PT \, . \, PT'}$$

where S' and T' are the inverse points of S and T with respect to the circle.

Solution :

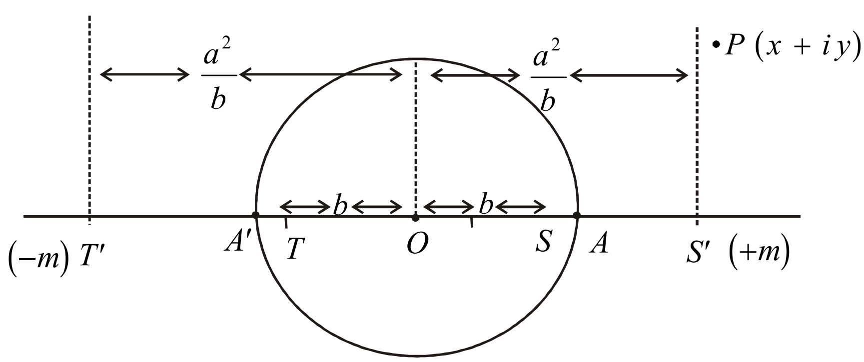

Figure 15.14

Let AOA' be the diameter of the circle of radius $OA = OA' = a$ with centre O. Let S' and T' be the inverse points of S and T, the distances are further been shown in the figure. At S and T be the source and sinks of equal strength. Image system will consists

(i) Source $(+m)$ at S (ii) Source $(+m)$ at S'

(iii) Sink $(-m)$ at T (iv) Sink $(-m)$ at T'

then the complex potential will be

$$w = -m\log(z-b) - m\log\left(z-a^2/b\right) + m\log(z+b) + m\log\left(z+a^2/b\right)$$

where $OS = b = OT$, $OS' = OT' = a^2/b$ and $OA = OA' = a$.

On differentiating, we obtain

$$\frac{dw}{dz} = \frac{-m}{(z-b)} - \frac{m}{\left(z-a^2/b\right)} + \frac{m}{(z+b)} + \frac{m}{\left(z+a^2/b\right)}$$

The velocity at any point P,

$$q = \left|\frac{dw}{dz}\right| = m\left|\frac{1}{(z-b)} + \frac{1}{\left(z-a^2/b\right)} - \frac{1}{(z+b)} - \frac{1}{\left(z+a^2/b\right)}\right|$$

$$= 2m \cdot \frac{\left(a^2+b^2\right)}{b}\left|\frac{(z-a)(z+a)}{(z-b)(z+b)\left(z-a^2/b\right)\left(z+a^2/b\right)}\right|$$

$$= 2m \cdot \frac{\left(b^2+a^2\right)}{b} \frac{|z-a|\,|z+a|}{|z-b| \cdot |z+b| \cdot \left|z-a^2/b\right| \cdot \left|z+a^2/b\right|}$$

$$q = 2m \cdot \frac{OS^2+OA^2}{OS} \frac{PA \cdot PA'}{PS \cdot PS' \cdot PT \cdot PT'}$$

which is the required result.

Example 8 : In the parts of an infinite plane bounded by a circular quadrant AB and the productions of the radii OA, OB there is two-dimensional motion due to the production of liquid at A, and its obsorption at B, at the uniform rate m. Find the velocity potential of the motion and show that the fluid which issuses from A in the direction making an angle μ with OA follows the path whose polar equation is

$$r = a\sin^{1/2} 2\theta\left[\cot\mu + \sqrt{\cot^2\mu + \operatorname{cosec}^2 2\theta}\right]^{1/2}$$

the positive sign being taken for all the square roots.

Solution : The equivalent image system of source $\frac{m}{2\pi}$ at A with respect to circular boundary consists of a source $\frac{m}{2\pi}$ at A, a source $\frac{m}{2\pi}$ at A because A is the inverse point of itself and a sink $-\frac{m}{2\pi}$ at O.

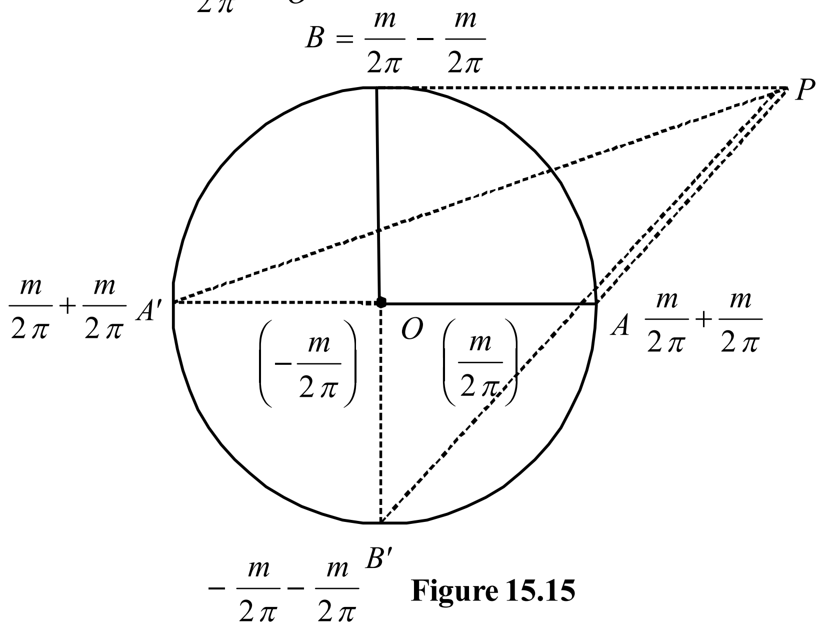

Figure 15.15

As per given the image system of $\frac{m}{2\pi}$ at A with respect to the lines OA and OB is

(i) Sources $\frac{m}{2\pi} + \frac{m}{2\pi}$ at A.

(ii) Sources $\frac{m}{2\pi} + \frac{m}{2\pi}$ at A' (image of the source A).

(iii) Sink $-\frac{m}{2\pi}$ at O.

Similarly considering the image system of $-\frac{m}{2\pi}$ at B with respect to circular boundary and OA, OB, we get.

(i) Sinks $-\frac{m}{2\pi} - \frac{m}{2\pi}$ at B

(ii) Sinks $-\frac{m}{2\pi} - \frac{m}{2\pi}$ at B'

(iii) Source $\frac{m}{2\pi}$ at O.

The complex potential given source and sinks at any point P is given by

$$w = -\frac{m}{\pi}\log(z-a) - \frac{m}{\pi}\log(z+a) + \frac{m}{\pi}\log(z-ai) + \frac{m}{\pi}\log(z+ai)$$

where the source and sink at centre of the circular boundary cancel each other.

$$w = \phi + i\psi = -\frac{m}{\pi}\log(z^2-a^2) + \frac{m}{\pi}\log(z^2+a^2) \qquad ...(1)$$

On equating real parts to both sides, we have

$$\phi = -\frac{m}{\pi}\log\left|z^2-a^2\right| + \frac{m}{\pi}\log\left|z^2-i^2a^2\right|$$

$$= -\frac{m}{\pi}\log\left[|z-a|\,.\,|z+a|\right] + \frac{m}{\pi}\log\left[|z-ia|\,.\,|z+ia|\right]$$

$$= -\frac{m}{\pi}\log\frac{BP\,.\,B'P}{AP\,.\,A'P} \qquad ...(2)$$

Putting $z = e^{i\theta}$ in (1) then equating imaginary parts, we get

$$\psi = -\frac{m}{\pi}\tan^{-1}\frac{r^2\sin 2\theta}{(r^2\cos 2\theta - a^2)} + \frac{m}{\pi}\tan^{-1}\frac{r^2\sin 2\theta}{(r^2\cos 2\theta + a^2)}$$

$$= -\frac{m}{\pi}\tan^{-1}\frac{2a^2r^2\sin 2\theta}{r^4-a^4} \qquad ...(3)$$

The required stream line that leaves A at an inclination μ is given by

$$\psi = -\frac{m}{\pi}\mu$$

Therefore $\qquad -\frac{m}{\pi}\mu = -\frac{m}{\pi}\tan^{-1}\frac{2a^2r^2\sin 2\theta}{(r^4-a^4)}$

or $\qquad r^4 - 2a^2r^2\sin 2\theta\cot\mu - a^4 = 0$

or $\qquad r^2 = \frac{1}{2}\left[2a^2\sin 2\theta\cot\mu + \sqrt{4a^4\sin^2 2\theta\cot^2\mu + 4a^4}\right]$

where (-) sign has been omitted because r^2 is non-negative quantity, Thus, we have

$$r = a\left[\sin 2\theta\left\{\cot\mu + \sqrt{\cot^2\mu + \operatorname{cosec}^2 2\theta}\right\}\right]^{1/2}$$

which is required result.

15.18 Summary

In this unit, we studied the complex potential of sources, sinks and doublets in two-dimensional motion. Now, we are also capable to obtain the complex potential of source, sinks, and doublet with the use of images with respect to straight line and circle.

15.19 Answers to self learning exercise

(1) Orthogonally

(2) Zero

(3) does not

(4) $w = -\, m \log(z-a) + m \log(z-ai)$

(5) Sources of strength m at origin and equal sinks at (1, 0) and (-1, 0).

(6) $w = \dfrac{\mu\, e^{i\alpha x}}{z}$

(7) $r^2 \sin(\theta + nt) = $ constant.

15.20 Exercise

1. In two dimensional irrotational motion, show that if the stream lines are confocal ellipses

$$\frac{x^2}{(a^2+\lambda)} + \frac{y^2}{(b^2+\lambda)} = 1$$

then $\psi = A \log\left(\sqrt{a^2+\lambda} + \sqrt{b^2+\lambda}\right) + B$ and the velocity at any point is inversly proportional to the square root of the rectangle under the focal readii of the point.

2. The sources, each of strength m are placed at the points $(-a, 0)$, $(a, 0)$ and a sink of strength $2m$ at the origin. Show that the stream lines are the curves $(x^2+y^2)^2 = a^2(x^2 - y^2 + \lambda xy)$ where λ is a variable parameter. Show also that the fluid speed at any point is $\dfrac{2ma^2}{r_1 r_2 r_3}$ where r_1, r_2, r_3 are the distances of the points from the sources and the sink.

3. Let there be a source of strength m at $(a, 0)$ and a sink $(-m)$ at $(-a, 0)$. Find ϕ, ψ, ω and velocity q. Ans. {See article 15.11}

4. If there are sources at $(a, 0)$, $(-a, 0)$ and sinks at $(0, a)$, $(0, -a)$ all of equal strength. Show that the circle through these four points is a stream line.

5. Find the stream function of two-dimensional motion due to two equal sources and an equal sink midway between them. In a region bounded by a quadrantal arc and its radii, deduce the motion due to a source and an equal sink situated at the ends of the bounding radii. Show that the stream line leaving either end at angle α with radius is

$$r^2 \sin(\alpha+\theta) = a^2 \sin(\alpha-\theta)$$

6. Use the method of images to prove that if there be a source m at a point z_0 in a fluid bounded by the lines $\theta = 0$ and $\theta = \frac{\pi}{3}$, the solution is

$$\phi + i\psi = -m \log\left[\left(z^3 - z_0^3\right)\left(z^3 - \bar{z}_0\right)\right]$$

where $z_0 = x_0 + i\,y_0$ and $\bar{z}_0 = x_0 - i\,y_0$.

7. Between the fixed boundaries $\theta = \frac{\pi}{4}$ and $\theta = -\frac{\pi}{4}$, there is a two-dimensional liquid motion due to a source of strength m at the point $(r = a,\ \theta = 0)$ and an equal sink at the point $(r = b,\ \theta = 0)$. Show that the stream function is

$$-\,m\,\tan^{-1}\left[\frac{r^4\left(a^4 - b^4\right)\sin 4\theta}{r^8 - r^4\left(a^4 + b^4\right)\cos 4\theta + a^4 b^4}\right]$$

Show also that the velocity at $(r,\ \theta)$ is

$$\frac{4\,m\left(a^4 - b^4\right) r^3}{\left(r^8 - 2\,a^4\,r^4\cos 4\theta + a^8\right)^{1/2}\left(r^8 - 2\,b^4\,r^4\cos 4\theta + b^8\right)^{1/2}}$$

8. In the two dimensional motion of an infinite liquid there is a rigid boundary consisting of that part of the circle $x^2 + y^2 = a^2$ which lies in the first and fourth quadrants and the part of y-axis which lie outside the circle. A simple source of strength m is placed at the point $(f,\ 0)$ where $f > a$. Prove that the speed of the fluid at the point $(a\cos\theta, a\sin\theta)$ of the semi-circular boundary is

$$4\,a\,m\,f^2 \sin 2\theta \,/\left(a^4 + f^4 - 2\,a^2 f^2 \cos 2\theta\right).$$

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