MA/MSc MT-04

Vardhaman Mahaveer Open University, Kota

Differential Geometry and Tensors-1

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Differential Geometry and Tensors-1

PREFACE

*The Present book entitled **"Differential Geometry and Tensors-1"** has been designed so as to cover the unit-wise syllabus of Mathematics-Fourth paper for M.A./M.Sc. (Previous) students of Vardhaman Mahaveer Open University, Kota. It can also be used for competitive examinations. The basic principles and theory have been explained in a simple, concise and lucid manner. Adequate number of illustrative examples and exercises have also been included to enable the students to grasp the subject easily. The units have been written by various experts in the field. The unit writers have consulted various standard books on the subject and they are thankful to the authors of these reference books.*

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UNIT 1 : Space Curves, Tangent, Contact of Curve and Surface, Osculating Plane

Structure of the Unit

1.0 Objectives

This unit provides a general overview of :

- Differential geometry
- Space curves
- Tangent
- Contact of curve and surface
- Osculating plane.

1.1 Introduction

Differential geometry is that part of geometry which is treated with the help of differential calculus. There are two branches of differential geometry :

Local differential geometry : In which we study the properties of curves and surfaces in the neighbourhood of a point.

Global differential geometry : In which we study the properties of curves and surfaces as a whole.

1.2 Space curves

A curve in space is defined as the locus of a point whose cartesian coordinates are the functions of a single variable parameter u, say.

We can represent a space curve in the following two ways :

As intersection of two surfaces :

Let $f_1 (x, y, z) = 0, f_2 (x, y, z) = 0$ be two surfaces then these equations together represent the curve of intersection of the above surfaces. If this curve lies in a plane then it is called a plane curve, otherwise it is called to be skew, twisted or tortous.

For example, if $f_1 (x, y, z) = 0$, represents a sphere and $f_2 (x, y, z) = 0$ represents a plane then these two equations together represent a circle.

Parametric representation :

If the coordinates of a point on a space curve be represented by the equations of the following form

$$x = f_1 (t), \quad y = f_2 (t), \quad z = f_3 (t) \qquad .....(1.2.1)$$

where f_1, f_2, f_3 are real valued functions of a single real variable t ranging over a set of values $a \le t \le b$.

The equation in (1.2.1) are called parametric equation of the space curve.

1.2.1 Vector representation of a space curve :

If $\vec{r}$ be the position vector of a current point A on the space curve whose cartesian coordianates be x, y, z then we know that

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

or $$\vec{r} = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$$

or $$\vec{r} = f(t)$$

or $$\vec{r} = (f_1(t), f_2(t), f_3(t)) \qquad .....(1.2.2)$$

where f is a vector valued function of a single variable t. Thus space curve may be defined as :

A space curve is the locus of a point whose position vector $\vec{r}$ with respect to a fixed origin may be expressed as a function of single parameter.

1.2.2 Unit tangent vector of a curve :

Consider two neighbouring points A (x, y, z) and B $(x + \delta x, y + \delta y\ z + \delta z)$ on a curve C whose position vectors are r and $r + \delta r$, respectively. We have

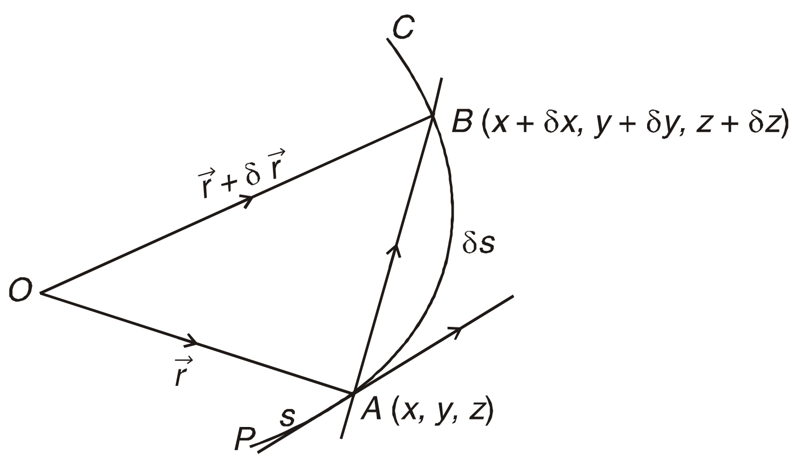


Fig. 1.1

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{r} + \delta\vec{r} - \vec{r} = \delta\vec{r} \cdot$$

Let δs be length of arc AB measured along the curve and arc $PA = s$ is measured from any convenient point P on the curve.

$$\text{Unit vector along chord } AB = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{\delta\vec{r}}{\text{Chord } AB}$$

$$= \frac{\delta\vec{r}}{\delta s} \cdot \frac{\text{Arc } AB}{\text{Chord } AB} \qquad .....(1.2.3)$$

But as B tends to A, then the chord AB tends to be tangent at P.

Also we know that $$\lim_{B \to A} \frac{\text{Arc } AB}{\text{Chord } AB} = 1$$

Hence, unit vector along tangent at A $= \lim_{B\to A} \frac{\delta\vec{r}}{\delta s}\cdot\frac{\text{Arc } AB}{\text{Chord } AB} = \frac{d\,\vec{r}}{ds}\cdot 1$

$$= \frac{d\,\vec{r}}{ds} = \vec{r}' \qquad .....(1.2.4)$$

Unit tangent vector at A is denoted by $\hat{t}$ and is taken in the direction of s increasing

If $\quad \vec{r} = (x, y, z)$ *i.e.* $\vec{r} = x\hat{i} + y\hat{i} + z\hat{k}$

then $\quad \hat{t} = \frac{d\vec{r}}{ds} = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$

i.e. $$\hat{t} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k} \qquad .....(1.2.5)$$

Since $\hat{t}$ is unit tangent vector, $|\hat{t}| = 1$.

$\therefore$ $$1 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2$$

or $$1 = \left(\frac{dx}{dt}\cdot\frac{dt}{ds}\right)^2 + \left(\frac{dy}{dt}\cdot\frac{dt}{ds}\right)^2 + \left(\frac{dz}{dt}\cdot\frac{dt}{ds}\right)^2$$

or $$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

or $$\dot{s}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2,$$

where $$\dot{s} = \frac{ds}{dt},\ \dot{x} = \frac{dx}{dt}\ \ etc. \qquad .....(1.2.6)$$

and t is any parameter.

1.2.3 The equation of tangent line to a curve at a given point :

The tangent line to a curve at any point A is defined as the limiting position of a straight line through the point A and a neighbouring point B on the curve as B tends to A along the curve.

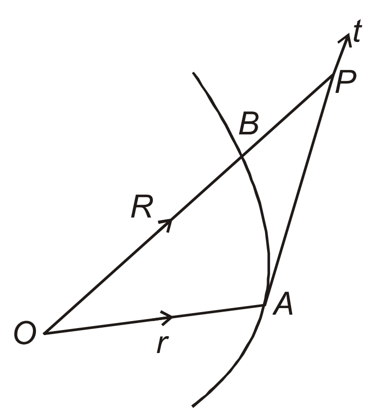


Fig. 1.2

Let $\vec{r} = \vec{r}(s)$ be the parametric equation of a curve and A be any point on it whose position vector is $\vec{r}$ and a unit tangent vector at A be denoted by $\hat{t} = \dfrac{d\vec{r}}{ds} = \vec{r}'$.

Let P be any point on the tangent line at A whose position vector is $\vec{R}$ (say).

Also $\overrightarrow{AP} = w\,\hat{t}$ where $\left|\overrightarrow{AP}\right| = w$

But $\quad \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$$\therefore \quad \vec{R} = \vec{r} + w\,\hat{t} \quad \text{or} \quad \vec{R} = \vec{r} + w\,\vec{r}' \qquad .....(1.2.7)$$

Equation (1.2.2) gives us the equation of tangent line at A.

Tangent line in cartesian form :

We may write $\quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\Rightarrow \quad \vec{r}' = x'\hat{i} + y'\,\hat{j} + z'\hat{k}$

and $\quad \vec{R} = X\hat{i} + Y\,\hat{j} + Z\hat{k}$

Substituting these value in equation (1.2.2) of tangent line, we get

$$X\hat{i} + Y\hat{j} + Z\hat{k} = x\hat{i} + y\hat{j} + z\hat{k} + c\left(x'\hat{i} + y'\hat{j} + z'\hat{k}\right)$$

or

$$X\hat{i} + Y\hat{j} + Z\hat{k} = \left(x + cx'\right)\hat{i} + \left(y + cy'\right)\hat{j} + \left(z + cz'\right)\hat{k},$$

where c is a non-zero constant.

Equating coefficients of $\hat{i}, \hat{j}, \hat{k}$ from both sides

$$X = x + cx',\ Y = y + cy',\ Z = z + cz'$$

i.e.

$$\frac{X - x}{x'} = \frac{Y - y}{y'} = \frac{Z - z}{z'} = c,$$

i.e.

$$\frac{X - x}{x'} = \frac{Y - y}{y'} = \frac{Z - z}{z'} \qquad .....(1.2.8)$$

This is the required equation of tangent line at (x, y, z) and direction cosines of the tangent line are proportional to x', y', z'.

1.2.4 Equation of tangent line when the equation of the curve is given as the intersection of two surfaces :

Let the equation of two surfaces are

$$F_1\,(x, y, z) = 0 \text{ and } F_2\,(x, y, z) = 0 \qquad .....(1.2.9)$$

where x, y, z are functions of a parameter.

Now

$$\frac{\partial F_1}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F_1}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F_1}{\partial z} \cdot \frac{dz}{dt} = 0 \qquad .....(1.2.10)$$

$$\frac{\partial F_2}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F_2}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F_2}{\partial z} \cdot \frac{dz}{dt} = 0 \qquad .....(1.2.11)$$

Hence from equation (1.2.3) and (1.2.4)

$$\frac{\dot{x}}{\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial z}-\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial y}}=\frac{\dot{y}}{\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial z}}=\frac{\dot{z}}{\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y}-\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x}}$$

which are the direction ratios of the tangent and dot represents differentiation w.r. to 't'.

Therefore, the equation of tangent line at a point (x, y, z) on the curve of intersection of the two given surfaces is given as

$$\frac{X-x}{\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial z}-\frac{\partial F_1}{\partial z}\cdot\frac{\partial F_2}{\partial y}}=\frac{Y-y}{\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial x}\cdot\frac{\partial F_2}{\partial z}}=\frac{Z-z}{\frac{\partial F_1}{\partial x}\cdot\frac{\partial F_2}{\partial y}-\frac{\partial F_1}{\partial y}\cdot\frac{\partial F_2}{\partial x}} \quad .....(1.2.13)$$

1.2.5 Direction-cosines of the tangent line :

Let $A\ (x, y, z)$ and $B\ (x + \delta x, y + \delta y, z + \delta z)$ be adjacent points on a given curve in rectangular coordinate axes. δr the measure of chord AB is given by

$$\delta\vec{r}^2 = \delta x^2 + \delta y^2 + \delta z^2$$

Let s be the length of the arc measure from some fixed point P to any point A on the curve.

If the measure of the arc AB of the curve be δs then

$$\left(\frac{\delta r}{\delta s}\right)^2=\left(\frac{\delta x}{\delta s}\right)^2+\left(\frac{\delta y}{\delta s}\right)^2+\left(\frac{\delta z}{\delta s}\right)^2$$

Since $$\lim_{B\to A}\frac{\text{Chord } AB}{\text{Arc } AB}=1$$

$$1=\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2,$$

or $$\left(\frac{ds}{dt}\right)^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2=|\,\dot{r}\,|^2$$

Hence $$\dot{s}^2=\dot{x}^2+\dot{y}^2+\dot{z}^2 \quad .....(1.2.14)$$

where x, y, z are functions of t and $\dot{x}=\frac{dx}{dt}$ etc.

But $\dot{x}, \dot{y}, \dot{z}$ are direction ratios of a tangent line therefore the direction cosines of the tangent line at A are

$$\frac{\dot{x}}{\dot{s}},\frac{\dot{y}}{\dot{s}},\frac{\dot{z}}{\dot{s}} \quad \text{or} \quad \frac{dx}{ds},\frac{dy}{ds},\frac{dz}{ds}$$

But $$\frac{d\vec{r}}{ds}=\frac{dx}{ds}\hat{i}+\frac{dy}{ds}\hat{j}+\frac{dz}{ds}\hat{k}.$$

The direction cosines of the tangent line are x', y', z' which are the components of r' where a prime denotes differentiation with respect to s. Clearly $|\,\vec{r}'\,|=1,$ *i.e.* $\vec{r}'$ is unit vector along the tangent.

1.2.6 Examples :

***Ex.*1.** *Find the equation to the tangent at the point* θ *on the circular helix*

$$x = a\cos\theta,\ y = a\sin\theta,\ z = C\theta$$

Sol. The vector equation of the helix is given by

$$\vec{r} = a\cos\theta\hat{i} + a\sin\theta\hat{j} + C\,\theta\,\hat{k}$$

$$\vec{r}' = -\,a\sin\theta\hat{i} + a\cos\theta\hat{j} + C\,\hat{k}$$

The equation of the tangent in given by

$$\vec{R} = \vec{r} + \lambda\vec{r}'$$

or
$$\vec{R} = \left(a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j} + C\,\theta\hat{k}\right) + \lambda\left(-a\sin\theta\,\hat{i} + a\cos\theta\,\hat{j} + C\,\hat{k}\right)$$

If
$$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k},$$

then
$$X\hat{i} + Y\hat{j} + Z\hat{k} = a(\cos\theta - \lambda\sin\theta)\,\hat{i} + a(\sin\theta + \lambda\cos\theta)\,\hat{j} + C(\theta + \lambda)\,\hat{k}$$

which gives
$$\frac{X - a\cos\theta}{-a\sin\theta} = \frac{Y - a\sin\theta}{a\cos\theta} = \frac{Z - C\theta}{C}.$$

It is the required equation of tangent line.

***Ex.*2.** *Show that the tangent at any point of the curve whose equations are*

$$x = 3t,\ y = 3t^2,\ z = 2t^3$$

makes a constant angle with line

$$y = z - x = 0.$$

Sol. The direction-rations of the tangent at 't' to the given curve are

$$3,\ 6t,\ 6t^2 \quad (i.e.,\ \ \dot{x}, \dot{y}, \dot{z})$$

The direction ratios of the given line are

$$1,\ 0,\ 1.$$

If θ be the angle between the tangent and the given line, than

$$\cos\theta = \frac{3\times 1 + 6t\times 0 + 6t^2\times 1}{\left(\sqrt{9 + 36t^2 + 36t^4}\right)\left(\sqrt{1+0+1}\right)}$$

$$= \frac{3\left(1+2t^2\right)}{\sqrt{2}\times 3\left(1+2t^2\right)} = \frac{1}{\sqrt{2}}$$

which is independent of t, hence θ is constant.

***Ex.*3.** *Show that the tangent at a point of the curve of the intersection of the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

and the confocal whose parameter λ is given by

$$\frac{x(X-x)}{a^2(b^2-c^2)(a^2-\lambda)}=\frac{y(Y-y)}{b^2(c^2-a^2)(b^2-\lambda)}=\frac{z(Z-z)}{c^2(a^2-b^2)(c^2-\lambda)}.$$

Sol. The equation of a confocal to the ellipsoid

$$F_1\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \qquad .....(1)$$

is

$$F_2\equiv\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}-1=0 \qquad .....(2)$$

Equations to a tangent line are

$$\frac{X-x}{\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial z}-\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial y}}=\frac{Y-y}{\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial z}}=\frac{Z-z}{\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y}-\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x}} \qquad .....(3)$$

Here $\quad \frac{\partial F_1}{\partial x}=\frac{2x}{a^2},\ \frac{\partial F_1}{\partial y}=\frac{2y}{b^2},\ \frac{\partial F_1}{\partial 3}=\frac{2z}{c^2},\ \frac{\partial F_2}{\partial x}=\frac{2x}{a^2-\lambda},\ \frac{\partial F_2}{\partial y}=\frac{2y}{b^2-\lambda},\ \frac{\partial F_2}{\partial z}=\frac{2z}{c^2-\lambda}$

Putting these values in equation (3), we get

$$\frac{x(X-x)}{a^2(b^2-c^2)(a^2-\lambda)}=\frac{y(Y-y)}{b^2(c^2-a^2)(b^2-\lambda)}=\frac{z(Z-z)}{c^2(a^2-b^2)(c^2-\lambda)}$$

which are the required equations of the tangent.

1.2.7 Self-learning exercise-1 :

1. Name the branches of differential Geometry.

2. If the curve lies in a plane then it is called ..... .

3. The intersection of two surfaces is called ....

4. Write the equation of a tangent line at a point.

5. Write the equation of tangent line when the equation of the curve is given as the intersection of two surfaces.

1.3 Contact of curve and surface

We know that in a plane curve the tangent at A is the limiting position of the chord AB when B coincides with A. In a similar manner if $A_1, A_2, ..., A_{n+1}$ be points on a given curve lying on a given surface and if $A_2, A_3, ..., A_{n+1}$ all coincide with A_1, than we say that a curve has a contact of nth order with the surface at A_1. We may also say that the curve and the surface has $(n + 1)$ points of contact.

1.3.1 Definition :

If $A, A_1, A_2,..., A_n$ points on a given curve lie on a given surface and $A_1, A_2, ..., A_n$ coincide with A, then curve and surface are said to have the contact of nth order at the point A.

1.3.2 To find the condition that a curve and a surface have a contact of *n*th order :

Let the equation of the curve C be given by

$$\vec{r} = \{x(t), y(t), z(t),\} \qquad .....(1.3.1)$$

and the equation of the surface S be given by

$$f(x, y, z) = 0 \qquad .....(1.3.2)$$

The values of t corresponding to the points of intersection of the curve C and surface S are the roots of the equation

$$F(t) = f\{x(t), y(t), z(t)\} = 0 \qquad .....(1.3.3)$$

Let $t = t_0$ be a root of the equation $F(t) = 0$ so that

$$F(t_0) = 0. \qquad .....(1.3.4)$$

Then $t = t_0$ give as a point of intersection of C and S.

Put $t = t_0 + h$ so that

$$F(t) = F(t_0 + h). \qquad .....(1.3.5)$$

Expanding $F(t)$ about t_0 by Taylor's theorem, we get

$$F(t) = F(t_0) + h\dot{F}(t_0) + \frac{h^2}{\lfloor 2}\ddot{F}(t_0) + \frac{h^3}{\lfloor 3}\dddot{F}(t_0) + \dots . \qquad .....(1.3.6)$$

Since t_0 is a solution of the equation (1.3.4) therefore $F(t_0) = 0$, then we have

$$F(t) = h\dot{F}(t_0) + \frac{h^2}{\lfloor 2}\ddot{F}(t_0) + \frac{h^3}{\lfloor 3}\dddot{F}(t_0) + \dots . \qquad .....(1.3.7)$$

We have the following cases :

(i) If $\dot{F}(t_0) \neq 0$, then we say that the curve and the surface have a simple intersection at $\vec{r}$ (t_0).

(ii) If $\dot{F}(t_0) = 0$, but $\ddot{F}(t_0) \neq 0$, then $F(t)$ is of second order of h and we say that t_0 is a double zero of $F(t)$ and in this case C and S have two points of contact (or contact of first order) at $\vec{r}$ (t_0).

(iii) If $\dot{F}(t_0) = 0, \ddot{F}(t_0) = 0$, but $\dddot{F}(t_0) \neq 0$, then $F(t)$ is of third order of h and we say that t_0 is a triple zero of $F(t)$ and in this case we say that C and S have three point contact or contact of second order.

(iv) In general if $\dot{F}(t_0) = 0, \ddot{F}(t_0) = 0, \dots, F^{n-1}(t_0) = 0$, but $f^n(t_0) \neq 0$, then $F(t)$ is of nth order of h and we say that C and S have a n point contact or contact of $(n - 1)$th order.

1.3.3 Inflexional tangent :

A straight line which meets the surface S in three coincident points *i.e.*, it has a second order point of contact is called inflexional tangent to the surface at that point.

1.3.4 Examples :

***Ex.*4.** *Find the plane that has three point contact at the origin with the curve*

$$x = t^4 - 1, \quad y = t^3 - 1, \quad z = t^2 - 1.$$

Sol. Let the equation of the plane at the origin be

$$l\,x + m\,y + n\,z = 0 \qquad .....(1)$$

The equations of the given curve are

$$x = t^4 - 1, \quad y = t^3 - 1, \quad z = t^2 - 1 \qquad .....(2)$$

At origin,

$$t^4 - 1 = 0, \quad t^3 - 1 = 0, \quad t^2 - 1 = 0.$$

Clearly, $t = 1$ satisfies all of these three equations. Hence, at the origin, we have $t = 1$.

Now the points of intersection of the curve (2) and the surface (1) are given by the zeroes of the function

$$F(t) = l\,(t^4 - 1) + m\,(t^3 - 1) + n\,(t^2 - 1)$$

or $$F(t) = l\,t^4 + m\,t^3 + n\,t^2 - l - m - n \qquad .....(3)$$

For three point contact, we should have

$$\dot{F}(t) = 0, \ddot{F}(t) = 0.$$

Now $$\dot{F} = 4l\,t^3 + 3m\,t^2 + 2nt = 0 \qquad .....(4)$$

and $$\ddot{F} = 12l\,t^2 + 6m\,t + 2n = 0 \qquad .....(5)$$

At the origin *i.e.* at $t = 1$, the equation (4) and (5) become

$$4l + 3m + 2n = 0, \; 12l + 6m + 2n = 0 \qquad .....(6)$$

Solving equation (6), we get

$$\frac{l}{3} = \frac{m}{-8} = \frac{n}{6}$$

Hence the required equation of plane is

$$3x - 8y + 6z = 0.$$

Ex.5. *Prove that if the circle*

$$lx + my + nz = 0, \quad x^2 + y^2 + z^2 = 2cz$$

has three point contact at the origin with the paraboloid

$$ax^2 + by^2 = 2z,$$

then $$c = (l^2 + m^2)/(bl^2 + am^2)$$

Sol. Let the parametric equation of the circle be

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t). \qquad .....(1)$$

Putting these values of x, y, z in the equation of the paraboloid, we get

$$F(t) = ax^2 + by^2 - 2z = 0 \qquad .....(2)$$

where x, y, z are functions of t.

For a three point contact at the origin we must have

$$F(t) = \dot{F}(t) = \ddot{F}(t) = 0, \quad \text{at the origin.}$$

$$F(t) = ax^2 + by^2 - 2z = 0 \qquad \text{.....(3)}$$

$$\dot{F}(t) = 2ax\dot{x} + 2by\dot{y} - 2\dot{z} = 0 \qquad \text{.....(4)}$$

$$\ddot{F}(t) = \left(a\dot{x}^2 + b\dot{y}^2 - \ddot{z}\right) + \left(ax\ddot{x} + by\ddot{y}\right) = 0 \qquad \text{.....(5)}$$

At the origin we have $\quad \dot{z} = 0,\ a\dot{x}^2 + b\dot{y}^2 - \ddot{z} = 0.$ .....(6)

Proceeding as above with the equations of the circle, we get

$$l\dot{x} + m\dot{y} + n\dot{z} = 0 \qquad x\dot{x} + y\dot{y} + z\dot{z} - c\dot{z} = 0 \qquad \text{.....(7)}$$

$$l\ddot{x} + m\ddot{y} + n\ddot{z} = 0 \qquad \dot{x}^2 + \dot{y}^2 + \dot{z}^2 + x\ddot{x} + y\ddot{y} + z\ddot{z} - c\ddot{z} = 0 \qquad \text{.....(8)}$$

At the origin from (7) $\quad c\dot{z} = 0 \ \ i.e.\ \ \dot{z} = 0 \ \text{ so that } \ l\dot{x} + m\dot{y} = 0$ .....(9)

At the origin from (8) and putting $\dot{z} = 0,$ we have

$$\dot{x}^2 + \dot{y}^2 - c\ddot{z} = 0 \text{ or } \dot{x}^2 + \dot{y}^2 = c\left(a\dot{x}^2 + b\dot{y}^2\right)$$

$$\therefore \qquad c = \frac{\dot{x}^2 + \dot{y}^2}{a\dot{x}^2 + b\dot{y}^2}, \text{ from (7) } = \frac{\dot{x}}{m} = \frac{-\dot{y}}{l} = k, \text{ say}$$

or

$$c = \frac{\left(l^2 + m^2\right)}{\left(am^2 + bl^2\right)}.$$

***Ex.*6.** *Find the lines that have four point contact at* (0, 0, 1) *with the surface*

$$x^4 + 3xyz + x^2 - y^2 - z^2 + 2yz - 3xy - 2y + 2z = 1$$

Sol. Any line through (0, 0, 1) is

$$\frac{x}{l} = \frac{y}{m} = \frac{z-1}{n} = k\,(\text{say}) \qquad \text{.....(1)}$$

Thus the parametric equations of the line are

$$x = lk, \quad y = mk, \quad z = nk + 1 \qquad \text{.....(2)}$$

Putting the values of x, y, z in the equation of the surface we get

$$F(k) = l^4 k^4 + 3lm\, k^2 (nk + 1) + l^2 k^2 - m^2 k^2 - n^2k^2$$
$$+ 2mk (nk + 1) - 3lm\, k^2 - 2mk + 2 (nk + 1) - 1 = 0$$
$$= l^4 k^4 + 3lmn\, k^3 + (l^2 - m^2 - n^2 + 2mn)\, k^2 = 0 \qquad \text{.....(3)}$$

For four point contact we must have

$$F(k) = 0, \dot{F}(k) = 0, \ddot{F}(k) = 0, \dddot{F}(k) = 0 \text{ at } k = 0 \text{ for } (0,0,1)$$

$F(k), \dot{F}(k)$ are clearly zero at $k = 0$ .....(4)

$\ddot{F}(k) = 0 \quad$ gives $\quad l^2 - m^2 - n^2 + 2mn = 0. \quad \dddot{F}(k) = 0 \quad$ gives $\quad lmn = 0.$

Thus the direction ratios satisfy the above two relations.

***Ex.*7.** *Find the inflexional tangents at* (x, y, z) *on the surface*

$$y^2 z = 4ax.$$

Sol. We know that the inflexional tangents are tangents which have three point contact with the given surface.

Any line through (x_1, y_1, z_1) is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = k\left(\text{say}\right) \qquad .....(1)$$

$$\therefore \qquad x = x_1 + lk, \quad y = y_1 + mk, \quad z = z_1 + nk \qquad .....(2)$$

Substituting the values of x, y, z in the given surface, we get

$$F(k) = (z_1 + nk)(y_1 + mk)^2 - 4a(x_1 + lk) \qquad .....(3)$$

$$\dot{F}\left(k\right) = n(y_1 + mk)^2 + 2m(y_1 + mk)(z_1 + nk) - 4al \qquad .....(4)$$

$$\ddot{F}\left(k\right) = 2nm(y_1 + mk) + 2m^2(z_1 + nk) + mn(y_1 + mk) \qquad .....(5)$$

For three point contact at (x_1, y_1, z_1) *i.e.* where $k = 0$, we must have

$$F\left(k\right) = 0, \ \dot{F}\left(k\right) = 0, \ \ddot{F}\left(k\right) = 0$$

Hence from (3), (4) and (5), we get

$$y_1^2 z_1 - 4ax_1 = 0 \qquad .....(6)$$

$$2my_1z_1 + ny_1^2 - 4al = 0 \qquad .....(7)$$

$$2mny_1 + 2m^2 z_1 = 0$$

or $$2ny_1 + mz_1 = 0$$

or $$n = -\frac{mz_1}{2y_1} \qquad .....(8)$$

Putting for n in (7), we get

$$2my_1z_1 - \frac{2mz_1}{2y_1} \cdot y_1^2 - 4al = 0$$

$$\therefore \qquad l = \frac{3my_1z_1}{8a} \qquad .....(9)$$

Putting for l and n in the equation (1), we get the required inflexional tangent as

$$\frac{x-x_1}{\left(3my_1z_1/8a\right)} = \frac{y-y_1}{m} = \frac{z-z_1}{\left(-mz_1/2y_1\right)}$$

or $$\frac{x-x_1}{\left(3y_1^2z_1/4a\right)} = \frac{y-y_1}{2y_1} = \frac{z-z_1}{-z_1} \quad \text{put} \quad y_1^2 z_1 = 4ax_1$$

$$\therefore \qquad \frac{x-x_1}{3x_1} = \frac{y-y_1}{2y_1} = \frac{z-z_1}{-z_1}.$$

***Ex.*8.** *Prove that the condition that four consecutive points of a curve should be coplanar is*

$$\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} = 0 \cdot$$

Sol. Let the parametric equation of the given curve be

$$x = f_1\,(t), \quad y = f_2\,(t), \quad z = f_3\,(t). \qquad \text{.....(1)}$$

Let the parameter for a given point $A\,(x_0, y_0, z_0)$ be t_0 so that the equation of the plane through A is

$$(x - x_0)\,l + (y - y_0)\,m + (z - z_0)\,n = 0. \qquad \text{.....(2)}$$

Putting for x, y, z from (1) and (2), we get

$$F\,(t) = [x\,(t) - x\,(t_0)]\,l + [y\,(t) - y\,(t_0)]\,m + [z\,(t) - z\,(t_0)]\,n = 0. \qquad \text{.....(3)}$$

The plane (2) passes through four consecutive points if it has four point contact, *i.e.,* if

$$F\left(t_0\right) = 0, \dot{F}\left(t_0\right) = 0, \ddot{F}\left(t_0\right) = 0, \dddot{F}\left(t_0\right) = 0\,. \qquad \text{.....(4)}$$

These conditions are equivalent to

$$\dot{x}l + \dot{y}m + \dot{z}n = 0 \qquad \text{.....(5)}$$

$$\ddot{x}l + \ddot{y}m + \ddot{z}n = 0 \qquad \text{.....(6)}$$

$$\dddot{x}l + \dddot{y}m + \dddot{z}n = 0 \quad \text{and clearly} \quad F\,(t_0) = 0. \qquad \text{.....(7)}$$

Eliminating l, m, n between the above equations, we have

$$\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} = 0.$$

(Here dashes and dots represent derivative.)

1.3.5 Self-learning exercise-2 :

1. Write the condition for simple intersection of a curve and surface.
2. Write the condition for two point contact of a curve and a surface.
3. Write the condition for three point contact of a curve and a surface.
4. Write the condition for n point contact of a curve and surface.
5. Define an inflexional tangent.

1.4 Osculating plane

Definition : The osculating plane at a point P of a curve C of class greater then or equal to two is the limiting position of the plane passing through the tangent line at P and a neighbouring point Q on the curve C as $Q \rightarrow P$. (or which contains the tangent line at P and is parallel to the tangent at Q as $Q \rightarrow P$).

Alternative : Let P, Q, R be three points on a curve C, the limiting position of the plane PQR, when Q and R tend to P, is called the osculating plane at the point P.

1.4.1 To find the equation of the osculating plane :

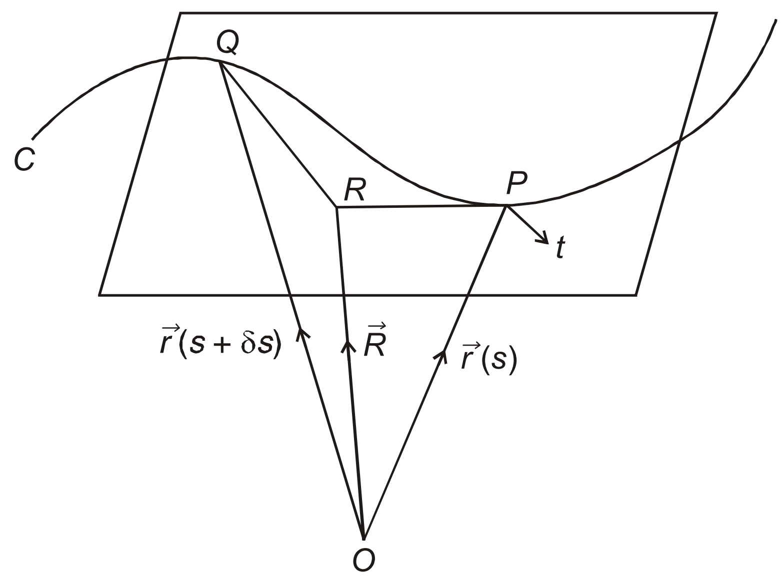


Fig. 1.3

Let $\vec{r} = \vec{r}(s)$ be the given curve C of class ≥ 2, in terms of parameter s, where s is the length of the arc of the curve measured from a fixed point on it. Let P and Q be two neighbouring points on the curve C with $\vec{r}(s)$ and $\vec{r}(s+\delta s)$ be their position vectors. Let $\vec{R}$ be the position vector of current point R on the plane containing the tangent line at P and the point Q.

Here $\qquad \overrightarrow{OP} = \vec{r}(s), \overrightarrow{OQ} = \vec{r}(s+\delta s), \overrightarrow{OR} = \vec{R}$

Hence $\qquad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \vec{r}(s+\delta s) - \vec{r}(s)$

and $\qquad \overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \vec{R} - \vec{r}(s)$

Again if $\hat{t}$ be the unit tangent vector at P,

then, $\qquad \hat{t} = \dfrac{d\vec{r}}{ds} = \vec{r}'(s).$

Now the vectors $\overrightarrow{PR}, \hat{t}$ and $\overrightarrow{PQ}$ are coplanar lying in the plane PQR and hence their scalar triple product is zero.

$$\left[\overrightarrow{PR},\ \hat{t},\ \overrightarrow{PQ}\right] = 0 \qquad .....(1.4.1)$$

or
$$\left[\vec{R} - \vec{r}(s), \vec{r}'(s), \vec{r}(s+\delta s) - \vec{r}(s)\right] = 0 \qquad .....(1.4.2)$$

but
$$\vec{r}(s+\delta s) - \vec{r}(s) = \vec{r}'(s)\delta s + \vec{r}''(s)\frac{(\delta s)^2}{\underline{|2}} + ... \qquad .....(1.4.3)$$

We know that $[a\ b\ c] = a \cdot (b \times c)$.

Equation (1.4.2) may be written as

$$\left[\vec{R}-\vec{r}(s)\right]\cdot\vec{r}'(s)\times\left[\vec{r}(s+\delta s)-\vec{r}(s)\right]=0 \qquad .....(1.4.4)$$

form (1.4.3) and (1.4.4)

$$\left[\vec{R}-\vec{r}(s)\right]\cdot\vec{r}'(s)\times\left[\vec{r}'(s)\delta s+\vec{r}''(s)\frac{(\delta s)^2}{|\underline{2}}+\ldots\right]=0 \qquad .....(1.4.5)$$

or $$\left[\vec{R}-\vec{r}(s)\right]\cdot\vec{r}'(s)\times\left[\vec{r}''(s)\frac{(\delta s)^2}{|\underline{2}}+\text{ terms of higher order of }\delta s\right]=0$$

or $$\left[\vec{R}-\vec{r}(s)\right]\cdot\vec{r}'(s)\times\left[\vec{r}''(s)+0\{\delta s\}\right]=0 \qquad .....(1.4.6)$$

The plane *PQR* tends to be the osculating plane when *Q* tends to *P* *i.e.* when $\delta s\to 0$, and hence the equation of the osculating plane is

$$\left[\vec{R}-\vec{r}(s)\right]\cdot\vec{r}'(s)\times\vec{r}''(s)=0$$

or $$\left[\vec{R}-\vec{r}(s),\vec{r}'(s),\vec{r}''(s)\right]=0 \qquad .....(1.4.7)$$

Equation (1.4.7) represents the equation of the osculating plane in terms of parameter *s* of the point *P*.

1.4.2 Equation of the osculating plane in terms of general parameter *t* :

Let *P* (*t*) and *Q* ($t+\delta t$) be the two neighbouring points on curve *C*. Let position vector of *P* and *Q* be $\vec{r}(t)$ and $\vec{r}(t+\delta t)$ with respect to origin, respectively.

The tangents at *P* and *Q* will be parallel to the vectors $\dot{\vec{r}}(t)$ and $\dot{\vec{r}}(t+\delta t)$, respectively.

Therefore the plane through the tangents at *P* (*t*) and *Q* ($t+\delta t$) is perpendicular to the vector

$$\dot{\vec{r}}(t)\times\dot{\vec{r}}(t+\delta t)$$

or to the vector $$\dot{\vec{r}}(t)\times\left[\dot{\vec{r}}(t+\delta t)-\dot{\vec{r}}'(t)\right] \quad \left[\because\ \dot{\vec{r}}(t)\times\dot{\vec{r}}(t)=0\right]$$

i.e. to the vector $$\dot{\vec{r}}(t)\times\frac{\dot{\vec{r}}(t+\delta t)-\dot{\vec{r}}'(t)}{\delta t} \qquad .....(1.4.8)$$

As $Q\to P$, $\delta t\to 0$ in this unit the osculating plane is perpendicular to the vector $\dot{\vec{r}}(t)\times\ddot{\vec{r}}(t)$.

If $\vec{R}$ be the position vector of any current point on the osculating plane, the equation of the osculating plane may be written as

$$\left(\vec{R}-\vec{r}\right)\cdot\dot{\vec{r}}\times\ddot{\vec{r}}=0 \quad \text{or} \quad \left[\vec{R}-\vec{r},\dot{\vec{r}},\ddot{\vec{r}}\right]=0 \qquad .....(1.4.9)$$

1.4.3 Equation of osculating plane in cartesian coordinates :

Let the coordinates of a point *P* on a given curve *C* be (*x, y, z*) and coordinates of any current point be (*X, Y, Z*), these are functions of a parameter *t*.

Then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

and $\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$

Substituting these values in (1.4.9) the equation of the osculating plane is given by

$$\left[(X-x)\hat{i}+(Y-y)\hat{j}+(Z-z)\hat{k}, \dot{x}\hat{i}+\dot{y}\hat{j}+\dot{z}\hat{k}, \ddot{x}\hat{i}+\ddot{y}\hat{j}+\ddot{z}\hat{k}\right]=0$$

or
$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0 \qquad \text{.....(1.4.10)}$$

which is the equation of the osculating plane at a point P (x, y, z).

Theorem : *To show that when the curve is analytic, there exists a definite osculating plane at a point of inflexion, provided the curve is not a straight line.*

Proof : We know that $\vec{r}'\left(=\hat{t}\right)$ is a unit tangent vector, therefore $\vec{r}'^2 = 1$. .....(1)

Differentiating w.r.t. '*s*' we get

$$2\vec{r}'\cdot\vec{r}'' = \vec{0} \quad \text{or} \quad \vec{r}'\cdot\vec{r}'' = \vec{0} \qquad \text{.....(2)}$$

Again differentiating, we get

$$\vec{r}''\cdot\vec{r}'' + \vec{r}'\cdot\vec{r}''' = \vec{0}$$

$$\vec{r}'\cdot\vec{r}''' = \vec{0} \qquad \text{.....(3)}$$

(At a point P where $\vec{r}'' = \vec{0}$, the tangent line is called inflexional and the point P is called the point of inflexion.)

If $\vec{r}''' \neq \vec{0}$, then r' is linearly independent of $\vec{r}'''$. Differentiating successively (3) and applying, above argument shall get

$$\vec{r}'\cdot\vec{r}^{m} = \vec{0}, \quad m \geq 2 \qquad \text{.....(4)}$$

where $\vec{r}^{m}$ is the first non-zero derivative of r at P.

Therefore if $\vec{r}^{m} \neq \vec{0}$, from equation (1.4.3), we get

$$\vec{r}(s+\delta s) - \vec{r}(s) = \frac{(\delta s)^m}{\lfloor m}\cdot\vec{r}^{m}(s) + 0\left\{(\delta s)^{m+1}\right\} \qquad \text{.....(5)}$$

Hence the equation of the osculating plane at P is

$$\left[\vec{R}-\vec{r}(s), \vec{r}'(s), \vec{r}^{m}(s)\right] = 0 \qquad \text{.....(6)}$$

Again if for all $m \geq 2$ the derivative $\vec{r}^{m} = 0$, we conclude $\vec{r}'\left(=\hat{t}\right)$ is constant (since the curve under consideration is analytic) *i.e.* the tangent vector is same at each point of the curve and hence the curve is a straight line.

Hence equation (6) is the equation of the osculating plane at a point of inflexion P when the curve is not straight line.

1.4.4 To find the osculating plane at a point of a space curve given by the intersection of two surfaces.

Let the equations of the surfaces be

$$f\left(\vec{r}\right)=0 \text{ and } g\left(\vec{r}\right)=0 \qquad .....(1.4.11)$$

The equations of the tangent planes of these surfaces are given by

$$\left(\vec{R}-\vec{r}\right)\cdot\nabla f=0 \text{ and } \left(\vec{R}-\vec{r}\right)\cdot\nabla g=0 \qquad .....(1.4.12)$$

where ∇f and ∇g are normal vectors to $f\left(\vec{r}\right)=0$ and $g\left(\vec{r}\right)=0$ respectively and $\vec{R}$ be the position vector of current point on the plane.

The equation of the plane through the tangent line to the curve of intersection of the two surfaces is

$$F\equiv\left(\vec{R}-\vec{r}\right)\cdot\nabla f-\lambda\left(\vec{R}-\vec{r}\right)\cdot\nabla g=0 \qquad .....(1.4.13)$$

If (1.4.13) be the equation of the osculating plane at P, it must have three point contact with the curve at P. Therefore the required conditions are

$$F=0, \dot{F}=0, \ddot{F}=0; \qquad .....(1.4.14)$$

when $\vec{R}=\vec{r}$ and dashes denote differentiation with respect to parameter 't'.

$\dot{F}=0$ gives

$$\dot{R}\cdot\nabla f+\left(\vec{R}-\vec{r}\right)\cdot\left(\nabla f\right)^{\cdot}-\lambda\,\dot{\vec{R}}\cdot\nabla g-\lambda\left(\vec{R}-\vec{r}\right)\cdot\left(\nabla g\right)^{\cdot}=0 \qquad .....(1.4.15)$$

At $P, \vec{R}=\vec{r},$ condition (1.4.12) reduces to

$$\dot{\vec{r}}\cdot\nabla f-\lambda\,\dot{\vec{r}}\cdot\nabla g=0 \qquad .....(1.4.16)$$

But we know that $\dot{\vec{r}}$ is a tangent vector and ∇f and ∇g are normal vectors to $f\left(\vec{r}\right)=0$ and $g\left(\vec{r}\right)=0$ and hence both

$$\dot{\vec{r}}\cdot\nabla f=0 \text{ and } \dot{\vec{r}}\cdot\nabla g=0 \qquad .....(1.4.17)$$

Hence $\dot{\vec{F}}=0$ reduces to an identity.

Now consider the condition $\ddot{\vec{F}}=0$ at $P, \vec{R}=\vec{r},$ we have

$$\ddot{\vec{r}}\cdot\nabla f=0-\lambda\,\ddot{\vec{r}}\cdot\nabla g=0$$

or $$\lambda=\frac{\ddot{\vec{r}}\cdot\nabla f}{\ddot{\vec{r}}\cdot\nabla g} \qquad .....(1.4.18)$$

Now differentiating the equation (1.4.17), we get

$$\ddot{\vec{r}}\cdot\nabla f+\dot{\vec{r}}\cdot\left(\nabla f\right)^{\cdot}=0,\ \ddot{\vec{r}}\cdot\nabla g+\dot{\vec{r}}\cdot\left(\nabla g\right)^{\cdot}=0, \qquad .....(1.4.19)$$

or $$\ddot{\vec{r}}\cdot\nabla f=-\dot{\vec{r}}\cdot\left(\nabla f\right)^{\cdot}$$

and $$\ddot{\vec{r}}\cdot\nabla g=-\dot{\vec{r}}\cdot\left(\nabla g\right)^{\cdot}$$

$$\therefore \qquad \frac{\ddot{\vec{r}}\cdot\nabla f}{\ddot{\vec{r}}\cdot\nabla g}=\frac{\dot{\vec{r}}\cdot(\nabla f)^{\cdot}}{\dot{\vec{r}}\cdot(\nabla g)^{\cdot}}=\lambda, \text{ from (1.4.18)} \qquad .....(1.4.20)$$

Putting the value of λ in (1.4.13), we get

$$\frac{\left(\vec{R}-\vec{r}\right)\cdot\nabla f}{\left(\vec{R}-\vec{r}\right)\cdot\nabla g}=\lambda=\frac{\dot{\vec{r}}\cdot(\nabla f)^{\cdot}}{\dot{\vec{r}}\cdot(\nabla g)^{\cdot}} \text{ form (1.4.20)}$$

or

$$\frac{\left(\vec{R}-\vec{r}\right)\cdot\nabla f}{\dot{\vec{r}}\cdot(\nabla f)^{\cdot}}=\frac{\left(\vec{R}-\vec{r}\right)\cdot\nabla g}{\dot{\vec{r}}\cdot(\nabla g)^{\cdot}} \qquad .....(1.4.21)$$

Above equation represents the equation of the osculating plane at P.

Cartesian form :

Let $\quad f\left(\vec{r}\right)=f\left(x,y,z\right),\ g\left(\vec{r}\right)=g\left(x,y,z\right)$

$$\vec{R}=Xi+Yj+Zk,\ \vec{r}=xi+yj+zk$$

$$\nabla f=\left(\frac{\partial f}{\partial x}\right)\hat{i}+\left(\frac{\partial f}{\partial y}\right)\hat{j}+\left(\frac{\partial f}{\partial z}\right)\hat{k}$$

$$\nabla f=f_x\,\hat{i}+f_y\,\hat{j}+f_z\,\hat{k}$$

$$\Rightarrow \qquad (\nabla f)^{\cdot}=\Sigma\left(f_{xx}\,\dot{x}+f_{xy}\,\dot{y}+f_{xz}\,\dot{z}\right)\hat{i}$$

substituting in equation (1.4.21) of the osculating plane, we get

$$\frac{\left(X-x\right)f_x+\left(Y-y\right)f_y+\left(Z-z\right)f_z}{\left(\dot{x}^2f_{xx}+...+2\dot{y}\dot{z}\ f_{yz}+...\right)}=\frac{\left(X-x\right)g_x+\left(Y-y\right)g_y+\left(Z-z\right)g_z}{\left(\dot{x}^2g_{xx}+...+2\dot{y}\dot{z}\ g_{yz}+...\right)} \qquad .....(1.4.22)$$

1.4.4 Examples :

Ex.9. *For the curve* $x = 3t$, $y = 3t^2$, $z = 2t^3$, *show that any plane meets it in three points and deduce the equation to the osculating plane at* $t = t_1$.

Sol. Let the equation of the plane be

$$Ax + By + Cz + D = 0 \qquad .....(1)$$

$$\therefore \qquad F(t) = 3At + 3Bt^2 + 2Ct^3 + D = 0 \qquad .....(2)$$

which is cubic in t. Hence the plane meets the given curve in three points.

Also $\quad \dot{x}=3,\ \dot{y}=6t,\ \dot{z}=6t^2$

$$\ddot{x}=0,\ \ddot{y}=6,\ \ddot{z}=12t \qquad .....(3)$$

Hence the equation of osculating plane at the point t_1 is

$$\begin{vmatrix} x-3t_1 & y-3t_1^2 & z-2t_1^3 \\ 3 & 6t_1 & 6t_1^2 \\ 0 & 6 & 12t_1 \end{vmatrix}=0$$

or $\quad 2t_1^2x-2t_1y+z=2t_1^3 \quad$ is the required equation of the osculating plane at $t = t_1$.

***Ex.*10.** *Find the osculating plane at the point 't' on the helix.*

$$x = a\cos t,\ \ y = a\sin t,\ \ z = ct$$

Sol. We know that the equation of the osculating plane is

$$\left[\vec{R}-\vec{r}, \dot{\vec{r}}, \ddot{\vec{r}}\right]=0$$

or $$\left(\vec{R}-\vec{r}\right)\cdot\left[\dot{\vec{r}}\times\ddot{\vec{r}}\right]=0 \qquad .....(1)$$

Here $$\vec{r}=\left(a\cos t, a\sin t, c\,t\right) \qquad .....(2)$$

$\therefore$ $$\dot{\vec{r}}=\left(-a\sin t, a\cos t, c\right) \qquad .....(3)$$

$$\ddot{\vec{r}}=\left(-a\cos t, -a\sin t, 0\right) \qquad .....(4)$$

$\therefore$ $$\dot{\vec{r}}\times\ddot{\vec{r}} = \begin{vmatrix} i & j & k \\ -a\sin t & a\cos t & c \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= ca\sin t\ i - ca\cos t\ j + a^2 k$$

$$=\left(ca\sin t, -ca\cos t, a^2\right) \qquad .....(5)$$

Also $$\vec{R}-\vec{r} = \left(X-a\cos t, Y-a\sin t, Z-ct\right) \qquad .....(6)$$

Hence $$\left(\vec{R}-\vec{r}\right)\cdot\left[\dot{\vec{r}}\times\ddot{\vec{r}}\right] = \left[X-a\cos t, Y-a\sin t, Z-ct\right]\cdot\left[ca\sin t, -ca\cos t, a^2\right]$$

or $$c\,(X\sin t - Y\cos t) + a\,Z - act = 0 \qquad .....(7)$$

Alternative method :

The equation of the osculating plane is

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix}=0$$

$$\begin{vmatrix} X-a\cos t & Y-a\sin t & Z-ct \\ -a\sin t & a\cos t & c \\ -a\cos t & -a\sin t & 0 \end{vmatrix}=0$$

or $$c\,(X\sin t - Y\cos t) + a\,Z - act = 0 \qquad .....(8)$$

***Ex.*11.** *Prove that the osculating plane at* (x_1, y_1, z_1) *on the curve of intersection of the cylinders* $x^2 + z^2 = a^2,\ y^2 + z^2 = b^2$ *is given by*

$$\frac{xx_1^3 - zz_1^3 - a^4}{a^2} = \frac{yy_1^3 - zz_1^3 - b^4}{b^2}.$$

Sol. We know that osculating plane at (x, y, z) a point on the curve of intersection of two surfaces is given by

$$\frac{\Sigma(X-x)f_x}{\Sigma\dot{x}^2 f_{xx}+\Sigma 2\dot{y}\dot{z}\,f_{yz}}=\frac{\Sigma(X-x)g_x}{\Sigma\dot{x}^2 g_{xx}+\Sigma 2\dot{y}\dot{z}\,g_{yz}} \qquad .....(1)$$

where $f = x^2 + z^2 - a^2$, $g = y^2 + z^2 - b^2$, x, y, z are functions of 't'.

$\therefore \qquad x\dot{x}+z\dot{z}=0, \quad y\dot{y}+z\dot{z}=0 \quad \text{or} \quad \frac{\dot{x}}{(1/x)}=\frac{\dot{y}}{(1/y)}=\frac{\dot{z}}{-(1/z)}$

$\therefore f_x = 2x, f_y = 0, f_z = 2z, f_{xx} = 2, f_{xy} = 0, f_{xz} = 0, f_{yy} = 0, f_{yz} = 0, f_{xz} = 2, g_x = 0, g_y = 2y,$
$g_z = 2z, g_{xx} = 0, g_{xy} = 0, g_{xz} = 0, g_{yy} = 2, g_{yz} = 0, g_{zz} = 2.$

Hence the equation of the osculating plane at (x_1, y_1, z_1) is

$$\frac{(x-x_1)2x_1+(z-z_1)2z_1}{2\left(\frac{1}{x_1^2}+\frac{1}{z_1^2}\right)}=\frac{(y-y_1)2y_1+(z-z_1)2z_1}{2\left(\frac{1}{y_1^2}+\frac{1}{z_1^2}\right)}$$

$$\frac{\left(xx_1+zz_1-x_1^2-z_1^2\right)x_1^2\,z_1^2}{x_1^2+z_1^2}=\frac{\left(yy_1+zz_1-y_1^2-z_1^2\right)y_1^2\,z_1^2}{\left(y_1^2+z_1^2\right)}$$

Put $\qquad x_1^2+z_1^2=a^2 \;\text{ and }\; y_1^2+z_1^2=b^2$

$$\frac{\left(xx_1+zz_1-a^2\right)x_1^2}{a^2}=\frac{\left(yy_1+zz_1-b^2\right)y_1^2}{b^2}$$

or $$\frac{xx_1^3+\left(zz_1-a^2\right)\left(a^2-z_1^2\right)}{a^2}=\frac{yy_1^3+\left(zz_1-b^2\right)\left(b^2-z_1^2\right)}{b^2}$$

or $$\frac{xx_1^3-zz_1^3-a^4+a^2\left(z_1^2+zz_1\right)}{a^2}=\frac{yy_1^3-zz_1^3-b^4+b^2\left(z_1^2+zz_1\right)}{b^2}$$

or $$\frac{xx_1^3-zz_1^3-a^4}{a^2}=\frac{yy_1^3-zz_1^3-b^4}{b^2}$$ Hence proved.

***Ex.*12.** *Show that the osculating plane at (x, y, z) on the curve* $x^2 + 2ax = y^2 + 2by = z^2 + 2cz$ *has the equation*

$$(b^2 - c^2)(x + a)^2 (X - x) + (c^2 - a^2)(y + b)^2 (Y - y) + (a^2 - b^2)(z + c)^2 (Z - z) = 0$$

Sol. Let $\qquad f = x^2 - y^2 + 2ax - 2by$

$\qquad g = x^2 - z^2 + 2ax - 2cz.$

Let $P(x, y, z)$ be any point on the curve of intersection of these surfaces and x, y, z are functions of 't'.

$\therefore \qquad x\dot{x}-y\dot{y}+a\dot{x}-b\dot{y}=0 \quad \text{or} \quad \dot{x}(x+a)-\dot{y}(y+b)=0$

$\qquad x\dot{x}-z\dot{z}+a\dot{x}-c\dot{z}=0 \quad \text{or} \quad \dot{x}(x+a)-\dot{z}(z+c)=0$

$\therefore \qquad \dot{x}(x+a)=\dot{y}(y+b)=\dot{z}(z+c)$

or $$\frac{\dot{x}}{(y+b)(z+c)}=\frac{\dot{y}}{(z+c)(x+a)}=\frac{\dot{z}}{(x+a)(y+b)} \qquad .....(2)$$

Again $\quad f_x = 2(x+a),\ \ f_y = -2(y+b), f_z = 0,$

$g_x = 2(x+a), g_y = 0, g_z = -2\,(z+c)$

$f_{xx} = 2, f_{yy} = -2, f_{zz} = 0, f_{xy} = f_{yz} = f_{zx} = 0$

$g_{xx} = 2, g_{yy} = 0, g_{zz} = -2, g_{xy} = g_{yz} = g_{zx} = 0.$

The equation of the osculating plane is

$$\frac{\Sigma(X-x)f_x}{\Sigma\dot{x}^2 f_{xx}+\Sigma 2\dot{y}\dot{z}\, f_{yz}}=\frac{\Sigma(X-x)g_x}{\Sigma\dot{x}^2 g_{xx}+\Sigma 2\dot{y}\dot{z}\, g_{yz}}$$

or $$\frac{(X-x)(2x+2a)+(Y-y)(-2g-2b)+(Z-z)0}{2(y+b)^2(z+c)^2-2(z+c)^2(x+a)^2}$$

$$=\frac{(X-x)(2x+2a)+(Y-y)0+(Z-z)(-2z-x)}{2(y+b)^2(z+c)^2-2(x+a)^2(y+b)^2}$$

or $\quad \Sigma\,(X-x)\,(x+a)^2\,(b^2-c^2)=0.$

1.4.5 Self-learning exercise-3 :

1. Define osculating plane.
2. Write the equation of osculating plane.
3. Write the equation of osculating plane in cartesian coordinates.

1.5 Answers to self-learning exercises

Self-learning exercise-1

1. **(*a*)** Local differential geometry **(*b*)** Global differential geometry
2. Plane curve
3. Curve
4. $\dfrac{X-x}{\dot{x}}=\dfrac{Y-y}{\dot{y}}=\dfrac{Z-z}{\dot{z}}$
5. $$\frac{X-x}{\frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial z}-\frac{\partial F_1}{\partial z}\cdot\frac{\partial F_2}{\partial y}}=\frac{Y-y}{\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial x}\cdot\frac{\partial F_2}{\partial z}}=\frac{Z-z}{\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y}-\frac{\partial F_1}{\partial y}\cdot\frac{\partial F_2}{\partial x}}$$

Self-learning exercise-2

1. If $\dot{F}\neq 0$
2. If $\dot{F}=0$ but $\ddot{F}\neq 0$

3. If $\dot{F} = 0, \ddot{F} = 0$ but $\dddot{F} \neq 0$

4. If $\dot{F} = 0, \ddot{F} = 0, \ldots, F^{n-1} = 0$ but $F^{n} \neq 0$

5. A straight line which meets the surface S in the three coincident.

Self-learning exercise-3

1. Let P, Q, R be three points on a curve C, the limiting position of the plane PQR, when Q and R tend to P, is called osculating plane at point R.

2. $\left[\vec{R} - \vec{r},\ \dot{\vec{r}}, \ddot{\vec{r}}\right] = 0$

3. $\begin{vmatrix} X - x & Y - y & Z - z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0$

1.6 Exercises

1. Find the equation of the tangent line to the curve $x = t, y = t^2,\ z = t^3$ at the point $t = 1$.

[Ans. $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3}$]

2. Find the equation of the tangent line at the point t = 1 to the curve $r = (1 + t, t^2, 1 + t^3)$.

[Ans. $\frac{x-2}{1} = \frac{y-1}{2} = \frac{z-2}{3}$]

3. Define a space curve and write its parametric equations.

4. Determine a, h, b so that the paraboloid $2z = ax^2 + 2hxy + by^2$ may have the closest possible contact at the origin with the curve $x = t^3 - 2t^2 + 1, y = t^3 - 1, z = t^2 - 2t + 1$. Find also the order of contact. **[Ans.** $\frac{a}{45} = \frac{h}{-3} = \frac{b}{5} = \frac{1}{54}$; Fourth]

5. Show that the curve $x = t, y = t^2, z = t^3$ has six point contact with the paraboloid $x^2 + y^2 = y$ at the origin.

6. Find the equation of the osculating plane of the curve given by

$$\vec{r} = (a \sin t + b \cos t, a \cos t + b \sin t, c \sin 2t).$$

[Ans. $2cx\ \{a \cos t\ (2 - \cos 2t) - b \sin t\ (2 + \cos 2t)\}$
$+ 2cy\ \{a \sin t\ (2 - \cos 2t) - b \cos t\ (2 - \cos 2t)\}$
$+ 2\ (b^2 - a^2) + 3c\ (b^2 - a^2) \sin 2t = 0$**]**

7. For the curve $x = 3t, y = 3t^2, z = 2t^3$ show that any plane meets it in three points and deduce the equation of the osculating plane at $t = t_1$. **[Ans.** $2t_1^2x - 2t_1y + z = 2t_1^3$**]**

□□□

Unit 2 : Principal Normal and Binormal, Curvature, Torsion, Serret-Frenet's Formulae, Osculating Circle and Osculating Sphere

Structure of the Unit

2.0 Objectives

This unit provides a general overview of

- Principal normal and Binormal
- Three fundamental planes
- Curative and Torsion
- Serret-Frenet's formula
- Osculating circle and osculating sphere

2.1 Introduction

In this unit we shall study principal normal and Binormal. Equations of principal normal and Binormal Curvature and Torsion. Formulae for radius of curvature and radius of Torsion, Serret-Frenet's formula. Theorem on curvature and Torsion. In the last of this unit detailed study is given on osculating circle and osculating sphere.

2.2 Principal Normal and Binormal

All the normals to a given curve at any point lie in the normal plane. Two nromals namely principal normal and binormal are significant and defined in this section.

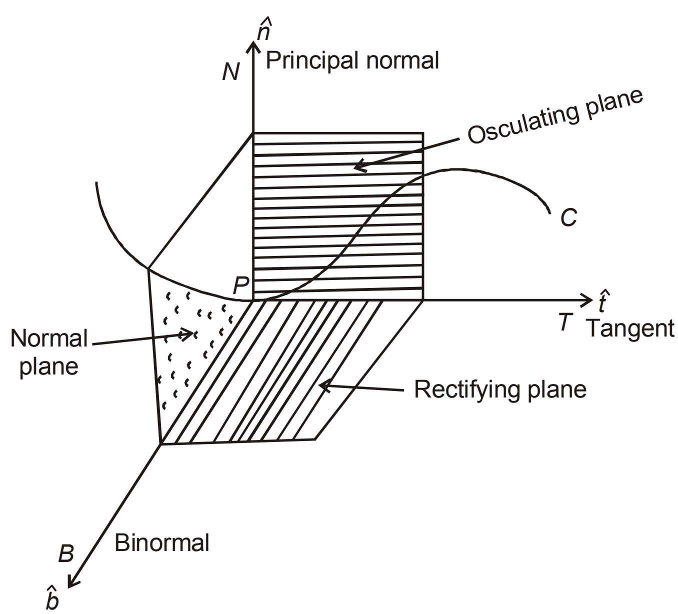


Fig. 2.1

2.2.1 Principal normal :

The principal normal at any point P of a given curve C is defined as the normal which lies in the osculating plane at P.

From the above definition it is clear that the principal normal is the line of intersection of the normal plane and osculating plane because being normal it must lie in normal plane and being principal normal it must lie in osculating plane.

The unit vector along principal normal shall be denoted by $\hat{n}$.

2.2.2 Binormal :

The binormal at any point P of *a* curve C is defined as the normal which is perpendicular to the osculating plane.

From the above definition it in clear that binormal is perpendicular to principal normal because the perpendicular to osculating plane and the latter lies in the osculating plane.

The unit vector along the binormal shall be denoted by $\hat{b}$.

2.2.3 The fundamental unit vectors $\hat{t}, \hat{n}, \hat{b}$:

We know that principal normal and binormal are perpendicular to each other and both these normals are perpendicular to $\hat{t}$. Hence these three form a triad of mutually perpendicular unit vectors such that $\hat{t}, \hat{n}, \hat{b}$ form a right handed orthogonal system of axes.

Therefore $\quad \hat{t}\times\hat{n}=\hat{b},\ \hat{n}\times\hat{b}=\hat{t},\ \hat{b}\times\hat{t}=\hat{n}$

$$\hat{n}\cdot\hat{b}=0,\ \hat{b}\cdot\hat{t}=0,\ \hat{t}\cdot\hat{n}=0 \qquad .....(2.2.1)$$

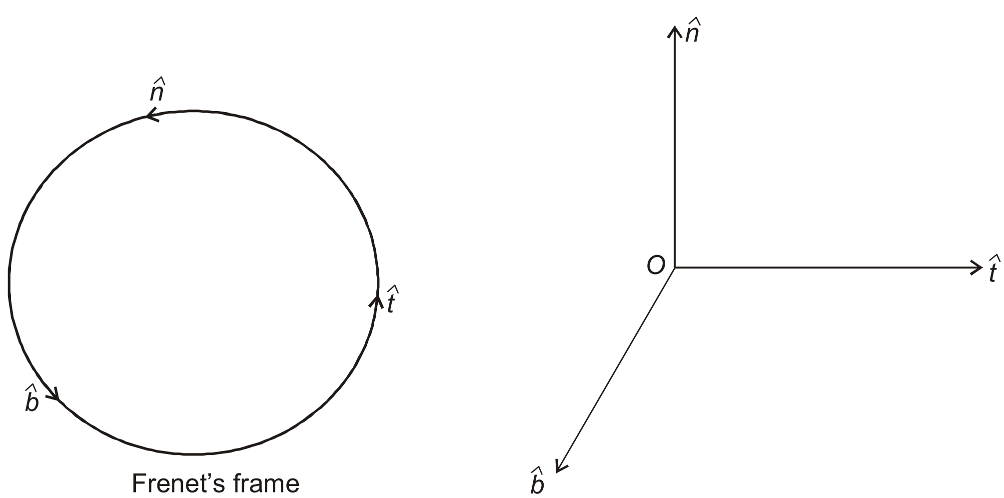


Fig. 2.2

2.2.4 Direction cosines of the tangent, principal normal and binormal :

We will denote the direction-cosines of tangent by (l_1, m_1, n_1) of principal normal by (l_2, m_2, n_2) and of binormal by (l_3, m_3, n_3).

(i) when the parameter is arc length 's'

We know that unit tangent vector $\hat{t}$ is given by r', we have

$$\hat{t} = r' = \frac{d\vec{r}}{ds} = \frac{dx}{ds}\,\hat{i} + \frac{dy}{ds}\,\hat{j} + \frac{dz}{dz}\,\hat{k} \qquad .....(2.2.2)$$

Hence $$l_1 = \frac{dx}{ds},\ m_1 = \frac{dy}{ds},\ n_1 = \frac{dz}{ds} \qquad .....(2.2.3)$$

The binormal is perpendicular to the osculating plane. Equation of osculating plane is given by

$$\left[\vec{R} - \vec{r},\ \vec{r}',\ \vec{r}''\right] = 0 \quad \text{or} \quad (\vec{R} - \vec{r}) \cdot (\vec{r}' \times \vec{r}'') = 0$$

Therefor the vector $\vec{r}' \times \vec{r}''$ is normal to the osculating plane. This implies that binormal is parallel to the vector $\vec{r}' \times \vec{r}''$.

Hence $$\hat{b} = \frac{\vec{r}' \times \vec{r}''}{|\vec{r}' \times \vec{r}''|} \qquad .....(2.2.4)$$

$\Rightarrow$ $$l_3 = \frac{y'z'' - z'y''}{\sqrt{\Sigma\,(y'z'' - z'y'')^2}},\quad m_3 = \frac{z'x'' - x'z''}{\sqrt{\Sigma\,(y'z'' - z'y'')^2}},$$

$$n_3 = \frac{x'y'' - y'x''}{\sqrt{\Sigma\,(y'z'' - z'y'')^2}} \qquad .....(2.2.5)$$

Further $$\hat{n} = \hat{b} \times \hat{t} = \frac{(\vec{r}' \times \vec{r}'') \times \vec{r}'}{|\vec{r}' \times \vec{r}''|} = \frac{(\vec{r}' \cdot \vec{r}')\,\vec{r}'' - (\vec{r}'' \cdot \vec{r}')\,\vec{r}'}{|\vec{r}' \times \vec{r}''|} = \frac{\vec{r}''}{|\vec{r}' \times \vec{r}''|}$$

$\left(\text{since } \vec{r}' \cdot \vec{r} = 1 \text{ and } \vec{r}' \cdot \vec{r}'' = 0\right)$

Hence, $$l_2 = \frac{x''}{\sqrt{\Sigma\,(y'z'' - z'y'')^2}},\quad m_2 = \frac{y''}{\sqrt{\Sigma\,(y'z'' - z'y'')^2}},$$

$$n_2 = \frac{z''}{\sqrt{\Sigma\,(y'z'' - z'y'')^2}}, \qquad .....(2.2.6)$$

(ii) when the parameter is 't'.

Here $$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r}' = x'\hat{i} + y'\hat{j} + z'\hat{k} = \Sigma\, x'\hat{i}$$

$$\vec{r}'' = \Sigma x''\hat{i}$$

$\therefore$ $$\vec{r}' \times \vec{r}'' = \Sigma\,(y'z'' - z'y'')\,\hat{i} \qquad .....(2.2.7)$$

Hence the principal normal being parallel to the vector

$$(\vec{r}' \times \vec{r}'') \times \vec{r}'$$

i.e. $$\left[\Sigma\,(y'z'' - z'y'')\,\hat{i}\right] \times \Sigma\,(x'\hat{i})$$

i.e. $$\Sigma\,\{z'(z'x'' - x'z'') - y'(x'y'' - y'x'')\}\,\hat{i}$$

The direction ratios of it are

$$z'(z'x''-x'z'')-y'(x'y''-y'x'');$$
$$x'(x'y''-y'x'')-z'(y'z''-z'y'');$$
and
$$y'(y'z''-z'y'')-x'(z'x''-x'z'')$$

Since the binormal is parallel to the vector $\vec{r}'\times\vec{r}''$ *i.e.,* to $\sum(y'z''-z'y'')\,\hat{i}$, its direction ratios are

$$y'z''-z'y'',\ z'x''-x'z'',\ x'y''-y'x''. \qquad .....(2.2.8)$$

2.2.5 Self-learning exercise-1

1. Principal normal lies in the .... plane

2. .... is perpendicular to the osculating plane.

3. Give the formula for $\hat{b}$ and $\hat{n}$.

2.3 The three fundamental planes

At each point of the curve there is a triad of orthogonal unit vectors which determine three fundamentals planes as shown in the figure (2.1) which contains two of these, the third being the normal to that plane and which are mutually perpendicular.

2.3.1 Osculating plane :

The plane through P containing $\hat{t}$ and $\hat{n}$ whose normal is therefor $\hat{b}$ is called osculating plane whose equation is given as $(\vec{R}-\vec{r})\cdot\hat{b}=0$.

In cartesian coordinates let (X, Y, Z) be a current point and (x, y, z) the point whose osculating plane is determined.

$$\vec{R}=X\hat{i}+Y\hat{j}+Z\hat{k},\ \vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

Again let (l_r, m_r, n_r) $(r = 1, 2, 3)$ be the direction ratios of the tangent, principal normal and binormal so that

$$\hat{t}=l_1\hat{i}+m_1\hat{j}+n_1\hat{k},$$
$$\hat{n}=l_2\hat{i}+m_2\hat{j}+n_2\hat{k},$$
$$\hat{b}=l_3\hat{i}+m_3\hat{j}+n_3\hat{k}.$$

Substituting the values of $\vec{R}$, $\vec{r}$ and $\hat{b}$ in $(\vec{R}-\vec{r})\cdot\hat{b}=0$

$$\left[(X-x)\hat{i}+(Y-y)\hat{j}+(Z-z)\hat{k}\right]\cdot\left[l_3\hat{i}+m_3\hat{j}+n_3\hat{k}\right]=0$$

or
$$l_3(X-x)+m_3(Y-y)+n_3(Z-z)=0 \qquad .....(2.3.1)$$

2.3.2 Normal plane :

The plane through P containing $\hat{b}$ and $\hat{n}$ whose normal is therefore $\hat{t}$ is called normal plane whose equation in given by $\left(\vec{R}-\vec{r}\right)\cdot\hat{t}=0$.

In cartesian coordinates : Substituting the values of $\vec{R}, \vec{r}$ and $\hat{t}$ in

$$(\vec{R}-\vec{r})\cdot\hat{t}=0$$

$$\left[(X-x)\hat{i}+(Y-y)\hat{j}+(Z-z)\hat{k}\right]\cdot\left[l_1\hat{i}+m_1\hat{j}+n_1\hat{k}\right]=0$$

or $$l_1(X-x)+m_1(Y-y)+n_1(Z-z)=0 \qquad .....(2.3.2)$$

2.3.3 Rectifying plane :

The plane through P containing $\hat{b}$ and $\hat{t}$ whose normal is therefore $\hat{n}$ is called rectifying plane whose equation is given by $(\vec{R}-\vec{r})\cdot\hat{n}=0$.

In cartesian coordinates : Substituting the values of $\vec{R}$, $\vec{r}$ and $\hat{n}$ in $(\vec{R}-\vec{r})\cdot\hat{n}=0$

$$\left[(X-x)\hat{i}+(Y-y)\hat{j}+(Z-z)\hat{k}\right]\cdot\left[l_2\hat{i}+m_2\hat{j}+n_2\hat{k}\right]=0$$

or $$l_2(X-x)+m_2(Y-y)+n_2(Z-z)=0 \qquad .....(2.3.3)$$

2.3.4 Equations of principal normal and binormal :

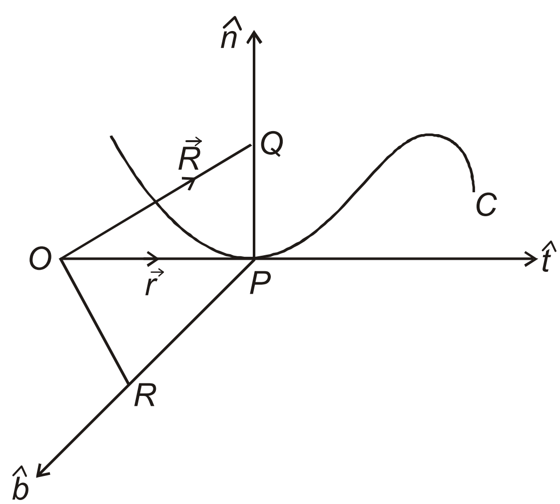


Fig. 2.3

Let $\vec{r}$ be the position vector of any point P on the curve C referred to O as origin. Also let $\vec{R}$ be the position vector of a current point Q on the principal normal.

We have $\overrightarrow{OP}=\vec{r}, \overrightarrow{OQ}=\vec{R}, \overrightarrow{PQ}=\lambda\hat{n},$ since $\hat{n}$ is the unit vector along the principal normal and λ is some scalar.

Now $$\overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{PQ}$$

i.e. $$\vec{R}=\vec{r}+\lambda\hat{n} \qquad .....(2.3.4)$$

which is the required equation of the principal normal at the point P of the curve C.

Similarly, if $\vec{R}$ is the position vector of a current point R on binormal, then equation of the binormal at the point P on the curve C is given by

$$\vec{R}=\vec{r}+\mu\hat{b}, \qquad .....(2.3.5)$$

where μ is a scalar.

2.3.5 Self-learning exercise-2

1. Define osculating plane.
2. Define normal plane.
3. Define rectifying plane.
4. Write equations of principal normal.
5. Write equation of binormal.

2.4 Curvature and Torsion

2.4.1 Curvature :

The rate of change of the direction of tangent with respect to the arc length '*s*' as the point $P(\vec{r})$ moves along the curve is called curvature vector of the curve whose magnitude is denoted by *k* (kappa) called the curvature at *P*.

Hence $$k = \left|\frac{d\hat{t}}{ds}\right| = |\hat{t}'| = |\vec{r}''| \qquad .....(2.4.1)$$

Radius of curvature : The reciprocal of the curvature is called the radius of curvature and is denoted by ρ.

$\therefore$ $$\rho = \frac{1}{k} \qquad .....(2.4.2)$$

Curvature at a point : Let *P* and *Q* be two neighbouring points on a curve such that *PQ* = δ*s*, where *OP* = *s* and the unit tangents at these point be denoted by $\hat{t}$ and $\hat{t} + \delta\hat{t}$ which makes angle ψ and ψ + δψ with a fixed direction.

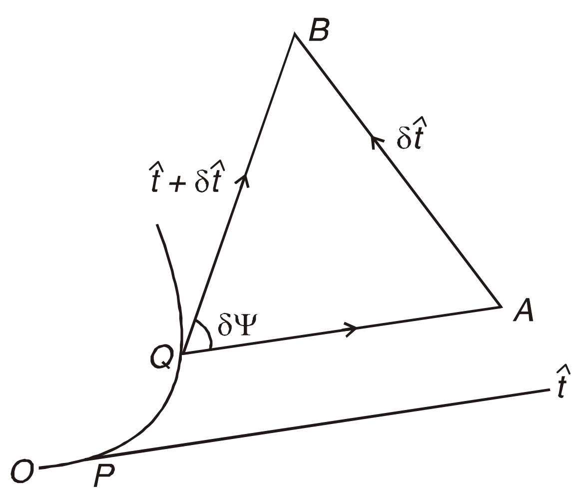


Fig. 2.4

Through *Q* draw vector $\overrightarrow{QA}$ parallel to $\hat{t}$. If the vectors $\overrightarrow{QA}$ and $\overrightarrow{QB}$ are respectively $\hat{t}$ and $\hat{t} + \delta\hat{t}$ then $|\overrightarrow{QA}| = |\overrightarrow{QB}| = 1$ and the angle between tham is δψ.

Also $\qquad \overrightarrow{QB} = \overrightarrow{QA} + \overrightarrow{AB}$ or $\overrightarrow{AB} = \overrightarrow{QB} - \overrightarrow{QA} = \delta\hat{t}$

Now from isosceles triangle QAB.

$$\overrightarrow{AB} = 2\overrightarrow{QA}\sin\frac{\delta\psi}{2}$$

$$|\overrightarrow{AB}| = 2|\overrightarrow{QA}|\sin\frac{\delta\psi}{2}$$

$\therefore$ $$|\delta\hat{t}| = 2\sin\frac{\delta\psi}{2} \quad \text{or} \quad \left|\frac{\delta\hat{t}}{\delta\psi}\right| = \frac{\sin\delta\psi/2}{\delta\psi/2}$$

$$\lim_{\delta\psi\to 0}\left|\frac{\delta\hat{t}}{\delta\psi}\right| = \lim_{\delta\psi\to 0}\frac{\sin\delta\psi/2}{\delta\psi/2} \quad \text{or} \quad \left|\frac{d\,\hat{t}}{d\psi}\right| = 1.$$

The curvature at point P

$$k = \lim_{\delta\psi\to 0}\frac{\delta\psi}{\delta s} = \frac{d\psi}{ds} = \left|\frac{d\psi}{dt}\right|\left|\frac{d\hat{t}}{ds}\right| = \left|\frac{d\hat{t}}{ds}\right| = |\hat{t}'| = |\vec{r}''|. \qquad .....(2.4.3)$$

2.4.2 Torsion :

The rate of change of the direction of binormal with respect to arc length as the point P moves along the curve is called the torsion vector of the curve whose magnitude is denoted by τ called the torsion at P.

Hence $$\tau = \left|\frac{d\hat{b}}{ds}\right| = |\hat{b}'| \qquad .....(2.4.4)$$

Radius of Torsion : The reciprocal of the torsion is called the radius of torsion and is denoted by $\sigma = \frac{1}{\tau}$.

Torsion at a point :

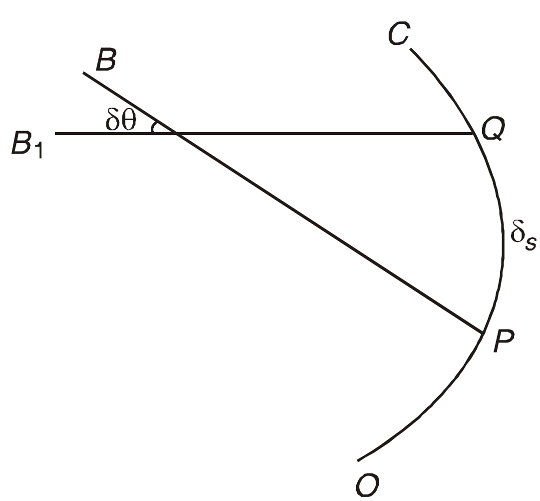


Fig. 2.5

Let P and Q be two neighbouring points on a curve such that $PQ = \delta s$ where $OP = s$ and the unit binormals at these points be denoted by $\hat{b}$ and $\hat{b} + \delta \hat{b}$ and $\delta\theta$ be the angle between these vectors.

Average rate of change of direction of the osculating plane over the arc $PQ = \dfrac{\delta\theta}{\delta s}$.

The torsion of the curve at P

$$\tau = \lim_{\delta s \to 0} \frac{\delta\theta}{\delta s} = \frac{d\theta}{ds}. \qquad \text{.....(2.4.5)}$$

Aliter :

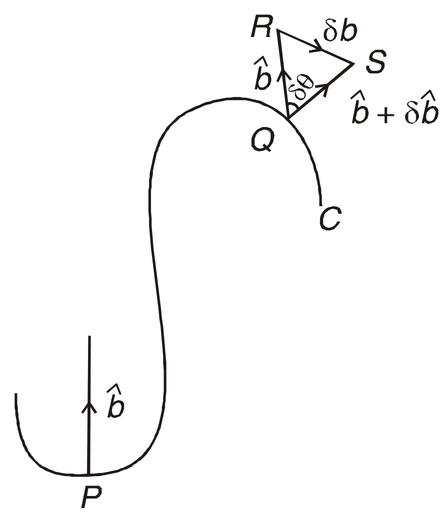


Fig. 2.6

From isosceles triangle QRS, we have

$$RS = \left| \overrightarrow{RS} \right| = 2 \left| \overrightarrow{QR} \right| \sin\left(\frac{\delta\theta}{2}\right)$$

$$\Rightarrow \qquad \left| \delta\hat{b} \right| = 2\sin\left(\frac{\delta\theta}{2}\right)$$

$$\Rightarrow \qquad \left| \frac{\delta\hat{b}}{\delta\theta} \right| = \left(\frac{\sin \delta\theta/2}{\delta\theta/2} \right)$$

Hence

$$\left| \frac{\delta\hat{b}}{\delta\theta} \right| = \lim_{\delta\theta \to 0} \left(\frac{\sin \delta\theta/2}{\delta\theta/2} \right) = 1$$

$\therefore$ Torsion τ at P is

$$\tau = \left| \frac{d\hat{b}}{ds} \right| = \left| \frac{d\hat{b}}{d\theta} \right| \cdot \left| \frac{d\theta}{ds} \right|$$

$$= \frac{d\theta}{ds} = \frac{1}{\sigma}. \qquad \text{.....(2.4.6)}$$

2.4.3 Skew-curvature :

The rate of change of the direction of principal normal with respect to arc length as the point P moves along the curve is called the skew curvature vector and its magnitude is given by

$$\left(\frac{d\hat{n}}{ds}\right)=\sqrt{\left(\kappa^2+\tau^2\right)}. \qquad .....(2.4.7)$$

2.4.4 Self learning exercise-3

1. Give the formula for curvature.

2. Give the formula for torsion.

2.5 Serret-Frenet Formulae

Arc derivative of three unit vectors $\hat{t},\hat{n},\hat{b}$ are known as Serret-Frenet formulae as given below

1. $\hat{t}'=\dfrac{d\hat{t}}{ds}=\kappa\,\hat{n}.$

2. $\hat{n}'=\dfrac{d\hat{n}}{ds}=\tau\,\hat{b}-\kappa\,\hat{t}.$

3. $\hat{b}'=\dfrac{d\hat{b}}{ds}=-\tau\,\hat{n}.$ .....(2.5.1)

Proof : 1. Since $\hat{t}\cdot\hat{t}=t^2=1.$

Different with respect to arc length s

$$2\hat{t}\cdot\frac{d\hat{t}}{ds}=0$$

$$\Rightarrow \qquad \hat{t}\cdot\frac{d\hat{t}}{ds}=0$$

$\Rightarrow$ $\dfrac{d\hat{t}}{ds}$ is perpendicular to $\hat{t}$. .....(2.5.2)

But $\dfrac{d\hat{t}}{ds}=\hat{t}'=\vec{r}''$

The equation of osculating plane at a point P on a curve is

$$\left[\vec{R}-\vec{r},\ \vec{r}',\vec{r}''\right]=0 \qquad .....(2.5.3)$$

$\Rightarrow$ $\vec{r}''=\hat{t}'$ in the osculating plane.

Therefore $\hat{t}'$ is perpendicular to $\hat{b}$ and it is also perpendicular to $\hat{t}$.

Hence, $\hat{t}'$ is parallel to $\hat{b}\times\hat{t}$ *i.e.* along the principal normal $\hat{n}$. Therefore $\hat{t}'$ is proportional to $\hat{n}$, *i.e.,*

$$\frac{d\hat{t}}{ds} = \hat{t}' = \pm\,\kappa\,\hat{n}$$

$$\Rightarrow \qquad \hat{t}' = \kappa\,\hat{n} \qquad .....(2.5.4)$$

(The direction of the principal normal is so chosen that curvature κ is always positive)

2. Since $\hat{b}\cdot\hat{b} = 1$.

Differentiating with respect to 's', we have

$$\hat{b}\cdot\frac{d\hat{b}}{ds} = 0 \qquad .....(2.5.6)$$

This implies $\frac{d\hat{b}}{ds}$ is perpendicular to $\hat{b}$ and thus $\frac{d\hat{b}}{ds}$ lies in osculating plane.

Also $\hat{b}\cdot\hat{t} = 0,$ on differentiating with respect to s, we get

$$\frac{d\hat{b}}{ds}\cdot\hat{t} + \hat{b}\cdot\frac{d\hat{t}}{ds} = 0$$

or $$\frac{d\hat{b}}{ds}\cdot\hat{t} + \hat{b}\cdot\kappa\,\hat{n} = 0$$

or $$\frac{d\hat{b}}{ds}\cdot\hat{t} = 0 \qquad \left(\text{as } \hat{b}\cdot\hat{n}\right)$$

$$\Rightarrow \qquad \frac{d\hat{b}}{ds} \text{ is perpendicular to } \hat{t}. \qquad .....(2.5.7)$$

Thus $\frac{d\hat{b}}{ds}$ is perpendicular to the vector $\hat{b}$. This implies that $\frac{d\hat{b}}{ds}$ is parallel to $\hat{b}\times\hat{t}$ *i.e.* $\frac{d\hat{b}}{ds}$ is parallel to $\hat{n}$.

Hence $$\frac{d\hat{b}}{ds} = \pm\,\tau\,\hat{n}. \qquad .....(2.5.8)$$

Taking $$\frac{d\hat{b}}{ds} = -\,\tau\,\hat{n} \qquad .....(2.5.9)$$

(In the right hand screw system, by convention, τ is negative).

3. Since $\hat{n} = \hat{b}\times\hat{t}$

$$\therefore \qquad \frac{d\hat{n}}{ds} = \frac{d\hat{b}}{ds}\times\hat{t} + \hat{b}\times\frac{d\hat{t}}{ds} \qquad .....(2.5.10)$$

Using $$\frac{d\hat{t}}{ds} = \kappa\,\hat{n} \qquad \text{(formula 1)}$$

and $$\frac{d\hat{b}}{ds} = -\tau\,\hat{n} \qquad \text{(formula 2)},$$

we get $\quad \dfrac{d\hat{n}}{ds} = -(\tau \hat{n}\hat{t} + \hat{b} \times \kappa\, \hat{n})$

$$= \tau\, \hat{b} - \kappa\, \hat{t} \qquad (\because\ \hat{n} \times \hat{t} = -\hat{b} \ \text{ and } \ \hat{b} \times \hat{n} = -\,\hat{t}) \qquad .....(2.5.11)$$

The Serret-Frenet formulae can be put in the matrix form as follows :

$$\begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} \hat{t}' \\ \hat{n}' \\ \hat{b}' \end{bmatrix} \qquad .....(2.5.12)$$

2.5.1 Theorems on curvature and rorsion :

Theorem 1. *The necessary and sufficient condition for the curve to be a straight line is that* $\kappa = 0$ *at all points of the curve.*

Proof : Necessary condition : Given the curve is a straight line.

To prove $\quad \kappa = 0$

We know that the vector equation of a straight line is

$$\vec{r} = \vec{a} + \vec{s}\,\vec{c}, \qquad .....(1)$$

where $\vec{a}$ and $\vec{c}$ are constant vectors and s be the measure of the length of the arc from the point whose position vector is $\vec{a}$.

Differentiating with respect to s, we get

$$\vec{r}' = \vec{c} \Rightarrow \vec{r}'' = 0 \qquad .....(2)$$

Also by definition $\quad \kappa = \left|\dfrac{d\hat{t}}{ds}\right| = |\,\hat{t}'\,| = |\,\vec{r}''\,| = 0$

Hence the condition is necessary.

Sufficient condition.

Given $\kappa = 0$.

To prove curve is a straight line.

Here $\quad \kappa = 0$

$\therefore \quad \vec{r}'' = 0 \qquad .....(3)$

On integration, we have

$$\vec{r}' = \vec{c} \ \text{ (a constant vector)} \qquad .....(4)$$

Again, integrating, we get

$$\vec{r} = \vec{a} + \vec{s}\,\vec{c}, \qquad .....(5)$$

where $\vec{a}$ is another constant vector.

Clearly $\vec{r} = \vec{a} + \vec{s}\,\vec{c}$ denotes a straight line.

Hence the condition is sufficient.

Theorem 2. *The necessary and sufficient condition that a given curve be a plane curve is that* $\tau = 0$ *at all point of the curve or in other words* $\left[\vec{r}'\,\vec{r}''\,\vec{r}'''\right] = 0$

Proof : Necessary condition :

Given the given curve is a plane curve.

To prove $\tau = 0$.

If the given curve be a plane curve then we know that tangent and normal at all points lie in the plane of the curve. (It means that plane of the curve is the osculating plane at all points of the curve.)

Hence the unit normal *i.e.* binormal $\hat{b}$ is same at all points which means that $\hat{b}$ is a constant vector both in magnitude and direction and as such

$$\hat{b}' = \frac{d\hat{b}}{ds} = \vec{0}$$

or $$-\tau\,\hat{n} = \vec{0}$$

or $$\tau = 0 \quad \text{(by Serret-Frenet formula)} \qquad .....(1)$$

Hence the condition is necessary.

Sufficient condition :

Given $\tau = 0$.

To prove the curve is a plane curve.

If $$\tau = 0$$

$$\Rightarrow \qquad \frac{d\hat{b}}{ds} = -\tau\,\hat{n} = 0 \qquad .....(2)$$

and hence $\hat{b}$ is a constant vector *i.e.* the direction of binormal is same at all points of the curve.

This means the osculating plane is same at all points of the curve *i.e.* osculating plane contains the curve. Hence the curve must be a plane curve.

Hence the condition is sufficient.

Theorem 3. *If the tangent and the binormal at a point of a curve make angles* θ, ϕ *respectively with a fixed direction, then*

$$\frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi} = \pm\frac{\kappa}{\tau}.$$

Proof : Let $\hat{a}$ denotes a unit vector in the fixed direction, then by the given condition, we have

$$\cos\theta = \hat{a}\cdot\hat{t} \qquad .....(1)$$

and $$\cos\phi = \hat{a}\cdot\hat{b} \qquad .....(2)$$

$$\therefore \qquad -\sin\theta\frac{d\theta}{ds} = \hat{a}\cdot\hat{t}'$$

$$= \hat{a}\cdot\kappa\,\hat{n} = \kappa\left(\hat{a}\cdot\hat{n}\right) \qquad .....(3)$$

and $$-\sin\phi\frac{d\phi}{ds}=\hat{a}\cdot\hat{b}' \;=\hat{a}\cdot(\pm\tau\,\hat{n}) \;=+\,\tau\,(\hat{a}\cdot\hat{n}) \qquad .....(2)$$

from (1) and (2) we have $$\frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi}=\pm\frac{\kappa}{\tau}.$$

Hence proved.

Theorem 4. *The principal normals at consecutive points of a curve do not intersect unless* $\tau = 0$.

Proof : Let the position vectors of two consecutive points P and Q on a curve be $\vec{r}$ and $\vec{r}+d\,\vec{r}$ and let the principal normals at these points be $\hat{n}$ and $\hat{n}+d\,\hat{n}$ respectively.

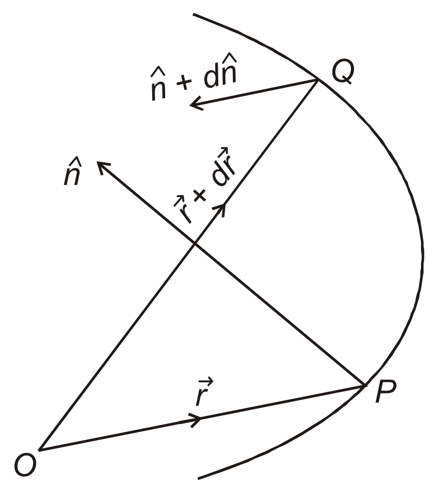


Fig. 2.7

In order to prove that the principal normals at these points may intersect, it is necessary that the three vectors $d\,\vec{r},\hat{n},\hat{n}+d\,\hat{n}$ are coplanar.

These vectors are coplanar if

$$\left[d\,\vec{r},\hat{n},\hat{n}+d\hat{n}\right]=0$$

or $$\left[d\,\hat{r},\hat{n},d\,\hat{n}\right]=0 \qquad .....(1)$$

or $$\left[\vec{r},\hat{n},n'\right]=0$$

or $$\left[\hat{t},\hat{n},\tau\,\hat{b}-\kappa\,\hat{t}\right]=0$$

or $$\left[\hat{t},\hat{n},\tau\,\hat{b}\right]=0$$

or $$\tau\left[\hat{t},\hat{n},\hat{b}\right]=0$$

$\Rightarrow$ $$\tau=0$$

since $$\left[\hat{t},\hat{n},\hat{b}\right]=1.$$

Theorem 5. *Prove that*

(i) $\kappa = \dfrac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$

(ii) $\tau = \dfrac{|\vec{r}'\ \vec{r}''\ \vec{r}'''|}{|\vec{r}' \times \vec{r}''|^2}$

Proof : *(i)* We know that $\vec{r}' = \hat{t}$ .....(1)

$\therefore$ $\vec{r}'' = \hat{t} = \kappa\,\hat{n}$ (by Serret-Frenet formula) .....(2)

$\therefore$ $\vec{r}' \times \vec{r}'' = \kappa\,(\hat{t} \times \vec{n}) = \kappa\,\hat{b}$ .....(3)

Again $\vec{r}' = \dfrac{d\,\vec{r}}{dt} = \dfrac{d\,\vec{r}}{ds} \cdot \dfrac{ds}{dt} = \dot{s}r'$ .....(4)

$\vec{r}'' = \ddot{s}\,\vec{r}' + \dot{s}\,\vec{r}''\,\dot{s}$

$= \ddot{s}\,\vec{r}' + \dot{s}^2\,\vec{r}''$ .....(5)

from (4) and (5) $\vec{r}' \times \vec{r}'' = \dot{s}\,\vec{r}' \times (\ddot{s}\,\vec{r}' + \dot{s}^2\,\vec{r}'')$

$\vec{r}' \times \vec{r}'' = \dot{s}^3 (\vec{r}' \times \vec{r}'')$ .....(6)

from (3) $|\vec{r}' \times \vec{r}''| = \kappa\,|\hat{b}| = \kappa$ .....(7)

$\therefore$ $\kappa = |\vec{r}' \times \vec{r}''|$

$$\frac{|\vec{r}' \times \vec{r}''|}{s'^3} = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$$

because from (4) $|\vec{r}| = \dot{s}\,|\vec{r}'| = \dot{s}.1 = \dot{s}$

or $\vec{r}' = \hat{t}$ is unit vector.

Again from equation (3) $\vec{r}' \times \vec{r}'' = \kappa\,\hat{b}$

Differentiating again with respect to s, we have

$\vec{r}'' \times \vec{r}'' + \vec{r}' \times \vec{r}''' = \kappa'\,\hat{b} + \kappa\,\hat{b}'$

or $0 + \vec{r}' \times \vec{r}''' = \kappa'\,\hat{b} - \kappa\,\tau\,\hat{n}$ $(\because\ \hat{b} = -\tau\,\hat{n})$ .....(8)

Now $\left[\vec{r}'\ \vec{r}''\ \vec{r}'''\right] = \vec{r}'' \cdot \left(\vec{r}''' \times \vec{r}'\right)$

$= \vec{r}'' \cdot \left(\vec{r}' \times \vec{r}'''\right)$

$= -\kappa\,\hat{n} \cdot \left(\kappa'\,\hat{b} - \kappa\,\tau\,\hat{n}\right)$ [from 8]

$\left[\vec{r}'\ \vec{r}''\ \vec{r}'''\right] = \kappa^2\,\tau$ .....(9)

$\Rightarrow$ $\tau = \dfrac{\left[\vec{r}'\ \vec{r}''\ \vec{r}'''\right]}{\kappa^2}$

$= \dfrac{\left[\vec{r}'\ \vec{r}''\ \vec{r}'''\right]}{\left[\vec{r}' \times \vec{r}''\right]^2}$

$$\therefore \qquad \tau = \frac{\left[\vec{r}'\ \vec{r}''\ \vec{r}'''\right] / \dot{s}^6}{\left[\vec{r}' \times \vec{r}''\right]^2 / \left(\dot{s}^3\right)^2}$$

$$\text{or} \qquad \tau = \frac{\left[\vec{r}'\ \vec{r}''\ \vec{r}'''\right]}{\left[\vec{r}' \times \vec{r}''\right]^2}$$

2.5.2 Examples :

***Ex.*1.** *For the curve* $x = 3u,\ y = 3u^2,\ z = 2u^3.$

Prove that $\rho = -\sigma = \frac{3}{2}\left(1+2u^2\right)^2$

Sol. Here $\vec{r} = (3u, 3u^2, zu^3)$

$$\therefore \qquad \dot{\vec{r}} = (3, 6u, 6u^2) = 3\,(1, 2u, 2u^2)$$

$$\ddot{\vec{r}} = 3\,(0, 2, 4u)$$

$$\text{or} \qquad \ddot{\vec{r}} = 6\,(0, 1, 2u)$$

$$\dddot{\vec{r}} = 6\,(0, 0, 2)$$

$$|\,\dot{\vec{r}}\,| = 3\sqrt{1+4u^2+4u^4} = 3\left(1+2u^2\right)$$

$$\begin{aligned}\dot{\vec{r}} \times \ddot{\vec{r}} &= 18\,[1, 2u, 2u^2] \times [0, 1, 2u] \\ &= 18\,[4u^2 - 2u^2, 0 - 2u, 1 - 0] \\ &= 18\,[2u^2, -2u, 1]\end{aligned}$$

$$\therefore \qquad |\,\dot{\vec{r}} \times \ddot{\vec{r}}\,| = 18\sqrt{4u^2+4u^4+1} = 18\left(1+2u^2\right)$$

$$\therefore \qquad \left[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}\right] = 3.6.6\begin{vmatrix} 1 & 2u & 2u^2 \\ 0 & 1 & 2u \\ 0 & 0 & 2 \end{vmatrix} = 216$$

$$\therefore \qquad \kappa = \frac{|\,\dot{\vec{r}} \times \ddot{\vec{r}}\,|}{|\,\dot{\vec{r}}\,|^3} = \frac{18\left(1+2u^2\right)}{27\left(1+2u^2\right)^3} = \frac{2}{3}\,\frac{1}{1\left(1+2u^2\right)^2}$$

$$\tau = \frac{|\,\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}\,|}{|\,\dot{\vec{r}} \times \ddot{\vec{r}}\,|^2} = \frac{216}{18.18\left(1+2u^2\right)^2} = \frac{2}{3}\,\frac{1}{\left(1+2u^2\right)^2}$$

$$\therefore \qquad \rho = \frac{1}{\kappa} = \frac{3}{2}\left(1+2u^2\right)^2;$$

$$\sigma = \frac{1}{\tau} = \frac{3}{2}\left(1+2u^2\right)^2$$

Now for a left hand system $\sigma = -\frac{3}{2}\left(1+2u^2\right)^2$

$$\therefore \qquad \rho = -\sigma = \frac{3}{2}\left(1+2u^2\right)^2$$

Ex.2. *For the curve* $x = a\,(3u - u^3),\ y = 3au^2,\ z = a\,(3u + u^3)$ *show that the curvature and torsion are equal.*

Sol. Here $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

or $\vec{r} = a\left(3u - u^3\right)\hat{i} + 3au^2\hat{j} + a\left(3u + u^3\right)\hat{k}$

Differentiating with respect to s, we get

$$\hat{t} = \frac{d\vec{r}}{ds} = \left[a\left(3-3u^3\right)\hat{i} + 3au\hat{j} + a\left(3+3u^3\right)\hat{k}\right]\frac{du}{ds} \qquad .....(1)$$

or $$\hat{t} = \frac{d\vec{r}}{ds} = a\,[(3-3u^2),\ 6u,\ (3+3u^2)]\ \frac{du}{ds}$$

$$= 3a\,[(1-u^2),\ 2u,\ (1+u^2)]\ \frac{du}{ds}$$

$$\therefore \qquad \hat{t}^2 = 9a^2\,[(1-u^2)^2 + 4u^2 + (1+u^2)^2]\left(\frac{du}{ds}\right)^2$$

or $$1 = 9a^2\,[2\,(1+u^4) + 4u^2]\left(\frac{du}{ds}\right)^2$$

or $$1 = 18a^2\,(1+u^2)^2\left(\frac{du}{ds}\right)^2$$

$$\therefore \qquad \left(\frac{du}{ds}\right) = \frac{1}{3\sqrt{2}\left[a\left(1+u^2\right)\right]}$$

$$\therefore \qquad \hat{t} = \frac{1}{\sqrt{2}}\left(\frac{1-u^2}{1+u^2},\ \frac{2u}{1+u^2},\ 1\right)$$

Differentiating again with respect to s, we get

$$\hat{t}' = \frac{1}{\sqrt{2}}\left[-\frac{4u}{\left(1+u^2\right)^2},\ \frac{2\left(1-u^2\right)}{\left(1+u^2\right)^2},\ 0\right]\left(\frac{du}{ds}\right)$$

$$= \frac{1}{6a}\left[-\frac{4u}{\left(1+u^2\right)^3},\ \frac{2\left(1-u^2\right)}{\left(1+u^2\right)^3},\ 0\right]$$

$\therefore$ $$\kappa^2 = |\hat{t}'|^2 = \frac{1}{36a^2}\left[\frac{16u^2 + 4\left(1-u^2\right)^2}{\left(1+u^2\right)^6}\right]$$

$$= \frac{1}{9a^2}\left[\frac{\left(1+u^2\right)^2}{\left(1+u^2\right)^6}\right] = \frac{1}{9a^2}\frac{1}{\left(1+u^2\right)^4}$$

$\therefore$ $$\kappa = \frac{1}{3a}\frac{1}{\left(1+u^2\right)^2}$$

$\therefore$ $$\rho = 3a\,(1+u^2)^2$$

Now $$\hat{t}' = \kappa\,\hat{n}$$

or $$\hat{n} = \frac{1}{\kappa}\hat{t}'$$

$$= \frac{3a\left(1+u^2\right)^2}{6a}\left[\frac{-4u}{\left(1+u^2\right)^3},\ \frac{2\left(1-u^2\right)^2}{\left(1+u^2\right)^3},\ 0\right]$$

$\therefore$ $$\hat{n} = \frac{1}{2}\left[\frac{-4u}{\left(1+u^2\right)},\ \frac{2\left(1-u^2\right)}{\left(1+u^2\right)},\ 0\right]$$

$$\hat{b} = \hat{t}\times\hat{n}$$

$$= \frac{1}{2\sqrt{2}}\left[\frac{1-u^2}{1+u^2},\ \frac{2u}{1+u^2},\ 1\right]\times\left[-\frac{4u}{1+u^2},\ \frac{2\left(1-u^2\right)}{1+u^2},\ 0\right]$$

$$= \frac{1}{2\sqrt{2}}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1-u^2}{1+u^2} & \frac{2u}{1+u^2} & 1 \\ \frac{-4u}{1+u^2} & \frac{2\left(1-u^2\right)}{1+u^2} & 0 \end{vmatrix}$$

$$= \frac{1}{2\sqrt{2}}\left[0 - \frac{2\left(1-u^2\right)}{1+u^2},\ \frac{-4u}{1+u^2},\ \frac{2\left(1-u^2\right)^2}{\left(1+u^2\right)^2} + \frac{8u^2}{\left(1+u^2\right)^2}\right]$$

or $$\hat{b} = \frac{1}{2\sqrt{2}}\left[-\frac{2(1-u^2)}{1+u^2},\ \frac{-4u}{1+u^2},\ 2\right]$$

$$= -\frac{1}{\sqrt{2}}\left[\frac{1-u^2}{1+u^2},\ \frac{2u}{1+u^2},\ 1\right].$$

Values of $\hat{b}$ and $\hat{t}$ are same except a change of sign.

***Ex.*3.** *Find the radii of curvature and torsion of the helix*

$$x = a\cos\theta,\ y = a\sin\theta,\ z = a\theta\tan\alpha.$$

Sol. Here $$\vec{r} = (a\cos\theta,\ a\sin\theta,\ a\theta\tan\alpha)$$

$$\dot{\vec{r}} = (-a\sin\theta,\ a\cos\theta,\ a\tan\alpha)$$

$$\ddot{\vec{r}} = (-a\cos\theta,\ -a\sin\theta,\ 0)$$

$$\dddot{\vec{r}} = (a\sin\theta,\ -a\cos\theta,\ 0)$$

$\therefore$ $$|\dot{\vec{r}}| = a\sqrt{\sin^2\theta+\cos^2\theta+\tan^2\alpha} = a\sec\alpha$$

$$\dot{\vec{r}}\ \ddot{\vec{r}} = a^2(-\sin\theta,\ \cos\theta,\ \tan\alpha)\times(-\cos\theta,\ -\sin\theta,\ 0)$$

$$= a^2(\sin\theta\tan\alpha,\ -\cos\theta\tan\alpha,\ \sin^2\theta+\cos^2\theta)$$

$$= a^2(\sin\theta\tan\alpha,\ -\cos\theta\tan\alpha,\ 1)$$

$$|\dot{\vec{r}}\ \ddot{\vec{r}}| = a^2\sqrt{\sin^2\theta\tan^2\alpha+\cos^2\theta\tan^2\alpha+1}$$

$$= a^2\sqrt{\tan^2\alpha+1} = a^2\sec\alpha$$

$$|\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}| = a^3\begin{vmatrix}-\sin\theta & \cos\theta & \tan\alpha\\ -\cos\theta & -\sin\theta & 0\\ \sin\theta & -\cos\theta & 0\end{vmatrix}$$

$$= a^3\tan\alpha$$

$\therefore$ $$\kappa = \frac{|\dot{\vec{r}}\times\ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} = \frac{a^2\sec\alpha}{a^3\sec^2\alpha} = \frac{1}{a}\cos^2\alpha$$

$\therefore$ $$\rho = a\sec^2\alpha$$

$$\tau = \frac{\left[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}\right]}{|\dot{\vec{r}}\ \ddot{\vec{r}}|^2}$$

$$= \frac{a^3\tan\alpha}{a^4\sec^2\alpha} = \frac{1}{a}\sin\alpha\cos\alpha$$

$\therefore$ $$\sigma = a\sec\alpha\,\mathrm{cosec}\,\alpha.$$

***Ex.*4.** *Determine the function* $f(\theta)$ *so that*

$$x = a\cos\theta,\ \ y = a\sin\theta,\ \ z = f(\theta);$$

shall be a plane curve.

Sol. Here $\vec{r} = (a\cos\theta, a\sin\theta, f(\theta))$,

$$\dot{\vec{r}} = (-a\sin\theta, a\cos\theta, \dot{f}(\theta)),$$

$$\ddot{\vec{r}} = (-a\cos\theta, -a\sin\theta, \ddot{f}(\theta)),$$

$$\dddot{\vec{r}} = (a\sin\theta, -a\cos\theta, \dddot{f}(\theta)),$$

The condition for a curve to be a plane curve is

$$\left[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}\right] = 0$$

i.e.

$$\begin{vmatrix} -a\sin\theta & a\cos\theta & \dot{f}(\theta) \\ -a\cos\theta & -a\sin\theta & \ddot{f}(\theta) \\ a\sin\theta & -a\cos\theta & \dddot{f}(\theta) \end{vmatrix} = 0$$

Applying $R_1 + R_3$, we get

$$\begin{vmatrix} 0 & 0 & \dot{f}(\theta)+\dddot{f}(\theta) \\ -a\cos\theta & -a\sin\theta & \ddot{f}(\theta) \\ a\sin\theta & -a\cos\theta & \dddot{f}(\theta) \end{vmatrix} = 0$$

or $$a^2\left[\dot{f}(\theta)+\dddot{f}(\theta)\right] = 0$$

or $$\dot{f}(\theta)+\dddot{f}(\theta) = 0$$

or $$f(\theta)+\ddot{f}(\theta) = A$$

or $$D^2 f(\theta) + f(\theta) = A$$

$\therefore$ $$f(\theta) = A + B\sin(\theta + C)$$

***Ex.*5.** Find the radii of curvature and torsion at a point of the curve

$$x^2 + y^2 = a^2,\ \ x^2 - y^2 = az,$$

Sol. The parametric equation of the curve may be given by

$$x = a\cos\theta,\ \ y = a\sin\theta,\ \ z = a\cos 2\theta$$

Therefore $\vec{r} = (a\cos\theta, a\sin\theta, a\cos 2\theta)$

$$\dot{\vec{r}} = a(-\sin\theta, \cos\theta, -2\sin 2\theta)$$

$$\ddot{\vec{r}} = a(-\cos\theta, -\sin\theta, -4\cos 2\theta)$$

$$\dddot{\vec{r}} = a(\sin\theta, -\cos\theta, 8\sin 2\theta)$$

$\therefore$ $$\dot{\vec{r}}\times\ddot{\vec{r}} = a^2(-\sin\theta, \cos\theta, -2\sin 2\theta)\times(-\cos\theta, -\sin\theta, -4\cos 2\theta)$$

or $$\dot{\vec{r}}\times\ddot{\vec{r}} = a^2(-4\cos^3\theta, 4\sin^3\theta, 1)$$

$$\left[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}\right] = a^3 \begin{vmatrix} -\sin\theta & \cos\theta & -2\sin 2\theta \\ -\cos\theta & -\sin\theta & -4\cos 2\theta \\ \sin\theta & -\cos\theta & 8\cos 2\theta \end{vmatrix}$$

Applying $R_1 + R_3$, we get

$$\left[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}\right] = a^3 \begin{vmatrix} 0 & 0 & 6\sin 2\theta \\ -\cos\theta & -\sin\theta & -4\cos 2\theta \\ \sin\theta & -\cos\theta & 8\cos 2\theta \end{vmatrix}$$

$$= a^3 (6 \sin 2\theta) \cdot 1 = 6 a^3 \sin 2\theta$$

Also $$|\dot{\vec{r}}|^2 = a^2 (\sin^2 \theta + \cos^2 \theta + 4 \sin^2 2\theta)$$

$$= a^2 [1 + 4 (1 - \cos^2 2\theta)]$$

$$= a^2 [5 - 4 \cos^2 2\theta]$$

$$|\dot{\vec{r}} \times \ddot{\vec{r}}|^2 = a^4 (5 + 12 \cos^2 2\theta)$$

Therefore $$\kappa^2 = \frac{1}{\rho} = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}{|\dot{\vec{r}}|^6}$$

$$= \frac{a^4\left(5+12z^2/a^2\right)}{a^2\left(5-4z^2/a^2\right)^3}$$

Hence $$\rho^2 = \frac{\left(5a^2-4z^2\right)^3}{a^2\left(5a^2+12z^2\right)}$$

and $$\frac{1}{\sigma} = \tau = \frac{\left[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}\right]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2} = \frac{6a^3 \sin 2\theta}{a^4\left(5+12\cos^2 2\theta\right)}$$

$\therefore$ $$\sigma = \frac{a\left(5+12z^2/a^2\right)}{6\sqrt{1-z^2/a^2}}$$

or $$\sigma = \frac{5a^2+12z^2}{6a\sqrt{a^2-z^2}}.$$

2.6 Osculating circle and osculating sphere

2.6.1 Definitions of osculating circle and osculating sphere :

Osculating circle : If P, Q, R are three points on a curve, the circle, PQR in its limiting position where Q, R tend to P is called the circle of curvature at P and radius of circle is the radius of curvature and is denoted by ρ.

or

The circle which has three point contact with the curve at P is called the osculating circle at a point P on a curve.

To find the centre and radius of circle of curvature :

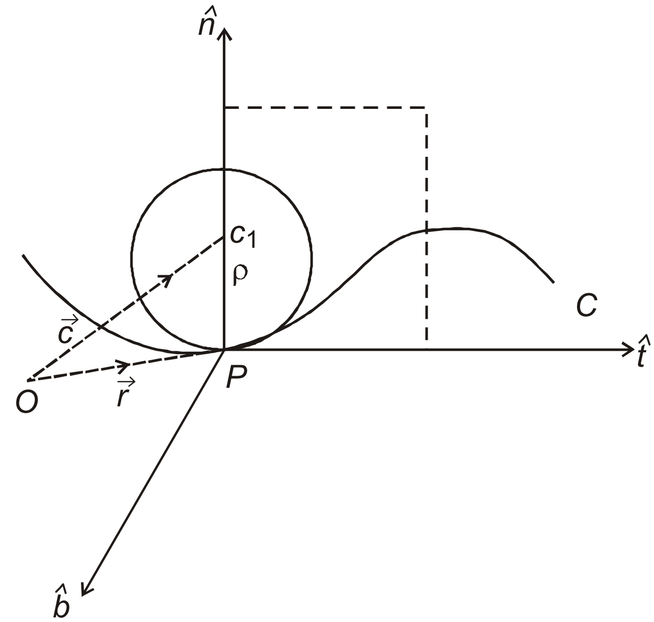


Fig. 2.8

Let $\vec{c}$ be the position vector of the centre C_1 of the osculating circle at point P to the curve C whose equation be $\vec{r} = \vec{r}\,(s)$. Let a be the radius of the circle

$$|\vec{r} - \vec{c}| = a \quad i.e. \quad |\vec{r} - \vec{c}|^2 = a^2 \qquad .....(1)$$

where $\vec{r}$ is the position vector of the point P. The osculating circle is the intersection of the sphere (1) and the osculating plane at P.

The point of intersection of the curve C and sphere (1) are given by

$$F\,(s) \equiv \left[\vec{r}\,(s) - \vec{c}\right]^2 - a^2 = 0$$

The curve will have three point contact if

$$F\,(s) = 0, \quad F'(s) = 0, \quad F''(s) = 0$$

$$F\,(s) = 0 \Rightarrow \left(\vec{r} - \vec{c}\right)^2 = a^2,$$

$$F'(s) = 0 \Rightarrow \left(\vec{r} - \vec{c}\right) \cdot \vec{r}' = 0$$

$$\text{or } \left(\vec{r} - \vec{c}\right) \cdot \hat{t} = 0 \qquad .....(2)$$

$$F''(s) = 0 \Rightarrow \left(\vec{r} - \vec{c}\right) \cdot \vec{r}'' + \vec{r}' \cdot \vec{r}' = 0$$

$$\text{or } \left(\vec{r} - \vec{c}\right) \cdot \kappa\hat{n} + 1 = 0$$

$$\text{or } \left(\vec{r} - \vec{c}\right) \cdot \hat{n} = -\frac{1}{\kappa}$$

or $$\left(\vec{r} - \vec{c}\right) \cdot \hat{n} = -\,\rho \qquad .....(3)$$

Result (2) shows that $\hat{t}$ is orthogonal to $(\vec{r}-\vec{c})$ and as such $(\vec{r}-\vec{c})$ lies in the normal plane at P. Also by definition $(\vec{r}-\vec{c})$ lies in the osculating plane at P. Hence $(\vec{r}-\vec{c})$ lies along the intersection of these two points *i.e.* along the principal normal at P.

$$\therefore \qquad (\vec{r}-\vec{c}) = a\hat{n} \qquad .....(4)$$

where a is a scalar.

Putting (4) in (3), we get

$$a\,\hat{n}\cdot\hat{n} = -\rho \Rightarrow a = -\rho$$

and hence $\qquad (\vec{r}-\vec{c}) = -\rho\hat{n}$

Squaring, we get $\qquad (\vec{r}-\vec{c})^2 = \rho^2$

or $\qquad a^2 = \rho^2$

$$\Rightarrow \qquad \rho = a \qquad .....(5)$$

Above equation gives the radius of curvature.

Also from $\vec{r}-\vec{c} = -\rho\hat{n}$, we get

$$\boxed{\vec{c} = \vec{r} + \rho\hat{n}} \qquad .....(6)$$

Above relation shows that the centre $\vec{c}$ lies on the principal normal at a distance ρ from the point P whose position vector in $\vec{r}$.

Results (5) and (6) give the radius and position vector of the centre of the circle of curvature. (It should also be noted the sign of ρ is always positive.)

Cartesian form : Let (α, β, γ) be the centre of the circle of curvature at a point (x, y, z) of a given curve, and a be its radius.

The equation of the circle can be written as the intersection of

$$\left.\begin{aligned} &\text{Sphere:} && (\xi-\alpha)^2+(\eta-\beta)^2+(\zeta-\gamma)^2 = a^2, \\ &\text{Osculating plane:} && l_3(\xi-\alpha)+m_3(\eta-\beta)+n_3(\zeta-\gamma)=0 \end{aligned}\right\} \qquad .....(1)$$

where l_3, m_3, n_3 being d.c.s of the binormal.

Since the circle (1) has three point contact at (x, y, z) therefore

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = a^2 \qquad .....(2)$$

$$(x-\alpha)\, l_3 + (y-\beta)\, m_3 + (z-\gamma)\, n_3 = 0 \qquad .....(3)$$

On differentiating of (3) with respect to 's', we get

$$\Sigma\,(x-\alpha)\,\frac{dx}{ds} = 0$$

or $$\Sigma\,(x-\alpha)\, l_1 = 0 \qquad .....(4)$$

Again differentiating (4) with respect to 's' we get

$$\sum (x-\alpha)\frac{dl_1}{ds} + \sum l_1\frac{dx}{ds} = 0.$$

Using Serret-Frenet formulae

$$\frac{dl_1}{ds} = \frac{l_2}{\rho} \text{ etc.}$$

in equation (4), we get $\Sigma (x - \alpha)\, l_2 = -\rho$ ......(5)

Squaring and adding equation (3), (4) and (5), we get

$$\Sigma (x - \alpha)^2 = \rho^2 \qquad .....(6)$$

From equation (2), and equation (6), we conclude that

$$a = \rho.$$

Now multiplying (3) by l_3, (4) by l_1 and (5) by l_2 and adding, we have

$$(x - \alpha) = - l_2\rho$$

Hence $\alpha = x + l_2\rho$,

similarly, $\beta = y + m_2\rho$ and $\gamma = n_2\rho + z$ .....(7)

Thus, it is clear that the centre of curvature lies on the principal normal.

2.6.2 Properties of the locus of the centre of circle of curvature :

Property 1 : The tangent to the locus of the centre of curvature lies in the normal plane of the original curve and is inclined to $\hat{n}$ at an angle $\tan^{-1}\left(\frac{\rho\tau}{\rho'}\right)$.

Proof : Let $\vec{r}\ (= \vec{c})$ be position vector of centre of c_1 then

$$\vec{r}_1 = \vec{r} + \rho\hat{n} \qquad .....(1)$$

$$\therefore \qquad \hat{t}_1 = \frac{d\,\vec{r}_1}{ds_1} = \left(\vec{r}\,' + \rho'\hat{n} + \rho\hat{n}'\right)\frac{ds}{ds_1}$$

or $$\hat{t}_1 = \left[\hat{t} + \rho'\hat{n} + \rho\left(\tau\hat{b} - \kappa\hat{t}\right)\right]\frac{ds}{ds_1} \quad \text{(By Serret-Frenet formulae)}$$

or $$\hat{t}_1 = \left(\rho'\hat{n} + \rho\tau\hat{b}\right)\frac{ds}{ds_1} \qquad (\because\ \rho\cdot\kappa = 1) \qquad .....(2)$$

Above relation shows that $\hat{t}_1$ lies in the plane containing $\hat{b}$ and $\hat{n}$ *i.e.* normal plane c.

If α be the angle made by $\hat{t}_1$ with $\hat{n}$, then

$$\hat{t}_1 = \cos\alpha\,\hat{n} + \sin\alpha\,\hat{b} \qquad .....(3)$$

Hence by comparing (2) and (3)

$$\rho'\frac{ds}{ds_1} = \cos\alpha \quad \text{and} \quad \rho\tau\frac{ds}{ds_1} = \sin\alpha;$$

$$\therefore \qquad \tan\alpha = \frac{\rho\tau}{\rho'} \Rightarrow \boxed{\alpha = \tan^{-1}\frac{\rho\tau}{\rho'}}.$$

Property 2 : *If the curvature κ of a curve c is constant, then the curvature κ_1 of c_1 is also constant and its torsion τ_1 varies inversely as τ of the curve c.*

Proof : Now if κ is constant *i.e.* $\rho = (1/\kappa) =$ const., then $\rho' = 0$ and hence from (2)

$$\hat{t}_1 = \rho\,\tau\,\hat{b}\frac{ds}{ds_1} \qquad .....(4)$$

squaring both sides $\quad 1 = \left(\rho\,\tau\,\hat{b}\frac{ds}{ds_1}\right)^2 \qquad \left(\hat{b}\cdot\hat{b} = 1\right)$

or $\quad \rho\,\tau\,\frac{ds}{ds_1} = 1$

Putting in (4) $\quad \boxed{\hat{t}_1 = \hat{b}}.$

Differentiating with respect to s_1,

$$\hat{t}_1' = \hat{b}'\frac{ds}{ds_1}$$

or $\quad \kappa_1\hat{n}_1 = (-\tau\,\hat{n})\left(\frac{1}{\rho\,\tau}\right)$ or $\kappa_1\hat{n}_1 = -\kappa\,\hat{n} \qquad .....(5)$

This relation show that n_1 is parallel to n and if we choose the direction of n_1 opposite to that of n *i.e.* $n_1 = -n$ then from (5) we get $\kappa_1 = \kappa =$ constant as κ is given to be constant.

Thus the curvature of c_1 is also constant.

Again, $\quad \hat{b}_1 = \hat{t}_1 \times \hat{n}_1$

or $\quad \hat{b}_1 = \hat{b}\times(-\hat{n}) = -\left(\hat{b}\times\hat{n}\right) = \hat{n}\times\hat{b} = \hat{t}$

$$\hat{b}_1 = \hat{t}.$$

Differentiating with respect to s_1, we get

$$\hat{b}_1' = \hat{t}'\frac{ds}{ds_1}$$

or $\quad -\tau_1\,n_1' = \kappa\,n\cdot\frac{1}{\rho\tau}$

or $\quad \tau_1\,n = \frac{\kappa^2}{\tau}n \qquad \left(\because\ \frac{1}{\rho} = \kappa,\ -n_1 = n\right)$

∴ $\quad \tau_1 = \frac{\kappa^2}{\tau} = \frac{\text{constant}}{\tau}$

∴ $\quad \tau\tau_1 = \text{constant}.$

Above shows that torsion of c_1 is inversely proportional to torsion of c.

Property 3 : *Principal normal to c is normal to c_1 at the points where curvature is stationary.*

Proof : We know that the position vector $\vec{c}$ of the center of curvature is given by

$$\vec{c} = \vec{r} + \rho\hat{n}$$

Let the locus of $\vec{c}\left(=\vec{r}_1\right)$ be given by

$$\vec{r}_1 = \vec{r} + \rho\hat{n}$$

Differentiating r_1 with respect to s_1, we have

$$\hat{t}_1 = \left\{\hat{t} + \rho'\hat{n} + \rho\hat{n}'\right\}\frac{ds}{ds_1}$$

$$= \left\{\hat{t} + \rho'\hat{n} + \rho\left(\tau\,\hat{b} - \kappa\,\hat{t}\right)\right\}\frac{ds}{ds_1}$$

$$= \left\{\hat{t} + \rho'\hat{n} + \rho\tau\,\hat{b} - \hat{t}\right\}\frac{ds}{ds_1}$$

$$= \left\{\rho'\hat{n} + \rho\tau\,\hat{b}\right\}\frac{ds}{ds_1}$$

$\therefore$ $$\hat{t}_1\cdot\hat{n} = \left\{\rho'\hat{n}\cdot\hat{n} + \rho\,\tau\,\hat{b}\cdot\hat{n}\right\}\frac{ds}{ds_1}$$

$$= \rho'\frac{ds}{ds_1} \qquad\qquad \hat{b}'\cdot\hat{n} = 0$$

$\therefore$ $$\hat{t}_1\cdot\hat{n} = \rho'\frac{ds}{ds_1}$$

If $\quad \kappa = \text{const} \Rightarrow \rho' = 0.$

Therefore $\quad \hat{t}_1\cdot\hat{n} = 0.$

which shows that principal normal is normal to the locus of center of curvature at those points where the curvature is stationary.

2.6.3 Osculating sphere or sphere of curvature :

A sphere which has a four point contact with the curve at a point P is called the osculating sphere at P.

Let c be the centre and R the radius of sphere so that its equation is

$$\left(\vec{r} - \vec{c}\right)^2 = R^2 \qquad .....(1)$$

where $\vec{r}$ is the position vector of point P on the curve.

The points of intersection of the sphere with the curve $\vec{r} = \vec{r}\left(s\right)$ are given by

$$F\left(s\right) = \left[\vec{r}\left(s\right) - \vec{c}\right]^2 - R^2 = 0 \qquad .....(2)$$

The sphere will have a four point contact with the curve if

$$F = 0, F' = 0, F'' = 0, F''' = 0,$$

$F = 0,$ $\quad (\vec{r} - \vec{c})^2 = R^2$

$F' = 0,$ $\quad (\vec{r} - \vec{c}) \cdot \vec{r}' = 0 \quad \text{or} \quad (\vec{r} - \vec{c}) \cdot \hat{t} = 0$ .....(3)

$F'' = 0,$ $\quad (\vec{r} - \vec{c}) \cdot \hat{t}' + \vec{r}' \cdot \hat{t} = 0$

or $\quad (\vec{r} - \vec{c}) \cdot \kappa\hat{n} + \hat{t} \cdot \hat{t} = 0$

or $\quad (\vec{r} - \vec{c}) \cdot \kappa\hat{n} + 1 = 0$

or $\quad (\vec{r} - \vec{c}) \cdot \hat{n} = -\dfrac{1}{\kappa} = -\rho$ .....(4)

$F'' = 0,$ $\quad (\vec{r} - \vec{c}) \cdot \hat{n}' + \vec{r}' \cdot \hat{n} = -\rho'$

or $\quad (\vec{r} - \vec{c}) \cdot (\tau\hat{b} - \kappa\hat{t}) + \hat{t} \cdot \hat{n} = -\rho'$

or $\quad \tau(\vec{r} - \vec{c}) \cdot \hat{b} + \kappa(\vec{r} - \vec{c}) \cdot \hat{t} + \hat{t} \cdot \hat{n} = -\rho'$

or $\quad \tau(\vec{r} - \vec{c}) \cdot \hat{b} + 0 + 0 = -\rho'$ [by (3)]

or $\quad (\vec{r} - \vec{c}) \cdot \hat{b} = \dfrac{-\rho'}{\tau} = -\sigma\,\rho'$ .....(5)

Result (3) shows that $\hat{t}$ is orthogonal to $(\vec{r} - \vec{c})$ and as such it lies in the normal plane at P which contains $\hat{n}$ and $\hat{b}$ and hence it can be expressed as a linear combination of $\hat{n}$ and $\hat{b}$.

$\therefore \quad \vec{r} - \vec{c} = \lambda\hat{n} + \mu\hat{b}$

$\therefore \quad (\vec{r} - \vec{c}) \cdot \hat{n} = \lambda \text{ or } -\rho = \lambda$ [by (4)]

$\quad (\vec{r} - \vec{c}) \cdot \hat{b} = \mu \text{ or } -\sigma\,\rho' = \mu$ [by (5)]

$\therefore \quad (\vec{r} - \vec{c}) = -\rho\hat{n} - \sigma\,\rho'\,\hat{b}$

or $\quad \boxed{\vec{c} = \vec{r} + \rho\hat{n} + \sigma\,\rho'\,\hat{b}}$ .....(6)

Above relation gives us the position vector of the center c of the osculating sphere.

Again $\quad (\vec{r} - \vec{c})^2 = R^2$

or $\quad (-\rho\hat{n} - \sigma\,\rho'\,\hat{b})^2 = R^2$

or $\quad \boxed{\rho^2 + \sigma^2\,\rho'^2 = R^2}$ .....(7)

Above relation gives us the radius R of the sphere of curvature.

Properties of the locus of the counter of sphere of curvature.

(i) $\hat{t}_1, \hat{n}_1, \hat{b}_1$ of c_1 are parallel respectively to $\hat{b}, \hat{n}, \hat{t}$ of c.

Proof : Let c be the original curve, c_1 the locus of the centre of spherical curvature. Let the suffix unity denote quantities belonging to c_1.

The position vector $\vec{c} = (\vec{r}_1 \text{ say})$ of centre of spherical curvature is given by

$$\vec{r}_1 = \vec{r} + \rho\hat{n} + \sigma\,\rho'\hat{b} \qquad .....(1)$$

differentiating with respect to s_1

$$\frac{d\vec{r}_1}{ds_1} = \left(\vec{r}' + \rho'\hat{n} + \rho\hat{n}' + \sigma'\rho'\hat{b} + \sigma\,\rho''\hat{b} + \sigma\,\rho'\hat{b}'\right)\frac{ds}{ds_1}$$

or $$\hat{t}_1 = \left[\hat{t} + \rho'\hat{n} + \rho\left(\tau\hat{b} - \kappa\hat{t}\right) + \left(\sigma'\rho' + \sigma\,\rho''\right)\hat{b} - \sigma\,\rho'\tau\hat{n}\right]\frac{ds}{ds_1}$$

or $$\hat{t}_1 = \left[\hat{t} + \rho\tau\hat{b} - t + \rho'\hat{n} + \left(\sigma'\rho' + \sigma\,\rho''\right)\hat{b} - \rho'\hat{n}\right]\frac{ds}{ds_1}$$

or $$\hat{t}_1 = \left[\left(\frac{\rho}{\sigma}\right) + \sigma'\rho' + \sigma\,\rho''\right]\hat{b}\frac{ds}{ds_1} \qquad .....(2)$$

which shows that t_1 (tangent to c_1) is $\|$ to b.

Squaring (2), we get

$$\left(\frac{ds_1}{ds}\right)^2 = \left[\frac{\rho}{\sigma} + \sigma'\rho' + \sigma\,\rho''\right]^2$$

or $$\frac{ds_1}{ds} = \left[\frac{\rho}{\sigma} + \sigma'\rho' + \sigma\,\rho''\right] = \frac{\rho}{\sigma} + \left(\sigma\,\rho'\right)' \qquad .....(3)$$

from (2) & (3) $$\hat{t}_1 = \hat{b} \qquad .....(4)$$

Differentiating equation (4) with respect to 's_1'.

$$\frac{d\hat{t}_1}{ds_1} = \hat{b}'\frac{ds}{ds_1}$$

or $$\kappa_1\hat{n}_1 = -\tau\hat{n}\frac{ds}{ds_1} \qquad .....(5)$$

which shows that $\hat{n}_1$ is parallel to $\hat{n}$.

Squaring (3), we get $\kappa^2 = \tau^2\left(\dfrac{ds}{ds_1}\right)^2$

or $$\left(\frac{ds}{ds_1}\right) = \frac{\kappa_1}{\tau} \qquad .....(6)$$

from (5) and (6), we get $\qquad \hat{n}_1 = -\hat{n}$ .....(7)

Equation (7) shows that directions of n_1 and n are opposite to each other.

Taking cross product of (4) and (7)

$$\hat{t}_1 \times \hat{n}_1 = \hat{b} \times (-\hat{n}) \quad \text{or} \quad \hat{b}_1 = \hat{t} \qquad .....(8)$$

which shows that $\hat{b}_1$ is parallel to $\vec{c}$.

(ii) The product of the torsion of c_1 at corresponding points is equal to the product of curvatures at these points.

Proof : Differentiating equation (8) with respect to s_1

$$\hat{b}_1' = \hat{t}' \frac{ds}{ds_1}$$

or
$$-\tau_1 \hat{n}_1 = \kappa \hat{n} \frac{ds}{ds_1} \qquad .....(9)$$

but from equation (7) $\hat{n}_1 = -\hat{n}$, hence from equation (9), we have

$$\tau_1 = \kappa \frac{ds}{ds_1} \qquad .....(10)$$

or
$$\tau_1 = \frac{\kappa_1 \kappa}{\tau} \qquad \text{[using (6)]}$$

or
$$\kappa\kappa_1 = \tau\tau_1 \qquad .....(11)$$

$\Rightarrow$
$$\boxed{\rho\rho_1 = \sigma\sigma_1}$$

(iii) If the curvature κ of c is constant then curvature κ_1 of c_1 is also constant.

Proof : The curvature κ of c is constant

i.e. $\qquad \rho = \text{ const.}, \rho' = 0, \rho'' = 0.$

equation (3) reduces to $\qquad \dfrac{ds}{ds_1} = \dfrac{\rho}{\sigma} = \dfrac{\tau}{\kappa}.$

Hence from (6), we have

$$\frac{\tau}{\kappa} = \frac{\tau}{\kappa_1} \quad \text{or} \quad \boxed{\kappa = \kappa_1} \qquad .....(12)$$

which shows that the curvature κ_1 of c_1 is also constant.

2.6.4 Examples :

Ex.6. *If a curve lies on a sphere show that ρ and σ are related by*

$$\frac{d}{ds}(\sigma\rho') + \frac{\rho}{\sigma} = 0$$

show that a necessary and sufficient condition that a curve lies on a sphere is that

$$\frac{\rho}{\sigma}+\frac{d}{ds}\left(\frac{\rho'}{\tau}\right)=0$$

at every point on the curve.

Sol. Necessary condition :

Let the curve lie on a sphere then we have to prove the given condition. Now the sphere will be osculating sphere for every point. The radius R of the osculating sphere is given by

$$R^2 = \rho^2 + \sigma^2 \rho'^2 \qquad \text{.....(1)}$$

Differentiating with respect to 's', we get

$$0 = \rho\rho' + \sigma^2 \rho' \rho'' + \sigma\sigma' \rho'^2$$

Dividing by $\rho'\sigma$, we get

$$0=\frac{\rho}{\sigma}+\rho''\sigma+\sigma'\rho'$$

or $$\frac{\rho}{\sigma}+\frac{d}{ds}(\sigma\,\rho')=0$$

or $$\frac{\rho}{\sigma}+\frac{d}{ds}\left(\frac{\rho'}{\tau}\right)=0$$

Sufficient condition : If $\frac{\rho}{\sigma}+\frac{d}{ds}\left(\frac{\rho'}{\tau}\right)=0$ to show that the curve lies on a sphere

$$\rho^2+\sigma^2\rho'^2=a^2 \qquad \text{[by (1)]}$$

showing that the radius of osculating sphere is independent of the point on the curve.

Again the centre of spherical curvature is given by

$$\vec{C}=\vec{r}+\rho\hat{n}+\sigma\,\rho'\hat{b}$$

$\therefore$ $$\frac{d\vec{c}}{ds}=\hat{t}+\rho'\hat{n}+\rho\left(\tau\hat{b}-\kappa\hat{t}\right)+\sigma'\,\rho'\hat{b}+\sigma\,\rho''\hat{b}-\sigma\,\rho'\tau\hat{n}$$

$$=\left(\frac{\rho}{\sigma}+\sigma'\,\rho'+\sigma\,\rho''\right)\hat{b}$$

But $\frac{\rho}{\sigma}+\frac{d}{ds}(\sigma\,\rho')$ *or* $\frac{\rho}{\sigma}+\sigma'\,\rho'+\sigma'\,\rho''$ is zero.

$\therefore$ $$\frac{d\vec{c}}{ds}=0$$

or $\vec{c}$ = constant vector

i.e., the centre of osculating sphere is independent of the point on the curve.

***Ex.*7.** *Prove that the curve given by*

$$x = a \sin u,\ \ y = 0,\ \ z = a \cos u$$

lies on a sphere.

Sol. Here $\vec{r} = a(\sin u, 0, \cos u)$ .....(1)

$$\hat{t} = \dot{\vec{r}} = a(\cos u, 0, -\sin u)\left(\frac{du}{ds}\right)$$

squaring $1 = a^2\left(\frac{du}{ds}\right)^2 \Rightarrow \frac{du}{ds} = \frac{1}{a}$

Hence $\hat{t} = (\cos u, 0, -\sin u)$ .....(2)

$\therefore$ $\dot{\hat{t}} = \kappa n = (-\sin u, 0, -\cos u)\left(\frac{du}{ds}\right)$

or $\kappa\hat{n} = (-\sin u, 0, -\cos u)\left(\frac{1}{a}\right)$

squaring $\kappa^2 = \frac{1}{a^2} \Rightarrow \kappa = \frac{1}{a}$

$\Rightarrow$ $\rho = a = \text{constant}$ .....(3)

Hence $\hat{n} = (-\sin u, 0, -\cos u)$ .....(4)

$$\hat{b} = \hat{t}\times\hat{n} = (0,1,0)$$

$\therefore$ $\dot{\hat{b}} = -\tau\,\hat{n} = (0,0,0)$

$\Rightarrow$ $\tau = 0$ (as $\hat{n} \neq 0$) .....(5)

We know that curve will lie on a sphere if

$$\frac{d}{ds}(\sigma\,\rho') + \rho\,\tau = 0$$

Here $\rho = a$

$\therefore$ $\rho' = 0$ and also $\tau = 0$.

Therefore, the relation $\frac{d}{ds}(\sigma\,\rho') + \rho\,\tau = 0$ is clearly satisfied. Hence the given curve lies on a sphere.

Ex.8. *Prove that* $x'''^2 + y'''^2 + z'''^2 = \frac{1}{\rho^2\sigma^2} + \frac{1+\rho'^2}{\rho^4}$

where dashes denote differentiation with respect to 's'.

Sol. Here $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{r}' = x'\hat{i} + y'\hat{j} + z'\hat{k}$$

$$\vec{r}'' = x''\hat{i} + y''\hat{j} + z''\hat{k}$$

$$\vec{r}''' = x'''\hat{i} + y'''\hat{j} + z'''\hat{k}$$

$$(\vec{r}''')^2 = (x''')^2 + (y''')^2 + (z''')^2 \quad .....(1)$$

Also $\qquad \vec{r}' = \hat{t},\ \vec{r}'' = \hat{t}' = \kappa\hat{n} = \dfrac{\hat{n}}{\rho}$

$$\vec{r}'' = \frac{\hat{n}''}{\rho} + \left(\frac{-\rho'}{\rho^2}\right)\hat{n} = \frac{\tau\hat{b} - \kappa\hat{t}}{\rho} - \frac{\rho'}{\rho^2}\hat{n}$$

$$= -\frac{1}{\rho^2}\hat{t} + \frac{1}{\rho\sigma}\hat{b} - \frac{\rho'}{\rho^2}\hat{n}$$

$$\therefore \qquad \left(\vec{r}'''\right)^2 = \frac{1}{\rho^4} + \frac{1}{\sigma^2\rho^2} + \frac{\rho'^2}{\rho^4}. \qquad \text{......(2)}$$

Hence from (1) and (2)

$$x'''^2 + y'''^2 + z'''^2 = \frac{1}{\sigma^2\rho^2} + \frac{1+\rho'^2}{\rho^4}.$$

2.7 Answers to self-learning exercises

Self-learning exercise-1

1. osculating plane.
2. Binormal
3. $\hat{b} = \dfrac{\vec{r}' \times \vec{r}''}{|\vec{r}' \times \vec{r}''|}, \hat{n} = \dfrac{\vec{r}''}{|\vec{r}' \times \vec{r}''|}.$

Self-learning exercise-2

1. If P, Q, R be three points on a curve, the limiting position of the plane PQR when Q and R tend to P, is called the osculating plane at P.
2. The plane through P and perpendicular to the tangent line at P is called the normal plane at P of the curve.
3. The plane through P and containing tangent and binormal is called rectifying plane.
4. $\vec{R} = \vec{r} + \lambda\hat{n}$
5. $\vec{R} = \vec{r} + \mu\,\hat{b}$

2.8 Exercises

1. Show that the tangent and binormal at any point of the curve

$$x = t^3 - 1,\ \ y = \sqrt{3}(t^2 - 1),\ \ \ z = 2\left(t - 1\right)$$

make the same angle with the line $\dfrac{x}{1} = \dfrac{y}{0} = \dfrac{z}{1}$ and that the three directions are coplanar.

2. Establish the Serret-Frenet formulae at a point of a space curve.

3. Find the radii of curvature and torsion of a helix

$$x = a \cos \theta, y = a \sin \theta, \; z = a \, \theta \tan \alpha.$$

$$\left[\textbf{Ans. } \rho = a \sec^2 \alpha, \;\; \sigma = -\frac{a}{\sin \alpha \cos \alpha}\right]$$

4. For the curve $\quad x = a\,(3t - t^3), \; y = 3at^2, \; z = a\,(3t + t^3)$

Show that $\quad \rho = \sigma = 3a\,(1 + t^2)^2.$

5. Find the osculating plane, curvature and torsion at any point 'θ' of the curve

$$x = a \cos 2\theta, \; y = a \sin 2\theta, \; z = 2a \sin \theta.$$

[Ans. $(\sin \theta + \sin 2\theta \cos \theta)\, x - 2 \cos^3 \theta y + 2z = 3a \sin \theta,$

$$\left. \sigma = \frac{a}{3}\left(5 \sec \theta + 3 \cos \theta\right), \;\; \rho = \frac{2a\left(1 + \cos^2 \theta\right)^{3/2}}{\left(5 + 3\cos^2 \theta\right)^{1/2}} \right]$$

6. For the curve $\quad x = 2abt, \; y = a^2 \log t, \; z = b^2 t^2.$

Show that $\quad \rho = \sigma = (a^2 + 2b^2 t^2) / 2abt.$

7. Find the equation of the osculating sphere and osculating circle at (1, 2, 3) on the curve

$$x = 2t + 1, \quad y = 3t^2 + 2, \quad z = 4t^3 + 3.$$

[Ans. $\;3\,(x^2 + y^2 + z^2) - 6x - 16y - 18z + 50 = 0,$

$\qquad 3\,(x^2 + y^2 + z^2) - 6x - 16y - 18z + 50 = 0, \;\; z = 3$**]**

8. Show that the radius of spherical curvature of a circular helix

$$x = a \cos \theta, y = a \sin \theta, z = a \, \theta \cot a$$

is equal to circular curvature.

9. If the radius of spherical curvature is constant show that the curve either lies on a sphere or has a constant curvature $R^2 = \rho^2 + (\sigma\rho')^2$ where R is constant.

10. Find the equation of the osculating sphere at origin of the curve

$$x = a_1 t^3 + 3b_1 t^2 + 3c_1 t, \quad y = a_2 t^3 + 3b_2 t^2 + 3c_2 t, \quad z = a_3 t^3 + 3b_3 t^2 + 3c_3 t.$$

$$\left[\textbf{Ans.} \begin{vmatrix} x^2 + y^2 + z^2 & 2x & 2y & 2z \\ 9\left(b_1 c_1 + b_2 c_2 + b_3 c_3\right) & a_1 & a_2 & a_3 \\ 3\left(c_1^2 + c_2^2 + c_3^2\right) & 2b_1 & 2b_2 & 2b_3 \\ 0 & c_1 & c_2 & c_3 \end{vmatrix} = 0\right]$$

□□□

UNIT 3 : Existence and Uniqueness Theorems, Bertrand Curves, Involute, Evolutes, Conoids, Inflexional Tangents, Singular Points, Indicatrix

Structure of the Unit

3.0 Objectives

This unit provides a general overview of :

- Existence and uniqueness theorems
- Bertrand curves
- Involute
- Evolute
- Conoids
- Inflexional tangents
- Singular points
- Indicatrix

3.1 Existence and uniqueness theorems

Existence and uniqueness theorem for space curves is also called fundamental theorem on space curves

3.1.1 Existence theorem :

If $k(s)$ and $\tau(s)$ are continuous functions of a real variable s $(s \geq 0)$ then there exists a space curve for which k is the curvature, τ is the torsion and s is the arc-lengths measured from some suitable base point.

Proof : The proof of this theorem depends on the existence theorem of the solution of differential equations which states that the linear differential equations

$$\frac{dx}{ds} = ky, \frac{dy}{ds} = \tau z - kx, \frac{dz}{ds} = -\tau y \qquad .....(3.1.1)$$

where k and τ are continuous functions of s in the interval $0 \le s \le a$.

Equation (1) admits a unique set of solutions for a given set of values of x, y, z at $s = 0$.

In particular, there exists a unique set (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) which have values $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ at $s = 0$, respectively.

Now,

$$\frac{d}{ds}\left(x_1^2 + y_1^2 + z_1^2\right) = 2\left(x_1 \frac{dx_1}{ds} + y_1 \frac{dy}{ds} + z_1 \frac{dz_1}{ds}\right)$$

$$= 2\left\{x_1\left(ky_1\right) + y_1\left(\tau z_1 - kx_1\right) - z_1 \tau y_1\right\} = 0 \qquad \text{[using (1)]}$$

Hence,

$$x_1^2 + y_1^2 + z_1^2 = \text{constant} = C_1 \text{ (say)}$$

Since at $s = 0, x_1(0) = 1, y_1(0) = 0, z_1(0) = 0$, therefore $C_1 = 1$.

Thus, we get $x_1^2 + y_1^2 + z_1^2 = 1, \forall s. \qquad .....(3.1.2)$

Similarly, we get

$$\left.\begin{aligned} x_2^2 + y_2^2 + z_2^2 &= 1, \\ x_3^2 + y_3^2 + z_3^2 &= 1, \quad \text{for } \forall s \end{aligned}\right\} \qquad .....(3.1.3)$$

Further, $\frac{d}{ds}\left(x_1x_2 + y_1y_2 + z_1z_2\right)$

$$= \left(x_1 \frac{dx_2}{ds} + x_2 \frac{dx_1}{ds}\right) + \left(y_1 \frac{dy_2}{ds} + y_2 \frac{dy_1}{ds}\right) + \left(z_1 \frac{dz_2}{ds} + z_2 \frac{dz_1}{ds}\right)$$

$$= x_1(ky_2) + x_2(ky_1) + y_1(\tau z_2 - kx_2) + y_2(\tau z_1 - kx_1) + z_1(-\tau y_2) + z_2(-\tau y_1) = 0.$$

Hence on integration,

$$x_1 x_2 + y_1 y_2 + z_1 z_2 = \text{const.} = d_1 \text{ (say)}$$

The value of the constant d_1, determined by the initial conditions and we get $d_1 = 0$.

Thus we get $x_1 x_2 + y_1 y_2 + z_1 z_2 = 0, \ \forall \ s. \qquad .....(3.1.4)$

Similarly, we get

$$\left.\begin{aligned} x_2x_3 + y_2y_3 + z_2z_3 &= 0 \\ x_3x_1 + x_3x_1 + x_3x_1 &= 0 \end{aligned}\right\} \text{for } \forall \ s \qquad .....(3.1.5)$$

Hence, we have six relations given by (3.1.2) to (3.1.5) in the elements of three sets namely (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) defined for each value of s.

If $r = \int_0^s t \ ds$, then $r = r(s)$

is the position vector of a point on the curve with (k, τ, s) curvature, torsion and arc-length respectively and $\left(\hat{t}, \hat{n}, \hat{b}\right)$ as unit tangent vector, unit principal normal vector and unit binormal respectively

Hence the existence of curve is proved.

3.1.2 Uniqueness theorem :

A curve is uniquely determined, except as to position in space, when its curvature and torsion are given functions of its arc-length.

Proof : If possible let there be two curves c_1 and c having equal curvature k and equal torsion τ for the same value of s. Let the suffix unity be used for quantities belonging to c_1. Now, if c_1 is moved (without deformation) so that the two points on c and c_1 corresponding to the same value of 's' coincide. We have

$$\frac{d}{ds}\left(\hat{t}\cdot\hat{t}_1\right) = \hat{t}\cdot k_1\hat{n}_1 + k\hat{n}\cdot\hat{t}_1$$

or $$\frac{d}{ds}\left(\hat{t}\cdot\hat{t}_1\right) = \hat{t}\cdot k\hat{n}_1 + k\hat{n}\cdot\hat{t}_1 \quad [\because\ k_1 = k \text{ given}] \qquad .....(3.1.6)$$

$$\frac{d}{ds}\left(\hat{n}\cdot\hat{n}_1\right) = \hat{n}\cdot\left(\tau\hat{b}_1 - k\hat{t}_1\right) + \left(\tau\hat{b} - k\hat{t}\right)\cdot\hat{n}_1 \qquad .....(3.1.7)$$

$$\frac{d}{ds}\left(\hat{b}\cdot\hat{b}_1\right) = \hat{b}\cdot\left(-\tau\hat{n}_1\right) + \left(-k\hat{n}\right)\cdot\hat{b}_1 \qquad .....(3.1.8)$$

Adding equations (1), (2) and (3), we get

$$\frac{d}{ds}\left(\hat{t}\cdot\hat{t}_1 + \hat{n}\cdot\hat{n}_1 + \hat{b}\cdot\hat{b}_1\right) = 0 \qquad .....(3.1.9)$$

which on integrating gives $$\hat{t}\cdot\hat{t}_1 + \hat{n}\cdot\hat{n}_1 + \hat{b}\cdot\hat{b}_1 = \text{constant}. \qquad .....(3.1.10)$$

If c_1 is moved in such a manner that at $s = 0$ the two triads $\left(\hat{t}, \hat{n}, \hat{b} \text{ and } \hat{t}_1, \hat{n}_1, \hat{b}_1\right)$ coincide. Then at the point $\hat{t} = \hat{t}_1, \hat{n} = \hat{n}_1, \hat{b} = \hat{b}_1$ and then the value of constant in equation (3.1.10) becomes 3.

Thus, $$\hat{t}\cdot\hat{t}_1 + \hat{n}\cdot\hat{n}_1 + \hat{b}\cdot\hat{b}_1 = 3 \qquad .....(3.1.11)$$

But the sum of three cosines is equal to 3 if each angle is zero or in an integral multiple of 2π.

Thus for each pair of corresponding points $\hat{t} = \hat{t}_1,\ \hat{n} = \hat{n}_1,\ \hat{b} = \hat{b}_1$.

Also, $\hat{t} = \hat{t}_1$ gives $\vec{r}' = \vec{r}_1'$

i.e. $$\frac{d}{ds}\left(\vec{r} - \vec{r}_1\right) = 0 \quad i.e.,\quad \vec{r} - \vec{r}_1 = a \text{ (constant vector)}$$

But when $s = 0$, $\vec{r} - \vec{r}_1 = 0$ or $\vec{r} = \vec{r}_1$ at all corresponding points and hence the two curves coincide or the two curves are congruent.

Hence the uniqueness theorem is proved.

3.2 Bertrand curves

3.2.1 Definitions :

Two curves c and c_1 are said to be Bertrand curves or conjugate if the principal normals to c are also principal normals to c_1.

3.2.2 Properties of Bertrand curves :

Property I. *The distance between corresponding points of the two curve is constant.*

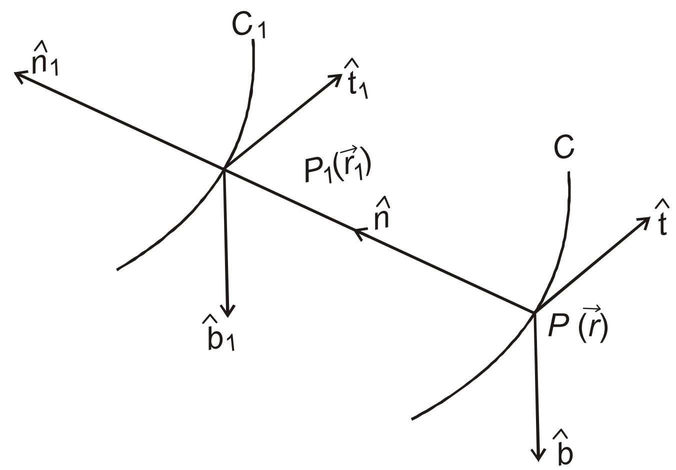


Fig 3.1

Proof : Let P and P_1 be the corresponding points on the Bertrand curves c and c_1. $\hat{n}$ and $\hat{n}_1$ be principal normals at P and P_1 on curves c and c_1.

Let the corresponding quantities for the curve c_1 be denoted by the suffix unity.

Let $PP_1 = \lambda$. Then the position vector $\vec{r}_1$ related to $\vec{r}$ as,

$$\vec{r}_1 = \vec{r} + \lambda\,\hat{n} \qquad .....(3.2.1)$$

where λ is function of 's'.

Differentiating (1) with respect to 's', we get

$$\frac{d\vec{r}_1}{ds_1}\frac{ds_1}{ds} = \hat{t} + \lambda\,\hat{n}' + \lambda'\,\hat{n}$$

or

$$\hat{t}_1\frac{ds_1}{ds} = \hat{t} + \lambda\left(\tau\,\hat{b} - k\,\hat{t}\right) + \lambda'\,\hat{n}$$

$$= \left(1-\lambda k\right)\hat{t} + \lambda'\,\hat{n} + \lambda\tau\,\hat{b} \qquad .....(3.2.2)$$

By definition $\hat{n}_1 = \hat{n}$ .....(3.2.3)

Taking the dot product of (3.2.2) and (3.2.3) and noting that $\hat{t}\cdot\hat{n} = \hat{b}\cdot\hat{n} = 0,\ \hat{n}\cdot\hat{n} = 1$

$$\therefore \qquad 0 = \lambda'$$

which on integration gives

$$\lambda = \text{constant}.$$

i.e. $\qquad PP_1 = \text{constant}.$

Property II. *The tangents at the corresponding points of the associate Bertrand curves are inclined at a constant angle.*

Proof : If α be the angle between $\hat{t}$ and $\hat{t}_1$ then we have to prove that α is constant.

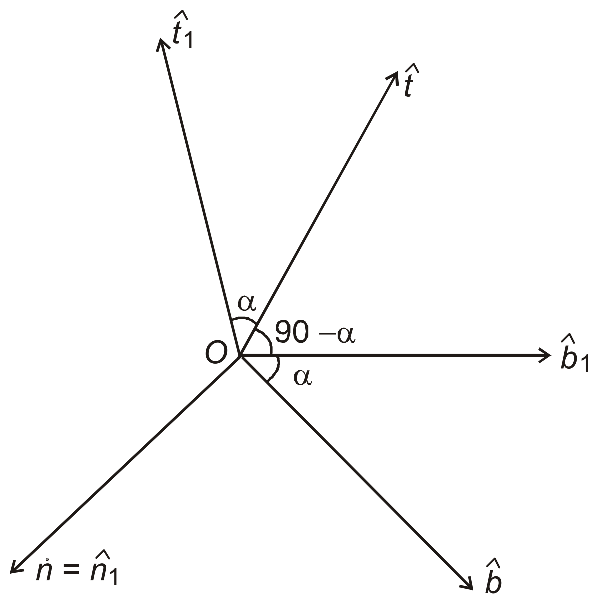


Fig 3.2

But $\hat{t} \cdot \hat{t}_1 = \cos\alpha$

and
$$\frac{d}{ds}\left(\hat{t} \cdot \hat{t}_1\right) = \hat{t}' \cdot \hat{t}_1 + \hat{t} \cdot \hat{t}_1' \frac{ds_1}{ds}$$

$$= K\ \hat{n}\ \hat{t}_1 + \hat{t} \cdot K_1\ \hat{n}_1\ \frac{ds_1}{ds}$$

$$= K\ \hat{n}_1 \cdot \hat{t}_1 + K_1 \frac{ds_1}{ds} \hat{t} \cdot \hat{n}. \qquad \because \hat{n} = \hat{n}_1$$

$$= 0 \qquad \text{.....(3.2.4)}$$

$\therefore \qquad \hat{n}_1 \cdot \hat{t}_1 = 0, \quad \hat{t} \cdot \hat{n} = 0$

Integrating we get $\quad \hat{t} \cdot \hat{t}_1 = \text{constant} = \cos\alpha$, say .....(3.2.5)

$\therefore \qquad \alpha = \text{constant}.$ .....(3.2.6)

Further, as the principal normals of the two curves coincide, it follows from the above that the binormals of the two curves are also inclined at the same constant angle.

Property III. *The curvature and torsion of either associate Bertrand curves are connected by a linear relation.*

Proof : From property *I*, we have $\lambda' = 0$. .....(3.2.7)

Therefore,

$$\hat{t}_1 \frac{ds_1}{ds} = \left(1 - \lambda k\right) \hat{t} + \lambda\tau\hat{b} \qquad \text{.....(3.2.8)}$$

This implies that $\hat{t}_1, \hat{t}$ and $\hat{b}$ are coplanar. On taking dot product with $\hat{b}_1$, we get

$$0 = (1 - \lambda k)\, \hat{t} \cdot \hat{b}_1 + \lambda\tau\, \hat{b} \cdot \hat{b}_1$$

But, $\hat{t} \cdot \hat{b}_1 = \cos(90 - \alpha) = \sin\alpha$ and $\hat{b} \,.\, \hat{b}_1 = \cos\alpha$ .....(3.2.9)

Hence, $(1 - \lambda k) \sin\alpha + \lambda\, \tau \cos\alpha = 0$

or $\tau = \left(k - \dfrac{1}{\lambda}\right) \tan\alpha$ for the curve c. .....(3.2.10)

This shows that τ and k are linearly related.

Again from (3.2.1), $\vec{r}_1 = \vec{r} + \lambda \hat{n}$

$\therefore$ $\vec{r} = \vec{r}_1 - \lambda \hat{n}$

Above shows that the point $P(\vec{r})$ is at a distance $-\lambda$ along the normal at $P_1(\vec{r}_1)$ and $\hat{t}$ is inclined at an angle $(-\alpha)$ with $\hat{t}_1$. Hence (3.2.10) takes the form

$$\tau_1 = \left(K_1 + \frac{1}{\lambda}\right) \tan(-\alpha) \quad \text{or} \quad c_1 = -\left(K_1 + \frac{1}{\lambda}\right) \tan\alpha \qquad .....(3.2.11)$$

which gives the linear relationship between τ_1 and K_1.

3.2.3 Theorem based on Bertrand curves :

Theorem. *The torsion of the two Bertrand curves have the same sign and their product is constant.*

Proof : From property (III), we have

$$\hat{t}_1 \frac{ds_1}{ds} = (1 - \lambda K)\hat{t} + \lambda\tau\, \hat{b} \qquad .....(3.2.12)$$

and

$$\hat{t}_1 = \hat{t}\cos\alpha - \hat{b}\sin\alpha \qquad .....(3.2.13)$$

On comparing the coefficients in (3.2.12) and (3.2.13), we get

$$\frac{(ds_1 / ds)}{1} = \frac{1 - \lambda K}{\cos\alpha} = \frac{\lambda\tau}{-\sin\alpha} \qquad .....(3.2.14)$$

then, we have $\lambda\tau = -\sin\alpha \dfrac{ds_1}{ds},\ 1 - \lambda K = \cos\alpha \dfrac{ds_1}{ds}$ .....(3.2.15)

Replacing λ by $-\lambda$, α by $-\alpha$ and s by $-s_1$ and s_1 by s, we have

$$-\lambda\tau_1 = \sin\alpha \frac{ds}{ds_1},\ 1 + \lambda K_1 = \cos\alpha \frac{ds}{ds_1} \qquad .....(3.2.16)$$

On multiplying, we have $\tau\tau_1 = 1/\lambda^2 \sin^2 K$ = constant as both λ and α are constants and

$$(1 - \lambda K)(1 + \lambda K_1) = \cos^2\alpha. \qquad .....(3.2.17)$$

3.4.4 Self-learning exercise-1.

1. How many properties of Bertrand curves are ?

2. In which relation the torsion and curvature of Bertrand curves are connected ?

3. The angle between tangents of Bertrand curve are ....

3.3 Involute

3.3.1 Definition :

If tangents to a give space curve c are normals to another curve c_1, then the curve c_1 is called involute of the curve c.

3.3.2 General equation of the involute of a given space curve :

Let c_1 be an involute of c and let equation of c be $\vec{r} = \vec{r}(s)$. Let the quantities belonging to c_1 be distinguished by the suffix unity.

Any point P_1 on c_1 is given by

$$OP_1 = OP + PP_1 \Rightarrow \vec{r}_1 = \vec{r} + \mu\,\hat{t} \qquad .....(3.3.1)$$

Where μ is to be determined. Differentiating equation (3.3.1)

$$\hat{t}_1 = \left(\hat{t} + \mu'\,\hat{t} + \mu\,K\,\hat{n}\right)\frac{ds}{ds_1} \qquad .....(3.3.2)$$

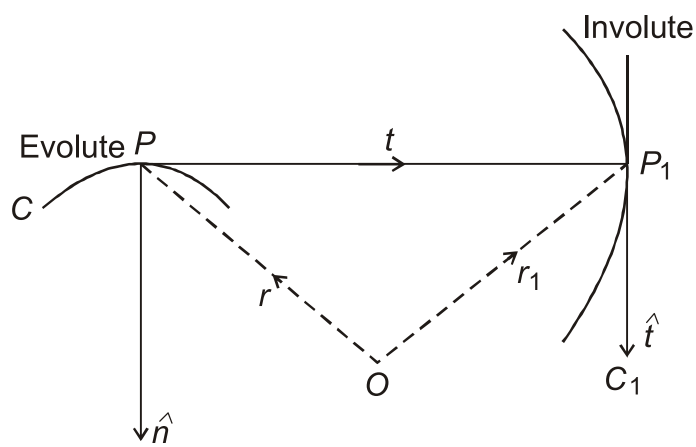


Fig 3.3

But $\hat{t}$ is perpendicular to t_1 for an involute, hence taking dot product of both sides of equation (3.3.2) with $\hat{t}$ and using $\hat{t}\cdot\hat{t}_1 = 0$, we get

$$\left(1+\mu'\right)\frac{ds}{ds_1} = 0 \;\; i.e.\;\; 1 + \mu' = 0 \;\; i.e.\;\; ds + d\mu = 0$$

hence on integration, we get $\quad s + \mu = c$ or $\mu = c - s \qquad .....(3.3.3)$

where c is constant of integration

$$\therefore \qquad \vec{r}_1 = \vec{r} + (c - s)\hat{t} \qquad .....(3.3.4)$$

This is the required equation of involute c_1 of the curve c.

3.3.3 Curvature of the involute :

From (3.3.2) and (3.3.3), we have

$$\hat{t}_1 \frac{ds_1}{ds} = \mu\, k\, \hat{n} \quad \text{where} \quad \mu = c - s. \qquad .....(3.3.5)$$

This shows that $\hat{t}_1$ is parallel to $\hat{n}$. Taking the direction of $\hat{t}_1$ same as that of $\hat{n}$, we get

$$\frac{ds_1}{ds} = \lambda\, K, \hat{t}_1 = \hat{n} \qquad .....(3.3.6)$$

Differentiating equation (3.3.6), we have

$$\frac{d\hat{t}_1}{ds_1} = \frac{d\hat{n}}{ds}\frac{ds}{ds_1} \Rightarrow K_1 \hat{n}_1 = \left(\tau\hat{b} - K\hat{t}\right)\cdot\frac{1}{\mu K} \qquad .....(3.3.7)$$

On squaring $$K_1^2 = \frac{\tau^2 + K^2}{\mu^2 K^2} \Rightarrow K_1 = \frac{\left(\tau^2 + K^2\right)^{1/2}}{\mu\, K} \quad \text{where} \quad \mu = c - s. \qquad .....(3.3.8)$$

Hence, equation (3.3.8) determine curvature of the involute.

3.3.4 Torsion of the involute :

From (3.3.7), we have

$$\hat{n}_1 = \frac{\tau\hat{b} - K\hat{t}}{\mu K\, K_1} \qquad .....(3.3.9)$$

Therefore using (3.3.6) and (3.3.8), we have

$$\hat{b}_1 = \hat{t}_1 \times \hat{n}_1 = \hat{n} \times \hat{n}_1 = \frac{\tau\hat{t} + K\hat{b}}{\mu\, K K_1} \quad \text{or} \quad \hat{b} = \frac{\tau\hat{t} + K\hat{b}}{\left(\tau^2 + K^2\right)^{1/2}} \qquad .....(3.3.10)$$

Differentiating (3.3.10) with respect to 's' and using Serret-Frenet formulae, we find

$$\frac{d\hat{b}_1}{ds_1}\frac{ds_1}{ds} = \frac{1}{\left(\tau^2 + K^2\right)^{1/2}}\left\{\tau\frac{d\hat{t}}{ds} + K\frac{d\hat{b}}{ds} + \tau'\,\hat{t} + K'\hat{b}\right\} - \frac{\left(\tau\,\hat{t} + K\hat{b}\right)}{\left(\tau^2 + K^2\right)^{3/2}}\left\{\tau\tau' + KK'\right\},$$

or $$-\tau_1 \mu \hat{n}_1 K = \frac{1}{\left(\tau^2 + K^2\right)^{1/2}}\left\{\tau\, K\hat{n} - K\tau\,\hat{n} + \tau'\,\hat{t} + K'\,\hat{b}\right\} - \frac{\left(\tau\,\hat{t} + K\hat{b}\right)}{\left(\tau^2 + K^2\right)^{3/2}}\left\{\tau\tau' + KK'\right\},$$

or $$-\tau_1 \mu \hat{n}_1 K = \frac{\left(\tau^2 + K^2\right)\left(\tau'\,\hat{t} + K'\,\hat{b}\right) - \left(\tau\,\hat{t} + K\,\hat{b}\right)\left(\tau\tau' + KK'\right)}{\left(\tau^2 + K^2\right)^{3/2}},$$

or $$-\tau_1 \mu K \hat{n}_1 = \frac{\left(K\tau' - \tau\, K'\right)\left(K\,\hat{t} - \tau\,\hat{b}\right)}{\left(\tau^2 + K^2\right)^{3/2}} \qquad .....(3.3.11)$$

Squaring both sides, we get

$$\tau_1^2 \mu^2 K^2 = \frac{\left(K\tau' - \tau K'\right)^2}{\left(\tau^2 + K^2\right)^2},$$

$\therefore$ $$\tau_1 = \pm \frac{\left(K\tau' - \tau K'\right)}{\mu K\left(\tau^2 + K^2\right)} \quad \text{where} \quad \mu = c - s. \qquad .....(3.3.12)$$

Hence equation (3.3.12) determines the torsion of the involute.

3.3.5 Example :

***Ex.*1.** *Find the involute of a circular helix given by.*

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = a\,\theta\tan\alpha.$$

Sol. Here $\vec{r} = (a\cos\theta,\ a\sin\theta,\ a\,\theta\tan\alpha)$.

Diff. with respect to θ

$\therefore$ $$\dot{\vec{r}}\cdot\frac{ds}{d\theta} = \left(-a\sin\theta,\ a\cos\theta,\ a\tan\alpha\right)$$

$\therefore$ $$\hat{t}\,\frac{ds}{d\theta} = \left(a\sin\theta,\ a\cos\theta,\ a\tan\alpha\right) \qquad .....(1)$$

On squaring, we have

$$\left(\frac{ds}{d\theta}\right)^2 = a^2\left(\sin^2\theta + \cos^2\theta\right) + a^2\tan^2\alpha = a^2\sec^2\alpha$$

$\therefore$ $$\frac{ds}{d\theta} = a\sec\alpha$$

or $$s = \int_0^\theta a\sec\alpha\, d\theta = a\,\theta\sec\alpha$$

Putting for $\frac{ds}{d\theta}$ in (1), we get

$$\hat{t}\, a\sec\alpha = a\left(-\sin\theta, \cos\theta, \tan\alpha\right)$$

$$\hat{t} = \cos\alpha\left(-\sin\theta, \cos\theta, \tan\alpha\right)$$

Now the equation of involute is,

$$\vec{r}_1 = \vec{r} + (c - s)\hat{t}$$

or $$\vec{r}_1 = \left(a\cos\theta,\ a\sin\theta,\ a\theta\tan\alpha\right) + \left(c - a\sec\theta\right)\cos\alpha\left(-\sin\theta, \cos\theta, \tan\alpha\right)$$

or $$x = a\cos\theta - (c - a\,\theta\sec\alpha)\cos\alpha\cdot\sin\theta$$

$$y = a\cos\theta - (c - a\,\theta\sec\alpha)\cos\alpha\cdot\cos\theta$$

$$z = a\,\theta\tan\theta + (c - a\,\theta\sec\alpha)\sin\alpha.$$

Ex.2. *Show that the distance between corresponding points of two involutes is constant.*

Sol. The equation of the involute is,

$$\vec{r}_1 = \vec{r} + (c - s)\hat{t}$$

where 'c' is arbitrary constant.

Let $c = c_1$ and $c = c_2$ be the values of constant for the two point P and Q on the involute whose position vectors are $\vec{r}_1$ and $\vec{r}_2$ say, so that

$$\vec{r}_1 = \vec{r} + (c_1 - s)\hat{t}, \quad \vec{r}_2 = \vec{r} + (c_2 - s)\hat{t}$$

$$PQ = \left|\overline{PQ}\right| = \left|\vec{r}_2 - \vec{r}_1\right| = \left|(c_1 - c_2)t\right| = (c_1 - c_2) = \text{constant.}$$

3.3.6 Self-learning exercise-2.

1. Write the formulae of curvature of an involute
2. Write the formulae of torsion of an involute.
3. Write the equation of involute.

3.4 Evolute

3.4.1 Definition :

If the tangents to a curve c are normals to another curve c_1, then c is called an evolute of c_1.

3.4.2 General equation of the evolute of a given space curve :

In other words, we are given the equation of the involute c and are required to find its evolute c_1.

Let $\vec{r} = \vec{r}(s)$ be a given curve c.

Let $\vec{r}_1$ be the position vector of any point Q on c_1 and that of the corresponding point P on c be $\vec{r}$.

Now, since the tangent to c_1 are normals to c, the point Q must lie in the normal plane to the curve c at P.

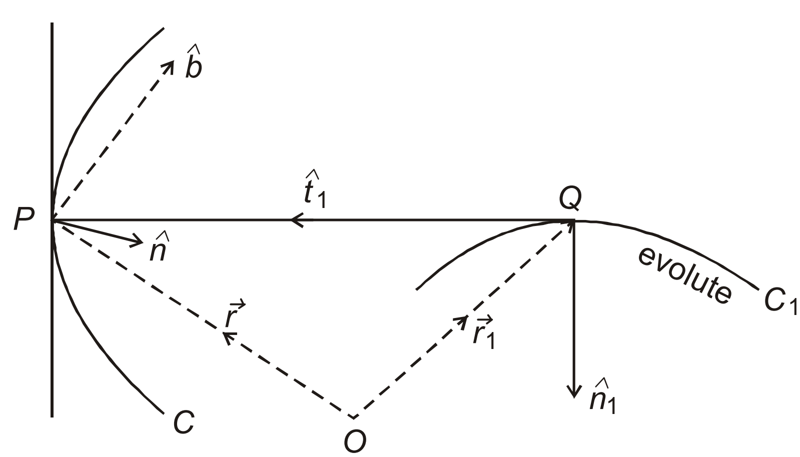


Fig 3.4

Thus, $$\vec{r}_1 = \vec{r} + \lambda\,\hat{n} + \mu\,\hat{b} \qquad .....(3.4.1)$$

where λ and μ are to be found out.

Differentiating with respect to 's', $\left[\text{as } PQ = \vec{r}_1 - \vec{r} = \lambda\hat{n} + \mu\,\hat{b}\right]$

$$\hat{t}_1 = \left[\hat{t} + \lambda\left(\tau\hat{b} - K\hat{t}\right) + \left(\lambda'\hat{n} + \mu'\hat{b} - \mu\tau\,\hat{n}\right)\right]\frac{ds}{ds_1}$$

$$= \left[\left(1 - K\lambda\right)\hat{t} + \left(\lambda' - \mu\tau\right)\hat{n} + \left(\mu' + \lambda\hat{t}\right)\hat{b}\right]\frac{ds}{ds_1} \qquad .....(3.4.2)$$

As $\hat{t}_1$ lies in the normal plane of c at P, therefore it must be parallel to $\lambda\hat{n} + \mu\hat{b}$, hence comparing this with the relation in (3.4.2), we obtain

$$1 - K\lambda = 0 \quad i.e.\ \lambda = \rho$$

and $$\frac{\lambda' - \mu\tau}{\lambda} = \frac{\mu' + \lambda\tau}{\mu} \quad i.e.,\quad \tau = \frac{\lambda'\mu - \lambda\mu'}{\lambda^2 + \mu^2} = \frac{d}{ds}\tan^{-1}\left(\frac{\lambda}{\mu}\right)$$

or $$\tau = \frac{d}{ds}\tan^{-1}\left(\frac{\lambda}{\mu}\right) \qquad .....(3.4.3)$$

Integrating equation (3.4.3), we get

$$a + \int \tau ds = \tan^{-1}\frac{\rho}{\mu} \qquad [\text{as } \rho = \lambda \text{ and } a \text{ is constant}]$$

$$= \cot^{-1}\frac{\mu}{\rho}$$

or $$\mu = \rho\cot\left(\int \tau\, ds + a\right)$$

Substituting values of λ and μ in equation (3.4.1), we get

$$\vec{r}_1 = \vec{r} + \rho\hat{n} + \rho\cot\left(\int \tau ds + a\right)\hat{b}. \qquad .....(3.4.4)$$

This is the required equation of evolute c_1 of the curve c. As, we give different value of a, we get infinite system of evolutes of the given curve, one evolute arising from each choice of a.

If we assume $$\int \tau\, ds = \psi \ \text{ and } \ a = c - \frac{1}{2\pi} \qquad .....(3.4.5)$$

so that $$\tau = \frac{d\psi}{ds}$$

Hence, equation (3.4.5) of the evolute becomes,

$$\vec{r}_1 = \vec{r} + \rho\hat{n} - \rho\tan\left(\psi + c\right)\hat{b}\,. \qquad .....(3.4.6)$$

3.4.3 Curvature of the evolute :

Differentiating equation (3.4.6) with respect to 's_1'

$$t_1 = \left[r + \rho\hat{n} - \rho\tan(\psi + c)\hat{b}\right]' \frac{ds}{ds_1}$$

$$= \left[t + \rho'\hat{n} + \rho\left(\tau\hat{b} - K\hat{t}\right) - \rho'\tan(\psi + c)\hat{b}\right.$$

$$\left. + \ \tau\rho\hat{n}\tan(\psi + c) - \rho\tau\sec^2(\psi + c)b\right]\frac{ds}{ds_1} \qquad \left[\because \ \frac{ds}{ds_1} = \tau\right]$$

$$= \left[\rho' + \rho\tau\tan(\psi + c)\right]\left[\hat{n} - \tan(\psi + c)b\right]\frac{ds}{ds_1}$$

$$= \left[\frac{K\tau\sin(\psi + c) - K'\cos(\psi + c)}{K^2\cos^2(\psi + c)}\right] \times \left[\hat{n}\cos(\psi + c) - \hat{b}\sin(\psi + c)\right]\frac{ds}{ds_1} \qquad \left[\because \ \rho' = \frac{-K'}{K^2}\right]$$

Hence the unit tangent to the evolute is,

$$t_1 = \hat{n}\cos(\psi + c) - \hat{b}\sin(\psi + c) \qquad .....(3.4.7)$$

where $$\frac{ds_1}{ds} = \left[\frac{K\tau\sin(\psi + c) - K'\cos(\psi + c)}{K^2\cos^2(\psi + c)}\right]$$

Differentiating equation (3.4.7) w.r.t. 's_1'

$$K_1\hat{n}_1 = \left[\left(\tau\hat{b} - K\hat{t}\right)\cos(\psi + c) - \hat{n}\sin(\psi + c)\tau + \ \tau\hat{n}\sin(\tau + c) - \hat{b}\cos(\psi + c)\tau\right]\frac{ds}{ds_1}$$

$$= \left[-K\cos(\psi + c)\hat{t}\right]\frac{ds}{ds_1} \qquad .....(3.4.8)$$

This equation shows that the principal normal to the evolute is parallel to $\hat{t}$.

We may choose the direction such that

$$\hat{n}_1 = -\hat{t} \qquad .....(3.4.9)$$

Therefore $$K_1 = K\cos(\psi + c)\frac{ds}{ds_1}$$

or $$K_1 = \frac{K^3\cos^3(\psi + c)}{K\tau\sin(\psi + c) - K'\cos(\psi + c)} \qquad .....(3.4.10)$$

Hence equation (3.4.10) determines the curvature of evolute.

3.4.4 Torsion of the evolute :

We know $$\hat{b}_1' = \hat{t}_1 \times \hat{n}_1 = \left[\hat{n}\cos(\psi + c) - \hat{b}\sin(\psi + c)\right] \times \left(-\hat{t}\right)$$

or $$\hat{b}_1 = \cos(\psi + c)\hat{b} + \sin(\psi + c)\hat{n} \qquad .....(3.4.11)$$

Differentiating this relation with respect to 's'

$$-\tau_1 \hat{n}_1 \frac{ds}{ds_1} = -\sin(\psi + c)\tau\,\hat{b} - \cos(\psi + c)\tau\hat{n} + \cos(\psi + c)\tau\hat{n} + \sin(\psi + c)(\tau\hat{b} - K\hat{t})$$

or
$$-\tau_1 \hat{n}_1 \frac{ds}{ds_1} = -K\sin(\psi + c)\cdot\hat{t}$$

or
$$\boxed{\tau_1 = \frac{-K^3\sin(\psi + c)\cos^2(\psi + c)}{Kt\sin(\psi + c) - K'\cos(\psi + c)}} \quad [\because n_1 = -t \text{ from } (3.4.9)] \quad .....(3.4.12)$$

Hence equation (3.4.12) determines the torsion of the evolute.

Now the relation between curvature and torsion is given by

$$\frac{\tau_1}{K_1} = -\tan(\psi + c) \quad .....(3.4.13)$$

3.4.5 Example :

Ex. 1. *Find the evolutes of the circular helix*

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = a\,\theta\tan\alpha$$

Sol. Here $\vec{r} = (a\cos\theta,\ a\sin\theta,\ a\theta\tan\alpha)$ .....(1)

Equation of evolute of space curve $\vec{r} = \vec{r}(s)$ is given by,

$$\vec{r}_1 = \vec{r} + \rho\hat{n} - \rho\tan(\psi + c)\hat{b} \quad .....(2)$$

where
$$\psi = \int \tau\, ds \quad .....(3)$$

Differentiating, (1) gives

$$\hat{t} = \vec{r}' = a(-\sin\theta,\ \cos\theta,\ \tan\alpha)\frac{d\theta}{ds} \quad .....(4)$$

Taking module of both sides

$$1 = \sqrt{a^2\sec^2\alpha}\left(\frac{d\theta}{ds}\right) \Rightarrow \frac{d\theta}{ds} = a\sec\alpha \quad .....(5)$$

Using (5), (4) gives

$$\hat{t} = \cos\alpha(-\sin\theta, \cos\theta, \tan\alpha)$$

$\therefore$
$$\hat{t}' = K\hat{n} = \frac{\cos^2\alpha}{a}(-\cos\theta,\ -\sin\theta, 0) \quad \text{[using (5)]}$$

which gives
$$K = \frac{\cos^2\alpha}{a} \quad i.e.\ \rho = a\sec^2\alpha$$

and
$$\hat{n} = (-\cos\theta, -\sin\theta, 0)$$

$\therefore$
$$\hat{b} = \hat{t}\times\hat{n} = \cos\alpha(\sin\theta\tan\alpha, -\cos\theta\tan\alpha, 1)$$

$\therefore \qquad \hat{b}' = -t\,\hat{n} = \dfrac{\cos^2\alpha}{a}\tan\alpha\left(\cos\theta, -\sin\theta, 0\right)$

which gives, $\quad \tau = \sin\alpha \dfrac{\cos\alpha}{a}$

$\therefore \qquad \psi = \int \dfrac{1}{a}\sin\alpha\cos\alpha\, ds = \dfrac{1}{a}\cdot s\cdot\sin\alpha\cos\alpha \qquad$ [Using (5)]

$= (\theta \sin\alpha).$

Hence the equation of evolute is given by

$$\left.\begin{aligned} x &= -a\cos\theta\tan^2\alpha - a\tan\alpha\sec\alpha\sin\theta\tan\left(c+\theta\sin\alpha\right) \\ y &= -a\sin\theta\tan^2\alpha - a\tan\alpha\sec\alpha\cos\theta\tan\left(c+\theta\sin\alpha\right) \\ z &= a\,\theta\tan\alpha - a\tan\alpha\sec\alpha\tan\left(c+\theta\sin\alpha\right) \end{aligned}\right\} \qquad .....(6)$$

Equation (6) gives required evolutes.

Ex. 2. *Prove that the locus of the centre of curvature is an evolute and is given only when the curve is plane.*

Sol. The equation of evolute of space curve $\vec{r} = \vec{r}\left(s\right)$ is given by

$$\vec{r}_1 = \vec{r} + \rho\hat{n} - \rho\tan\left(\psi + c\right)\hat{b} \qquad .....(1)$$

where 'c' is arbitrary constant.

The locus of the centre of curvature is given by the equation as

$$\vec{r}_1 = \vec{r} + \rho\hat{n} \qquad .....(2)$$

If equations (1) and (2) represent the same curves, then on comparison, we get

$$\tan\left(\psi + c\right) = 0 \Rightarrow \psi + c = n\,\pi\,;\, n \text{ is an integer} \qquad .....(3)$$

On differentiating with respect to 's', we get

$$\psi' = 0 \Rightarrow \tau = 0 \qquad \left(\because \psi' = \frac{d\psi}{ds} = \tau\right) \qquad .....(4)$$

Hence the curve should be a plane curve.

3.4.6 Self-learning exercise-3.

1. Write down the equation of the evolute.
2. Write the formulae of the curvature of the evolute.
3. Write down the formulae of the torsion of the evolute.
4. Given relation between curvature and torsion of the evolute.

3.5 Conoids

3.5.1 Definition :

The surfaces generated by a moving straight line under certain conditions are called ruled surfaces.

Cone and cylinder are examples of the ruled surface.

A conoid, is defined as the locus of a line which always intersects a fixed line (a given line) and a given curve and is parallel to a given plane.

Right conoid : If the given line is at right angles to the given plane, the locus is a right conoid.

3.5.2 Equation of a conoid :

Let the coordinate axes be so chosen as fixed line be z-axis and xy-plane be the given plane.

In such a case, the generators of the conoid will project the given curve on the plane $x = 1$ in a curve, whose equation be taken as,

$$x = 1, \quad z = f(y) \qquad \text{.....(3.5.1)}$$

Let $P(1, y, z)$ be any point on this curve, therefore

$$z_1 = f(y_1) \qquad \text{.....(3.5.2)}$$

Let $Q(0, 0, z_1)$ be the corresponding point on the fixed line. The generator of the conoid through P is the line joining P and Q, whose equation is,

$$\frac{x-0}{1} = \frac{y-0}{y_1} = \frac{z-z_1}{0} \qquad \text{.....(3.5.3)}$$

Eliminating y_1 and z_1 between the eqn (3.5.2) and (3.5.3), we obtain the required equation of the conoid *i.e.,*

$$z = f(y/x) \qquad \text{.....(3.5.4)}$$

3.5.3 Examples :

***Ex.*1.** *Find the equation to the conoid generated by lines parallel to the plane XOY, are drawn to intersect OZ and the curve*

$$x^2 + y^2 = r^2, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}.$$

Sol. The generators of the conoid are parallel to the plane XOY and intersect OZ, therefore their equations may be written as

$$Z = \mu \quad \text{and} \quad x = \lambda y \qquad \text{.....(1)}$$

say (x, y, μ) is a point lying on the curve through which the generator of the conoid passes, then the other point will be $(0, 0, \mu)$.

Therefore the equation to the generators are,

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{z-\mu}{0} \qquad \text{.....(2)}$$

Also, $$x_1^2 + y_1^2 = r^2, \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{2\mu}{c} \qquad \text{.....(3)}$$

or $$cr^2\left(\frac{\lambda^2}{a^2} + \frac{1}{b^2}\right) = 2\mu\left(1+\lambda^2\right) \qquad [\because x_1 = \lambda y_1] \qquad \text{.....(4)}$$

Eliminating unknown constants λ and μ between (1) and (4), the required locus is

$$cr^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) = 2z\left(x^2+y^2\right).$$

Ex.2. *Find the equation to the right conoid generated by lines which meet OZ, are parallel to the plane XOY and intersect the circle*

$$x = a, y^2 + z^2 = r^2.$$

Sol. The generators of the conoid will project the given curve on the plane $x = a$ is the circle $y^2 + z^2 = r^2$.

Let (x, y, z) or (a, y, z) is point on the circle through which the generators of the conoid pass.

Since the lines meet oz, therefore the other point will be $(0, 0, z)$.

Therefore, the equations to the generating lines are,

$$\frac{x}{a} = \frac{y}{y_1} = \frac{z-z_1}{0} \qquad .....(1)$$

Also, $\qquad y_1^2 + z_1^2 = r^2$

Eliminating y_1, z_1, between (1) and (2), we have

$$x^2 (z^2 - r^2) + a^2 y^2 = 0$$

which is the required equation of the right conoid.

3.5.6 Self-learning exercise-4.

1. Define right conoid.
2. Write equation of conoid.
3. How can we obtain the equation of conoid ?

3.6 Inflexional tangents

3.6.1 Definition :

Let the equation to the line through a point (x_1, y_1, z_1) on a given surface be

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = (u) \qquad .....(3.6.1)$$

The inflexional tangents are the lines which have three point contact inside the given surface where $u = 0$.

Another definition : At a point P where $\vec{r}'' = \vec{0},$ the tangent line is called inflexional and the point P is called point of inflexion.

3.6.2 The equation of the inflexional tangents at a point on given surface :

Let $\zeta = f(\xi, \eta)$ be the equation to the surface, the point of intersection of the line

$$\frac{\xi-x}{l} = \frac{\eta-y}{m} = \frac{\zeta-z}{n} = (\rho) \qquad .....(3.6.2)$$

are given by $z + \eta\,\rho = f(x + l\,\rho, y + m\,\rho)$

$$= f(x,y) + \rho\left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}\right)f + \frac{\rho^2}{2!}\left(l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y}\right)^2 f + \ldots$$

or
$$= f(x,y) + \rho(pl + qm) + \frac{\rho^2}{2!}\left(rl^2 + 2slm + m^2k\right) + \ldots$$

where
$$p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$$

Therefore the equation of the tangent plane at (x, y, z) is

$$(\xi - x)\,p + (\eta - y)q = \zeta - z \qquad .....(3.6.3)$$

and the inflexional tangents are the lines of intersection of the tangent plane and the pair of planes given by

$$r\,(\xi - x)^2 + 2s\,(\eta - x)\,(\eta - y) + t\,(\eta - y)^2 = 0. \qquad .....(3.6.4)$$

3.6.3 Examples :

***Ex.*1.** *Find the inflexional tangent at* (x_1, y_1, z_1) on the surface $y^2z = 4ax$

Sol. The equation to a line through (x_1, y_1, z_1) is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = u\,(\text{say}) \qquad .....(1)$$

The inflexional tangents are the lines which have three point contact inside the surface where $u = 0$.

From equation (1) substituting the values of x, y, z in the equation of surface $y^2z = 4ax$, we get

$$F(u) = (mu + y_1)^2\,(nu + z_1) - 4a\,(lu + x_1) = 0 \qquad .....(2)$$

For three point contact, we have

$$F'(u) = (mu + y_1)\,2m\,(nu + z_1) + (mu + y_1)^2\,n - 4al = 0 \qquad .....(3)$$

$$F''(u) = 2m^2\,(nu + z_1) + 2mn\,(mu + y_1) + 2mn\,(mu + y_1) = 0 \qquad .....(4)$$

At $u = 0$, the above equations (2), (3) and (4) are reduced to

$$y_1^2 z_1 - 4ax_1 = 0 \qquad ....(5)$$

$$2my_1z_1 - ny_1^2 - 4al = 0 \qquad .....(6)$$

$$2m^2z_1 + 2mny_1 + 2mny_1 = 0$$

or
$$mz_1 + 2ny_1 = 0 \qquad .....(7)$$

Using (7), (6) become

$$2my_1z_1 - \frac{mz_1}{2y_1}y_1^2 - 4al = 0 \text{ or } l = \frac{3my_1z_1}{8a}$$

Substituting value of l and n in (1), we get

$$\frac{x - x_1}{(3my_1z_1/8a)} = \frac{y - y_1}{m} = \frac{z - z_1}{-(mz_1/2y_1)}$$

or $$\frac{x-x_1}{\left(3y_1^2 z_1/4a\right)}=\frac{y-y_1}{2y_1}=\frac{z-z_1}{-z_1}$$

or $$\frac{x-x_1}{3x_1}=\frac{y-y_1}{2y_1}=\frac{z-z_1}{-z_1} \qquad \text{[using (5)]}$$

which is the required equation of the inflexional tangent. **Answer**

3.7 Singular points

3.7.1 Definition :

If at a point P (x, y, z) of the surface F $(x, y, z) = 0$

$$\frac{\partial F}{\partial x}=\frac{\partial F}{\partial y}=\frac{\partial F}{\partial z}=0 \qquad .....(3.7.1)$$

then every straight line through P (x, y, z) will meet surface in two coincident points, such a point is called a singular point or the first order on the surface.

3.7.2 Tangents at the singular point :

The straight lines through P (x, y, z) whose direction ratio satisfy the equation

$$\left(l\frac{\partial}{\partial x}+m\frac{\partial}{\partial y}+n\frac{\partial}{\partial z}\right)^2 F=0 \qquad .....(3.7.2)$$

meet the surface in three coincident points at P (x, y, z) and are called the tangents at the singular point.

Eliminating l, m, n from the equations of the straight lines and (2), we get the locus of the system of tangents through P (x, y, z) as the surface

$$(\xi-x)^2\frac{\partial^2 F}{\partial x^2}+...+2(\eta-y)(\zeta-z)\frac{\partial^2 F}{\partial y\partial z}+...=0 \qquad .....(3.7.3)$$

Singular points are classified according to the nature of the locus of the tangent lines represented by (3.7.3) :

(i) if this locus is a proper cone, then the point P is called a conical point or conic node.

(ii) when it is a pair of distinct planes, then the point P is called a biplaner node or binode.

(iii) when the pairs of planes coincide, then the point P is called uniplanar node or unode.

3.7.3 Examples :

***Ex.*1.** *Find and classify the singular points of the surface*

$$xyz - a^2(x + y + z) + 2a^3 = 0$$

Sol. The equation of the surface can be written as,

$$F(x, y, z) = xyz - a^2(x + y + z) + 2a^3 = 0 \qquad .....(1)$$

Differentiating (1) with respect to x, y and z respectively, we get

$$\frac{\partial F}{\partial x} = 0 \Rightarrow yz - a^2 = 0 \text{ or } yz = a^2, \quad \text{.....(2)}$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow xz - a^2 = 0 \text{ or } xz = a^2, \quad \text{.....(3)}$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow xy - a^2 = 0 \text{ or } xy = a^2. \quad \text{.....(4)}$$

From (2), (3) and (4), we get (a, a, a) which is a singular point.

Now, shifting the origin at (a, a, a) by substituting $x = X + a$, $y = Y + a$, $z = Z + a$ the equation of the surface reduce to

$$(X + a)(Y + a)(Z + a) - a^2 (X + a + Y + a + Z + a) + 2a^3 = 0$$

$$= XYZ + a(XY + YZ + ZX) = 0.$$

The locus of the inflexional tangents are

$$a(XY + YZ + ZX) = 0$$

which is an equation of a cone therefore (a, a, a) is a conic node.

Ex.2. *Prove that the z-axis is a nodal line with unodes at the points* (0, 0, – 2) *and* (0, 0, 2/3) *for the surface*

$$2xy + x^3 - 3x^2y - 3xy^2 + y^3 + z(x^2 - xy + y^2) = 0.$$

Sol. The origin is singular point and the locus of the inflexional tangent is

$$2xy = 0 \Rightarrow \text{pair of planes } x = 0 \text{ and } y = 0.$$

Therefore, the origin is binode.

But $x = 0 = y$ is the z-axis and origin lies on oz *i.e.* z-axis is the nodal line. Consider a point (0, 0, – 2) on z-axis and shifting the origin at (0, 0, – 2) by substituting

$$x = X, y = Y, z = Z - 2.$$

The equation of surface is reduced to

$$-2X^2 - 2Y^2 + 4XY + X^3 - 3X^2Y - 3XY^2 + Y^3 + Z(X^2 - XY + Y^2) = 0$$

The locus of inflexional tangents is,

$$-2X^2 - 2Y^2 + 4XY = 0$$

or $$(X - Y)^2 = 0 \Rightarrow X - Y = 0 \quad \text{and} \quad X - Y = 0.$$

These are two coincident planes, therefore (0, 0, – 2) is a unode.

By similar treatment, we can prove (0, 0, 2/3) is also a unode.

3.7.4 Self-learning exercise-5.

1. If the locus is proper cone then singular point is called ..... .
2. Write the other name of singular point.

3.8 Indicatrix

3.8.1 Definition :

Let the plane $z = 0$ be taken as the tangent plane and the z-axis as normal at a given point of the surface.

If $z = f(x, y)$ is the equation of surface, expanding it by Maclaurin's theorem we get

$$z = px + qy + \frac{1}{2}\left(rx^2 + 2sxy + ty^2\right) + \ldots \qquad .....(3.8.1)$$

where $p = \dfrac{\partial z}{\partial x},\ q = \dfrac{\partial z}{\partial y},\ r = \dfrac{\partial^2 z}{\partial x^2},\ s = \dfrac{\partial^2 z}{\partial y\,\partial x},\ t = \dfrac{\partial^2 z}{\partial y^2}$ are the values at the origin.

Since the tangent plane at the origin is $z = 0$, we have $p = 0$ and $q = 0$ and therefore at the origin

$$2z = rx^2 + 2sxy + ty^2 + \ldots \qquad .....(3.8.2)$$

If we neglect the third and higher powers of x and y, the shape of the surface in the neighbourhood of the origin is approximately a conicoid given by

$$2z = rx^2 + 2sxy + ty^2 \qquad .....(3.8.3)$$

This conicoid is a paraboloid or parabolic cylinder according as $rt \neq s^2$ or $rt = s^2$, respectively.

The section of the surface by the plane $z = h$ is the same as the section of the conicoid therefore it is a conic, given by

$$z = h,\ 2h = rx^2 + 2sxy + ty^2 \qquad .....(3.8.4)$$

and is called the indicatrix.

Thus the conic in which a surface is cut by a parallel plane at an infinitesimal distance near the tangent plane at any point is called the indicatrix at the point.

3.8.2 Examples :

***Ex.*1.** *Prove that the indicatrix at a point of the surface $z = f(x, y)$ is a rectangular hyperbola if*

$$(1 + p^2)\,t + (1 + q^2)\,r - 2pq\,s = 0.$$

Sol. The equation of the surface is given by

$$z = f(x, y) \qquad .....(1)$$

The direction cosines of the inflexional tangents are given by

$$lp + mq - n = 0 \qquad .....(2)$$

and

$$l^2 r + 2lms + m^2 t = 0 \qquad .....(3)$$

where $p = \dfrac{\partial z}{\partial x},\ q = \dfrac{\partial z}{\partial y},\ r = \dfrac{\partial^2 z}{\partial x^2},\ s = \dfrac{\partial^2 z}{\partial y\,\partial x},\ t = \dfrac{\partial^2 z}{\partial y^2}.$

Equation (3) may be written as

$$\left(\frac{l}{m}\right)^2 r + 2\left(\frac{l}{m}\right)s + t = 0$$

which gives $$\frac{l_1\, l_2}{m_1\, m_2} = \frac{t}{r}. \qquad .....(4)$$

Also eliminating l between (2) and (3), we get

$$\left(\frac{n-mq}{p}\right)^2 r + 2ms\left(\frac{n-mq}{p}\right) + m^2 t = 0$$

or $$m^2\,(rq^2 + tp^2 - 2pqs) + 2nm\,(-\,qr + ps) + n^2 r = 0$$

for which $$\frac{m_1\, m_2}{n_1\, n_2} = \frac{r}{q^2 r + p^2 t - 2pqs} \qquad .....(5)$$

from equations (4) and (5), we have

$$\frac{l_1\, l_2}{t} = \frac{m_1\, m_2}{r} = \frac{n_1\, n_1}{q^2 r + p^2 t - 2pqs} \qquad .....(6)$$

These inflexional tangents will be right angle if

$$l_1\, l_2 + m_1\, m_2 + n_1\, n_2 = 0 \qquad .....(7)$$

Substituting (6) into (7), we get

$$t + r + q^2 r + p^2 t - 2pqs = 0$$

or $$(1 + p^2)t - 2pqs + (1 + q^2)\, r = 0$$

which is the required condition for the indicatrix to be rectangular hyperbola.

Ex.2. *Prove that the points of the surface*

$$xyz - a\,(yz + zx + xy) = 0$$

at which the Indicatrix is a rectangular hyperbola, lie on the cone

$$x^4\,(y + z) + y^4\,(z + x) + z^4\,(x + y) = 0.$$

Sol. The given surface equation can be written as,

$$z = \frac{axy}{(xy - ay - ax)} \qquad .....(1)$$

Therefore, $$p = \frac{\partial z}{\partial x} = \frac{(xy - ay - ax)(ay) - axy(y - a)}{(xy - ay - ax)^2} = \frac{-z^2}{x^2} \qquad .....(2)$$

Similarly $$q = \frac{-z^2}{y^2},\ r = \frac{\partial^2 z}{\partial x^2} = \frac{2z^2(z + x)}{x^4}$$

$$s = \frac{\partial^2 z}{\partial x\, \partial y} = \frac{2z^3}{x^2 y^2} \text{ and } t = \frac{\partial^2 z}{\partial y^2} = \frac{2z^2(z + y)}{y^4} \qquad .....(3)$$

Substituting these values into the equation of indicatrix, we have

$$\left(1+\frac{z^4}{x^4}\right)\frac{2z^2(z+y)}{y^4}+\left(1+\frac{z^4}{y^4}\right)\frac{2z^2(z+x)}{x^4}-\frac{2z^2}{x^2}\times\frac{z^2}{y^2}\times\frac{2z^3}{x^2y^2}=0$$

or $$2z^2\{x^4(y+z)+y^4(z+x)+z^4(x+z)\}=0$$

Therefore the required locus is

$$x^4(y+z)+y^4(z+x)+z^4(x+y)=0 \qquad (\text{since } z\neq 0) \qquad .....(4)$$

which is a cone.

3.9 Answers to self-learning exercises

Self-learning exercise-1

1. Three
2. Linear relation
3. Constant angle

Self-learning exercise-2

1. $K_1=\frac{\left(\tau^2+K^2\right)^{1/2}}{\mu K}$, where $\mu=c-s$
2. $\tau_1=\pm\frac{\left(K\tau'-\tau K'\right)^{1/2}}{\mu K\left(\tau^2+K^2\right)}$, where $\mu=c-s$
3. $\vec{r}_1=\vec{r}+(c-s)\hat{t}$

Self-learning exercise-3

1. $\vec{r}_1=\vec{r}+\rho\hat{n}-\rho\tan(\psi+c)\hat{b}.$
2. $K_1=\frac{K^3\cos^3(\psi+c)}{K\tau\sin(\psi+c)-K'\cos(\psi+c)}$
3. $\tau_1=\frac{-K^3\sin(\psi+c)\cos^2(\psi+c)}{Kt\sin(\psi+c)-K'\cos(\psi+c)}$
4. $\frac{\tau_1}{K_1}=-\tan(\psi+c).$

Self-learning exercise-4

1. **Right conoid :** If the given line is at right angles to the given plane, the locus is a right conoid.
2. $z=f(y/x)$.
3. By intersection of fixed line and given plane.

Self-learning exercise-5

1. Conic node.

2. First order on the surface.

3.10 Exercises

1. State and prove existence and uniqueness theorems.

2. Prove that the distance between corresponding points of two curves is constant.

3. Show that the involutes of a circular helix are plane curves.

4. Write down the equation of conoid.

5. Find the equation to the conoid generated by lines parallel to the plane *XOY*, which are drawn to intersect *OZ* and the curve

$$x^2 + y^2 + z^2 = b^2, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

[Ans. $\left(b^2 - z^2\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = \left(1 - \frac{z^2}{c^2}\right)\left(x^2 + y^2\right)$]

6. Find and classify the singular points of the surfaces

(i) $xyz = ax^2 + by^2 + cz^2$ **[Ans.** (0, 0, 0) is a conic node**]**

(ii) $xyz - a^2 (x + y + z) + 2a^3 = 0.$ **[Ans.** (a, a, a) is a conic node**]**

7. Prove that the indicatrix at every point of the helicoid $z = c \tan^{-1} (y/x)$ is a rectangular hyperbola.

8. Prove that every point on a cone or cylinder is a parabolic point.

□ □ □

UNIT 4 : Envelope, Edge of Regression, Ruled Surfaces, Developable Surface, Tangent Plane to a Ruled Surface

Structure of the Unit

4.0 Objectives

After studying this unit you will be able to understand :

1. characteristic, envelope and edge of regression of family of surfaces,
2. ruled surfaces, their classification and associated properties.

4.1 Introduction

Family of surfaces admit certain geometrical features such as characteristic and edge of regression which are in fact curves lying on the surface. Their study is of vital importance in the theory of differential geometry. Similarly, envelope of family of surfaces has a unique property that it touches each member of the family of surfaces.

There are many surfaces which are generated due to motion of straight lines. Such surfaces are called ruled surfaces. This includes their classification as developable, skew surfaces and associated properties.

4.2 Envelope

4.2.1 Family of surfaces (One parameter) :

An equation

$$F(x, y, z, a) = 0 \qquad .....(4.2.1)$$

where a is a parameter, represents a family of surfaces. By assigning different real values to the parameter a we get different surfaces belonging to family given by (4.2.1). For specific value to a, we get a specific surface of the family and is called member of the family of the surfaces.

4.2.2 Characteristic of a family of surfaces :

Characteristic of a surfaces is the curve of intersection of two consecutive surfaces.

Let $$F(x, y, z, \alpha) = 0, \quad F(x, y, z, \alpha + \delta\alpha) = 0 \qquad .....(4.2.2)$$

be two consecutive surfaces of the family given by (4.2.1). Then the curve of intersection of the consecutive surfaces (4.2.2) is given by

$$F(x, y, z, \alpha) = 0, \frac{F(x, y, z, \alpha + \delta\alpha) - F(x, y, z, \alpha)}{\delta\alpha} = 0 \qquad .....(4.2.3)$$

The limiting position of the curve as $\delta\alpha \to 0$ is obtained as

$$F(x, y, z, \alpha) = 0, \frac{\partial F}{\partial \alpha} = 0, \qquad .....(4.2.4)$$

which determines the characteristic curve corresponding to the value α.

4.2.3 Envelope :

The concept of envelope of a family of surface is very important. The envelope of a family of surfaces touches every member of the family, at all points of its characteristic. Geometrically, the envelope of the family of surfaces is the locus of characteristic for all values of the parameter. Hence, the envelope is obtained from the equation

$$F = 0, \frac{\partial F}{\partial \alpha} = 0, \qquad .....(4.2.5)$$

by eliminating α.

Theorem 1. *The envelope of a family of surfaces touches each member of the family at all points of its characteristic.*

Proof : Let $F(x, y, z, a) = 0$ be the family of surfaces, where a being the parameter.

Let $(x, y, z, \alpha) = 0$, $F(x, y, z, \alpha + \delta\alpha) = 0$ be any two consecutive surfaces of the given family. Then the envelope is obtained by eliminating α from the equations

$$F = 0, \ \frac{\partial F}{\partial \alpha} = 0.$$

Consider $(x, y, z, \alpha) = 0$ as equation of the envelope, where α is not merely a constant but a function of x, y, z satisfying $\dfrac{\partial F}{\partial \alpha} = 0.$

Now, the normal to the envelope

$$F(x,y,z,\alpha)=0 \quad \left(\text{where } \frac{\partial F}{\partial \alpha}=0\right)$$

is parallel to the vector ∇F *i.e.* parallel to the vector

$$\left(\frac{\partial F}{\partial x}+\frac{\partial F}{\partial \alpha}\frac{\partial \alpha}{\partial x}\right)\hat{i}+\left(\frac{\partial F}{\partial y}+\frac{\partial F}{\partial \alpha}\frac{\partial \alpha}{\partial y}\right)\hat{j}+\left(\frac{\partial F}{\partial z}+\frac{\partial F}{\partial \alpha}\frac{\partial \alpha}{\partial z}\right)\hat{k}$$

or
$$\left(\frac{\partial F}{\partial x}\right)\hat{i}+\left(\frac{\partial F}{\partial y}\right)\hat{j}+\left(\frac{\partial F}{\partial z}\right)\hat{k}. \qquad \left[\because \frac{\partial F}{\partial \alpha}=0\right]$$

The vector $\sum\left(\frac{\partial F}{\partial x}\right)\hat{i}$ is parallel to normal to the surface $F(x, y, z, \alpha) = 0$. This reveals that at all common points, the surface and the envelope admit the same normal, and consequently the same tangent plane. This concludes that surface and envelope touch each other at all points of the characteristic.

Note : Characteristic of the envelope is the curve in which two consecutive surfaces intersect. Thus, each characteristic lies on the envelope.

4.2.4 Edge of regression :

Edge of regression is a curve that lies on the envelope. We have seen that the characteristic is the curve in which two consecutive surfaces intersect.

Two consecutive characteristics meet in one or more points. The locus of points of intersection of consecutive characteristics is called the edge of regression of the envelope. Obviously, the edge of regression (a curve) lies on the envelope simply because every characteristic lies on the envelope. Edge of regression may have the following formal definition :

"Edge of regression is the locus of the ultimate points of intersection of consecutive characteristics of one parameter family of surfaces".

Equation of the edge of regression of the envelope :

Let
$$F(x, y, z, a) = 0 \qquad \text{......(4.2.6)}$$
be the family of surfaces, a being the parameter.

Let $F(x, y, z, \alpha) = 0$ and $F(x, y, z, \alpha + \delta\alpha) = 0$,

be two consecutive surfaces. Then the characteristic to the surface $F(x, y, z, \alpha) = 0$ is given by

$$F(x,y,z,\alpha)=0, \quad \frac{\partial F}{\partial \alpha}=0 \qquad \text{......(4.2.7)}$$

The characteristic to the surface $F(x, y, z, \alpha + \delta\alpha) = 0$ is given by

$$F(x,y,z,\alpha+\delta\alpha)=0, \quad \frac{\partial F(x,y,z,\alpha+\delta\alpha)}{\partial \alpha}=0 \qquad \text{......(4.2.8)}$$

Expanding the equations (4.2.8) be Taylor's series, we get

$$F(x,y,z,\alpha)+\delta\alpha\,\frac{\partial F}{\partial \alpha}+.....=0$$

$$\Rightarrow \qquad \frac{\partial F}{\partial \alpha}+\delta\alpha\,\frac{\partial^2 F}{\partial \alpha^2}+....=0 \qquad .....(4.2.9)$$

From equations (4.2.7) and (4.2.9), we obtain

$$F=0,\ \frac{\partial F}{\partial \alpha}=0,\ \frac{\partial^2 F}{\partial \alpha^2}=0 \qquad .....(4.2.10)$$

the edge of regression is obtained by eliminating α from the equations (4.2.10).

Theorem 2. *Each characteristic touches the edge of regression.*

Proof : Let $(x, y, z, a) = 0$, be family of surfaces. Then for $a = \alpha$ and $a = \alpha + \delta\alpha$,

$$F(x, y, z, \alpha) = 0 \qquad .....(4.2.11)$$

and

$$F(x, y, z, \alpha + \delta\alpha) = 0 \qquad .....(4.2.12)$$

are two consecutive surfaces.

Thus the characteristic curve corresponding to the surface $F(x, y, z, \alpha) = 0$ is given by

$$F=0,\ \frac{\partial F}{\partial \alpha}=0 \qquad .....(4.2.13)$$

and the edge of regression is given by

$$F=0,\ \frac{\partial F}{\partial \alpha}=0,\ \frac{\partial^2 F}{\partial \alpha^2}=0 \qquad .....(4.2.14)$$

We can consider edge of regression given by

$$F=0,\ \frac{\partial F}{\partial \alpha}=0$$

provided α is a function of x, y, z given by

$$\frac{\partial^2 F}{\partial \alpha^2}=0.$$

Note that the tangent at any point $P(x, y, z)$ to the edge of regression is nothing but the line of intersection of the tangent planes to the surface. Consequently the tangent is normal to the vectors ∇F and ∇F_α, where α is function of x, y, z.

Thus this tangent is perpendicular to the vectors

$$\sum\left(\frac{\partial F}{\partial x}+\frac{\partial F}{\partial \alpha}\,\frac{\partial \alpha}{\partial x}\right)\hat{i}, \qquad .....(4.2.15)$$

and

$$\sum\left(\frac{\partial F_\alpha}{\partial x}+\frac{\partial F_\alpha}{\partial \alpha}\,\frac{\partial \alpha}{\partial x}\right)\hat{i}, \qquad .....(4.2.16)$$

Using $\frac{\partial F}{\partial \alpha} = 0,\ \frac{\partial^2 F}{\partial \alpha^2} = 0,$ the equations (4.2.15) and (4.2.16) are reduces to

$$\sum \left(\frac{\partial F}{\partial x} \right) \hat{i}, \qquad .....(4.2.17)$$

and

$$\sum \left(\frac{\partial^2 F}{\partial x \partial \alpha} \right) \hat{i}. \qquad .....(4.2.18)$$

Vectors (4.2.17) and (4.2.18) are perpendiculars to tangent planes at $P(x, y, z)$ to the characteristic $F(x, y, z, \alpha) = 0,\ \frac{\partial F(x, y, z, \alpha)}{\partial \alpha} = 0.$ This concludes that the tangent to the edge of regression is parallel to the tangent to the characteristic and consequently the two curves touch at their common points.

4.2.5 Family of surfaces (two parameters) :

We now proceed for the case of envelope of two-parameter family of surfaces.

Envelope of two parameter family of surfaces.

Let $$F(x, y, z, a, b) = 0 \qquad .....(4.2.19)$$

where a, b are parameters, denote a family of surfaces.

Then the consecutive surfaces for $a = \alpha$, $b = \beta$ are

$$F(x, y, z, \alpha, \beta) = 0 \qquad .....(4.2.20)$$

$$F(x, y, z, \alpha + \delta\alpha, \beta + \delta\beta) = 0 \qquad .....(4.2.21)$$

On expanding (4.2.21) by Taylor's series, we get

$$F(x, y, z, \alpha, \beta) + \left(\frac{\partial F}{\partial \alpha} \delta\alpha + \frac{\partial F}{\partial \beta} \delta\beta \right) + ... = 0$$

when $\delta\alpha \to 0$, $\delta\beta \to 0$ we ought to have at a point of intersection in the limiting case :

$$F = 0,\ \frac{\partial F}{\partial \alpha} \delta\alpha + \frac{\partial F}{\partial \beta} \delta\beta = 0. \qquad .....(4.2.22)$$

Further since $\delta\alpha$, $\delta\beta$ are mutually independent then the identity (4.2.22) is line if

$$F = 0,\ \frac{\partial F}{\partial \alpha} = 0,\ \frac{\partial F}{\partial \beta} = 0. \qquad .....(4.2.23)$$

Thus, we conclude that the criterion given by (4.2.23) is mandatory for two consecutive surfaces given by (4.2.20) and (4.2.21) to interest. On elimination of α, β we get the equation of the envelope of two parameter family of surfaces.

***Ex.*1.** *Suppose that a tangent plane to the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

meets the coordinate axes in points P, Q, R. Prove that the envelope of the sphere OPQR is

$$(ax)^{2/3} + (by)^{2/3} + (cz)^{2/3} = (x^2 + y^2 + z^2)^{2/3}$$

where 0 *is the origin*.

Sol. Let us consider

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1 \qquad .....(1)$$

to be the tangent plane to the ellipsoid

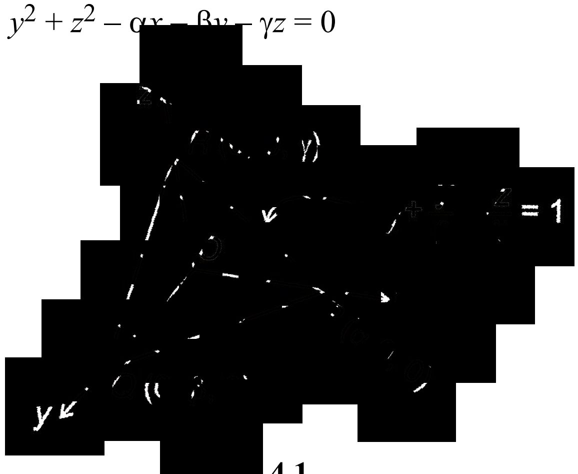
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad .....(2)$$

Then the condition of tangency ensures that

$$\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2} = 1 \qquad .....(3)$$

Given that the tangent plane (1) meets the axes in points P, Q, R, then the equation to the sphere $OPQR$ is

$x^2, y^2 + z^2 -$ [redacted] $\gamma z = 0$ .....(4)



[redacted] **4.1**

Note that for variable values of α, β, γ we would have different tangent planes to the ellipsoid and consequently different spheres of the form (4), *i.e.*, (4) constitutes a family of surfaces, where α, β, γ are parameters.

We denote $$F(x, y, z, \alpha, \beta, \gamma) \equiv x^2 + y^2 + z^2 - \alpha x - \beta y - \gamma z = 0 \qquad .....(5)$$

and $$\phi(\alpha, \beta, \gamma) \equiv \frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2} - 1 = 0 \qquad .....(6)$$

The equation of the surfaces given by (5) is obtained on elimination of the parameters α, β, and γ from the equations

$$\frac{F_\alpha}{\phi_\alpha} = \frac{F_\beta}{\phi_\beta} = \frac{F_\gamma}{\phi_\gamma}.$$

This gives $$\frac{-x}{\left(-2a^2/\alpha^3\right)} = \frac{-y}{\left(-2b^2/\beta^3\right)} = \frac{-z}{\left(-2c^2/\gamma^3\right)}$$

or $$\frac{\alpha x^{1/3}}{a^{2/3}} = \frac{\beta y^{1/3}}{b^{2/3}} = \frac{\gamma z^{1/3}}{c^{2/3}} = k \ \text{(say)}$$

or $$\alpha = \frac{k\,a^{2/3}}{x^{1/3}},\ \beta = \frac{k\,b^{2/3}}{y^{1/3}},\ \gamma = \frac{k\,c^{2/3}}{z^{1/3}} \qquad \text{......(7)}$$

The values of α, β, γ as obtained in (7) are now put in (3) and (5) to yield

$$\frac{a^2 x^{2/3}}{a^{4/3}} + \frac{b^2 y^{2/3}}{b^{4/3}} + \frac{c^2 z^{2/3}}{c^{4/3}} = K^2$$

or $$(ax)^{2/3} + (by)^{2/3} + (cz)^{2/3} = K^2 \qquad \text{......(8)}$$

and $$x^2 + y^2 + z^2 - x\left(\frac{k\,a^{2/3}}{x^{1/3}}\right) - y\left(\frac{k\,b^{2/3}}{y^{1/3}}\right) - z\left(\frac{k\,c^{2/3}}{z^{1/3}}\right) = 0$$

or $$x^2 + y^2 + z^2 = k\left\{(ax)^{2/3} + (by)^{2/3} + (cz)^{2/3}\right\}$$

or $$K = \frac{x^2 + y^2 + z^2}{(ax)^{2/3} + (by)^{2/3} + (cz)^{2/3}}. \qquad \text{.....(9)}$$

From (8) and (9), we get the required result.

***Ex.*2.** *Find the envelope of the family of planes*

$$F(x, y, z, \theta, \phi) \equiv \frac{x}{a}\cos\theta\sin\phi + \frac{y}{b}\sin\theta\sin\phi + \frac{z}{c}\cos\phi - 1 = 0.$$

Sol. We have,

$$F(x, y, z, \theta, \phi) \equiv \frac{x}{a}\cos\theta\sin\phi + \frac{y}{b}\sin\theta\sin\phi + \frac{z}{c}\cos\phi - 1 = 0 \qquad \text{.....(1)}$$

The required envelope is obtained by the elimination of the parameters θ and ϕ from the equations

$$F = 0,\ \frac{\partial F}{\partial \theta} = 0,\ \frac{\partial F}{\partial \phi} = 0$$

On differentiating (1) partially with respect to θ and ϕ, respectively, we obtain.

$$\frac{\partial F}{\partial \theta} = -\frac{x}{a}\sin\theta\sin\phi + \frac{y}{b}\cos\theta\sin\phi = 0$$

$$= -\frac{x}{a}\sin\theta + \frac{y}{b}\cos\theta = 0 \qquad [\because\ \sin\phi \neq 0] \qquad \text{......(2)}$$

$$\frac{\partial F}{\partial \phi} = \frac{x}{a}\cos\theta\cos\phi + \frac{y}{b}\sin\theta\cos\phi - \frac{z}{c}\sin\phi = 0 \qquad \text{......(3)}$$

Equation (2) gives, $$\tan\theta = \frac{ay}{bx} \qquad \text{......(4)}$$

Equation (3) gives, $$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = \frac{z}{c}\tan\phi \qquad \text{......(5)}$$

The equation (1) can be rewritten as

$$\left(\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta\right)\sin\phi=1-\frac{z}{c}\cos\phi \qquad ......(6)$$

using (5) in (6), we get
$$\left(\frac{z}{c}\tan\phi\right)\sin\phi=1-\frac{z}{c}\cos\phi$$

or
$$\left(\frac{z}{c}\frac{\sin\phi}{\cos\phi}\right)\sin\phi=1-\frac{z}{c}\cos\phi$$

or
$$\frac{z}{c}\left(\frac{\sin^2\phi+\cos^2\phi}{\cos\phi}\right)=1$$

or
$$\cos\phi=\frac{z}{c} \qquad ......(7)$$

Now using (7) in (5), we get

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=\frac{z}{c}\sqrt{\frac{c^2}{z^2}-1} \qquad \left[\because\ \cos\phi=\frac{z}{c}\Rightarrow\tan\phi=\sqrt{\frac{c^2}{z^2}-1}\right] \qquad ......(8)$$

On squaring (2) and (8) and then on adding we get

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}\left(\frac{c^2}{z^2}-1\right)$$

or
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

as the required envelope of the given family of planes.

Ex.3. *Find the equation of the developable surface whose generating line passes through the curve $y^2 = 4ax$, $z = 0$; $x^2 = 4ay$, $z = c$ and show that its edge of regression is given by*

$$cx^2-3ayz=0=cy^2-3ax\,(c-z).$$

Sol. Recall that a developable surface is generated by one parameter family of planes. In order to find the equation of the required developable surface we first find the family of planes $\boldsymbol{F\,(m)=0}$ (where m is parameter). The developable is obtained eliminating m from $F\,(m) = 0$ and $\dot{F}\,(m)=0.$ We proceed as follows.

The equation to the tangent to the curve $y^2 = 4ax$, $z = 0$ is

$$y=mx+\frac{a}{m},\ z=0 \qquad .....(1)$$

Then any plane touching the parabola $y^2 = 4ax$, $z = 0$ is

$$\left(y-mx-\frac{a}{m}\right)+\lambda z=0, \quad \text{(where } \lambda \text{ being scalar)} \qquad .....(2)$$

The section of the plane (2) by the plane $z = c$ is

$$y + \lambda c - mx - \frac{a}{m} = 0 \qquad ......(3)$$

or
$$x = \frac{y}{m} + \left(\frac{\lambda c}{m} - \frac{a}{m^2}\right)$$

If (3) touches the parabola $x^2 = 4ay, z = c,$ then the equation

$$\left[\frac{y}{m} + \left(\frac{\lambda c}{m} - \frac{a}{m^2}\right)\right]^2 = 4ay,$$

must have equal roots

i.e.,
$$\frac{\lambda^2 c^2}{m^2} + \left(\frac{y}{m} - \frac{a}{m^2}\right)^2 + 2\lambda\frac{c}{m}\left(\frac{y}{m} - \frac{a}{m^2}\right) - 4ay = 0,$$

must have equal roots.

This gives
$$\lambda = \frac{am^2}{c} + \frac{a}{mc}.$$

Putting this value of λ in (2), we get the plane touching both the given curves and it is

$$F(m) = y - mx - \frac{a}{m} + \left(\frac{am^2}{c} + \frac{a}{cm}\right)z = 0$$

or
$$F(m) \equiv \left(am^3 + a\right)\frac{z}{c} + my - m^2 x - a = 0 \qquad .....(4)$$

This gives
$$\dot{F}(m) = 3am^2\frac{z}{c} + y - 2mx = 0. \qquad .....(5)$$

Elimination of m from (4) and (5) will give the required developable surface.

Edge of regression :

Differentiating (5) partially with respect to m, we get

$$\ddot{F}(m) \equiv 6am\frac{z}{c} - 2x = 0$$

This gives
$$m = \frac{cx}{3az}. \qquad ......(6)$$

Note that the edge of regression is given by F (m) = 0, $\dot{F}(m) = 0,$ $\ddot{F}(m) = 0.$ Hence putting the value of m From (6) in $\dot{F}(m) = 0,$ we find

$$3a\frac{z}{c}\left[\frac{cx}{3az}\right]^2 + y - 2\left[\frac{cx}{3az}\right]x = 0$$

On simplification, we get

$$6x^2 - 3ayz = 0 \qquad ......(7)$$

Again, on putting the value of m in (4) and performing simplification, we get

$$27a^3z^3 + 9azxyc^2 - 2c^3x^3 - 27ca^3z^2 = 0.$$

Using (7) in the above equation, we obtain

$$27a^3z^2 (z - c) - 2c^2x (3ayz) + 9azxyc^2 = 0.$$

or $$27a^3 \cdot \frac{c^2x^4}{9a^2y^2}(z-c)+(3ayz)c^2x=0$$

or $$\frac{3ac^2x^4}{y^2}(z-c)+\left(cx^2\right)c^2x=0$$

or $$cy^2 - 3ax (c - z) = 0$$

Hence the edge of regression is given by

$$cx^2 - 3ayz = 0 = cy^2 - 3ax (c - z).$$

Ex.4. *Find the equation of the developable surface which contains the two curves*

$$y^2 = 4ax, \quad z = 0 \quad and \quad (y - b)^2 = 4cz, \quad x = 0$$

and show that its edge of regression lies on the surface

$$(ax + by + cz)^2 = 3abx (b + y).$$

Sol. The given curves are

$$y^2 = 4ax,\ z = 0 \qquad .....(1)$$

$$(y - b)^2 = 4cz,\ x = 0 \qquad ......(2)$$

The equation to the tangent to the curve (1) is

$$y = mx + \frac{a}{m},\ z = 0, \text{(where } m \text{ is the slope)} \qquad .....(3)$$

Now, the equation to the plane that touches the parabola (1) is

$$\left(y - mx - \frac{a}{m}\right) + \lambda z = 0, \text{ (where } \lambda \text{ is a scalar)} \qquad .....(4)$$

If the plane (4) touches the curve (2), then it means that the line $y = \frac{a}{m} - \lambda z$ touches the curve

$$(y - b)^2 = 4cz.$$

That is, $\left(\frac{a}{m} - \lambda z - b\right)^2 = 4cz$ must have equal roots.

$\Rightarrow$ $$\lambda^2 z^2 - z\left[4c + 2\lambda\left(\frac{a}{m} - b\right)\right] + \left(\frac{a}{m} - b\right)^2 = 0 \text{ must have equal roots.}$$

i.e. $$\left[4c + 2\lambda\left(\frac{a}{m} - b\right)\right]^2 = 4\lambda^2\left(\frac{a}{m} - b\right)^2$$

On simplification, we get

$$\lambda = \frac{mc}{bm - a} \qquad \text{.....(5)}$$

On putting the value of λ from (5), in (4), we find the equation of the plane touching the curve (2), and it is

$$y - mx - \frac{a}{m} + \frac{mcz}{bm - a} = 0 \qquad \text{.....(6)}$$

or on simplifying it becomes

$$bm^3 x - m^2 (ax + by + cz) + am (y + b) - a^2 = 0 \qquad \text{.....(7)}$$

We denote the surface (7) by $F(m)$.

Developable surface :

We know that a developable surface of the surface $F(m) = 0$ is obtained by eliminating the parameter m from the equations $F(m) = 0$ and $\dot{F}(m) = 0$.

Differentiating (6) with respect to m, we get

$$\dot{F}(m) = 3bm^2 x - 2m(ax + by + cz) + a(b + y) = 0 \qquad \text{.....(8)}$$

From (7) and (8), we obtain

$$m = \frac{a(b + y)(ax + by + cz) - 9a^2 bx}{2(ax + by + cz)^2 - 6abx(y + b)} \qquad \text{.....(9)}$$

Using this value of m in (7), we get the required developable surface.

Edge of regression :

We know that edge of regression for the surface $F(m) = 0$ is obtained on elimination of m from, $F(m) = 0$, $\dot{F}(m) = 0$ and $\ddot{F}(m) = 0$, we have

$$\ddot{F}(m) = 6bmx - 2(ax + by + cz) = 0 \qquad \text{[from (7)]}$$

This gives, $$m = \frac{ax + by + cz}{3bx}. \qquad \text{.....(10)}$$

Putting this value of m is (7), we get

$$3bx\left[\frac{ax + by + cz}{3bx}\right]^2 - 2\left[\frac{ax + by + cz}{3bx}\right](ax + by + cz) + a(b + y) = 0$$

or $$(ax + by + cz)^2 = 3ab(b + y), \qquad \text{.....(11)}$$

which gives the surface on which edge of regression lies.

***Ex.*5.** *Find the developable surface which passes through the curves*

$$y^2 = 4ax, \quad z = 0 \quad \textit{and} \quad y^2 = 4bz, x = 0.$$

Sol. The equation to the tangent to the curves

$$y^2 = 4ax, z = 0 \qquad \text{.....(1)}$$

is $$y = mx + \frac{a}{m},\ z = 0 \qquad .....(2)$$

Therefore, equation to the plane touching (1) is

$$\left(y - mx - \frac{a}{m}\right) + \lambda z = 0 \qquad .....(3)$$

[Remember that here *m* is the slope of the tangent and is a parameter].

Equation (3) touches the curve

$$y^2 = 4bz,\ \ x = 0 \qquad .....(4)$$

That means $$y = \frac{a}{m} - \lambda z \qquad \text{[on putting } x = 0 \text{ in (4)]}$$

and $\left[\frac{a}{m} - \lambda z\right]^2 = 4bz$ has equal roots *i.e.*, discriminant is zero. [putting $y = \frac{a}{m} - \lambda z$ in the equation $y^2 = 4bz$]

or $$\lambda^2 z^2 - \left(4b + \frac{2a\lambda}{m}\right)z + \frac{a^2}{m^2} = 0,$$

has equal roots. That is $$\left[4b + \frac{2a\lambda}{m}\right]^2 = \frac{4\lambda^2 a^2}{m^2}$$

or $$\lambda = -\frac{bm}{a} \qquad .....(5)$$

Putting the value of λ in the equation (3), we get the equation of the plane touching the given curve as

$$F(m) \equiv y - mx - \frac{a}{m} - \frac{bm}{a}z = 0. \qquad .....(6)$$

Developable surface :

We know that the developable surface is obtained by eliminating *m* from the equations $F(m) = 0,\ \dot{F}(m) = 0.$ Now, differentiating (6) partially with respect to *m*, we obtain

$$\dot{F}(m) = -x + \frac{a}{m^2} - \frac{bz}{a} = 0 \qquad .....(7)$$

Equation (6) can be written as

$$y - m\left[x + \frac{a}{m^2} + \frac{bz}{a}\right] = 0$$

or $$y - m\left[\frac{a}{m^2} + \frac{a}{m^2}\right] = 0 \qquad \left[\because \text{ From (7)} \quad x + \frac{bz}{a} = \frac{a}{m^2}\right]$$

or $$y - \frac{2a}{m} = 0$$

or $$m = \frac{2a}{y} \qquad \text{.....(8)}$$

On putting the value of m in equation (7), we get the developable surface as

$$y^2 = 4ax + 4bz.$$

Ex.6. *Show that the edge of regression of the developable that passes through the parabolas $x = 0$, $z^2 = 4ay$; $y^2 = 4az$, $x = a$ is given by*

$$\frac{3x}{y} = \frac{y}{z} = \frac{z}{3(a-x)}.$$

Sol. In order to find the required edge of regression we have to first find the plane that touches both the given curves.

Equation to the tangent to the parabola $x = 0$, $z^2 = 4ay$ is

$$z = my + \frac{a}{m}, \; x = 0.$$

Then the plane through this tangent (*i.e.* touching the parabola $x = 0$, $z^2 = 4ay$) is

$$\left(z - my - \frac{a}{m}\right) + \lambda x = 0 \qquad \text{.....(1)}$$

or $$y = \frac{z}{m} - \frac{a}{m^2} + \frac{\lambda x}{m} = 0 \qquad \text{.....(2)}$$

Equation (2) meets the parabola $y^2 = 4az$, $x = a$ therefore its section by $x = a$ is

$$y = \frac{z}{m} - \frac{a}{m^2} + \frac{\lambda a}{m} \qquad \text{......(3)}$$

Now, if equation (3) touches the parabola $y^2 = 4az$, $x = a$.

Then the equation $$\left(\frac{z}{m} - \frac{a}{m^2} + \frac{\lambda a}{m}\right)^2 = 4az$$

must have equal roots, *i.e.* $(mz - a + \lambda am)^2 = 4azm^2$

must have equal roots, *i.e.* $m^2 z^2 + [2\lambda am^2 - 2ma - 4am^4]\, z + a^2 (m\lambda - 1)^2 = 0$ has equal roots.

Thus we must have $[2\lambda am^2 - 2ma - 4am^4]^2 = 4a^2 m^2 (m\lambda - 1)^2.$

On solving we get $$\lambda = \frac{1 + m^3}{m} \qquad \text{.....(4)}$$

Putting this value of λ in (2) we find the plane that touches both the given parabolas as

$$y = \frac{z}{m} - \frac{a}{m^2} + \frac{1 + m^3}{m} \cdot x \qquad \text{.....(5)}$$

or $$F(m) = m^3 x - m^2 y + mz + (x - a) = 0 \qquad \text{.....(6)}$$

Edge of regression :

The edge of regression is given by

$$F(m)=0,\ \dot{F}(m)=0,\ \ddot{F}(m)=0$$

From equ. (4), we find $\quad \dot{F}(m)=3m^2x-2my+z=0 \qquad$ .....(7)

$$\ddot{F}(m)=6mx-2y=0$$

$\Rightarrow \qquad m=\dfrac{y}{3x} \qquad$ .....(8)

Equation (7) can be written as

$$m^2-2m\frac{y}{3x}+\frac{z}{3x}=0$$

or $\qquad m^2-2m\cdot(m)+\dfrac{z}{3x}=0 \qquad$ [putting $m=\dfrac{y}{3x}$]

or $\qquad m^2=\dfrac{z}{3x} \qquad$ .....(9)

Note that to obtain the answer in required form we have to perform some tricky mathematical manipulation as follows.

Dividing (4) by x, we obtain

$$m^3-m^2\frac{y}{x}+m\frac{z}{x}+1-\frac{a}{x}=0$$

or $\qquad m^3-3m^3+3m^3+1-\dfrac{a}{x}=0 \qquad$ [using (6), (7)]

or $\qquad m^3=\dfrac{a-x}{x} \qquad$ .....(10)

Now, we write $\qquad m^3=m^2m$

$\Rightarrow \qquad \dfrac{a-x}{x}=\dfrac{z}{3x}\cdot\dfrac{y}{3x} \qquad$ [using (6), (7), (8)]

or $\qquad \dfrac{3x}{y}=\dfrac{z}{3(a-x)} \qquad$ .....(11)

Again, $\qquad m^3\cdot m=(m^2)^2$

$\Rightarrow \qquad \left(\dfrac{a-x}{x}\right)\left(\dfrac{y}{3x}\right)=\left(\dfrac{z}{3x}\right)^2$

$\Rightarrow \qquad \dfrac{y}{z}=\dfrac{z}{3(a-x)}. \qquad$ .....(12)

From (11) and (12), the required edge of regression is obtained as

$$\frac{3x}{y}=\frac{y}{z}=\frac{z}{3(a-x)}.$$

4.2.6 Self-learning exercises-1

1. Characteristic of the family of surfaces $F(x, y, z, \alpha) = 0$ is given by

(a) $F = 0,\ \dfrac{\partial^2 F}{\partial \alpha^2} = 0$ (b) $\dfrac{\partial^2 F}{\partial \alpha^2} = 0,\ \dfrac{\partial^3 F}{\partial \alpha^3} = 0$

(c) $F = 0$ (d) $F = 0,\ \dfrac{\partial F}{\partial \alpha} = 0$

2. Which of the following is not true ?

(a) characteristic may not lie on the envelope

(b) envelope touches each member of the family

(c) edge of regression is a curve

(d) edge of regression lies on the envelope.

3. Find the envelope of the plane $lx + my + nz = 0$, where $al^2 + bm^2 + cn^2 = 0$.

4.3 Ruled surface

You are familiar with the surfaces such as cones, cylinders, hyperboloid of one sheet and hyperbolic paraboloid. All these surfaces are generated by single parameter family of straight lines. But things are not that simple and we need a further analysis. Hence we define ruled surfaces.

Ruled surface : A ruled surface is a surface which is generated by single parameter family of straight lines. The line is called the **generating line** or **ruling** or **generator** of the ruled surface. All the surfaces mentioned above are obviously ruled surfaces. Ruled surfaces are classified into two categories depending upon intersection/non intersection of their consecutive generators.

Ruled surface on which consecutive generators intersect is called **developable surface**. Cones, cylinders and conicoid are developable surfaces. A ruled surface on which two consecutive generators do not intersect is called a **skew surface** or a scroll. Hyperboloid of one sheet and hyperbolic paraboloid are scrolls.

4.3.1 Equation to a ruled surface :

To find the equation of a ruled surface let us first explain directrix or base curve of a ruled surface. A curve C on the ruled surface is called the **base curve** if it meets each generator exactly once. Note that a ruled surface has many base curves. Now note that a ruled surface is determined by a base curve C, say and the direction of the generator at the point of intersection of the generator and the base curve C. After understanding the above now we find expression for a ruled surface.

Let P be a general point on the ruled surface and Q be a point on the base curve C. Let $\vec{r}(s)$ and $\vec{R}$ be position vectors of Q and P respectively with respect to the origin. Further we assume that $\hat{g}(s)$ be the unit vector along the generator at Q then the equation to the ruled surface is

$$\vec{R} = \vec{r}(s) + \lambda \hat{g}(s) \qquad .....(4.3.1)$$

where λ is a parameter that determines the directed distance along the generator from C.

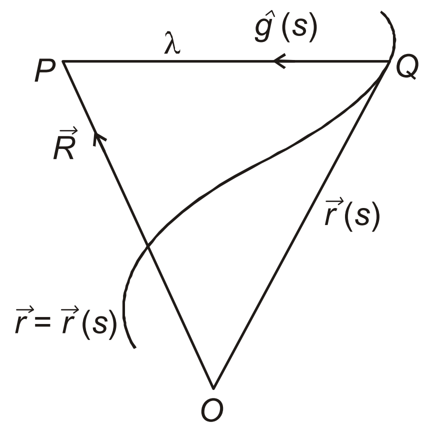


Fig. 4.2

The equation (4.3.1) can also be written in the Cartesian form.

Let (x, y, z) and (X, Y, Z) be the coordinates of the points Q and P, respectively. Then we can write

$$\vec{r}(s) = x\hat{i} + y\hat{j} + z\hat{k}, \quad \hat{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$$

$$\hat{g}(s) = g_1\hat{i} + g_2\hat{j} + g_3\hat{k}$$

where

$$|\hat{g}(s)| = \sqrt{g_1^2 + g_2^2 + g_3^2} = 1$$

using above in the equation (4.3.1) we find the equation of the ruled surface as

$$\frac{X-x}{g_1} = \frac{Y-y}{g_2} = \frac{Z-z}{g_3} (= \lambda) \qquad ......(4.3.2)$$

Equation (4.3.2) emphasises that a ruled surface is determined by single parameter (λ) family of straight lines. Note that the equation (4.3.2) can also be written as

$$X = aZ + \alpha, \quad Y = bZ + \beta$$

where a, b, α, β are functions of λ.

Uptil now we have gone through the idea of ruled surface and its further classifications as developable and skew surface. The following theorem is the criterion to determine whether the ruled is developable or skew.

4.3.2 Criterion for a surface to be developable :

Theorem 1. *A ruled surface is developable or skew if and only if* $[\hat{t}, \hat{g}', \hat{g}] = 0$ *or* $\neq 0$ *accordingly where* $\hat{t}$ *is the unit tangent vector at a point on the base curve, and* $\hat{g}$ *is unit vector along the generator through the point.*

Proof : The condition is necessary :

Let C be a base curve given by $\vec{r} = \vec{r}(s)$ on the ruled surface. Let RS be an arc on the curve C such that $R(\vec{r})$ and $S(\vec{r} + d\vec{r})$ be two consecutive points. Let arc $RS = ds$.

For the neighbouring point S of R, we have the position vector

$$\vec{r} + d\vec{r} = \vec{r} + \frac{d\vec{r}}{ds}ds = \vec{r} + \hat{t}\,ds. \qquad \text{(where } \hat{t} = \frac{d\vec{r}}{ds} = \text{ unit tangent vector at } R\text{)}$$

Let g_1 and g_2 be generators through R and S, respectively (*i.e.* g_1 and g_2 are consecutive generators since R, S are consecutive points on the base curve C) and $\hat{g}$ and $\hat{g} + d\hat{g}$ are unit vectors along the generators g_1 and g_2. Let MN be shortest distance between g_1 and g_2 then MN is perpendicular to both g_1 and g_2. Then MN is parallel to $(\hat{g} + d\hat{g}) \times \hat{g}$ or to

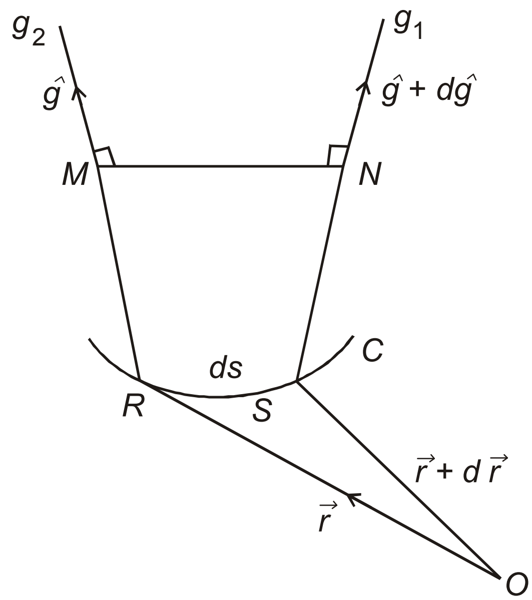


Fig. 4.3

$$(\hat{g} + \hat{g}'ds) \times \hat{g} \qquad \left[\textit{Note } \hat{g} + d\hat{g} = \hat{g} + \frac{d\hat{g}}{ds}ds = \hat{g} + \hat{g}'\,ds\right]$$

or $\qquad$ MN is parallel to $(\hat{g}' \times \hat{g})ds \qquad [\because\ \hat{g} \times \hat{g} = 0]$

Shortest distance MN = Projection of RS on MN

$$= d\vec{r}.\text{(unit vector along } MN\text{)}$$

$$= d\vec{r} \cdot \frac{\hat{g}' \times \hat{g}}{|\hat{g}' \times \hat{g}|} = \frac{d\vec{r}}{ds} \cdot \frac{\hat{g}' \times \hat{g}}{|\hat{g}' \times \hat{g}|}ds$$

$$= [\hat{t}, \hat{g}', \hat{g}]\frac{ds}{|\hat{g}'|} \qquad \text{......(4.3.3)}$$

[**Note :** Since $\hat{g}'$ is perpendicular to $\hat{g}$ and $|\hat{g}| = 1 \Rightarrow |\hat{g}' \times \hat{g}| = |\hat{g}'||\hat{g}|\sin 90° = |\hat{g}'|$]

Recall that in developable surface two consecutive generators intersect that is the shortest distance between the generators is zero. Thus we find that if the surface is developable, shortest distance MN is zero.

Hence $\qquad [\hat{t}, \hat{g}', \hat{g}] = 0 \qquad \left[\because\ \frac{ds}{|\hat{g}'|} \neq 0\right] \qquad$ .....(4.3.4)

This is the necessary condition for the ruled surface to be developable. In ruled surface to be developable.

In the case when the surface is skew, the consecutive generators don't interest and therefore the shortest distance is not zero. Thus, $\left[\hat{t},\, \hat{g}',\, \hat{g}\right] \neq 0$ is the necessary condition for a ruled surface to be skew.

The condition is sufficient :

If $\left[\hat{t},\, \hat{g}',\, \hat{g}\right] = 0 \Rightarrow$ the shortest distance between the consecutive generators is zero, hence the surface is developable.

Similarly $\left[\hat{t},\, \hat{g}',\, \hat{g}\right] \neq 0$ consecutive generators don't intersect, hence the surface is skew.

Theorem 2. *A ruled surface generated by* $x = az + \alpha$, $y = bz + \beta$ *is developable or skew if* $\alpha' b' - \beta' a' = 0$ *or* $\neq 0$ *respectively.*

Proof. The ruled surface is given to be generated by single parameter family of straight lines

$$x = az + \alpha, \qquad y = bz + \beta \qquad .....(4.3.5)$$

where a, b, α and β are functions of single parameter λ (say). The equation (4.3.5) can be written as

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-0}{1} \qquad .....(4.3.6)$$

Equation of ruled surface in vector form is

$$\vec{R} = \vec{r} + \lambda \hat{g} \qquad .....(4.3.7)$$

Then $\qquad \vec{r} = \alpha\hat{i} + \beta\hat{j} + 0\hat{k} = (\alpha,\, \beta,\, 0)$

$\Rightarrow \qquad \dfrac{d\vec{r}}{ds} = \hat{t} = (\alpha',\, \beta',\, 0)$

and $\qquad \hat{g} = a\hat{i} + b\hat{j} + 1\cdot\hat{k} = (a,\, b,\, 1)$

$\Rightarrow \qquad g' = \dfrac{d\hat{g}}{ds} = (a',\, b',\, 0)$

Thus $\qquad \left[\hat{t},\, \hat{g}',\, \hat{g}\right] = \left(\alpha'\hat{i} + \beta'\hat{j} + 0\cdot\hat{k}\right)\cdot\left(g' \times \hat{g}\right)$

$$= \left(\alpha'\hat{i} + \beta'\hat{j} + 0.\hat{k}\right)\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a' & b' & 0 \\ a & b & 1 \end{vmatrix}$$

$$= \left(\alpha'\hat{i} + \beta'\hat{j} + 0.\hat{k}\right)\cdot\left\{b'\hat{i} - \hat{j}a' + \hat{k}(a'b - ab')\right\}$$

$$= \alpha' b' - \beta' a'. \qquad .....(4.3.8)$$

Thus the surface is developable if $\alpha' b' - \beta' a' = 0$ or skew if

$$\alpha' b' - \beta' a' \neq 0. \qquad .....(4.3.9)$$

Aliter : The given generator can be written as

$$\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-0}{1} \qquad \text{......(4.3.10)}$$

Let $$\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z}{1}$$

and $$\frac{x-(\alpha+\delta\alpha)}{a+\delta a}=\frac{y-(\beta+\delta\beta)}{b+\delta b}=\frac{z-0}{1}$$

be consecutive generators of the surface. If the surface is developable then these generators intersect hence shortest distance between them is zero.

i.e. $$\begin{vmatrix} \delta\alpha & \delta\beta & 0 \\ a+\delta a & b+\delta b & 1 \\ a & b & 1 \end{vmatrix}=0$$

or $$\begin{vmatrix} \delta\alpha & \delta\beta & 0 \\ \delta a & \delta b & 0 \\ a & b & 1 \end{vmatrix}=0$$

or $$\delta\alpha\,\delta b-\delta\beta\,\delta a=0$$

or $$\left(\frac{\delta\alpha}{\delta t}\frac{\delta b}{\delta t}-\frac{\delta\beta}{\delta t}\frac{\delta a}{\delta t}\right)(\delta t)^2=0$$

$\Rightarrow$ $$\alpha' b'-\beta' a'=0 \quad [\because \delta t\neq 0] \qquad (4.3.11)$$

Ex. *Find the equation to the edge of regression of the developable*

$$y=xt-t^3, \quad z=t^3y-t^6.$$

Sol. In the given surface "t" is the parameter. The find the edge of regression recall that the point of intersection of the two consecutive generators of a developable surface is a point on the edge of regression.

The given equation of the developable is nothing but composed of the generators

$$y=xt-t^3, \quad z=t^3y-t^6.$$

The two consecutive generators are

$$y=xt-t^3,\ z=t^3y-t^6 \qquad \text{.....(1)}$$

and $$y=x(t+\delta t)-(t+\delta t)^3, \quad z=(t+\delta t)^3y-(t+\delta t)^6 \qquad \text{.....(2)}$$

Now, solving $y=xt-t^3$ and $y=x(t+\delta t)-(t+\delta t)^3$ and on neglecting higher powers of δt, we get

$$(x-3t^2)\,\delta t=0$$

$\Rightarrow$ $$x=3t^2 \qquad [\because \delta t\neq 0]$$

Again solving $z=t^3y-t^6$ and $z=(t+\delta t)^3y-(t+\delta t)^6$, we get on neglecting higher powers of δt

$$y=2t^3.$$

Using this value of y in the given generator we get

$$z = t^3 (2t^3) - t^6 = t^6.$$

Thus $x = 3t^2$, $y = 2t^3$, $z = t^6$ is the required edge of regression.

4.3.3 Self-learning exercises-2

1. Developable surface is generated by :

(a) cones *(b)* cyclinders

(c) spheres *(d)* straight lines

2. Prove that the line $x = 3t^2 z + 2t(1 - 3t^4)$, $y = t^2(3 + 4t^2) - 2tz$ generates a skew surface.

3. Prove that $xyz = 2$ is a developable surface.

4. Explain that a developable surface can be found to pass through two given curves.

5. Name two skew surfaces.

4.3.4 Equation of a tangent plane to a ruled surface :

(A) Equation in the vectorial notation :

Let $\vec{r}$ be the position vector of any point P on the directrix, $\hat{g}$ be unit vector along the generator at P and $\vec{r}$ and $\hat{g}$ are function of single parameter μ. Then the ruled surface is given by

$$\vec{R} = \vec{r} + \lambda \hat{g} \qquad \text{......(4.3.13)}$$

where $\vec{R}$ is the position vector of the current point on the ruled surface. Note that $\vec{R}$ is function of two independent parameters λ and μ.

The equation to the tangent plane to the ruled surface (4.3.13) is given by

$$\left(\vec{R}^* - \vec{R}\right) \cdot \left(\vec{R}_1 \times \vec{R}_2\right) = 0 \qquad \text{.....(4.3.14)}$$

where suffixes '1' and '2' denote differentiation of $\vec{R}$ with respect to μ and λ respectively and R^* is the position vector of the current point on the tangent plane. Thus (4.3.14) can be written as

$$\left[\left(\vec{R}^* - \vec{R}\right), \vec{r}_1 + \lambda \hat{g}_1, \hat{g}_2\right] = 0 \qquad \text{......(4.3.15)}$$

(B) Equation in cartesian notation :

Let $$x = az + \alpha,\ y = bz + \beta \qquad \text{.....(4.3.16)}$$

be the generator of the ruled surface, where a, b, α and β are functions of single parameter s.

Let us assume that (ξ, η, ζ) be a point on the ruled surface generated by (4.3.13). Then obviously ξ, η, ζ can be regarded as functions of s and z

where $$\xi = az + \alpha, \quad \eta = bz + \beta,\ \zeta = z. \qquad \text{......(4.3.17)}$$

Thus, the equation of the tangent plane at the point (s, z) is given by

$$\begin{vmatrix} \xi - x & \eta - y & \zeta - z \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} & \dfrac{\partial z}{\partial s} \\ \dfrac{\partial x}{\partial z} & \dfrac{\partial y}{\partial z} & \dfrac{\partial z}{\partial z} \end{vmatrix} = 0. \qquad \text{.....(4.3.18)}$$

Now $\qquad x = az + \alpha, y = bz + \beta$

$$\frac{\partial x}{\partial s} = a'z + \alpha', \ \frac{\partial y}{\partial s} = b'z + \beta' \quad \text{[Note that } a, b, \ \alpha, \ \beta \text{ are functions of } s \text{ only]}$$

where a', b', α', β' denote differentiation of respective quantities with respect to s.

Thus (4.3.18) becomes

$$\begin{vmatrix} \xi - az - \alpha & \eta - bz - \beta & \zeta - z \\ a'z + \alpha' & b'z + \beta' & 0 \\ a & b & 1 \end{vmatrix} = 0. \qquad \text{......(4.3.19)}$$

Performing column operations $c_1 - ac_3$ and $c_2 - bc_3$ in (4.3.19), we get

$$\begin{vmatrix} \xi - a\zeta - \alpha & \eta - b\zeta - \beta & \zeta - z \\ a'z + \alpha' & b'z + \beta' & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0. \qquad \text{.....(4.3.20)}$$

On simplifying the above determinant we get

$$(\xi - a\zeta - \alpha)(b'z + \beta') - (\eta - b\zeta - \beta)(a'z + \alpha') = 0$$

or
$$\xi - a\zeta - \alpha = \frac{a'z + \alpha'}{b'z + \beta'} (\eta - b\zeta - \beta) \qquad \text{.....(4.3.21)}$$

Note that the equation (4.3.21) represents the equation of the plane passing through the line $\xi = az + \alpha$, $\eta = bz + \beta$. Recall that the line $\xi = az + \alpha$, $\eta = bz + \beta$ is a generator of the ruled surface at the point (s, z). The above discussion reveals an important fact that the tangent plane at any point of a ruled surface contains the generator through that point.

Note 1 : If the ruled surface is developable, then the tangent plane is same at all points of the generator and involves only one parameter. This is evident from the following discussion :

Let the ruled surface is developable then $\alpha' b' - \beta' a' = 0$

$$\Rightarrow \qquad \frac{a'}{b'} = \frac{\alpha'}{\beta'} \Rightarrow \frac{a'}{b'} = \frac{\alpha'}{\beta'} = \frac{a'z + \alpha'}{b'z + \beta'} = K \text{ (say)} \qquad \text{.....(4.3.22)}$$

where K is function of s. In view of this, the equation (4.3.21) takes the form

$$\xi - az - \alpha = K(\eta - b\zeta - \beta) \qquad \text{......(4.3.23)}$$

and involves only one parameter s and is independent of z. Further note that the parameter s has a fixed value for a particular generator, therefore the tangent plane will be the same at all point of the generator.

Note 2 : If the surface is skew, then at different points of a generator, we have different tangent planes.

Let the surface is skew, then $\alpha' b' - \beta' a' \neq 0$.

That means the equation (4.3.21) contains both s and z. If s is kept fixed (for a particular generator) then (4.3.21) would give different tangent planes for different values of z. That means, the tangent planes are different at different points of the generator of skew surface.

Theorem. *Prove that the generators of a developable surface are tangents to curve.*

Proof. Let $\qquad x = az + \alpha,\ y = bz + \beta \qquad$ .....(4.3.24)

be generator of the developable surface, where a, b, α, β are functions of single parameter s.

Let $\qquad x = az + \alpha,\ \ y = bz + \beta \qquad$ .....(4.3.25)

and $\qquad x = (a + \delta a)\, z + (\alpha + \delta\alpha),\ \ y = (b + \delta b)\, z + (\beta + \delta\beta), \qquad$ .....(4.3.26)

be two consecutive generators of the developable surface. We know that, two consecutive generators of a developable surface do intersect, hence point of intersection of (4.3.25) and (4.3.26) is given by

$$x = \alpha - a\frac{\delta\alpha}{\delta a},\ y = \beta - b\,\frac{\delta\beta}{\delta b},\ z = -\frac{\delta\alpha}{\delta a} = -\frac{\delta\beta}{\delta b} \qquad \text{.....(4.3.27)}$$

Here note that (4.3.27) represents a curve since a, b, α', β' are functions of parameter s only. Thus x, y, z are functions of single parameter, hence (4.3.27) is a curve.

We have to show that the generators to the developable surface are tangents to the curve given by (4.3.27).

The equation to the tangent to the curve (4.3.27) at point (x, y, z) is given by

$$\frac{\xi - x}{\dot{x}} = \frac{\eta - y}{\dot{y}} = \frac{\zeta - z}{\dot{z}} \qquad \text{......(4.3.28)}$$

where dot denotes differentiation with respect to s.

From (4.3.27), we have

$$x = \alpha - a\frac{\delta\alpha/\delta s}{\delta a/\delta s},\quad y = \beta - b\,\frac{\delta\beta/\delta s}{\delta a/\delta s},\quad z = -\frac{\delta\alpha/\delta s}{\delta a/\delta s} = -\frac{\delta\beta/\delta s}{\delta b/\delta s} \qquad \text{[note]}$$

$$x = \alpha - \frac{a\dot{\alpha}}{\dot{a}},\ \ y = \beta - \frac{b\dot{\beta}}{\dot{a}},\ \ z = -\frac{\dot{\alpha}}{\dot{a}} = -\frac{\dot{\beta}}{\dot{a}} \qquad \text{.....(4.3.29)}$$

Now differentiation of (4.3.29), with respect to s, we get

$$\dot{x} = \frac{(\dot{\alpha}\,\ddot{a} - \dot{a}\,\ddot{\alpha})\,a}{\dot{a}^2} = a\,\dot{z},\quad \dot{y} = \frac{(\dot{\beta}\,\ddot{b} - \ddot{\beta}\,\dot{b})\,b}{\dot{b}^2} = b\,\dot{z} \qquad \text{.....(4.3.30)}$$

using $\dot{x} = a\dot{z},\ \dot{y} = b\dot{z}$ in equation (4.3.28), we get

$$\frac{\xi - x}{a\,\dot{z}} = \frac{\eta - y}{b\,\dot{z}} = \frac{\zeta - z}{\dot{z}}$$

or $\xi = a\zeta - az + x,\ \eta = b\zeta - bz + y$

or $\xi = a\zeta + \alpha,\ \ \eta = b\zeta + \beta$ .....(4.3.31)

Equation (4.3.30) is nothing but the generator through (x, y, z).

4.4 Summary

In this unit, you learnt that characteristic and edge of regression are the curves that lie on the envelope. An envelope of family of the surfaces is that surface that touches every member of the family. In the process of learning you came across with idea of ruled surfaces (developable and skew surfaces) and associated issues.

4.5 Answers to self-learning exercises

Self-learning exercise-1

1. (*d*) **2.** (*d*) **3.** (*a*) **4.** $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0$

Self-learning exercise-2

1. (*a*) **5.** Hyperboloid of one sheet, hyperbolic paraboloids.

4.6 Exercises

1. Find the envelope of the plane $\frac{x}{a+u}+\frac{y}{b+u}+\frac{z}{c+u}=1$ where u is the parameter.

[Ans. $(9\upsilon - \mu\lambda)^2 = 4\,(\mu^2 - 3\lambda\upsilon)\,(\lambda^2 - 3\mu)$, where $\lambda = a + b + c - (x + y + z)$, $\mu = ab + bc + ca - x(b + c) - y\,(a + c) - z\,(a + b),\ \upsilon = abc - (bcx - acy - abz)$**]**

2. Find the envelope of the surface $lx + my + nz = p$ where $a^2l^2 + b^2 m^2 + 2np = 0$.

$$\left[\textbf{Ans. } \frac{x^2}{a^2}+\frac{y^2}{b^2}=2z\right]$$

3. Find the envelope of the plane $\frac{(\mu-\lambda)}{a}x+\frac{(1+\lambda\mu)}{b}y+\frac{(1-\lambda\mu)}{c}z=\mu+\lambda$ where λ and μ are the parameters.

$$\left[\textbf{Ans. } \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1\right]$$

4. From a point P on the conicoid $a^2x^2 + b^2y^2 + c^2z^2 = 1$ perpendiculars PL, PM, PN are drawn to the coordinate planes. Find the envelope of the plane LMN.

[Ans. $(ax)^{2/3} + (by)^{2/3} + (cz)^{2/3} = 2^{2/3}$**]**

□□□

UNIT 5 : Necessary and Sufficient Condition that a Surface $\zeta = F(\xi, \eta)$ should Represent a Developable Surface, Metric of a Surface

Structure of the Unit

5.0 Objectives

After reading this unit you will be able to understand :

1. derivation of necessary and sufficient condition for the surface $\zeta = F(\xi, \eta)$ to be developablem
2. some important concepts such as parametric transformation,
3. oarametric equations of a few surfaces,
4. notion of metric of a surface.

5.1 Introduction

Last unit aimed to present an idea of developable surface. In this unit we would find a criterion for the surface of the form $\zeta = F(\xi, \eta)$ to be developable. Sometimes this criterion proves to be quite handy. Many surfaces can be written in parametric form and therefore parametric transformation and parametric representations of many surfaces have been dealt with in this unit. The above notions are useful in theory of metric of surface. A metric of a surface is the measure of an arc lying on the surface.

5.2 Necessary and sufficient condition that a surface $\zeta = F(\xi, \eta)$ should represent a developable surface

Let $$\zeta = F(\xi, \eta) \qquad .....(5.2.1)$$

be a given surface. In order to seek condition that the given surface is developable, we have to use the fact "if the surface is developable, then the tangent plane is same at all points of the generator and contains only one parameter".

We now proceed as follows :

The equation to the tangent plane at the point $P(x, y, z)$ on the given surface (5.2.1) is given by

$$(\xi - x)\frac{\partial z}{\partial x} + (\eta - y)\frac{\partial z}{\partial y} - (\zeta - z) = 0, \qquad .....(5.2.2)$$

Denoting $$p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y},$$

then (5.2.2) is reduced to

$$\xi p + \eta q - \zeta = px + qy - z$$

or $$\xi p + \eta q - \zeta = \phi, \qquad .....(5.2.3)$$

where $$\phi \equiv px + qy - z.$$

We will find the required condition making use of the tangent plane (5.2.3).

Necessary condition :

Let $\zeta = F(\xi, \eta)$ be the developable surface, then the tangent plane (5.2.3) involves only one parametric t (say). Thus we can write p, q and ϕ as functions of t. Thus, let

$$p = f(t),\ \ q = g(t),\ \phi = h(t) \qquad .....(5.2.4)$$

On elimination of t in the equation (5.2.4), p and ϕ can be written in terms of q, as given below

$$p = f_1(q), \quad \phi = f_2(q) \qquad .....(5.2.5)$$

Thus $$\frac{\partial p}{\partial x} = \frac{\partial f_1}{\partial q}\cdot\frac{\partial q}{\partial x} = f_1' s.$$

or $$r = f_1' s \qquad .....(5.2.6)$$

$$\left[\text{where } r = \frac{\partial p}{\partial x} = \frac{\partial^2 z}{\partial x^2},\quad s = \frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y}\right]$$

and $$\frac{\partial p}{\partial y} = \frac{\partial f_1}{\partial q}\frac{\partial q}{\partial y}$$

or $$s = f_1' t, \qquad \left[\text{where} \quad t = \frac{\partial q}{\partial y} = \frac{\partial^2 z}{\partial y^2}\right] \qquad .....(5.2.7)$$

From (5.2.6) and (5.2.7), on elimination of f_1', we obtain

$$r\,t = s^2 \qquad .....(5.2.8)$$

Thus, the condition given by equation (5.2.8) is the required necessary condition.

Sufficient condition :

Let $$r\,t = s^2$$

Then we will show that the tangent plane (5.2.3) involves only one parameter.

For this we shall show that the Jacobian $\frac{\partial(\phi, q)}{\partial(x, y)} = 0$ implying that ϕ is function of parameter (single parameter) q.

We now consider,

$$\frac{\partial(\phi, q)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial \phi}{\partial x} & \frac{\partial q}{\partial x} \\ \frac{\partial \phi}{\partial y} & \frac{\partial q}{\partial y} \end{vmatrix}. \qquad .....(5.2.9)$$

Since $$\phi = px + qy - z,$$

Therefore $$\frac{\partial \phi}{\partial x} = p + x\frac{\partial p}{\partial x} + \frac{\partial q}{\partial x} y - \frac{\partial z}{\partial x} = x\,r + s\,y,$$

and $$\frac{\partial \phi}{\partial y} = \frac{\partial p}{\partial y} x + \frac{\partial q}{\partial y} y + q - \frac{\partial z}{\partial y} = x\,s + t\,y. \qquad .....(5.2.10)$$

Using the above expressions in the Jacobian (5.2.9), we find

$$\frac{\partial(\phi, q)}{\partial(x, y)} = \begin{vmatrix} rx + sy & s \\ sx + ty & t \end{vmatrix} = x\begin{vmatrix} r & s \\ s & t \end{vmatrix} + y\begin{vmatrix} s & s \\ t & t \end{vmatrix} = x\,(rt - s^2) \qquad .....(5.2.11)$$

$$\Rightarrow \qquad \frac{\partial(\phi, q)}{\partial(x, y)} = 0 \qquad [\because\ r\,t - s^2 = 0]$$

$\Rightarrow$ ϕ is function of q.

$\Rightarrow$ Tangent plane (5.2.3) to the given surface (5.2.1) involves only one parameter.

$\Rightarrow$ Surface $\zeta = F(\xi, \eta)$ is developable.

Hence, we conclude that $r\,t - s^2 = 0$ is the necessary and sufficient condition for the surface $\zeta = F(\xi, \eta)$ to be developable, where r, t and s have their usual meanings.

***Ex.*1.** *Examine whether the surface $z = y \sin x$ is developable*

Sol. Given that $$z = y \sin x. \qquad .....(1)$$

The surface (1) is developable iff $r\,t - s^2 = 0$.

Now $\quad r=\dfrac{\partial^2 z}{\partial x^2}=-y\sin x,\quad t=\dfrac{\partial^2 z}{\partial y^2}=0,\quad s=\dfrac{\partial^2 z}{\partial x\partial y}=\cos x$

Thus $\quad r\,t-s^2=0-\cos^2 x.$ .....(2)

The surface is developable if

$$r\,t-s^2=-\cos^2 x=0$$

$$\Rightarrow \qquad x=\left(\frac{2n+1}{2}\right)\pi,\ n \text{ being integer.}$$

Hence, the surface is developable when $x=\left(\dfrac{2n+1}{2}\right)\pi$, n being integer.

Ex.2. *Show that the surface* $z-c=\sqrt{x\,y}$ *is developable.*

Sol. Given that $\qquad z=c+\sqrt{x\,y}$ .....(1)

We compute $\qquad r=\dfrac{\partial^2 z}{\partial x^2}=-\dfrac{1}{4}\sqrt{y}\,x^{-3/2},\ t=\dfrac{\partial^2 z}{\partial y^2}=-\dfrac{1}{4}\sqrt{x}\,y^{-3/2}$

$$s=\frac{\partial^2 z}{\partial x\partial y}=\frac{1}{4\sqrt{xy}}$$

Then, $\qquad rt-s^2=\dfrac{1}{16xy}-\dfrac{1}{16xy}=0$ .....(2)

Therefore, the surface (1) is developable.

5.3 Metric of a surface

In the coming test, you will see that a surface has two family of curves on it. You are familiar with the idea of arc length. A metric of a surface is the measure of the arc lengths of the curves on the surface. In order to derive the formula for the metric, we need to go through some concepts and terminology as follows :

We know that a curve is the locus of a point $P(x, y, z)$ whose cartesian coordinates are functions of single parameter t (say). On the same line, we define a surface as the locus of point $P(x, y, z)$ whose cartesian coordinates are functions of two independent parametric u and v (say). If $\vec{r}$ is the position vector of the point $P(x, y, z)$. Then the surface in vectorial notation is represented as

$$\vec{r}=\vec{r}(u, v). \qquad .....(5.3.1)$$

5.3.1 Curves on a surface and curvilinear coordinates :

Let $\vec{r}=\vec{r}(u, v)$ be the surface. Then by keeping one of the parameters among u and v fixed and varying the other we get a family of curves on the surface. If we keep $u=u_0$ (constant), then v will vary and the locus of the point $p(x, y, z)$ as v varies would give a parametric curve called v-curve on the surface. For different values of v, we get different v-curves. This constitutes a system of curves u = constant. Similarly, we get a system of curves v = constant *i.e.* u-curves on the surface $\vec{r} = \vec{r}(u, v)$.

Notes :

1. If u_0, v_0 be fixed values of u and v, then (u_0, v_0) is a point on the surface $\vec{r} = \vec{r}(u, v)$
2. (u_0, v_0) is called curvilinear coordinates of the surface
3. Through every point of a surface, there passes one and only one curve of each system
4. No two curves of the same system interest and two curves of different system meet only once.

5.3.2 Parametric transformation :

Let $\vec{r} = \vec{r}(u, v)$ be a surface whose parameters u and v be transformed to another set of parameters u^*, v^* as given below

$$u^* = u^*(u, v), \quad v^* = v^*(u, v) \qquad .....(5.3.2)$$

where u^*, v^* are single valued and derivable.

The above transformation is called proper if the Jacobian

$$\frac{\partial(u^*, v^*)}{\partial(u, v)} \neq 0. \qquad .....(5.3.3)$$

We, now denote $$\vec{r}_1 = \frac{\partial \vec{r}}{\partial u}, \vec{r}_2 = \frac{\partial \vec{r}}{\partial v}. \qquad .....(5.3.4)$$

These partial derivatives are important since $\vec{r}_1$ and $\vec{r}_2$ have tangential directions to u-curves and v-curves, respectively (in the sense of u and v increasing).

5.3.3 Regular and singular points :

Behaviour of the surface $\vec{r} = \vec{r}(u, v)$ in the neighbourhood of the point $p\,(x, y, z)$ or $p(\vec{r})$ is closely dependent on $\vec{r}_1$ and $\vec{r}_2$.

The point $p(\vec{r})$ on the surface is called regular point or ordinary point, if

$$\vec{r}_1 \times \vec{r}_2 \neq \vec{0} \qquad .....(5.3.5)$$

If $\vec{r}_1 \times \vec{r}_2 = \vec{0}$, then the point is called *singular* point or *singularity* of the surface.

5.3.4 Parametric equations of some surfaces :

This section pertains to equations of some surfaces in parametric form, you are advised to remember these as it would help you in solving the questions.

***(i)* Sphere :** The equation of a sphere with radius a and centre at the origin is

$$x = a \sin\phi \cos\theta, \; y = a \sin\phi \sin\theta, \; z = a \cos\phi$$

where θ and ϕ are parameters

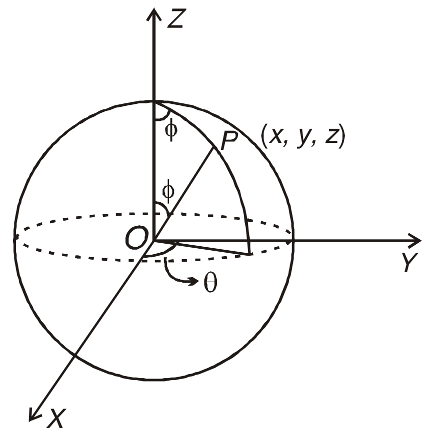


Fig 5.1

(ii) **Surface of revolution :** Note that a surface of revolution is the surface which is generated by revolving a plane curve about an axis in the plane of the curve.

As an illustration, let us consider a plane curve

$$z = f(y), \quad x = 0 \ \text{ in } YZ\text{-plane.}$$

Let p $(0, w, z)$ be any point on this curve. Then obviously $z = f(w)$.

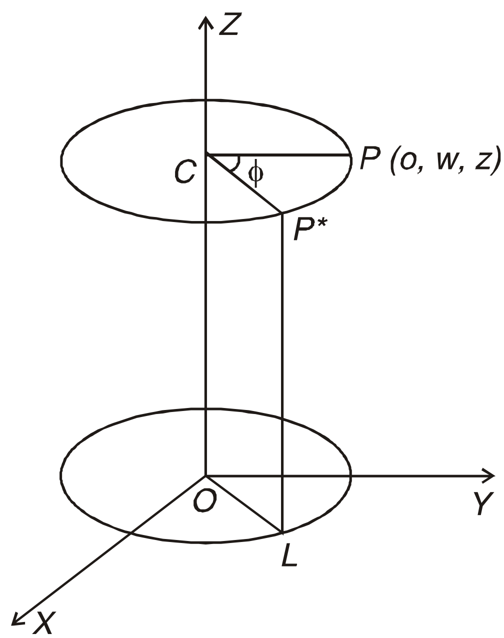


Fig 5.2

Let us suppose that the curve is revolved about Z-axis, then the point p will traverse a circle in a plane normal to Z-axis with the centre on the Z-axis (the axis of rotation).

Let c be the centre of the circle and p^* be new position of the point p, then the point p^* (x, y, z) is obtained as

$$x = OL \cos\phi = CP^* \cos\phi = w\cos\phi$$

$$y = w\sin\phi, \qquad z = f(w) \qquad .....(5.3.6)$$

where ϕ is the angle between XZ-plane and CP^*LO.

Note that here w and ϕ are the parameters.

Note : On the above surface, we have two family of curves. The parametric curves u = constant are called parallels and the curves ϕ = constant are called meridians.

Geometrically, the parallels are curves of intersection of surface of revolution and the planes perpendicular to the axis of revolution. The meridians are the curve of section by the planes through the axis of revolution.

(iii) **Anchor sing :** Anchor ring is a surface generated by revolution of a circle in a particular setup.

Consider a circle of radius a and with center c on the Y-axis in YZ-plane. Let the centre c is at a distance d from the origin, then the coordinates of any point P on the circle are $(0, d + a \cos\theta, a \sin\theta)$, where $\angle PCY = \theta$ and $OC = d$.

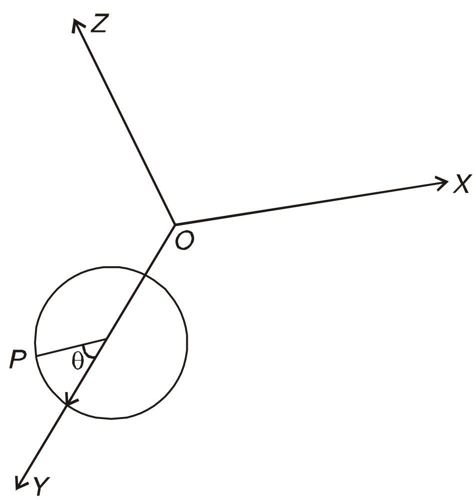


Fig 5.3

When this circle is revolved about Z-axis, then it would give rise to the surface

$$\vec{r} = ((d + a \cos\theta) \cos\phi, (d + a \cos\theta) \sin\phi, a \sin\theta) \qquad .....(5.3.7)$$

known as anchor ring. Here θ and ϕ are parameters.

(iv) **Helicoid :** Helicoid is the surface generated by the screw motion of a curve about a fixed line. In generation of a helicoid, a curve undergoes with two motions–curve is first translated through a distance μ (say) parallel to the fixed line (the axis) and subsequently it is revolved through an angle ϕ (say) about the axis such that $\frac{\mu}{\phi}$ is constant.

Let $\frac{\mu}{\phi} = b$ (constant), then $2\pi b$ is called the pitch of the helicoid.

As an illustration, let us consider a curve in XZ-plane given by

$$x = f(v), y = 0, z = g(v).$$

Let Z-axis be the axis of helicoid, then the equation of the helicoid is

$$\vec{r} = (f(v)\cos\phi, \quad f(v)\sin\phi, \quad g(v) + b\,\phi) \qquad .....(5.3.8)$$

where v and ϕ are the parameters.

***(v)* Right helicoid :** Right helicoid is generated by screw motion of a straight line about the fixed line (axis) such that the straight line meets the axis at right angle.

Let X-axis be the straight line generating the right helicoid, then its equation is given by

$$\vec{r} = (v\cos\phi, v\sin\phi, a\,\phi), \qquad .....(5.3.9)$$

a being the pitch

***(vi)* Right circular cone :** A right circular cone is a locus of a variable straight line passing through a fixed point (vertex) and making a constant angle α (semi-vertical angle) with a line (axis) through the vertex.

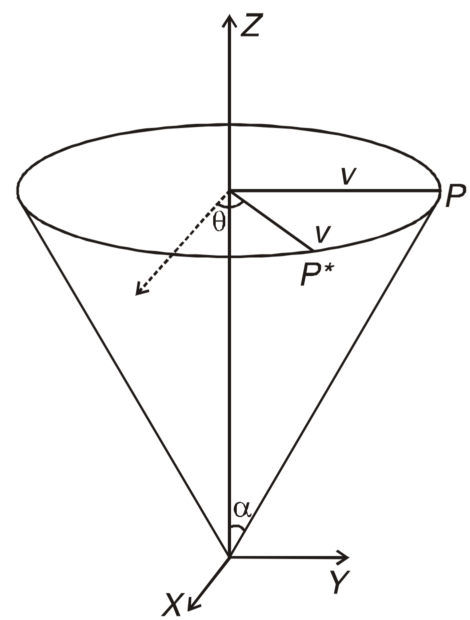


Fig. 5.4

As an illustration,

$$\vec{r} = (v\cos\theta, v\sin\theta, v\cot\alpha) \qquad .....(5.3.10)$$

represents a right circular cone, whose vertex is at the origin, Z-axis being the axis of the cone and the variable straight line lies in the YZ-plane as shown in the figure.

5.3.5 Metric of a surface : A metric of a surface is the measure of the arc lengths of the curves on the surface.

Let $\vec{r} = \vec{r}(u, v)$ be a surface on which a curve $u = u(t)$, $v = v(t)$ lies. Let P and Q be two neighbouring points on this curve such that the position vectors of P and Q are $\vec{r}$ and $\vec{r} + d\vec{r}$, respectively. Let A be a fixed point on the curve such that arc $AP = s$ and arc $PQ = ds$ where s is the measure of arc length. Since P, Q are neighbouring points, therefore ds is the infinitesimal distance between these two points.

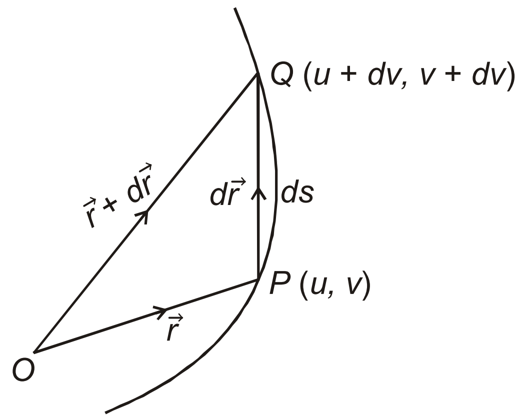


Fig. 5.5

Now $$d\vec{r} = \frac{\partial \vec{r}}{\partial u} du + \frac{\partial \vec{r}}{\partial v} dv$$

$$= \vec{r}_1\, du + \vec{r}_2\, dv, \qquad .....(5.3.11)$$

where $$\vec{r}_1 = \frac{d\vec{r}}{du}, \vec{r}_2 = \frac{d\vec{r}}{dv}.$$

Since P and Q are neighbouring points therefore,

$$PQ = \text{chord } PQ \text{ i.e. } ds = |\, d\vec{r}\, |$$

$\Rightarrow$ $$(ds)^2 = (d\vec{r})^2. \qquad .....(5.3.12)$$

Then, $$(ds)^2 = (\vec{r}_1\, du + \vec{r}_2\, dv)^2.$$

$$= (\vec{r}_1)^2 du^2 + (\vec{r}_2)^2 dv^2 + 2\vec{r}_1 \cdot \vec{r}_2\, du\, dv$$

$$(ds)^2 = E\, du^2 + 2F\, du\, dv + G\, dv^2, \qquad .....(5.3.13)$$

where $$E = (\vec{r}_1)^2, F = \vec{r}_1 \cdot \vec{r}_2, G = (\vec{r}_2)^2.$$

Equation (5.3.13) is quadratic differential form in du and dv, and is called **metric** or **first fundamental form**. The quantities E, F and G are called **first fundamental coefficients**. Note that the values of E, F and G vary, in general, for different points on the surface simply because these are functions of surface parameters u and v. Alternatively we can say that the metric is the relation between the differentials of the arc of the curve and curvilinear coordinates u, v. A metric is also referred to as **linear element**.

***Ex.*1.** *Prove that for the curve* $x = r\cos\theta, y = r\sin\theta, z = 0, \quad ds^2 = dr^2 + r^2 d\theta^2$

Sol. $\because \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ or $\vec{r} = (r\cos\theta, r\sin\theta, 0)$.

Here r and θ are two parameters *i.e.* "u" $\equiv r$, "v" $= \theta$

$\therefore$ $$\vec{r}_1 = \frac{d\vec{r}}{dr} = (\cos\theta, \sin\theta, 0); \vec{r}_2 = \frac{d\vec{r}}{d\theta} = (-r\sin\theta, r\cos\theta, 0).$$

Here $$E = (\vec{r}_1)^2 = \cos^2\theta + \sin^2\theta = 1; F = \vec{r}_1 \cdot \vec{r}_2 = -r\cos\theta\sin\theta + r\cos\theta\sin\theta$$

$$G = (\vec{r}_2)^2 = r^2\left(\sin^2\theta + \cos^2\theta\right) = r^2.$$

Then $\quad ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 = dr^2 + r^2\,d\theta^2. \qquad [\because u = r,\ v = \theta]$

5.3.6 Theorem : *The metric of a surface is invariant under parametric transformation.*

Proof : Let $\vec{r} = \vec{r}(u, v)$ be a surface and the parameters (u, v) undergo parametric transformation as given below

$$u^* = u^*(u, v)\ ;\ v^* = v^*(u, v) \qquad .....(5.3.14)$$

Now,
$$\vec{r}_1^{\,*} = \frac{\partial \vec{r}}{\partial u^*} = \frac{\partial \vec{r}}{\partial u}\frac{\partial u}{\partial u^*} + \frac{\partial \vec{r}}{\partial v}\frac{\partial v}{\partial u^*}$$

$$\Rightarrow \quad \vec{r}_1^{\,*} = \vec{r}_1\frac{\partial u}{\partial u^*} + \vec{r}_2\frac{\partial v}{\partial u^*} \qquad .....(5.3.15)$$

$$\vec{r}_2^{\,*} = \frac{\partial \vec{r}}{\partial v^*} = \frac{\partial \vec{r}}{\partial u}\frac{\partial u}{\partial v^*} + \frac{\partial \vec{r}}{\partial v}\frac{\partial v}{\partial v^*}$$

$$\Rightarrow \quad \vec{r}_2^{\,*} = \vec{r}_1\frac{\partial u}{\partial v^*} + \vec{r}_2\frac{\partial v}{\partial v^*} \qquad .....(5.3.16)$$

From (5.3.14), $u = u(u^*, v^*)$, $v = v(u^*, v^*)$,

then
$$du = \frac{\partial u}{\partial u^*}du^* + \frac{\partial u}{\partial v^*}dv^* \qquad .....(5.3.17)$$

and
$$dv = \frac{\partial v}{\partial u^*}du^* + \frac{\partial v}{\partial v^*}dv^*. \qquad .....(5.3.18)$$

Let $E^*\,du^{*2} + 2F^*\,du^*\,dv^* + G^*\,dv^{*2}$ be the metric of the given surface for the parameters (u^*, v^*). Then

$$E^*\,du^{*2} + 2F^*\,du^*\,dv^* + G^*\,dv^{*2} = \vec{r}_1^{\,*2}\,du^{*2} + 2\vec{r}_1^{\,*}\cdot\vec{r}_2^{\,*}\,du^*\,dv^* + \vec{r}_2^{\,*2}\,dv^{*2}$$

$$= \left(\vec{r}_1^{\,*}du^* + \vec{r}_2^{\,*}dv^*\right)^2$$

$$= \left[\left(\vec{r}_1\frac{\partial u}{\partial u^*} + \vec{r}_2\frac{\partial v}{\partial u^*}\right)du^* + \left(\vec{r}_1\frac{\partial u}{\partial v^*} + \vec{r}_2\frac{\partial u}{\partial v^*}\right)dv^*\right]^2$$

$$= \left[\vec{r}_1\left(\frac{\partial u}{\partial u^*}du^* + \frac{\partial u}{\partial v^*}dv^*\right) + \vec{r}_2\left(\frac{\partial v}{\partial u^*}du^* + \frac{\partial v}{\partial v^*}dv^*\right)\right]^2$$

$$= \left[\vec{r}_1\,du + \vec{r}_2\,dv\right]^2$$

$$= \vec{r}_1^{\,2}\,du^2 + 2\vec{r}_1\cdot\vec{r}_2\,du\,dv + \vec{r}_2^{\,2}dv^2$$

$$= E\,du^2 + 2F\,du\,dv + G\,dv^2. \qquad .....(5.3.19)$$

Thus we have shown that if the parameters u and v are transformed to new set of parameters u^* and v^*, then the metric does not change *i.e.,* the metric is invariant.

Self-learning exercise-1

1. A point $P(\vec{r})$ on a surface is a regular point if

(a) $\vec{r}_1 \cdot \vec{r}_2 = 0$ *(b)* $\vec{r}_1 \times \vec{r} = \vec{0}$ *(c)* $\vec{r}_2 \times \vec{r}_2 = \vec{0}$ *(d)* $\vec{r}_1 \times \vec{r}_2 \neq \vec{0}$

2. Parametric transformation $u^* = u^*(u, v)$, $v^* = v^*(u, v)$ is proper if

(a) $\dfrac{\partial u^*}{\partial u} = \dfrac{\partial v^*}{\partial v}$ *(b)* $\dfrac{\partial^2 u^*}{\partial u^2} - \dfrac{\partial^2 v^*}{\partial v^2} = 0$ *(c)* $\dfrac{\partial(u^*, v^*)}{\partial(u, v)} \neq 0$ *(d)* it is conviently defined.

3. With usual notations, $\vec{r}_1$ signifies for the surface $\vec{r} = \vec{r}(u, v)$

(a) unit normal vector to the surface *(b)* normal vector to the skew surface

(c) normal vector to the envelope *(d)* None of these.

4. In the expression $ds^2 = E\,du^2 + 2\,F\,du\,dv + G\,dv^2$, F stands for.

(a) $\vec{r}_1 \cdot \vec{r}_1$ *(b)* $\vec{r}_2 \cdot \vec{r}_2$ *(c)* $(\vec{r}_1 \cdot \vec{r}_2)^{1/2}$ *(d)* None of these

5.4 Summary

In this unit you came across with the criterion of surface $\zeta = F(\xi, \eta)$ to be developable. This criterion is an important tool to examine the surface being developable or screw. Further, you learnt that a surface has two distinct family of curves on it. Before going to the core topic of metric of a surface, you have learnt many essential concepts such as-parametric equation of some standard surfaces.

5.5 Answers to self-learning exercises

1. *(d)* 2. *(c)* 3. *(d)* 4. *(d)*

5.6 Exercises

1. Explain curvilinear equation of a curve lying on a surface

2. Derive a formula for metric of a surface

3. Find metric of a point $P(x, y, 0)$ in XY-plane. **[Ans.** $ds^2 = dx^2 + dy^2$]

4. Find metric of a point $P(x, y, 0)$ in XY-plane where $x = r\cos\theta$, $y = r\sin\theta$.

[Ans. $ds^2 = dr^2 + r^2\,d\theta^2$]

□□□

UNIT 6 : First, Second and Third Fundamental Forms, Fundamental Magnitudes of Some Important Surfaces, Orthogonal Trajectories, Normal Curvature

Structure of the Unit

6.0 Objectives

In this unit you will study about :

1. fundamental forms of a surface,
2. fundamental magnitudes of some standard surfaces,
3. directors and orthogonal trajectories,
4. normal curvature.

6.1 Introduction

A surface is associated with there important forms which are infold quadratic differential expressions in du, dv. Each form has its definite geometrical significance and serves as a founding stone in the development of differential geometry. To distinguish, these are called first, second and third fundamental forms. This unit aims to discuss the forms and the related properties.

6.2 Fundamental forms

6.2.1 First fundamental form :

In the last unit you have studied about the metric. The metric of a surface determines the first fundamental form of the surface. Thus the quadratic differential from

$$E\,du^2 + 2F\,du\,dv + G\,dv^2$$

is called the first fundamental form and the quantities *E, G, H* are called the first order fundamental magnitudes or first fundamental coefficients. Here it should be noted that since the quantities *E, F, G* depend on u and v therefore, in general, they vary from point to point on the surface.

6.2.2 Second fundamental form :

The second fundamental from of a surface $\vec{r} = \vec{r}(u,v)$ is a quadratic differential in *du* and *dv* together with the resolved parts of the second order partial derivatives of $\vec{r}$ (with respect to parameter u and v) in the direction of the normal at the point $p(\vec{r})$ on the surface.

Let $\vec{r}$ be position vector of any arbitrary point p on the surface $\vec{r} = \vec{r}(u,v)$. Then we denote

$$\vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial u^2}, \vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial u \partial v}, \vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial v^2} \qquad .....(6.2.1)$$

Let *L, M, N* be the resolved parts of $\vec{r}_{11}, \vec{r}_{12}, \vec{r}_{22}$, respectively in the direction of normal vector $\hat{N}$ at the point $p(\vec{r})$, then

$$L = \vec{r}_{11} \cdot \hat{N}, M = \vec{r}_{12} \cdot \hat{N}, N = \vec{r}_{22} \cdot \hat{N} \qquad .....(6.2.2)$$

The quantities *L, M, N* are called the second order fundamental magnitudes or second fundamental coefficients.

The quadratic differential form in *du* and *dv*

$$L\,du^2 + 2M\,du\,dv + N\,dv^2 \qquad .....(6.2.3)$$

is called the second fundamental form.

Note : Since the normal at the point $p(\vec{r})$ on the surface is parallel to the vector $\vec{r}_1 \times \vec{r}_2$, therefore unit normal vector $\hat{N}$ at point $p(\vec{r})$ is given by

$$\hat{N} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} = \frac{\vec{r}_1 \times \vec{r}_2}{H}, \qquad \text{where} \qquad H = |\vec{r}_1 \times \vec{r}_2|. \qquad .....(6.2.4)$$

The second fundamental coefficients *L, M, N* given by (6.2.3) can be expressed alternatively as follows

We know that the unit normal vector $\hat{N}$ is parallel to the vector $\vec{r}_1 \times \vec{r}_2$, therefore $\hat{N}$ is perpendicular to both $\vec{r}_1$ and $\vec{r}_2$.

Thus, $$\hat{N} \cdot \vec{r}_1 = 0, \qquad .....(6.2.5)$$

and $$\hat{N} \cdot \vec{r}_2 = 0. \qquad .....(6.2.6)$$

On differentiating (6.2.5) with respect to *u*, we find

$$\frac{\partial \hat{N}}{\partial u} \cdot \vec{r}_1 + \hat{N} \frac{\partial \vec{r}_1}{\partial u} = 0, \qquad \text{or} \qquad \hat{N}_1 \cdot \vec{r}_1 + \hat{N} \cdot \vec{r}_{11} = 0, \qquad .....(6.2.7)$$

where $$\vec{N}_1 = \frac{\partial \hat{N}}{\partial u} \quad \text{and} \quad \vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial \vec{r}}{\partial u}\right) = \frac{\partial \vec{r}_1}{\partial u}$$

or $$\vec{r}_{11} \cdot \hat{N} = -\vec{N}_1 \cdot \vec{r}_1$$

$$\Rightarrow \qquad L = -\vec{N}_1 \cdot \vec{r}_1, \qquad \left(\because L = \vec{r}_{11} \cdot \hat{N}\right) \qquad .....(6.2.8)$$

Similarly on differentiating (6.2.6) with respect to *v*, are get

$$\frac{\partial \hat{N}}{\partial v} \cdot \vec{r}_2 + \hat{N} \frac{\partial \vec{r}_2}{\partial v} = 0$$

or $$\vec{N}_2 \cdot \vec{r}_2 + \hat{N} \cdot \vec{r}_{22} = 0$$

or $$\hat{N} \cdot \vec{r}_{22} = -\vec{N}_2 \cdot \vec{r}_2$$

or $$N = -\vec{N}_2 \cdot \vec{r}_2 \qquad \left(\because N = \hat{N} \cdot \vec{r}_{22}\right) \qquad .....(6.2.9)$$

Further on differentiating (6.2.5) and (6.2.6) with respectively with *v* and *u* respectively, we get

$$M = -\vec{N}_2 \cdot \vec{r}_1 \quad \text{and} \quad M = -\vec{N}_1 \cdot \vec{r}_2. \qquad .....(6.2.10)$$

The alternative expressions for *L, M, N* obtained above give rise to another expression for the second fundamental form as follows :

We have, $$d\vec{r} = \vec{r}_1 du + \vec{r}_2 \, dv \qquad .....(6.2.11)$$

$$d\hat{N} = \frac{\partial \hat{N}}{\partial u} du + \frac{\partial \hat{N}}{\partial v} dv$$

$$\Rightarrow \qquad d\hat{N} = \vec{N}_1 du + \vec{N}_2 dv, \qquad .....(6.2.12)$$

where $$\vec{N}_1 = \frac{\partial \hat{N}}{\partial u}, \qquad \vec{N}_2 = \frac{\partial \hat{N}}{\partial v}.$$

We compute $$d\hat{N} \cdot d\vec{r} = \left(\vec{N}_1 du + \vec{N}_2 dv\right) \cdot \left(\vec{r}_1 du + \vec{r}_2 dv\right)$$

$$= \left(\vec{N}_1 \cdot \vec{r}_1\right) du^2 + \left(\vec{N}_1 \cdot \vec{r}_2 + \vec{N}_2 \cdot \vec{r}_1\right) du\, dv + \left(\vec{N}_2 \cdot \vec{r}_2\right) dv^2$$

$$= -[L\, du^2 + 2M\, du\, dv + N\, dv^2]. \quad \text{[using (6.2.8), (6.2.9), (6.2.10)]}$$

Thus the second fundamental form

$$L\, du^2 + 2m\, du\, dv + N\, dv^2 = -\, d\hat{N} \cdot d\vec{r} \qquad .....(6.2.13)$$

6.2.3 Geometrical significance of second fundamental form :

The following theorem entails the geometrical interpretation of the second fundamental form :

Theorem. *Let P (u, v) and Q $(u + du, v + dv)$ be two neighbouring points on the surface $\vec{r} = \vec{r}(u, v)$. Then second fundamental form $L\, du^2 + 2M\, dudv + N\, dv^2$ at the point $P(u, v)$ is twice the length of perpendicular from Q $(u + du, v + dv)$ on the tangent plane at P (u, v), to a second order approximation in du, dv.*

Proof : Let $\vec{r}$ and $\vec{r} + d\vec{r}$ be the position vectors of two neighbouring points P (u, v) and Q $(u + du, v + dv)$ of the surface $\vec{r} = \vec{r}(u, v)$.

Now, $$\vec{r} + d\vec{r} = \vec{r}(u + du, v + dv).$$

Then by Taylor's series for two variables, we have

$$\vec{r}(u + du, v + dv) = \vec{r}(u, v) + \left(\frac{\partial \vec{r}}{\partial u} du + \frac{\partial \vec{r}}{\partial v} dv\right) + \frac{1}{2}\left(\frac{\partial^2 \vec{r}}{\partial u^2} du^2 + \frac{\partial^2 \vec{r}}{\partial u \partial v} du\, dv + \frac{\partial^2 \vec{r}}{\partial v^2} dv^2\right) + \ldots$$

or $$\vec{r} + d\vec{r} = \vec{r} + \left(\vec{r}_1\, du + \vec{r}_2\, dv\right) + \frac{1}{2}\left(\vec{r}_{11} du^2 + 2\vec{r}_{12} du\, dv + \vec{r}_{22} dv^2\right) + \text{higher order terms}$$

or $$d\vec{r} = \left(\vec{r}_1\, du + \vec{r}_2\, dv\right) + \frac{1}{2}\left(\vec{r}_{11} du^2 + 2\vec{r}_{12} du\, dv + \vec{r}_{22} dv^2\right) \qquad .....(6.2.14)$$

(on neglecting higher order terms)

Let QA be perpendicular from Q on the tangent plane at the point P (u, v) to the surface $\vec{r} = \vec{r}(u, v)$, then we have

QA = Projection of PQ on normal vector $\hat{N}$ at P

$$= \hat{N} \cdot d\vec{r} = \hat{N} \cdot \left[\left(\vec{r}_1\, du + \vec{r}_2\, dv\right) + \frac{1}{2}\left(\vec{r}_{11} du^2 + 2\vec{r}_{12} du\, dv + \vec{r}_{22} dv^2\right)\right]$$

$$= \left(\hat{N} \cdot r_1\right) du + \left(\hat{N} \cdot r_2\right) dv + \frac{1}{2}\left[\left(\hat{N} \cdot r_{11}\right) du^2 + 2\left(\hat{N} \cdot r_{12}\right) du\, dv + \left(\hat{N} \cdot r_{22}\right) dv^2\right]$$

$$= 0 + 0 + \frac{1}{2}\left[L\, du^2 + 2M\, du\, dv + N\, dv^2\right] \quad \left[\begin{array}{l} \because\ \hat{N} \text{ is perpendicular to both } \vec{r}_1 \text{ and } \vec{r}_2 \\ \Rightarrow \hat{N} \cdot r_1 = 0 = \hat{N} \cdot r_2 \text{ and } L = r_{11} \cdot \hat{N} \text{ etc.} \end{array}\right]$$

$$\Rightarrow \qquad 2QA = L\, du^2 + 2M\, du\, dv + N\, dv^2 \qquad .....(6.2.15)$$

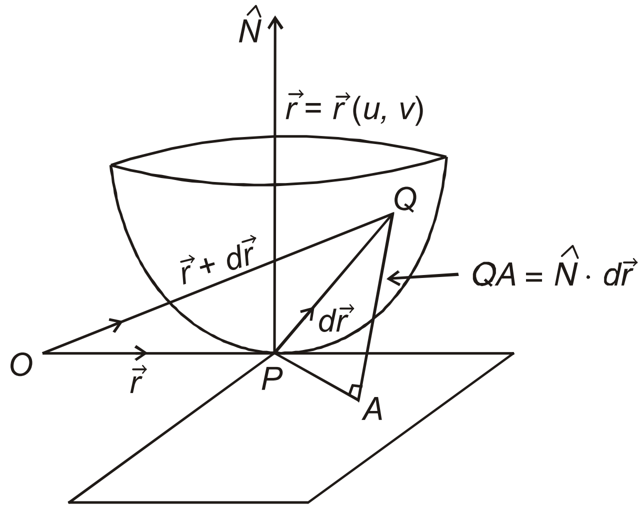


Fig. 6.1

6.2.4 Weingarten equations :

Let $\vec{r} = \vec{r}(u, v)$ be the surface, $\hat{N}$ be the unit normal at a point $p(\vec{r})$ on the surface then we denote

$$\hat{N}_1 = \frac{\partial \hat{N}}{\partial u}, \hat{N}_2 = \frac{\partial \hat{N}}{\partial v}.$$

Here it is emphasized that $\hat{N}$ is perpendicular to both $\hat{N}_1$ and $\hat{N}_2$, which means that the vectors $\hat{N}_1$, $\hat{N}_2$ are tangential to the surface. Therefore, the vectors $\hat{N}_1$ and $\hat{N}_2$, are spanned by the vectors $\vec{r}_1$ and $\vec{r}_2$ and hence there exist scalars *a, b, c, d* such that

$$\hat{N}_1 = a\,\vec{r}_1 + b\,\vec{r}_2 \quad .....(6.2.16)$$

$$\hat{N}_2 = c\,\vec{r}_1 + d\,\vec{r}_2 \quad .....(6.2.17)$$

From (6.2.16), we find

$$\vec{r}_1 \cdot \hat{N}_1 = a\,\vec{r}_1 \cdot \vec{r}_1 + b\,\vec{r}_1 \cdot \vec{r}_2$$

or
$$-L = a\,E + b\,F \quad .....(6.2.18)$$

From (6.2.16), we find

$$\vec{r}_2 \cdot \hat{N}_1 = a\,\vec{r}_1 \cdot \vec{r}_2 + b\,\vec{r}_2 \cdot \vec{r}_2$$

or
$$-M = aF + bG \quad .....(6.2.19)$$

Thus solving the system of linear equations given by (6.2.18) and (6.2.19), we get

$$a = \frac{FM - GL}{EG - F^2}, b = \frac{FL - EM}{EG - F^2}$$

or
$$a = \frac{FM - GL}{H^2}, b = \frac{FL - EM}{H^2} \quad .....(6.2.20)$$

Using the values of a, b in (1), we get

$$\hat{N}_1 = \left(\frac{FM - GL}{H^2}\right)\vec{r}_1 + \left(\frac{FL - EM}{H^2}\right)\vec{r}_2$$

or $$H^2\hat{N}_1 = (FM - GL)\vec{r}_1 + (FL - EM)\vec{r}_2 \qquad .....(6.2.21)$$

Similarly by taking the scalar product of (6.2.17) with $\vec{r}_1$ and $\vec{r}_2$ successively, and solving the resultant linear system, we get the values of c and d as follows :

$$c = \left(\frac{FN - GM}{H^2}\right), d = \frac{FM - EN}{H^2}. \qquad .....(6.2.22)$$

Thus from (6.2.17) we get

$$H^2\hat{N}_2 = (FN - GM)\vec{r}_1 + (FM - EN)\vec{r}_2. \qquad .....(6.2.23)$$

6.3 Fundamental magnitudes of some important surfaces

***(i)* Anchor ring :**

The parametric equation of an anchor ring is

$$\vec{r} = ((b + a\cos\theta)\cos\phi, (b + a\cos\theta)\sin\phi, a\sin\theta) \qquad .....(6.3.1)$$

Then we compute,

$$\vec{r}_1 = \frac{\partial\vec{r}}{\partial\theta} = (-a\sin\theta\cos\phi, -a\sin\theta\sin\phi, a\cos\theta)$$

$$\vec{r}_2 = \frac{\partial\vec{r}}{\partial\phi} = (-(b + a\cos\theta)\sin\phi, (b + a\cos\theta)\cos\phi, 0)$$

$$\vec{r}_{11} = \frac{\partial^2\vec{r}}{\partial\theta^2} = (-a\cos\theta\cos\phi, -a\cos\theta\sin\phi, -a\sin\theta)$$

$$\vec{r}_{12} = \frac{\partial^2\vec{r}}{\partial\theta\partial\phi} = (a\sin\theta\sin\phi, -a\cos\theta\cos\phi, 0)$$

$$\vec{r}_{22} = \frac{\partial^2\vec{r}}{\partial\phi^2} = (-(b + a\cos\theta)\cos\phi, -(b + a\cos\theta)\sin\phi, 0) \qquad .....(6.3.2)$$

Thus we have

$$\begin{aligned} E = \vec{r}_1^{\,2} &= a^2\sin^2\theta\cos^2\phi + a^2\sin^2\theta\sin^2\phi + a^2\cos^2\theta \\ &= a^2\sin^2\theta\,(\cos^2\phi + \sin^2\phi) + a^2\cos^2\theta \\ &= a^2\,(\sin^2\theta + \cos^2\theta) = a^2 \end{aligned}$$

$$F = \vec{r}_1\cdot\vec{r}_2 = 0, G = \vec{r}_2^{\,2} = (b + a\cos\theta)^2 \qquad .....(6.3.3)$$

$$H^2 = EG - F^2 = a^2(b + a\cos\theta)^2 \qquad .....(6.3.4)$$

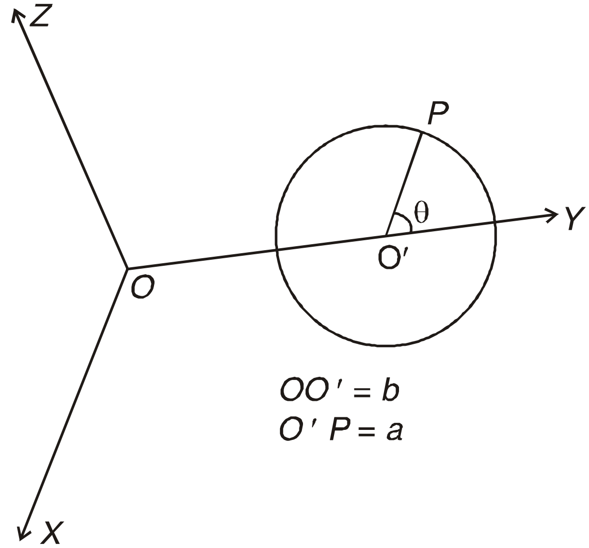


Fig. 6.2

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} i & j & k \\ -a\sin\theta\cos\phi & -a\sin\theta\sin\phi & a\cos\theta \\ -(b+a\cos\theta)\sin\phi & (b+a\cos\theta)\cos\phi & 0 \end{vmatrix}$$

$$= -i\,(b + a\cos\theta)\cos\phi\cos\theta + j\,(b + a\cos\theta)\sin\phi\cos\phi$$

$$+ k - a\,(b + a\cos\theta)\sin\theta\cos^2\phi + a\sin\theta\,(b + a\cos\theta)\sin^2\phi \quad .....(6.3.5)$$

Thus $\qquad \hat{N} = \dfrac{\vec{r}_1 \times \vec{r}_2}{H} = (-\cos\theta\cos\phi, -\cos\theta\sin\phi, -\sin\theta)$

Hence $L = \hat{N}\cdot\vec{r}_{11} = a, M = \hat{N}\cdot\vec{r}_{12} = 0$ and $N = \hat{N}\cdot\vec{r}_{22} = (b + a\cos\theta)\cos\theta$ .....(6.3.6)

***(ii)* Conoidal surface :**

Let the surface of revolution be

$$\vec{r} = [u\cos v, u\sin v, f(v)] \quad .....(6.3.7)$$

Then $\qquad \vec{r}_1 = \dfrac{\partial\vec{r}}{\partial u} = (\cos v, \sin v, 0)$

$$\vec{r}_2 = \frac{\partial\vec{r}}{\partial v} = (-u\cos v, u\cos v, f')$$

$$\vec{r}_{11} = \frac{\partial^2\vec{r}}{\partial u^2} = (0,0,0),$$

$$\vec{r}_{12} = \frac{\partial^2 r}{\partial u\partial v} = (-\sin v, \cos v, 0)$$

$$\vec{r}_{22} = \frac{\partial^2\vec{r}}{\partial v^2} = (-u\cos v, -u\sin v, f'') \quad .....(6.3.8)$$

Therefore $E=\vec{r}_1^{\,2}=\cos^2 u+\sin^2 v=1$

$$F=\vec{r}_1\cdot\vec{r}_2=-u\cos v\sin v+u\cos v\sin v+0$$

$$G=\vec{r}_2^{\,2}=u^2\sin^2 v+u^2\cos^2 v+f''^2=u^2+f'^2 \qquad .....(6.3.9)$$

$$H=\sqrt{EG-F^2}=\sqrt{\left(u^2+f^{12}\right)-0}=\sqrt{u^2+f'^2} \qquad .....(6.3.10)$$

$$\hat{N}=\frac{\vec{r}_1\times\vec{r}_2}{H}=\frac{\left(f'\sin v,-f'\cos v,u\right)}{u^2+f'^2} \qquad .....(6.3.11)$$

Further $L=\hat{N}\cdot\vec{r}_{11}=\hat{N}\cdot(0,0,0)=0$

$$M=\hat{N}\cdot\vec{r}_{12}=\frac{\left(f'\sin v,-f'\cos v,u\right)}{\sqrt{u^2+f'^2}}\cdot\left(-\sin v,\cos v,0\right)$$

$$=\frac{-f'}{\sqrt{u^2+f'^2}}$$

and $$N=\hat{N}\cdot\vec{r}_{22}=\frac{\left(f'\sin v,-f'\cos v,u\right)}{\sqrt{u^2+f'^2}}\cdot\left(-u\cos v,-u\sin v,f''\right)$$

$$=\frac{uf''}{\sqrt{u^2+f'^2}} \qquad .....(6.3.12)$$

***(iii)* Monge's form surface :**

The equation of surface given in the form $z=f(x,y)$ is called Monge's form.

Let the position vector of a current point on the surface be given by

$$\vec{r}=\left(x,y,z\right), \qquad .....(6.3.13)$$

where $z=f(x,y)$.

Since z in a function of x and y, therefore the equation (6.3.13) may be regarded as the parametric equation of the surface with parameters x and y.

Hence $\vec{r}_1=\left(1,0,p\right),\quad \vec{r}_2=\left(0,1,q\right),\quad \vec{r}_{11}=\left(0,0,t\right),$

$$\vec{r}_{12}=\left(0,0,s\right) \quad \text{and} \quad \vec{r}_{22}=\left(0,0,t\right) \qquad .....(6.3.14)$$

where $$p=\frac{\partial z}{\partial x},q=\frac{\partial z}{\partial y},r=\frac{\partial^2 z}{\partial x^2},s=\frac{\partial^2 z}{\partial y\partial x},t=\frac{\partial^2 z}{\partial y^2}.$$

and suffixes 1 and 2 denote differentiation w.r. to x and y, respectively.

Therefore $E=\vec{r}_1^{\,2}=1+p^2,\quad F=\vec{r}_1\cdot\vec{r}_2=pq,\quad G=r_2^2=1+q^2$

$$H=\sqrt{EG-f^2}=\sqrt{1+p^2+q^2},$$

and $$\hat{N} = \frac{\vec{r}_1 \times \vec{r}_1}{H} = \frac{(-p,-q,1)}{\sqrt{1+p^2+q^2}}. \qquad .....(6.3.15)$$

Also $$L = \hat{N} \cdot \vec{r}_{11} = \frac{r}{\sqrt{1+p^2+q^2}},$$

$$M = \hat{N} \cdot \vec{r}_{12} = \frac{s}{\sqrt{1+p^2+q^2}},$$

$$N = \hat{N} \cdot \vec{r}_{22} = \frac{t}{\sqrt{1+p^2+q^2}}. \qquad .....(6.3.16)$$

6.4 Orthogonal trajectories

6.4.1 Direction coefficients :

Analogous to concept of direction coefficients in analytical geometry, we have the same notion in differential geometry.

The discussion undertaken earlier has shown that at a point $p(\vec{r})$ of the surface $\vec{r} = \vec{r}(u,v)$, these exist three independent vectors $\vec{r}_1, \vec{r}_2$ and $\hat{N}$. Among these vectors, $\vec{r}_1$ and $\vec{r}_2$ lie in the tangent plane to the surface at $p(\vec{r})$ and the vector $\hat{N}$ is along the normal direction at $p(\vec{r})$. Consequently, we conclude that any vector at $p(\vec{r})$, can be expressed uniquely as a linear combination of $\vec{r}_1, \vec{r}_2$ and $\hat{N}$. Let $\hat{b}$ be any vector at $p(\vec{r})$, then we can write

$$\vec{b} = \lambda\,\vec{r}_1 + \mu\,\vec{r}_2 + \nu\hat{N} \qquad .....(6.4.1)$$

where λ, μ and ν are scalars. Here λ and μ are called tangential components of $\vec{b}$ and ν is called normal component of $\vec{b}$.

Here, we are concerned with tangential vectors *i.e.* the vector in which normal component ν is zero.

Let $\vec{b} = \lambda\,\vec{r}_1 + \mu\,\vec{r}_2$ be a vector along the tangent at $p(\vec{r})$.

Then $$|\vec{b}|^2 = |\lambda\,\vec{r}_1 + \mu\,\vec{r}_2|^2$$

$$= \lambda^2 |\vec{r}_1|^2 + 2\lambda\,\mu\,|\vec{r}_1 \cdot \vec{r}_2| + \mu^2 |\vec{r}_2|^2$$

$$= E\lambda^2 + 2F\lambda\mu + G\mu^2$$

$$\Rightarrow \quad |\vec{b}| = \left(E\lambda^2 + 2F\lambda\mu + G\mu^2\right)^{1/2} \qquad .....(6.4.2)$$

Eqn. (6.4.2) provides magnitude of the tangential vector $\vec{b}$.

Direction coefficients, determine the direction, and these are determined by using a unit vector.

Let $\vec{e} = l\vec{r}_1 + m\vec{r}_2$ be a unit vector in a tangential direction at $p(\vec{r})$ of the surface, then the components l and m are called direction coefficients. Since $\vec{e}$ is unit vector, therefore

$$|\vec{e}|^2 = 1 = |l\vec{r}_1 + m\vec{r}_2|^2 = l^2|\vec{r}_1|^2 + m^2|\vec{r}_2|^2 + 2lm|\vec{r}_1 \cdot \vec{r}_2|$$

$$\Rightarrow \qquad |\vec{e}| = 1 = \left(El^2 + 2lmF + Gm^2\right)^{1/2} \qquad \text{.....(6.4.3)}$$

We know that a metric on the curve is given by

$$ds^2 = E\,du^2 + 2F\,du\,dv + G dv^2$$

$$\text{or} \qquad 1 = E\left(\frac{du}{ds}\right)^2 + 2F\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + G\left(\frac{dv}{ds}\right)^2 \qquad \text{.....(6.4.4)}$$

On comparing the equation (6.4.3) and (6.4.4), we find that

$$l = \frac{du}{ds} \quad \text{and} \quad m = \frac{dv}{ds}.$$

Hence, we may conclude that $\frac{du}{ds}$ and $\frac{dv}{ds}$ are the actual direction coefficients of the tangent at the point $p(\vec{r})$ to the curve $\phi(u, v) = c$ lying on the surface $\vec{r} = \vec{r}(u, v)$.

Note : If (l, m) constitute the direction coefficients of the direction, then the direction coefficients of the direction opposite to the given direction are $(-l, -m)$.

6.4.2 Direction ratios :

Direction ratios are the quantities which are proportional to the direction coefficients.

Let (λ, μ) be the numbers which are proportional to the direction coefficients (l, m). Then we may find expressions for λ and μ as follows :

From (6.4.2), we have $\vec{e} = l\vec{r}_1 + m\vec{r}_2$ .....(6.4.5)

$$\vec{e} = \frac{\vec{b}}{|\vec{b}|} = \frac{\lambda\vec{r}_1 + \mu\vec{r}_2}{\left(E\lambda^2 + 2F\lambda\mu + G\mu^2\right)^{1/2}} \qquad \text{.....(6.4.6)}$$

On comparing (6.4.5) and (6.4.6), we find that

$$l = \frac{\lambda}{\left(E\lambda^2 + 2F\lambda\mu + G\mu^2\right)^{1/2}}, m = \frac{\mu}{\left(E\lambda^2 + 2F\lambda\mu + G\mu^2\right)^{1/2}}$$

$$\left.\begin{aligned} \text{or} \quad \lambda &= \left(E\lambda^2 + 2F\lambda\mu + G\mu^2\right)^{1/2} l \\ \text{and } \mu &= \left(E\lambda^2 + 2F\lambda\mu + G\mu^2\right)^{1/2} m \end{aligned}\right\} \qquad \text{.....(6.4.7)}$$

Equation (6.4.7) provides relation between direction coefficients (l, m) and associated direction ratios (λ, μ).

The above expressions are very useful in determining the directions and are readily used. To illustrate, let us consider a surface $\vec{r} = \vec{r}(u, v)$. Recall that we have two distinct family of curves (u-curves and v-curves) on the surface. We know that $\vec{r}_1$ and $\vec{r}_2$ are the vectors along the tangents to u-curves and v-curve respectively.

Then we have

$$\vec{r}_1 = 1 \cdot \vec{r}_1 + 0 \cdot \vec{r}_2 \qquad .....(6.4.8)$$

$\Rightarrow$ $\lambda = 1, \mu = 0.$

Consequently, $l = \dfrac{1}{\sqrt{E}}, m = 0$ (using (6.4.7))

Thus the unit vector in the tangential direction to u-curve is $\dfrac{\vec{r}_1}{\sqrt{E}}$ .....(6.4.9)

Similarly, the unit vector along the tangential direction to the v-curve is $\dfrac{\vec{r}_2}{\sqrt{G}}$ .....(6.4.10)

6.4.3 Orthogonal curves :

The parametric curves u = const., v = const. are said to be orthogonal if they intersect at right angle *i.e.* the angle between their tangents at the point of intersection of the curves is 90°. Before finding the condition for orthogonality, we find formula for the angle between parametric curves.

Let the curves u = const., v = const. do intersect in a point $p(\vec{r})$ at an angle α $(0 \le \alpha \le \pi)$.

Then $$\cos\alpha = \frac{\vec{r}_1 \cdot \vec{r}_2}{|\vec{r}_1||\vec{r}_2|} = \frac{F}{\sqrt{EG}} \qquad .....(6.4.11)$$

and $$\sin\alpha = \frac{|\vec{r}_1 \times \vec{r}_2|}{|\vec{r}_1||\vec{r}_2|} = \frac{H}{\sqrt{EG}} \qquad .....(6.4.12)$$

and obviously, $$\tan\alpha = \frac{H}{F} \qquad .....(6.4.13)$$

From the above we see that when $\alpha = 90^\circ$, then cos α and a is zero *i.e.* $F = 0$. Conversely if $F = 0$ then $\alpha = 90^\circ$. Thus we have $F = 0$ to be the necessary and sufficient condition for the parametric curves to be orthogonal.

6.4.4 Angle between two tangential directions on the surface :

Let θ be the angle between two tangential directions (l_1, m_1) and (l_2, m_2) on the surface $\vec{r} = \vec{r}(u, v)$, where l_i, m_i, $l = 1, 2$ are actual directions coefficient. Let $\hat{t}_1$ and $\hat{t}_2$ denote the unit vectors in the tangential directions, then we have,

$$\hat{t}_1 = l_1\vec{r}_1 + m_1\vec{r}_2,$$

$$\hat{t}_2 = l_2\vec{r}_1 + m_2\vec{r}_2,$$

Then $$\hat{t}_1 \cdot \hat{t}_2 = \left|\hat{\vec{t}}_1\right| \left|\hat{\vec{t}}_2\right| \cos\theta \qquad .....(6.4.14)$$

$$= 1 \cdot 1 \cos\theta \quad \left[\because \hat{t}_1 \cdot \hat{t}_2 \text{ and unit vectors therefore } \left|\hat{t}_1\right| \left|\hat{t}_2\right| = 1\right]$$

$\Rightarrow$ $$\cos\theta = \hat{t}_1 \cdot \hat{t}_2 \qquad .....(6.4.15)$$

or $$\cos\theta = \left(l_1\vec{r}_1 + m_1\vec{r}_2\right) \cdot \left(l_2\vec{r}_1 + m_2\vec{r}_2\right)$$

$$= \left(l_1\, l_2\right)\left(\vec{r}_1 \cdot \vec{r}_1\right) + \left(l_1 m_2 + m_1 l_2\right)\vec{r}_1 \cdot \vec{r}_2 + \left(m_1 m_2\right)\left(\vec{r}_2 \cdot \vec{r}_2\right)$$

$$= l_1\, l_2 E + \left(l_1 m_2 + m_1 l_2\right) F + m_1 m_2\, G \qquad .....(6.4.16)$$

Similarly, $$\sin\theta = \left|\,\hat{t}_1 \times \hat{t}_2\,\right| \qquad \left[\because\ \vec{A} \times \vec{B} = |\vec{A}||\,\vec{B}|\,\sin\theta\hat{n}\right]$$

$$= \left|\,\left(l_1\vec{r}_1 + m_1\vec{r}_2\right) \times \left(l_2\vec{r}_1 + m_2\vec{r}_2\right)\,\right|$$

$$= \left(l_1 m_1 - m_1 l_2\right)\left|\,\vec{r}_1 \times \vec{r}_2\,\right| \qquad \left[\because \vec{r}_1 \times \vec{r}_1 = 0,\quad \vec{r}_2 \times \vec{r}_2 = 0\right]$$

$$= \left(l_1 m_1 - m_1 l_2\right) H \qquad \left[\because H = \left|\,\vec{r}_1 \times \vec{r}_2\,\right|\right] \qquad .....(6.4.17)$$

Thus, $$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{H\left(l_1 m_2 - m_1 l_2\right)}{l_1 l_2 E + \left(l_1 m_2 + m_1 l_2\right) F + m_1 m_2 G} \qquad .....(6.4.18)$$

The angle between tangential directions can also be determined in terms of direction ratios.

Let (λ_1, μ_1) and (λ_2, μ_2) be the direction ratios corresponding to direction coefficients (l_1, m_1) and (l_2, m_2), then we have

$$l_1 = \frac{\lambda_1}{\left(E\lambda_1^2 + 2F\lambda_1\mu_1 + G\mu_2^2\right)^{1/2}},\ m_1 = \frac{\mu_1}{\left(El_1^2 + 2F\lambda_1\mu_1 + G\mu_1^2\right)^{1/2}}$$

$$l_2 = \frac{\lambda_2}{\left(E\lambda_2^2 + 2F\lambda_2\mu_2 + G\mu_2^2\right)^{1/2}},\ m_2 = \frac{\mu_2}{\left(E\lambda_2^2 + 2F\lambda_2\mu_2 + G\mu_2^2\right)^{1/2}} \qquad .....(6.4.19)$$

Using the above we find

$$\cos\theta = \frac{\lambda_1\lambda_2 E + F\left(\lambda_1\mu_2 + \mu_1\lambda_2\right) + \mu_1\mu_2 G}{\left(E\lambda_1^2 + 2F\lambda_1\mu_1 + G\mu_1^2\right)^{1/2}\left(E\lambda_2^2 + 2F\lambda_2\mu_2 + G\mu_2^2\right)^{1/2}} \qquad .....(6.4.20)$$

$$\sin\theta = \frac{H\left(\lambda_1\mu_2 - \mu_1\lambda_2\right)}{\left(E\lambda_1^2 + 2F\lambda_1\mu_1 + G\mu_1^2\right)^{1/2}\left(E\lambda_2^2 + 2F\lambda_2\mu_2 + G\mu_2^2\right)^{1/2}} \qquad .....(6.4.21)$$

and consequently,

$$\tan\theta = \frac{H(\lambda_1\mu_2 - \mu_1\lambda_2)}{\lambda_1\lambda_2 E + F(\lambda_1\mu_2 + \mu_1\lambda_2) + \mu_1\mu_2 G}. \quad .....(6.4.22)$$

From the above discussions, we conclude that the two directions be at right angles (orthogonal) *i.e.* $\theta = \pi/2$ if

$$\lambda_1\lambda_2 E + (\lambda_1\mu_2 + \mu_1\lambda_2)F + \mu_1\mu_2 G = 0 \quad .....(6.4.23)$$

or
$$l_1 l_1 E + (l_1 m_2 + m_1 l_2) F + m_1 m_2 G = 0. \quad [\because \cos\pi/2 = 0] \quad .....(6.4.24)$$

Ex. *Prove that the equation $Edu^2 - Gdv^2 = 0$ denote the curves bisecting the angles between the parametric curves u = constant, v = constant on a surface $\vec{r} = \vec{r}(u, v)$*

Sol. Let α_1 be the angle between the direction $\left(\frac{du}{ds}, \frac{dv}{ds}, \right)$ and the *u*-curve (*i.e.* v = const.).

Let (l_2, m_2) denote the direction (tangent) to the *u*-curve, then

$$l_2 = \frac{1}{\sqrt{E}}, m_2 = 0 \cdot$$

Then, "$\cos\alpha_1 = l_1 l_2 E + (l_1 m_2 + m_1 l_2) F + m_1 m_2 G$" gives

$$\cos\alpha_1 = \frac{du}{ds}\cdot\frac{1}{\sqrt{E}}E + \left(0 + \frac{dv}{ds}\cdot\frac{1}{\sqrt{E}}\right)F + 0 \qquad \left[\because l_1 = \frac{du}{ds}, m_1 = \frac{dv}{ds}\right]$$

$$= \frac{1}{\sqrt{E}}\left(E\frac{du}{ds} + F\frac{dv}{ds}\right)\cdot$$

Let α_2 be angle between the direction $\left(\frac{du}{ds}, \frac{dv}{ds}, \right)$ and *v*-curve (*i.e.* u = const.) and we suppose that (l_2, m_2) denote the direction of *v*-curve, then

$$l_2 = 0, m_2 = \frac{1}{\sqrt{E}}.$$

Then, we have
$$\cos\alpha_2 = l_1 l_2 E + (l_1 m_2 + l_2 m_1) F + m_1 m_2 G$$

$$= 0 + \left(\frac{du}{ds}\cdot\frac{1}{\sqrt{G}} + 0\right)F + \frac{dv}{ds}\cdot\frac{1}{\sqrt{G}} + G$$

$$= \frac{1}{\sqrt{G}}\left(F\frac{du}{ds} + G\frac{dv}{ds}\right).$$

Given that the direction $\left(\frac{du}{ds}, \frac{dv}{ds}\right)$ bisects the angle between the parametric curves, therefore we must have,

$$\cos\alpha_1 = \pm\cos\alpha_2. \quad .....(3)$$

Here + and – signs correspond to internal and external bisector respectively.

Putting the values of $\cos\alpha_1$ and $\cos\alpha_2$ from the equation (1) and (2) in equation (3), we get

$$\frac{1}{\sqrt{E}}\left(E\frac{du}{ds}+F\frac{dv}{ds}\right)=\pm\frac{1}{\sqrt{G}}\left(F\frac{du}{ds}+G\frac{dv}{ds}\right)$$

or $$\sqrt{G}\left(E\frac{du}{ds}+F\frac{dv}{ds}\right)=\pm\sqrt{E}\left(F\frac{du}{ds}+G\frac{dv}{ds}\right)$$

or $$G\left(E\frac{du}{ds}+F\frac{dv}{ds}\right)^2=E\left(F\frac{du}{ds}+G\frac{dv}{ds}\right)^2$$

or $$GE^2\,du^2+GF^2\,dv^2+2EFG\,dudv=EF^2\,du^2+EG^2\,dv^2+2EFG\,dudv$$

or $$E\,(EG-F^2)\,du^2-G\,(EG-F^2)\,dv^2=0$$

or $$(EG-F^2)\,(E\,du^2-G\,dv^2)=0$$

$\Rightarrow$ $$E\,du^2-G\,dv^2=0. \qquad [\because\ EG-F^2>0].$$

6.4.5 Family of curves and associated differential equations :

For a surface $$\vec{r}=\vec{r}\,(u,v) \qquad .....(6.4.25)$$

an implicit relation of the form $$\phi\,(u,\,v)=c \qquad .....(6.4.26)$$

give rise to family of curves on the surface (6.4.25), where c is a parameter, $\phi\,(u,\,v)$ is single valued function of $u,\,v$ and $\phi\,(u,\,v)$ possesses continuous derivatives $\frac{\partial\phi}{\partial u},\frac{\partial\phi}{\partial v}$ which don't vanish simultaneously. By assigning different values to c, we have different members of the family of curves given by (6.4.26). Here it should be noted that through every point of the surface, there passes one and only one curve of the family of curves given by (6.4.26).

The family of curves can be expressed in the form of differential equation as explained below.

Consider equation (6.4.26) as family of curves on the surface (6.4.25). Differentiating (6.4.26), we get

$$\frac{\partial\phi}{\partial u}du+\frac{\partial\phi}{\partial v}dv=0, \qquad .....(6.4.27)$$

$\Rightarrow$ $$\phi_1\,du+\phi_2\,dv=0,$$

where $$\phi_1=\frac{\partial\phi}{\partial u} \quad \text{and} \quad \phi_2=\frac{\partial\phi}{\partial v}.$$

Obviously on integrating $\phi_1\,du+\phi_2\,dv=0$ we would get (6.4.26), we assume

$$\phi_1=P\,(u,\,v) \quad \text{and} \quad \phi_2=Q\,(u,\,v) \qquad .....(6.4.28)$$

Thus equation (6.4.27) becomes

$$P\,du+Q\,dv=0 \qquad .....(6.4.29)$$

The equation (6.4.29) constitutes a differential equation of the family of curves (6.4.26).

From (6.4.29), we have

$$\frac{du}{-Q}=\frac{dv}{P} \qquad \text{.....(6.4.30)}$$

equation (6.4.30) emphasises that tangent at (u, v) to the curve has $(-Q, P)$ as its direction ratios.

6.4.6 Orthogonal trajectories :

Trajectory : A trajectory of the given family of curves is a curve which intersects every member of the family of curves by following some definite law. If the trajectory intersects the members of the family of curves at a constant angle α, then it is called α-trajectory. If $\alpha = 90°$, then it is called an orthogonal trajectory. Following proposition underlines the fact that every family of curves on a surface has orthogonal trajectories.

Differential equation of orthogonal trajectory :

Let $\qquad \phi(u, v) = c \qquad$ .....(6.4.31)

be a family of curves on the surface $\qquad \vec{r}=\vec{r}(u,v). \qquad$ .....(6.4.32)

Recall that here ϕ has continuous derivatives $\frac{\partial\phi}{\partial u},\frac{\partial\phi}{\partial v}$ which don't vanish simultaneously. On differentiating (6.4.31), we have

$$\frac{\partial\phi}{\partial u}du+\frac{\partial\phi}{\partial v}dv=0 \quad \text{or} \quad \phi_1\,du+\phi_2\,dv=0, \qquad \text{.....(6.4.33)}$$

where $\phi_1=\frac{\partial\phi}{\partial u}, \phi_2=\frac{\partial\phi}{\partial v}$.

We prescribe $\qquad \phi_1 = P(u, v), \quad \phi_2 = Q(u, v)$

Then (6.4.33), becomes

$$P\,du+Q\,dv=0 \quad \text{or} \quad \frac{du}{-Q}=\frac{dv}{P}. \qquad \text{.....(6.4.34)}$$

Then $(-Q, P)$ are direction ratios of tangent at any point (u, v) of a member of family of curves given by (6.4.31).

Let $(\partial u, \partial v)$ be direction ratios of the tangent at the point (u, v) of a member of orthogonal trajectories of (6.4.31).

Recalling that the directions (u_1, v_1) (u_2, v_2) are orthogonal if

$$E\,u_1u_2+F(u_1\,v_2+v_1\,u_2)+G\,v_1v_2=0 \qquad \text{.....(6.4.35)}$$

We have for the present case,

$$E(-Q)\,\partial u+F(-Q\,\partial v+P\partial u)+G\,P\partial v=0$$

On simplification, it reduces to

$$(FP-EQ)\,\partial u+(GP-FQ)\,\partial v=0 \qquad \text{.....(6.4.36)}$$

Equation (6.4.36) represents the differential equation of the orthogonal trajectories of the family of curves.

Note that equation (6.4.36) is integrable simply because ($FP - EQ$) and ($GP - FQ$) (the coefficients of ∂u, ∂v) in (6.4.36)} are continuous.

On integrating (6.4.36), we get the equation of orthogonal trajectory.

Theorem. *On a given surface, a family of curves and their orthogonal trajectories can always be chosen as parametric curves.*

Proof : We know that the differential equation

$$P\,du + Q\,dv = 0 \qquad .....(1)$$

where P, Q are functions of u and v, represents a family of curves on the surface $\vec{r} = \vec{r}(u, v)$

Let $\phi(u, v) = c_1$ (c_1 being constant) .....(2)

be the solution of (1). Then $P = \lambda \dfrac{\partial \phi}{\partial u}$, $Q = \lambda \dfrac{\partial \phi}{\partial v}$, (where $\lambda \neq 0$)

As discussed earlier, we know that the differential equation

$$(FP - EQ)\,\delta u + (GP - FQ)\,\delta v = 0 \qquad .....(3)$$

gives the orthogonal trajectories of the family of curves given by (1).

Let $\psi(u, v) = c_2$ (c_2 being constant) .....(4)

is the solution of (3), then we may have

$$FP - EQ = \mu \frac{\partial \psi}{\partial u},$$

$$GP - FQ = \mu \frac{\partial \psi}{\partial v} \qquad (\text{where } \mu \neq 0) \qquad .....(5)$$

In order to prove the theorem, we will show that

$$\frac{\partial(\phi, \psi)}{\partial(u, v)} \neq 0. \qquad .....(6)$$

That means that the two family of curves $\phi(u, v) = c_1$ and $\psi(u, v) = c_2$ are mutually independent.

Thus we examine, $$\frac{\partial(\phi, \psi)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial \phi}{\partial u} & \dfrac{\partial \psi}{\partial u} \\ \dfrac{\partial \phi}{\partial v} & \dfrac{\partial \psi}{\partial v} \end{vmatrix} = \frac{1}{\mu\lambda} \begin{vmatrix} P & FP - EQ \\ Q & GP - FQ \end{vmatrix}$$

$$= \frac{1}{\mu\lambda}\left[P(GP - FQ) - Q(FP - EQ)\right]$$

$$= \frac{1}{\mu\lambda}\left[EQ^2 - 2FPQ + GP^2\right] \neq 0 \cdot \qquad .....(7)$$

Thus $\dfrac{\partial(\phi,\psi)}{\partial(u,v)} \neq 0$ ensures that a family of curves and their orthogonal trajectories can always be chosen as parametric curves. Since ϕ is independent of ψ, hence proper transformation $u^* = \phi(u, v)$, $v^* = \psi(u, v)$ transforms the given family of curves and their orthogonal trajectories into the two families of parametric curves.

Double family of curves :

We have seen that the equation $P\,du + Q\,dv = 0$ give rise to a family of curves on the surface $\vec{r} = \vec{r}(u,v)$. Similarly, the quadratic equation

$$P\,du^2 + 2Q\,du\,dv + R\,dv^2 = 0 \qquad .....(1)$$

where P, Q, R are continuous functions of the parameters u and v and do not vanish together and are such that $Q^2 - PR > 0$, then the equation (1) give rise to two distinct family of curves as illustrated below.

Equation (1) can be written as

$$P\left(\frac{du}{dv}\right)^2 + 2Q\left(\frac{du}{dv}\right) + R = 0 \qquad .....(2)$$

Equation (2) is quadratic in $\dfrac{du}{dv}$ and has two solutions

$$\frac{du}{dv} = \frac{-Q \pm \sqrt{Q^2 - PR}}{P} \qquad .....(3)$$

which infact correspond to directions of the tangents to two distinct family of curves.

Let $\dfrac{l_1}{m_1}, \dfrac{l_2}{m_2}$ be the two directions, then

$$\frac{l_1}{m_1} = \frac{-Q + \sqrt{Q^2 - PR}}{P}, \frac{l_2}{m_2} = \frac{-Q - \sqrt{Q^2 - PR}}{P} \qquad .....(4)$$

Thus we have $$\frac{l_1}{m_1} + \frac{l_2}{m_2} = \frac{-2Q}{P}, \frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{R}{P}. \qquad .....(5)$$

If θ is the angle between these two directions then

$$\tan\theta = \frac{2H\sqrt{Q^2 - PR}}{(ER - 2FQ + GP)} \qquad .....(6)$$

Obviously these directions are orthogonal if $\theta = 90°$, *i.e.,*

$$ER - 2\,PQ + GP = 0. \qquad .....(7)$$

Note that (7) is the necessary and sufficient condition for the curves described by (1) to be orthogonal.

***Ex.*1.** *Examine whether the parametric curves*

$$x = b \sin u \cos v, \quad y = b \sin u \sin v, \quad z = b \cos u$$

on a sphere of radius b constitute an orthogonal system.

Sol. The given sphere is

$$\vec{r} = (b \sin u \cos v, b \sin u \sin v, b \cos u) \qquad .....(1)$$

The parametric curves would constitute an orthogonal system if $F = 0$ *i.e.* $\vec{r}_1 \cdot \vec{r}_2 = 0$,

we compute $\vec{r}_1 = (b \cos u \cos v, \quad b \cos u \sin v, \quad -b \sin u)$

$$\vec{r}_2 = (-b \sin u \sin v, \quad b \sin u \cos v, \quad 0)$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = -b^2 \cos u \sin u \cos v \sin v + b^2 \cos u \sin u \sin v \cos v$$

$$= 0. \qquad .....(2)$$

Thus the given parametric curves are orthogonal.

***Ex.*2.** *On the paraboloid $x^2 - y^2 = z$, find the orthogonal trajectories of the sections by the planes z = constant.*

Sol. Let $x = u$ and $y = v$, then for the given paraboloid we have $u^2 - v^2 = z$.

Thus $$\vec{r} = (u, v, u^2 - v^2) \qquad .....(1)$$

is the parametric equation of the given paraboloid.

The curves of section by the planes z = constant on (1) are given by

$$u^2 - v^2 = \text{constant} \qquad [\text{note}] \qquad .....(2)$$

Differentiating (2), we get

$$u\, du - v\, dv = 0. \qquad .....(3)$$

Thus, we have to find out the equation of trajectories orthogonal to family of curves given by (3).

Recall that if $$P\, du + Q\, dv = 0 \qquad .....(4)$$

is the given family of curves, then its orthogonal trajectories are given by

$$(PF - QE)\, \delta u + (PG - QF)\, \delta v = 0 \qquad .....(5)$$

where $(\delta u, \delta v)$ are direction ratios of the orthogonal trajectories.

Comparing (3), (4) we find

$$P = u, \; Q = -v \qquad .....(6)$$

$$\left.\begin{array}{l} \vec{r}_1 = (1, 0, 2u), \quad E = \vec{r}_1^{\,2} = 1 + 4u^2 \\ \vec{r}_2 = (0, 1, -2v), \quad F = \vec{r}_1 \cdot \vec{r}_2 = -4uv, \; G = \vec{r}_2^{\,2} = 1 + 4v^2 \end{array}\right\}. \qquad .....(7)$$

Using (6) and (7) in equation (5), we get the orthogonal trajectories as

$$(-4u^2v + v + 4u^2v)\, \delta u + (u + 4uv^2 - 4uv^2)\, \delta v = 0$$

or $$v\delta u + 4\delta v = 0$$

or $\delta(uv) = 0$

or $uv = \text{constant}$

or $xy = \text{constant}.$ $[\because x = u, y = v]$ .....(8)

Thus the hyperbolic cylinders xy = constant are the required orthogonal trajectories.

***Ex.*3.** *Let $v^2\, du^2 + u^2\, dv^2$ be the metric of a given surface. Then find*

(i) The family of curves orthogonal to the curves uv = constant

(ii) The metric corresponding to the new parameters so that these two families are parametric curves.

Sol. *(i)* Let s represent the arc length on the given surface. Then as given, we have

$$ds^2 = v^2\, du^2 + u^2\, dv^2 \qquad .....(1)$$

Thus, $E = v^2,\ F = 0,\ G = u^2$

We have to find family of curves orthogonal to the curves

$$uv = \text{constant}. \qquad .....(2)$$

Differentiating (2), we get

$$u\, dv + v\, du = 0 \Rightarrow \frac{du}{dv} = -\frac{u}{v}. \qquad .....(3)$$

Hence, the direction ratios of the tangent to the curve (2) are $(-u, v)$.

Let (du, dv) be the direction orthogonal to the direction, then the condition of orthogonality *i.e.*

$$\lambda_1 \lambda_2 E + (\lambda_1 \mu_2 + \lambda_2 \mu_1) F + \mu_1, \mu_2 G = 0.$$

or $E(-u)\, du + G\, v\, dv = 0,$ $[\because F = 0]$

$$-uv^2\, du + u^2 v\, dv = 0 \Rightarrow \frac{du}{u} = \frac{dv}{v} \qquad .....(4)$$

On integration of equation (4), we get

$$\log u = \log v + \log(\text{const}) \Rightarrow \frac{u}{v} = \text{const}. \qquad .....(5)$$

Hence equation (5) is the equation of the orthogonal trajectory of family of curves (2).

(ii) If the family of curves (2) and their orthogonal trajectories (5) are taken as parametric curves, then the new parameters u^* and v^* are given by

$$u^* = \frac{u}{v} \text{ and } v^* = uv \Rightarrow u^2 = u^*\, v^* \text{ and } v^2 = \frac{v^*}{u^*}. \qquad .....(6)$$

Now $$\vec{r}_1{}^* = \frac{\partial \vec{r}}{\partial u^*} = \frac{\partial \vec{r}}{\partial u}\frac{\partial u}{\partial u^*} + \frac{\partial \vec{r}}{\partial v}\frac{\partial v}{\partial u^*} = \frac{v^*}{2u}\vec{r}_1 - \frac{u^*}{\partial v u^{*2}}\vec{r}_2. \qquad .....(7)$$

and $$\vec{r}_2{}^* = \frac{\partial \vec{r}}{\partial v^*} = \frac{\partial \vec{r}}{\partial u}\frac{\partial u}{\partial v^*} + \frac{\partial \vec{r}}{\partial v}\frac{\partial v}{\partial v^*} = \frac{u^*}{2u}\vec{r}_1 + \frac{1}{2vu^*}\vec{r}_2. \qquad .....(8)$$

The new coefficients E^*, F^*, G^* are given by

$$E^* = \vec{r}_1^{\,*2} = \frac{u^{*2}}{4u^2}\vec{r}_1^{\,2} + \frac{v^{*2}}{4v^2u^{*2}}\vec{r}_2^{\,2}, \because \vec{r}_1 \cdot \vec{r}_2 = F = 0$$

$$= E\frac{u^{*2}}{4u^2} + G\frac{v^{*2}}{4v^2u^{*2}} = \frac{1}{2}\frac{v^{*2}}{u^{*2}},$$

$$F^* = \vec{r}_1{}^* \cdot \vec{r}_2{}^* = 0,$$

$$G^* = \vec{r}_2^{\,*2} = \left(\frac{u^*}{2u}\vec{r}_1 + \frac{1}{2vu^*}\vec{r}_2\right)^2 = \frac{1}{2} \qquad .....(9)$$

Therefore, the metric referred to new parameters u^*, v^* is given by

$$ds^{*2} = E^*\, du^2 + 2F^*\, du^*\, dv^* + G^*\, dv^{*2}$$

$$= \frac{1}{2}\frac{v^{*2}}{u^{*2}}du^{*2} + \frac{1}{2}dv^{*2}.$$

***Ex.*4.** *Show that the curves* $du^2 - (u^2 + c^2)\, dv^2 = 0$ *from an orthogonal system on the right helicoid*

$$\vec{r} = (u \cos v,\ u \sin v,\ cv)$$

Sol. The given surface is $\vec{r} = (u \cos v,\ u \sin v,\ cv)$ .....(1)

Then $$E = \vec{r}_1^{\,2} = \left(\frac{\partial \vec{r}}{\partial u}\right)^2 = \cos^2 v + \sin^2 v = 1$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = (\cos v, \sin v, 0) \cdot (-u \sin v, u \cos v, c)$$

$$= -u \cos v \sin v + u \cos v \sin v + 0 = 0$$

$$G = \vec{r}_2^{\,2} = \left(\frac{\partial \vec{r}}{\partial v}\right)^2 = (-u \sin v, u \cos v, c)$$

$$= u^2 (\sin^2 v + \cos^2 v) + c^2 = u^2 + c^2 \qquad .....(2)$$

Recall that the two family of curves given by the quadratic differential equation

$$P\, du^2 + 2Q\, du\, dv + R\, dv^2 = 0 \qquad .....(3)$$

constitute an orthogonal system if and only if

$$ER - 2FQ + GP = 0 \qquad .....(4)$$

The given curves are $du^2 - (u^2 + c^2)\, dv^2 = 0$ .....(5)

On comparing (3) and (5), we get

$$P = 1,\ Q = 0,\ R = -(u^2 + c^2). \qquad .....(6)$$

Using these values and the values of E, F, G computed above in the equation (4) we see that (4) is identically satisfied

$$1 \cdot [-(u^2 + c^2)] - 0 + (u^2 + c^2) \cdot 1 = 0.$$

Ex.5. *Show that on a right helicoid, the family of curves orthogonal to the curves $u \cos v$ = constant is the family $(u^2 + a^2) \sin^2 v$ = constant*

Sol. Let the given right helicoid be

$$\vec{r} = (u \cos v, u \sin v, cv) \qquad .....(1)$$

Then from previous example

$$E = 1, F = 0, G = u^2 + c^2 \qquad .....(2)$$

The given family of curves is

$$u \cos v = \text{constant}. \qquad .....(3)$$

On differentiating equation (3), we get

$$\cos v \, du - u \sin v \, dv = 0. \qquad .....(4)$$

Equation (3) implies that the direction ratios of the tangent to given curve at the point (u, v) is $(u \sin v, \cos v)$. Let (du, dv) be direction of the required orthogonal curves. Then by the condition of orthogonality we have

$$E (u \sin v) \, du + F (u \sin v \, dv + \cos v \, du) + G \cos v \, dv = 0 \qquad .....(5)$$

Putting the values $E = 1$, $F = 0$, $G = u^2 + c^2$ in (4), we get

$$u \sin v \, du + (u^2 + c^2) \cos v \, dv = 0$$

or
$$\frac{u\,du}{u^2 + c^2} + \frac{\cos v \, dv}{\sin v} = 0$$

On integration, it gives

$$\frac{1}{2}\log\left(u^2 + c^2\right) + \log\left(\sin v\right) = \text{constant}.$$

or
$$\log (u^2 + c^2) \; 2 \log \sin v = \text{constant} \qquad .....(6)$$

Equation (6) represents the required orthogonal curves family.

6.5 Normal curvature

Before embarking on the idea of normal curvature, we first have to go through some basic things as follows :

(i) **Plane section of a surface :**

A plane drawn through a point on a surface cuts the surface, in general, in a plane curve. This plane curve is called the **plane section** of the surface.

(ii) **Normal section of the surface :**

If the plane section of the surface is such that it contains the normal to the surface at that point, the section is called the **normal section**. The section, which is not normal section is called the **oblique section**.

(iii) **Curvature at a point on a given surface :**

The curvature at a point on a given surface is closely related to the plane section at the point.The curvature at a point $P(u, v)$ of the given surface $\vec{r}=\vec{r}(u,v)$ in a direction (du, dv) is the curvature of the plane section (curve) of the surface which passes through the point $P(u, v)$ and contains the direction (du, dv).

6.5.1 Curvature of normal section :

Let $\vec{r}=\vec{r}(u,v)$ be the given surface and $P(u, v)$ be any point on it. Let k_n denotes the curvature of the normal section. By convention, we presume that the sense of the unit principal normal to the curve *i.e.* $\hat{n}$ and the unit surface normal *i.e.,* $\hat{N}$ are the same. Further note that k_n is considered positive when the curve is concave on the side towards which $\hat{N}$ points out.

We, now, have

$$\vec{r}''=\frac{dt}{ds}=k_n\,\hat{n}=k_n\,\hat{N} \qquad \left[\because \text{ Here } \quad \hat{n}=\hat{N}\right]$$

Therefore,

$$k_n=\hat{N}\cdot\vec{r}''. \qquad .....(6.5.1)$$

Again, we know that

$$\vec{r}'=\frac{d\vec{r}}{ds}=\frac{\partial\vec{r}}{du}\frac{du}{ds}+\frac{\partial\vec{r}}{dv}\frac{dv}{ds}=\vec{r}_1\frac{du}{ds}+\vec{r}_2\frac{dv}{ds}$$

$$=\vec{r}_1\,u'+\vec{r}_2\,v', \qquad \left[\text{where } u'=\frac{du}{ds}, v'=\frac{dv}{ds}\right] \qquad .....(6.5.2)$$

Again differentiating with respect to s, we get

$$\vec{r}''=\frac{d}{ds}\left[\vec{r}_1\,u'+\vec{r}_2\,v'\right]$$

$$=\vec{r}_1\,u''+\frac{d\vec{r}_1}{ds}u'+\vec{r}_2v''+\frac{d\vec{r}_2}{ds}v'$$

$$=\vec{r}_1\,u''+\vec{r}_2v''+\left(\frac{\partial\vec{r}_1}{\partial u}\frac{du}{ds}+\frac{\partial\vec{r}_1}{\partial s}\frac{dv}{ds}\right)u'+\left(\frac{\partial\vec{r}_2}{\partial u}\frac{du}{ds}+\frac{\partial\vec{r}_2}{\partial v}\frac{dv}{ds}\right)v'$$

$$=\vec{r}_1\,u''+\vec{r}_2v''+\vec{r}_{11}u'^2+\vec{r}_{12}u'v'+\vec{r}_{21}u'v'+\vec{r}_{22}v'^2$$

$$=\vec{r}_1\,u''+\vec{r}_2v''+\vec{r}_{11}\,u'^2+2\vec{r}_{12}u'v'+\vec{r}_{22}v'^2. \qquad .....(6.5.3)$$

Thus, we have

$$k_n=\hat{N}\cdot r''$$

$$=\hat{N}\cdot\left[\vec{r}_1\,u''+\vec{r}_2v''+\vec{r}_{11}\,u'^2+2\vec{r}_{12}u'v'+\vec{r}_{22}v'^2\right]$$

$$= \left(\hat{N}\cdot\vec{r}_1\right)u'' + \left(\hat{N}\cdot\vec{r}_2\right)v'' + \left(\hat{N}\cdot\vec{r}_{11}\right)u'^2 + 2\left(\hat{N}\cdot\vec{r}_{12}\right)u'v' + \left(\hat{N}\cdot\vec{r}_{22}\right)v'^2$$

$$= 0 + 0 + Lu'^2 + 2Mu'v' + N\cdot v'^2 \qquad .....(6.5.4)$$

$$\left[\because\ \hat{N}\cdot\vec{r}_1 = \hat{N}\cdot\vec{r}_2 = 0 \quad \text{and} \quad \hat{N}\cdot\vec{r}_{11} = L \quad \text{etc.}\right]$$

Thus, $$k_n = L\left(\frac{du}{ds}\right)^2 + 2M\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + N\left(\frac{dv}{ds}\right)^2$$

or $$k_n = \frac{L\,du^2 + 2M\,du\,dv + N\,du^2}{ds^2}$$

or $$k_n = \frac{L\,du^2 + 2M\,du\,dv + N\,du^2}{Edu^2 + 2F\,du\,dv + G\,dv^2} \qquad .....(6.5.5)$$

$$[\because\ ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2]$$

Equations (6.5.5) provides the curvature of the normal section, parallel to the direction (*du, dv*) in terms of fundamental magnitudes.

Notes :

1. k_n depends purely on the direction (*du, dv*) of the curve drawn on the surface and the quantities *E, F, G, L, M, N* which are determined at given point *P*.

This reasoning helps us to conclude that all the curves tangent to the same direction on the given surface have the same normal curvature, since normal curvature at a point on the surface is the property of the surface which depends on the direction at the point on the surface.

2. The reciprocal of k_n is called the **radius of normal curvature**, and is denoted by ρ_n.

6.6 Summary

In this unit you came across with the notion of the fundamental forms which are quadratic equation in du and dv. Each form has its definite geometrical significance. Further the directions on the surface where explained and the criterion was extended to the orthogonal trajectories on the surface. Lastly the notion of normal curvature was given.

6.7 Self-learning exercises

1. Write down first and second fundamental forms

2. Define direction coefficients.

3. Define direction ratios.

4. What are orthogonal trajectories ?

5. Define oblique section and normal section.

6.8 Exercises

1. Compute the fundamental magnitudes for the surface

$$\vec{r} = (u \cos v,\ u \sin v,\ f(u) + cv).$$

2. Prove that the curves $du^2 - (u^2 + c^2)\, dv^2 = 0$ form an orthogonal system on the right helicoid

$$\vec{r} = (u \cos v,\ u \sin v,\ cv).$$

3. Compute E, F, G, H for the surface

(i) $x = u,\ y = v,\ z = u^2 - v^2$,

(ii) $2z = ax^2 + 2hxy + by^2$.

4. Prove that : $\quad H\hat{N} \times N_1 = M\,\vec{r}_1 - L\,\vec{r}_2.$

□ □ □

Unit 7 : Meunier's Theorem, Principal direction and Principal curvatures, First Curvature, Mean Curvature, Gaussian Curvature, Umbilics, Radius of Curvature of any Normal Section at an Umbilic on $z = f(x, y)$. Radius of Curvature of a given Section through any Point on $z = f(x, y)$, Lines of Curvature

Structure of the Unit

7.1 Objectives

This unit provides a general overview of the following and after reading this unit you will be able to learn

1. about Meunier's theorem,
2. about Principal direction and Principal curvature of the surfaces,
3. about first curvature, Mean curvature and Gaussian curvature,
4. about Umbilics, radius of curvature of any normal section at an umbilics $z = f(x, y)$, radius of curvature of a given section through any point on $z = f(x, y)$,
5. about lines of curvatures.

7.2 Introduction

In this unit we shall study local non-intrinsic properties of a surface. We shall also study curvature of surfaces, plane section of surfaces and oblique section of surfaces. After that we shall establish a relationship between curvature of normal section (κ_n) and curvature of oblique section (κ), which is known as Meunier's theorem. In the end of the unit we shall study about lines of curvature.

7.3 Definitions

***(i)* Intrinsic property**

Property of a surface deducible from the metric alone, without using the surface equation $\vec{r} = \vec{r}(u, v)$ is called an intrinsic property.

***(ii)* Plane section of a surface**

A plane drawn through a point A of a given surface intersects it in a plane curve, known as the plane section of the surface. In Fig. 7.1 ABC and ADE are two plane section of the surface $\vec{r} = \vec{r}(u, v)$.

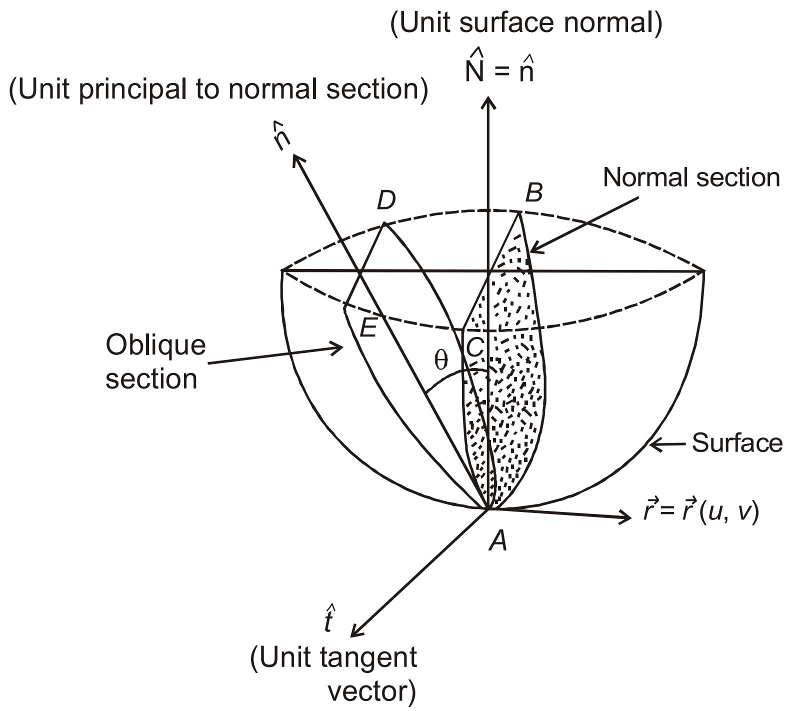


Fig. 7.1

***(iii)* Normal section**

The plane is so drawn that it contains the normal $\hat{N}$ to the surface at the point (say *A*), then the curve of intersection is called normal section. In the Fig. 7.1 the curve bounding shaded area *ABC* is the normal section. Thus the normal section is parallel to the normal $\hat{N}$ to the surface.

***(iv)* Oblique section**

The plane is so drawn that it does not contain the normal $\hat{N}$ to the surface at the point (say *A*), then the curve of intersection is called oblique section.

Note : There exist infinite number of planes of normal sections through the principal normal at point *A*, but there will be only one such plane of normal section having directions (*du, dv*).

Principal normal $\hat{n}$ for normal section is parallel to surface normal $\hat{N}$ and principal normal $\hat{n}$ for oblique section is inclined at angle θ to surface normal $\hat{N}$.

We adopt the convention that vector $\hat{n}$ has the same direction as that of vector $\hat{N}$, and with this convention $\hat{n} = \hat{N}$.

***(v)* Curvature at a point on a given surface**

Let *A* be a point with position vector $\vec{r}\left(u,v\right)$ on the surface $\vec{r} = \vec{r}\left(u,v\right)$ the normal section at *A* in the direction (*dn, dv*) is equal to the curvature at *A* of the normal section at *A* parallel to the direction (*dn, dv*) of $\hat{t}$.

***(vi)* Fundamental Magnitudes**

Let $\vec{r}=\vec{r}(u,v)$ be the equation of the surface and let $\vec{r}_1=\dfrac{\partial\vec{r}}{\partial u}, \vec{r}_2=\dfrac{\partial\vec{r}}{\partial v}$ then $E=\vec{r}_1^{\,2}=\vec{r}_1\cdot\vec{r}_1$, $F=\vec{r}_1\cdot\vec{r}_2$, $G=\vec{r}_2^{\,2}=\vec{r}_2\cdot\vec{r}_2$ are called first order fundamental magnitudes and if $\hat{N}$ be unit normal vector at $A(\vec{r})$, then $L=\vec{r}_{11}\cdot\hat{N}$, $M=\vec{r}_{12}\cdot\hat{N}$ and $N=\vec{r}_{22}\cdot\hat{N}$ are called second order fundamental magnitudes, where $\vec{r}_{11}=\dfrac{\partial^2\vec{r}}{\partial u^2}$, $\vec{r}_{12}=\dfrac{\partial^2\vec{r}}{\partial u\partial v}$, $\vec{r}_{22}=\dfrac{\partial^2\vec{r}}{\partial v^2}$.

7.4 Curvature of section (*i.e.*, normal curvature)

Let $\vec{r}=\vec{r}(u,v)$ be the equation of a given surface and $A\ (u,v)$ is any point on the surface.

Here we assume that $$\hat{N}=\hat{n} \qquad .....(7.4.1)$$

Let κ_n represents the curvature of normal section, which will be positive when the curve is concave on side towards which $\hat{N}$ points out.

Now $$\vec{r}'=\hat{t}=\frac{d\vec{r}}{ds} \qquad .....(7.4.2)$$

where s is arc length.

Again differentiating, we have

$$\frac{d\hat{t}}{ds}=\vec{r}''=\kappa_n\hat{n}$$

or $$\vec{r}''=\kappa_n\hat{N} \quad [\because \hat{N}=\hat{n} \text{ from equation (1)}] \qquad .....(7.4.3)$$

Taking dot product by $\hat{N}$, we get

$$\hat{N}\cdot\vec{r}''=\kappa_n\left(\hat{N}\cdot\hat{N}\right)$$

$$\Rightarrow \quad \hat{N}\cdot\vec{r}''=\kappa_n \qquad \left(\because \hat{N}\cdot\hat{N}=1\right) \qquad .....(7.4.4)$$

Also we know that

$$\vec{r}'=\frac{d\vec{r}}{ds}=\frac{\partial\vec{r}}{\partial u}\frac{\partial u}{\partial s}+\frac{\partial\vec{r}}{\partial v}\frac{\partial v}{\partial s}$$

$$=\vec{r}_1\frac{du}{ds}+\vec{r}_2\frac{dv}{ds}=\vec{r}_1u'+\vec{r}_2v' \text{ (where } u'=\frac{du}{ds} \text{ etc.)} \qquad .....(7.4.5)$$

Differentiating this relation again with respect to s, we have

$$\vec{r}''=\vec{r}_1u''+\frac{d\vec{r}_1}{ds}u'+\vec{r}_2v''+\frac{d\vec{r}_2}{ds}v' \ (u''=\frac{d^2u}{ds^2} \text{ etc.})$$

or $$\vec{r}''=\left(\vec{r}_1u''+\vec{r}_2v''\right)+\left(\frac{\partial\vec{r}_1}{\partial u}\frac{du}{ds}+\frac{\partial\vec{r}_1}{\partial v}\frac{dv}{ds}\right)u'+\left(\frac{\partial\vec{r}_2}{\partial u}\frac{du}{ds}+\frac{\partial\vec{r}_2}{\partial v}\frac{dv}{ds}\right)v'$$

$$=\vec{r}_1u''+\vec{r}_2v''+\vec{r}_{11}u'^2+\vec{r}_{12}u'v'+\vec{r}_{21}u'v'+\vec{r}_{22}v'^2 \qquad .....(7.4.6)$$

where $\vec{r}_{12} = \dfrac{\partial \vec{r}_1}{\partial v}, \quad \vec{r}_{11} = \dfrac{\partial \vec{r}_1}{\partial u}$ etc.

Now taking dot product by $\hat{N}$, we get

$$\vec{r}'' \cdot \hat{N} = \left(\vec{r}_1 u'' + \vec{r}_2 v'' + \vec{r}_{11} u'^2 + \vec{r}_{12} u'v' + \vec{r}_{21} u'v' + \vec{r}_{22} v'^2\right) \cdot \hat{N}$$

$$= \left(\vec{r}_1 \cdot \hat{N}\right) u'' + \left(\vec{r}_2 \cdot \hat{N}\right) v'' + \left(\vec{r}_{11} \cdot \hat{N}\right) u'^2 + \left(\vec{r}_{12} \cdot \hat{N}\right) u' \cdot v'$$

$$+ \left(\vec{r}_{21} \cdot \hat{N}\right) u'v' + \left(\vec{r}_{22} \cdot \hat{N}\right) v'^2 \quad .....(7.4.7)$$

But we know that

$$\vec{r}_1 \cdot \hat{N} = 0, \ \vec{r}_2 \cdot \hat{N} = 0 \quad .....(7.4.8)$$

($\because$ unit normal vector $\hat{N}$ is $\perp$ to both direction vectors of the tangents r_1 and r_2)

and
$$\left.\begin{aligned} \vec{r}_{11} \cdot \hat{N} &= L \\ \vec{r}_{12} \cdot \hat{N} &= M \\ \vec{r}_{22} \cdot \hat{N} &= N \end{aligned}\right\} \quad .....(7.4.9)$$

(The resolved parts of r_{11}, r_{12}, r_{22} in the direction of normal to the surface are *L, M, N* called second order fundamental magnitudes)

Using values from equation (7.4.8) and (7.4.8) in equation (7.4.7), we get

$$\vec{r}'' \cdot \hat{N} = \kappa_n = Lu'^2 + 2Mu'v' + Nv'^2$$

or
$$\kappa_n = L\left(\frac{du}{ds}\right)^2 + 2M\frac{du}{ds}\frac{dv}{ds} + N\left(\frac{dv}{ds}\right)^2 = \frac{Ldu^2 + 2Mdu\,dv + Ndv^2}{ds^2}$$

or
$$\kappa_n = \frac{Ldu^2 + 2Mdu\,dv + Ndv^2}{Edu^2 + 2Fdu\,dv + Gdv^2} \quad .....(7.4.10)$$

($\because ds^2 = Edu^2 + 2Fdu\,dv + Gdv^2$, the first fundamental form)

Equation (7.4.10) gives the curvature of the normal section, usually called normal curvature parallel to the direction (*du, dv*) in terms of the fundamental magnitudes.

Remark : Since κ_n depends only on the direction (*du, dv*) of the curve drawn on the surface, as fundamental magnitudes *E, F, G, L, M, N* are determined by the given point *A*. So normal curvature at a point on a surface is a property of the surface which depend on the direction at the point on the surface. Hence all curves tangent to the same direction on a surface have the same normal curvature.

7.4.1. Radius of normal curvature

Reciprocal of the normal curvature (κ_n) is called the radius of normal curvature and it is denoted by ρ_n *i.e.*

$$\rho_n = \frac{1}{\kappa_n}. \quad .....(7.4.11)$$

7.4.2. Normal curvature definition

Let A (u, v) be a point on the surface $\vec{r} = \vec{r}(u, v)$. The normal curvature at A in the direction (du, dv) is equal to the curvature of the normal section at A, parallel to the direction (du, dv).

7.5 Meunier's Theorem (or Meusnier's theorem)

Statement : If κ and κ_n are the curvatures of oblique and normal sections through the same tangent line and θ be the angle between these sections, then $\kappa_n = \kappa \cos \theta$.

Proof : Refer Fig. 7.1, let $\hat{t}$ be the tangent vector to the normal section of the given surface. Let ADE be the oblique section of the surface by a plane through $\hat{t}$. Here $\hat{N}$ is the surface normal which is also principal normal of the normal section. Let $\hat{n}$ be the principal normal of the oblique section, then we have

$$\cos \theta = \hat{n} \cdot \hat{N}, \qquad \text{.....(7.5.1)}$$

because θ is the angle between the planes of sections as shown in the fig. 7.1

But, if κ is the curvature of oblique section

then, for any section $\qquad \vec{r}'' = \kappa \hat{n}$. $\qquad$ .....(7.5.2)

Now taking dot product by $\hat{N}$, we have

$$\vec{r}'' \cdot \hat{N} = \kappa \hat{n} \cdot \hat{N}$$

$$= \kappa \cos \theta \qquad \text{[by equation (7.5.1)]} \qquad \text{.....(7.5.3)}$$

Now $\qquad \vec{r}'' \cdot \hat{N}$ = normal curvature at A in the direction

(du, dv) = curvature of the normal section at A parallel to be direction

$(du, dv) = \kappa_n$

$\therefore$ by equation (7.5.3),

$$\kappa_n = \vec{r}'' \cdot \hat{N} = \kappa \cos \theta$$

$$\Rightarrow \qquad \kappa_n = \kappa \cos \theta.$$

Hence Proved.

7.5.1. Important result

If a sphere is described with ρ_n as diameter then all centers of curvature lie on this sphere, provided unit tangent vector $\hat{t}$ is the same.

Proof : Let C be center of oblique section and C_n that of normal section for all planes containing $\hat{t}$, as shown in the Fig. 7.2. From figure, we have

$$AC_n = \rho_n \quad \text{and} \quad AC = \rho \quad \text{and} \quad \angle C_n AC = \theta.$$

Join C_n and C.

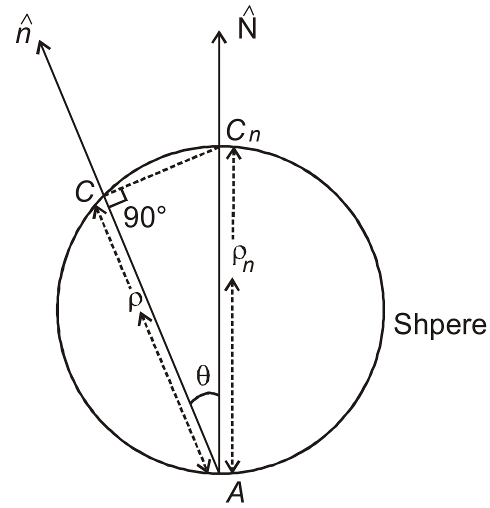


Fig. 7.2

Now, we know that $\kappa_n = \kappa \cos\theta$ (by Meunier's Theorem)

or $$\frac{1}{\rho_n} = \frac{1}{\rho}\cos\theta \qquad \left(\because \rho_n = \frac{1}{\kappa_n}, \rho = \frac{1}{\kappa}\right)$$

$$\Rightarrow \qquad \rho = \rho_n \cos\theta. \qquad .....(7.5.4)$$

From this we conclude from $\Delta\, AC_nC$, that $\angle C_nAC = 90^\circ$.

This result is interpreted as if a sphere is described with ρ_n as diameter, all centres of curvature lie on this sphere, provided $\hat{t}$ is the same.

7.5.2. Self-learning exercise-1

1. Define intrinsic property of a surface.
2. Define plane section of a surface.
3. Define normal section and oblique section of a surface.
4. Define curvature at a point on a given surface.
5. Write formula for curvature of normal section in terms of fundamental magnitudes.
6. Define normal curvature and radius of normal curvature.
7. Write the statement of Meunier's Theorem.

7.5.3. Illustrative Examples

***Ex.*1.** *Find the curvature of a normal section of the right helicoid*

$$x = u\cos\phi,\ y = u\sin\phi,\ z = c\,\phi.$$

Sol. The curvature of normal section is given by

$$\kappa_n = \frac{Ldu^2 + 2Mdu\,dv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2} \qquad .....(1)$$

To find κ_n, we shall first evaluate fundamental magnitudes *E, F, G, L, M, N.*

Let $\vec{r} = (u\cos\phi,\ u\sin\phi,\ c\,\phi)$ .....(2)

with u and ϕ as parameters, C is constant.

Let suffixes 1 and 2 represent partial differentiations of $\vec{r}$ with respect to u and ϕ $\left(i.e.\ \vec{r}_1 = \frac{\partial \vec{r}}{\partial u}, \vec{r}_2 = \frac{\partial \vec{r}}{\partial \phi}\right)$.

Then on differentiating, (2) with respect to u

$$\vec{r}_1 = (\cos\phi,\ u\sin\phi,\ 0) \qquad .....(3)$$

Now differentiating equation (2) with respect to ϕ, we have

$$\vec{r}_2 = (-u\sin\phi,\ u\cos\phi,\ c) \qquad .....(4)$$

Now differentiating (3) with respect to u, we get

$$\vec{r}_{11} = (0,0,0) \qquad .....(5)$$

Differentiating (3) with respect to ϕ, we get

$$\vec{r}_{12} = (-\sin\phi,\ \cos\phi,\ 0) \qquad .....(6)$$

Now differentiating equation (4) with respect to ϕ

$$\vec{r}_{22} = (-u\cos\phi, -u\sin\phi, 0) \qquad .....(7)$$

$\because$ $$E = \vec{r}_1 \cdot \vec{r}_2 = r_1^2 = \cos^2\phi + \sin^2\phi = 1, \qquad .....(8)$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = (\cos\phi, \sin\phi, 0)\cdot(-u\sin\phi, u\cos\phi, 0)$$

$$= (-u\sin\phi\cos\phi + u\sin\phi\cos\phi + 0) = 0 \qquad .....(9)$$

and $$G = \vec{r}_2, \vec{r}_2 = (-u\sin\phi, u\cos\phi, c)\cdot(-u\sin\phi, u\cos\phi, c)$$

$$= u^2\sin^2\phi + u^2\cos^2\phi + c^2$$

or $$G = u^2 + c^2 \qquad .....(10)$$

Now $$H^2 = EG - F^2 = 1\cdot(u^2 + c^2) - 0^2 = u^2 + c^2 \qquad .....(11)$$

and $$\hat{N} = \frac{\vec{r}_1 \times \vec{r}_2}{H} = \frac{1}{H}\left(\cos\phi\,\hat{i} + \sin\phi\,\hat{j} + 0\hat{k}\right)\times\left(-u\sin\phi\,\hat{i} + u\cos\phi\,\hat{j} + c\hat{k}\right)$$

$$= \frac{1}{H}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\phi & \sin\phi & 0 \\ -u\sin\phi & u\cos\phi & c \end{vmatrix}$$

$$= \frac{1}{\sqrt{(u^2 + c^2)}}\left\{\hat{i}(c\sin\phi - 0) + \hat{j}(0 - c\cos\phi) + \hat{k}\left(u\cos^2\phi + u\sin^2\phi\right)\right\}$$

$$= \frac{1}{\sqrt{u^2 + c^2}}\left\{c\sin\phi\,\hat{i} - c\cos\phi\,\hat{j} + u\,\hat{k}\right\}$$

or $$\hat{N} = \frac{(c\sin\phi, -c\cos\phi, u)}{H = \sqrt{(u^2 + c^2)}} \qquad .....(12)$$

Now $\qquad L = \hat{N}\cdot\vec{r}_{11} = \hat{N}\cdot(0,0,0) = 0 \quad$ [From equation (5)] $\qquad$ .....(13)

$$M = \hat{N}\cdot\vec{r}_{12} = (-\sin\phi,\ \cos\phi,\ 0)\cdot\frac{(c\sin\phi,\ -c\cos\phi,\ u)}{\sqrt{(u^2+c^2)}}$$

$$= \frac{-c\sin^2\phi - c\cos^2\phi + 0}{\sqrt{(u^2+c^2)}}$$

or $$M = \frac{-c}{\sqrt{u^2+c^2}} \qquad .....(14)$$

$$N = \hat{N}\cdot\vec{r}_{22} = \frac{(c\sin\phi,\ -c\cos\phi,\ u)}{\sqrt{u^2+c^2}}\cdot(-u\cos\phi,\ -u\sin\phi,\ 0)$$

$$= \frac{-cu\sin\phi\cos\phi + uc\sin\phi\cos\phi + 0}{\sqrt{(u^2+c^2)}} = 0\cdot \qquad .....(15)$$

Now using values from equations (8), (9), (10), (13),(14) and (15) in equation (1), we have

$$\kappa_n = \frac{0 + 2\left(\frac{-c}{\sqrt{u^2+c^2}}\right)du\,d\phi + 0}{du^2 + 0 + (u^2+c^2)\,d\phi^2}$$

$$= \frac{-2c\,du\,d\phi}{\sqrt{u^2+c^2}\left(du^2 + (u^2+c^2)\,d\phi^2\right)}\cdot$$

***Ex.*2.** *Show that the curvature* κ *at any point P of the curve of intersection of two surfaces is given by* $\kappa^2\sin^2\alpha = \kappa_1^2 + \kappa_2^2 - 2\kappa_1\kappa_2\cos\alpha$, *where* κ_1 *are* κ_2 *the normal curvatures of the surfaces in the direction of the curve at P and* α *is the angle between their normals at that point.*

Sol. Let S_1 and S_2 be the two given surfaces and $\hat{N}_1$ and $\hat{N}_2$ be the unit normals to them at any common point *P*, respectively.

Curve of intersection of S_1 and S_2.

Let $\hat{n}$ be the unit principal normal to the curve of intersection at *P*, as shown in the Fig. 7.3.

Since $\hat{N}_1$, $\hat{n}$ and $\hat{N}_2$ are to be drawn through the same tangent line, so clearly $\hat{N}_1, \hat{n}, \hat{N}_2$ are coplanar.

Let β be angle between $\hat{n}$ and $\hat{N}_1$, then $(\alpha - \beta)$ is the angle between $\hat{N}_2$ and $\hat{n}$. Now using Meunier's theorem

$$\kappa_1 = \kappa\cos\beta \qquad .....(1)$$

and $$\kappa_2 = \kappa \cos(\alpha - \beta) \quad \text{.....(2)}$$

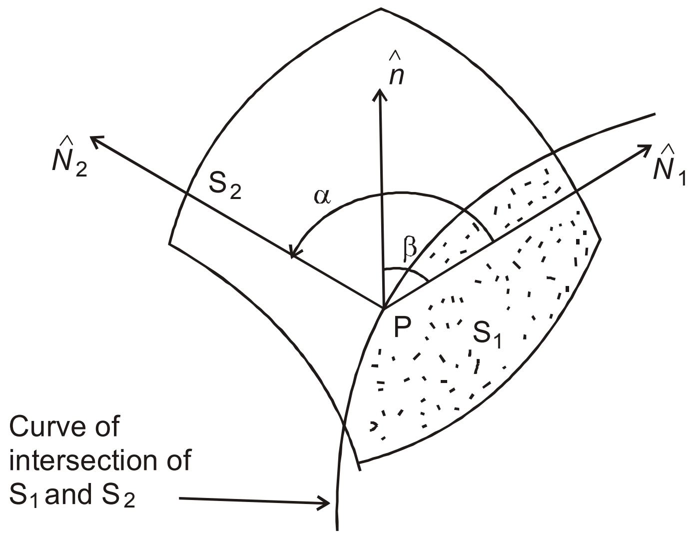


Fig. 7.3

From equation (1),

$$\cos\beta = \frac{\kappa_1}{\kappa}$$

and $$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{\left(1-\frac{\kappa_1^2}{\kappa^2}\right)}. \quad \text{.....(3)}$$

Now from equation (2),

$$\kappa_2 = \kappa\,[\cos\alpha\cos\beta + \sin\alpha\sin\beta]$$
$$= \cos\alpha\,(\kappa\cos\beta) + \kappa\sin\alpha\sin\beta$$

$$\kappa_2 = \cos\alpha\,\kappa_1 + \kappa\sin\alpha\sqrt{1-\frac{\kappa_1^2}{k^2}} \quad \text{[From (1) and (3)]}$$

or $$(\kappa_2 - \kappa_1\cos\alpha) = \sin\alpha\sqrt{\kappa^2 - \kappa_1^2}$$

Squaring both sides

$$\kappa_2^2 + \kappa_1^2\cos^2\alpha - 2\kappa_2\kappa_1\cos\alpha = \sin^2\alpha\left(\kappa^2 - \kappa_1^2\right)$$

or $$\kappa_2^2 + \left(\kappa_1^2\cos^2\alpha + \kappa_1^2\sin^2\alpha\right) - 2\kappa_2\kappa_1\cos\alpha = \kappa^2\sin^2\alpha$$

or $$\kappa_1^2 + \kappa_2^2 - 2\kappa_1\kappa_2\cos\alpha = \kappa^2\sin^2\alpha$$

which is the required result.

7.6 Principal direction and principal curvatures

We have seen that curvature of normal section of a surface at a point varies with the direction (du, dv) on the surface $\vec{r} = \vec{r}(u,v)$. Among all the normal sections there are two directions for which the curvature is maximum or minimum.

7.6.1. Definitions :

(i) **Principal section**

The normal section of a surface through a given point having maximum or minimum curvatures at the point are called principal sections of the surface at that point.

(ii) **Principal Direction**

Tangents to the principal section, *i.e.,* directions of the principal section are called principal directions at the given point. We shall see that, in general, there are two principal directions at a point and these are orthogonal.

(iii) **Principal Curvature**

The maximum and minimum curvatures of the two principal sections of a surface are called the principal curvatures.

(iv) **Principal radius of curvature (*i.e.* principal radii)**

Radii of curvatures of principal curvatures are called principal radius of curvature.

(v) **Surface of centres**

The locus of centres of principal curvatures at all points of a given surface called surface of centres.

7.7 Equation giving the principal directions at a point of surface and to derive the differential equation of the principal sections

We know that normal curvature κ_n at point A (u, v) in the direction (du, dv) of surface $\vec{r} = \vec{r}(u, v)$ is given by

$$\kappa_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2} \qquad \text{.....(7.7.1)}$$

If (l, m) be actual direction coefficients of the direction (du, dv),

where $$l = \frac{du}{ds},\ m = \frac{dv}{ds},$$

Then from (7.7.1), we have

$$\kappa_n = \frac{\left(\dfrac{Ldu^2 + 2Mdudv + Ndv^2}{ds^2}\right)}{\left(\dfrac{Edu^2 + 2Fdudv + Gdv^2}{ds^2}\right)} = \frac{L\left(\dfrac{du}{ds}\right)^2 + 2M\dfrac{du}{ds}\cdot\dfrac{dv}{ds} + N\left(\dfrac{dv}{ds}\right)^2}{E\left(\dfrac{du}{ds}\right)^2 + 2F\dfrac{du}{ds}\cdot\dfrac{dv}{ds} + G\left(\dfrac{dv}{ds}\right)^2}$$

or $$\kappa_n = \frac{Ll^2 + 2Mlm + Nm^2}{El^2 + 2Flm + Gm^2} \quad \left(\because l = \frac{du}{ds},\ m = \frac{du}{ds}\right) \qquad \text{.....(7.7.2)}$$

or $$\kappa_n = \left(Ll^2 + 2Mlm + Nm^2\right) \qquad .....(7.7.3)$$

$\because$ $$El^2 + 2Flm + Gm^2 = 1 \qquad .....(7.7.4)$$

Since L, M, N are fixed at A, so value of κ_n at A depends upon the values l, m at A. Hence κ_n is a function of two variables l, m, which are connected by equation (7.7.3) Taking l as a function of m, we find for stationary values by

$$\frac{d\kappa_n}{dm} = 0 \Rightarrow 2Ll\frac{dl}{dm} + 2M\left(l + m\frac{dl}{dm}\right) + 2Nm = 0$$

[on differentiating equation (3)]

or $$Ll\frac{dl}{dm} + M\left(l + m\frac{dl}{dm}\right) + Nm = 0 \qquad .....(7.7.5)$$

and by differentiating (7.7.4), we have

$$2El\frac{dl}{dm} + 2F\left(l + m\frac{dl}{dm}\right) + 2Gm = 0$$

or $$El\frac{dl}{dm} + F\left(l + m\frac{dl}{dm}\right) + Gm = 0 \qquad .....(7.7.6)$$

By equation (7.7.5), rearranging the terms

$$\frac{dl}{dm}\left(Ll + Mm\right) + \left(Ml + Nm\right) = 0 \qquad .....(7.7.7)$$

and by equation (7.7.6)

$$\frac{dl}{dm}\left(El + Fm\right) + \left(Fl + Gm\right) = 0 \qquad .....(7.7.8)$$

Eliminating $\frac{dl}{dm}$, 1 between equation (7.7.7) and (7.7.8), we get

$$\begin{vmatrix} Ll + Mm & Ml + Nm \\ El + Fm & Fl + Gm \end{vmatrix} = 0$$

or $$\left(Ll + Mm\right)\left(Fl + Gm\right) - \left(El + Fm\right)\left(Ml + Nm\right) = 0 \qquad .....(7.7.9)$$

Simplifying, we get

$$\left(LF - EM\right)l^2 + \left(GL - EN\right)lm + \left(MG - FN\right)m^2 = 0$$

or $$\left(EM - LF\right)l^2 + \left(EN - GL\right)lm + \left(FN - MG\right)m^2 = 0 \qquad .....(7.7.10)$$

This equation determines the principal directions of the principal section.

To obtain the differential equation of the principal section replace l, m by direction ratios du, dv in equation (7.7.10), we get

$$(EM - LF)\, du^2 + (EN - GL)\, du\, dv + (FN - GM)\, du^2 = 0,$$

which can be expressed in the form of determinant as follows :

$$\begin{vmatrix} dv^2 & -du\,dv & du^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0. \qquad .....(7.7.11)$$

This equation gives differential of the principal section.

7.8 There are two principal directions at every point on a surface which are mutually orthogonal

We know that the equation determining the principal directions at a point $A\ (u,\ v)$ of the surface $\vec{r} = \vec{r}\left(u, v\right)$ is given by

$$\left(EM - LF\right)l^2 + \left(EN - GL\right)lm + \left(FN - MG\right)m^2 = 0 \qquad \text{[equation (7.7.10)]}$$

which can be expressed as

$$\left(EM - LF\right)\left(\frac{l}{m}\right)^2 + \left(EN - GL\right)\left(\frac{l}{m}\right) + \left(FN - MG\right) = 0 \qquad .....(7.8.1)$$

(on dividing above equation by m^2)

This being a quadratic equation in $\frac{l}{m}$, which provides two directions, *i.e.* there are two roots (say) $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$ of equation (7.8.1)

Then $\qquad$ sum of roots $= \frac{l_1}{m_1} + \frac{l_2}{m_2} = \frac{-\left(EN - GL\right)}{\left(EM - LF\right)} = \left(\frac{GL - EN}{EM - LF}\right) \qquad ......(7.8.2)$

and $\qquad$ product of roots $= \frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \left(\frac{FN - GM}{EM - LF}\right) \qquad .....(7.8.3)$

or $\qquad \frac{l_1 l_2}{FN - GM} = \frac{m_1 m_2}{EM - LF}. \qquad .....(7.8.4)$

Now by equation (7.8.2)

$$\frac{l_1 m_2 + l_2 m_1}{m_1 m_2} = \frac{GL - EN}{\left(EM - LF\right)} \Rightarrow \frac{l_1 m_2 + l_2 m_1}{GL - EN} = \frac{m_1 m_2}{EM - LF} \qquad .....(7.8.5)$$

From equation (7.8.4) and (7.8.5), we get

$$\frac{l_1 m_2 + l_2 m_1}{GL - EN} = \frac{l_1 l_2}{FN - GM} = \frac{m_1 m_2}{EM - LF} = c_1 \text{ (say)}$$

$\Rightarrow \qquad \left(l_1 m_2 + l_2 m_1\right) = c_1\left(GL - EN\right), \quad l_1 l_2 = c_1\left(FN - GM\right),$

$$m_1 m_2 = c_1\left(EM - LF\right) \qquad .....(7.8.6)$$

Now if θ is the angle between these directions, than

$$\tan\theta = \frac{H(l_1m_2 - m_1l_2)}{El_1l_2 + F(l_1m_2 + m_1l_2) + Gm_1m_2}$$

which can be expressed as

$$\tan\theta = \frac{H\sqrt{(l_1m_2 + m_1l_2)^2 - 4l_1l_2m_1m_2}}{El_1l_2 + F(l_1m_2 + m_1l_2) + Gm_1m_2}$$

or $$\tan\theta = \frac{H\sqrt{c_1^2(GL - EN)^2 - 4c_1(FN - GM)\cdot c_1(EM - LF)}}{Ec_1(FN - GM) + Fc_1(GL - EN) + Gc_1(EM - LF)} \quad \text{[by equation (7.8.6)]}$$

or $$\tan\theta = \frac{H\sqrt{(GL - EN)^2 - 4(EM - FL)(FN - GM)}}{E(FN - GM) + F(GL - EN) + G(EM - LF)}$$

$$= \frac{H\sqrt{(GL - EN)^2 - 4(EM - FL)(FN - GM)}}{EFN - EGM + FGL - FEN + GEM - GLF}$$

or $$\tan\theta = \frac{H\sqrt{(GL - EN)^2 - 4(EM - FL)(FN - GM)}}{0} = \infty$$

$\therefore\ \theta = \frac{\pi}{2}$. Hence the two principal direction are mutually orthogonal.

7.9 Umbilics

To derive the condition that a point be umbilic on the surface $\vec{r} = \vec{r}(u,v)$.

The equation determining the principal directions at a point of surface $\vec{r} = \vec{r}(u,v)$ is given by

$$(EM - LF)l^2 + (EN - GL)lm + (FN - MG)m^2 = 0 \quad .....(7.9.1)$$

in this, if

$$\left.\begin{aligned} EM - LF = 0 &\Rightarrow \frac{E}{L} = \frac{F}{M} \\ EN - GL = 0 &\Rightarrow \frac{E}{L} = \frac{G}{N} \\ \text{and}\quad FN - MG = 0 &\Rightarrow \frac{F}{M} = \frac{G}{N} \end{aligned}\right\} \Rightarrow \frac{E}{L} = \frac{F}{M} = \frac{G}{N} \quad .....(7.9.2)$$

then sum of roots $\frac{l_1}{m_1} + \frac{l_2}{m_2}$ and product of roots $\frac{l_1}{m_1}\cdot\frac{l_2}{m_2}$ of the equation

$$(EM - LM)\left(\frac{l}{m}\right)^2 + (EN - GL)\left(\frac{l}{m}\right) + (FN - MG) = 0 \quad .....(7.9.3)$$

becomes $\frac{0}{0}$, *i.e.* values of $\frac{l}{m}$ becomes inderminate, which means that in this situation the normal curvature becomes independent of directions (*du, dv*) and so has the same value for all directions through the given point *A* (*u,v*) of the surface. Such a point is called an *umbilic* or a *navel point* on the surface $\vec{r}=\vec{r}(u,v)$.

Definition : A point *A* (*u,v*) on the surface $\vec{r}=\vec{r}(u,v)$ is called an umbilic, if at the point

$$\frac{E}{L}=\frac{F}{M}=\frac{G}{N}=\frac{\sqrt{EG-F^2}}{\sqrt{LN-M^2}}=\frac{H}{T} \qquad .....(7.9.4)$$

An umbilic can also be taken as a circular section of zero radius.

Since at each point of a sphere, the normal curvature is same, so every point of a sphere is an umbilical point.

7.10 The equation giving the principal curvatures at a point *A* (*u, v*) of the surface $\vec{r}=\vec{r}(u,v)$

By the equation (7.7.9) which determines the principal directions, we have

$$(Ll+Mm)(Fl+Gm)-(El+Fm)(Ml+Nm)=0 \qquad .....(7.10.1)$$

$$\Rightarrow \qquad \frac{Ll+Mm}{El+Fm}=\frac{Ml+Nm}{Fl+Gm}$$

$$\Rightarrow \qquad \frac{Ll+Mm}{El+Fm}=\frac{Ml+Nm}{Fl+Gm}=\frac{l(Ll+Mm)+m(Ml+Nm)}{l(El+Fm)+m(Fl+Gm)}$$

$$=\frac{Ll^2+2Mml+Nm^2}{El^2+2Flm+Gm^2}=\kappa_n$$

$$\Rightarrow \qquad \frac{Ll+Mm}{El+Fm}=\frac{Ml+Nm}{Fl+Gm}=\kappa_n \qquad .....(7.10.2)$$

Hence $\qquad \frac{Ll+Mm}{El+Fm}=\kappa_n \Rightarrow Ll+Mm=\kappa_n(El+Fm) \qquad .....(7.10.3)$

and $\qquad \frac{Ml+Nm}{Fl+Gm}=\kappa_n \Rightarrow (Ml+Nm)=\kappa_n(Fl+Gm) \qquad .....(7.10.4)$

Equation (7.10.3) and (7.10.4) can be rewritten as

$$(L-E\kappa_n)l+(M-F\kappa_n)m=0 \qquad .....(7.10.5)$$

and $\qquad (M-F\kappa_n)l+(N-G\kappa_n)m=0 \qquad .....(7.10.6)$

On eliminating *l, m*, we have

$$\begin{vmatrix} L-E\kappa_n & M-F\kappa_n \\ M-F\kappa_n & N-G\kappa_n \end{vmatrix}=0$$

$$\Rightarrow \qquad (L - E\kappa_n)(N - G\kappa_n) - (M - F\kappa_n)(M - F\kappa_n) = 0$$

or $$(EG - F^2)\kappa_n^2 - (EN + LG - 2FM)\kappa_n + (LN - M^2) = 0$$

or $$H^2\kappa_n^2 - (EN + LG - 2FM)\kappa_n + T^2 = 0, \qquad .....(7.10.7)$$

where $H^2 = EG - F^2$ and $T^2 = LN - M^2$.

This equation being quadratic in κ_n gives two roots say $\kappa_n = \kappa_a$ and $\kappa_n = \kappa_b$, which are called two principal curvatures. Thus from equation (7.10.7), we have

$$\text{sum of roots} = (\kappa_a + \kappa_b) = \frac{EN + LG - 2FM}{H^2\left(= EG - F^2\right)} \qquad .....(7.10.8)$$

and $$\text{product of roots} = \kappa_a \cdot \kappa_b = \frac{T^2}{H^2} = \frac{LN - M^2}{EG - F^2}. \qquad .....(7.10.9)$$

7.11 Same important definitions

***(i)* Mean curvature or mean normal curvatures**

The arithmetic mean of the principal curvatures at a point is called the mean curvature. It is denoted by symbol μ

i.e. $$\mu = \frac{1}{2}(\kappa_a + \kappa_b) = \frac{EN - LG - 2FM}{2\left(EG - F^2\right)} \qquad .....(7.11.1)$$

***(ii)* Amplitude of normal curvatures**

Amplitude of normal curvature is denoted by A is defined as

$$A = \frac{1}{2}(\kappa_a - \kappa_b) \qquad .....(7.11.2)$$

***(iii)* First curvature**

The sum of principal curvatures at a point is called the first curvature at the point, denoted by J and given by

$$J = (\kappa_a + \kappa_b) = \frac{EN - LG - 2FM}{\left(EG - F^2\right)} \qquad .....(7.11.3)$$

then clearly $\mu = \frac{1}{2}J$.

***(iv)* Gaussian curvature (or second curvatures)**

The product of the principal curvatures at a point is called the Gaussian curvature at the point, denoted by symbol κ

and given by $$\kappa = \kappa_a \cdot \kappa_b = \frac{LN - M^2}{EG - F^2}. \qquad .....(7.11.4)$$

It is also called, specific curvature or total curvature.

***(v)* Minimal surface**

The surface for which the first curvature is zero (or the mean curvature is zero) at all points, is called a minimal surface.

Hence the surface will be minimal, if and only if

$$EN + GL - 2FM = 0 \;\Rightarrow \mu = 0$$
$$\Rightarrow (\kappa_a + \kappa_b) = 0$$
$$\Rightarrow \rho_1 + \rho_2 = 0$$
$$\Rightarrow \rho_1 = -\rho_2 \qquad .....(7.11.5)$$

where ρ_1 and ρ_2 are radius of normal curvatures.

***(vi)* Developable surface**

The surfaces for which Gaussian curvature is zero, are called developable surfaces.

Thus for developable surfaces

$$\kappa = 0 \Rightarrow LN - M^2 = 0 \qquad .....(7.11.6)$$

Remarks :

(a) The necessary and sufficient condition for a surface to be developable is that its Gaussian curvature should be zero.

(b) If there is a surface of minimum area passing through a closed space curve, it is necessarily a minimal surface, *i.e.*, a surface of zero mean curvature.

7.12 Radius of curvature at an umbilic on the surface $z = f(x, y)$

The Principal radii of curvature are given by the equation (7.10.7), when κ_n is replaced by $1/\rho$ (principal radius of curvature) *i.e.*,

$$\frac{H^2}{\rho^2} - (EN + LG - 2FM)\frac{1}{\rho} + T^2 = 0 \qquad .....(7.12.1)$$

For the surface $z = f(x, y)$, we know that

$$\left.\begin{array}{llll} E = 1 + p^2, & F = pq, & G = 1 + q^2, & H = \sqrt{1 + p^2 + q^2} \\ L = \dfrac{r}{H}, & M = \dfrac{s}{H}, & N = \dfrac{t}{H}, & T^2 = \dfrac{rt - s^2}{H^2}, \end{array}\right\} \qquad .....(7.12.2)$$

where $p = \dfrac{\partial z}{\partial x}, \; q = \dfrac{\partial z}{\partial y}, \; r = \dfrac{\partial^2 z}{\partial x^2}, \; t = \dfrac{\partial^2 z}{\partial y^2}, \; s = \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}.$

Then substituting values from equation (7.12.2) in (7.12.1) we get

$$\frac{H^2}{\rho^2} - \frac{1}{H}\left\{t(1 + p^2) + r(1 + q^2) - 2pqs\right\} \cdot \frac{1}{\rho} + \frac{rt - s^2}{H^2} = 0 \cdot \qquad .....(7.12.3)$$

But when equation of surface is given in the Monge's form $z=f(x,y)$, the condition for a point to be umbilic is

$$\frac{1+p^2}{r}=\frac{pq}{s}=\frac{1+q^2}{t}=\frac{\sqrt{1+p^2+q^2}}{\sqrt{rt-s^2}}=\frac{H}{\sqrt{rt-s^2}}. \qquad .....(7.12.4)$$

From this,

$$\frac{1+p^2}{r}=\frac{H}{\sqrt{rt-s^2}} \Rightarrow \frac{1+p^2}{H}=\frac{r}{\sqrt{rt-s^2}}, \qquad .....(7.12.5)$$

$$\frac{1+q^2}{t}=\frac{H}{\sqrt{rt-s^2}} \Rightarrow \frac{r\left(1+q^2\right)}{H}=\frac{rt}{\sqrt{rt-s^2}}, \qquad .....(7.12.6)$$

and $$\frac{pq}{s}=\frac{H}{\sqrt{rt-s^2}} \Rightarrow \frac{pq}{H}=\frac{s}{\sqrt{rt-s^2}}. \qquad .....(7.12.7)$$

Using values from equation (7.12.5) to (7.12.7) in (7.12.3), we get

$$\frac{H^2}{\rho^2}-\frac{1}{\rho}\left\{t\cdot\frac{r}{\sqrt{rt-s^2}}+r\cdot\frac{t}{\sqrt{rt-s^2}}-\frac{2s\cdot s}{\sqrt{rt-s^2}}\right\}+\frac{rt-s^2}{H^2}$$

or $$\frac{H^2}{\rho^2}-\frac{1}{\rho}\left\{\frac{2\left(rt-s^2\right)}{\sqrt{rt-s^2}}\right\}+\frac{rt-s^2}{H^2}=0,$$

or $$\left(\frac{H}{\rho}-\frac{\sqrt{rt-s^2}}{H}\right)^2=0$$

or $$\frac{H}{\rho}=\frac{\sqrt{rt-s^2}}{H} \Rightarrow \rho=\frac{H^2}{\sqrt{rt-s^2}}=\frac{1+p^2+q^2}{\sqrt{rt-s^2}}. \qquad .....(7.12.8)$$

Thus from (7.12.4) and (7.12.8), it follows that for an umbilic

$$\frac{1+p^2}{r}=\frac{pq}{s}=\frac{1+q^2}{t}=\frac{\rho}{H}. \qquad .....(7.12.9)$$

7.12.1. Self-learning exercise-2

1. Define the following :
   (i) Principal sections of the surface.
   (ii) Principal direction.
   (iii) Principal curvature.
   (iv) Principal radius of curvature.
   (v) Surface of centres.
2. Write the differential equation of the principal section of the surface $\vec{r}=\vec{r}(u,\, v)$.

3. Write the equation giving the principal directions at a point of a surface $\vec{r} = \vec{r}(u, v)$.
4. Are the two principal directions at every point on a surface $\vec{r} = \vec{r}(u, v)$, mutually orthogonal ?
5. Write the Condition that a point be umbilic on the surface $\vec{r} = \vec{r}(u, v)$.
6. Define the following :
   (i) Mean curvature at a point of a surface.
   (ii) First curvature.
   (iii) Amplitude of normal curvature.
   (iv) Gaussian curvature.
   (v) Minimal surface.
   (vi) Developable surface.
7. Write the formula for radius of curvature at are umbilic on the surface $z = f(x, y)$.

7.12.2. Illustrative Examples

***Ex.*1.** *Find the principal sections and principal curvatures of the surface*

$$x = a(u + v),\ y = b(u - v),\ z = uv.$$

Sol. The position vector $\vec{r}$ of any point on the surface is given by

$$\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$$

or $$\vec{r} = a\ (u + v)\ \hat{i} + b(u - v)\ \hat{j} + uv\hat{\kappa} \qquad \text{[vector equation of surface]}$$

or $$\vec{r} = [a(u + v), b(u - v), uv] \qquad \text{......(1)}$$

Here a and b are constants and u, v are parameters.

Differentiating (1) partially with respect to u, we get

$$\frac{\partial \vec{r}}{\partial u} = \vec{r}_1 = (a, b, v). \qquad \text{.....(2)}$$

Again differentiating with respect to u, we get

$$\frac{\partial^2 \vec{r}}{\partial u^2} = \vec{r}_{11} = (0, 0, 0) \qquad \text{.....(3)}$$

Now differentiating equation (2) partially with respect to v, we get

$$\frac{\partial^2 \vec{r}}{\partial v \partial u} = \vec{r}_{12} = (0, 0, 1) \qquad \text{.....(4)}$$

Now differentiating equation (1) partially with respect to v, we get

$$\frac{\partial \vec{r}}{\partial v} = \vec{r}_2 = (a, -b, u) \qquad \text{.....(5)}$$

Differentiating this with respect to v again, we get

$$\frac{\partial^2 \vec{r}}{\partial v^2} = \vec{r}_{22} = (0,0,0) \qquad .....(6)$$

Now $\qquad \vec{r}_1 \times \vec{r}_2 = (a,b,v) \times (a,-b,u)$

$$= \left(a\hat{i} + b\hat{j} + v\hat{k}\right) \times \left(a\hat{i} - b\hat{j} + u\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & v \\ a & -b & u \end{vmatrix}$$

$$= \hat{i}\,(bu + bv) + \hat{j}\,(av - au) + \hat{k}\,(-ab - ab)$$

or $\qquad \vec{r}_1 \times \vec{r}_2 = \left[b(u+v), a(v-u), -2ab\right] \qquad .....(7)$

Now $\quad E = \vec{r}_1 \cdot \vec{r}_1 = \vec{r}_1^{\,2} = a^2 + b^2 + v^2 \qquad .....(8)$

$\quad F = \vec{r}_1 \cdot \vec{r}_2 = (a,b,v),(a,-b,u) = a^2 - b^2 + uv \qquad .....(9)$

and $\quad G = \vec{r}_2 \cdot \vec{r}_2 = \vec{r}_2^{\,2} = \left(a^2 + b^2 + u^2\right) \qquad .....(10)$

Now $\quad H^2 = \left(EG - F^2\right) = \left(a^2 + b^2 + v^2\right)\left(a^2 + b^2 + u^2\right) - \left(a^2 - b^2 + uv\right)^2$

or $\quad H^2 = a^4 + a^2b^2 + a^2u^2 + a^2b^2 + b^4 + b^2u^2 + a^2v^2 + b^2v^2 + u^2v^2$

$$-\left\{a^4 + b^4 + u^2v^2 - 2a^2b^2 + 2a^2uv - 2b^2uv\right\}$$

$$= a^2u^2 + b^2u^2 + a^2v^2 + b^2v^2 - 2a^2uv + 2b^2uv$$

$$= a^2\left(u^2 + v^2 - 2uv\right) + b^2\left(u^2 + v^2 + 2uv\right) + 4a^2b^2 .$$

or $\quad H^2 = a^2(u-v)^2 + b^2(u+v)^2 + 4a^2b^2 .$

or $\quad H = \left\{a^2(u-v)^2 + b^2(u+v)^2 + 4a^2b^2\right\}^{1/2} \qquad .....(11)$

Now, $\quad \hat{N} = \frac{\vec{r}_1 \times \vec{r}_2}{H} = \frac{1}{H}\left\{\left(a\hat{i} + b\hat{j} + v\hat{k}\right) \times \left(a\hat{i} - b\hat{j} + u\hat{k}\right)\right\}$

$$= \frac{1}{H}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & v \\ a & -b & u \end{vmatrix}$$

$$= \frac{1}{H}\left\{\hat{i}\,(bu + bv) + \hat{j}\,(av - au) + \hat{k}\,(-ab - ab)\right\}$$

or $\qquad \hat{N} = \frac{1}{H}\left\{b(u+v), a(v-u), -2ab\right\} \qquad .....(12)$

Now $$L = \hat{N} \cdot \vec{r}_{11} = \hat{N} \cdot (0,0,0) = 0 \qquad .....(13)$$

$$M = \hat{N} \cdot \vec{r}_{12} = \frac{1}{H}\{(b(u+v), a(v-u), -2ab) \cdot (0,0,1)\} = -\frac{2ab}{H} \qquad .....(14)$$

and $$N = \hat{N} \cdot \vec{r}_{22} = \hat{N} \cdot (0,0,0) = 0 \qquad .....(15)$$

$$T^2 = LN - M^2 = 0 - \left(\frac{-2ab}{H}\right)^2 = -\frac{4a^2b^2}{H^2} \qquad .....(16)$$

***(i)* Principal sections**

The differential equation determining the principal sections is

$$(EM - LF)du^2 + (EN - LG)dudv + (FN - GM)dv^2 = 0 \qquad .....(17)$$

Using values of *E, F, G, L, M, N* from above equations, we get

$$\left[(a^2+b^2+v^2)\left(\frac{-2ab}{H}\right) - 0\right]du^2 + [0-0]dudv + \left[0 - (a^2+b^2+u^2)\left(\frac{-2ab}{H}\right)\right]dv^2 = 0$$

Simplifying

$$\frac{du}{\sqrt{a^2+b^2+u^2}} = \pm\frac{dv}{\sqrt{a^2+b^2+v^2}}$$

On integrating, we get

or $$\int \frac{du}{\sqrt{\left(\sqrt{a^2+b^2}\right)^2 + (u)^2}} = \pm\int \frac{dv}{\sqrt{\left(\sqrt{a^2+b^2}\right)^2 + (v)^2}} + c_1$$

or $$\sinh^{-1}\frac{u}{\sqrt{a^2+b^2}} = \pm\sinh^{-1}\frac{v}{\sqrt{a^2+b^2}} + c_1 \qquad .....(18)$$

where c_1 is constant of integration.

Equation (18) is the equation of principal section.

***(ii)* Principal curvatures**

The differential equation determining the principal curvatures is

$$H^2\kappa^2 - (EN + LG - 2FM)\kappa + T^2 = 0 \qquad .....(19)$$

Using values of *E, F, G* and *L, N, M* and T^2, we get

$$H^2\kappa^2 - \left[0 + 0 - 2(a^2 - b^2 + uv)\left(\frac{-2ab}{H}\right)\right]\kappa + \left(\frac{-4a^2b^2}{H^2}\right) = 0$$

or $$H^4\kappa^2 - 4abH(a^2 - b^2 + uv)\kappa - 4a^2b^2 = 0 \qquad .....(20)$$

where $$H^2 = EG - F^2 = (a^2+b^2+v^2)(a^2+b^2+u^2) - (a^2-b^2+uv)^2$$

or $$H^2 = a^2(u-v)^2 + b^2(u+v)^2 + 4a^2b^2 \qquad .....(21)$$

On using equation (21) in (20), we get a quadratic equation in κ. We can find principal curvatures.

***Ex.*2.** *For the hyperboloid* $2z = 7x^2 + 6xy - y^2$, *prove that the principal radii at the origin are* $\frac{1}{8}$ *and* $-\frac{1}{2}$, *and that the principal sections are* $x = 3y$, $3x = -y$.

Sol. The given surface is

$$z = \frac{1}{2}\left(7x^2 + 6xy - y^2\right), \qquad .....(1)$$

which is the Monge's form of equation of surface.

Here first we shall calculate p, q, r, s, t at the origin and then we shall calculate fundamental magnitudes of this surface. Here x, y will be treated as parameters.

Now differentiating (1) partially with respect to x and y, we get

$$p = \frac{\partial z}{\partial x} = (7x + 3y), \quad q = \frac{\partial z}{\partial y} = (3x - y)$$

and $$r = \frac{\partial^2 z}{\partial x^2} = 7, \quad s = \frac{\partial^2 z}{\partial x \partial y} = 3, \quad t = \frac{\partial^2 z}{\partial y^2} = -1$$

We shall find these values at the origin (0, 0, 0), so

$$p = \left(\frac{\partial z}{\partial x}\right)_{(0,0,0)} = 0, \quad q = \left(\frac{\partial z}{\partial y}\right)_{(0,0,0)} = 0, \qquad .....(2)$$

$$r = \left(\frac{\partial^2 z}{\partial x^2}\right)_{(0,0,0)} = 7, \quad s = \left(\frac{\partial^2 z}{\partial x \partial y}\right)_{(0,0,0)} = 3, \quad t = \left(\frac{\partial^2 z}{\partial y^2}\right)_{(0,0,0)} = -1 \qquad .....(3)$$

Here consider $$\vec{r} = (x, y, f(x, y) = z)$$

$$\therefore \quad \vec{r}_1 = \frac{\partial z}{\partial x} = (1, 0, p), \quad \vec{r}_2 = \frac{\partial z}{\partial y} = (1, 0, q), \quad \vec{r}_{11} = \frac{\partial^2 z}{\partial x^2} = (0,\ 0,\ r)$$

$$\vec{r}_{12} = \frac{\partial^2 z}{\partial x \partial y} = (0,\ 0,\ s), \quad \vec{r}_{22} = \frac{\partial^2 z}{\partial y^2} = (0,\ 0,\ t).$$

$$\therefore \quad \left.\begin{array}{l} E = \vec{r}_1 \cdot \vec{r}_1 = 1 + p^2, \ F = \vec{r}_1 \cdot \vec{r}_2 = pq, \ G = \vec{r}_2 \cdot \vec{r}_2 = 1 + q^2 \\ L = \dfrac{r}{H}, \ M = \dfrac{s}{H}, \ N = \dfrac{t}{H} \quad \text{and} \quad H = \sqrt{1 + p^2 + q^2} \end{array}\right\} \qquad .....(4)$$

Using values of p, q, r, s, t from equations (2) and (3) in equation (4), we get

$$E = 1, F = 0, G = 1, H = 1, L = 1, M = 3, N = -1 \qquad .....(5)$$

Now equation giving the principal curvatures is

$$\left(EG - F^2\right)\kappa_n^2 - \left(EN - 2FM + LG\right)\kappa_n + \left(LN - M^2\right) = 0 \qquad .....(6)$$

Putting the values of E, F, G, L, M, N and H from equation (5), we get

$$\kappa_n^2 - 6\kappa_n - 16 = 0 \Rightarrow (\kappa_n - 8)(\kappa_n + 2) = 0$$

$$\therefore \qquad \kappa_n = 8, -2 \qquad .....(7)$$

Hence principal radii are $\rho_1 = \dfrac{1}{\kappa_n} = \dfrac{1}{8}$ and $\rho_2 = \dfrac{1}{\kappa_n} = \dfrac{1}{-2} = -\dfrac{1}{2}$

Again the equation of principal section is

$$(EM - FL)dx^2 + (EN - GL)dxdy + (FN - GM)dy^2 = 0 \qquad .....(8)$$

or $\qquad 3dx^2 - 8dxdy - 3dy^2 = 0 \qquad$ [on using values of E, F, G, L, M and N]

or $\qquad (3dx + dy)(dx - 3dy) = 0$

$\therefore \qquad 3dx + dy = 0$ or $dx - 3\,dy = 0$

On integrating $\qquad 3x + y = c_1$, $x - 3y = c_2 \qquad .....(9)$

But at the origin (0,0,0), using $x = 0$, $y = 0$, we get $c_1 = 0$, $c_2 = 0$ [form (9)]

$\therefore$ Principal sections of the origin are

$$3x + y = 0, \quad x - 3y = 0 \qquad \text{[From (9)]}$$

or $\qquad 3x = -y, \quad x = 3y \qquad .....(10)$

Ex.3. *Show that the points of intersection of the surface $x^m + y^m + z^m = a^m$ and the line $x = y = z$ are umbilics and that the radius of curvature at an umbilic is given by*

$$\rho = \frac{a}{m-1} \cdot 3^{(m-2)/2m}.$$

Sol. The equation of given surface is

$$x^m + y^m + z^m = a^m, \qquad .....(1)$$

where a is a constant.

This surface may be regarded as Monge's from [$z = f(x, y)$] by taking z as a function of x and y.

Differentiating equation (1) partially with respect to x and y respectively, we get

$$mx^{m-1} + 0 + mz^{m-1}\frac{\partial z}{\partial x} = 0 \Rightarrow -\frac{\partial z}{\partial x} = -p = \left(\frac{x}{z}\right)^{m-1} \qquad .....(2)$$

$$0 + my^{m-1} + mz^{m-1}\frac{\partial z}{\partial y} = 0 \Rightarrow -\frac{\partial z}{\partial y} = -q = \left(\frac{y}{z}\right)^{m-1} \qquad .....(3)$$

Now, from equation (2), on taking log

$$\log(-p) = \log\left(\frac{x}{z}\right)^{m-1} \Rightarrow \log(-p) = (m-1)[\log x - \log z] \qquad .....(4)$$

Differentiating this partially with respect to x, we get

$$\frac{1}{(-p)}\frac{\partial(-p)}{\partial x} = (m-1)\left[\frac{1}{x} - \frac{1}{z}\frac{\partial z}{\partial x}\right]$$

or $$\frac{1}{p}\frac{\partial p}{\partial x}=(m-1)\left(\frac{1}{x}-\frac{1}{z}\cdot p\right) \qquad \left(\because \frac{\partial z}{\partial x}=p\right)$$

or $$\frac{r}{p}=(m-1)\left(\frac{1}{x}-\frac{p}{z}\right) \qquad \left[\because r=\frac{\partial p}{\partial x}=\frac{\partial^2 z}{\partial x^2}\right]$$

Now differentiating equation (4) partially with respect to y

$$\frac{1}{(-p)}\frac{\partial(-p)}{\partial y}=(m-1)\left[0-\frac{1}{z}\frac{\partial z}{\partial y}\right]$$

or $$\frac{1}{p}\frac{\partial p}{\partial y}=\frac{-(m-1)\dfrac{\partial z}{\partial y}}{z}$$

or $$\frac{s}{p}z=-\frac{(m-1)q}{z} \qquad \left[\because s=\frac{\partial p}{\partial y}=\frac{\partial^2 z}{\partial y\partial x}\right] \qquad ......(6)$$

From equation (3), on taking log

$$\log(-q)=\log\left(\frac{y}{z}\right)^{m-1} \Rightarrow \log(-q)=(m-1)\left[\log(y)-\log(z)\right]$$

Differentiating this partially with respect to y, we get

$$\frac{1}{(-q)}\frac{\partial(-q)}{\partial y}=(m-1)\left[\frac{1}{y}-\frac{1}{z}\frac{\partial z}{\partial y}\right]$$

or $$\frac{1}{q}\frac{\partial q}{\partial y}=(m-1)\left(\frac{1}{y}-\frac{1}{z}\frac{\partial z}{\partial y}\right)$$

or $$\frac{t}{q}=(m-1)\left(\frac{1}{y}-\frac{1}{z}q\right) \qquad ......(7)$$

Now for an umbilic $$\frac{1+p^2}{r}=\frac{pq}{s} \qquad ......(8)$$

But from (6), $$\frac{pq}{s}=-\frac{z}{m-1} \qquad ......(9)$$

$\therefore$ From equation (8), $$\frac{1+p^2}{r}=\frac{pq}{s}=\frac{-z}{m-1}$$

or $$\frac{1+p^2}{r}=\frac{-z}{m-1}$$

or $$\left(1+p^2\right)=\frac{-z}{(m-1)}\cdot r$$

$$=\frac{-z}{(m-1)}\cdot(m-1)p\left(\frac{1}{x}-\frac{p}{z}\right) \qquad \text{[From equation (5)]}$$

or $$\left(1+p^2\right)=\frac{-pz}{x}+p^2$$

$\Rightarrow$ $$1=-\frac{pz}{x}$$

or $$p=-(x/z) \qquad .....(10)$$

Using this value of p in equation (2), we get

$$\frac{x}{z}=\left(\frac{x}{z}\right)^{m-1} \Rightarrow 1=\left(\frac{x}{z}\right)^{m-2} \Rightarrow z^{m-2}=x^{m-2} \qquad .....(11)$$

Similarly we can find, $y^{m-2}=z^{m-2}$ .....(12)

Therefore from equation (11) and (12), we get

$$x^{m-2}=y^{m-2}=z^{m-2} \Rightarrow x=y=z$$

Therefore for an umbilic, $x=y=z$ .....(13)

Then from equation (1),

$$z^m+z^m+z^m=a^m \Rightarrow 3z^m=a^m \qquad (\because x=z,\ y=z)$$

$$\Rightarrow z=\left(\frac{a}{3^{1/m}}\right) \qquad .....(14)$$

Now from equation (2),

$$-p=\left(\frac{z}{z}\right)^{m-1} \Rightarrow p=-1 \qquad (\because x=z)$$

Similarly from equation (3),

$$-q=\left(\frac{z}{z}\right)^{m-1} \Rightarrow q=-1 \qquad (\because y=z)$$

From equation (6),

$$\frac{s}{p}=\frac{-(m-1)q}{z} \Rightarrow s=\frac{-(m-1)}{z} \qquad [\because p=-1=q]$$

or $$s=\frac{-(m-1)}{\left(\frac{a}{3^{1/m}}\right)}=\frac{-(m-1)3^{1/m}}{a} \qquad \text{[From equation (14)]}$$

and $$H=\sqrt{1+p^2+q^2}=\sqrt{1+(-1)^2+(-1)^2}=\sqrt{3}$$

Then radius of curvature is given by

$$\rho=\frac{pqH}{s}=\frac{(-1)(-1)\sqrt{3}}{-\frac{(m-1)}{a}\cdot 3^{1/m}}=\frac{-a}{(m-1)}3^{(1/2)-(1/m)}$$

or $$\rho = \frac{-a}{(m-1)} 3^{(m-2)/2m}$$

or $$\rho = \frac{a}{(m-1)} \cdot 3^{(m-2)/2m}. \qquad \text{(numerically)}$$

Ex.4. *Show that the surface* $e^z \cos x = \cos y$ *is minimal surface.*

Sol. The given surface can be expressed as

$$e^z = \frac{\cos y}{\cos x} \text{ or } z = \log(\cos y) - \log(\cos x) \text{ [on taking log]} \qquad .....(1)$$

i.e., the equation of the surface is in the form

$$z = f(x, y) \qquad \text{[Monge's form]}$$

The position vector $\vec{r}$ of any current point (x, y, z) on this surface is given by

$$\vec{r} = (x, y, z) = (x, y, \log\cos y - \log\cos x) \qquad .....(2)$$

On differentiating it partially with respect to x and y respectively

$$\frac{\partial \vec{r}}{\partial x} = \vec{r}_1 = (1, 0, \tan x) \qquad .....(3)$$

and $$\frac{\partial \vec{r}}{\partial y} = \vec{r}_2 = (0, 1, -\tan y) \qquad .....(4)$$

Again differentiating (3) partially with respect to x and y, we get

$$\vec{r}_{11} = (0, 0, \sec^2 x), \ \vec{r}_{12} = (0, 0, 0) \qquad .....(5)$$

Differentiating (4) partially with respect to y

$$\vec{r}_{22} = (0, 0, -\sec^2 y) \qquad .....(6)$$

Now, $$\left.\begin{aligned} E &= \vec{r}_1^{\,2} = 1 + \tan^2 x = \sec^2 x, \\ F &= \vec{r}_1 \cdot \vec{r}_2 = -\tan x \tan y, \\ G &= \vec{r}_2^{\,2} = 1 + \tan^2 y = \sec^2 y \end{aligned}\right\} \qquad .....(7)$$

$$\vec{r}_1 \times \vec{r}_2 = (1, 0, \tan x) \times (0, 1, -\tan y) = (-\tan x, \tan y, 1)$$

Now $$\hat{N} = \frac{\vec{r}_1 \times \vec{r}_2}{H} = \frac{(-\tan x, \tan y, 1)}{H}$$

Hence $$\left.\begin{aligned} L &= \hat{N} \cdot \vec{r}_{11} = \frac{(-\tan x, \tan y, 1)}{H} \cdot (0, 0, \sec^2 x) = \frac{\sec^2 x}{H} \\ M &= \hat{N} \cdot \vec{r}_{12} = \frac{(-\tan x, \tan y, 1)}{H} \cdot (0, 0, 0) = 0 \quad \text{and} \\ N &= \hat{N} \cdot \vec{r}_{22} = \frac{(-\tan x, \tan y, 1)}{H} \cdot (0, 0, -\sec^2 y) = \frac{-\sec^2 y}{H} \end{aligned}\right\} \qquad .....(8)$$

The condition for the surface to be minimal is

$$EN - 2FM + GL = 0 \qquad .....(9)$$

Putting values of *E, F, G* and *L, M, N* form equation (7) and (8) in (9), we get

$$EN - 2FM + GL = \frac{\sec^2 x\left(-\sec^2 y\right)}{H} - 0 + \frac{\sec^2 y \sec^2 x}{H} = 0$$

Hence the given surface is minimal.

Ex.5. *Find the values of (i) First curvature. (ii) Gaussian curvature, at any point of right helicoid $x = u\cos\theta$, $y = u\sin\theta$, $z = c\theta$.*

Hence show that a right helicoid is a minimal surface.

Sol. The position vector *r* of any current point (*x, y, z*) on this surface is given by

$$\vec{r} = \left(x\hat{i} + y\hat{j} + z\hat{k}\right) = (x, y, z) = (u\cos\theta, u\sin\theta, c\theta) \qquad .....(1)$$

Then on differentiating *r* partially with respect to *u* and θ, we get

$$\vec{r}_1 = \frac{\partial\vec{r}}{\partial u} = (\cos\theta, \sin\theta, 0), \quad \vec{r}_2 = \frac{\partial\vec{r}}{\partial\theta} = (-u\sin\theta, u\cos\theta, c) \qquad .....(2)$$

Similarly

$$\vec{r}_{11} = \frac{\partial^2\vec{r}}{\partial u^2} = (0,0,0), \quad \vec{r}_{12} = \frac{\partial^2\vec{r}}{\partial u\partial\theta} = (-\sin\theta, \cos\theta, 0)$$

$$\vec{r}_{22} = \frac{\partial^2\vec{r}}{\partial\theta^2} = (-u\cos\theta, -u\sin\theta, 0). \qquad .....(3)$$

Then the fundamental magnitudes are

$$E = \vec{r}_1\cdot\vec{r}_1 = \vec{r}_1^{\,2} = 1, \quad F = \vec{r}_1\cdot\vec{r}_2 = 0, \; G = \vec{r}_2\cdot\vec{r}_2 = u^2 + c^2$$

and $\quad H^2 = EG - F^2 = 1\cdot\left(u^2 + c^2\right) - 0 = u^2 + c^2 \Rightarrow H = \sqrt{u^2 + c^2}$

$$\hat{N} = \frac{\vec{r}_1\times\vec{r}_2}{H} = \frac{(c\sin\theta, \, -c\cos\theta, \, u)}{H}, \quad L = \hat{N}\times\vec{r}_{11} = 0$$

$$M = \hat{N}\cdot\vec{r}_{12} = \frac{-c}{\sqrt{u^2 + c^2}}, \quad N = \hat{N}\cdot\vec{r}_{22} = 0, \; T^2 = -\frac{a^2}{H^2}\cdot \qquad .....(4)$$

***(i)* First curvature**

The first curvature *J* of the given surface at any point (*u*, θ) is obtained by

$$J = \frac{EN + GL - 2FM}{EG - F^2} = \frac{0 + 0 - 2\times 0}{1\cdot\left(u^2 + c^2\right) - 0} = 0, \qquad .....(5)$$

which shows that first curvature for right helicoid is zero, hence it is a minimal surface.

***(ii)* Gaussian curvature**

The Gaussian curvature K at any point (u, θ) is obtained by

$$K = \frac{T^2}{H^2} = \frac{LN - M^2}{h^2} = \frac{-c^2}{\left(u^2 + c^2\right)^2}. \qquad .....(6)$$

7.13 Radius of curvature of a given section through any point of a surface $z = f(x, y)$.

Suppose the surface is $z = f(x, y)$.

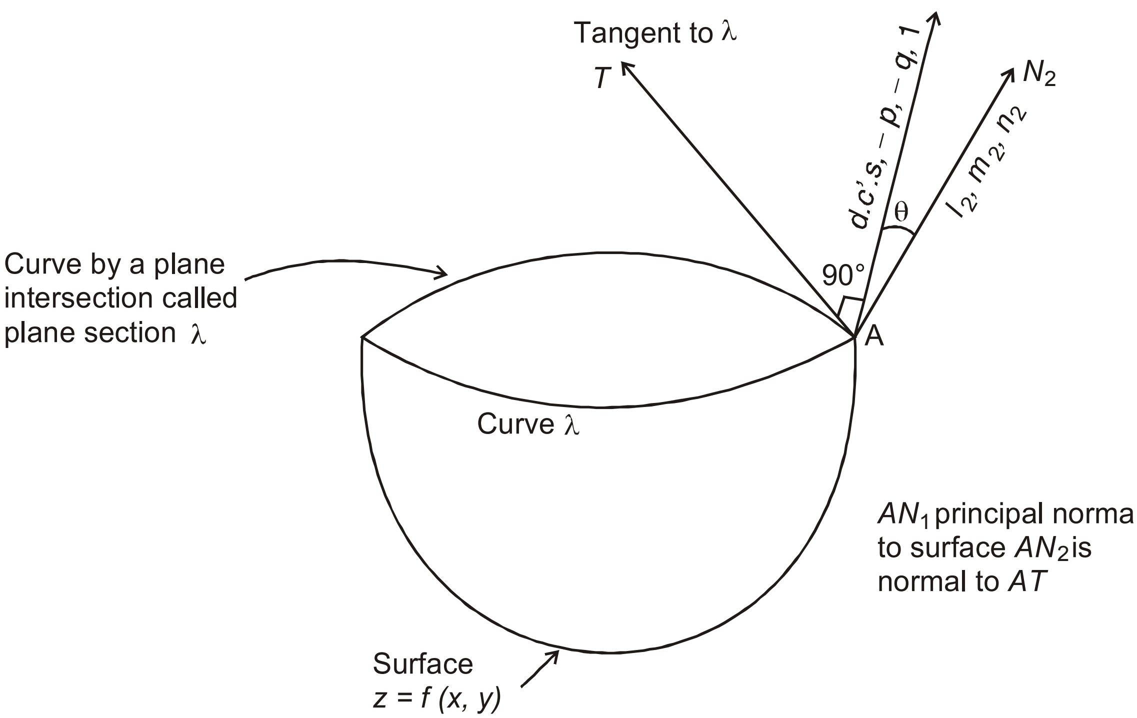


Fig 7.4

A plane cut in it is a curve λ. Suppose A be a point of λ, AT be tangent to λ at A having direction cosines (say) l_1, m_1, n_1. Let AN_2 be one of the normal to AT lying in the plane of section. Let d.c.'s of AN_2 be l_2, m_2, n_2. Also let AN_1 be principal normal to surface at A. If equation of surface is taken in the form $F(x, y, z) = 0$, then direction ratios of the normal to surface are

$$\frac{\partial f}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \quad i.e., \quad -p, -q, 1, \quad \text{where} \quad p = \frac{\partial z}{\partial x}, \; q = \frac{\partial z}{\partial y}.$$

Therefore d.c.'s of principal normal are

$$\frac{-p}{\sqrt{p^2 + q^2 + 1}}, \frac{-q}{\sqrt{p^2 + q^2 + 1}}, \frac{1}{\sqrt{1 + p^2 + q^2}} \qquad .....(7.13.1)$$

Let angle between the plane of the section and normal section through AT be θ, then

$$\cos\theta = \frac{-pl_2 - qm_2 + n_2}{\sqrt{1+p^2+q^2}} \qquad \text{.....(7.13.2)}$$

Now, AT and AN_1 are perpendicular, therefore

$$pl_1 + qm_1 - n_1 = 0 \qquad \text{.....(7.13.3)}$$

Differentiating this equation with respect to s, we get

$$\left(\frac{dp}{ds}l_1 + p\frac{dl_1}{ds}\right) + \left(\frac{dq}{ds}m_1 + q\frac{dm_1}{ds}\right) - \frac{dn_1}{ds} = 0$$

or
$$p\frac{dl_1}{ds} + q\frac{dm_1}{ds} - \frac{dn_1}{ds} + \left(l_1\frac{dp}{ds} + m_1\frac{dq}{ds}\right) = 0 \qquad \text{.....(7.13.4)}$$

Now we shall find values of $\frac{dp}{ds}$ and $\frac{dq}{ds}$

$$\frac{dp}{ds} = \frac{\partial p}{\partial x}\frac{dx}{ds} + \frac{\partial p}{\partial y}\frac{dy}{ds} = (rl_1 + sm_1) \qquad \text{.....(7.13.5)}$$

$$\left[\because r = \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ etc. } l_1 = \frac{dx}{ds},\ m_1 = \frac{dy}{ds}\right]$$

Similarly
$$\frac{dq}{ds} = \frac{\partial q}{\partial x}\frac{dx}{ds} + \frac{\partial q}{\partial y}\frac{dy}{ds} = (sl_1 + tm_1) \qquad \text{.....(7.13.6)}$$

multiplying equation (7.13.5) by l_1 and equation (7.13.6) by m_1 and adding, we get

$$\left(l_1\frac{dp}{ds} + m_1\frac{dq}{ds}\right) = l_1\,(rl_1 + sm_1) + m_1\,(sl_1 + tm_1)$$

$$= \left(rl_1^2 + sl_1m_1 + sl_1m_1 + tm_1^2\right)$$

$$= \left(rl_1^2 + 2sl_1m_1 + tm_1^2\right) \qquad \text{.....(7.13.7)}$$

Now using Serret-Frenet formulae, we find

$$p\frac{dl_1}{ds} + q\frac{dm_1}{ds} - \frac{dn_1}{ds} = p\left(\frac{l_2}{\rho}\right) + q\left(\frac{m_2}{\rho}\right) - \frac{n_2}{\rho} = \frac{1}{\rho}\left(pl_1 + qm_2 - n_2\right) \qquad \text{.....(7.13.8)}$$

Using values from equation (7.13.7) and (7.13.8) in (7.13.4), we get

$$\frac{1}{\rho}\left(pl_2 + qm_2 - n_2\right) = \left(rl_1^2 + 2sl_1m_1 + tm_1^2\right) = 0$$

or
$$\left(pl_2 + qm_2 - n_2\right) = -\rho\left(rl_1^2 + 2sl_1m_1 + tm_1^2\right) \qquad \text{.....(7.13.9)}$$

Using this value in equation (7.13.2), we have

$$\cos = \frac{\rho\,\left(rl_1^2 + 2sl_1m_1 + tm_1^2\right)}{\sqrt{1+p^2+q^2}} \Rightarrow \frac{\cos\theta}{\rho} = \frac{rl_1^2 + 2sl_1m_1 + tm_1^2}{\sqrt{1+p^2+q^2}} \qquad \text{.....(7.13.10)}$$

This relation gives radius curvature ρ for surface $z = f(x, y)$ at a point A of it by a plane section, where l_1, m_1, n_1 are direction cosines of its tangent at the point (x, y, z). If $\theta = 0$, then from (7.3.10)

$$\frac{1}{\rho} = \frac{rl_1^2 + 2sl_1m_1 + tm_1^2}{\sqrt{1+p^2+q^2}} \qquad .....(7.13.11)$$

then this expression gives the principal radius of curvature corresponding to a given principal section.

Ex. *Find the principal radii at the origin of the surface*

$$2z = 5x^2 + 4xy + 2y^2.$$

Find also the radius of curvature of the section $x = y$.

Sol. Equation of surface can be expressed as

$$z = \frac{1}{2}\left(5x^2 + 4xy + 2y^2\right) \qquad .....(1)$$

Deafferentating it partially with respect to x and y respectively.

$$\frac{\partial z}{\partial x} = p = 5x + 2y, \quad \frac{\partial z}{\partial y} = q = 2x + 2y$$

and
$$r = \frac{\partial^2 z}{\partial x^2} = 5, \quad s = \frac{\partial^2 z}{\partial x\, \partial y} = 2, \quad t = \frac{\partial^2 z}{\partial y^2} = 2 \qquad .....(2)$$

$$p = 0, \quad q = 0, \quad r = 5, \quad s = 2, \quad t = 2 \quad \text{and} \quad H^2 = 1 + p^2 + q^2 = 1 \qquad .....(3)$$

Now the principal curvatures for a surface in Monge's form is given by

$$H^4\kappa^2 - H\,[(1+p^2)\,t + (1+q^2)\,r - 2spq]\,\kappa + (rt - s^2) = 0$$

Using values form equation (2), we get

$$\kappa^2 - 7\kappa + 6 = 0 \quad \text{or} \quad (\kappa - 1)(\kappa - 6) = 0 \quad \Rightarrow \quad \kappa = 1, 6 \qquad .....(4)$$

then principal radii of curvature (say), ρ_1 and ρ_2 are

$$\rho_1 = \frac{1}{\kappa} = \frac{1}{1} = 1, \text{ and } \rho_2 = \frac{1}{\kappa} = \frac{1}{6}. \qquad .....(5)$$

Second part : The given point is origin (0, 0, 0) and the plane of the section is $x - y = 0$. The equation of tangent plane of the surface $z = f(x, y)$ at (0, 0, 0) is $z = 0$

Hence the direction cosines of the line

$$x - y = 0, \quad z = 0$$

or
$$\frac{x}{1} = \frac{y}{1} = \frac{z-0}{0} \quad \text{or} \quad \frac{x}{\frac{1}{\sqrt{2}}} = \frac{y}{\frac{1}{\sqrt{2}}} = \frac{z}{0} \qquad .....(6)$$

are
$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = (l_1, m_1, n_1) \qquad .....(7)$$

where l_1, m_1, n_1, are the d.c.'s of tangent through origin for curve of section.

Now, equation of normal plane to the surface at origin (0, 0, 0) through the tangent line is,

$$x - y = 0 \qquad .....(8)$$

which is same as given plane.

So, $\theta = 0$, then radius of curvature is given by

$$\frac{1}{\rho} = \frac{rl_1^2 + 2sl_1m_1 + tm_1^2}{\sqrt{1+p^2+q^2}} = \frac{\frac{5}{2} + 2\cdot 2\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} + 2\cdot\frac{1}{2}}{\sqrt{1+0+0}}$$

or $\quad \dfrac{1}{\rho} = \dfrac{(5/2)+3}{1} = \dfrac{11}{2} \Rightarrow \rho = \text{radius of curvature} = \dfrac{2}{11}. \qquad .....(9)$

7.14 Lines of curvature

Definition : A curve on a surface is called a line of curvature if the tangent at any point of it is along the principal direction at that point.

7.14.1 There are two systems of lines of curvature

At each point of the surface there are two principal directions which are at right angles. Hence, we have two orthogonal systems of lines of curvature on the surface and through each point on the surface there pass two lines of curvature, one corresponding to each system.

7.14.2 To find the differential equation of lines of curvature at point (u, v) of the surface $\vec{r} = \vec{r}(u,v)$.

By definition of line of curvature, the direction of line of curvature at any point is along the principal direction at that point, so the differential equation of the two systems of line of curvature is the same as the differential equation of the principal section and is given by

$$(EM - FL)\, du^2 + (EN - GL)\, du\, dv + (FN - GM)\, dv^2 = 0 \qquad .....(7.14.1)$$

this equation can be expressed in the following form

$$Pdu^2 + 2Qdu\, dv + Rdv^2 = 0 \qquad .....(7.14.2)$$

where $\quad P = EM - LF, \quad 2Q = EN - GL,$ and $R = FN - GM.$

Here $\quad ER - 2FQ + GP = E\,(FN - GM) - F\,(EN - GL) + G\,(EM - LF) = 0, \quad ...(7.14.3)$

which shows that lines of curvature cut orthogonally at a point on the surface $\vec{r} = \vec{r}(u,v)$.

Remark :

(a) Equation (1) can be put in the following determinant form

$$\begin{vmatrix} dv^2 & -du\, dv & du^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0 \text{ (Here } u \text{ and } v \text{ are parameters)} \qquad .....(7.14.3)$$

(b) If we take the parameters x, y in place of (u, v), the above equation (7.14.3) of line of curvature through a point on $z = f(x, y)$ is given by

$$\begin{vmatrix} dy^2 & -dx\,dy & dx^2 \\ 1+p^2 & pq & 1+q^2 \\ r & s & t \end{vmatrix} = 0 \qquad .....(7.14.4)$$

Q $\qquad E = 1 + p^2,\ \ F = pq,\ \ G = 1 + p^2,\ \ H^2 = 1 + p^2 + q^2$

and $\qquad L = \dfrac{r}{H},\ \ M = \dfrac{s}{H},\ \ N = \dfrac{t}{H}.$

7.14.3 Illustrative examples

Ex.1. *Prove that the cone* $\kappa xy = z\,\{(x^2 + z^2)^{1/2} + (y^2 + z^2)^{1/2}\}$ *passes through a line of curvature of the paraboloid* $xy = az$.

Sol. From the given equation of paraboloid, we have

$$z = \frac{1}{a}xy \qquad .....(1)$$

Differentiating partially with respect to x and y respectively

$$\frac{\partial z}{\partial x} = p = \frac{y}{a}. \qquad .....(2)$$

$$\frac{\partial z}{\partial y} = q = \frac{x}{a} \qquad .....(3)$$

and again differentiating $\qquad r = \dfrac{\partial^2 z}{\partial x^2} = 0,\ \ s = \dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{1}{a},\ \dfrac{\partial^2 z}{\partial y^2} = r = 0 \qquad .....(4)$

Here parameters are x and y, so differential equation of lines of curvature is given by

$$\begin{vmatrix} dy^2 & -dx\,dy & dx^2 \\ 1+p^2 & pq & 1+q^2 \\ r & s & t \end{vmatrix} = 0 \qquad .....(5)$$

Using value of p, q, r, s, t, we get

$$\begin{vmatrix} dy^2 & -dx\,dy & dx^2 \\ 1+\dfrac{y^2}{a^2} & \dfrac{xy}{a^2} & 1+\dfrac{x^2}{a^2} \\ 0 & \dfrac{1}{a} & 0 \end{vmatrix} = 0 \qquad .....(6)$$

expanding the determinant, we get

$$dy^2\left(0-\frac{1}{a}\left(1+\frac{x^2}{a^2}\right)\right)+dx\,xy(0)+dx^2\left(\frac{1}{a}\left(1+\frac{y^2}{a^2}\right)-0\right)=0$$

or $$-\left(a^2+x^2\right)dy^2+\left(a^2+y^2\right)dx^2=0$$

or $$\frac{dx}{\sqrt{a^2+x^2}}\pm\frac{dy}{\sqrt{a^2+y^2}}=0 \qquad .....(7)$$

On integrating, we get $$\sinh^{-1}\frac{x}{a}\ \pm\ \sinh^{-1}\frac{y}{a}=c_1\left(\text{constant}\right) \qquad .....(8)$$

where c_1 is arbitrary constant if integration.

This relation in equation (8) provides us the surface on which the lines of curvature lie.

Equation (8) can be expressed as

$$\sinh^{-1}\left[\frac{x}{a}\sqrt{1+\frac{y^2}{a^2}}\pm\frac{y}{a}\sqrt{1+\frac{x^2}{a^2}}\right]=c_1$$

or $$\frac{x}{a}\sqrt{1+\frac{y^2}{a^2}}\pm\frac{y}{a}\sqrt{1+\frac{x^2}{a^2}}=\sinh\left(c_1\right)=\left(\text{constant}\right) \qquad .....(9)$$

Now from equation $az=xy$ of the paraboloid, we have

$$\frac{x}{a}=\frac{z}{y},\ \frac{y}{a}=\frac{z}{x}, \qquad \text{[using these in equation (9)]}$$

we get $$\frac{z}{y}\sqrt{1+\frac{z^2}{x^2}}\pm\frac{z}{x}\sqrt{1+\frac{z^2}{y^2}}=\left(\text{constant}\right)=K\left(\text{say}\right)$$

or $$\frac{z\left(x^2+z^2\right)^{1/2}\pm z\left(y^2+z^2\right)^{1/2}}{xy}=K\left(\text{constant}\right)$$

or $$z\left[\left(x^2+z^2\right)^{1/2}\pm\left(y^2+z^2\right)^{1/2}\right]=Kxy \qquad .....(10)$$

which is a can passing through a line of curvature of $xy=az$.

Ex.2. *Prove that in general three lines of curvature pass through an umbilic.*

Sol. Let the umbilical point be taken as origin (0, 0, 0); and xy-plane as the tangent plane at origin, and z-axis normal at the origin.

Now tangent plane is xy-plane, so we expect that first degree terms to be zero in the equation of surface $z=f(x, y)$. Further, the section at umbilic being circle, so the equation of surface $z=f(x, y)$ will have, coefficient of x^2 = coefficient of y^2 and no term of xy will be present. Under these restrictions we express the equation of surface $z=f(x, y)$ as follows :

$$2z = \frac{x^2 + y^2}{\rho} + \frac{1}{3}\left(ax^3 + 3bx^2y + 2cxy^2 + dy^3\right) + \ldots \qquad \ldots..(1)$$

Now, the condition that the normal (0, 0, 0) and other point (x, y, z) should intersect, *i.e.*, the two lines whose equation are

$$\frac{\xi - 0}{0} = \frac{\eta - 0}{0} = \frac{\zeta - 0}{1} \qquad \ldots..(2)$$

and
$$\frac{\xi - x}{p} = \frac{\eta - y}{q} = \frac{\zeta - z}{-1} \qquad \ldots..(3)$$

should be coplanar is
$$\begin{vmatrix} x & y & z \\ p & q & -1 \\ 0 & 0 & 1 \end{vmatrix} = 0 \cdot \qquad \ldots..(4)$$

On expanding, we get $\qquad qx = py \qquad \ldots..(5)$

Now, differentiating equation (1) partially with respect to x and y respectively we get

$$\frac{2\partial z}{\partial x} = 2p = \frac{2x}{\rho} + ax^2 + 2bxy + cy^2 + \ldots \qquad \ldots..(6)$$

and
$$\frac{2\partial z}{\partial x} = 2q = \frac{2y}{\rho} + bx^2 + 2cxy + dy^2 + \ldots \qquad \ldots..(7)$$

Using values of p and q from (5) and (6) in (4), we get

$$\frac{2xy}{\rho} + ax^2y + 2bxy^2 + cy^3 + \ldots = \frac{2xy}{\rho} + bx^3 + 2cx^2y + dxy^2 + \ldots$$

or
$$bx^3 + x^2y\,(2c - a) + xy^2\,(d - 2b) - cy^3 = 0$$

divide by $-x^3$, we get
$$c\left(\frac{y}{x}\right)^3 + (2b - d)\left(\frac{y}{x}\right)^2 + (a - 2c)\left(\frac{y}{x}\right) - b = 0 \qquad \ldots..(8)$$

Let θ be the angle which tangent to a line of curvature makes with z-axis, then $\tan\theta = \lim\ (y/x)$ therefore from equation (8), we have

$$c\tan^3\theta + (2b - d)\tan^2\theta - (2c - a)\tan\theta - b = 0. \qquad \ldots..(9)$$

This is a cubic equation in $\tan\theta$, so it given three lines of curvature through the umbilic, corresponding to three values of $\tan\theta$.

7.14.4 Self-learning exercise-3

1. Write formula for radius of curvature of a given section through any point of a surface $z = f(x, y)$.
2. Define lines of curvature.
3. What is the differential equation of lines of curvature at point (u, v) of the surface $\vec{r} = \vec{r}(u, v)$?
4. In general how many lines of curvature pass through an umbilic ?

7.15 Summary

1. In this unit you have studied Meunier's theorem, Principal direction and principal curvature at point (u, v) on surface $\vec{r} = \vec{r}(u,v)$, about first curvature, mean curvature, and Gaussian Curvatures at point (u, v) on surface $\vec{r} = \vec{r}(u,v)$, about umbilic point, about formula of radius of curvature of any normal section at an umbilic on $z = f(x, y)$, about radius of curvature of a given section through any point on surface of the form $z = f(x, y)$, about lines of curvature and its differential equation.
2. Sufficient number of examples have been solved in the unit.
3. Partial derivative p, q, r, s, t and fundamental magnitudes E, F, G and L, M, N, H, T will help the students to easily understand the text of the unit.
4. Examples in the text have been inserted frequently to help students to understand the text of the unit.

7.16 Answers to self-learning exercises

Self-learning exercise-1

1. See definitions of §7.3

5. $\kappa_n = \dfrac{Ldu^2 + 2Mdu\,dv + Ndv^2}{Edu^2 + 2Fdu\,du + Gdv^2}$

6. See §7.4 *(i)* & *(ii)*

7. See Meurier's Theorem.

Self-learning exercise-2

1. See §7.6.1 *(i)* to *(v)*

2. $(EM - LF)\,du^2 + (EN - GL)\,du\,dv + (FN - GM)\,dv^2 = 0$

3. $(EM - LF)\,l^2 + (EN - GL)\,lm + (FN - GM)\,m^2 = 0$

4. Yes

5. $\dfrac{E}{L} = \dfrac{F}{M} = \dfrac{G}{N}$

6. See §7.11 *(i)* to *(vi)*

7. $\rho = \left(\dfrac{1+p^2}{r}\right)H = \dfrac{Hpq}{s} = \dfrac{H\left(1+q^2\right)}{t}.$

Self-learning exercise-3

1. $\dfrac{\cos\theta}{\rho} = \dfrac{rl_1^2 + 2sl_1m_1 + tm_1^2}{\sqrt{1+p^2+q^2}}$

2. See 7.14.2

3. $Pdu^2 + 2Qdu\,dv + Rdv^2 = 0$, where $P = Em - LF$, $Q = EN - GL$, $R = FN - GM$.

4. Three.

7.17 Exercises

1. Find the curvature of the normal section of the helicoid $x = u\cos\theta,\ y = u\sin\theta,\ z = f(x) + c\theta$.

$$\textbf{[Ans. } \kappa_n = \frac{(uf'')\,du^2 - 2c\,du\,dv + (u^2 f')\,fv^2}{H\left\{(1+f'^2)\,du^2 + 2cf'\,du\,dv + (u^2+c^2)\,dv^2\right\}}\textbf{]}$$

2. For the surface $\dfrac{x}{a} = \dfrac{u+v}{2},\ \dfrac{y}{b} = \dfrac{u-v}{2},\ z = \dfrac{uv}{2}$; prove that the principal radii are given by

$$a^2b^2\rho^2 + \lambda ab\rho\,(a^2 - b^2 + uv) - \lambda^4 = 0$$

where $4\lambda^2 = 4a^2b^2 + a^2(u-v)^2 + b^2(u+v)^2$ and that the line of curvature are given by

$$\frac{du}{\sqrt{a^2+b^2+u^2}} = \pm\frac{dv}{\sqrt{a^2+b^2+v^2}}.$$

3. Prove that for helicoid $x = u\cos\theta,\ y = u\sin\theta,\ z = c\,\theta$,

$$\rho_1 = -\rho_2 = \frac{u^2+c^2}{c}, \text{ where } u^2 = x^2 + y^2$$

and that the lines of curvature are given by

$$d\theta = \pm\frac{du}{\sqrt{u^2+c^2}}$$

4. Find the umbilics of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. If P is an umbilic of this ellipsoid, then prove that the curvature at P of any normal section through P is $\left(\dfrac{ac}{b^3}\right)$.

$$\textbf{[Ans. } \frac{x/a}{\sqrt{a^2-b^2}} = \frac{(y/b)}{0} = \frac{(z/z)}{\pm\sqrt{b^2-c^2}}\textbf{]}$$

5. Find the Gaussian curvature at point (u, v) of the anchor ring,

$$x = (b + a\cos u)\cos v, \quad y = (b + a\cos u)\sin v, \quad z = a\sin u,$$

where the domain of u, v is $0 \le u \le 2\pi,\ 0 \le v \le 2\pi$. Verify that the total curvature of the whole surface is zero. **[Ans.** Gaussian curvature $\kappa = \dfrac{\cos u}{a(b + a\cos u)}$**]**

□□□

UNIT 8 : Principal Radii, Relation between Fundamental Forms, Asymptotic Lines, Differential Equation of an Asymptotic line, Curvature and Torsion of an Asymptotic Line

Structure of the Unit

8.0 Objectives

This unit provides a general overview of Principal Radii, relation between three fundamental forms of surface, Asymptotic lines and their differential equations, also curvature and torsion of an asymptotic line. After reading this unit you will be able to learn :

1. about Principal Radii.
2. about relation between fundamental forms.
3. about Asymptotic lines and their differential equations.
4. about curvature and torsion of asymptotic lines.

8.1 Introduction

In this unit we shall study principal radii through a point of the surface $z = f(x, y)$, relation between three fundamental forms, asymptotic lines and differential equation of asymptotic lines. In the end of unit we shall study curvature and torsion of an asymptotic lines.

In the earlier Unit-7 we have already obtained the principal curvatures and principal sections for the Monge's surface $z = f(x, y)$ from which expression for principal radii can be directly written. We shall discuss here an alternative method.

8.2 Principal radii through a point of the surface $z = f(x, y)$

Equation of surface is $z = f(x, y)$ .....(8.2.1)

If l_1, m_1, n_1 are the direction cosines of the tangent to a normal section of the surface through the point (x, y, z), then the radius of curvature of the section is given by

$$\frac{1}{\rho} = \frac{rl_1^2 + 2sl_1m_1 + tm_1^2}{\sqrt{1+p^2+q^2}}, \qquad .....(8.2.2)$$

where $$p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{\partial y},\ r=\frac{\partial^2 z}{\partial x^2},\ s=\frac{\partial^2 z}{\partial x\partial y},\ t=\frac{\partial^2 z}{\partial y^2}.$$

Equation (8.2.2) can be expressed as

$$1=\frac{rl_1^2\rho}{H}+\frac{2s\rho l_1 m_1}{H}+\frac{m_1^2 t\rho}{H},\ \text{where}\ \ H=\sqrt{1+p^2+q^2}$$

or $$l_1^2\left(\frac{r\rho}{H}\right)+l_1m_1\left(\frac{2s\rho}{H}\right)+m_1^2\left(\frac{t\rho}{H}\right)=1 \qquad .....(8.2.3)$$

Now, $-p, -q$, 1 are the direction ratios of the surface normal at (x, y, z), so by the condition of perpendicularity

$$pl_1+qm_1-n_1=0$$

or $$n_1=pl_1+qm_1. \qquad .....(8.2.4)$$

On squaring, we get

$$n_1^2=(pl_1+qm_1)^2$$

or $$1-(l_1^2+m_1^2)=p^2l_1^2+q^2m_1^2+2pql_1m_1 \qquad [\because\ l_1^2+m_1^2+n_1^2=1]$$

or $$(1+p^2)\,l_1^2+2pql_1m_1+m_1^2\,(1+q^2)=1 \qquad .....(8.2.5)$$

Subtracting equation (8.2.3) from equation (8.2.5), we get

$$l_1^2\left(1+p^2-\frac{r\rho}{H}\right)+2l_1m_1\left(pq-\frac{s\rho}{H}\right)+m_1^2\left(1+q^2-\frac{t\rho}{H}\right)=0. \qquad .....(8.2.6)$$

The equation (8.2.6) is quadratic equation in $\frac{l_1}{m_1}$, therefore it gives two values of $\frac{l_1}{m_1}$, corresponding to a given radius of curvature. If ρ is a principal radius, these sections coincide (*i.e.,* values coincide). Therefore for the principal radii (by condition of roots both equal $B^2-2AC=0$).

$$4\left(pq-\frac{s\rho}{H}\right)^2-4\left(1+p^2-\frac{r\rho}{H}\right)\left(1+q^2-\frac{t\rho}{H}\right)=0. \qquad .....(8.2.7)$$

On simplifying, we get

$$\rho^2\,(rt-s^2)-H\rho\ \{(1+p^2)\,t+(1+q^2)\,r-2pqs\}+H^4=0. \qquad .....(8.2.8)$$

This equation gives the principal radii.

8.3 Fundamental forms

In the earlier units, we have studied first and second fundamental forms. Here we shall define third fundamental form and then we will derive relation between them.

Three fundamental forms are given by first fundamental form

$$I=d\vec{r}\cdot d\vec{r}=E\,(du)^2+2F\,du\,dv+G\,(dv)^2=(ds)^2, \qquad .....(8.3.1)$$

where equation of surface is regarded as $\vec{r} = \vec{r}\,(u, v)$, u, v are its parameters, and E, F, G are first order fundamental magnitudes.

Second fundamental form

$$II = -d\,\vec{r}\cdot d\hat{N} = L\,du^2 + 2M du dv + N dv^2 = (\kappa_n ds^2), \qquad .....(8.3.2)$$

where L, M, N are second order fundamental magnitudes.

Third fundamental form.

The quadric $\quad A\,du^2 + 2B\,dudv + C\,dv^2, \qquad .....(8.3.3)$

is called the third fundamental form for surface $\vec{r} = \vec{r}\,(u, v)$ and is denoted by III, where

$$A = \hat{N}_1^{\,2},\ B = \hat{N}_1\cdot\hat{N}_2 \text{ and } C = \hat{N}_2^{\,2} \text{ and } \hat{N}_1 = \frac{\partial\hat{N}}{\partial u},\ \hat{N}_2 = \frac{\partial\hat{N}}{\partial v}.$$

$$\therefore \qquad III = d\hat{N}\cdot d\hat{N} = A\,du^2 + 2B\,dudv + C\,dv^2. \qquad .....(8.3.4)$$

8.4 Relation between three fundamental forms

If K is the total curvature and J is the first curvature then to show that

$$KI - J.II + III = 0 \qquad .....(8.4.1)$$

Proof. Suppose that the lines of curvature be the parametric curves, then

$$F = M = 0 \qquad .....(8.4.2)$$

Then, the two fundamental forms are reduced to

$$I = E\,du^2 + G\,dv^2 \qquad .....(8.4.3)$$

and $\quad II = L\,du^2 + N\,dv^2. \qquad .....(8.4.4)$

If κ_a, κ_b be the principal curvatures at the point (u, v) of surface $\vec{r} = \vec{r}\,(u, v)$ we have for two parametric curves by Rodrigue's formula

$$\kappa_a \frac{dr}{du} + \frac{d\hat{N}}{du} = 0 \Rightarrow \kappa_a\vec{r}_1 + \hat{N}_1 = 0 \Rightarrow \hat{N}_1 = -\kappa_a\vec{r}_1 \qquad .....(8.4.5)$$

and $\quad \kappa_b \dfrac{dr}{dv} + \dfrac{d\hat{N}}{dv} = 0 \Rightarrow \kappa_b\vec{r}_2 + \hat{N}_2 = 0 \Rightarrow \hat{N}_2 = -\kappa_b\vec{r}_2 \qquad .....(8.4.6)$

So, the third fundamental form is given by

$$\begin{aligned} III = (d\hat{N})^2 &= \left(\hat{N}_1 du + \hat{N}_2 dv\right)^2 \\ &= \hat{N}_1^{\,2} du^2 + 2\hat{N}_1\cdot\hat{N}_2\,dudv + \hat{N}_2^{\,2} dv^2 \\ &= \left(-\kappa_a\vec{r}_1\right)^2 du^2 + 2\left[\left(-\kappa_a\vec{r}_1\right)\cdot\left(-\kappa_b\vec{r}_2\right)\right]dudv + \left(-\kappa_b\vec{r}_2\right)^2 dv^2. \end{aligned}$$

Using equations (8.4.5) & (8.4.6)

$$III = \kappa_a^2 \vec{r}_1^{\,2} du^2 + 2\kappa_a\kappa_b (\vec{r}_1 \cdot \vec{r}_2)\, dudv + \kappa_b^2 \vec{r}_2^{\,2}\, dv^2$$

$$= \kappa_a^2 E du^2 + 2\kappa_a\kappa_b\, F\, dudv + \kappa_b^2 G\, dv^2$$

$$\Rightarrow \qquad III = \kappa_a^2 E du^2 + \kappa_b\, G\, dv^2 \quad (\because\ F = 0) \qquad .....(8.4.7)$$

Now $$K.I. - J.II = \kappa_a\kappa_b \left(Edu^2 + Gdv^2\right) - \left(\kappa_a + \kappa_b\right)\left(L\,du^2 + N\,dv^2\right) \qquad .....(8.4.8)$$

Using $K = \kappa_a\kappa_b$, and $J = \kappa_a + \kappa_b$

$$\because \qquad \kappa_a = \frac{L}{E} \Rightarrow L = E\kappa_a,\ \kappa_b = \frac{N}{G} \Rightarrow N = \kappa_b G. \qquad .....(8.4.9)$$

Using these values of L and N in equation (8.4.8), we get

$$K.I. - J.II = \kappa_a\kappa_b E du^2 + \kappa_a\kappa_b G dv^2 - \left(\kappa_a + \kappa_b\right)\left(E\kappa_a du^2 + \kappa_b G dv^2\right)$$

$$= -\left(\kappa_a E du^2 + \kappa_b G dv^2\right) = -III. \qquad .....(8.4.10)$$

Hence $$K.I - J.II + III = 0. \qquad .....(8.4.11)$$

This equation gives a relation between three fundamental forms of the surface $\vec{r} = \vec{r}\,(u, v)$. This can also be expressed in the form

$$III - 2\mu \cdot II + K \cdot I = 0. \quad \left(\because\ \frac{J}{2} = \mu\right) \qquad .....(8.4.12)$$

8.5 Asymptotic lines

Before defining the asymptotic lines we shall define conjugate directions at a point of a surface and we shall derive expressions for conjugate directions.

Definition. Conjugate directions

Conjugate directions at a given point (u, v) on a surface $\vec{r} = \vec{r}\,(u, v)$ are defined as follows :

Let P be any point (u, v) on a surface (say) S and Q be a neighbouring point on it [*i.e.*, Q is ($u + du$, $v + dv$)].

Let tangent planes at P and Q to the surface S intersect in a line (say) LM. Then the limiting directions of the line PQ and LM as $Q \to P$ are called conjugate directions at P.

8.6 To show that to a given direction there is one and only one conjugate direction. Also derive the condition for the two directions (*du, dv*) and (*Du, Dv*) to be conjugate

Proof. Let $\vec{r} = \vec{r}\,(u, v)$ be equation of surface. Let $\vec{r}$ be position vector of the point P (u, v) on this surface and $\vec{r} + d\vec{r}$ be position vector of point Q ($u + du$, $v + dv$), a point adjacent to P in the direction (du, dv). Let $\hat{N}$ and $\hat{N}+d\hat{N}$ be unit normals at the points $P(\vec{r})$ and $Q(\vec{r}+d\vec{r})$ on the surface, respectively. Then clearly $d\vec{r}$ is the limiting position of the vector $\overrightarrow{PQ}$ as $Q \to P$. Let $D\vec{r}$ be the vector along the limiting position of the line LM as $Q \to P$.

(where D is used to denote the other direction in place of d)

then $$\overrightarrow{PQ} = d\vec{r} = \frac{\partial \vec{r}}{\partial u} du + \frac{\partial \vec{r}}{\partial v} dv = \vec{r}_1 du + \vec{r}_2 dv$$

or $$d\vec{r} = \vec{r}_1 du + \vec{r}_2 dv \qquad .....(8.6.1)$$

and $$d\hat{N} = \frac{\partial \hat{N}}{\partial u} du + \frac{\partial \hat{N}}{\partial v} dv = \hat{N}_1 du + \hat{N}_2 dv \qquad .....(8.6.2)$$

Also $$D\vec{r} = \frac{\partial \vec{r}}{\partial u} Du + \frac{\partial \vec{r}}{\partial v} Dv = \vec{r}_1 Du + \vec{r}_2 Dv \qquad .....(8.6.3)$$

Now the vector $D\vec{r}$ lies in the tangent plane of $P(\vec{r})$ as well as that of $Q(\vec{r}+d\vec{r})$ so it will be $\perp$ to both normals $\hat{N}$ and $\hat{N}+d\hat{N}$.

$\therefore$ $$D\vec{r} \cdot \hat{N} = 0 \qquad .....(8.6.4)$$

and $$D\vec{r} \cdot \left(\hat{N} + d\hat{N}\right) = 0 \qquad .....(8.6.5)$$

or $$D\vec{r} \cdot \hat{N} + D\vec{r} \cdot d\hat{N} = 0 \quad \text{or} \quad D\vec{r} \cdot d\hat{N} = 0 \qquad .....(8.6.6)$$

Using values from (8.6.2) and (8.6.3), we get

$$\left(\vec{r}_1\, Du + \vec{r}_2\, Dv\right) \cdot \left(\hat{N}_1 du + \hat{N}_2 dv\right) = 0$$

$$\left(\vec{r}_1 \cdot \hat{N}_1\right) Dudu + \left(\vec{r}_1 \cdot \hat{N}_2\right) Dudv + \left(\vec{r}_2 \cdot \hat{N}_1\right) Dvdu + \left(\vec{r}_2 \cdot \hat{N}_2\right) Dvdv = 0$$

$\Rightarrow$ $$L\, Dudu + M\, Dudv + M\, Dvdu + N\, Dvdv = 0$$

or $$L\, Dudu + M\,(Dudv + Dvdu) + N\, Dvdv = 0, \qquad .....(8.6.7)$$

where $$\hat{N}_1 \cdot \vec{r}_1 = L,\ \hat{N}_1 \cdot \vec{r}_2 = M = \hat{N}_2 \cdot \vec{r}_1,\ \vec{r}_2 \cdot \hat{N}_2 = N.$$

The equation (8.6.7) is the required condition for the two directions (du, dv) and (Dv, Du) to be conjugate.

Also the equation (8.6.7) is linear in each of the ratio $\frac{du}{dv}$ and $\frac{Du}{Dv}$ which shows that to a given direction there is one and only one conjugate direction.

Remarks.

(i) The symmetry of equation (8.6.7) shows that if the direction $\frac{du}{dv}$ is conjugate to directions $\frac{Du}{Dv}$, then $\frac{Du}{Dv}$ is conjugate to direction $\frac{du}{dv}$ *i.e.,* the property of conjugate direction is reciprocal.

(ii) It follows from equation (8.6.7) that the directions (l_1, m_1) and (l_2, m_2) at point P are conjugate iff $Ll_1l_2 + M(l_1m_2 + m_1l_2) + Nm_1m_2 = 0$.

8.7 To show that the directions given by $Pdu^2 + 2Qdudv + Rdv^2 = 0$, are conjugate if $LR - 2MQ + NP = 0$.

Proof. Given
$$LR - 2MQ + NP = 0. \qquad .....(8.7.1)$$

The equation $Pdu^2 + 2Qdudv + Rdv^2 = 0$ can be expressed as

$$P\left(\frac{du}{dv}\right)^2 + 2Q\frac{du}{dv} + R = 0. \qquad .....(8.7.2)$$

This equation being quadratic in $\frac{du}{dv}$, gives two roots, say $\frac{du}{dv}, \frac{Du}{Dv}$ then

$$\text{sum of roots} = \left(\frac{du}{dv} + \frac{Du}{Dv}\right) = -\frac{2Q}{P}, \qquad .....(8.7.3)$$

and
$$\text{product of roots} = \frac{du}{dv} \cdot \frac{Du}{Dv} = \frac{R}{P}. \qquad .....(8.7.4)$$

Now from equation (8.6.7), we have

$$L\, Dudv + M\,(Dudv + Dvdu) + NDvdv = 0.$$

which can be expressed as

$$L\frac{Du}{Dv}\frac{du}{dv} + M\left(\frac{Du}{Dv} + \frac{du}{dv}\right) + N = 0. \qquad .....(8.7.5)$$

Using values from equation (8.7.3) and (8.7.4), we get

$$L\left(\frac{R}{P}\right) + M\left(-\frac{2Q}{P}\right) + N = 0$$

$$\Rightarrow \qquad LR - 2MQ + NP = 0, \qquad .....(8.7.6)$$

which is the required condition.

8.8 Family conjugate to the family of curves $P\,du + Q\,dv = 0$

The equation of family of curves given by

$$Pdu + Qdv = 0, \qquad .....(8.8.1)$$

can be expressed as
$$Pdu = -\,Qdv$$

$$\Rightarrow \qquad \frac{du}{Q} = \frac{dv}{-P} = k \text{ (say)}$$

$$\Rightarrow \qquad du = kQ,\ dv = -\,kP. \qquad .....(8.8.2)$$

Now condition for two direction to be conjugate is

$$L\, Du\, du + M\,[Du\,(-dv) + Dv du] + N\, Dv dv = 0 \qquad .....(8.8.3)$$

Using equation (8.8.2) into the equation (8.8.3), we get

$$L\, Du\,(kQ) + M\,[Du\,(-kP) + Dv\,(kQ)] + N\, Dv\,(-kP) = 0$$

$\Rightarrow$ $k\,[LQ\, Du - MP Du + MQ\, Dv - NP Dv] = 0$

or $(LQ - MP)\, Du + (MQ - NP)\, Dv = 0$ .....(8.8.4)

which is the required result.

8.9 Conjugate directions and parametric curves.

Necessary and sufficient condition that the parametric curves through a point to have conjugate directions is that $M = 0$.

Proof : The condition is necessary : The parametric curves, $u = c_1$ (constant) and $v = c_2$ (constant), in the differential form will be

$$du\, dv = 0, \qquad .....(8.9.1)$$

which can be expressed as $0\, du^2 + 1\, du\, dv + 0\, dv^2 = 0.$ .....(8.9.2)

Comparing it with $P du^2 + 2Q\, du\, dv + R dv^2 = 0,$

we find that $P = 0,\ 2Q = 1,\ R = 0.$ .....(8.9.3)

Now the direction of parametric curves are conjugate if

$$LR - 2MQ + NP = 0 \qquad .....(8.9.4)$$

Using equation (8.9.3), we get

$$0 - M + 0 \Rightarrow M = 0. \qquad .....(8.9.5)$$

Thus the parametric curves have conjugate direction when $M = 0$.

The condition is sufficient : If $M = 0$ .....(8.9.6)

and for parametric curves $P = 0,\ R = 0.$ .....(8.9.7)

Then the condition $LR - 2MQ + NP = 0$ is satisfied by using (8.9.6) and (8.9.7).

Hence, if $M = 0$, the directions are conjugate for parametric curves.

8.10 Principal directions (line of curvature) at a point are always orthogonal and conjugate.

Proof : We know that parametric curves u = const. and v = const., whose combined equation is given by

$$du\, dv = 0.$$

The necessary and sufficient condition that these parametric curves be line of curvature are

$$F = 0, \quad M = 0 \qquad \text{.....(8.10.1)}$$

and condition for parametric curves to be conjugate is

$$M = 0. \qquad \text{.....(8.10.2)}$$

Also the necessary and sufficient condition for parametric curves to be orthogonal is that

$$F = 0. \qquad \text{...(8.10.3)}$$

From these we conclude that the direction of line of curvature are always orthogonal as well as conjugate.

8.11 A characteristic property of conjugate direction.

To show that conjugate direction at a point *P* on a surface are parallel to conjugate diameters of the indicatrix at *P*.

Proof : Take the lines of curvature as parametric curves, so that

$$F = 0, \text{ and } M = 0. \qquad \text{.....(8.11.1)}$$

Then by equation $(EG - F^2)\, k_n^2 - (EN + LG - 2FM)\, k_n + (LN - M^2) = 0$

becomes $EG\, k_n^2 - (EN + LG)\, k_n + LN = 0$

or $E\,(Gk_n - N)\, k_n - L\,(Gk_n - N) = 0$

$$(Ek_n - L)\,(Gk_n - \mathrm{N}) = 0$$

$$\Rightarrow \qquad k_n = \frac{L}{E}, \frac{N}{G} \text{ (say)} \qquad k_a = \frac{L}{E} \text{ and } k_b = \frac{N}{G} \qquad \text{.....(8.11.2)}$$

are the principal curvatures at the point P.

Now by setting point P as origin, x-axis along the principal direction v = const. at P, y-axis along the principal direction u = const. at P and z-axis along the normal to the surface at P, then equation of indicatrix (Dupin's indicatrix) at P is given by

$$\frac{x^2}{(2h/k_a)} + \frac{y^2}{(2h/k_b)} = 1, \quad z = h. \qquad \text{.....(8.11.3)}$$

But we know from coordinate geometry that the lines $y = m_1 x$ and $y = m_2 x$ are conjugate diameters of the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{if} \quad m_1 m_2 = \pm\frac{b^2}{a^2}, \qquad \text{.....(8.11.4)}$$

so the directions of conjugate diameters of curve given by equation (8.11.3) are given by

$$m_1 m_2 = -\frac{k_a}{k_b} \qquad \text{.....(8.11.5)}$$

$$\Rightarrow \qquad m_1 m_2 = -\frac{(L/E)}{(N/G)} = -\frac{LG}{EN}.$$

Let θ_1 and θ_2 be the angles which the conjugate directions (du, dv) and (Du, Dv) make with the parametric curve v = const., then

$$\text{and} \qquad m_1 = \tan\theta_1 = \sqrt{\frac{G}{E}}\frac{dv}{du} \qquad \text{.....(8.11.6)}$$

$$m_2 = \tan\theta_2 = \sqrt{\frac{G}{E}}\frac{Dv}{Du}. \qquad \text{.....(8.11.7)}$$

Using these values of m_1, m_2 in to equation (8.11.5), we get

$$\sqrt{\frac{G}{E}}\frac{dv}{du}\cdot\sqrt{\frac{G}{E}}\frac{Dv}{Du} = -\frac{LG}{EN}$$

$$\frac{G}{E}\frac{dv}{du}\frac{Dv}{Du} = -\frac{LG}{EN}$$

$$N\,dv\,D = -L\,du\,Du$$

$$\text{or} \qquad L\,du\,Du + N\,dv\,Dv = 0, \qquad \text{.....(8.11.8)}$$

which is the equation determining conjugate direction, when $M = 0$. Hence the result.

Alternate definition of conjugate directions :

Two directions at a point of a surface are said to be conjugate if they are parallel to the conjugate diameters of the indicatrix of the surface at that point.

8.11.1 Self-learning exercise-1.

1. Write a formula to find principal radii through a point of the surface $z = f(x, y)$.
2. Write the third fundamental form of the surface.
3. Write the relation between three fundamental forms of a surface $\vec{r} = \vec{r}(u,v)$.
4. Define conjugate directions at a given point (u, v) on a surface $\vec{r} = \vec{r}(u,v)$.
5. Write the condition for two directions (du, dv) and (Du, Dv) to be conjugate.
6. What is the necessary condition for parametric curves through a point to have conjugate direction ?

8.12 Definitions

(i) **Asymptotic directions :** A self conjugate direction on a surface $\vec{r} = \vec{r}(u,v)$ is called asymptotic direction.

(ii) **Asymptotic lines :** A curve on a surface $\vec{r} = \vec{r}(u,v)$ whose direction at each point is self conjugate, is called an asymptotic line.

Alternate definition of asymptotic line : A curve drawn on a surface $\vec{r} = \vec{r}(u,v)$ so as to touch at each point one of the inflexional tangents through the point, is called an asymptotic line on the surface.

8.13 Differential equation of the asymptotic lines at a point (*u*, *v*) on the surface $\vec{r} = \vec{r}(u,v)$.

Let (du, dv) be directions of an asymptotic line at any point (u, v) on the surface $\vec{r} = \vec{r}(u,v)$. Hence (du, dv) is a self-conjugate direction.

So for asymptotic line $\quad \dfrac{du}{dv} = \dfrac{Du}{Dv}, \qquad$ .....(8.13.1)

where (du, dv) and (Du, Dv) are conjugate directions.

The condition of conjugacy of these two directions are

$$L\,du\,Du + M\,(du\,Dv + dv\,Du) + N\,dv\,Dv = 0$$

or
$$L\frac{du}{dv}\frac{Du}{Dv} + M\left(\frac{du}{dv} + \frac{Du}{Dv}\right) + N = 0. \qquad .....(8.13.2)$$

On using (8.13.1), we have

$$L\left(\frac{du}{dv}\right)^2 + 2M\frac{du}{dv} + N = 0$$

$$\Rightarrow \qquad L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0, \qquad .....(8.13.3)$$

this is differential equation of the asymptotic lines at a point (u, v) on the surface $\vec{r} = \vec{r}(u,v)$ in curvilinear coordinates. Equation (8.13.3) can also be expressed in the following form

$$d\hat{N}\cdot d\vec{r} = 0. \qquad .....(8.13.4)$$

8.14 Conditions for two asymptotic directions at a point to be real and district, coincident or imaginary

Differential equation of asymptotic line is

$$L\left(\frac{du}{dv}\right)^2 + 2M\frac{du}{dv} + N = 0 \qquad .....(8.14.1)$$

which is quadratic in $\dfrac{du}{dv}$. It follows that in general asymptotic direction through every point of surface are self conjugate.

(i) These directions will be real and distinct if the discriminant $(B^2 - 4\,Ac) = 4\,(M^2 - LN)$ or $(M^2 - LN)$ is positive or $(LN - M^2) < 0$.

But the Gaussian curvature is

$$K = \frac{LN - M^2}{H^2} \Rightarrow \quad K = \frac{(-\text{ve})}{H^2} = (-\text{ve}). \qquad .....(8.14.2)$$

Hence asymptotic lines are real and distinct if

$$K < 0 \qquad .....(8.14.3)$$

and when $K < 0$, the point is called hyperbolic point and the asymptotic lines at the point are parallel to the asymptotes of the indicatrix at the point.

(ii) The directions given by equation (8.14.1) will be real and coincident, if the discriminant of equation (8.14.1) (say)

$$(B^2 - 4\,Ac) = 4\,(M^2 - LN) = 0 \quad \text{or} \quad M^2 - LN = 0. \qquad .....(8.14.4)$$

Then Gaussian curvature

$$K = \frac{LN - M^2}{H^2} = 0,$$

then the point is called parabolic point.

(iii) The directions given by equation (8.14.1) will be imaginary if the discriminant

"$B^2 - 4\,Ac < 0$" *i.e.,* $4\,(M^2 - LN) < 0$

$\Rightarrow \qquad (M^2 - LN) < 0$

$\Rightarrow \qquad (LN - M^2) > 0. \qquad .....(8.14.5)$

Then the Gaussian curvature

$$K = \frac{LN - M^2}{H^2} \frac{(+ve)}{H^2} > 0$$

i.e. $K > 0$, then the point is called an elliptic point.

In this case the asymptotic lines are imaginary.

8.14.1 Asymptotic lines are orthogonal if the surface is minimal

Proof : When both roots of the differential equation

$$Ldu^2 + 2M\,du\,dv + N\,dv^2 = 0 \qquad .....(8.14.6)$$

of the asymptotic lines are real and district, then

$(M^2 - LN) > 0$

$\Rightarrow \qquad (LN - M^2) < 0. \qquad .....(8.14.7)$

Then Gaussian curvature

$$K = \frac{LN - M^2}{H^2} < 0, \qquad .....(8.14.8)$$

then the point is hyperbolic and the indicatrix at the point is rectangular hyperbola and asymptotic lines are orthogonal. In this condition, the first curvature

$$J = 0 \Rightarrow EN - 2MF + GL = 0,$$

which is condition of minimal surface.

Hence asymptotic lines are orthogonal if surface is minimal.

8.15 To show that an asymptotic line is a curve on a surface such that the normal curvature of the surface in its direction is zero.

Proof. Differential equation of asymptotic line is

$$L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0 \qquad .....(8.15.1)$$

and normal curvature at any point (u, v) on surface $\vec{r} = \vec{r}\,(u, v)$ is given by

$$K_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}. \qquad .....(8.15.2)$$

Using equation (8.15.1), we get

$$K_n = 0. \qquad .....(8.15.3)$$

Hence in case of asymptotic lines $K_n = 0$, an asymptotic line is a curve on a surface such that the normal curvature of the surface in its direction is zero.

Remarks.

1. A curve drawn on a surface so that its osculating plane at any point contains the binormal to the curve at the point is an asymptotic line on the surface.
2. Two asymptotic lines through any point have the same osculating plane.
3. When the principal curvatures k_a and k_b equal and opposite *i.e.,* $k_a = -k_b$, the indicatrix is a rectangular hyperbola and so asymptotic lines are at right angles.
4. The normals to a surface at points of an asymptotic line generate a skew surface whose line of striction is the asymptotic line.

8.15.1 Illustrative Examples

***Ex.*1.** *Prove that on the surface $z = f(x, y)$ (Monge's form) the equation of asymptotic lines are*

$$r\,dx^2 + 2s\,dxdy + t\,dy^2 = 0.$$

Sol. The given surface is $z = f(x, y)$, then position vector $\vec{r}$ of the current point is $r = (x, y, z)$, where $z = f(x, y)$, x and y are taken parameters.

On differentiating partially with respect to x and y respectively

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial x} = (1,\ 0,\ p), \vec{r}_2 = \frac{\partial \vec{r}}{\partial y} = (0,\ 1,\ q) \qquad .....(1)$$

where $p = \dfrac{\partial z}{\partial x},\ q = \dfrac{\partial z}{\partial y}$ etc.

and $\quad \left.\begin{aligned} &\vec{r}_{12} = \frac{\partial^2 r}{\partial y \partial x} = (0,\ 0,\ s),\ \vec{r}_{11} = \frac{\partial^2 r}{\partial x^2} = (0,\ 0,\ r), \\ &\vec{r}_{22} = \frac{\partial^2 r}{\partial y^2} = (0,\ 0,\ t), \end{aligned}\right\} \qquad .....(2)$

Therefore $E = \vec{r}_1 \cdot \vec{r}_1 = 1 + p^2, F = \vec{r}_1 \cdot \vec{r}_2 = pq, G = \vec{r}_2 \cdot \vec{r}_2 = 1 + q^2 \qquad .....(3)$

$$\hat{N} = \frac{\vec{r}_1 \times \vec{r}_2}{H} = \frac{(-p, -q, 1)}{H}, H^2 = EG - F^2 = 1 + p^2 + q^2. \qquad .....(4)$$

$$\left.\begin{aligned} &L = \hat{N} \cdot \vec{r}_{11} = \frac{r}{H} = \frac{r}{\sqrt{1+p^2+q^2}}, M = \hat{N} \cdot \vec{r}_{12} = \frac{s}{H} = \frac{s}{\sqrt{1+p^2+q^2}} \\ &N = \hat{N} \cdot \vec{r}_{22} = \frac{t}{\sqrt{1+p^2+q^2}} \end{aligned}\right\} \qquad .....(5)$$

Now differential equation of asymptotic line is

$$L du^2 + 2M\, du\, dv + N\, dv^2 = 0 \qquad .....(6)$$

Here parameters $u = x$, and $v = y$, then differential equation (6) of the asymptotic lines at point (x, y) on the surface $z = f(x, y)$ reduces to the form

$$L dx^2 + 2M\, dx\, dy + N dy^2 = 0. \qquad .....(7)$$

Using values from equation (5), we get

$$\frac{r dx^2}{\sqrt{1+p^2+q^2}} + \frac{2s\, dx\, dy}{\sqrt{1+p^2+q^2}} + \frac{t dy^2}{\sqrt{1+p^2+q^2}} = 0$$

or $\qquad r dx^2 + 2s\, dx\, dy + t dy^2 = 0, \qquad .....(8)$

which is the required equation of asymptotic line.

***Ex.*2.** *Find the asymptotic lines on the surface*

$$z = y \sin x.$$

Sol. Equation of the surface is of the form $z = f(x, y)$ [monge's form], so differential equation of asymptotic line will be

$$r\, dx^2 + 2s\, dx\, dy + t\, dy^2 = 0, \qquad .....(1)$$

where $r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}.$

Surface is $\qquad z = y \sin x. \qquad .....(2)$

Differentiating partially with respect to x and y, we get

$$p = \frac{\partial z}{\partial x} = y \cos x \qquad .....(3)$$

and $$q=\frac{\partial z}{\partial y}=\sin x \qquad .....(4)$$

and $$\left.r=\frac{\partial^2 z}{\partial x^2}=-y\sin x, s=\frac{\partial^2 z}{\partial x\partial y}=\cos x, t=\frac{\partial^2 z}{\partial y^2}=0\right\} \qquad .....(5)$$

Using values of *r, s* and *t* form equation (5) in equation (1), we get

$$(-y\sin x)\, dx^2 + 2\cos x\, dx\, dy + 0 = 0$$

$$\Rightarrow \qquad \left(-\tan x\, dx+\frac{2}{y}dy\right)dx=0$$

Then either $\qquad dx = 0 \quad$ or $\quad -\tan x\, dx+\frac{2}{y}dy=0$

when $\qquad dx = 0 \quad \Rightarrow \quad x = \text{constant} = c_1$ (say) $\qquad .....(6)$

and when $\qquad -\tan x\, dx+\frac{2}{y}dy=0$

$\Rightarrow \qquad \log(\cos x) + 2\log y = (\text{constant}) = \log c^2$ (say) $\qquad$ [on integrating]

or $\qquad \log(\cos x) + \log(y^2) = \log(c^2)$

or $\qquad \log(\cos x \cdot y^2) = \log(c^2)$

or $\qquad \cos x \cdot y^2 = c^2 \qquad .....(7)$

Hence form equation (6) and (7) are the equations of asymptotic lines as given $x = c_1$ and $y_2 \cos x = c_2$.

Ex.3. *Prove that for the surface $x = 3u(1 + v^2) - u^3$, $y = 3v(1 + u^2) - v^3$, $z = 3u^2 - 3v^2$, the asymptotic line are $u \pm v$ = constant.*

Sol. Let $\vec{r}$ be the position vector of any current point on the surface, then

$$\vec{r} = (3u(1+v^2) - u^3, 3v(1+u^2) - v^3, 3u^2 - 3v^2) \qquad .....(1)$$

On differentiating with respect to u, we get

$$\vec{r}_1=\frac{\partial \vec{r}}{\partial u}=3\left(1+v^2-u^2, 2uv, 2u\right) \qquad .....(2)$$

On differentiating (1) partially with respect to v

$$\vec{r}_2=\frac{\partial \vec{r}}{\partial v}=3\left(2uv, 1+u^2-v^2, -2u\right) \qquad .....(3)$$

From equation (2), on differentiating with respect to x again

$$\left.\begin{array}{l}\vec{r}_{11}=6(-u,v,1), \text{ similarly } \vec{r}_{12}=6(u,v,0)\\ \vec{r}_{22}=6(u,-v,-1)\end{array}\right\} \qquad .....(4)$$

Now $$\hat{N} = \frac{\vec{r}_1 \times \vec{r}_2}{H} = \frac{1}{H}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3\left(1+v^2-u^2\right) & 6uv & 6u \\ 6uv & 3\left(1+u^2-v^2\right) & -6v \end{vmatrix}$$

or $$\hat{N} = \frac{9}{H}\left(-2u\left(1+u^2+v^2\right), 2v\left(1+u^2+v^2\right), 1-u^4-v^4-2u^2v^2\right)$$

$$\left.\begin{aligned} \therefore \quad L &= \hat{N}\cdot\vec{r}_{11} = \frac{54\left(u^2+v^2+1\right)^2}{H}, M = \hat{N}\cdot\vec{r}_{12} = 0 \\ \text{and } N &= \hat{N}\cdot\vec{r}_{22} = \frac{-54\left(u^2+v^2+1\right)^2}{H} \end{aligned}\right\} \qquad .....(5)$$

Now differential equation of asymptotic line is

$$L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0. \qquad .....(6)$$

Using values of L, M and N from equation (5) into equation (6), we get

$$\frac{54}{H}\left(u^2+v^2+1\right)du^2 + 0 + \left(\frac{-54}{H}\left(u^2+v^2+1\right)^2\right)dv^2 = 0$$

or $\quad du^2 - dv^2 = 0$ or $du = \pm\, dv \Rightarrow du \pm dv = 0$

On integrating $(u \pm v)$ = constant which is the required equation. Hence proved.

8.16 Necessary and sufficient condition for the parametric curves to be asymptotic lines

Proof : The parametric curves are u = constant, and v = constant, then the combined differential equation of the parametric curves is given by

$$du\,dv = 0. \qquad .....(8.16.1)$$

And differential equation of asymptotic lines is

$$L du^2 + 2M\,du\,dv + N\,dv^2 = 0 \qquad .....(8.16.2)$$

***(i)* The condition is necessary :** When the parametric curves are asymptotic lines, the above two equations (1) and (2) must be same, so on comparing these we get

$$L = 0, N = 0, M \neq 0. \qquad .....(8.16.3)$$

***(ii)* The condition is sufficient :** When $L = 0$, $N = 0$ and $M \neq 0$, then equation (2) reduces to

$$0 + 2M\,du\,dv + 0 = 0 \Rightarrow du\,dv = 0. \qquad .....(8.16.4)$$

which is equation (1), this shows that the asymptotic lines are parametric curves.

8.17 If the parametric curves are asymptotic lines, then find differential equation of line of curvature and show that first and second curvatures are $-\dfrac{2FM}{H^2}$ and $\dfrac{-M^2}{H^2}$.

Proof : If the parametric curves are asymptotic lines, then

$$L = 0, N = 0, M \neq 0 \qquad \text{.....(8.17.1)}$$

and the differential equation $\quad (EM - LF)\, du^2 + (EN - GL)\, du\, dv + (FN - GM)\, dv^2 = 0$

of lines of curvature reduces to the following form

$$EMdu^2 + 0 + (0 - GM)\, dv^2 = 0$$

$$\Rightarrow \qquad EMdu^2 - GM\, dv^2 = 0 \qquad (\because M \neq 0) \qquad \text{.....(8.17.2)}$$

Now the equation of the principal curvatures

$$H^2 \kappa^2 - (EN + GL - 2FM)\, \kappa + (LN - M^2) = 0$$

reduces to the following form $\quad H^2 \kappa^2 + 2FM\kappa - M^2 = 0 \qquad \text{.....(8.17.3)}$

which is a quadric in κ, so it gives two roots. *i.e.,* κ_a, κ_b, say.

Hence, $\qquad$ sum of roots $= (\kappa_a + \kappa_b) = \dfrac{-2FM}{H^2} = J =$ first curvature

and $\qquad$ product of roots $= (\kappa_a \cdot \kappa_b) = K = \dfrac{-M^2}{H^2} =$ second curvature.

Hence the result.

8.18 Osculating plane at any point of a curved asymptotic line is the tangent plane to the surface.

Proof : Let equation of surface be $\vec{r} = \vec{r}(u, v)$, and let $\vec{r} = \vec{r}(s)$ be curved asymptotic line lying on this surface. Let $\hat{t}\left(= \dfrac{d\vec{r}}{ds}\right)$ be the unit tangent to the asymptotic line at any point $P(r)$ and $\hat{N}$ be the unit normal to surface at point P, then

$$\hat{N} \cdot \hat{t} = 0. \qquad \text{.....(8.18.1)}$$

Differentiating with respect to s, we get

$$\frac{d\hat{N}}{ds} \cdot \hat{t} + \hat{N} \cdot \frac{d\hat{t}}{ds} = 0 \quad \text{or} \quad \frac{d\hat{N}}{ds} \cdot \frac{d\vec{r}}{ds} + \hat{N} \cdot \kappa \hat{n} = 0 \qquad \text{.....(8.18.2)}$$

$$\left(\because \frac{d\hat{t}}{ds} = \kappa\, \hat{n} \quad \text{and} \quad \hat{t} = \frac{d\vec{r}}{ds}, \ \text{where } \kappa = \text{curvature}, \hat{n} = \text{unit principal normal vector}\right)$$

But by Serret-Frenet formulae. The differential equation of asymptotic line is

$$d\hat{N}\cdot d\vec{r}=0 \quad \text{or} \quad \frac{d\hat{N}}{ds}\cdot\frac{d\vec{r}}{ds}=0 \qquad .....(8.18.3)$$

then equation (8.18.2) reduces to

$$\kappa\left(\hat{N}\cdot\hat{n}\right)=0 \Rightarrow \hat{N}\cdot\hat{n}=0 \qquad (\because \kappa\neq 0) \qquad .....(8.18.4)$$

By equation (8.18.1) and (8.18.4), we have

$$\hat{N}\cdot\hat{t}=0 \text{ and } \hat{N}\cdot\hat{n}=0,$$

this means $\hat{N}$ is perpendicular to both $\hat{t}$ and $\hat{n}$ so $\hat{N}$ is parallel to

$$\left(\hat{t}\times\hat{n}\right)=\hat{b} \quad \text{or} \quad \hat{N}=\pm\hat{b} \qquad .....(8.18.5)$$

where $\hat{b}$ is unit vector along binormal.

From this, we conclude that at any point of a curved asymptotic line, the binormal is also normal to the surface.

Let $$\hat{N}=\hat{b}. \qquad .....(8.18.6)$$

Now the equation of osculating plane to curve $\vec{r}=\vec{r}(s)$ at a point $P(s)$ is given by

$$\left(\vec{R}-\vec{r}\right)\cdot\hat{b}=0 \qquad .....(8.18.7)$$

and equation of tangent plane to the surface at this point is given by

$$\left(\vec{R}-\vec{r}\right)\cdot\hat{N}=0\cdot \qquad .....(8.18.8)$$

The relation (8.18.6) makes the above planes coincident at the point on the curve on the surface.

Hence the result.

8.19 Torsion of an asymptotic line $\vec{r}=\vec{r}(s)$ on the surface $\vec{r}=\vec{r}(u,v)$.

We know that the unit vector along binormal $\hat{b}$ to an asymptotic line is the unit surface normal $\hat{N}$

i.e. $$\hat{N}=\hat{b} \qquad .....(8.19.1)$$

Differentiating both sides with respect to arc length s, we get

$$\frac{d\hat{N}}{ds}=\frac{d\hat{b}}{ds}\Rightarrow\frac{d\hat{N}}{ds}=-\tau\,\hat{n}, \qquad \text{[by Serret-Frenet formulae]}$$

$$\Rightarrow \hat{N}'=-\tau\,\hat{n}, \qquad .....(8.19.2)$$

where $\hat{N}'=\dfrac{dN}{ds}$.

Now taking dot product of both sides of equation (8.19.2) by $\hat{n}$, we have

$$\hat{N}' \cdot \hat{n} = -\tau \, \hat{n} \cdot \hat{n}$$

or $$\hat{N}' \cdot \hat{n} = -\tau \qquad (\because \hat{n} \cdot \hat{n} = 1) \qquad .....(8.19.3)$$

now $$\hat{n} = \hat{b} \times \hat{t}$$

or $$\hat{n} = \hat{N} \times \hat{t} \qquad .....(8.19.4)$$

Now, using value of $\hat{n}$ in equation (8.19.3), we get

$$\tau = -\hat{N}' \cdot \hat{N} \times \hat{t}$$

or $$\tau = -\left[\hat{N}' \; \hat{N} \; \hat{t}\right]$$

or $$\tau = \left[\hat{N} \; \hat{N}' \; \hat{t}\right]$$

or $$\tau = \left[\hat{N} \; \hat{N}' \; \vec{r}'\right], \qquad .....(8.19.6)$$

which is torsion of an asymptotic line.

8.20 Curvature of an asymptotic line $\vec{r} = \vec{r}(s)$ on the surface $\vec{r} = \vec{r}(u,v)$.

$\hat{t} = \dfrac{d\vec{r}}{ds}$ is the unit vector along the tangent then we know that

$$\frac{d\hat{t}}{ds} = \hat{t}' = \kappa \hat{n}. \qquad .....(8.20.1)$$

Now, taking dot product of both sides by $\hat{n}$

$$\hat{t}' \cdot \hat{n} = \kappa \, \hat{n} \cdot \hat{n}$$

or $$\hat{t}' \cdot \hat{n} = \kappa \quad (\because \hat{n} \cdot \hat{n} = 1)$$

$$\kappa = \hat{t}' \cdot \hat{N} \times \hat{t} \qquad \left(\because \hat{n} = \hat{b} \times \hat{t} = \hat{N} \times \hat{t}\right)$$

or $$\kappa = \left[\hat{t}' \; \hat{n} \; \hat{t}\right] = \left[\hat{N} \; \hat{t} \; \hat{t}'\right]$$

or $$\kappa = \left[\hat{N} \; \vec{r}' \; \vec{r}''\right], \qquad .....(8.20.2)$$

where $\hat{t} = \vec{r}'$ and $\hat{t}' = \vec{r}''$, is curvature of an asymptotic line.

8.21 Beltrami-Enneper theorem

Statement : *At a point on a surface, where the Gaussian curvature is negative and equal to K, the torsion of the asymptotic lines is* $\pm\sqrt{-K}$.

Proof : The torsion τ of an asymptotic line $\vec{r} = \vec{r}(s)$ is

$$\tau = \left[\hat{N} \; \hat{N}' \; \vec{r}'\right] \qquad .....(8.21.1)$$

or $$\tau = \hat{N} \cdot \left(\hat{N}' \times \vec{r}' \right) \qquad .....(8.21.2)$$

Now $$\hat{N}' = \frac{d\hat{N}}{ds} = \frac{\partial \hat{N}}{\partial u}\frac{du}{ds} + \frac{\partial \hat{N}}{\partial v}\frac{dv}{ds} = \hat{N}_1 u' + \hat{N}_2 v' \qquad .....(8.21.3)$$

and $$\vec{r}' = \frac{d\vec{r}}{ds} = \frac{\partial \vec{r}}{\partial v}\frac{du}{ds} + \frac{\partial \vec{r}}{\partial v}\frac{dv}{ds} = \vec{r}_1\, u' + \vec{r}_2 v' \qquad .....(8.21.4)$$

$\therefore$ $$\hat{N}' \times \vec{r}' = \left(\hat{N}_1 u' + \hat{N}_2 v' \right) \times \left(\vec{r}_1\, u' + \vec{r}_2 v' \right)$$

$$= \left(\hat{N}_1 \times \vec{r}_1 \right) u'^2 + \left\{ \left(\hat{N}_1 \times \vec{r}_2 \right) + \left(\hat{N}_2 \times \vec{r}_1 \right) \right\} u'v' + \left(\hat{N}_2 \times \vec{r}_2 \right) v'^2. \qquad .....(8.21.5)$$

Using value of $\left(\hat{N}' \times \vec{r}' \right)$ from equation (8.21.5) in equation (8.21.2), we have

$$\tau = \hat{N} \cdot \left(\hat{N}_1 \times \vec{r}_1 \right) u'^2 + \left\{ \hat{N} \cdot \left(\hat{N}_1 \times \vec{r}_2 \right) + \hat{N} \cdot \left(\hat{N}_2 \times \vec{r}_1 \right) \right\} u'v' + \hat{N} \cdot \left(\hat{N}_2 \times \vec{r}_2 \right) v'^2.$$

$\Rightarrow$ $$\tau = \left[\hat{N}\ \hat{N}_1\ \vec{r}_1 \right] u'^2 + \left\{ \left[\hat{N}\ \hat{N}_1\ \vec{r}_2 \right] + \left[\hat{N}\ \hat{N}_2\ \vec{r}_1 \right] \right\} u'v' + \left[\hat{N}\ \hat{N}_2\ \vec{r}_2 \right] v'^2$$

or $$\tau = \frac{EM - FL}{H} u'^2 + \frac{EN - GL}{H} u'v' + \frac{FN - GM}{H} v'^2 \qquad .....(8.21.6)$$

by using Weingarton equations, where $\left[\hat{N}\ \hat{N}_1\ \vec{r}_1 \right] = \dfrac{EM - FL}{H}$ etc.

Now let the asymptotic lines be taken as parametric curves, then

$$L = 0,\ N = 0,\ M \neq 0. \qquad .....(8.21.7)$$

Using $L = 0$, $N = 0$ into equation (8.21.6), we get

$$\tau = \left(\frac{EM}{H} - 0 \right) u'^2 + 0 + \left(\frac{0 - GM}{H} \right) v'^2$$

or $$\tau = \frac{M}{H} \left(Eu'^2 - Gv'^2 \right) \qquad .....(8.21.8)$$

First for asymptotic line $\quad u = \text{constant},\ u' = 0 \qquad .....(8.21.9)$

then from equation (8.21.8), $$\tau = \frac{-MG}{H} v'^2 \qquad .....(8.21.10)$$

Now from the first fundamental form, we have

$$Eu'^2 + 2Fu'\ v' + Gv'^2 = 1 \qquad .....(8.21.11)$$

From this for the curve u = constant $\Rightarrow u' = 0$, we have from equation (8.21.11),

$$0 + 0 + Gv'^2 = 1 \Rightarrow Gv'^2 = 1,$$

then from equation (8.21.12), we get

$$\tau = \frac{-M}{H} \qquad .....(8.21.12)$$

Again for the asymptotic line v = constant, we have $v' = 0$ and from equation (8.21.11),

$$Eu'^2 = 1. \qquad .....(8.21.13)$$

And from (8.21.8),

$$\tau = \frac{ME}{H}u'^2, \Rightarrow \tau = \frac{M}{H}. \qquad .....(8.21.14)$$

The Gaussian curvature

$$\kappa = \frac{LN - M^2}{H^2} = \frac{-M^2}{H^2} \qquad [\because \text{ Here } L = 0, N = 0]$$

$$\Rightarrow \qquad \frac{M^2}{H^2} = -\kappa \Rightarrow \frac{M}{H} = \pm\sqrt{-\kappa}$$

or $\qquad \tau == \pm\sqrt{-\kappa} \qquad$ [by equation (8.21.12) and (8.21.14)]

Hence the theorem is proved.

8.21.1 Illustrative examples

***Ex.*4.** *Prove that on the surface $z = f(x, y)$ torsion of the asymptotic lines are*

$$\pm\frac{\sqrt{(s^2 - rt)}}{(1 + p^2 + q^2)}.$$

Sol. By Example 1, for this surface

$$z = f(x, y).$$

Let $\vec{r} = (x, y, z = f(x, y))$ be position vector of a point then

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial x} = (1, 0, p), \vec{r}_2 = \frac{\partial \vec{r}}{\partial y} = (0, 1, q)$$

$$\vec{r}_{12} = (0, 0, s), \vec{r}_{11} = (0, 0, r), \vec{r}_{22} = (0, 0, t)$$

therefore $\qquad E = \vec{r}_1 \cdot \vec{r}_1 = 1 + p^2, F = \vec{r}_1 \cdot \vec{r}_2 = pq, G = \vec{r}_2 \cdot \vec{r}_2 = 1 + q^2$

$$\hat{N} = \frac{\vec{r}_1 \times \vec{r}_2}{H} = \frac{(-p, -q, 1)}{H}, H^2 = EG - F^2 = 1 + p^2 + q^2$$

and $\quad L = \hat{N} \cdot \vec{r}_{11} = \dfrac{r}{\sqrt{1 + p^2 + q^2}}, M = \hat{N} \cdot \vec{r}_{12} = \dfrac{s}{\sqrt{1 + p^2 + q^2}}, N = \hat{N} \cdot \vec{r}_{22} = \dfrac{t}{\sqrt{1 + p^2 + q^2}}.$

Then on using values of L, M and N in the differential equation

$$Ldu^2 + 2M\,du\,dv + Ndv^2 = 0,$$

of the asymptotic line, we get $\qquad rdx^2 + 2s\,dx\,dy + tdy^2 = 0, \qquad$ (Here parameters are x, y)

Also $\quad K$ = Gaussian curvature $= \dfrac{LN - M^2}{H^2} = \dfrac{(rt - s^2)}{H^4}$

$\therefore$ Torsion $\tau = \pm\sqrt{-K} = \pm\sqrt{\dfrac{s^2 - st}{H^4}} = \pm\dfrac{\sqrt{s^2 - st}}{H^2}$

or $\tau = \pm\dfrac{\sqrt{s^2 - st}}{(1 + p^2 + q^2)}$,

which is the required result.

***Ex.*5.** *Show that the curvature of an asymptotic line may be expressed as*

$$\frac{(\vec{r}_1 \cdot \vec{r}\,')(\vec{r}_2 \cdot \vec{r}\,'') - (\vec{r}_2 \cdot \vec{r}\,')(\vec{r}_1 \cdot \vec{r}\,'')}{H}.$$

Sol. We know that for an asymptotic line curvature is given by

$$\kappa = \left[\hat{N}\,\vec{r}\,'\,\vec{r}\,''\right] \qquad \text{(box product)}$$

$$= \hat{N} \cdot (\vec{r}\,' \times \vec{r}\,'') = \frac{(\vec{r}_1 \times \vec{r}_2)}{H} \cdot (\vec{r}\,' \times \vec{r}\,'')$$

$$= \frac{1}{H}\{(\vec{r}_1 \times \vec{r}_2) \cdot (\vec{r}\,' \times \vec{r}\,'')\} = \frac{1}{H}\begin{vmatrix} \vec{r}_1 \cdot \vec{r}\,' & \vec{r}_1 \cdot \vec{r}\,'' \\ \vec{r}_2 \cdot \vec{r}\,' & \vec{r}_2 \cdot \vec{r}\,'' \end{vmatrix} \qquad \text{[By Lagrange's identity]}$$

$$= \frac{1}{H}\{(\vec{r}_1 \cdot \vec{r}\,')\,(\vec{r}_2 \cdot \vec{r}\,'') - (\vec{r}_2 \cdot \vec{r}\,')\,(\vec{r}_1 \cdot \vec{r}\,'')\}$$

Hence the result.

***Ex.*6.** *Prove that on the surface of revolution $x = u \cos v$, $y = u \sin v$, $z = f(u)$, the asymptotic lines are $f_{11}\, du^2 + u f_1\, dv^2 = 0$. Also show that the values of their torsions are*

$$\pm\frac{\sqrt{\left(-u^3 f_1 f_{11}\right)}}{u^2\left(1 + f_1^2\right)}.$$

Sol. The position vector $\vec{r}$ of any point on the given surface is

$$\vec{r} = (u \cos v, u \sin v, f(u))$$

Differentiating partially with respect to u and v

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial u} = (\cos v,\ \sin v,\ f_1(u)),\quad \vec{r}_2 = \frac{\partial \vec{r}}{\partial v} = (-u \sin v,\ u \cos v,\ 0)$$

Similarly $\dfrac{\partial^2 \vec{r}}{\partial u^2} = \vec{r}_{11} = (0,\ 0,\ f_{11}(u)),\quad \vec{r}_{12} = (-\sin v,\ \cos v,\ 0)$

and $\vec{r}_{22} = (-u \cos u, -u \sin v, 0)$

$$\vec{r}_1 \times \vec{r}_2 = (-u \cos v\, f_1,\ -u \sin v\, f_1,\ u)$$

$\therefore$ $E = \vec{r}_1^{\,2} = 1 + f_1^2,\quad F = \vec{r}_1 \cdot \vec{r}_2 = (-u \cos v \sin v + u \cos v \sin v) = 0$

$$G = \vec{r}_2^{\,2} = u^2,\ \hat{N} = \frac{\vec{r}_1 \times \vec{r}_2}{H} = \frac{(-u\cos v\, f_1,\ -u\sin v\, f_1,\ u)}{H}$$

where $\qquad H^2 = EG - F^2 = u^2\left(1 + f_1^2\right)$

Also $\qquad L = \hat{N}\cdot\vec{r}_{11} = \dfrac{uf_{11}}{H},\ M = \hat{N}\cdot\vec{r}_{12} = 0,\ N = \hat{N}\cdot\vec{r}_{22} = \dfrac{u^2 f_1}{H}$

Therefore the equation of asymptotic lines is

$$Ldu^2 + 2M\,du\,dv + N\,dv^2 = 0$$

which on using values of L, M, N reduces to the following form

$$\frac{uf_{11}}{H}du^2 + \frac{u^2 f_1}{H}dv^2 = 0 \quad \text{or} \quad f_{11}du^2 + uf_1 dv^2 = 0$$

which is the required equation.

Again $\qquad$ torsion $\tau = \pm\sqrt{-K} = \pm\sqrt{-\left(\dfrac{LN - M^2}{H^2}\right)}$ $\quad$ or $\quad \tau = \pm\dfrac{\sqrt{\left(-u^3 f_1 f_{11}\right)}}{u^2\left(1 + f_1^2\right)}.$

8.21.2 Self-learning exercise-2.

1. Define :

(i) Asymptotic direction on a surface $\vec{r} = \vec{r}(u,v)$.

(ii) asymptotic lines.

2. Write the differential equation of the asymptotic lines at a point (u, v) on the surface $\vec{r} = \vec{r}(u,v)$.

3. Write the condition for asymptotic lines to be orthogonal.

4. For Monge's form of surface $z = f(x, y)$ write the equation of asymptotic line.

5. Write formula for torsion and curvature of an asymptotic line $\vec{r} = \vec{r}(s)$ on surface $\vec{r} = \vec{r}(u,v)$.

6. State Beltrami-Enneper theorem.

8.22 Summary

1. In this unit you have studied about principal radii through a point of surface $z = f(x, y)$, relation between three fundamental forms, asymptotic lines and differential equation of asymptotic lines at a point (u, v) on surface $\vec{r} = \vec{r}(u,v)$ in curvilinear coordinates, curvature (κ) and torsion (τ) of an asymptotic line $\vec{r} = \vec{r}(s)$ on the surface $\vec{r} = \vec{r}(u,v)$.

2. Sufficient number of examples have been solved in the unit.

3. Fundamental magnitudes *E, F, G* and *L, M, N, H* and *T* along with differential equation which gives the principal radii, expressions of curvature and torsion will help the students to easily understand the text of the unit.

8.23 Answers to self-learning exercises

Self-learning exercise-1

1. $\rho^2 (rt - s^2) - H\rho \{1 + p^2) t + (1 + q^2) r - 2 pqs\} + H^4 = 0.$
2. $III = Adu^2 + 2B\, du\, dv + c\, du^2$, where $A = \hat{N}_1^2, B = \hat{N}_1 \cdot \hat{N}_2, C = \hat{N}_1^2$
3. $K \cdot I - J \cdot II + III = 0$
4. See §8.5.
5. $LDu\, du + M\,(Du\, dv + Dv\, du) + N\, Dv\, dv = 0.$
6. $M = 0$

Self-learning exercise-2

1. See §8.12 *(i)* and *(ii)*
2. $Ldu^2 + 2M\, du\, dv + N\, dv^2 = 0$
3. First curvature $J = 0$; *i.e.* surface is minimal.
4. $rdx^2 + 2s\, dx\, dy + tdy^2 = 0$
5. Torsion $\tau = \left[\hat{N}\ \hat{N}'\ \vec{r}'\right]$ (box product) and curvature $\kappa = \left[\hat{N}\ \vec{r}'\ \vec{r}''\right]$ (box product).
6. See §8.21.

8.24 Exercises

1. Prove that the asymptotic lines of the surface $x = v - 2u - e^{-u}$, $y = e^{v-u}$, $z = e^{u-v}$ lie on the cylinders $yz + ay - e^a = 0$, $xy + by + e^{-b} = 0$, where a, b are arbitrary constant.
2. Show that the asymptotic lines of helicoid $x = u \cos\theta$, $y = v \sin\theta$, $z = c\theta$ consist of the generators and the curves of intersection with coaxial right cylinders.
3. Derive the formula for torsion and curvature of an asymptotic line $\vec{r} = \vec{r}(s)$ on surface $\vec{r} = \vec{r}(u, v)$.
4. Derive the differential equation of the asymptotic lines at a point (u, v) on the surface $\vec{r} = \vec{r}(u, v)$ in curvilinear coordinates.

□ □ □

Unit 9 : Geodesics, Differential Equation of a Geodesic, Single Differential Equation of a Geodesic, Geodesic on a Surface of Revolution, Geodesic Curvature and Torsion, Gauss-Bonnet Theorem

Structure of the Unit

9.0 Objectives

This unit provides a general overview of geodesics, differential equation of a geodesic, single differential equation of a geodesic, geodesic on a surface of revolution, geodesic curvature and torsion, Gauss-Bonnet theorem. After reading this unit you will be able to learn :

1. about geodesics,
2. about the general differential equation of a geodesic on a surface,
3. about single differential equation of a geodesic when curve on surface is given by a single relation between the parameters u and v (either $v = v/u$ or $u = u/v$),
4. about geodesic on a surface of revolution,
5. about geodesic curvature and torsion on a surface,
6. about Gauss-Bonnet theorem, which gives us the relation between torsion τ of a curve c and torsion of its geodesic tangent.

9.1 Introduction

We know that in Euclidean space curves of shortest distance between any two points are straight lines. But curves on a surface $\vec{r} = \vec{r}\,(u, v)$ having shortest length are called geodesics. So roughly speaking "a geodesic on a surface may be defined as a curve of shortest distance between two points on that surface"

But to find the arc of shortest distance between two points on a given surface is a very complicated affair.

9.2 Normal property of a geodesic

The normal to the surface coincides with the principal normal to the curve (geodesic).

Consider a tightly stretched string on the smooth convex side of the surface to lie along the curve (geodesic) joining two points on the surface $\vec{r} = \vec{r}(u, v)$, very close to each other. The forces which keep this small string in equilibrium are the tensions at its extremities and the reaction normal to the surface. Because the string is very small so these tensions are in the osculating plane of the string and therefore for equilibrium the force of reaction must also lie in the same plane, which implies that the normal to the surface coincides with the principal normal to the curve (geodesic). This property is termed as the normal property of a geodesic.

9.3 Definition:

Geodesic (or geodesic curve) : Geodesic on a surface is defined as the curve of stationary length (rather than strictly shortest distance) on a surface between any two points in its plane.

or

A geodesic on a surface is a curve whose osculating plane at each point contains the normal to the surface at that point.

Hence the normal to the surface coincides with the principal normal to the geodesic.

9.4 General differential equations of geodesics on a surface $\vec{r} = \vec{r}(u, v)$

Let $\vec{r}$ be the position vector of any point $P\ (u, v)$ on the geodesic drawn on the surface $\vec{r} = \vec{r}(u, v)$. Let $\hat{n}$ and $\hat{N}$ be the principal normals to the geodesic curve at P and the normal to the given surface at the same point P, respectively. Then by definition of geodesic

$$\hat{n} = \hat{N} \qquad .....(9.4.1)$$

We know that $$\vec{r}' = \frac{d\vec{r}}{ds} = \hat{t},$$

so again differentiating with respect to s,

$$\vec{r}'' = \hat{t}' = \kappa\hat{n} \Rightarrow \vec{r}'' = \kappa\hat{N}, \qquad .....(9.4.2)$$

where κ is the curvature of geodesic at point P.

Now $\vec{r} = \vec{r}(u, v)$.

On differentiating with respect to s

$$\vec{r}' = \frac{d\vec{r}}{ds} = \frac{\partial \vec{r}}{\partial u}\frac{du}{ds} + \frac{\partial \vec{r}}{\partial v}\frac{dv}{ds} = \left(\vec{r}_1 u' + \vec{r}_2 v'\right). \qquad .....(9.4.3)$$

Again differentiating, we get

$$\vec{r}'' = \left\{\vec{r}_1 u'' + \frac{d(\vec{r}_1)}{ds}u'\right\} + \left\{\vec{r}_2 v'' + \frac{d(\vec{r}_2)}{ds}v'\right\}$$

$$= \left[\vec{r}_1 u'' + \left\{\frac{\partial \vec{r}_1}{\partial u}\frac{du}{ds} + \frac{\partial \vec{r}_1}{\partial v}\frac{dv}{ds}\right\}u'\right] + \left[\vec{r}_2 v'' + \left\{\frac{\partial \vec{r}_2}{\partial u}\frac{du}{ds} + \frac{\partial \vec{r}_2}{\partial v}\frac{dv}{ds}\right\}v'\right]$$

$$= \vec{r}_1 u'' + \left(\vec{r}_{11}u' + \vec{r}_{12}v'\right)u' + \vec{r}_2 v'' + \left(\vec{r}_{21}u' + \vec{r}_{22}v'\right)v'$$

$$= \vec{r}_1 u'' + \vec{r}_{11}u'^2 + \vec{r}_{12}u'v' + \vec{r}_2 u'' + \vec{r}_{12}u'v' + \vec{r}_{22}v'^2$$

$$\vec{r}'' = \vec{r}_1 u'' + \vec{r}_{11}u''^2 + 2\vec{r}_{12}u'v' + \vec{r}_2 v'' + \vec{r}_{22}v'^2. \qquad .....(9.4.4)$$

Using value of r'' from equation (9.4.2) in (9.4.4), we get

$$\kappa \hat{N} = \vec{r}_1 u'' + \vec{r}_{11}u'^2 + 2\vec{r}_{12}u'v' + \vec{r}_2 v'' + \vec{r}_{22}v'^2. \qquad .....(9.4.5)$$

Now taking scalar product of equation (9.4.5) with $\vec{r}_1$, we get

$$\kappa\left(\vec{r}_1 \cdot \hat{N}\right) = \left(\vec{r}_1 \cdot \vec{r}_1\right)u'' + \left(\vec{r}_1 \cdot \vec{r}_{11}\right)u'^2 + 2\left(\vec{r}_1 \cdot \vec{r}_{12}\right)u'v' + \left(\vec{r}_1 \cdot \vec{r}_2\right)v'' + \left(\vec{r}_1 \cdot \vec{r}_{22}\right)v'^2$$

or $$0 = Eu'' + \frac{1}{2}E_1 u'^2 + E_2 u'v' + Fv'' + \left(F_2 - \frac{1}{2}G_1\right)v'^2 \qquad .....(9.4.6)$$

$$\left(\because \vec{r}_1 \cdot \hat{N} = 0,\ E_1 = \frac{\partial E}{\partial u} = \frac{\partial}{\partial u}\left(\vec{r}_1^{\,2}\right) = 2\vec{r}_1 \cdot \frac{\partial \vec{r}_1}{\partial u} = 2\left(\vec{r}_1 \cdot \vec{r}_{11}\right) \atop E_2 = \frac{\partial E}{\partial v} = \frac{\partial}{\partial v}\left(\vec{r}_1^{\,2}\right) = 2\vec{r}_1 \cdot \frac{\partial \vec{r}_1}{\partial v} = 2\left(\vec{r}_1 \cdot \vec{r}_{12}\right) \text{ etc.}\right)$$

Now taking scalar product of equation (9.4.5) with $\vec{r}_2$, we get

$$\kappa\left(\vec{r}_2 \cdot \hat{N}\right) = \left(\vec{r}_2 \cdot \vec{r}_1\right)u'' + \left(\vec{r}_2 \cdot \vec{r}_{11}\right)u'^2 + 2\left(\vec{r}_2 \cdot \vec{r}_{12}\right)u'v' + \left(\vec{r}_2 \cdot \vec{r}_2\right)v'' + \left(\vec{r}_2 \cdot \vec{r}_{22}\right)v'^2$$

$$0 = Fu'' + \left(F_1 - \frac{1}{2}E_2\right)u'^2 + G_1 u'v' + Gv'' + \frac{1}{2}G_2 v'^2 \qquad .....(9.4.7)$$

$$\left(\because \vec{r}_2 \cdot \hat{N} = 0,\ \vec{r}_2 \cdot \vec{r}_{11} = F_1 - \frac{1}{2}E_2,\ G_1 = \frac{\partial G}{\partial u} = \left(2\vec{r}_2 \cdot \vec{r}_{12}\right) \text{ etc.}\right)$$

Hence equation (9.4.6) and (9.4.7) are the general differential equation of geodesics on a surface $\vec{r} = \vec{r}(u, v)$.

9.5 Canonical equations of a geodesic on the surface $\vec{r} = \vec{r}\,(u, v)$

The general differential equations of geodesics on surface $\vec{r} = \vec{r}\,(u, v)$ are

$$Eu'' + Fv'' + \frac{1}{2}E_1u'^2 + E_2u'v' + \left(F_2 - \frac{1}{2}G_1\right)v'^2 = 0 \qquad .....(9.5.1)$$

and
$$Fu'' + Gv'' + \left(F_1 - \frac{1}{2}E_2\right)u'^2 + G_1u'v' + \frac{1}{2}G_2v'^2 = 0 \qquad .....(9.5.2)$$

where
$$E_1 = \frac{\partial E}{\partial u}, E_2 = \frac{\partial E}{\partial v},\ G_1 = \frac{\partial G}{\partial u},\ G_2 = \frac{\partial G}{\partial v} \ \text{ etc.}$$

and
$$u' = \frac{\partial u}{\partial s},\ u'' = \frac{\partial^2 u}{\partial s^2}\ ,\ v' = \frac{\partial v}{\partial s},\ v'' = \frac{\partial^2 v}{\partial s^2} \ \text{ etc.}$$

These equation (9.5.1) and (9.5.2) may be written in a more compact form if we denote

$$T = \frac{1}{2}\left(Eu'^2 + 2Fu'v' + Gv'^2\right) \qquad .....(9.5.3)$$

where $u' = \dfrac{\partial u}{\partial s}$ and $v' = \dfrac{\partial v}{\partial s}$.

Differentiating partially equation (9.5.3) with respect to u' and v' respectively, we get

$$\frac{\partial T}{\partial u'} = Eu' + Fv' \qquad .....(9.5.4)$$

and
$$\frac{\partial T}{\partial v'} = Fu' + Gv' \qquad .....(9.5.5)$$

Now differentiating (9.5.3) partially with respect to u and v, we get

$$\frac{\partial T}{\partial u} = \frac{1}{2}\left(E_1u'^2 + 2F_1\,u'v' + G_1v'^2\right) \qquad .....(9.5.6)$$

and
$$\frac{\partial T}{\partial v} = \frac{1}{2}\left(E_2u'^2 + 2F_2\,u'v' + G_2v'^2\right) \qquad .....(9.5.7)$$

Differentiating equation (9.5.4) with respect to s, we get

$$\frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) = \frac{d}{ds}\left(Eu' + Fv'\right) = Eu'' + \left(\frac{\partial E}{\partial u}\frac{du}{ds} + \frac{\partial E}{\partial v}\frac{dv}{ds}\right)u' + Fv'' + \left(\frac{\partial F}{\partial u}\frac{du}{ds} + \frac{\partial F}{\partial v}\frac{dv}{ds}\right)v'$$

$$= Eu'' + \left(E_1u' + E_2v'\right)u' + Fv'' + \left(F_1u' + F_2\,v'\right)v'$$

or
$$\frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) = \left(Eu'' + Fv'' + E_1u'^2 + E_2u'v' + F_1u'v' + F_2u'^2\right) \qquad .....(9.5.8)$$

Similarly differentiating equation (9.5.5) with respect to s, we get

$$\frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) = \left(Fu'' + Gv'' + F_1u'^2 + F_2u'v' + G_1u'v' + G_2v'^2\right) \qquad .....(9.5.9)$$

Now subtracting equation (9.5.6) from equation (9.5.8), we get

$$\frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right)-\frac{\partial T}{\partial u}=Eu''+Fv''+\frac{1}{2}E_1u'^2+E_2u'v'+\left(F_2-\frac{1}{2}G_1\right)v'^2=0$$

$$\Rightarrow \qquad \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right)-\frac{\partial T}{\partial u}=0\,. \qquad .....(9.5.10)$$

Similarly subtracting equation (9.5.7) from equation (9.5.9), we get

$$\frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right)-\frac{\partial T}{\partial v}=\left(Fu''+Gv''+\left(F_1-\frac{1}{2}E_2\right)u'^2+G_1u'v'+\frac{1}{2}G_2v'^2\right)=0$$

$$\Rightarrow \qquad \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right)-\frac{\partial T}{\partial v}=0\,. \qquad .....(9.5.11)$$

Hence, the equations (9.5.10) and (9.5.11) are called the canonical equations of a geodesic.

9.6 Differential equations of a geodesic in Gauss coefficients

It will be another simple form of the general differential equations of a geodesic given below

$$Eu''+Fv''+\frac{1}{2}E_1\,u'^2+E_2\,u'v'+\left(F_2-\frac{1}{2}G_1\right)v'^2=0 \qquad .....(9.6.1)$$

and $$Fu''+Gv''+\left(F_1-\frac{1}{2}E_2\right)u'^2+G_1\,u'v'+\frac{1}{2}G_2\,v'^2=0 \qquad .....(9.6.2)$$

Multiplying equation (9.6.1) by G and equation (9.6.2) by F and subtracting, we get

$$\left(EG-F^2\right)u''+\frac{1}{2}\left(GE_1-2FF_1+FE_2\right)u'^2$$

$$+\left(GE_2-FG_1\right)u'v'+\frac{1}{2}\left(2GF_2-GG_1-FG_2\right)v'^2=0 \qquad .....(9.6.3)$$

Dividing whole equation by $(E\,G-F^2)$ $(=H^2)$, we get

$$u''+\frac{1}{2}\frac{\left(GE_1-2FF_1+FF_2\right)}{H^2}u'^2+\frac{\left(GE_2-FG_1\right)}{H^2}\,u'v'+\frac{1}{2}\frac{\left(2GF_2-GG_1-FG_2\right)}{H^2}v'^2=0$$

.....(9.6.4)

Using the Gauss coefficients into the equation (9.6.4), we get

and $$u''+lu'^2+2m\,u'v'+n\,v'^2=0, \qquad .....(9.6.5)$$

where $$l=\frac{1}{2H^2}\left(GE_1-2FF_1+FF_2\right),$$

$$m=\frac{1}{2H^2}\left(GE_2-FG_1\right),$$

$$n=\frac{1}{2H^2}\left(2GF_2-GG_1-FG_2\right).$$

Similarly multiplying equation (9.6.1) by F and (9.6.2) by E and subtracting, we get

$$u'' + \lambda u'^2 + 2\mu u'v' + \nu v'^2 = 0, \qquad .....(9.6.6)$$

where λ, μ, ν are Gauss coefficients given by

$$\lambda = \frac{1}{2H^2}\left(2EF_1 - EF_2 - FE_1\right),\ \mu = \frac{1}{2H^2}\left(EG_1 - FE_2\right),$$

$$\nu = \frac{1}{2H^2}\left(EG_2 - 2FE_2 + FG_1\right).$$

Hence equation (9.6.5) and (9.6.6) are differential equation of a geodesic in Gauss coefficients.

9.7 Single differential equation of geodesics on surface

When a curve on the surface $\vec{r} = \vec{r}\,(u, v)$ may be determined by a single relation between the parameters u and v either by $v = v\,(u)$ or by $u = u\,(v)$.

Proof : Let the equation of surface be $\vec{r} = \vec{r}\,(u, v)$, u, v are parameters. A curve on a surface may be determined by a single relation between the parameters u and v, which paves us our way to develope a single relation between the parameters from pair of following differential equation of a geodesic in Gauss coefficients given by

$$u'' + l u'^2 + 2m u'v' + n u'^2 = 0 \qquad .....(9.7.1)$$

and
$$v'' + \lambda u'^2 + 2\mu u'v' + \nu v'^2 = 0 \qquad .....(9.7.2)$$

First taking a single relation $\quad v = v\,(u). \qquad .....(9.7.3)$

then we shall reduce pair of equation (9.7.1) and (9.7.2) in to single differentiating equation.

On a geodesics, on differentiating equation (9.7.3) with respect to s

$$\frac{dv}{ds} = v' = \frac{dv}{du}\cdot\frac{du}{ds} = u'\frac{dv}{du}. \qquad .....(9.7.4)$$

Again differentiating, we get

$$v'' = u''\frac{dv}{du} + u'^2\frac{d^2v}{du^2}. \qquad .....(9.7.5)$$

Putting values from equation (9.7.4) and (9.7.5) in to equation (9.7.2), we get

$$\left(u''\frac{dv}{du} + u'^2\frac{d^2v}{du^2}\right) + \lambda\ u'^2 + 2\mu\left(u'\frac{dv}{du}\right)u' + \nu\left(u'\frac{dv}{du}\right)^2 = 0$$

or
$$u'^2\frac{d^2v}{du^2} + u''\frac{dv}{du} + \lambda u'^2 + 2\mu u'^2\frac{dv}{du} + \nu u'^2\left(\frac{dv}{du}\right)^2 = 0. \qquad .....(9.7.6)$$

Now from equation (9.7.1),

$$u'' = \left(-lu'^2 - 2mu'v' - nv'^2\right),\quad v' = u'\frac{dv}{du}$$

Using it in equation (9.7.6), we get

$$u'^2 \frac{d^2v}{du^2} - lu'^2 \frac{dv}{du} - 2mu'^2 \left(\frac{dv}{du}\right)^2 - nu'^2 \left(\frac{dv}{du}\right)^3 + \lambda u'^2 + 2\mu u'^2 \frac{dv}{du} + \nu u'^2 \left(\frac{dv}{du}\right)^2 = 0$$

.....(9.7.7)

Cancelling u'^2 throughout, we get

$$\frac{d^2v}{du^2} = n\left(\frac{dv}{du}\right)^3 + (2m - \nu)\left(\frac{dv}{du}\right)^2 + (l - 2\mu)\frac{dv}{du} - \lambda \quad (\text{as } u' \neq 0) \qquad .....(9.7.8)$$

which is single differential equation of geodesic.

Equation (9.7.8) is a second order ordinary non-linear differential equation, so it has a unique solution of v [$v = v(u)$] which takes a given value v_0 when $u = u_0$, also $\frac{dv}{du}$ takes a value

$$\left(\frac{dv}{du}\right)_{u=u_0} = \left(\frac{dv}{du}\right)_0 \text{ at } u = u_0.$$

Therefore through each point of a surface there passes a unique geodesic in a given direction.

If we start by taking the relation between parameters u and v, of the from $u = u(v)$, then we get the single differential of geodesic as

$$\frac{d^2u}{dv^2} = \lambda\left(\frac{du}{dv}\right)^3 + (2\mu - l)\left(\frac{du}{dv}\right)^2 + (\nu - 2m)\frac{du}{dv} - n. \qquad .....(9.7.9)$$

On comparing equation (9.7.9) with equation (9.7.8), we see that equation (9.7.9) can be obtained from (9.7.8) on interchanging u and v, and changing l, m, n by ν, μ, λ and vice-versa.

Remark :

1. Note that unlike lines of curvature and asymptotic lines, geodesics are not determined uniquely by the nature of surface.
2. Through any point there passes an infinite number of geodesic so each geodesic being decided by its direction at the point.
3. A geodesic is uniquely determined by an initial point and tangent at that point of the surface.
4. The differential equation of geodesic [equation (9.4.6) and (9.4.7)] are in terms of E, F, G and $E_1, E_2, F_1, F_2, G_1, G_2$, (derivatives of E, F, G). Therefore if a surface is deformed without stretching such that the length ds of each arc element does not change, then the geodesics remain geodesics on the deformed surface.

9.8 On the general surface, a necessary and sufficient condition that the curve $v = c$ (const.) be a geodesic is $EF_2 + FE_1 - 2EF_1 = 0$, when $v = c$, for all values of u.

On the curve $v = c$ (constant), we may take u as a parameter, so that $v = c$, $u = t$ (say)

$$\dot{v} = 0,\ \dot{u} = 1. \qquad \text{.....(9.8.1)}$$

On differentiating with respect to t

$$\dot{v} = 0,\ \dot{u} = 1 \qquad \left(\dot{u} = \frac{du}{dt}\right) \qquad \text{.....(9.8.2)}$$

Hence $$T = \frac{1}{2}\left(E\dot{u}^2 + 2F\dot{u}\,\dot{v} + G\dot{v}^2\right). \qquad \text{.....(9.8.3)}$$

Now differentiating (9.8.3) partially with respect to u and $\dot{u}$, we get

$$\frac{\partial T}{\partial u} = \frac{1}{2}\left[\frac{\partial E}{\partial u}\dot{u}^2 + 2\frac{\partial F}{\partial u}\dot{u}\,\dot{v} + \frac{\partial G}{\partial u}\dot{v}^2\right]$$

$$= \frac{1}{2}\left(E_1 \cdot 1 + 2F_1 \times 1 \times 0 + G_1 \times 0\right) \qquad \left(\because \dot{u} = 1,\ \dot{v} = 0,\ E_1 = \frac{\partial E}{\partial u} \text{ etc.}\right)$$

or $$\frac{\partial T}{\partial u} = \frac{1}{2}E_1 \qquad \text{.....(9.8.4)}$$

and $$\frac{\partial T}{\partial \dot{u}} = \frac{1}{2}\left(2E\dot{u} + 2F \cdot 1 \cdot \dot{v}\right) = E \qquad (\because \dot{u} = 1,\ \dot{v} = 0) \qquad \text{.....(9.8.5)}$$

Now differentiating equation (9.8.3) partially with respect to v and $\dot{v}$, we get

$$\frac{\partial T}{\partial v} = \frac{1}{2}\left(\frac{\partial E}{\partial v}\dot{u}^2 + 2\frac{\partial F}{\partial v}\dot{u}\,\dot{v} + \frac{\partial F}{\partial v}\dot{v}^2\right) = \frac{1}{2}E_2 \qquad \text{.....(9.8.6)}$$

[by equation (9.8.2) and $E_2 = \dfrac{\partial E}{\partial v}$]

$$\frac{\partial T}{\partial \dot{v}} = \frac{1}{2}\left(2F\dot{u} + 2G\dot{v}\right) = F \qquad \text{.....(9.8.7)}$$

Now $$U\text{ (say)} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \left(\frac{\partial T}{\partial u}\right) = \frac{d(E)}{dt} - \frac{1}{2}E_1 = \frac{dE}{du} - \frac{1}{2}E_1 = E_1 - \frac{E_1}{2}$$

$\Rightarrow$ $$U = \frac{E_1}{2} \qquad \text{.....(9.8.8)}$$

and $$V\text{ (say)} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{dF}{dt} - \frac{1}{2}E_2 = \left(F_1 - \frac{1}{2}E_2\right) \qquad \text{.....(9.8.9)}$$

$$\left(F_1 = \frac{dF}{dt} = \frac{dF}{du}\right)$$

Now the necessary and sufficient condition for the curve $u = u(t)$, $v = v(t)$ (t-parameter) to be geodesic is

$$V\frac{\partial T}{\partial \dot{u}} - U\frac{\partial T}{\partial \dot{v}} = 0, \qquad \text{for all values of } t. \qquad .....(9.8.10)$$

On using values from equation (9.8.5), (9.8.7), (9.8.8) and (9.8.9), we get

$$V\frac{\partial T}{\partial \dot{u}} - U\frac{\partial T}{\partial \dot{v}} = \left(F_1 - \frac{1}{2}E_2\right)E - \frac{1}{2}E_1 \cdot F = 0$$

$$\Rightarrow \qquad EF_2 + FE_1 - 2EF_1 = 0. \qquad .....(9.8.11)$$

Hence the result.

Note : If the parametric curves are orthogonal then $F = 0$ and obviously $F_1 = 0$, then from above equation (9.8.11) we get, $EE_2 + 0 - 0 = 0$

$\Rightarrow$ $E_2 = 0$, when $v = c$, for all values of u,

$\Rightarrow$ E is independent of v, so E is function of u only *i.e.,* $E = E(u)$.

9.9 The curve $u = c$ (constant) is a geodesic if and only if $GG_1 + FG_2 - 2GF_2 = 0$.

On the curve $u = c$, we may take v as a parameter, so

$u = c, u = t$ t-parameter, then

$\dot{u} = 0, \dot{v} = 1$.

Now proceeding exactly on the same lines as in 9.8, we get

$$GG_1 + FG_2 - 2\,GF_2 = 0. \qquad .....(9.9.1)$$

Note : If the parametric curves are orthogonal then $F = 0$ and $F_2 = 0$, then from above equation (9.9.1), we get

$$GG_1 = 0 \quad \Rightarrow \quad G_1 = 0 \quad \Rightarrow \quad \frac{\partial G}{\partial u} = 0 \quad \Rightarrow \quad \frac{dG}{dc} = 0 \quad \Rightarrow \quad G \text{ is independent of } u.$$

Hence G is function of v only *i.e.,* $G = G(v)$.

9.9.1 Self-learning exercise-1

1. What is normal property of geodesic ?
2. Define geodesic.
3. Write the general differential equations of geodesics on a surface $\vec{r} = \vec{r}(u, v)$.
4. Write the canonical equations of a geodesic on the surface $\vec{r} = \vec{r}(u, v)$.
5. Like lines of curvature and asymptotic lines, can geodesic be determined uniquely by the nature of surface ?
6. On the general surface $\vec{r} = \vec{r}(u, v)$ what is the necessary and sufficient condition that the curve $v = c$ (constant) be a geodesic ?

9.9.2 Illustrative Examples

***Ex.*1.** *Prove that the curves $u + v$ = constant are geodesics on a surface with metric* $(1 + u^2)\, du^2 - 2uv$ *and* $v + (1 + v^2)\, dv^2$.

Sol. The parametric equation of the given curve $u + v$ = constant, can be taken as $u = t$, $v = c - t$. .....(1)

So on differentiating these with respect to t

$$\dot{u} = 1, \quad \dot{v} = -1 \qquad .....(2)$$

Here $$E = 1 + u^2, \quad F = -uv, \quad G = (1 + v^2) \qquad .....(3)$$

Now we know that

$$T = \frac{1}{2}\left(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2\right) \qquad .....(4)$$

Using values of E, F and G from equation (3), we get

$$T = \frac{1}{2}\left[\left(1+u^2\right)\dot{u}^2 - 2uv\dot{u}\dot{v} + 1\left(1+v^2\right)\dot{v}^2\right] \qquad .....(5)$$

On differentiating (5) with respect to u and v, we get

$$\frac{\partial T}{\partial u} = u\dot{u}^2 - v\dot{u}\,\dot{v} = t\cdot 1 - (c-t)\cdot(-1) = c \qquad ......(6)$$

$$\frac{\partial T}{\partial v} = -u\,\dot{u}\,\dot{v} + v\dot{v}^2 = t + c - t = c \qquad ......(7)$$

Now differentiating (5) with respect to $\dot{u}$ and $\dot{v}$, we get

$$\frac{\partial T}{\partial \dot{u}} = \left(1+u^2\right)\dot{u} - uv\dot{v} = \left(1+t^2\right) - t(c-t)(-1) = 1 + ct \qquad ......(8)$$

$$\frac{\partial T}{\partial \dot{v}} = -uv\dot{u} + \left(1+v^2\right)\dot{v} = -t(c-t) + \left\{1 + (c-t)^2\right\}(-1) = \left(ct - 1 - c^2\right) \qquad ......(9)$$

Now $$U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{d}{dt}(1+ct) - c = c - c = 0 \qquad ......(10)$$

and $$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{d}{dt}\left(ct - 1 - c^2\right) - c = 0 \qquad ......(11)$$

then $$V\frac{\partial T}{\partial \dot{u}} - U\frac{\partial T}{\partial \dot{v}} = 0(1-ct) - 0\left(ct - 1 - c^2\right) = 0$$ [by equation (8), (9), (10) and (11)]

Hence the relation $$V\frac{\partial T}{\partial \dot{u}} - U\frac{\partial T}{\partial \dot{v}} = 0,$$

for all values of t.

Therefore the given curve $u + v$ = constant is a geodesic.

Ex.**2.** *Prove that the curves of the family* $\dfrac{v^3}{u^2}$ = *constant, are geodesics on a surface with metric* $v^2\,du^2 - 2\,uv\,dv\,du + 2\,u^2\,dv^2$; $(u > 0,\ v > 0)$.

Sol. The parametric equation of the given curve can be conveniently chosen to be

$$u = ct^3,\quad v = ct^2,\quad c = \text{(constant)} \qquad .....(1)$$

The differentiating equation (1) with respect to t

$$\dot{u} = 3ct^2,\quad \dot{v} = 2ct \qquad .....(2)$$

then $$E = v^2,\quad F = -\,uv,\quad G = 2u^2 \qquad .....(3)$$

Therefore, $$T = \frac{1}{2}\left(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2\right) \qquad .....(4)$$

Using values of E, F and G from (3) in (4), we get

$$T = \frac{1}{2}\left(v^2\dot{u}^2 - 2uv\,\dot{u}\dot{v} + 2u^2\dot{v}^2\right) \qquad .....(5)$$

On differentiating equation (5) partially with respect to u

$$\frac{\partial T}{\partial u} = -\,v\dot{u}\dot{v} + 2u\dot{v}^2 = -\left(ct^2\right)\left(3ct^2\right)\left(2ct\right) + 2\left(ct^3\right)\left(2ct\right)^2$$

$$= -6c^3t^5 + 8c^3t^5 = 2c^3t^5 \qquad \text{[by equation (1) and (2)]}$$

Now partially differentiating equation (5) with respect to v, we get

$$\frac{\partial T}{\partial v} = v\dot{u}^2 - u\dot{u}\dot{v} = 3c^3t^6 \qquad \text{[by (1), (2)]}$$

Now differentiating equation (5) partially with respect to $\dot{u}$ and $\dot{v}$, we get

$$\frac{\partial T}{\partial \dot{u}} = v^2\dot{u} - uv\dot{v} = c^3t^6$$

and $$\frac{\partial T}{\partial \dot{v}} = -uv\dot{u} + 2u^2\dot{v} = c^3t^7$$

Now $$U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{d}{dt}\left(c^3t^6\right) - 2c^3t^5 = 4c^3t^5$$

and $$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{d}{dt}\left(c^3t^7\right) - 3c^3t^6 = 4c^3t^6$$

then $$\left(V\frac{\partial T}{\partial \dot{u}} - U\frac{\partial T}{\partial \dot{v}}\right) = \left(4c^3t^6\right)\left(c^3t^6\right) - \left(4c^3t^5\right)\left(c^3t^7\right) = 4c^6t^{12} - 4c^6t^{12} = 0$$

Hence $$\left(V\frac{\partial T}{\partial \dot{u}} - U\frac{\partial T}{\partial \dot{v}}\right) = 0, \text{ for all values of } t.$$

This shows that the given family $\dfrac{v^3}{u^2}$ = constant (c) is a geodesic for all values of c.

9.10 Differential equation of a geodesics on the surface $F(x, y, z) = 0$.

We have discussed about the differential equation of a geodesic on a surface whose equation was given in parametric form. Now we shall find equation of geodesic when equation of surface is given in the implicit form $F(x, y, z) = 0$.

Equation of surface is $$F(x, y, z) = 0. \qquad .....(9.10.1)$$

We know that if $\vec{r}$ be the position vector of a point

then $$\vec{r}' = \frac{dr}{ds} = \hat{t}$$

Again differentiating with respect to s

$$\vec{r}'' = \hat{t}' = \kappa n'$$

or $$\vec{r}'' = \kappa \hat{n} = \kappa \hat{N}, \quad (\because \text{ for geodesic } \hat{n} = \hat{N}) \qquad .....(9.10.2)$$

where κ is the curvature.

But $$r = \left(x\hat{i} + y\hat{j} + z\hat{k}\right) \Rightarrow \vec{r}'' = \left(x''\hat{i} + y''\hat{j} + z''\hat{k}\right), \qquad .....(9.10.3)$$

where $x'' = \dfrac{d^2x}{ds^2}$ etc.

and $\hat{N}$ be normal at a point on the surface $F(x, y, z) = 0$, so

$$\hat{N} = \frac{\hat{i}\dfrac{\partial F}{\partial x} + \hat{j}\dfrac{\partial F}{\partial y} + \hat{k}\dfrac{\partial F}{\partial z}}{\sqrt{(\partial F/\partial x)^2 + (\partial F/\partial y)^2 + (\partial F/\partial z)^2}} = \frac{\left(\hat{i}\dfrac{\partial F}{\partial x} + \hat{j}\dfrac{\partial F}{\partial y} + \hat{k}\dfrac{\partial F}{\partial z}\right)}{\sqrt{F_x^2 + F_y^2 + F_z^2}} \qquad .....(9.10.4)$$

where $F_x = (\partial F/\partial x)$ etc.

Therefore $$r'' = \kappa \hat{N} \Rightarrow \left(\frac{d^2x}{ds^2}\hat{i} + \frac{d^2y}{ds^2}\hat{j} + \frac{d^2z}{ds^2}\hat{k}\right) = \frac{\kappa\left(\hat{i}\dfrac{\partial F}{\partial x} + \hat{j}\dfrac{\partial F}{\partial y} + \hat{k}\dfrac{\partial F}{\partial z}\right)}{\sqrt{F_x^2 + F_y^2 + F_z^2}} \qquad .....(9.10.5)$$

Comparing coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$, we get

$$\frac{d^2x}{ds^2} = \frac{\kappa(\partial F/\partial x)}{\sqrt{F_x^2 + F_y^2 + F_z^2}} \Rightarrow \frac{\left(d^2x/ds^2\right)}{(\partial F/\partial y)} = \frac{\kappa}{\sqrt{F_x^2 + F_y^2 + F_z^2}} \qquad .....(9.10.6)$$

Similarly $$\frac{\left(d^2y/ds^2\right)}{(\partial F/\partial y)} = \frac{\kappa}{\sqrt{F_x^2 + F_y^2 + F_z^2}} \qquad .....(9.10.7)$$

$$\frac{\left(d^2z/ds^2\right)}{(\partial F/\partial y)} = \frac{\kappa}{\sqrt{F_x^2 + F_y^2 + F_z^2}}. \qquad .....(9.10.8)$$

Hence from equation (9.10.6) to (9.10.8), we get

$$\frac{\left(d^2x/ds^2\right)}{F_x}=\frac{\left(d^2y/ds^2\right)}{F_y}=\frac{\left(d^2z/ds^2\right)}{F_z}. \qquad .....(9.10.9)$$

If the integral of one of these equations is found, it will contain two arbitrary constants and with the equation to the surface $F(x, y, z) = 0$, will represent the geodesics.

9.11 Differential equation of geodesics on the surface $z = f(x, y)$, the Monge's form.

Taking x, y as parameters, let $\vec{r}$ be the position vector of any point on the surface $z = f(x, y)$, then

$$\vec{r} = (x,\ y,\ f(x, y) = z). \qquad .....(9.11.1)$$

On differentiating with respect to x and y partially. we get

$$\vec{r}_1=\frac{\partial \vec{r}}{\partial x}=(1,0,p), \qquad \vec{r}_2=\frac{\partial r}{\partial y}=(0,1,q), \qquad p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{\partial y}$$

$$\vec{r}_{11}=\frac{\partial^2 \vec{r}}{\partial x^2}=(0,0,r), \quad \vec{r}_{12}=\frac{\partial^2 r}{\partial x\partial y}=(0,\ 0,\ s), \quad \vec{r}_{22}=\frac{\partial^2 r}{\partial y^2}=(0,0,t). \qquad .....(9.11.2)$$

Hence $E = \vec{r}_1\cdot\vec{r}_1 = 1 + p^2, F = \vec{r}_1\cdot\vec{r}_2 = pq, G = \vec{r}_2\cdot\vec{r}_2 = 1 + q^2$

$$H^2 = EG - F^2 = (1 + p^2)(1 + q^2) - p^2 q^2 = (1 + p^2 + q^2) \qquad .....(9.11.3)$$

Further, the Gauss coefficients are obtained as

$$l=\frac{pr}{H^2},\ m=\frac{ps}{H^2},\ n=\frac{pt}{H^2}$$

and $$\lambda=\frac{qr}{H^2},\ \mu=\frac{qs}{H^2},\ \upsilon=\frac{qt}{H^2}, \qquad .....(9.11.4)$$

where $$l=\frac{1}{2}H^2\left(GE_1-2FE_1+FE_2\right) \text{ etc.}$$

Hence the single differential equation

$$\frac{d^2v}{du^2}=n\left(\frac{dv}{du}\right)^3+(3m-\upsilon)\left(\frac{dv}{du}\right)+(l-2\mu)\frac{dv}{du}-\lambda$$

with $u = x,$ and $v = y$ becomes

$$H^2\frac{d^2y}{dx^2}=pt\left(\frac{dy}{dx}\right)^3+(2ps-qt)\left(\frac{dy}{dx}\right)^2+(pr-2qs)\frac{dy}{dx}-qr$$

$$\left(1+p^2+q^2\right)\frac{d^2y}{dx^2}=\left(p\frac{dy}{dx}-q\right)\cdot\left\{t\left(\frac{dy}{dx}\right)^2+2s\frac{dy}{dx}+r\right\} \qquad .....(9.11.5)$$

which is differential equation of geodesic for surface $z = f(x, y)$.

9.12 Geodesic on a surface of revolution

Let the surface of revolution be

$$x = u\cos\theta,\ y = u\sin\theta\ ,\ z = f(u) \qquad \text{.....(9.12.1)}$$

Let $\vec{r}$ be position vector of a point on this surface

then $$\vec{r} = (u\cos\theta,\ u\sin\theta, f(u)). \qquad \text{.....(9.12.2)}$$

Differentiating with respect u and θ partially, we get

$$\vec{r}_1 = \frac{\partial\vec{r}}{\partial u} = (\cos\theta, \sin\theta, f'),\quad \vec{r}_2 = \frac{\partial\vec{r}}{\vec{r}\theta} = (-u\sin\theta, u\cos\theta, 0)$$

and $$\vec{r}_{11} = \frac{\partial^2\vec{r}}{\partial u^2} = (0, 0, f''),\quad \vec{r}_{12} = \frac{\partial^2\vec{r}}{\partial r\partial\theta} = (-\sin\theta, \cos\theta, 0),$$

$$\vec{r}_{22} = \frac{\partial^2\vec{r}}{\partial\theta^2} = (-u\cos\theta, -u\sin\theta, 0) \qquad \text{.....(9.12.3)}$$

then $$E = \vec{r}_1\cdot\vec{r}_1 = (1+f'^2),\quad F = \vec{r}_2\cdot\vec{r}_2 = 0,\quad G = \vec{r}_2\cdot\vec{r}_2 = u^2$$

$$\vec{r}_1\times\vec{r}_2 = u(-f'\cos\theta, -f'\sin\theta, 1),\ H^2 = EG - F^2 = u^2\,(1+f'^2) \qquad \text{......(9.12.4)}$$

Also $$\lambda = \frac{1}{2H^2}(2EF_1 - EE_2 + FE_1) = -\frac{EE_2}{2H^2} \qquad (\because F = 0)$$

but $E_2 = (\partial E/\partial\theta) = 0$

$$\therefore\quad \lambda = 0 \text{ and } \mu = \frac{1}{2H^2}(EG_1 - FE_2) = \frac{1}{u},\ \nu = \frac{1}{2H^2}(EG_2 - 2FF_2 - FG_1) = 0. \qquad \text{......(9.12.5)}$$

For the present form of geodesic, we use the equation of geodesics given below

$$v'' + \lambda u'^2 + 2\mu u'v' + \nu v'^2 = 0,$$

$$\frac{d^2\theta}{ds^2} + 0 + \frac{2}{u}\frac{du}{ds}\frac{d\theta}{ds} + 0 = 0, \qquad \text{where } v'' = \frac{d^2v}{ds^2} \text{ etc.} \qquad \text{......(9.12.6)}$$

Multiplying by u^2, we get $$u^2\frac{d^2\theta}{ds^2} + 2u\frac{du}{ds}\frac{d\theta}{ds} = 0,$$

which can be expressed as $$\frac{d}{ds}\left(u^2\frac{d\theta}{ds}\right) = 0\cdot \qquad \text{......(9.12.7)}$$

On integrating, we get $$u^2\frac{d\theta}{ds} = h_1.\ \text{(say)} \qquad \text{......(9.12.8)}$$

which is called the first integral of the equation of geodesic, where h_1 is constant of integration.

It is independent of form of $f(u)$.

Now we shall find the complete integral of equation (9.12.8), for this we proceed as follows :

The metric of surface $\vec{r} = \vec{r}(u, v)$ is

$$ds^2 = Edu^2 + 2Fdu\,dv + Gdv^2. \quad \text{......(9.12.9)}$$

In the present case, $u = u$, $v = \theta$, then above equation reduces to,

$$ds^2 = Edu^2 + 2Fdu\,d\theta + G\,d\theta^2. \quad \text{......(9.12.10)}$$

Using values of E, F and G from equation (9.12.4), we get

$$ds^2 = (1 + f'^2)\,du^2 + u^2\,d\theta^2 \quad \text{......(9.12.11)}$$

Now equation (9.12.8) may be expressed as

$$u^2\,d\theta = h_1 ds$$

or $$u^4\,d\theta^2 = h_1^2 ds^2$$

or $$u^4\,d\theta^2 = h_1^2\left\{\left(1+f'^2\right)du^2 + u^2 d\theta^2\right\}$$

$$u^4\,d\theta^2 - h_1^2 u^2\,d\theta^2 = h_1^2\left(1+f'^2\right)du^2$$

$$u^2\,(u^2 - h_1^2)\,d\theta^2 = h_1^2\left(1+f'^2\right)du^2$$

or $$u\left(u^2 - h_1^2\right)^{1/2} d\theta = \pm h_1\left(1+f'^2\right)^{1/2} du$$

$\Rightarrow$ $$d\theta = \pm\frac{h_1}{u}\left(\frac{1+f'^2}{u^2-h_1^2}\right)^{1/2} du \quad \text{......(9.12.12)}$$

On integrating, we get

$$\theta = \pm\frac{h_1}{1}\int\frac{1}{u}\left(\frac{1+f'^2}{u^2-h_1^2}\right)^{1/2} du + c_1 \quad \text{......(9.12.13)}$$

where c_1 is again another arbitrary constant of integration.

As the differential equation of geodesic is of second order and its solution (9.12.13) involves two arbitrary constants h_1 and c_1.

Hence, it is the complete integral of the differential equation of geodesic on the surface of revolution.

Note : If the arbitrary constant $h_1 = 0$ in equation (9.12.13), then $\theta = \pm 0 + c_1 \Rightarrow \theta = c_1$ (constant), which in this case is geodesics and are the meridians.

Hence every meridian is a geodesic on the surface of revolution.

9.13 Clairut's theorem

If the geodesic on the surface of revolution intersects the meridian (θ = constant) at any point P at an angle ψ, then $u \sin\psi$ is constant, where u is the distance of point P from the axis.

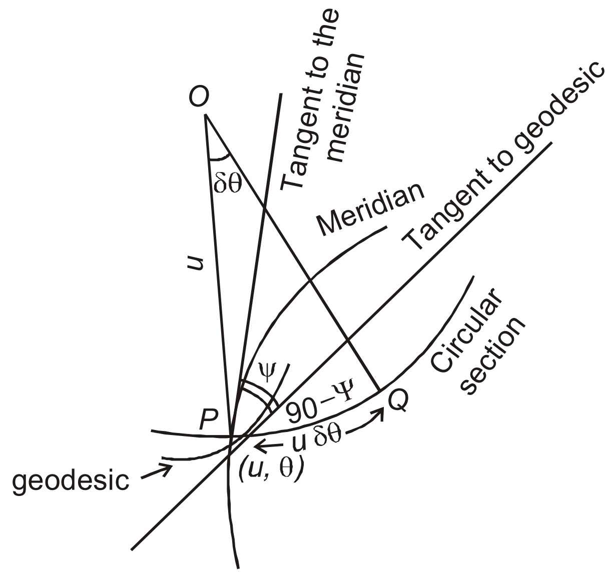


Fig . 9.1

Geometrical derivation of the result :

The projection of arc ds of the geodesic on the circular section (See Fig. 9.1) through point P is

$$ds \cos (90 - \psi) = PQ \qquad .....(9.13.1)$$

This arc PQ subtend an angle $d\theta$ at the centre O of the circular section of radius $\hat{u}$, but by formula,

$$\text{angle} = \frac{\text{arc}}{\text{radius}}$$

$$d\theta = \frac{arc\,PQ}{u} \Rightarrow \text{arc}\,PQ = u\,d\theta \qquad .....(9.13.2)$$

Using (9.13.2) into (9.13.1), we get

$$ds \sin \psi = u\,d\,\theta$$

$$\Rightarrow \qquad \sin \psi = u \frac{d\theta}{ds} \qquad .....(9.13.3)$$

But $$u^2 \frac{d\theta}{ds} = h_1 \text{ (constant)}$$

$$\Rightarrow \qquad u\left(u \frac{d\theta}{ds}\right) = h_1$$

$$\Rightarrow \qquad u \sin \psi = h_1 = \text{constant}, \qquad .....(9.13.4)$$

this is called Clairut's theorem.

It may be stated explicitly as follows :

At every point of a geodesic on a surface of revolution, the radius (u) of the circle of latitude multiplied by the sine of the angle between the geodesic and the meridian is constant.

Remark : From above equation (9.13.4), we draw one important conclusion that h_1 is the minimum distance from the axis of a point on the geodesic, and is attained at the point where the geodesic cuts a meridian at right angles.

9.14 A geodesic on a surface of revolution cuts the meridian at a constant angle, then the surface is a right circular cylinder.

The equation of the surface of revolution is given by

$$x = u\cos\theta, \quad y = u\sin\theta, \quad z = f(u). \qquad .....(9.14.1)$$

Then by Clairut's theorem, if a geodesic on a surface of revolution cuts the medium through any point P on it at angle ψ, then we have

$$u\sin\psi = h_1 \text{ (constant)}, \qquad .....(9.14.2)$$

where $u = \sqrt{x^2 + y^2}$, is the distance of the point P from the axis.

But, it is given that a geodesic cuts all the meridians at constant angle, so ψ = constant and

$$u = \frac{h_1}{\sin\psi} = \frac{\text{constant}}{(\text{constant})} = (\text{constant}) = \text{a (say)}$$

$$\Rightarrow \qquad u^2 = a^2$$

$$\Rightarrow \qquad x^2 + y^2 = a^2. \qquad .....(9.12.4)$$

which is equation of a right circular cylinder, whose axis is z-axis and radius is a.

9.15 A curve on sphere is a geodesic if and only if it is a great circle.

***(i)* The condition is necessary :** Let C be a geodesic curve on a sphere. Let $\hat{n}$ be the normal to curve C at a point P and let $\hat{N}$ be the normal to the surface of the sphere at P. Then by the normal property of the geodesic

$$\hat{n} = \hat{N} \cdot \qquad .....(9.15.1)$$

At point of the sphere, the normals pass through the centre of sphere, also the principal normal at every point of C will pass through the centre of sphere, which is a fixed point for the sphere.

Let $\vec{r}$ be the position vector of point P on the geodesic and let $\hat{a}$ be the position vector of the centre of the sphere.

Then $$\vec{r} - \hat{a} = \lambda\hat{n}, \qquad .....(9.15.2)$$

where λ is a scalar parameter which is function of arc lengths.

Now differentiating equation (9.15.2) with respect to s, we have

$$\frac{d\vec{r}}{ds} = \lambda\frac{d\hat{n}}{ds} + \hat{n}\frac{d\lambda}{ds}$$

$$\Rightarrow \qquad \hat{t}' = \lambda\left(\tau\hat{b} - \kappa\hat{t}\right) + \hat{n}\frac{d\lambda}{ds} \qquad \text{[by Serret-Frenet's formulae]}$$

or $$\hat{t}' = (\lambda\tau)\hat{b} - \lambda\kappa\hat{t} + \hat{n}\frac{d\lambda}{ds} \qquad .....(9.15.3)$$

Equating the coefficients of $\hat{b}$ on both sides, we get

$$\lambda\tau = 0 \Rightarrow \tau = 0 \text{ as } \lambda \neq 0 \Rightarrow \text{torsion} = 0. \qquad .....(9.15.4)$$

Hence the curve C is a plane curve. So the curve C (geodesic) on the sphere is a plane curve whose normals at each point of it pass through the centre of the sphere. Hence C is a section of the sphere by plane passing through its center. That is, C is a great circle.

***(ii)* The condition is sufficient :** If C is a great circle on the sphere, then at each point of C the principal normal to C coincides with the surface normal to the sphere. Therefore by normal property curve C is a geodesic.

9.16 Geodesic curvature and torsion of a geodesic

Geodesic (tangential) curvature : Let S be any surface $\vec{r} = \vec{r}(u, v)$ and C be a curve on this surface. Let $\vec{r}$ be position vector of a point P on the curve C.

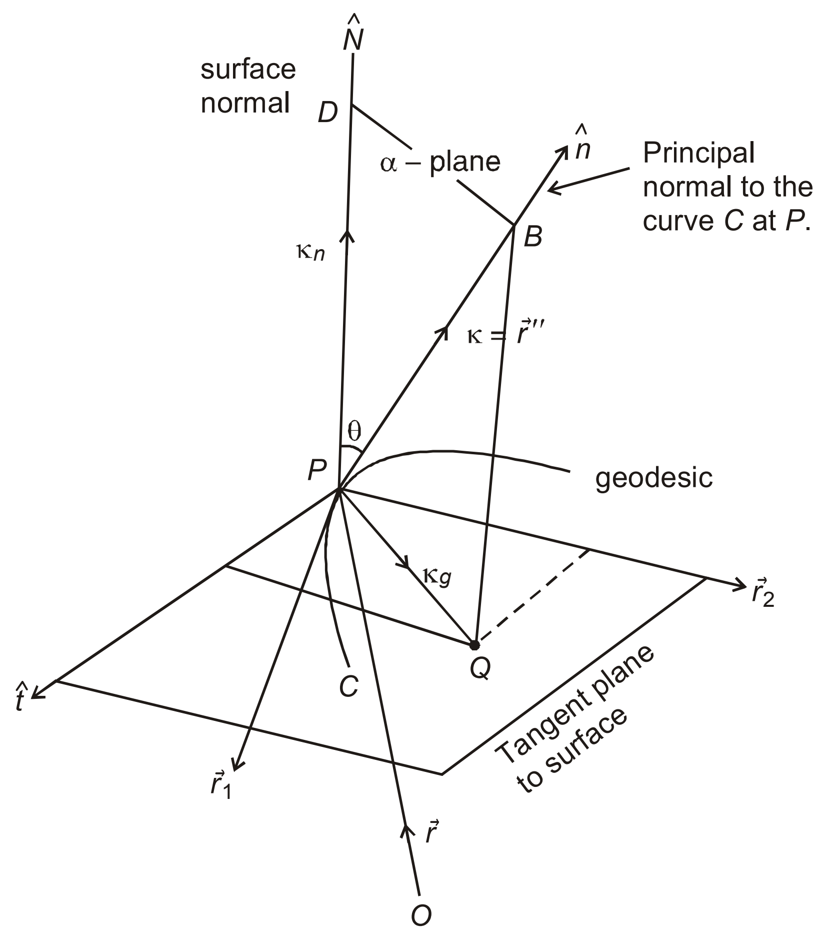


Fig 9.2

Then the curvature vector of a curve C on surface at a point P, with the tangent direction $\hat{t}$ is

$$\frac{d\hat{t}}{ds} = \vec{r}'' = \kappa\hat{n} = \vec{\kappa} \qquad .....(9.16.1)$$

and it lies in the plane (see α-plane in Fig. 9.2), through P perpendicular to $\hat{t}$, this plane also contains the surface normal $\hat{N}$.

Then according to the Meusnier's theorem, the projection of the curvature vector $\vec{\kappa}$ on this surface normal $\hat{N}$ is, the curvature vector of the normal section in direction $\hat{t}$. It is represented by $\vec{\kappa}_n$. Our main aim is to study the projection of κ on the tangent plane, which is called the vector of tangential curvature. It has been denoted by symbol κ_g. So we have the relation

$$\vec{\kappa} = \vec{\kappa}_n + \vec{\kappa}_g. \qquad .....(9.16.2)$$

The equation (9.16.2) implies that the curvature vector is the sum of the normal curvature and tangential curvature vectors.

The tangential curvature vector $\left(\vec{\kappa}_g\right)$ is generally called as geodesic curvature vector.

9.17 An expression for κ_g and that it is intrinsic

We know that the curvature vector $\vec{r}''$ at any point P on a curve C can be expressed as a linear combination of vectors

$$\hat{N}, \vec{r}_1\left(=\frac{\partial \vec{r}}{\partial u}\right), \quad \text{and} \quad \vec{r}_2\left(=\frac{\partial \vec{r}}{\partial v}\right)$$

as given below $\qquad \vec{r}'' = \kappa_n\hat{N} + \lambda\vec{r}_1 + \mu\vec{r}_2, \qquad .....(9.17.1)$

where λ and μ are scalars.

Taking dot product by $\hat{N}$, we get

$$\vec{r}''\cdot\hat{N} = \kappa_n\left(\hat{N}\cdot\hat{N}\right) + \lambda\left(\vec{r}_1\cdot\hat{N}\right) + \mu\left(\vec{r}_2\cdot\hat{N}\right)$$

$$\Rightarrow \qquad \vec{r}''\cdot\hat{N} = \kappa_n + \lambda\cdot 0 + \mu\cdot 0 \qquad \left[\because \hat{N}\cdot\hat{N} = 1,\ \vec{r}_1\cdot\hat{N} = 0 = \vec{r}_2\cdot\hat{N}\right]$$

$$\Rightarrow \qquad \kappa_n = \vec{r}''\cdot\hat{N} \qquad .....(9.17.2)$$

So it is deduced that $\kappa_g = \lambda\,\vec{r}_1 + \mu\,\vec{r}_2 \qquad .....(9.17.3)$

Now taking dot product of equation (9.17.1) by $\vec{r}_1$, we get

$$\vec{r}''\cdot\vec{r}_1 = \kappa_n\left(\hat{N}\cdot\vec{r}_1\right) + \lambda\left(\vec{r}_1\cdot\vec{r}_1\right) + \mu\left(\vec{r}_2\cdot\vec{r}_1\right)$$

$$= 0 + \lambda\,\vec{r}_1^{\,2} + \mu\left(\vec{r}_2\cdot\vec{r}_1\right)$$

$$\text{or} \qquad \left(\vec{r}''\cdot\vec{r}_1\right) = \lambda E + \mu F \qquad \left[\because E = \vec{r}_1^{\,2},\ F = \vec{r}_1\cdot\vec{r}_2\right] \qquad .....(9.17.4)$$

$$\text{but} \qquad \vec{r}''\cdot\vec{r} = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = U,$$

then from equation (9.17.4),

$$U = \lambda E + \mu F. \qquad \text{.....(9.17.5)}$$

Now taking dot product of equation (9.17.1) by $\vec{r}_2$, we get

$$\vec{r}'' \cdot \vec{r}_2 = \kappa_n \left(\hat{N} \cdot \vec{r}_2\right) + \lambda\left(\vec{r}_1 \cdot \vec{r}_2\right) + \mu\left(\vec{r}_2 \cdot \vec{r}_2\right)$$

or $\quad \vec{r}'' \cdot \vec{r} = 0 + \lambda F + \mu G \qquad \left[\because G = \vec{r}_2 \cdot \vec{r}_2\right]$

and $\quad \vec{r}'' \cdot \vec{r} = \dfrac{d}{ds}\left(\dfrac{\partial T}{\partial v'}\right) - \dfrac{\partial T}{\partial v} = V$

$\therefore$
$$U = \lambda F + \mu G. \qquad \text{.....(9.17.6)}$$

Solving equations (9.17.5) and (9.17.6), we get

$$\lambda = \frac{1}{H^2}\left(GU - FV\right), \quad \mu = \frac{1}{H^2}\left(EV - FU\right). \qquad \text{.....(9.17.7)}$$

Equation (9.17.7) shows that values of λ and μ are intrinsic.

Hence the geodesic curvature κ_g is intrinsic.

Now in case of geodesic $\vec{r}''$ is parallel to $\hat{N}$, therefore geodesic curvature vector (λ , μ) is zero for geodesic.

9.18 The geodesic curvature vector of any curve is orthogonal to the curve

Proof : $\because \vec{r}' = \dfrac{d\vec{r}}{ds} = \hat{t}$, again differentiating

$$\vec{r}'' = \hat{t}' = \kappa \hat{n} \qquad \text{.....(9.18.1)}$$

and $\vec{r}''$ can be expressed as a linear combination of the vectors $\hat{N}$, $\vec{r}_1$ and $\vec{r}_2$, so we can write

$$\vec{r}'' = \kappa_n \hat{N} + \lambda \vec{r}_1 + \mu \vec{r}_2. \qquad \text{.....(9.18.2)}$$

Now, form (9.18.1) and (9.18.2), we have

$$\kappa_n \hat{n} = \kappa_n \hat{N} + \lambda \vec{r}_1 + \mu \vec{r}_2 \qquad \text{.....(9.18.3)}$$

Taking to product by $\hat{t}$ $\quad \kappa_n \hat{n} \cdot \hat{t} = \kappa_n \hat{N} \cdot \hat{t} + \left(\lambda \vec{r}_1 + \mu \vec{r}_2\right) \cdot \hat{t}$

$$0 = 0 + \left(\lambda \vec{r}_1 + \mu \vec{r}_2\right) \cdot \hat{t} \qquad \left(\because \hat{n} \cdot \hat{t} = 0, \hat{N} \cdot \hat{t} = 0\right)$$

$\Rightarrow \quad 0 = \vec{\kappa}_g \cdot \hat{t}$

$\Rightarrow \quad \vec{\kappa}_g$ is orthogonal to $\hat{t}$. $\qquad \left(\because \vec{\kappa}_g = \lambda \vec{r}_1 + \mu \vec{r}_2\right)$

Hence geodesic curvature vector κ_g is orthogonal to the curve.

Hence the result.

9.19 Formulae for geodesic curvature

If parameter *s* (arc length), then show that geodesic curvature $\kappa_g = \left[\hat{N}\ \vec{r}'\ \vec{r}''\right]$, and if we replace parameter *s* by *t*, then show that

$$\kappa_g = \frac{1}{H\dot{s}^3}\left\{\frac{\partial T}{\partial \dot{u}}V(t) - \frac{\partial T}{\partial \dot{v}}U(t)\right\}.$$

Proof : In the article 9.18, we have proved that the geodesic curvature vector $\vec{\kappa}_g$ of a curve is orthogonal to the curve. Also we know that vector $\vec{\kappa}_g$ lies in the tangent plane (See Fig. 9.2), so it is perpendicular to the surface normal vector $\hat{N}$ also. Thus $\vec{\kappa}_g$ is orthogonal to both the unit vectors $\vec{r}'\left(=\hat{t}\right)$ and $\hat{N}$. Hence it is parallel to the unit vector $\hat{N}\times\vec{r}'$. Hence

$$\vec{\kappa}_g = \kappa_g\left(\hat{N}\times\vec{r}'\right), \quad \text{where} \quad \left|\vec{\kappa}_g\right| = \kappa_g. \qquad .....(9.19.1)$$

Now $$\vec{r}'' = \kappa_n\hat{N} + \vec{\kappa}_g$$

$$\vec{r}'' = \kappa_n\hat{N} + \kappa_g\left(\hat{N}\times\vec{r}'\right) \qquad .....(9.19.2)$$

Now taking scalar product by $\left(\hat{N}\times\vec{r}'\right)$, we get

$$\left(\hat{N}\times\vec{r}'\right)\cdot\vec{r}'' = \kappa_n\left\{\left(\hat{N}\times\vec{r}'\right)\cdot\hat{N}\right\} + \kappa_g\left\{\left(\hat{N}\times\vec{r}'\right)\cdot\left(\hat{N}\times\vec{r}'\right)\right\}$$

or $$\left[\hat{N}\ \vec{r}'\ \vec{r}''\right] = 0 + \kappa_g\cdot 1 \qquad \left\{\because \left(\hat{N}\times\vec{r}'\right)\cdot\hat{N} = 0 \text{ and} \left(\hat{N}\times\vec{r}'\right)\cdot\left(\hat{N}\times\vec{r}'\right) = 1\right\}$$

$\therefore$ $$\kappa_g = \left[\hat{N}\ \vec{r}'\ \vec{r}''\right] \qquad .....(9.19.3)$$

which is the required result.

Now, if we replace the parameter *s* by *t*,

then $$\vec{r}' = \frac{d\vec{r}}{ds} = \frac{\left(d\vec{r}/dt\right)}{\left(ds/dt\right)} = \frac{\dot{\vec{r}}}{\dot{s}} \qquad .....(9.19.4)$$

Again differentiating with respect to '*s*', we get

$$\vec{r}'' = \frac{d}{ds}\left(\frac{\dot{\vec{r}}}{\dot{s}}\right) = \frac{d}{dt}\left(\frac{\dot{\vec{r}}}{\dot{s}}\right)\cdot\frac{dt}{ds} = \left(\frac{\dot{s}\,\ddot{\vec{r}} - \dot{\vec{r}}\,\ddot{s}}{\dot{s}^3}\right) = \frac{\dot{s}\,\ddot{\vec{r}} - 0}{\dot{s}^3} = \frac{\ddot{\vec{r}}}{\dot{s}^2} \qquad [\text{Q } \&\&= 0] \qquad .....(9.19.5)$$

Using equation (9.19.4) and (9.19.5) in (9.19.3), we get

$$\kappa_g = \left[\hat{N}\ \frac{\dot{\vec{r}}}{s}\ \frac{\ddot{\vec{r}}}{\dot{s}^2}\right] = \frac{1}{\dot{s}^3}\left[\hat{N}\ \dot{\vec{r}}\ \ddot{\vec{r}}\right] \qquad .....(9.19.6)$$

But we know that $\qquad \hat{N} = \dfrac{\vec{r}_1 \times \vec{r}_2}{H}$

$$\therefore \qquad \kappa_g = \frac{1}{\dot{s}^3}\left[\frac{\vec{r}_1 \times \vec{r}_2}{H}\, \dot{\vec{r}}\ \ddot{\vec{r}}\right] = \frac{1}{H\dot{s}^3}\left[\vec{r}_1 \times \vec{r}_2\ \dot{\vec{r}}\ \ddot{\vec{r}}\right]$$

or $$\kappa_g = \frac{1}{H\dot{s}^3}\left\{(\vec{r}_1 \times \vec{r}_2)\cdot(\dot{\vec{r}} \times \ddot{\vec{r}})\right\} = \frac{1}{H\dot{s}^3}\begin{vmatrix} \vec{r}_1 \cdot \dot{\vec{r}} & \vec{r}_1 \cdot \ddot{\vec{r}} \\ \vec{r}_2 \cdot \dot{\vec{r}} & \vec{r}_2 \cdot \ddot{\vec{r}} \end{vmatrix}$$

or $$\kappa_g = \frac{1}{H\dot{s}^3}\left\{(\vec{r}_1 \cdot \dot{\vec{r}})(\vec{r}_2 \cdot \ddot{\vec{r}}) - (\vec{r}_1 \cdot \ddot{\vec{r}})(\vec{r}_2 \cdot \dot{\vec{r}})\right\}$$ (by Lagrange's Identity) .....(9.19.7)

Now, if we take $\quad T = \dfrac{1}{2}\left(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2\right) \qquad$ (where $\dot{u} = \dfrac{du}{dt}$ etc.) .....(9.19.8)

then $$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \frac{\partial \vec{r}}{\partial u}\cdot\frac{du}{dt} + \frac{\partial \vec{r}}{\partial v}\cdot\frac{dv}{dt} = \vec{r}_1\,\dot{u} + \vec{r}_2\,\dot{v}$$ .....(9.19.9)

Squaring, we get $\quad \dot{\vec{r}}^2 = (\vec{r}_1\,\dot{u} + \vec{r}_2\,\dot{v})^2,$

then
$$\begin{aligned}
\dot{\vec{r}}^2 &= (\vec{r}_1\,\dot{u} + \vec{r}_2\,\dot{v})\cdot(\vec{r}_1\,\dot{u} + \vec{r}_2\,\dot{v}) \\
&= (\vec{r}_1 \cdot \vec{r}_1)\dot{u}^2 + (\vec{r}_1 \cdot \vec{r}_2)\dot{u}\,\dot{v} + (\vec{r}_2 \cdot \vec{r}_1)\dot{u}\,\dot{v} + (\vec{r}_2 \cdot \vec{r}_2)\dot{v}^2 \\
&= \left(E\dot{u}^2 + F\dot{u}\,\dot{v} + F\dot{u}\,\dot{v} + G\dot{v}^2\right)
\end{aligned}$$

$\Rightarrow \qquad = \left(E\dot{u}^2 + 2F\dot{u}\,\dot{v} + G\dot{v}^2\right)$

$\Rightarrow \qquad \dot{r}^2 = 2T \qquad$ [by equation (9.19.8)]

or $$T = \frac{1}{2}\dot{r}^2$$ .....(9.19.10)

Differentiating with respect to $\dot{u}$

$$\begin{aligned}
\frac{\partial T}{\partial \dot{u}} &= \frac{1}{2}2\dot{\vec{r}}\cdot\frac{\partial \dot{\vec{r}}}{\partial \dot{u}} = \dot{r}\cdot\frac{\partial}{\partial \dot{u}}(\vec{r}_1\,\dot{u} + \vec{r}_2\,\dot{v}) \qquad \text{[by equation (9.19.9)]} \\
&= \dot{\vec{r}}\cdot(\vec{r}_1 + 0) = (\dot{\vec{r}}\cdot\vec{r}_1)
\end{aligned}$$

or $$\frac{\partial T}{\partial \dot{u}} = (\dot{\vec{r}}\cdot\vec{r}_1)$$ .....(9.19.11)

Similarly, $$\frac{\partial T}{\partial \dot{v}} = (\dot{\vec{r}}\cdot\vec{r}_2)$$ .....(9.19.12)

Now differentiating equation (9.19.10) with respect to u, we get

$$\begin{aligned}
\frac{\partial T}{\partial u} &= \frac{1}{2}2\dot{\vec{r}}\cdot\frac{\partial \dot{\vec{r}}}{\partial u} = \dot{\vec{r}}\cdot\frac{\partial}{\partial u}(\vec{r}_1\,\dot{u} + \vec{r}_2\,\dot{v}) \qquad \text{[by equation (9.19.9)]} \\
&= \dot{\vec{r}}\cdot(\vec{r}_{11}\,\dot{u} + \vec{r}_{21}\,\dot{v}) = \dot{\vec{r}}\cdot\frac{d}{dt}(\vec{r}_1) \qquad \text{[just as equation (9.19.9)]}
\end{aligned}$$

or $$\frac{\partial T}{\partial u} = \dot{\vec{r}} \cdot \frac{d}{dt}(\vec{r}_1) \qquad .....(9.19.13)$$

Similarly, $$\frac{\partial T}{\partial v} = \dot{\vec{r}} \cdot \frac{d}{dt}(\vec{r}_2) \qquad .....(9.19.14)$$

Now we know that $U(t) = \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{u}}\right) - \dfrac{\partial T}{\partial u}$

$$= \frac{d}{dt}\left(\dot{\vec{r}} \cdot \vec{r}_1\right) - \dot{\vec{r}} \cdot \frac{d}{dt}(\vec{r}_1)$$

or $$U(t) = \ddot{\vec{r}} \cdot \vec{r}_1 + \dot{\vec{r}} \cdot \frac{d\,\vec{r}_1}{dt} - \dot{\vec{r}} \cdot \frac{d(\vec{r}_1)}{dt} = \ddot{\vec{r}} \cdot \vec{r}_1 \qquad .....(9.19.15)$$

Similarly, $$V(t) = \ddot{\vec{r}} \cdot \vec{r}_2 \qquad .....(9.19.16)$$

Now, using equation (9.19.11), (9.19.12) and (9.19.15), (9.19.16) in equation (9.19.7), we get

$$\kappa_g = \frac{1}{H\dot{s}}\left\{\frac{\partial T}{\partial \dot{u}} V(t) - \frac{\partial T}{\partial v'} U(t)\right\} \qquad .....(9.19.17)$$

If parameter $t = s$, so that $\dot{s} = 1$ then we get

$$\kappa_g = \frac{1}{H}\left\{\frac{\partial T}{\partial u'} V(s) - \frac{\partial T}{\partial v'} U(s)\right\} \qquad .....(9.19.18)$$

Hence, equation (9.19.17) and (9.19.18) are formulae for κ_g.

Another form : We know that $u' U(s) + v' V(s) = 0$

$\therefore$ $$U(s) = -\frac{v'}{u'} V(s) \qquad .....(9.19.19)$$

Putting this value in (9.19.18), we get

$$\kappa_g = \frac{1}{H}\left\{\frac{\partial T}{\partial u'} V(s) + \frac{v'}{u'} V(s) \frac{\partial T}{\partial v'}\right\}$$

$$= \frac{V(s)}{H \cdot u'}\left\{u' \frac{\partial T}{\partial u'} + v' + \frac{\partial T}{\partial v'}\right\} = \frac{V(s)}{Hu'} \qquad \left[\because \; u' \frac{\partial T}{\partial u'} + v' \frac{\partial T}{\partial v'} = 2T = 1\right]$$

or $$\kappa_g = \frac{V(s)}{Hu'} \qquad .....(9.19.20)$$

which is value of κ_g in terms of $V(s)$.

Similarly in terms of $U(s)$, κ_g will be

$$\kappa_g = -\frac{U(s)}{Hv'}. \qquad .....(9.19.21)$$

9.20 Geodesic curvature in terms of Gauss coefficients.

We know that $$\vec{r}' = \vec{r}_1 u' + \vec{r}_2 v' \qquad \text{.......(9.20.1)}$$

Again differentiating with respect to 's', we get

$$\vec{r}'' = \vec{r}_{11}u'^2 + 2\vec{r}_{12}\, u'v' + \vec{r}_{22}v'^2 + \vec{r}_1\, u'' + \vec{r}_2\, v'' \qquad \text{......(9.20.2)}$$

then $$\vec{r}'\times\vec{r}_1'' = \left(\vec{r}_1 u' + \vec{r}_2 v'\right)\times\left(\vec{r}_{11}\, u'^2 + 2\vec{r}_{12}\, u'v' + \vec{r}_{22}\, v'^2 + \vec{r}_1\, u'' + \vec{r}_2 v''\right)$$

$$= \Big[\left(\vec{r}_1\times\vec{r}_{11}\right)u'^3 + \left(2\vec{r}_1\times 2\vec{r}_{12} + \vec{r}_2\times\vec{r}_{11}\right)u'^2 v' + \left(\vec{r}_1\times\vec{r}_{22} + 2\vec{r}_2\times\vec{r}_{12}\right)$$

$$u'v'^2 + \left(\vec{r}_2\times\vec{r}_{22}\right)v'^3 + \left(\vec{r}_1\times\vec{r}_2\right)\left(u'v'' - u''v'\right)\Big] \qquad \text{.....(9.20.3)}$$

Now, $$\kappa_g = \hat{N}\cdot\left(\vec{r}'\times\vec{r}''\right)$$

$$= \hat{N}\cdot\left(\vec{r}'\times\vec{r}_{11}\right)u'^3 + \left\{\hat{N}\cdot\left(2\vec{r}_1\times\vec{r}_{12}\right) + \hat{N}\cdot\left(\vec{r}_2\times\vec{r}_{11}\right)\right\}u'^2 v'$$

$$+\left\{\hat{N}\cdot\left(\vec{r}_1\times\vec{r}_{22}\right) + \hat{N}\cdot\left(2\vec{r}_2\times\vec{r}_{12}\right)\right\}u'v'^2 + \hat{N}\cdot\left(\vec{r}_2\times\vec{r}_{22}\right)v'^3 + \hat{N}\cdot\left(\vec{r}_1\times\vec{r}_2\right)\left(u'v'' - u''v'\right)$$

$$= Hu'^3 + \left(2\mu H - l\right)u'^2 v' + \left\{\nu H + 2\left(-mH\right)\right\}u'v'^2 + \left(-nH\right)v'^3 + H\left(u'v'' - v'u''\right)$$

or $$k_g = Hu'\left(v'' + \lambda u'^2 + 2\mu u'v' + \upsilon v'^2\right) - Hv'\left(u'' + lu'^2 + 2mu'v' + nv'^2\right) \qquad \text{......(9.20.4)}$$

which is the formula for geodesic curvature in terms of Gauss coefficients λ, μ, ν, l, m and n.

Since $$\hat{N}\cdot\left(\vec{r}_1\times\vec{r}_2\right) = \hat{N}\cdot H\hat{N} = H$$

$$\hat{N}\cdot\left(\vec{r}_1\times\vec{r}_{11}\right) = \frac{r_1\times r_2}{H}\cdot\left(\vec{r}_1\times\vec{r}_{11}\right) = \frac{1}{H}\left[\left(\vec{r}_{11}\cdot\vec{r}_2\right)\vec{r}_1^{\,2} - \left(\vec{r}_1\cdot\vec{r}_{11}\right)\left(\vec{r}_2\cdot\vec{r}_1\right)\right]$$

$$= \frac{1}{H}\left[\left(F_1 - \frac{1}{2}E_2\right)E - \frac{1}{2}E_1 F\right] = \frac{1}{2H}\left[2EF_1 - EE_2 - FE_1\right] = \lambda H$$

Similarly

$$\hat{N}\cdot\left(\vec{r}_1\times\vec{r}_{12}\right) = \mu H,\ \hat{N}\cdot\left(\vec{r}_1\times\vec{r}_{22}\right) = \nu H$$

$$\hat{N}\cdot\left(\vec{r}_2\times\vec{r}_{11}\right) = -lH,\ \hat{N}\cdot\left(\vec{r}_2\times\vec{r}_{12}\right) = -mH \ \text{ and}$$

$$\hat{N}\cdot\left(\vec{r}_2\times\vec{r}_{22}\right) = -nH. \qquad \text{......(9.20.5)}$$

9.21 Geodesic curvature for parametric curves.

(*i*) For the parametric curve v = c (constant) *i.e.,* u = curve, we have

$$v' = 0, \quad v'' = 0 \qquad \text{......(9.21.1)}$$

But $$k_g = Hu'\left(v'' + \lambda u'^2 + 2\mu u'v' + \upsilon v'^2\right) - Hv'\left(u'' + lu'^2 + 2mu'v' + nv'^2\right)$$

$$\left(k_g\right)_u = \left(k_g\right)_{v=c} = Hu'\left(0 + \lambda u'^2 + 0 + 0\right) - H\times 0\left(u'' + lu'^2 + 0 + 0\right)$$

$$\left(k_g\right)_u = \left(Hu'^3\lambda\right) \qquad \text{........(9.21.2)}$$

Now $\quad ds^2 = E du^2 \Rightarrow \dfrac{1}{E} = \left(\dfrac{du}{ds}\right)^2 \Rightarrow u'^2 = \dfrac{1}{E} \Rightarrow u' = \dfrac{1}{\sqrt{E}} \qquad$ ....(9.21.3)

Then equation (9.21.2) reduces to

$$\left(k_g\right)_u = \left(H\lambda E^{-3/2}\right). \qquad \text{......(9.21.4)}$$

(ii) For the parametric curve u = (constant) c (say) *i.e.,* v = curve, then $u' = 0$, $u'' = 0$ and proceeding as above we get

$$\left(k_g\right)_v = \left(k_g\right)_{u=c} = \left(-nHv'^3\right).$$

$\Rightarrow \qquad \left(k_g\right)_v = -\,nHG^{-3/2} \qquad \left[\because\ v' = G^{-1/2}\right] \qquad$ ....(9.21.5)

Remark : If the parametric curves are orthogonal, we have

$$F = 0,\ \text{this gives}\ H^2 = EG,\ \lambda = \frac{-E_2}{2G}\ \text{and}\ n = \frac{-G_1}{2E}$$

Hence $\qquad \left(k_g\right)_u = -\dfrac{E_2}{2E\sqrt{G}} = -\dfrac{1}{\sqrt{EG}}\dfrac{\partial}{\partial v}\left(\sqrt{E}\right) \qquad$ ....(9.21.6)

and $\qquad \left(k_g\right)_v = -\dfrac{-G_1}{2G\sqrt{E}} = -\dfrac{1}{\sqrt{EG}}\dfrac{\partial}{\partial u}\left(\sqrt{G}\right). \qquad$ .....(9.21.7)

***Ex.*3.** *Find the geodesic curvature of the curve u = constant, on the surface*

$$x = u\cos\theta, \quad y = u\sin\theta, \quad z = \frac{1}{2}au^2.$$

Sol. Let $\vec{r}$ be position vector of any point on the surface then

$$\vec{r} = \left(u\cos\theta, u\sin\theta, \frac{1}{2}au^2\right)$$

Differentiating with respect to u and θ

$$\frac{\partial \vec{r}}{\partial u} = \vec{r}_1 = \left(\cos\theta, \sin\theta, au\right), \frac{\partial r}{\partial \theta} = r_2 = \left(-u\sin\theta, u\cos\theta, 0\right)$$

then $\quad E = \vec{r}_1 \cdot \vec{r}_1 = \left(1 + a^2u^2\right),\ F = \vec{r}_1 \cdot \vec{r}_2 = 0 \qquad$ .....(1)

$$G = \vec{r}_2 \cdot \vec{r}_2 = u^2,\ \text{then}\ \frac{\partial G}{\partial u} = G_1 = 2u \qquad \text{.....(2)}$$

Since $\left(k_g\right)_{u=\text{consant}} = \dfrac{-Hn}{G^{3/2}}$, but $n = -\dfrac{G_1}{2E}$

$$\therefore \qquad \left(k_g\right)_{u=c} = \frac{G_1}{2G\sqrt{E}} = \frac{2u}{2u^2\sqrt{1 + a^2u^2}}$$

$$= \frac{u}{\sqrt{1 + a^2u^2}},\quad \text{which is the required result.}$$

9.22 Normal angle

Definition : The angle between the principal normal $\hat{n}$ and the surface normal $\hat{N}$ is known as normal angle, it is denoted by symbol $\bar{w}$.

(*i*) Here angle $\bar{w}$ is positive if the rotation from $\hat{n}$ to $\hat{N}$ is in the sense from $\hat{n}$ to binormal $\hat{b}$.

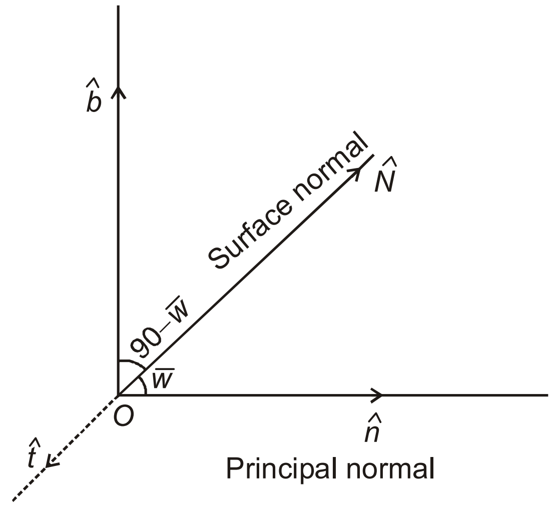


Fig. 9.3

(*ii*) Angle $\bar{w}$ is negative if the rotation from $\hat{n}$ to $\hat{N}$ is in the sense from $\hat{b}$ to $\hat{n}$.

All three vectors $\hat{N}, \hat{n}$ and $\hat{b}$ lie in the same plane. So angle between $\hat{N}$ and $\hat{b}$ is $\left(\frac{\pi}{2} - \bar{w}\right)$.

$\therefore$ $$\hat{N} \cdot \hat{b} = \cos(90 - \bar{w}) = \sin \bar{w} \qquad \text{......(9.22.1)}$$

and $$\hat{N} \cdot \hat{n} = \cos \bar{w}. \qquad \text{......(9.22.2)}$$

9.23 Geodesic curvature in terms of normal angle.

We know that $$\kappa_g = \left[\hat{N}\, \vec{r}'\, \vec{r}''\right]$$

or $$\kappa_g = \left[\hat{N}\, \hat{t}\, \hat{t}'\right] \qquad \left[\because \vec{r}' = \hat{t},\ \vec{r}'' = \vec{t}'\right]$$

or $$\kappa_g = \left[\hat{N}\, \hat{t}\, \kappa\, \hat{n}\right] \qquad \left[\because \hat{t}' = \kappa\, \hat{n}\right]$$

$$= \kappa\left[\hat{N}\, \hat{t}\, \hat{n}\right]$$

$$= \kappa\, \hat{N} \cdot (\hat{t} \times \hat{n}) = \kappa\, \hat{N} \cdot \hat{b} \qquad \left[\because \hat{t} \times \hat{n} = \hat{b}\right]$$

or $$\kappa_g = \kappa \sin \bar{w} \qquad \text{.....(9.23.1)}$$

or $$\kappa_g = \frac{1}{\rho} \sin \bar{w}, \qquad \text{.....(9.23.2)}$$

where $\rho = \frac{1}{\kappa}$ is the radius of curvature of the curve.

Remark : We also know that

$$\vec{r}'' = \kappa_n \hat{N} + \vec{\kappa}_g$$

Taking dot product with $\hat{N}$, we gets

$$\vec{r}'' \cdot \hat{N} = \kappa_n \hat{N} \cdot \hat{N} + \vec{\kappa}_g \cdot \hat{N}$$

$$\kappa \cos \overline{w} = \kappa_n \cdot 1 + 0 \left[\because \kappa_g \cdot \hat{N} = 0 \right] \quad \left[\vec{r}'' \cdot \hat{N} = \kappa\, \hat{n} \cdot \hat{N} = \kappa \cos \overline{w} \right]$$

or $$\kappa_n = \kappa \cos \overline{w}. \qquad .....(9.23.3)$$

Now, on dividing equation (9.23.1) by (9.23.3), we get

$$\frac{\kappa_g}{\kappa_n} = \tan \overline{w} \Rightarrow \kappa_g = \kappa_n \tan \overline{w} \qquad .....(9.23.4)$$

On squaring and adding equations (9.23.1) and (9.23.3), we get

$$\kappa_g^2 + \kappa_n^2 = \kappa^2. \qquad .....(9.23.5)$$

9.24 Expression of the torsion of a geodesic on any surface, and that the torsion of an asymptotic line is equal to the torsion of its geodesic tangent.

Let c be a curve on a surface S, let $\vec{r}$ be position vector of any point P of curve c, then

$$\hat{b} = \hat{t} \times \hat{n}. \qquad .....(9.24.1)$$

Differentiating with respect to s (arc length), we get

$$\frac{d\hat{b}}{ds} = \frac{d\hat{t}}{ds} \times \hat{n} + \hat{t} \times \frac{d\hat{n}}{ds} \qquad .....(9.24.2)$$

By Serret-Frenet formulae, equation (9.24.2) is reduced to

$$-\tau \hat{n} = \kappa\, \hat{n} \times \hat{n} + \hat{t} \times \frac{d\hat{n}}{ds}$$

or $$-\tau \hat{n} = 0 + \hat{t} \times \frac{d\hat{n}}{ds} \qquad .....(9.24.3)$$

Now, if the curve C is geodesic on surface S, then $\hat{n} = \hat{N}$, we also denote τ by τ_g as the torsion of the geodesic, then by equation (9.24.3), we get

$$-\tau_g\ \hat{N} = \hat{t} \times \frac{d\hat{N}}{ds} \cdot$$

Taking dot product by $\hat{N}$, we get

$$-\tau_g\ \hat{N} \cdot \hat{N} = \hat{N} \cdot \left(\hat{t} \times \hat{N}' \right) = -\hat{N} \cdot \left(\hat{N}' \times \hat{t} \right)$$

or $$\tau_g = \hat{N} \cdot \left(\hat{N}' \times \hat{t} \right) = \left[\hat{N}\ \hat{N}'\ \hat{t} \right], \qquad .....(9.24.4)$$

which is the basic expression for the torsion of a geodesic on a surface S.

Torsion found in equation (9.24.4) is same as the torsion of an asymptotic line. The geodesic which touches a curve at any point is often called its geodesic tangent at the point. Hence the torsion of an a asymptotic line is equal to the torsion of its geodesic tangent.

9.25 Expressions for the torsion of a geodesic in terms of fundamental magnitudes and also in terms of principal curvatures.

We know that Torsion of a geodesic is

$$\tau_g = \left[\hat{N} \;\; \hat{N}' \;\; \vec{r}'\right] \qquad .....(9.25.1)$$

Now
$$\hat{N}' = \frac{d\hat{N}}{ds} = \frac{\partial \hat{N}}{\partial u}\frac{du}{ds} + \frac{\partial \hat{N}}{\partial v}\frac{dv}{ds} = \hat{N}_1\, u' + \hat{N}_2\, v' \qquad .....(9.25.2)$$

and
$$\vec{r}' = \frac{d\vec{r}}{ds} = \frac{\partial \vec{r}}{\partial u}\frac{du}{ds} + \frac{\partial \vec{r}}{\partial v}\frac{dv}{ds} = \vec{r}_1 u' + \vec{r}_2 v' \qquad ......(9.25.3)$$

$\therefore$
$$\begin{aligned}\hat{N}' \times \vec{r}' &= \left(\hat{N}_1\, u' + \hat{N}_2\, v'\right) \times \left(\vec{r}_1\, u' + \vec{r}_2\, v'\right) \\ &= \left(\hat{N}_1 \times \vec{r}_1\right)u'^2 + \left(\hat{N}_1 \times \vec{r}_2\right)u'v' + \left(\hat{N}_2 \times \vec{r}_1\right)u'v' + \left(\hat{N}_2 \times \vec{r}_2\right)v'^2\end{aligned}$$

Taking dot product of both sides with vector $\hat{N}$, we get

$$\hat{N}\cdot\left(\hat{N}' \times \vec{r}'\right) = \left[\hat{N}\cdot\left(\hat{N}_1 \times \vec{r}_1\right)u'^2 + \hat{N}\cdot\left(\hat{N}_1 \times \vec{r}_2\right)u'v' + \hat{N}\cdot\left(\hat{N}_2 \times \vec{r}_1\right)u'v' + \hat{N}\cdot\left(\hat{N}_2 \times \vec{r}_2\right)v'^2\right]$$

or
$$\left[\hat{N}\;\hat{N}'\;\vec{r}'\right] = \tau_g = \left[\hat{N}\;\hat{N}_1\;\vec{r}_1\right]u'^2 + \left[\hat{N}\;\hat{N}_1\;\vec{r}_2\right]u'v' + \left[\hat{N}\;\hat{N}_2\;\vec{r}_1\right]u'v' + \left[\hat{N}\;\hat{N}_2\;\vec{r}_2\right]v'^2 \qquad .....(9.25.4)$$

But
$$\left[\hat{N}\;\hat{N}_1\;\vec{r}_1\right] = \frac{EM - FL}{H}, \left[\hat{N}\;\hat{N}_1\;\vec{r}_2\right] = \frac{FM - GL}{H}$$

$$\left[\hat{N}\;\hat{N}_2\;\vec{r}_1\right] = \frac{EN - FM}{H}, \left[\hat{N}\;\hat{N}_2\;\vec{r}_2\right] = \frac{FN - GM}{H}. \qquad .....(9.25.5)$$

Using equation (9.25.5) in to (9.25.4), we get

$$\tau_g = \frac{1}{H}\left[(EM - FL)u'^2 + \{(FM - GL) + (EN - FM)\}u'v' + (FN - GM)v'^2\right]$$

or
$$\tau_g = \frac{1}{H}\left\{(EM - GL)u'^2 + (EN - GL)u'v' + (FN - GM)v'^2\right\}, \qquad .....(9.25.6)$$

which is an expression for τ_g in terms of fundamental magnitudes.

To find expression in terms of principal curvatures chose the lines of curvature as parametric curves so that $F = 0$, $M = 0$ and $H^2 = EG \Rightarrow H = \sqrt{EG}$, then above equation (9.25.5) reduces to the following form

$$\tau_g = \frac{1}{\sqrt{EG}}\left\{(EN - GL)u'v'\right\} = \sqrt{EG}\left(\frac{N}{G} - \frac{L}{E}\right)u'v'. \qquad .....(9.25.7)$$

Now let ψ be the angle which the geodesic makes with the parametric curve $v = c$ (constant), then

$$\cos\psi = u'\sqrt{E} \text{ and } \sin\psi = v'\sqrt{G} \qquad .....(9.25.8)$$

Then using these in equation (9.25.7), we get

$$\tau_g = \left(\frac{N}{G} - \frac{L}{E}\right)\sin\psi\cos\psi \qquad .....(9.25.9)$$

But the principal curvatures κ_a and κ_b are given by

$$\kappa_a = \frac{L}{E} \text{ and } \kappa_b = \frac{N}{G}, \qquad .....(9.25.10)$$

So that (9.25.9) can be written as

$$\tau_g = \frac{1}{2}(\kappa_a - \kappa_b)\sin 2\psi, \qquad .....(9.25.11)$$

which is expression of τ_g in terms of principal curvatures.

From equation (9.25.11), it follows that τ_g is maximum when

$$2\psi = 90° \Rightarrow \psi = \pi/4,$$

hence the geodesic bisecting the angle between the line of curvature has maximum torsion.

Remark : If τ_g and τ'_g be the torsions of two orthogonal geodesics then from equation (9.25.11) above, we have

$$\tau_g = \frac{1}{2}(\kappa_b - \kappa_a)\sin 2\psi \qquad .....(A)$$

and $$\tau'_g = \frac{1}{2}(\kappa_b - \kappa_a)\sin 2\left(\psi + \frac{\pi}{2}\right) = \frac{1}{2}(\kappa_b - \kappa_a)(-\sin 2\psi)$$

$$\Rightarrow \quad \tau'_g = -\frac{1}{2}(\kappa_b - \kappa_a)\sin 2\psi \qquad .....(B)$$

In adding equation (A) and (B), we get

$$\tau_g + \tau'_g = 0 \Rightarrow \tau_g = -\tau'_g,$$

which shows that two orthogonal geodesics have their torsion equal but opposite in sign.

9.26 Some important definition

***(i)* Simply connected region *R* :** The region R, in which every closed curve lying in the region R on a surface can be contracted continuously into a point without leaving R, is called simply connected region [See Fig 9.4].

***(ii)* Excess of a closed curve *C* [*ex* (*c*)] :** Suppose a simply connected region R (See Fig 9.3) be enclosed by a closed curve (say) C, consisting of n arcs $A_0 A_1$, $A_1 A_2$, ..., $A_{n-1} A_n$, where

$A_0 = A_n$, making at the vertices exterior angles $\alpha_1, \alpha_2, ..., \alpha_n$; then the excess of closed curve C is denoted by $ex\,(c)$ is defined as

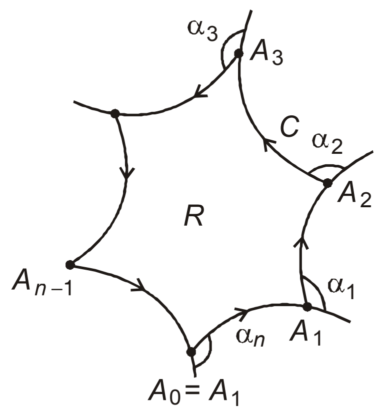


Fig.9.4

$$ex\,(c) = 2\pi - (\alpha_1 + \alpha_2 + ... + \alpha_n) - \int_c \kappa_g\, ds$$

or $$ex\,(c) = 2\pi - \sum_{r=1}^{n} \alpha_r - \int_c \kappa_g\, ds \qquad .....(9.26.1)$$

where κ_g being the geodesic curvature of the arcs.

(iii) **Total curvature of *R* :** The total curvature or Gaussian curvature of an arc on a surface is denoted by K and is given by

$$K = \frac{LN - M^2}{EG - F^2} = \frac{T^2}{H^2} \qquad .....(9.26.2)$$

or $$K = \frac{1}{2H}\frac{\partial}{\partial u}\left(\frac{FE_2}{EH} - \frac{G_1}{H}\right) + \frac{1}{2H}\frac{\partial}{\partial v}\left(\frac{2F_1}{H} - \frac{E_2}{H} - \frac{FE_1}{EH}\right) \qquad .....(9.26.3)$$

Therefore, the total curvature of a simply connected region R is given by

$$\iint_R \kappa\, ds. \qquad .....(9.26.4)$$

9.27 Gauss-Bonnet theorem

Statement : Any curve which encloses a simply connected region R, the excess of the closed curve C is equal to the total curvature of R, *i.e.*, $ex(c) = \iint_R \kappa\, ds.$

Proof : Let us consider a surface $\vec{r} = \vec{r}\,(u, v)$ of class 3 with u, v as parameters, let c be a closed curve, which is boundary of a simply connected region R on the surface. (see Fig 9.4). Let c consists of n (finite) smooth arcs

$$A_0 A_1, A_1 A_2, ..., A_{n-1} A_n; \qquad \text{(where } A_0 = A_n\text{)}$$

such that each arc is of class 2 and these are positively described in anti-clockwise direction. Let $\alpha_r (r = 1, 2, 3,..., n)$ be the exterior angle between the tangents to the arcs $A_{r-1} A_r$ and $A_r A_{r+1}$ at the vertex A_r, measured with usual convention. So that $-\pi < \alpha_r < \pi$.

The geodesic curvature κ_g exists at every point of c except possibly at the vertices A_r $(r = 1, 2, ..., n)$.

Then by Liuville's formula, we have

$$\kappa_g = \frac{d\theta}{ds} + \frac{H}{E}\left(\lambda \frac{du}{ds} + \mu \frac{dv}{ds}\right) \qquad .....(9.27.1)$$

or

$$\int_c \kappa \, ds = \int_c \left(d\theta + P\, du + Q\, dv\right) \quad \text{(on integrating over } c) \qquad .....(9.27.2)$$

where θ is the angle between the curve c and the parametric curve v = constant (u-curve) and P, Q are functions of parameter, u, v, which are given by

$$P(u, v) = \frac{\lambda H}{E} = \frac{1}{2HE}\left(2EF_1 - EE_2 - FE_1\right). \qquad .....(9.27.3)$$

and

$$Q(u, v) = \frac{\mu H}{E} = \frac{1}{2HE}\left(EG_1 - FE_2\right). \qquad .....(9.27.4)$$

Now, the parametric curves $v = c$ (constant) form a family in the region R enclosed by curve C, the tangent to C turns through 2π relative to these curves, so that

$$\int_c d\theta + \sum_{r=1}^{n} \alpha_r = 2\pi$$

or

$$\left(2\pi - \sum_{r=1}^{n} \alpha_r\right) = \int_c d\theta \qquad .....(9.27.5)$$

But by definition of excess of a closed curve C, we have

$$ex(c) = 2\pi - \sum_{r=1}^{n} \alpha_r - \int_c \kappa_g ds$$

or

$$ex(c) = \int_c d\theta - \int_c \kappa_g ds \qquad \text{[by equation (9.27.5)]}$$

$$= -\int_c \left(P\, du + Q\, dv\right) \qquad \text{[by equation (9.27.2)]}$$

$\Rightarrow$

$$ex(c) = -\iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) du\, dv \qquad \text{[using Green's theorem]} \qquad .....(9.27.6)$$

But in curvilinear coordinates, area of surface element ds (say) is given by

$$ds = \left|\vec{r}_1\, du \times \vec{r}_2\, dv\right| = \left|\vec{r}_1 \times \vec{r}_2\right| du dv = H\, du dv \qquad .....(9.27.7)$$

or $$\frac{ds}{H} = du\, dv.$$

Using it in (9.27.6), we get

$$ex\,(c) = -\iint_R \frac{1}{H}\left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) ds. \qquad .....(9.27.8)$$

But we know that the intrinsic formula for Gaussian curvature or total curvature κ at any point (u, v) on the surface is obtained by

$$K = \frac{1}{H}\frac{\partial}{\partial u}\left(\frac{1}{2HE}\left(FE_2 - EG_1\right)\right) + \frac{1}{H}\frac{\partial}{\partial v}\left(\frac{1}{2HE}\left(2EF_1 - EF_2 - FE_1\right)\right) \qquad .....(9.27.9)$$

or $$K = \frac{-1}{H}\left\{\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right\}.$$

Using this value in (9.27.8), we get

$$ex\,(c) = \iint_R K ds, \qquad .....(9.27.10)$$

which shows that the excess of the closed curve c is equal to the total curvature of R.

Hence the theorem.

9.27.1 Self-learning exercise-2

1. What is differential equation of a geodesic on a surface $F(x, y, z) = 0$?

2. Write the differential equation of geodesic on the surface $z = f(x, y)$.

3. State Clairut's theorem.

4. If a geodesic on a surface of revolution cuts the meridian at a constant angle, is surface a right cylinder ?

5. Is a curve on sphere a geodesic if it is a great circle ?

6. Define geodesic curvature.

7. Define normal angle.

8. Write geodesic curvature in terms of normal angle $\overline{w}$.

9. Are two orthogonal geodesic have their torsion equal in magnitude and sign ?

10. Define excess of a closed curve C.

11. State Gauss Bonnet theorem.

9.27.2 Illustrative examples

Ex.4. *Geodesic are drawn on a catenoid so as to cross the meridians at an angle whose sine is c/u, where u is the distance of the point of crossing from the axis. Prove that the polar equation to their projections on the xy-plane is* $\frac{u-c}{u+c} = e^{2(\theta+\alpha)}$, *where* α *is an arbitrary constant.*

Sol. A catenoid is a surface of revolution, obtained by revolving the catenary about its directrix. Let its equation be

$$x = u \cos \theta, y = u \sin \theta, z = c \cos h^{-1}\left(\frac{u}{c}\right) \qquad .....(1)$$

Then for a geodesic curve on its surface

$$u^2 \cdot \frac{d\theta}{ds} = h_1 \qquad .....(2)$$

It is given that geodesic cuts the meridians at an angle, say ψ,

whose sine is $\frac{c}{u}$, so $\qquad \sin \psi = \frac{c}{u}. \qquad .....(3)$

We know that $\qquad \sin \psi = u\frac{d\theta}{ds} \qquad .....(4)$

Using (3) in (4), we get $\qquad \frac{c}{u} = u\frac{d\theta}{ds} \Rightarrow u^2\frac{d\theta}{ds} = c \qquad .....(5)$

Then form equation (2) and (5), we have $h_1 = c$.

Now we find that the equation of the geodesic becomes

$$\theta + \alpha = c\int \frac{du}{u^2 - c^2} = \frac{1}{2}\log\frac{u-c}{u+c}$$

or $\qquad 2(\theta + \alpha) = \log\frac{u-c}{u+c} \Rightarrow e^{2(\theta+\alpha)} = \frac{u-c}{u+c}$

where α is constant of integration.

Ex.5. *A geodesic on the ellipsoid of revolution* $\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1$, *crosses a meridian at an angle* θ *at a distance* u *from the axis. Prove that at the point of crossing it makes an angle* $\cos^{-1}\left[\frac{cu\cos\theta}{\sqrt{\left\{a^4 - u^2\left(a^2 - c^2\right)\right\}}}\right]$ *with the axis.*

Sol. The given equation of the ellipsoid of revolution may be expressed as

$$x = u \cos v, y = u \sin v, z = c\sqrt{\left(1 - \frac{u^2}{a^2}\right)} \qquad .....(1)$$

where u, v are parameters. Let $\vec{r}$ be position vector of a point, then

$$\vec{r} = \left(u\cos v, \quad u\sin v, \quad c\sqrt{\left(1 - \frac{u^2}{a^2}\right)}\right).$$

$$\therefore \qquad \vec{r}_1 = \left(\cos v, \ \sin v, \ -\frac{uc}{a^2 \sqrt{\left[1-\left(u^2/a^2\right)\right]}} \right) = \frac{\partial r}{\partial u}$$

and $$\vec{r}_2 = \frac{\partial r}{\partial v} = \left(-u \sin v, \ u \cos v, \ 0\right)$$

and $$E = \vec{r}_1 \cdot \vec{r}_1 = \vec{r}_1^{\,2} = 1 + \frac{c^2 u^2}{a^2\left(a^2-u^2\right)}, F = \vec{r}_1 \cdot \vec{r}_2 = 0, G = \vec{r}_2 \cdot \vec{r}_2 = u^2 \qquad .....(2)$$

Then surface element ds is given by

$$ds^2 = Edu^2 + 2\ Fdu\ dv + G\ dv^2$$

$$1 = E\left(\frac{du}{ds}\right)^2 + 2F\frac{du}{ds}\frac{dv}{ds} + G\left(\frac{dv}{ds}\right)^2$$

or $$\left\{1 + \frac{c^2u^2}{a^2\left(a^2-u^2\right)}\right\}\left(\frac{du}{ds}\right)^2 + u^2\left(\frac{dv}{ds}\right)^2 = 1 \qquad (\because F = 0) \qquad .....(3)$$

Now we know that the first integral of a geodesic on a surface of revolution [of the form $x = u \sin v$, $y = u \sin v$, $z = f(u)$] is

$$u^2 \frac{dv}{ds} = G_1 \qquad .....(4)$$

Also, it is given that the geodesic crosses a meridian at an angle θ, therefore

$$\sin\theta = u\frac{dv}{ds}. \qquad .....(5)$$

On using equation (5) in (3), we get

$$\left\{1 + \frac{c^2u^2}{a^2\left(a^2-u^2\right)}\right\}\left(\frac{du}{ds}\right)^2 + \sin^2\theta = 1$$

$$\Rightarrow \qquad \left\{1 + \frac{c^2u^2}{a^2\left(a^2-u^2\right)}\right\}\left(\frac{du}{ds}\right)^2 = \cos^2\theta \qquad .....(6)$$

Now $z^2 = c^2\left(1 - \frac{u^2}{a^2}\right)$, on differentiating with respect to s, we get

$$2z\frac{dz}{ds} = c^2\left(0 - \frac{2u}{a^2}\frac{du}{ds}\right) \Rightarrow \frac{dz}{ds} = -\frac{c^2u}{a^2z}\frac{du}{ds} \qquad .....(7)$$

or $$\cos\phi \frac{dz}{ds} = -\frac{c^2 u}{a^2 z}\frac{du}{ds} \qquad .....(8)$$

where angle ϕ (say) is the angle which geodesic makes with z-axis.

From equation (8), $\dfrac{du}{ds} = -\dfrac{a^2 z \cos\phi}{c^2 u}$ using in equation (6), we get

$$\left\{\frac{1}{1} + \frac{c^2 u^2}{a^2\left(a^2 - z^2\right)}\right\}\frac{a^4 z^2}{c^4 u^2}\cos^2\phi = \cos^2\theta$$

or $$\frac{a^4 - a^2u^2 + c^2u^2}{a^2\left(a^2 - u^2\right)} \times \frac{a^4 z^2}{c^4 u^2} \times \cos^2\phi = \cos^2\theta$$

$$\Rightarrow \quad \cos^2\phi = \frac{c^2u^2\left(a^2 - u^2\right)}{a^2 z^2\left(a^4 - a^2u^2 + c^2u^2\right)}\cos^2\theta = \frac{c^2u^2\cos^2\theta}{a^4 - u^2\left(a^2 - c^2\right)} \qquad \left[\because\ z^2 = \frac{c^2\left(a^2 - u^2\right)}{a^2}\right]$$

or $$\cos\phi = \frac{cu\cos\theta}{\sqrt{a^4 - u^2\left(a^2 - c^2\right)}} \Rightarrow \phi = \cos^{-1}\frac{cu\cos\theta}{\sqrt{a^4 - u^2\left(a^2 - c^2\right)}}$$

which is the required result.

Ex.6. *Prove that the projection on the xy-plane of the geodesics on the catenoid $u = c\cos h\left(\dfrac{z}{c}\right)$ are given by*

$$d\phi = \frac{a\,du}{\sqrt{\left(u^2 - c^2\right)\left(u^2 - a^2\right)}},$$

where a is an arbitrary constant.

Sol. A catenoid is obtained by revolving a catenary about its directrix, hence its equation is

$$x = u\cos\theta,\ y = u\sin\theta,\ z = c\cosh^{-1}\left(\frac{u}{c}\right) = f(u) \text{ (say)} \qquad .....(1)$$

The equation of geodesics on the surface of revolution is given by

$$d\theta = \pm\frac{a}{u}\left\{\frac{1 + f'^2}{u^2 - a^2}\right\}du \qquad \text{[by equation (9), §9.12]} \qquad .....(2)$$

where $f' = \dfrac{df}{du}$.

Now by $\quad u = c\cos h\left(\dfrac{z}{c}\right),\ \dfrac{du}{dz} = c \times \sin h\left(\dfrac{z}{c}\right)\cdot\dfrac{1}{c} = \sin h\left(\dfrac{z}{c}\right)$

or $$\frac{dz}{du}=\frac{1}{\sin h\left(z/c\right)}\ \Rightarrow \frac{df}{du}=f'=\frac{1}{\sin h\ \left(z/c\right)} \qquad .....(3)$$

then by equation (2), we have

$$d\theta=\pm\frac{a}{c\cos h\ \left(z/c\right)}\ \frac{\sqrt{\left(1+\dfrac{1}{\sin h^2\,(z/c)}\right)}}{\sqrt{u^2-a^2}}du \qquad \text{[on using values of } u \text{ and } f'\text{]}$$

$$=\pm\frac{a}{c\sin h\ \left(z/c\right)}\times\frac{du}{\sqrt{u^2-a^2}} \qquad \left[\because 1+\sin h^2\frac{z}{c}=\cos h^2\frac{z}{c}\right]$$

or $$d\theta=\pm\frac{a\ du}{\sqrt{\left(u^2-c^2\right)\left(u^2-a^2\right)}} \qquad \left[\because \sin h\left(\frac{z}{c}\right)=\sqrt{\cos b^2\frac{z}{c}-1}=\sqrt{\left(\frac{u^2}{c^2}-1\right)}\right]$$

which is the required result.

***Ex.*7.** *Show that for a geodesic*

$$\tau^2=\left(\kappa-\kappa_a\right)\left(\kappa_b-\kappa\right)\ or\ \frac{1}{\sigma^2}=\left(\frac{1}{\rho}-\frac{1}{\rho_a}\right)\left(\frac{1}{\rho_b}-\frac{1}{\rho}\right).$$

Sol. The torsion and curvature of a geodesic are given by

$$\tau=\frac{1}{2}\left(\kappa_b-\kappa_a\right)\sin 2\psi \qquad .....(1)$$

and $$\kappa=\kappa_a\cos^2\psi+\kappa_b\sin^2\psi \qquad \text{(by Euler's theorem)} \quad .....(2)$$

where ψ is the angle between the line of curvature and the geodesic tangent.

Now $$(\kappa_b-\kappa)=\kappa_b-(\kappa_a\cos^2\phi+\kappa_b\sin^2\psi)$$
$$=\kappa_b\,(1-\sin^2\psi)-\kappa_a\cos^2\psi=\kappa_b\cos^2\psi-\kappa_b\cos^2\psi$$

or $$(\kappa_b-\kappa)=(\kappa_b-\kappa_a)\cos^2\psi \qquad .....(3)$$

Similarly $$(\kappa-\kappa_a)=(\kappa_a\cos^2\psi+\kappa_b\sin^2\psi)-\kappa_a$$

or $$(\kappa-\kappa_a)=(\kappa_b-\kappa_a)\sin^2\psi \qquad .....(4)$$

multiplying equation (3) and (4), we get

$$(\kappa_b-\kappa)\,(\kappa-\kappa_a)=(\kappa_b-\kappa_a)^2\sin^2\psi\cos^2\psi=\tau^2$$

or $(\kappa_b-\kappa)\,(\kappa-\kappa_a)=\tau^2$, which is the required result.

But $\kappa=\dfrac{1}{\rho},\ \kappa_a=\dfrac{1}{\rho_a},\ \kappa_b=\dfrac{1}{\rho_b}$, then above equation reduces to

$$\left(\frac{1}{\rho_b}-\frac{1}{\rho}\right)\left(\frac{1}{\rho}-\frac{1}{\rho_a}\right)=\frac{1}{\sigma^2} \qquad \left(\text{where } \tau=\frac{1}{\sigma}\right) \qquad .....(5)$$

which is another form of the result.

***Ex.*8.** *Prove that at the origin the geodesic curvature of the section of the surface* $2z = ax^2 + by^2$, *by the plane* $lx + my + nz = 0$, *is*

$$n\,(bl^2 + am^2)/(l^2 + m^2)^{3/2}.$$

Sol. The given point is (0, 0, 0) and the plane of the given section is

$$lx + my + nz = 0. \qquad \text{.....(1)}$$

Then the equation of the tangent plane of the surface

$$2z = ax^2 + by^2 \quad \text{at the origin is,} \quad z = 0$$

i.e.,

$$0\cdot x + 0\cdot y + z = 0. \qquad \text{......(2)}$$

Therefore the direction cosines of the line of intersection of planes (1) and (2), which will be the direction cosines of a tangent through origin to the given section of surface, are obtained as

$$l_1 = \frac{-m}{\sqrt{l^2 + m^2}},\ m_1 = \frac{1}{\sqrt{l^2 + m^2}},\ n_1 = 0\cdot \qquad \text{.....(3)}$$

Let the equation of the normal plane to the given surface at the origin through the tangent line be

$$\lambda x + \mu y + \nu z = 0 \qquad \text{.....(4)}$$

then

$$\lambda l_1 + \mu m_1 + \nu n_1 = 0$$

or

$$-\lambda m + \mu l = 0 \ \Rightarrow \frac{\lambda}{l} = \frac{\mu}{m} \qquad \text{.....(5)}$$

Also

$$\nu = 0. \qquad \text{.....(6)}$$

Now plane given in equation (4) passes through z-axis.

Hence

$$\frac{\lambda}{l} = \frac{\mu}{m} = \frac{\nu}{0}$$

or

$$\frac{\lambda}{l} = \frac{\mu}{m} = \frac{\nu}{0} = \frac{\sqrt{\lambda^2 + \mu^2 + \nu^2}}{\sqrt{l^2 + m^2 + 0^2}} = \frac{1}{\sqrt{l^2 + m^2}}$$

$$\Rightarrow \qquad \lambda = \frac{l}{\sqrt{l^2 + m^2}},\ \mu = \frac{m}{\sqrt{l^2 + m^2}}, \nu = 0. \qquad \text{.....(7)}$$

Now the direction cosines of the normal to the given section (1) are

$$\frac{l}{\sqrt{l^2 + m^2 + n^2}}, \frac{m}{\sqrt{l^2 + m^2 + n^2}}, \frac{n}{\sqrt{l^2 + m^2 + n^2}} \qquad \text{.....(8)}$$

If θ be the angle between (7) and (8) then

$$\cos\theta = \sqrt{\frac{l^2 + m^2}{l^2 + m^2 + n^2}}. \qquad \text{.....(9)}$$

The radius of curvature of a given section through any point of a surface is obtained by the expression

$$\frac{\cos\theta}{\rho} = \frac{rl_1^2 + 2sl_1m_1 + tm_1^2}{\sqrt{1+p^2+q^2}} \qquad .....(10)$$

For the given surface $2z = ax^2 + by^2$, we have at the origin $p = 0$, $q = 0$, $r = a$, $s = 0$ and $t = b$. Using value of $\cos\theta$ form (9) and these values of p, q, r, s, t in equation (10), we get

$$\frac{1}{\rho}\frac{\sqrt{l^2+m^2}}{\sqrt{l^2+m^2+n^2}} = \frac{am^2+bl^2}{l^2+m^2}$$

or $\qquad \rho = (l^2 + m^2)^{3/2}\,(am^2 + bl^2)^{-1}\,(l^2 + m^2 + n^2)^{-1/2}$

but $\qquad \kappa = \dfrac{1}{\rho}$, so $\kappa = \left(l^2+m^2\right)^{-3/2}\left(am^2+bl^2\right)\left(l^2+m^2+n^2\right)^{1/2} \qquad .....(11)$

Also, if ϕ is the angle between the given section and the normal section, then

$$\cos\phi = (l^2 + m^2)^{1/2}\,(l^2 + m^2 + n^2)^{-1/2}$$

Hence by Meunier's theorem

$$\kappa_n = \kappa\cos\phi = (l^2 + m^2)^{-1}\,(am^2 + bl^2) \qquad ......(12)$$

therefore, the geodesic curvature κ_g of the required section is given by $\kappa^2 = \kappa_n^2 + \kappa_g^2$

$$\Rightarrow \qquad \kappa_g^2 = \kappa^2 - \kappa_n^2 = \frac{\left(am^2+bl^2\right)^2\left(l^2+m^2+n^2\right)}{\left(l^2+m^2\right)^3} - \frac{\left(am^2+bl^2\right)^2}{\left(l^2+m^2\right)^2}$$

$$= \frac{\left(am^2+bl^2\right)^2 n^2}{\left(l^2+m^2\right)^3}$$

or $$\kappa_g = \frac{n\left(am^2+bl^2\right)}{\left(l^2+m^2\right)^{3/2}},$$

which is the required result.

***Ex.*10.** *Find the Gaussian curvature at the point* (u, v) *of the anchor ring*

$$\vec{r} = \left(g(u)\cos v,\ g(u)\sin v, f(u)\right).$$

where $g(u) = (b + a\cos u)$, $f(u) = a\sin u$ *and the domain of* u, v *is*

$$0 < u < 2\pi,\ 0 < v < 2\pi,$$

verify that the total curvature of the whole surface is zero.

Sol. We have position vector of a point (say) $\vec{r}$,

$$\vec{r} = \left(\left(b+a\cos u\right)\cos v,\ \left(b+a\cos u\right)\sin v,\ a\sin u\right)$$

then by finding $\vec{r}_1, \vec{r}_2, \vec{r}_{11}, \vec{r}_{12}, \vec{r}_{22}$ we can get

$$E = a^2, F = 0, G = (b + a\cos u)^2, H^2 = a^2(b + a\cos u)^2 \qquad .....(1)$$

$$L = a, M = 0, N = \cos u\,(b + a\cos u) \qquad .....(2)$$

Then the Gaussian curvature κ is given by

$$\kappa = \frac{LN - M^2}{H^2} = \frac{\cos u}{a(b + a\cos u)} \qquad .....(3)$$

Then the total curvature of the whole surface is

$$= \int_s K ds$$

$$= \int_{u=0}^{2\pi}\int_{v=0}^{2\pi} \kappa H \, du\, dv$$

$$= \int_0^{2\pi}\int_0^{2\pi} \frac{\cos u}{a(b + a\cos u)} a(b + a\cos u)\, du\, dv \qquad \text{[on using values form (3) and (1)]}$$

$$= \int_0^{2\pi}\int_0^{2\pi} \cos u\, du\, dv = \int_0^{2\pi} \cos u\, (v)_0^{2\pi}\, du$$

$$= 2\pi \int_0^{2\pi} \cos u\, du = 2\pi(0) = 0$$

∴ total curvature = 0. Hence verified.

9.28 Summary

1. In this unit you have studied about geodesic, differential equation of a geodesic, single differential equation of a geodesic, when the relation between parameters u and v be of the form $u = u(v)$ or $v = v(u)$, geodesic on a surface of revolution. About geodesic curvature and torsion and their expressions in different forms, about Gauss-Bonnet theorem.
2. Sufficient number of examples have solved in the unit.
3. Differential equation of geodesic in different forms and formulae for geodesic curvature and torsion will help the students to easily understand the text of the unit.
4. Examples in the text have been inserted frequently to help students to understand the text of the unit.

9.29 Answers to self-learning exercises

Self-learning exercise-1

1. See §9.2.

2. See §9.3.

3. $Eu'' + \frac{1}{2}E_1 u'^2 + E_2\, u'v' + Ev'' + \left(F_2 - \frac{1}{2}G_1\right)v'^2 = 0$ and

$$Fu'' + \left(F_1 - \frac{1}{2}E_2\right)u'^2 + G_1\, u'v' + Gv'' + \frac{1}{2}G_2 v'^2 = 0$$

4. $\frac{d}{ds}\left(\frac{\partial \tau}{\partial u'}\right) - \frac{\partial \tau}{\partial u} = 0$ and $\frac{d}{ds}\left(\frac{\partial \tau}{\partial v'}\right) - \frac{\partial \tau}{\partial v} = 0$

5. $u'' + l\, u'^2 + 2m\, u'v' + nu'^2 = 0$ and $v'' + \lambda\, u'^2 + 2\mu\, u'v' + \nu\, v'^2 = 0$

6. $\frac{d^2u}{dv^2} = \lambda\left(\frac{du}{dv}\right)^3 + (2\mu - l)\left(\frac{du}{dv}\right)^2 + (\nu - 2m)\frac{du}{dv} - n$

7. No.

8. $EF_2 + FE_1 - 2EF_1 = 0$

Self-learning exercise-2

1. Integral of one of the equation $\frac{\left(\frac{d^2x}{ds^2}\right)}{F_x} = \frac{\left(\frac{d^2y}{ds^2}\right)}{F_y} = \frac{\left(\frac{d^2z}{ds^2}\right)}{F_z}$, with the equation $F(x, y, z) = 0$

2. $\left(1 + p^2 + q^2\right)\frac{d^2y}{dx^2} = \left(p\frac{dy}{dx} - q\right)\left\{t\left(\frac{dy}{dx}\right)^2 + 2s\frac{dy}{dx} + r\right\}$

3. See §9.13.

4. Yes. (see §9.14)

5. Yes. (see §9.15)

6. See §9.16.

7. See §9.22.

8. $\kappa_g = \frac{1}{l}\sin \overline{w}$

9. No (see §9.25)

10. See §9.26*(ii)*.

11. See §9.27.

9.30 Exercises

1. Derive the general differential equations of geodesic on a surface $\vec{r} = \vec{r}(u, v)$. **[Ans.** See §9.4**]**

2. Derive the canonical equation of a geodesic on the surface $\vec{r} = \vec{r}(u, v)$. **[Ans.** See §9.5**]**

3. Derive the differential equations of a geodesic in Gauss coefficient. **[Ans.** See §9.6**]**

4. Find the single differential equation of geodesics, on surface $r = r(u, v)$, when a curve on the surface may be determined by a single relation between the parameters, u and v either by $v = v(u)$ or by $u = u(v)$. **[Ans.** See §9.7**]**

5. Show that for surface, a necessary and sufficient condition that the curve $v = c$ (constant) be a geodesic is $EF_2 + FE_1 - 2EF_1 = 0$, when $v = c$, for all values of u. **[Ans.** See §9.8**]**

6. Find the differential equation of geodesics on the surface $z = f(x, y)$, the Monge's form.

[Ans. See §9.11**]**

7. State and prove Clairant's theorem. **[Ans.** See §9.13**]**

8. Find an expression of κ_g (geodesic curvature) and show that it is intrinsic. **[Ans.** See §9.17**]**

9. Derive the formula for geodesic curvature of the form $\kappa_g = \left[\hat{N}\ \vec{r}'\ \vec{r}''\right]$ **[Ans.** See §9.19**]**

10. Find the geodesic curvature in terms of normal angle. **[Ans.** $\kappa_g = \frac{1}{\rho}\sin\overline{w}$**]**

11. Derive the basic expression for the torsion of a geodesic on a surface. **[Ans.** $\tau_g = \left[\hat{N}\ \hat{N}'\ \hat{t}\right]$**]**

12. State and prove Gauss-Bonnet theorem. **[Ans.** See §9.27**]**

13. Prove that on a surface with metric $ds^2 = a^2\, du^2 + b^2\, dv^2$ the geodesic curvature of the curve

$$u = c \quad \text{is} \quad (ab)^{-1}\left(\frac{\partial b}{\partial u}\right).$$

□□□

Unit 10 : Gauss Formulae, Gauss's Characteristic Equation Weingarten Equations, Mainardi-Codazzi Equations. Fundamental Existence Theorem for Surfaces, Parallel Surfaces, Gaussian and Mean Curvature for a Parallel Surface, Bonnet's Theorem on Parallel Surfaces.

Structure of the Unit

10.0 Objectives

Six fundamental magnitudes *E, F, G* and *L, M, N* and their partial derivatives play an important role in the surface theory. Gauss's formulae and Gauss's characteristic equations are some of the relations between them.

10.1 Introduction

This unit is devoted to the study of some relations between *E, F, G* and *L, M, N* and their partial derivatives. We also study the fundamental existence theorem for surfaces and Bonnets theorem on parallel surfaces.

10.2 Gauss's formulae

The Gauss's formulae or Gauss's equations are given below

$$\left.\begin{aligned} r_{11} &= L\,\hat{N} + l\,r_1 + \lambda\,r_2 \\ r_{12} &= M\,\hat{N} + m\,r_1 + \mu\,r_2 \\ r_{22} &= N\,\hat{N} + n\,\vec{r}_1 + \nu\,r_2 \end{aligned}\right\} \qquad .....(10.2.1)$$

where l, m, n; λ, μ, ν are called Christoffel symbols and are suitable functions of E, F, G and their partial derivatives with respect to u and v.

Proof : Second order partial derivative of $\vec{r}$ w. r. to u and v can be expressed linearly in terms of $\vec{r}_1, \vec{r}_2$ and $\hat{N}$ as given below.

$$\vec{r}_{11} = A\,\hat{N} + l\,\vec{r}_1 + \lambda\,\vec{r}_2, \qquad .....(10.2.2)$$

$$\vec{r}_{12} = B\,\hat{N} + m\,\vec{r}_1 + \mu\,\vec{r}_2, \qquad .....(10.2.3)$$

$$\vec{r}_{22} = C\,\hat{N} + n\,\vec{r}_1 + \nu\,\vec{r}_2, \qquad .....(10.2.4)$$

where A, B, C ; l, m, n ; λ, μ, ν are the coefficients to be determined.

Taking scalar multiplication of (10.2.2) by $\hat{N}$, we get

$$\vec{r}_{11}\cdot\hat{N} = A\,\hat{N}\cdot\hat{N} = A \text{ as } \vec{r}_1\cdot\hat{N} = 0 = \vec{r}_2\cdot\hat{N}$$

$\therefore \quad A = L$ as $\vec{r}_{11}\cdot\hat{N} = L$.

Similarly $\qquad B = M,\ C = N. \qquad .....(10.2.5)$

Hence the relation (10.2.2) to (10.2.3) assume the form (10.2.1).

Now to determine l, m, n ; λ, μ, ν we proceed as follow :

Clearly

$$\left.\begin{aligned} &E_1 = \frac{\partial E}{\partial u} = \frac{\partial}{\partial u}\left(\vec{r}_1^{\,2}\right) = 2\vec{r}_1\cdot\vec{r}_{11}\ ;\ E_2 = \frac{\partial E}{\partial v} = 2\vec{r}_1\cdot\vec{r}_{12}\ ; \\ &F_1 = \frac{\partial F}{\partial u} = \vec{r}_{11}\cdot\vec{r}_2 + \vec{r}_1\cdot\vec{r}_{12}\ ;\ F_2 = \frac{\partial F}{\partial v} = \vec{r}_{12}\cdot\vec{r}_2 + \vec{r}_1\cdot\vec{r}_{22} \\ &G_1 = \frac{\partial G}{\partial u} = 2\vec{r}_2\cdot\vec{r}_{12}\ ;\ G_2 = 2\,\vec{r}_2\cdot\vec{r}_{22} \end{aligned}\right\} \qquad .....(10.2.6)$$

Hence

$$\left.\begin{aligned} &\vec{r}_1\cdot\vec{r}_{11} = \frac{1}{2}E_1 \quad ; \quad \vec{r}_{11}\cdot\vec{r}_2 = F_1 - \frac{1}{2}E_1 \\ &\vec{r}_1\cdot\vec{r}_{12} = \frac{1}{2}E_2 \quad ; \quad \vec{r}_2\cdot\vec{r}_{12} = \frac{1}{2}G_1 \\ &\vec{r}_1\cdot\vec{r}_{22} = F_2 - \frac{1}{2}G_1 \quad ; \quad \vec{r}_2\cdot\vec{r}_{22} = \frac{1}{2}G_2 \end{aligned}\right\} \qquad .....(10.2.7)$$

Multiplying (10.2.2) by $\vec{r}_1$ and $\vec{r}_2$ successively scalarly, we get

$$\left.\begin{aligned}\vec{r}_1\cdot\vec{r}_{11} &= l\,E+\lambda F\\ \vec{r}_2\cdot\vec{r}_{11} &= l\,F+\lambda G\end{aligned}\right\} \qquad .....(10.2.8)$$

Using (10.2.7) in (10.2.8) and then on solving we get

$$\frac{1}{2}E_1 = l\,E+\lambda F \quad \text{and} \quad F_1-\frac{1}{2}E_2 = l\,F+\lambda G\,, \qquad .....(10.2.9)$$

where values of l and λ are given by

$$l=\frac{1}{2H^2}\left(GE_1-2FF_1+FE_2\right),\ \lambda=\frac{1}{2H^2}\left(2EF_1-EE_2+FE_1\right).$$

Proceeding exactly with (10.2.3) and (10.2.4), we get

$$m=\frac{1}{2H^2}\left(GF_1-FG_1\right)\ ;\ \ \mu=\frac{1}{2H^2}\left(EG_1-FE_2\right)$$

$$n=\frac{1}{2H^2}\left(2GF_2-GG_1-FG_2\right),\ \nu=\frac{1}{2H^2}\left(EG_2-2FF_2+FG_1\right). \qquad .....(10.2.10)$$

Corollary : In the case, the parametric curves are orthogonal, then $F = 0$ and $H^2 = EG$.

Hence $$l=\frac{GE_1}{2H^2}=\frac{E_1}{2E},\ \ m=\frac{E_2}{2E}\ ,\ n=-\frac{G_1}{2E};$$

$$\lambda=-\frac{E_2}{2G},\ \ \mu=\frac{G_1}{2G}\ ,\ \nu=\frac{G_2}{2G}. \qquad .....(10.2.11)$$

Thus Gauss's formulae become

$$r_{11}=L\,\hat{N}+\frac{1}{2}\frac{E}{E_1}r_1-\frac{E_2}{2G}r_2,$$

$$r_{12}=M\,\hat{N}+\frac{E_2}{2E}r_1+\frac{G_1}{2G}r_2,$$

and $$r_{22}=N\,\hat{N}-\frac{G_1}{2E}\,r_1+\frac{G_2}{2G}r_2 \qquad .....(10.2.12)$$

10.3 Gauss's characteristic equations

It states that

$$T^2=\frac{1}{2}\left(2F_{12}-E_{22}-G_{11}\right)+\left(m^2E+2m\mu F+\mu^2G\right)\ -\left(ln\ E+\left(l\nu+\lambda n\right)F+\lambda\nu G\right)$$

where $T^2 = LN - M^2$.

Proof : Taking scalar product of first and third Gauss's formulae and subtracting the square of the second, we get

$$\vec{r}_{11}\cdot\vec{r}_{22}-\vec{r}_{12}^{\,2} = \left(L\hat{N}+l\vec{r}_1+\lambda\vec{r}_2\right)\cdot\left(N\hat{N}+n\vec{r}_1+\nu\vec{r}_2\right)-\left(M\hat{N}+m\vec{r}_1+\mu\vec{r}_2\right)^2$$

$$= LN-M^2-\left(ln-m^2\right)\vec{r}_2^{\,2}+\left(l\nu+\lambda n-2m\mu\right)\vec{r}_1\cdot\vec{r}_2+\left(\lambda\nu-\mu^2\right)\vec{r}_2^{\,2}$$

$$= T^2+\left(ln-m^2\right)E+\left(l\nu-2m\mu+\lambda n\right)F+\left(\lambda\nu-\mu^2\right)G \qquad .....(10.3.1)$$

But $\quad E=\vec{r}_1^{\,2}$ *i.e.*, $\dfrac{1}{2}E_2=\vec{r}_1\cdot\vec{r}_{12}$ and $\dfrac{1}{2}E_{22}=\vec{r}_{12}^{\,2}+\vec{r}_1\cdot\vec{r}_{122}$

$F=\vec{r}_1\cdot\vec{r}_2$ *i.e.*, $F_2=\vec{r}_{12}\cdot\vec{r}_2+\vec{r}_1\cdot\vec{r}_{22}$ and $F_{12}=\vec{r}_{12}\cdot\vec{r}_{21}+\vec{r}_{121}\cdot\vec{r}_2+\vec{r}_{11}\cdot\vec{r}_{22}+\vec{r}_1\cdot\vec{r}_{221}$

$G=\vec{r}_2^{\,2}\Rightarrow\dfrac{1}{2}G_1=\vec{r}_2\cdot\vec{r}_{12}$ and $\dfrac{1}{2}G_{11}=\vec{r}_{21}\cdot\vec{r}_{12}+\vec{r}_2\cdot\vec{r}_{121}$

Hence $\dfrac{1}{2}\left(E_{22}+G_{11}-2F_{12}\right)=\vec{r}_{12}^{\,2}-\vec{r}_{11}\cdot\vec{r}_{22}$. $\qquad$ .....(10.3.2)

From (10.3.1) and (10.3.2), we get desired characteristic equation as

$$T^2=\frac{1}{2}\left(2F_{12}-E_{22}-G_{11}\right)+\left(m^2E+2m\mu F+\mu^2G\right)-\left(\ln E+\left(l\nu+\lambda n\right)F+\lambda\nu G\right).$$

Corollary 1. Gauss's characteristic equation can be put in the following form using the values of l, m, n ; λ, μ, ν form 10.2

$$LN-M^2=\frac{1}{2}H\frac{\partial}{\partial u}\left(\frac{F}{EH}E_2-\frac{1}{H}G_1\right)+\frac{1}{2}H\frac{\partial}{\partial v}\left(\frac{2F_1}{H}-\frac{E_2}{H}-\frac{FE_1}{EH}\right) \qquad .....(10.3.3)$$

Corollary 2. Suppose $\vec{r}=\vec{r}\left(u,v\right)$ represents a surface. We know that at any point (u, v) on the surface, the Gaussian curvature K is given by

$$\kappa=\frac{LN-M^2}{H^2}=\frac{1}{2H}\frac{\partial}{\partial u}\left(\frac{FE_2}{EH}-\frac{G_1}{H}\right)+\frac{1}{2H}\frac{\partial}{\partial v}\left(\frac{2F_1}{H}-\frac{E_2}{H}-\frac{FE_1}{EH}\right)$$

[Using (10.3.3) equation of corollary 1]

$$=\frac{1}{H}\frac{\partial}{\partial u}\left(\frac{FE_2-EG_1}{2HE}\right)+\frac{1}{H}\frac{\partial}{\partial v}\left(\frac{2EF_1-FE_1-EE_2}{2HE}\right) \qquad .....(10.3.4)$$

where $H^2 = EG - F^2$.

The formula (10.3.4) gives Gaussian curvature K in terms of first fundamental magnitudes E, F, G and their partial derivatives with respect to u and v.

Thus equation (10.3.4) is intrinsic formula for Gaussian curvature. In the case, the parametric curves are orthogonal, $F = 0$ and then $F_1 = 0$.

$$\Rightarrow \qquad LN-M^2=2\frac{1}{H}\frac{\partial}{\partial u}\left(-\frac{G_1}{H}\right)+\frac{1}{2}H\frac{\partial}{\partial v}\left(-\frac{E_2}{H}\right) \qquad [\because F=0]$$

But $\quad H^2 = EG - F^2 = EG.$

$$\therefore \qquad \kappa = \frac{LN - M^2}{H^2} = -\frac{1}{2H}\left[\frac{\partial}{\partial u}\left(\frac{G_1}{H}\right) + \frac{\partial}{\partial v}\left(\frac{E_2}{H}\right)\right]$$

$$= -\frac{1}{H}\left[\frac{1}{2}\frac{\partial}{\partial u}\left(\frac{G_1}{\sqrt{EG}}\right) + \frac{1}{2}\frac{\partial}{\partial v}\left(\frac{E_2}{\sqrt{EG}}\right)\right]$$

$$= -\frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\left(\sqrt{G}\right)}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\cdot\frac{\partial\left(\sqrt{E}\right)}{\partial v}\right)\right].$$

10.4 Weingarten equations

The formulae $\qquad H^2 N_1 = (FM - GL)\ \vec{r}_1 + (FL - EM)\ \vec{r}_2$

and $\qquad H^2 N_2 = (FN - GM)\ \vec{r}_1 + (FM - EN)\ \vec{r}_2$

are known as Weingarten formulae or Weingarten equations.

Proof : Since $\qquad \hat{N} = 1 \Rightarrow \hat{N}\cdot\hat{N}_1 = 0, \quad \hat{N}\cdot\hat{N}_2 = 0.$

Thus $\vec{N}_1$ and $\vec{N}_2$ are perpendicular to $\hat{N}$. Hence, these lie in the plane of $\vec{r}_1$ and $\vec{r}_2$.

Hence $$\vec{N}_1 = A\vec{r}_1 + B\vec{r}_2 \qquad .....(10.4.1)$$

Taking scalar multiplication of (10.4.1) with $\vec{r}_1$ and $\vec{r}_2$ successively, we get

$$-L = AE + BF, \quad -M = AF + BG.$$

Solving for A and B, we get

$$A = (FM - GL)/(EG - F^2),\ B = (FL - EM)/(EG - F^2)$$

But $EG - F^2 = H^2$, therefore by (10.4.1)

$$H^2\ \vec{N}_1 = (FM - GL)\ \vec{r}_1 + (FL - EM)\ \vec{r}_2$$

Similarly we can get second equation.

10.5 Mainardi–Codazzi equations

The three fundamental magnitudes L, M, N are not functionally independent. They are related through the equations.

$$L_2 - M_1 = mL - (l - \mu)\,M - \lambda\,N$$

$$M_2 - N_1 = nL - (m - \nu)M - \mu N.$$

which are called Mainardi-Codazzi equations.

Proof : Consider the identity

$$\frac{\partial}{\partial v}(\vec{r}_{11}) = \frac{\partial}{\partial u}(\vec{r}_{12}) \qquad .....(10.5.1)$$

Putting the values of $\vec{r}_{11}$ and $\vec{r}_{12}$ from Gauss's formula in (10.5.1), we get

$$\frac{\partial}{\partial v}\left(L\,\hat{N}+l\,\vec{r}_1+\lambda\,\vec{r}_2\right)=\frac{\partial}{\partial u}\left(M\,\hat{N}+m\,\vec{r}_1+\mu\,\vec{r}_2\right)$$

$$\Rightarrow \qquad L_2\hat{N}+l_2\vec{r}_1+\lambda_2\,\vec{r}_2+L\,N_2+l\,\vec{r}_{12}+\lambda\,\vec{r}_{22}=M_1\hat{N}+m_1\,\vec{r}_1+\mu_1\,\vec{r}_2+M\hat{N}_1+m\,\vec{r}_{11}+\mu\,\vec{r}_{12} \quad .$$

....(10.5.2)

Now substituting the values of $\vec{r}_{11},\vec{r}_{22},\vec{r}_{12}$ form Gauss's formulae and the values of $\hat{N}_1$ and $\hat{N}_2$ form Weingarten equations, namely

$$\hat{N}_1=\frac{1}{H^2}\left(FM-GL\right)\vec{r}_1+\frac{1}{H^2}\left(FL-EM\right)\vec{r}_2$$

and
$$\hat{N}_2=\frac{1}{H^2}\left(FN-GM\right)\vec{r}_1+\frac{1}{H^2}\left(FM-EN\right)\vec{r}_2 \qquad .....(10.5.3)$$

The identity (10.5.2) is expressed in terms of vectors $\hat{N}$, r_1, r_2 (non-coplaner vectors).

On equating the coefficients of $\hat{N}$ on both sides, we get

$$L_2+l\,M+\lambda N=M_1+m\,L+\mu M$$

or
$$L_2-M_1=m\,L-\left(l-\mu\right)M-\lambda N \qquad .....(10.5.4)$$

which is the first Mainardi-Codazzi equation.

Now consider the identity

$$\frac{\partial}{\partial v}\left(\vec{r}_{12}\right)=\frac{\partial}{\partial u}\left(\vec{r}_{22}\right). \qquad .....(10.5.5)$$

From Gauss's formula, we have

$$\vec{r}_{12}=M\hat{N}+m\,\vec{r}_1+\mu\,\vec{r}_2\,;\;\vec{r}_{22}=N\hat{N}+n\,\vec{r}_1+\nu\,\vec{r}_2 \qquad .....(10.5.6)$$

Putting these values in (10.5.5), we get

$$M_2\hat{N}+m_1\,\vec{r}_1+M\,\vec{N}_2+m\,\vec{r}_{12}+\mu\,\vec{r}_{22}=N_1\hat{N}+n_1\,\vec{r}_1+\nu\vec{r}_2+N\vec{N}_1+n\,\vec{r}_{11}+\nu\,\vec{r}_{12} \qquad .....(10.5.7)$$

Substituting in (10.5.7), the values of $\vec{r}_{11},\vec{r}_{22},\vec{r}_{12}$ from Gauss's formulae and for $\hat{N}_1,\hat{N}_2$ from Weingarten equation, we get a vector identity and then equating coefficients of $\hat{N}$ on both sides of the identity, we get

$$M_2-mN+\mu N=N_1+nL+\nu M$$

or
$$M_2-N_1=nL-(m-\nu)\,M+\mu N, \qquad .....(10.5.8)$$

which is second Mainardi-Codazzi equation.

10.6 Illustrative examples

Ex.1. *Show that for the surface $z=f(x, y)$ with x, y are parameters.*

$$l=\frac{pr}{H^2},\; m=\frac{ps}{H^2},\; n=\frac{pt}{H^2}\;\;;\; \lambda=\frac{qr}{H^2},\; \mu=\frac{qs}{H^2}\;,\; \nu=\frac{qt}{H^2}.$$

Sol. We have for the surface $z = f(x, y)$.

$$E = 1 + p^2, \qquad F = pq, \qquad G = 1 + q^2,$$

$$E_1 = 2pr, \qquad F_1 = rq + pq, \qquad G_1 = 2qs,$$

and $$E_2 = 2ps, \qquad F_2 = sq + pt, \qquad G_2 = 2qt. \qquad \text{.....(1)}$$

Again $$l = \frac{2}{H^2}\left(GE_1 - 2FF_1 + FE_2\right)$$

$$= \frac{1}{2H^2}\left\{\left(1+q^2\right)2pr - 2pq\left(rq + ps\right) + pq\left(2ps\right)\right\}$$

$$= \frac{1}{2H^2}\left(2pq + 2prq^2 - 2prq^2 - 2p^2qs + 2p^2qs\right) = \frac{pq}{H^2} \qquad \text{.....(2)}$$

Similarly, on putting these values in experience for m, n, λ, μ, ν, we get the required results.

***Ex.*2.** *For any surface prove that*

$$\frac{\partial}{\partial u}\left(\log H\right) = l + \mu, \ \frac{\partial}{\partial v}\left(\log H\right) = m + \nu .$$

where u and v are parameters and symbols have their usual meaning.

Sol. We have $H^2 = EG - F^2$

$$\therefore \qquad \frac{\partial}{\partial u}\left(\log H\right) = \frac{\partial}{\partial u}\left(\frac{1}{2}\log H^2\right) = \frac{1}{2}\cdot\frac{1}{H^2}\cdot\frac{\partial}{\partial u}\left(H^2\right)$$

$$= \frac{1}{2H^2}\frac{\partial}{\partial u}\left(EG - F^2\right) = \frac{E_1G + G_1E - 2FF_1}{2H^2}. \qquad \text{.....(1)}$$

Using values of Christoffel symbols from §10.2, we get

$$l + \mu = \frac{1}{2H^2}\left(GE_1 + FE_2 - 2FE_1 + EG_1 - FE_2\right)$$

$$= \frac{1}{2H^2}\left(E_1G + EG_1 - 2FF_1\right). \qquad \text{.....(2)}$$

From (1) and (2), $$\frac{\partial}{\partial u}\left(\log H\right) = l + \mu .$$

Similarly, we can prove the second result.

***Ex.*3.** *Show that for the right helicoid* $\vec{r} = (u \cos v, u \sin v, cv)$.

$$l = 0, \ m = 0, \ n = -u \ ; \ \lambda = 0, \ \mu = \frac{u}{\left(n^2 + c^2\right)}, \ \nu = 0.$$

Sol. We have $\vec{r} = (u \cos v, u \sin v, cv)$

$$\therefore \qquad \vec{r}_1 = \left(\cos v, \sin v, 0\right), \ \vec{r}_2 = \left(-u\sin v, u\cos v, c\right)$$

$$\therefore \qquad E = \vec{r}_1^{\,2} = 1, \ F = \vec{r}_1 \cdot \vec{r}_2 = 0, \ G = \vec{r}_2^{\,2} = u^2 + c^2$$

$$H^2 = EG - F^2 = u^2 + c^2 \qquad \text{.....(1)}$$

Therefore $E_1 = 0, E_2 = 0, F_1 = 0 = F_2$; $G_1 = 2u, G_2 = 0$.

Again $$l = \frac{1}{2H^2}\left(GE_1 - 2FF_1 + FF_2\right) = 0$$

$$\lambda = \frac{1}{2H^2}\left(2EF_1 - EE_2 - FE_1\right) = 0$$

$$m = \frac{1}{2H^2}\left(GE_2 - FG_1\right) = 0, \ \mu = \frac{1}{2H^2}\left(EF_1 - FE_2\right) = \frac{u}{u^2+c^2}$$

$$n = \frac{1}{2H^2}\left(2GF_2 - GG_1 - FG_2\right) = \frac{1}{2\left(u^2+c^2\right)}\ [-(u^2+c^2)2u] = -u,$$

$$\nu = \frac{1}{2H^2}\left(EG_2 - 2FF_2 + EG_1\right) = 0. \qquad .....(2)$$

Ex.4. *From the Gauss's characteristic equation deduce that, when the parametric curves are orthogonal*

$$\kappa = -\frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)\right].$$

Sol. In case parametric curves are orthogonal, we have

$$F = 0, \quad \therefore\ F_1 = 0 \qquad .....(1)$$

Hence equation (3) of §10.3 corollary 1, we have

$$LN - M^2 = \frac{1}{2}H\frac{\partial}{\partial u}\left(-\frac{G_1}{H}\right) + \frac{1}{2}H\frac{\partial}{\partial v}\left(-\frac{E_2}{H}\right) \qquad .....(2)$$

But $H^2 = EG - F^2 = EG$.

$$\therefore \quad \kappa = \frac{LN - M^2}{H^2}$$

$$= -\frac{1}{2H}\left[\frac{\partial}{\partial u}\left(\frac{G_1}{H}\right) + \frac{\partial}{\partial v}\left(\frac{E_2}{H}\right)\right]$$

$$= \frac{1}{H}\left[\frac{1}{2}\frac{\partial}{\partial u}\left\{\frac{G_1}{\sqrt{EG}}\right\} + \frac{1}{2}\frac{\partial}{\partial v}\left\{\frac{E_2}{\sqrt{EG}}\right\}\right]$$

$$= -\frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial}{\partial u}\sqrt{G}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial}{\partial v}\sqrt{E}\right)\right]. \qquad .....(3)$$

Ex.5. *Show that the surface whose metric is given by*

$$ds^2 = du^2 + D^2\,dv^2,$$

$l = 0, m = 0, n = -DD_1, \quad \lambda = 0, \ \mu = \frac{D_1}{D}, \quad \upsilon = \frac{D_2}{D}$.

Sol. Comparing $ds^2 = du^2 + D^2\, dv^2$ with

$$ds^2 = E\, du^2 + 2F\, dudv + G\, dv^2 \qquad .....(1)$$

Here, we have $E = 1,\quad F = 0,\quad G = D^2,\quad H^2 = EG - F^2 = D^2$

$\therefore \qquad E_1 = 0,\quad G_1 = 2DD_1;\quad E_2 = 0,\ G_2 = 2DD_1$

$$\lambda = \frac{E_2}{2G} = 0,\ \mu = \frac{G_1}{2G} = \frac{D_1}{2D},\ \upsilon = \frac{G_2}{2G} = \frac{D_2}{D} \qquad .....(2)$$

Ex.6. *For a surface given by $ds^2 = \phi\,(du^2 + dv^2)$ prove that*

$$l = \frac{\phi_1}{2\phi},\ m = \frac{\phi_2}{2\phi},\ n = -\frac{\phi_1}{2\phi},\ \lambda = \frac{\phi_2}{2\phi},\ \mu = \frac{\phi_1}{2\phi},\ \upsilon = \frac{\phi_2}{2\phi}$$

and further show that Mainardi-Codazzi relations become

$$L_2 - M_1 = \frac{1}{2}\frac{\phi_2}{\phi}(L+N),\ N_1 - M_2 = \frac{1}{2}\frac{\phi_1}{\phi}(L+N)$$

Also show that the Gauss characteristics equation then

$$LN - M^2 = \frac{1}{2}\phi\left(\phi_1^{\,2} + \phi_2^{\,2}\right) - \frac{1}{2}(\phi_{11} + \phi_{22}).$$

Sol. Comparing $ds^2 = \phi\,(du^2 + dv^2)$ with

$$ds^2 = E\, du^2 + 2F\, dudv + G\, dv^2$$

$$E = \phi,\quad F = 0,\quad G = \phi.$$

As $\phi = 0$, the Gauss's coefficient reduces to

$$l = \frac{1}{2}\frac{\phi_1}{\phi},\ m = \frac{1}{2}\frac{\phi_2}{\phi},\ n = -\frac{1}{2}\frac{\phi_1}{\phi},\ \lambda = -\frac{1}{2}\frac{\phi_2}{\phi},\ \mu = \frac{1}{2}\frac{\phi_1}{\phi},\ \upsilon = \frac{1}{2}\frac{\phi_2}{\phi}.$$

Hence, the Mainardi-Codazzi relations become

$$L_2 - M_1 = \frac{1}{2}\frac{\phi_2}{\phi}L - \left(\frac{1}{2}\frac{\phi_1}{\phi} - \frac{1}{2}\frac{\phi_1}{\phi}\right)M + \frac{1}{2}\frac{\phi_2}{\phi}N = \frac{1}{2}\frac{\phi_2}{\phi}(L+N)$$

and $$M_2 - N_1 = -\frac{1}{2}\frac{\phi_1}{\phi}L - \left(\frac{1}{2}\frac{\phi_2}{\phi} - \frac{1}{2}\frac{\phi_2}{\phi}\right)M - \frac{1}{2}\frac{\phi_1}{\phi}N = \frac{1}{2}\frac{\phi_1}{\phi}(L+N).$$

Ex.7. *Show that when the lines of curvature are chosen as parametric curves, the Codazzi relations expressed in terms of E, G, L, N and their derivatives are*

$$L_2 = \frac{1}{2}E_2\left(\frac{L}{E} + \frac{N}{G}\right),\ N_1 = \frac{1}{2}G_1\left(\frac{L}{E} + \frac{N}{G}\right).$$

Show also that the equation of Gauss may be written as

$$\frac{LN}{\sqrt{EG}} + \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right) = 0.$$

Sol. The Codazzi equations are

$$L_2 - M_1 = mL - (l - \mu)\, M - \lambda N \qquad .....(1)$$

$$M_2 - N_1 = nL - (m - \upsilon)\, M - \mu N \qquad .....(2)$$

Since lines of curvature are parametric curves, we have $F = 0$, $M = 0$.

Also then the Gaussian coefficients are

$$l = \frac{E_2}{2E},\ m = \frac{E_2}{2E},\ n = -\frac{G_2}{2E},\ \lambda = -\frac{E_2}{2G},\ \mu = \frac{G_1}{2G},\ \upsilon = \frac{G_2}{2G}.$$

Thus, the equation (1) reduces to

$$L_2 = \frac{E_2}{2E}L + \frac{E_2}{2G}N = \frac{1}{2}E_2\left(\frac{L}{E} + \frac{N}{G}\right)$$

and equation (2) reduces to

$$-N_1 = -\frac{G_1}{2E}L - \frac{G_1}{2G}N \quad \text{or} \quad N_1 = \frac{1}{2}G_1\left(\frac{L}{E} + \frac{N}{G}\right).$$

For the second part, from Ex.4 above, we have

$$\kappa = \frac{1}{\sqrt{LG}}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)\right]. \qquad .....(3)$$

In this case $\kappa = \dfrac{LN - M^2}{EG - F^2} = \dfrac{LN}{EG}.$ $\qquad [\because\ F = 0 = M]$

Putting this value of k in (3), we get the required result.

10.7 Fundamental existence theorem for surfaces

Statement : When the coefficients of the two quadratic differential forms

$$E\,du^2 + 2\,F\,dvdu + G\,dv^2 \quad \text{and} \quad L\,du^2 + 2M\,dudv + N\,dv^2$$

are such that the first form is positive definite and the six coefficients satisfy the Gauss' characteristic equation and the Mainardi-Codazzi equations, then there exists a surface, uniquely determined to within a Euclidean displacement, for which these forms are respectively the first and second fundamental forms.

Proof. The proofs of this theorem depends on the existence and uniqueness theorems of the first order differential equations which can be obtained from the two given fundamental quadratic differential forms.

It is easy, if we choose the principal directions and the normal $\hat{N}$ to the surface $\vec{r} = \vec{r}(u, v)$ in the curvilinear coordinates, as the coordinate axes. In this case $F = 0 = M$ and quadratic differential forms reduces to

$$E\,du^2 + G\,dv^2 \quad \text{and} \quad L\,du^2 + N\,dv^2 \qquad .....(10.7.1)$$

Let $\hat{\alpha}(u, v)$, $\hat{\beta}(u, v)$ denote the unit tangent vectors along the parametric curves v = constant (u-curve) and u = constant (v-curve), respectively.

Then $$\hat{\alpha} = \frac{1}{\sqrt{E}}\vec{r}_1, \quad \hat{\beta} = \frac{1}{\sqrt{G}}\vec{r}_2. \qquad .....(10.7.2)$$

In this statement of the theorem, we have given that the fundamental coefficients E, F, G and L, M, N satisfy Gauss's characteristics equation and the Mainardi-Codazzi equations and the Weingarten equations which are Gauss's equations

$$\vec{r}_{11} = \frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial u}\vec{r}_1 = \frac{\sqrt{E}}{G}\frac{\partial\sqrt{E}}{\partial v}\vec{r}_2 + L\hat{N}$$

$$\vec{r}_{12} = \frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial v}\vec{r}_1 + \frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial u}\vec{r}_2$$

$$\vec{r}_{22} = -\frac{\sqrt{G}}{E}\frac{\partial\sqrt{G}}{\partial u}\vec{r}_1 + \frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial v}\vec{r}_2 + N\hat{N} \qquad .....(10.7.3)$$

Weingarten equations

$$N_1 = -\frac{L}{E}\vec{r}_1, \quad N_2 = -\frac{N}{G}\vec{r}_2. \qquad .....(10.7.4)$$

Differentiating (10.7.2) partially with respect to u and v. Using (10.7.3) and (10.7.4), we find

$$\frac{\partial\hat{\alpha}}{\partial u} = -\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\hat{\beta} + \frac{L}{\sqrt{E}}\hat{N}, \quad \frac{\partial\hat{\alpha}}{\partial v} = \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\hat{\beta};$$

$$\frac{\partial\hat{\beta}}{\partial u} = \frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\hat{\alpha}, \quad \frac{\partial\hat{\beta}}{\partial v} = -\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\hat{\alpha} + \frac{N}{\sqrt{G}}\hat{N};$$

$$\frac{\partial\hat{N}}{\partial u} = -\frac{L}{\sqrt{E}}\hat{\alpha} \quad \text{and} \quad \frac{\partial\hat{N}}{\partial v} = -\frac{N}{\sqrt{G}}\hat{\beta}. \qquad .....(10.7.5)$$

Thus six first order partial differential equations for the triad $\{\hat{\alpha}(u,v), \hat{\beta}(u,v), \hat{N}(u,v)\}$ are necessary conditions to be satisfied so that their exists a surface $\vec{r} = \vec{r}(u,v)$ exists with $\vec{r}_1^{\,2} = E$, $\vec{r}_1 \cdot \vec{r}_2 = 0$, $\vec{r}_2^{\,2} = G$ and the equations (10.7.5) admits at least one solution $(\hat{\alpha}, \hat{\beta}, \hat{N})$ which assumes the prescribed values u_0, v_0 for u and v in a given interval such that $u(0) = u_0$, $v(0) = v_0$.

Now, if there are two surfaces S, S^* with the same prescribed fundamental forms then an Euclidean displacement we can arrange the triad $(\hat{\alpha}, \hat{\beta}, \hat{N})$ and $(\hat{\alpha}^*, \hat{\beta}^*, \hat{N}^*)$ to coincide when $u = u_0$, $v = v_0$ and therefore they all coincide for all values of u and v.

Hence, the surfaces S and S^* differ by atmost a Euclidean motion. This proves the uniqueness of the solutions limited within the Euclidean displacement.

10.8 Parallel surfaces

A surface S is said to be parallel to another surface S^* if the points of S^* are at a constant distance along the normal to S.

Clearly if $P(\vec{r})$ is any point on S, then corresponding point Q on S^* is given by

$$\vec{r}^* = \vec{r} - c\hat{N} \qquad .....(10.8.1)$$

where c is a scalar constant whose magnitude represents the distance along normal *i.e.,*

$$\left|\overrightarrow{PQ}\right| = c.$$

Fundamental magnitudes of S^* :

Clearly on differentiating (10.8.1) with respect to u and v successively we can get fundamental magnitudes provided we know $\vec{N}_1, \vec{N}_2$. $\vec{N}_1, \vec{N}_2$ are given by Weingarten formula.

As a particular case if lines of curvature are parametric curves *i.e.,* $F = 0 = M$, the Weingarten formulae reduces to

$$EG\vec{N}_1 = -GL\vec{r}_1, \quad \left[\because H^2 = EG \text{ when } F = 0 = M\right] \qquad .....(10.8.2)$$

Writing $\dfrac{L}{E} = \kappa_a$, $\dfrac{N}{G} = \kappa_b$, we shall have fundamental magnitudes $E^*, F^*, G^*, L^*, M^*, N^*$ as

$$E^* = E\left(1 + c\kappa_a\right)^2, \; F^* = 0, \; G^* = G\left(1 + c\kappa_b\right)^2$$

$$L^* = \pm\left(1 + c\kappa_a\right)L, \; M^* = 0, \; N^* = \pm\left(1 + c\kappa_b\right)^2$$

Also $$H^* = H\left(1 + 2\mu c + \kappa c^2\right). \qquad .....(10.8.3)$$

It at once follows that curves on S^* corresponding to lines of curvature of S are also lines of curvature on S^*.

10.9 Gaussian and mean curvature for the parallel surfaces

We have
$$\kappa^* = \frac{L^* N^* - M^{*2}}{H^{*2}}$$

$$= \frac{LN\left(1 + c\kappa_a\right)\left(1 + c\kappa_b\right) - 0}{H^2\left(1 + c\kappa_a\right)^2\left(1 + c\kappa_b\right)^2}$$

$$= \frac{\kappa}{\left(1 + c\kappa_a\right)\left(1 + c\kappa_b\right)}$$

$$= \frac{\kappa}{1 + 2\mu c + \kappa c^2}. \quad \left[\because \kappa = \frac{LN - 0}{H^2}\right] \qquad .....(10.9.1)$$

Remark : In the above proof, we have not taken into account the case when $c = -\dfrac{1}{\kappa_a}$ or $-\dfrac{1}{\kappa_b}$.

This happens in case parallel surfaces are spheres of radii $2B$ concentric with S. Then parallel surfaces degenerate into points.

Thus first part of the Bonnet's theorem can be modified into the following :

"For every surface with constant positive Gaussian curvature k there exists at least one (non-singular) parallel surface with constant mean curvature.

10.10 Bonnet's theorem on parallel surfaces

For every surface with constant positive Gaussian curvature B^{-2}, in general, there are associated two surfaces of constant mean curvatures $(\pm 2B)^{-1}$, which are parallel to the surface S and distant $\pm B$ from it, and for every surface S with constant mean curvature $(2B)^{-1}$ there is a parallel surface of constant Gaussian curvature B^{-2} distant B from it.

Proof. Here $\kappa = B^{-2}$ and $c = \pm B$.

$$\therefore \qquad \mu^* = \frac{1}{2}\left(\frac{\kappa_a^* + \kappa_b^*}{2}\right) = \frac{1}{2}\left(\frac{L^*}{E^*} + \frac{N^*}{G^*}\right)$$

$$= \pm\frac{1}{2}\left(\frac{\mu + c\kappa}{1 + 2\mu c + c^2\kappa}\right), \quad \text{from above}$$

$$= \pm\frac{1}{2}\frac{\left(\mu + B^{-1}\right)}{(1 \pm \mu B)} = \pm(2B)^{-1} \qquad .....(10.10.1)$$

Conversely, using $\mu = (2B)^{-1}$, $c = -B$, we have

$$\kappa^* = \frac{L^*N^* - M^{*2}}{H^{*2}} = \frac{\kappa}{\left(1 + 2\mu c + \kappa c^2\right)}, \qquad \left[\because \ \kappa = \frac{LN}{H^2}\right]$$

$$= \frac{\kappa}{\left(1 - 2(2B)^{-1} \cdot B + \kappa B^2\right)}$$

$$= \frac{1}{B^2} = B^{-2} \ \text{(constant)}. \qquad .....(10.10.2)$$

Hence proved.

Remark : In the above proof also, we have not taken into account the case when

$$c = -\frac{1}{\kappa_a} \text{ or } -\frac{1}{\kappa_b}.$$

10.11 Self-learning exercises

1. Write Gauss's characteristic equation.
2. What are the Weingarten formulae ?

3. Define parallel surfaces.
4. Write Mainardi-Codazzi equations.
5. State Bonnets theorem on parallel surfaces.

10.12 Summary

In this unit we have derived. Gauss's formulae in the form of partial differential equations and Gauss's characteristic equation. We have also studied the Weingarten formula and Mainardi-Codazzi equations. Fundamental existence theorem for parallel surfaces has also been studied in this unit. For parallel surfaces Bonnet's theorem has also been discussed. Some solved question on above theorems have been given in the exercises.

10.13 Answers to self-learning exercises

1. § 10.3
2. § 10.4
3. § 10.8
4. § 10.5
5. § 10.10

10.14 Exercises

1. Obtain the fundamental equation of surface theory.
2. Obtain the equation of Weingarten and use them to establish Mainardi-Codazzi equations.
3. Prove that the Gaussian curvature at a point is expressible in terms of the fundamental magnitudes of the first order and their derivatives of the first two orders.
4. Prove the Gauss characteristic equation and deduces that, when parametric curves are orthogonal

$$\kappa = -\frac{1}{\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial}{\partial u}\left(\sqrt{G}\right)\right)+\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial}{\partial v}\left(\sqrt{E}\right)\right)\right].$$

□ □ □

Unit 11 : Tensor Analysis, Kronecker Delta, Contravariant and Covariant Tensors, Symmetric Tensors, Quotient law of Tensors, Relative tensor

Structure of the Unit

11.0 Objective

Tensor calculus is the generalisation of the differential geometry of Gauss and Riemann. Einstien used it as a most suitable tool for the study of his general theory of relativity. The reason behind it is that a physicist wants to formulate the laws of physics which remain same (*i.e.* invariant) when we go from one frame of reference to another. The objective of this unit is to define the tensorial quantities and their properties. We also study the algebra of tensorial quantities in this unit.

11.1 Introduction

The tensor formulation became popular when Einstien (1879-1955) used it as an excellent tool for the presentation of his general theory of relativity. It has now become an excellent tool in the study of many branches of theoretical physics, such as mechanics, Fluid Mechanics, Elasticity, Plasticity, Electromagnetic theory etc.

Tensor analysis is the generalization of vector calculus. It handles the answers to the questions such as :

(i) are all basic physical laws expressible in terms of scalars and vectors ?

(ii) which transformation is suitable for the invariant character of physical laws ?

(iii) how a certain physical law be written if wider class of transformation is introduced ?

It is a basic principal of tensor analysis that we should not tie ourselves down to any our system of coordinates, we seek statements which are true, not for one system of coordinates but for all. The transformation laws for the components of an entity from one coordinate system to another are the basic criteria to determine the tensor character of that entity. In other words :

"A tensor is an entity whose components, when are being transformed from one coordinate system to another, obey certain basic transformation laws." The study of these laws is the prime aim of this unit.

11.2 Space of N-dimensions

We know that in the three dimensional rectangular space, the coordinates of a point are given by triplets in the form (x, y, z) where x, y, z are three numbers. But this representation is not suitable if we want to generalizes the concept of space from three dimension to N-dimensions. That is why it is advisable to use a triplet (x^1, x^2, x^3) in place of (x, y, z) where 1, 2, 3 are the superscripts not power indices. In general, the coordinate of a point in N-dimensional space are given by the N-tuples of the form $(x^1, x^2, x^3, \dots x^N)$ where 1, 2, ... N are not powers of x but are the superscripts of x and $N \geq 2$. This type of N-dimension space is denoted by V_N.

11.3 Coordinate transformation

Consider two different frames of references of N-dimensions. Let the coordinates of a point with respect to these frames be respectively $(x^1, x^2, \ldots x^N)$ and $\left(\overline{x}^1, \overline{x}^2, \ldots \overline{x}^N\right)$. Suppose these coordinates of the two systems have the following independent relations :

$$\overline{x}^i = \overline{x}^i\left(x^1, x^2, \ldots x^N\right) \quad (i = 1, 2, \ldots, N) \qquad \text{.....(11.3.1)}$$

where $\overline{x}^i$ are single valued, continuous functions and have continuous derivatives for certain ranges of $x^1, x^2, \ldots x^N$. Under these conditions equations (11.3.1) can be solved for x^i as functions of $\overline{x}^i$ given by

$$x^i = x^i\left(\overline{x}^1, \overline{x}^2, \ldots \overline{x}^N\right) \qquad (i = 1, 2, 3, \ldots N) \qquad \text{.....(11.3.2)}$$

The relations given by (11.3.1) and (11.3.2) define a transformation of coordinates from one frame of reference to another.

Differentiating (11.3.1), we get

$$d\overline{x}^i = \frac{\partial \overline{x}^i}{\partial x^1} dx^1 + \frac{\partial \overline{x}^i}{\partial x^2} dx^2 + \ldots + \frac{\partial \overline{x}^i}{\partial x^N} dx^N = \sum_{j=1}^{N} \frac{\partial \overline{x}^i}{\partial x^j} dx^j \,. \qquad \text{.....(11.3.3)}$$

11.4 Summation convention

We know that the expression

$$a_1 x^1 + a_2 x^2 + a_3 x^3 + \ldots + a_N x^N \qquad \text{.....(11.4.1)}$$

is represented by $\sum_{i=1}^{N} a_i x^i$.

According to summation convention we drop sigma sign and merely write the above sum as $a_i x^i$.

(a) Thus by summation convention we mean that if a small latin index (superscripts or subscripts) is repeated in a term then it is understood that we are to sum over this index from 1 to N unless otherwise stated. This summation convention was first used by Einstien.

(b) **Indicial (or Range) convention :** When a small latin index is used either as superscript or subscripts occurs unrepeated in a term, it takes all values from 1 to N unless otherwise stated, N being the number of dimensions of the space.

The unrepeated latin index used in a term is called free or real index and takes all values from 1 to N. For example 'i' is the free index in the following expressions :

$$x^i = x^i\left(\overline{x}^1, \overline{x}^2, \overline{x}^3, \ldots \overline{x}^N\right), \qquad \text{.....(11.4.2)}$$

and

$$\overline{x}^i = \overline{x}^i\left(x^1, x^2, x^3, \ldots x^N\right), \qquad \text{.....(11.4.3)}$$

(c) **Dummy index :** Any index, which is repeated in a given term so that the summation convention applies, is called a dummy index or dummy suffix. This is also called umbral or dextral index.

11.5 Kronecker delta

The Kronecker delta which is denoted by δ^i_j, is defined as :

$$\delta^i_j = \begin{cases} 1 & , \ if\ i = j \\ 0 & , \ if\ i \neq j \end{cases}. \qquad .....(11.5.1)$$

Thus, we have $\delta^1_1 = \delta^2_2 = \delta^3_3 = ... = \delta^N_N = 1$ (no summation over N)

$$\delta^1_2 = \delta^2_3 = ... = 0 \qquad ......(11.5.2)$$

and $\delta^i_i = \delta^1_1 + \delta^2_2 + ... + \delta^N_N = 1 + 1 + ... + 1 = N.$ ......(11.5.3)

An important properly of Kronecker delta is that

$$\delta^i_j\, A^j = A^i, \qquad .....(11.5.4)$$

since in the L.H.S. summation is carried over j.

If may also be noted that $\dfrac{\partial x^i}{\partial x^j} = \delta^i_j$, since the coordinates $x^1, x^2, ... x^N$ are independent.

Similarly $$\frac{\partial \overline{x}^i}{\partial \overline{x}^j} = \delta^i_j.$$

***Ex.*1.** *Use Einstien is summation convention to write the following* :

(i) $A^k_1 B^1 + A^k_2 B^2 + ... + A^k_N B^N$

(ii) $ds^2 = g_{11}\left(dx^1\right)^2 + g_{22}\left(dx^2 dx^2\right) + ... + g_{NN}\left(dx^N\right)^2$

$$+ g_{12} dx^1 dx^2 + g_{21} dx^2 dx^1 + ... + g_{1N} dx^1 + g_{N1} dx^N dx^1$$

Sol. ***(i)*** $A^k_1 B^1 + A^k_2 B^2 + ... + A^k_N B^N = A^k_i B^i$

(ii) $ds^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} dx^1 dx^2.$

***Ex.*2.** *Show that*

(i) $\delta^i_j \delta^j_k = \delta^i_k$ *(ii)* $\dfrac{\partial x^k}{\partial \overline{x}^i} \cdot \dfrac{\partial \overline{x}^i}{\partial x^j} = \delta^k_j$

Sol. ***(i)***

$$\delta^i_j\, \delta^j_k = \delta^i_1\, \delta^1_k + \delta^i_2\, \delta^2_k + \delta^i_3\, \delta^3_k + ... + \delta^i_k\, \delta^k_k + ... + \delta^i_N\, \delta^N_k$$

$$= 0 + 0 + ... + \delta^i_k(1) + 0 ... + 0 \qquad \text{[no summation over } k]$$

$$= \delta^i_k.$$

(ii)

$$\frac{\partial x^k}{\partial \overline{x}^i} \frac{\partial \overline{x}^i}{\partial x^j} = \frac{\partial x^k}{\partial \overline{x}^1} \frac{\partial \overline{x}^1}{\partial x^j} + \frac{\partial x^k}{\partial \overline{x}^2} \frac{\partial \overline{x}^2}{\partial x^j} + ... + \frac{\partial x^k}{\partial \overline{x}^N} \frac{\partial \overline{x}^N}{\partial x^j}$$

$$= \frac{\partial x^k}{\partial x^j} \qquad \text{[By chain rule]}$$

$$= \delta^k_j.$$

11.6 Contravariant vectors

If a set of N quantities A^i in a coordinate system x^i are transformed to the set of another N quantities $\overline{A}^j$ in the coordinate system $\overline{x}^j$ by the equations

$$\overline{A}^p = \frac{\partial \overline{x}^p}{\partial x^q} A^q$$

then A^i are said to be components of a contravariant vector or contravariant tensor of the first order or first rank.

Note : It is a convention that contravariant tensors are denoted by superscripts, with the exception of the coordinates x^i, which may behave as contravariant vector in special conditions (see Theorem 2)

Theorem 1. *The law of transformation of a contravariant vector is transitive.*

Proof. Let A^i be the components of a contravariant vector in the coordinate system x^i and they are related to the components $\overline{A}^j$ of same vector in the coordinate system $\overline{x}^j$, then we have by the law of transformation

$$\overline{A}^j = \frac{\partial \overline{x}^j}{\partial x^i} A^i \qquad .....(11.6.1)$$

Now, a further change of coordinates from ...... to x^{*k}, the new components A^{*k} by contravariant law is given by

$$A^{*k} = \frac{\partial x^{*k}}{\partial \overline{x}^j} \overline{A}^j \qquad .....(11.6.2)$$

Combining (11.6.1) and (11.6.2), we get

$$A^{*k} = \frac{\partial x^{*k}}{\partial \overline{x}^j} \cdot \frac{\partial \overline{x}^j}{\partial x^i} A^i$$

$$= \frac{\partial x^{*k}}{\partial x^i} A^i \qquad .....(11.6.3)$$

This shows that the law of transformation of contravariant vector is transitive.

Theorem 2. *The coordinates x^i behave like a contravariant vector with respect to linear transformation of the type $\overline{x}^j = a^i_j x^i$, where a^j_i are a set of N^2 constants.*

Proof. We have $\overline{x}^j = a^j_i \cdot x^i$ .....(11.6.4)

Differentiating, we get $\frac{\partial \overline{x}^j}{\partial x^i} = a^j_i .$ .....(11.6.5)

Combining (11.6.4) and (11.6.5), we get

$$\overline{x}^j = \frac{\partial \overline{x}^j}{\partial x^i} \cdot x^i , \qquad .....(11.6.6)$$

which shows that x^i behaves like a contravariant vector.

***Ex*.3.** *If a vector has components* $\dot{x}, \dot{y}\left(\dot{x}=\frac{dx}{dt}, \dot{y}=\frac{dy}{dt}\right)$ *in rectangular cartesian coordinates then* $\dot{r}, \dot{\theta}$ *are its components in polar coordinates.*

Sol. Here, the space is two dimensional.

Let for rectangular cartesian coordinates $x^1 = x, x^2 = y$ for polar coordinates

$$\bar{x}^1 = r, \bar{x}^2 = \theta$$

where $x^2 + y^2 = r^2, \tan^{-1}\frac{y}{x} = \theta$.

$$\therefore \qquad x\dot{x} + y\dot{y} = r\dot{r}, \qquad .....(1)$$

$$\Rightarrow \qquad x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 = r\ddot{r} + \dot{r}^2. \qquad .....(2)$$

$$\frac{1}{1+\frac{y^2}{x^2}}\frac{x\dot{y}-y\dot{x}}{(x)^2} = \dot{\theta} \Rightarrow r^2\dot{\theta} = x\dot{y} - y\dot{x}. \qquad .....(3)$$

Using (1) and (3)

$$r^2\dot{r}^2 + r^4\dot{\theta}^2 = \left(x^2+y^2\right)\left(\dot{x}^2+\dot{y}^2\right)$$

$$\Rightarrow \qquad \dot{r}^2 + r^2\dot{\theta}^2 = \dot{x}^2 + \dot{y}^2. \qquad .....(4)$$

Using contravariant law

$$\bar{A}^j = \frac{\partial \bar{x}^j}{\partial x^i}A^i = \frac{\partial \bar{x}^j}{\partial x^1}A^1 + \frac{\partial \bar{x}^j}{\partial x^2}A^2. \qquad .....(5)$$

$$\therefore \qquad \bar{A}^1 = \frac{\partial \bar{x}^1}{\partial x^1}A^1 + \frac{\partial \bar{x}^1}{\partial x^2}A^2 = \frac{\partial r}{\partial x}\dot{x} + \frac{\partial r}{\partial y}\cdot\dot{y}$$

$$= \frac{x}{r}\dot{x} + \frac{y}{r}\dot{y} = \frac{r\dot{r}}{r} = \dot{r}. \qquad .....(6)$$

$$\bar{A}^2 = \frac{\partial \bar{x}^2}{\partial x^1}A^1 + \frac{\partial \bar{x}^2}{\partial x^2}A^2$$

$$= \frac{\partial \theta}{\partial x}\dot{x} + \frac{\partial \theta}{\partial y}\dot{y} = -\frac{y}{r^2}\dot{x} + \frac{x}{r^2}\dot{y}$$

$$= \frac{x\dot{y}-y\dot{x}}{r^2} = \dot{\theta}. \qquad .....(7)$$

***Ex*.4.** *A vector has components* $\ddot{x}, \ddot{y}$ *in rectangular cartesian coordinates then its respective components in polar coordinates are*

$$\ddot{r} - r\dot{\theta}^2, \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta}.$$

Sol. Assuming $\qquad A^1 = \ddot{x}, A^2 = \ddot{y} \qquad .....(1)$

We find

$$\bar{A}^1 = \frac{\partial r}{\partial x}\ddot{x} + \frac{\partial r}{\partial y}\ddot{y}$$

$$= \frac{x\ddot{x} + y\ddot{y}}{r} = \frac{r\ddot{r} - r^2\dot{\theta}^2}{r} = r\ddot{r} - r\dot{\theta}^2. \qquad .....(2)$$

$$\bar{A}^2 = \frac{\partial \theta}{\partial x}\ddot{x} + \frac{\partial \theta}{\partial y}\ddot{y}$$

$$= \frac{x\ddot{y} - y\ddot{x}}{r^2} = \frac{r^2\ddot{\theta} - 2r\dot{\theta}\dot{r}}{r^2}$$

$$= \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta}. \qquad .....(3)$$

Note : It is noted form above examples that the velocity and acceleration components are contravariant vectors.

11.7 Covariant vectors

If a set of N quantities A_i in a coordinate system x^i are transformed to a set of another N quantities $\bar{A}_j$ in the coordinate system $\bar{x}^j$ by the equations

$$\bar{A}_p = \frac{\partial x^q}{\partial \bar{x}^p} A_q, \qquad .....(11.7.1)$$

then A^i are said to be the components of a covariant vector or covariant tensor of first order or first rank.

Note : The components of covariant vectors are denoted by subscript as a convention.

Theorem 3. *The law of transformation for a covariant vector is transitive.*

Proof. Let the components of a covariant vector in the coordinate system x^i be A_i and components of same vector in coordinate system $\bar{x}^j$ be $\bar{A}_j$, then by covariant law of transformation.

$$\bar{A}_j = \frac{\partial x^i}{\partial \bar{x}^j} A_i. \qquad .....(11.7.2)$$

Now for the further change of coordinates from system $\bar{x}^j$ to x^{*k} the new components A_k^* by covariant law are given by

$$A_k^* = \frac{\partial \bar{x}^j}{\partial x^{*k}} \bar{A}_j. \qquad .....(11.7.3)$$

Combining (11.7.2) and (11.7.3)

$$A_k^* = \frac{\partial \bar{x}^j}{\partial x^{*k}} \cdot \frac{\partial x^i}{\partial \bar{x}^j} A_i$$

$$= \frac{\partial x^i}{\partial x^{*k}} A_i, \qquad .....(11.7.4)$$

which shows that the law of transformation of covariant vectors is transitive.

Theorem 4. *These exists no distinction between contravariant and covariant vectors when we restrict ourselves to coordinate transformations of the type*

$$\bar{x}^i = a^i_m x^m + b^i,$$

where b^i are N constants which do not necessarily form the components of a contravariant vector and a^i_m are N^2 constants which do not necessarily form the components of a tensor such that

$$a^i_r\, a^i_m = \delta^r_m.$$

Proof. We have $$\bar{x}^i = a^i_m x^m + b^i. \qquad \text{....(11.7.5)}$$

Multiplying by a^i_r and summing over index i, we get

$$a^i_r\, \bar{x}^i = a^i_r\, a^i_m\, x^m + a^i_r b^i \qquad \text{.....(11.7.6)}$$

Using given relation $$a^i_r\, a^i_m = \delta^r_m,$$

we have $$a^i_r\, \bar{x}^i = \delta^r_m\, x^m + a^i_r\, b^i = x^r + a^i_r\, b^i. \qquad \text{.....(11.7.7)}$$

Now, replacing the free index r by m on both sides, we obtain

$$x^m = a^i_m\, \bar{x}^i - a^i_m\, b^i. \qquad \text{.....(11.7.8)}$$

From (11.7.5) and (11.7.6), it follows that

$$\frac{\partial \bar{x}^i}{\partial x^m} = a^i_m = \frac{\partial x^m}{\partial \bar{x}^i},$$

which sows that transformation laws for contravariant and covariant vectors respectively, define the same type of entity in the present ease.

11.8 Invariant

A function I of N coordinates x^i $[I = I(x^i)]$ is called an in variant or a scalar or tensor of zero order with respect to coordinate transformations if $I = \bar{I}$, where $\bar{I}\left[\bar{I} = \bar{I}\left(\bar{x}^j\right)\right]$ is the value of I in new coordinate system $\bar{x}^j$.

Ex.5. *A covariant tensor of first order has components xy, $2y - z^2$, xz in rectangular coordinates. Determine its covariant components in spherical polar coordinates.*

Sol. Here we have three dimensional space

$$x^1 = x, \quad x^2 = y, \quad x^3 = z,$$

$$\bar{x}^1 = r, \bar{x}^2 = \theta, \bar{x}^3 = \phi,$$

where $$x = r \sin\theta \cos\phi, \quad y = r\sin\theta \sin\phi, \quad z = r\cos\theta. \qquad \text{.....(1)}$$

Taking $$A_1 = xy, \quad A_2 = 2y - z^2, \quad A_3 = xz. \qquad \text{.....(2)}$$

Using covariant transformation law

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j$$

$$= \frac{\partial x^1}{\partial \bar{x}^i} A_1 + \frac{\partial x^2}{\partial \bar{x}^i} A_2 + \frac{\partial x^3}{\partial \bar{x}^i} A_3. \qquad \text{.....(3)}$$

We find that $\overline{A}_1 = \dfrac{\partial x^1}{\partial \overline{x}^1} A_1 + \dfrac{\partial x^2}{\partial \overline{x}^1} A_2 + \dfrac{\partial x^3}{\partial \overline{x}^1} A_3$

$$= \frac{\partial x}{\partial r} xy + \frac{\partial y}{\partial r}(2y - z^2) + \frac{\partial z}{\partial r}(xz)$$

$$= (\sin\theta\cos\phi)\, r^2 \sin^2\theta \sin\phi\cos\phi + \sin\theta\sin\phi\,(2r\sin\theta\sin\phi - r^2\cos^2\theta)$$
$$+ \cos\theta\, r^2 \sin\theta\cos\phi\cos\theta. \qquad \text{....(4)}$$

Similarly from (3)

$$\overline{A}_2 = \frac{\partial x^1}{\partial \overline{x}^2} A_1 + \frac{\partial x^2}{\partial \overline{x}^2} A_2 + \frac{\partial x^3}{\partial \overline{x}^2} A_3$$

$$= \frac{\partial x}{\partial \theta}(xy) + \frac{\partial y}{\partial \theta}(2y - z^2) + \frac{\partial z}{\partial \theta}(xz)$$

$$= (r\cos\theta\cos\phi)\, r^2\sin^2\theta\sin\phi\cos\phi + (r\cos\theta\sin\phi)\,(2r\sin\theta\sin\phi - r^2\cos^2\theta)$$
$$-\,(r\sin\theta\,(r^2\sin\theta\cos\theta\cos\phi) \qquad \text{.....(5)}$$

and $\overline{A}_3 = \dfrac{\partial x^1}{\partial \overline{x}^3} A_1 + \dfrac{\partial x^2}{\partial \overline{x}^3} A_2 + \dfrac{\partial x^3}{\partial \overline{x}^3} A_3$

$$= (-\,r\sin\theta\cos\phi)\, r^2\sin^2\theta\sin\phi\cos\phi + (r\sin\theta\cos\phi)\,(2r\sin\theta\sin\phi - r^2\cos^2\theta)$$
$$\text{.....(6)}$$

11.9 Second order tensors

***(a)* Contravariant tensor of rank two :** If a set of N^2 quantities A^{ij} in a coordinate system x^i are trans formed to another set of N^2 quantities $\overline{A}^{kl}$ in coordinate system $\overline{x}^j$ by the equations

$$\overline{A}^{kl} = \frac{\partial \overline{x}^k}{\partial x^i}\frac{\partial \overline{x}^l}{\partial x^j} A^{ij}, \qquad \text{.....(11.9.1)}$$

then A^{ij} are called components of a contravariant tensor of rank two or second order.

***(b)* Covariant tensor of second order :** If a set of N^2 quantities A_{ij} in a coordinate system x^i are transformed to another set of N^2 quantities $\overline{A}_{kl}$ in a coordinate system $\overline{x}^j$ by the relations

$$\overline{A}_{kl} = \frac{\partial x^i}{\partial \overline{x}^k}\frac{\partial x^j}{\partial \overline{x}^l} A_{ij}, \qquad \text{.....(11.9.2)}$$

then A_{ij} are said to be the components of a covariant tensor of rank two or second order.

***(c)* Mixed tensor of second order :** If a set of N^2 quantities A^i_j in a coordinate system x^i are transformed to another set of N^2 quantities $\overline{A}^k_l$ in the coordinate system $\overline{x}^j$ by the relations

$$\overline{A}^k_l = \frac{\partial \overline{x}^k}{\partial x^i}\frac{\partial x^j}{\partial \overline{x}^l} A^i_j, \qquad \text{.....(11.9.3)}$$

then A^i_j are said to be the components of a mixed tensor (contravariant rank one and covariant rank one) of second order or second rank.

Theorem 5. *The Kronecker delta is a mixed tensor of second order whose components in any other coordinate system again constitute the Kronecker delta.*

Proof. The Kronecker delta is

$$\delta^i_j = \begin{cases} 1 & , \quad if\ i = j, \\ 0 & , \quad if\ i \neq j. \end{cases} \qquad .....(11.9.4)$$

Let δ^i_j be the components in coordinate system x^i and corresponding components in $\overline{x}^j$ be $\overline{\delta}^k_l$

we have $$\frac{\partial \overline{x}^k}{\partial x^i}\frac{\partial x^j}{\partial \overline{x}^l}\delta^i_j = \frac{\partial \overline{x}^k}{\partial x^j}\frac{\partial x^j}{\partial \overline{x}^l} = \frac{\partial \overline{x}^k}{\partial \overline{x}^l} = \overline{\delta}^k_l \qquad .....(11.9.5)$$

$\Rightarrow$ δ^i_j behaves like a mixed tensor (contravariant rank one and covariant of rank one) of second order.

Theorem 6. *If A_{ij} be a covariant tensor of second order and B^i, C^i are contravariant vectors, prove that $A_{ij}\, B^i\, C^i$ is an invariant.*

Proof. We have A_{ij} a covariant tensor of rank two

$$\therefore \qquad \overline{A}_{pq} = \frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q}A_{ij}, \qquad .....(11.9.6)$$

and B^i, C^i are contravariant vectors

$$\overline{B}^p = \frac{\partial \overline{x}^p}{\partial x^k}B^k, \qquad .....(11.9.7)$$

$$\overline{C}^q = \frac{\partial \overline{x}^q}{\partial x^l}B^l, \qquad .....(11.9.8)$$

Multiplying equations (11.9.6), (11.9.7) and (11.9.8), we get

$$\begin{aligned} \overline{A}_{pq}\overline{B}^p\overline{C}^q &= \frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q}\frac{\partial \overline{x}^p}{\partial x^k}\frac{\partial \overline{x}^q}{\partial x^l}A_{ij}B^kC^l \\ &= \delta^i_k\ \delta^j_l A_{ij}B^kC^l \\ &= A_{ij}\, B^k\, C^l \qquad .....(11.9.9) \end{aligned}$$

which shows invariant character of $A_{ij}\, B^i\, C^i$.

11.10 Higher order tensors

If a set of N^{m+n} quantities $A^{i_1 i_2 .....i_m}_{j_1 j_2 ... j_n}$ in a coordinate system x^i are transformed to another set of N^{m+n} quantities $\overline{A}^{p_1 p_2 ... p_m}_{q_1 q_2 .....q_n}$ in, the coordinate system $\overline{x}^j$ by the relations

$$\overline{A}^{p_1 p_2 ... p_m}_{q_1 q_2 .....q_n} = \frac{\partial \overline{x}^{p_1}}{\partial x^{i_1}}\frac{\partial \overline{x}^{p_2}}{\partial x^{i_2}}\cdots\frac{\partial \overline{x}^{p_m}}{\partial x^{i_m}}\frac{\partial x^{j_1}}{\partial \overline{x}^{q_1}}\frac{\partial x^{j_2}}{\partial \overline{x}^{q_2}}\cdots\frac{\partial x^{j_n}}{\partial \overline{x}^{q_n}}\overline{A}^{i_1 i_2 ... i_m}_{j_1 j_2 ... j_n}, \qquad .....(11.10.1)$$

then $A^{i_1 i_2 .....i_m}_{j_1 j_2 ... j_n}$ are said to be the components of a mixed tensor of $(m + n)^{\text{th}}$ order contravariant of m^{th} order and covariant of n^{th} order.

Theorem 7. *The transformation of the tensors form a group i.e. the law of transformation of tensors possesses transitive property.*

Sol. Without loss of generality, we can consider a mixed tensor A_j^i in a coordinate system x^i and consider the transformation of coordinates from x^i to $\bar{x}^j$ and then x^{*l}. Let the corresponding components of the tensor be $\bar{A}_l^k$ and A_q^{*p}, then

$$\bar{A}_l^k = \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^l} A_j^i, \qquad \text{.....(11.10.2)}$$

and
$$A_q^{*p} = \frac{\partial x^{*p}}{\partial \bar{x}^k} \frac{\partial \bar{x}^l}{\partial x^{*q}} \bar{A}_l^r. \qquad \text{.....(11.10.3)}$$

Combining (11.10.2) and (11.10.3), we get

$$A_q^{*p} = \frac{\partial x^{*p}}{\partial \bar{x}^k} \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial \bar{x}^l}{\partial x^{*q}} \frac{\partial x^j}{\partial \bar{x}^l} A_j^i = \frac{\partial x^{*p}}{\partial x^i} \frac{\partial x^j}{\partial x^{*q}} A_j^i, \qquad \text{.....(11.10.4)}$$

which shows that the transformation of tensors possesses transitive property *i.e.* transformation of tensors form a group.

11.11 Zero tensor

A tensor whose components relatively to every coordinate system are all zero is known as zero tensor.

11.12 Symmetric tensor

A tensor is called symmetric with respect to two contravariant or two covariant in dices if its components remain unaltered upon interchange of the indices.

e.g. the tensor A_{st}^{pqr} is said to be symmetric in p and q if

$$A_{st}^{pqr} = A_{st}^{qpr}, \qquad \text{.....(11.12.1)}$$

and it is said to be symmetric in s and t if

$$A_{st}^{pqr} = A_{ts}^{pqr}. \qquad \text{.....(11.12.2)}$$

Theorem 8. *A symmetric tensor of the second order has atmost* $\frac{N(N+1)}{2}$. *different components in* V_N.

Sol. Let A_{ij} be a symmetric tensor of order two. The total number of its components in an array, in a V_N

$$\begin{array}{l} A_{11}\, A_{12} \,\ldots\ldots\ldots\, A_{1N} \\ A_{21}\, A_{22} \,\ldots\ldots\ldots\, A_{2N} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ A_{N1}\, A_{N2} \,\ldots\ldots\, A_{NN}, \end{array} \qquad \text{.....(11.12.3)}$$

are N^2, out of which all the N diagonal terms will be different and the rest $(N^2 - N)$ will be equal in pairs due to symmetric property. The number of such pairs will be $\frac{(N^2-N)}{2}$. Hence the total number of independent components

$$= N + \frac{N^2 - N}{2}$$

$$= \frac{1}{2} N(N+1). \qquad .....(11.12.4)$$

Theorem 9. *If a tensor is symmetric with respect to two contravariant indices (or covariant indices) in any coordinate system it remains symmetric with respect to these two indices in any other coordinate system.*

Proof. Due to involvement of only two indices in symmetric property, there in no loss of generality if we take contravariant tensor viz. $A^{ij} = A^{ji}$, it is symmetric in i, j.

We have
$$\bar{A}^{pq} = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} A^{ij}.$$

$$= \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} A^{ji} \qquad \text{[due to symmetry]}$$

$$= \frac{\partial \bar{x}^q}{\partial x^j} \frac{\partial \bar{x}^p}{\partial x^i} A^{ji}$$

$$= \bar{A}^{qp}.$$

Hence the proposition.

11.13 Skew symmetric tensor

A tensor is called skew symmetric with respect to two contravariant or two covariant indices if its components change sign upon interchange of the indices. *e.g.* $A^{ijk}_{pq} = -A^{jik}_{pq}$ is skew symmetric in *i* and *j* and if

$$A^{ijk}_{pq} = -A^{ijk}_{qp}$$

is said to be skew symmetric in *p* and *q*.

If a tensor is skew symmetric with respect to any two contravariant indices and also any two covariant indices, then it is called skew-symmetric tensor.

Notes :

(i) The property of skew symmetry (like that of symmetry) in also independent of the choice of the coordinate system.

(ii) Skew-symmetry, like symmetry cannot be defined with respect to the indices of which one denotes contravariance and the other covariance.

(iii) A skew-symmetric tensor A^{ij} of second order has at most $\frac{N(N-1)}{2}$ different arithmetical components, as all the N diagonal terms A^{ii} (no summation) are zero in this case.

11.14 Algebraic operations with tensors

***(i)* Addition :** The sum of two or more tensors of the same rank and same type is a tensor of same rank and same type.

Let
$$\bar{A}^i_j = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \bar{x}^j} A^p_q, \qquad .....(11.14.1)$$

and
$$\bar{B}^i_j = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \bar{x}^j} B^p_q. \qquad .....(11.14.2)$$

Adding (11.14.1) and (11.14.2) we get

$$\left(\bar{A}^i_j + \bar{B}^i_j\right) = \frac{\partial \bar{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \bar{x}^j}\left(A^p_q + B^p_q\right). \qquad .....(11.14.3)$$

This shows that $A^p_q + B^p_q = C^p_q$ (say) is a tensor of same rank and type.

Remark : It can easily be verified that the addition of tensors is commutative and associative.

***(ii)* Subtraction :** The difference of two tensors of the same rank and same type is also a tensor of the same rank and same type.

It immediately follows from above equations (11.14.1) and (11.14.2) that $A^p_q - B^p_q = D^p_q$ is also a tensor.

***(iii)* Outer multiplication :** The product of two tensors is a tensor whose rank is the sum of the ranks of given tensors.

This process involving ordinary multiplication of the components of the tensor is called open product or outer product of the two tensors, for example : the outer product of a tensor A^{ij}_{lmn} by a tensor B^k_{pq} is a tensor C^{ijk}_{lmnpq} is a mixed tensor of rank 8, contravariant of rank 3 and covariant of rank 5.

Notes :

(i) The converse of above product rule is not always true *i.e.* not every tensor can be written as a product of two tensors of lower ranks, for this, the reason is that the division of tensors is not always possible.

(ii) The division, in usual sense, of one tensor by another is not defined.

Theorem 10. *Outer multiplication of tensors is commutative and associative.*

Proof. Commutative law : Let A^{ij}_l and B^p_q be two tensors,

then
$$\bar{A}^{hk}_m = \frac{\partial \bar{x}^h}{\partial x^i}\frac{\partial \bar{x}^k}{\partial x^j}\frac{\partial x^l}{\partial \bar{x}^m} A^{ij}_l, \qquad .....(11.14.4)$$

$$\bar{B}^r_s = \frac{\partial \bar{x}^r}{\partial x^p}\frac{\partial x^q}{\partial \bar{x}^s} B^p_q. \qquad .....(11.14.5)$$

Multiplying (11.14.4) and (11.14.5), we get

$$\bar{A}^{hk}_{m}\bar{B}^{r}_{s} = \frac{\partial \bar{x}^h}{\partial x^i}\frac{\partial \bar{x}^k}{\partial x^j}\frac{\partial x^l}{\partial \bar{x}^m}A^{ij}_{l}.\bar{B}^{p}_{q}\cdot\frac{\partial \bar{x}^r}{\partial x^p}\cdot\frac{\partial x^q}{\partial \bar{x}^s}$$

$$= \left(\frac{\partial \bar{x}^h}{\partial x^i}\frac{\partial \bar{x}^k}{\partial x^j}\frac{\partial x^l}{\partial \bar{x}^m}\cdot\frac{\partial \bar{x}^r}{\partial x^p}\frac{\partial x^q}{\partial \bar{x}^s}\right)A^{ij}_{l}.\bar{B}^{p}_{q}. \qquad .....(11.14.6)$$

Now multiplying (11.14.5) and (11.14.4)

$$\bar{B}^{r}_{s}\ \bar{A}^{hk}_{m} = \frac{\partial \bar{x}^r}{\partial x^p}\frac{\partial x^q}{\partial \bar{x}^s}B^{p}_{q}\cdot\frac{\partial \bar{x}^h}{\partial x^i}\cdot\frac{\partial \bar{x}^k}{\partial x^j}\frac{\partial x^l}{\partial \bar{x}^m}A^{ij}_{l}.$$

$$= \left(\frac{\partial \bar{x}^h}{\partial x^i}\cdot\frac{\partial \bar{x}^k}{\partial x^j}\frac{\partial x^l}{\partial \bar{x}^m}\frac{\partial \bar{x}^r}{\partial x^p}\frac{\partial x^q}{\partial \bar{x}^s}\right)B^{p}_{q}A^{ij}_{l}. \qquad .....(11.14.7)$$

Equations (11.14.6) and (11.14.7) show that the expression within brackets in the R.H.S. are same therefore we can say

$$A^{ij}_{l}B^{p}_{q} = B^{p}_{q}\ A^{ij}_{l}. \qquad .....(11.14.8)$$

Associative law : Here we are to prove that

$$\left(A^{ij}_{l}\ B^{p}_{q}\right)C^{k}_{m} = A^{ij}_{l}\left(B^{p}_{q}C^{k}_{m}\right).$$

Proceed as usual.

(iv) **Contraction :** If one contravariant and one covariant index of tensor (mixed tensor) are set equal, the result indicates that a summation over the equal indices (dummy indices) is to be taken according to the summation convention. This resulting sum is a tensor of rank two less than that of the original tensor. The process is called contraction.

Consider a tensor A^{ij}_{pqr} of rank five. If we put $j = r$, we get A^{ir}_{pqr} a tensor of rank 3 obtained by contracting A^{ij}_{pqr}.

$$\bar{A}^{kl}_{stu} = \frac{\partial \bar{x}^k}{\partial x^i}\cdot\frac{\partial \bar{x}^l}{\partial x^j}\frac{\partial x^p}{\partial \bar{x}^s}\frac{\partial x^q}{\partial \bar{x}^t}\frac{\partial x^r}{\partial \bar{x}^u}A^{ij}_{pqr}. \qquad .....(11.14.9)$$

Putting $j = r$

$$\bar{A}^{kl}_{stu} = \frac{\partial \bar{x}^k}{\partial x^i}\cdot\frac{\partial \bar{x}^l}{\partial x^r}\frac{\partial x^p}{\partial \bar{x}^s}\frac{\partial x^q}{\partial \bar{x}^t}\frac{\partial x^r}{\partial \bar{x}^u}A^{ir}_{pqr}$$

$$= \frac{\partial \bar{x}^k}{\partial x^i}\cdot\frac{\partial x^p}{\partial \bar{x}^s}\frac{\partial x^q}{\partial \bar{x}^t}\delta^{l}_{u}A^{ir}_{pqr} \qquad .....(11.14.10)$$

$\Rightarrow$

$$\bar{A}^{ku}_{stu} = \frac{\partial \bar{x}^k}{\partial x^i}\cdot\frac{\partial x^p}{\partial \bar{x}^s}\frac{\partial x^q}{\partial \bar{x}^t}A^{ir}_{pqr}$$

$\Rightarrow$

$$\bar{A}^{k}_{st} = \frac{\partial \bar{x}^k}{\partial x^i}\cdot\frac{\partial x^p}{\partial \bar{x}^s}\frac{\partial x^q}{\partial \bar{x}^t}A^{i}_{pq} \qquad .....(11.14.11)$$

This is a law of transformation of a tensor of rank 3.

Thus after contraction we get a tensor of rank 3. Contravariant rank (2 – 1) and covariant rank (3 – 1).

Notes :

(i) We never contracts the indices of same type as the resulting sum is not necessarily a tensor.

(ii) The process of contraction reduces the order by two and may be repeatedly used, if so desired, to contract new tensors, whose order will always be non-negative.

***(v)* Inner multiplication :** The process of outer multiplication, followed by a contraction, we obtain a new tensor called inner product of the given tensor. The process is called inner multiplication.

For example : The outer product of A_l^{ij} and B_q^p is $A_l^{ij} B_q^p$. Putting $p = l$, we get the inner product

$$A_l^{ij}\ B_q^l = C_q^{ij}.$$

Again if we put $p = l, j = q$, we have another inner product $A_l^{iq}\ B_q^l = D^i$.

Note : It can easily be verified that inner multiplication of tensors is commutative and associative.

11.15 Illustrative examples

***Ex.*5.** *If $\phi = a_{ij} A^i A^j$, then we can always write $\phi = b_{ij} A^i A^j$ where b_{ij} is symmetric.*

Sol. $\phi = a_{ij} A^i A^j$. .....(1)

On interchanging the dummy indices

$\phi = a_{ji} A^j A^i$. .....(2)

Adding (1) and (2), we get $2\phi = (a_{ij} + a_{ji}) A^i A^j$

or $\phi = b_{ij} A^i A^j$, ...(3)

where $b_{ij} = \frac{1}{2}\left(a_{ij} + a_{ji}\right)$,

which is symmetric, *i.e.* $b_{ij} = b_{ji}$.

***Ex.*6.** *If A^{rs} is skew-symmetric and B_{rs} is symmetric, prove that $A^{rs} B_{rs} = 0$*

Sol Given that $A^{rs} = -A^{sr}$ and $B_{rs} = B_{sr}$ on changing the dummy indices in $A^{rs} B_{rs}$, we get

$$A^{rs} B_{rs} = A^{sr} B_{sr} = -A^{rs} B_{rs}$$

or $2A^{rs} B_{rs} = 0 \;\Rightarrow\; A^{rs} B_{rs} = 0$.

***Ex.*7.** *If a_{ij} is a symmetric covariant tensor and b_i a covariant vector which satisfy the relation* $a_{ij} b_k + a_{jk} b_i + a_{ki} b_j = 0$, prove that either

$$a_{ij} = 0 \quad \text{or} \quad b_i = 0.$$

Sol. Let $a_{ij} b_k = A_{ijk}$, ...(1)

then A_{ijk} is a third order covariant tensor which is symmetric with respect to the pair of indices *i* and *j* due to symmetric property of a_{ij}. Also replacing the indices *i, j* and *k* by *i* , *k* and *i* respectively on both sides, we find

$a_{ji} b_i = A_{jki}$, ...(2)

is symmetric with respect to j and k and similarly

$$a_{ki}\, b_j = A_{kij}, \qquad .....(3)$$

is symmetric with respect to k and i.

Hence a_{ijk} is a symmetric tensor.

Adding (1), (2) and (3), we get

$$A_{ijk} + A_{jki} + A_{kij} = 0$$

$\Rightarrow$ $$3A_{ijk} = 0$$

$\Rightarrow$ $$a_{ij}\, b_k = 0$$

$\Rightarrow$ $$a_{ij} = 0$$

or $$b_k = 0 \quad i.e. \quad b_i = 0. \qquad .....(4)$$

Ex.8. *If* $u_{ij} \neq 0$ *are the components of a tensor of the type* $(0, 2)$ *and if the equation*

$$fu_{ij} + gu_{ji} = 0.$$

holds, then prove that either $f = g$ *and* u_{ij} *is skew symmetric or* $f = -g$ *and* u_{ij} *is symmetric.*

Sol. Given that $$fu_{ij} + gu_{ji} = 0. \qquad .....(1)$$

Changing the free indices, we may write it as

$$fu_{ji} + gu_{ij} = 0. \qquad .....(2)$$

Adding (1) and (2), we get $$(f + g)\,(u_{ij} + u_{ji}) = 0. \qquad .....(3)$$

$\Rightarrow$ *(i)* either $u_{ij} + u_{ji} = 0$ *i.e.* u_{ij} is skew symmetric, and then from (1) it follows that $f = g$.

(ii) or $f = -g$ and then from (1) it follows that u_{ij} is symmetric.

11.16 Quotient law of tensors

In the study of tensor analysis some times it becomes necessary that whether a given entity is a tensor or not. Theoretically we may say that if components of an entity obey tensor transformation laws, then it is a tensor otherwise not. However in practice it is troublesome and a simple test is provided by a law known as Quotient law, which is as follows :

Theorem 11. *An entity whose inner product with an arbitrary tensor is a tensor, is itself a tensor.*

Proof. Let $A\,(i, j, k)$ be given entity in a coordinate system x^i, and B_m^{ij} be an arbitrary tensor whose inner product with $A\,(i, j, k)$ is a tensor C_{mk} *i.e.*

$$A(i,j,k)\, B_m^{ij} = C_{mk}. \qquad .....(11.16.1)$$

We have to show that $A\,(i, j, k)$ is a tensor.

In the coordinate system $\bar{x}^i$, we have

$$\bar{A}(p,q,r)\, \bar{B}_n^{pq} = \bar{C}_{nr}. \qquad .....(11.16.2)$$

But we have $$\bar{B}_n^{pq} = \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} \frac{\partial x^m}{\partial \bar{x}^n} B_m^{ij}, \qquad .....(11.16.3)$$

$$\overline{C}_{nr} = \frac{\partial x^m}{\partial \overline{x}^n}\frac{\partial x^k}{\partial \overline{x}^r}C_{mk}. \qquad .....(11.16.4)$$

Using (11.16.3) and (11.16.4) in (11.16.2), we get

$$\overline{A}(p,q,r)\frac{\partial \overline{x}^p}{\partial x^i}\frac{\partial \overline{x}^q}{\partial x^j}\frac{\partial x^m}{\partial \overline{x}^n}B^{ij}_m = \frac{\partial x^m}{\partial \overline{x}^n}\frac{\partial x^k}{\partial \overline{x}^r}C_{mk}$$

$$= \frac{\partial x^m}{\partial \overline{x}^n}\frac{\partial x^k}{\partial \overline{x}^r}A(i,j,k)B^{ij}_m. \qquad \text{[using (11.16.1)]}$$

$$\Rightarrow \qquad \frac{\partial x^m}{\partial \overline{x}^n}\left[\overline{A}(p,q,r)\frac{\partial \overline{x}^p}{\partial x^i}\frac{\partial \overline{x}^q}{\partial x^j} - \frac{\partial x^k}{\partial \overline{x}^r}A(i,j,k)\right]B^{ij}_m = 0$$

On inner multiplication by $\dfrac{\partial \overline{x}^n}{\partial x^s}$

$$\delta^m_s\left[\overline{A}(p,q,r)\frac{\partial \overline{x}^p}{\partial x^i}\frac{\partial \overline{x}^q}{\partial x^j} - \frac{\partial x^k}{\partial \overline{x}^r}A(i,j,k)\right]B^{ij}_m = 0$$

$$\Rightarrow \qquad \left[\overline{A}(p,q,r)\frac{\partial \overline{x}^p}{\partial x^i}\frac{\partial \overline{x}^q}{\partial x^j} - \frac{\partial x^k}{\partial \overline{x}^r}A(i,j,k)\right]B^{ij}_s = 0 \qquad .....(11.16.5)$$

Form above equation we cannot jump to the conclusion that the expression within bracket vanishes. Since here i and j are dummy indices which imply summation and it is the sum which is zero. However since B^{ij}_s is an arbitrary tensor we can arrange that only one of its components is non-zero. Now each component of B^{ij}_s may be chosen in turn as that one which does not vanish. Therefore the expression within brackets is identically zero.

Hence $$\overline{A}(p,q,r)\frac{\partial \overline{x}^p}{\partial x^i}\frac{\partial \overline{x}^q}{\partial x^j} - \frac{\partial x^k}{\partial \overline{x}^r}A(i,j,k) = 0. \qquad .....(11.16.6)$$

Taking inner multiplication with $\dfrac{\partial x^i}{\partial \overline{x}^m}\dfrac{\partial x^j}{\partial \overline{x}^n}$, we get

$$\overline{A}(p,q,r)\delta^p_m\delta^q_n = \frac{\partial x^k}{\partial \overline{x}^r}\frac{\partial x^i}{\partial \overline{x}^m}\frac{\partial x^j}{\partial \overline{x}^n}A(i,j,k)$$

$$\Rightarrow \qquad \overline{A}(m,n,r) = \frac{\partial x^k}{\partial \overline{x}^r}\frac{\partial x^i}{\partial \overline{x}^m},\frac{\partial x^j}{\partial \overline{x}^n}A(i,j,k), \qquad .....(11.16.7)$$

which shows that A (i, j, k) is a tensor of third order, and is covariant in i, j, and k and therefore may be written as A_{ijk}.

Ex.9. *Use Quotient law to prove that Knonecker delta is a mixed tensor of order two.*

Sol. Let A^j be an arbitrary contravariant vector, then by property of Kronecker delta, we have

$$\delta^i_j A^j = A^i, \qquad .....(1)$$

which is again a tensor of order one (contravariant).

Hence by Quotient law δ^i_j is a mixed tensor of order two.

Ex.10. *If A^i and B^i are arbitrary contravariant vectors and $C_{ij} A^i B^j$ is an invariant, show that C_{ij} is a covariant tensor of second order.*

Sol. Given that $C_{ij} A^i B^j$ is invariant, we have

$$C_{ij} A^i B^j = \overline{C}_{pq}\, \overline{A}^p\, \overline{B}^q. \qquad .....(1)$$

Further, A^i and B^i are contravariant vectors, therefore

$$\overline{A}^p = \frac{\partial \overline{x}^p}{\partial x^i} A^i, \qquad .....(2)$$

and

$$\overline{B}^q = \frac{\partial \overline{x}^q}{\partial x^j} B^j. \qquad .....(3)$$

Substituting (2) and (3) in (1), we get

$$\left(C_{ij} - \overline{C}_{pq} \frac{\partial \overline{x}^p}{\partial x^i} \frac{\partial \overline{x}^q}{\partial x^j} \right) A^i B^j = 0 \qquad .....(4)$$

$\because$ A^i, B^j are arbitrary vectors, therefore

$$C_{ij} = \overline{C}_{pq} \frac{\partial \overline{x}^p}{\partial x^i} \frac{\partial \overline{x}^q}{\partial x^j}, \qquad .....(5)$$

which shows that C_{ij} is a covariant tensor of rank two.

Ex.11. *If A^i is an arbitrary contravariant vector and $C_{ij} A^i A^j$ is an invariant, show that $C_{ij} + C_{ji}$ is a covariant tensor of second order.*

Sol. Proceeding as in Example 10, equation (4) in the present case may be written as

$$\left(C_{ij} - \overline{C}_{pq} \frac{\partial \overline{x}^p}{\partial x^i} \frac{\partial \overline{x}^q}{\partial x^j} \right) A^i A^j = 0. \qquad .....(1)$$

This quadratic form, vanishes for arbitrary A^i, but we can not jump to the conclusion that the expression within bracket is zero. We remember that in the form $b_{ij} A^i A^j$, the coefficient of the product $A^1 A^2$ is mixed up with the coefficient of $A^2 A^1$, it is in fact $b_{12} + b_{21}$. Thus interchanging the dummy indices i and j, and adding these two results, we can deduce only that

$$C_{ij} + C_{ji} = \overline{C}_{pq} \frac{\partial \overline{x}^p}{\partial x^i} \frac{\partial \overline{x}^q}{\partial x^j} + \overline{C}_{pq} \frac{\partial \overline{x}^p}{\partial x^j} \frac{\partial \overline{x}^q}{\partial x^i}. \qquad .....(2)$$

On changing the dummy indices p and q in last term, equation (2) becomes

$$\left(C_{ij}+C_{ji}\right)=\left(\bar{C}_{pq}+\bar{C}_{qp}\right)\frac{\partial \bar{x}^{p}}{\partial x^{i}}\frac{\partial \bar{x}^{q}}{\partial x^{j}}, \quad .....(3)$$

which establishes the tensor character of $(C_{ij} + C_{ji})$ as covariant tensor of the order two.

11.18 Relative tensor

If the component of a tensor $A^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s}$ transform according to the equation

$$\bar{A}^{u_1 u_2 \ldots u_r}_{v_1 v_2 \ldots v_s} = \left|\frac{\partial x}{\partial \bar{x}}\right|^{w} A^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots q_s} \frac{\partial \bar{x}^{u_1}}{\partial x^{p_1}} \cdots \frac{\partial \bar{x}^{u_r}}{\partial x^{p_r}} \frac{\partial x^{q_1}}{\partial \bar{x}^{v_1}} \cdots \frac{\partial x^{q_s}}{\partial \bar{x}^{v_s}}$$

then $A^{p_1 p_2 \ldots p_r}_{q_1 q_2 \ldots\ldots q_s}$ is called a tensor of weight w, where $\left|\frac{\partial x}{\partial \bar{x}}\right|$ is the Jacobian of transformation. If $w = 1$, the relative tensor is called a tensor density. If $w = 0$, the tensor is said to be absolut or simply tensor.

Note : If the rank of relative tensor is one then it is called relative vector. Hence if

$$\bar{A}^{p} = \left|\frac{\partial x}{\partial \bar{x}}\right|^{w} A^{u} \frac{\partial \bar{x}^{p}}{\partial x^{u}},$$

then A^p is a relative vector of weight w. If $w = 1$, the relative vector is called a vector density. If $w = 0$, the relative vector is called absolute vector or simply vector.

11.19 Conjugate (or Reciprocal) symmetric tensor

Consider a covariant symmetric tensor A_{ij} of rank two. Let d denotes the determinant $| A_{ij} |$ with elements A_{ij} *i.e.* $d = | A_{ij} |$ and $d \neq 0$. We define A^{ij} by

$$A^{ij} = \frac{\text{cofactor of } A_{ij} \text{ in the determinant } | A_{ij} |}{d}, \quad .....(11.19.1)$$

A^{ij} is a contravariant symmetric tensor of rank two and is said to be conjugate (or reciprocal) tensor of A_{ij}.

Theorem 12. *If $\xi\ (i, j)$ is the cofactor of A_{ij} in the determinant $d = | A_{ij} | \neq 0$ and A_{ij} is defined by*

$$A^{ij} = \frac{\xi(i,j)}{d},$$

then show that $\quad A_{ij}A^{rj} = \delta^{r}_{i}.$

Proof. From the properties of determinants we have following two results :

(i) $\quad A_{ij}\,\xi\,(i, j) = d \Rightarrow A_{ij}\dfrac{\xi(i,j)}{d} = 1,$

$$\Rightarrow A_{ij}A^{ij} = 1. \quad .....(11.19.2)$$

(ii) $A_{ij}\,\xi(r, j) = 0$

$$\Rightarrow \quad A_{ij}\frac{\xi(r,j)}{d} = 0 \qquad \because \; d \neq 0$$

$$\Rightarrow \quad A_{ij}\,A^{rj} = 0 \qquad \text{if } i \neq r$$

(i) and *(ii)* $\Rightarrow$ $$A_{ij}A^{rj} = \begin{cases} 1 & if\ i = r \\ 0 & if\ i \neq r \end{cases}$$

i.e. $$A_{ij}A^{rj} = \delta_i^r. \qquad .....(11.19.3)$$

Theorem 13. *Prove that* A^{ij} (*defined as above in theorem* 12) *is a symmetric contravariant tensor of rank two.*

Proof. Given $$A^{ij} = \frac{\xi(i,j)}{d}, \qquad .....(11.19.4)$$

where $\xi\,(i, j)$ is a cofactor of A_{ij} in $d = |\,A_{ij}\,|$.

Since A_{ij} is covariant symmetric tensor, so $\xi\,(i, j)$ is symmetric and hence $\dfrac{\xi(i,j)}{d} = A^{ij}$ is symmetric.

Now it remains to prove that A_{ij} is a tensor.

we know $$A_{ij}A^{rj} = \delta_i^r. \qquad .....(11.19.5)$$

We cannot apply the quotient law directly to this equation to establish the tensor character of A^{rj} because A_{ij} is not arbitrary.

Now consider the arbitrary contravariant vector ζ^k. Then $B_p = A_{kp}\,\zeta^k$ is an arbitrary covariant tensor.

Multiplying this equation by A^{ip}, we have

$$A^{ip}\,B_i = A^{ip}\,A_{kp}\,\zeta^k$$

$$= \delta_k^i\,\zeta^k = \zeta^i$$

$$\Rightarrow \qquad A^{ip}\,B_p = \zeta^i. \qquad .....(11.19.6)$$

Since B_p in arbitrary vector, hence by quotient law A^{ip} is a contravariant tensor of rank two.

Hence A^{ij} is a contravariant tensor of rank two.

11.20 Illustrative example

Ex.12. *If* A_{ij} *is a symmetric covariant tensor of rank two and* B^{ij} *is formed by dividing the cofactor of* A_{ij} *in the determinant* $|\,A_{ij}\,| = a$ *(say) by* $|\,A_{ij}\,|$ *itself, show that*

(i) $\left|\,B^{ij}\,\right| = \dfrac{1}{a}$ *and* *(ii)* $A_{ij}\,B^{ij} = N$

Sol. By theory of determinants

$$A_{ij}B^{ik} = \delta_j^k. \qquad .....(1)$$

(i) $$|A_{ij}||B^{ik}| = |\delta^k_j|$$

$$\Rightarrow \qquad a \cdot |B^{ik}| = 1$$

$$\Rightarrow \qquad |B^{ik}| = \frac{1}{a}. \qquad .....(2)$$

(ii) Again, form (1) identifying j and k, we get

$$A_{ij}B^{ij} = \delta^j_j = N.$$

***Ex.*13.** *If $A_{ij} = 0$ for $i \neq j$, show that the conjugate tensor $B^{ij} = 0$ for $i \neq j$ and $B^{ii} = \dfrac{1}{A^{ii}}$ (no summation).*

Sol. We have $$A_{ij}B^{ik} = \delta^k_j. \qquad .....(1)$$

(i) Let $k \neq j$, then
$$\begin{aligned} 0 &= A_{ij}B^{ik} \\ &= A_{1j}B^{1k} + A_{2j}B^{2k} + ... + A_{jj}B^{jk} + ... + A_{Nj}B^{Nk} \\ &= 0 + 0 + ... + A_{jj}B^{jk} + ... + 0 = A_{jj}B^{jk} \qquad \text{(No summation over } j\text{).} \end{aligned}$$

But $$A_{jj} \neq 0 \qquad \text{(No summation over } j\text{)}$$

Hence $$B^{jk} = 0, j \neq k.$$

i.e. $$B^{ij} = 0, i \neq j$$

(ii) Let $k = j$, then from (1)
$$\begin{aligned} 1 &= A_{ij}B^{ij} \\ &= A_{j1}B^{i1} + A_{i2}B^{i2} + ... + A_{ii}B^{ii} + ... + A_{iN}B^{iN}. \\ &= 0 + 0 + ... A_{ii}B^{ii} + ... + 0 = A_{ii}B^{ii} \qquad \text{(No summation over } i\text{).} \end{aligned}$$

But $$A_{ii} \neq 0 \qquad \text{(No summation over } i\text{)}$$

Hence $$B^{ii} = \frac{1}{A^{ii}} \qquad \text{(No summation).}$$

11.21 Self-learning exercises

1. What do you mean by Eienstien summation convention ?
2. What are dummy and free indices ?
3. Define Kronecker delta.
4. Define contravariant and covariant vectors.

11.22 Summary

The unit starts with the introduction of tensors in the space of N dimensions. By giving the concepts of indicial and summation convention we have defined the covariant and contravariant tensors of one or more ranks. Here we study the different properties of tensor entities. In algebra of tensors we define addition, subtraction, outer multiplications, contraction and inner-multiplication. Some theorems and examples on above concepts are given. The symmetric and skew-symmetric tensors have also been studied in this unit. To test whether a given quantity is a tensor or not, the quotient law of tensors is given. In the end conjugate tensors have been defined.

11.23 Answers to self-learning exercises

1. § 11.4*(a)*
2. § 11.4*(b), (c)*
3. § 11.5
4. § 11.6, § 11.7.

11.24 Exercise

1. Prove that the transformation of tensors form a group.
2. Show that a second rank covariant (or contravariant) tensor is expressible as a sum of two tensors one of which is symmetric and other is antisymmetric.
3. Prove that the contracted tensor A^i_j is a scalar.
4. Show that the tensor equation $a^i_j \lambda_i = \alpha \lambda_j$ where α is an invariant and λ_j an arbitrary tensor, demands that

$$a^i_j = \delta^i_j \alpha.$$

5. Show that the contraction of the outer product of the tensors A^p and B_q is an invariant.

□ □ □

Unit 12 : Riemannian Space, Metric Tensor, Indicator, Permutation Symbol and Permutation Tensors, Christoffel Symbols and their Properties

Structure of the Unit

12.0 Objective

In this unit our objective is to generalize the concept of distance between any two neighboring points from three dimensional space to N-dimensional Riemannian space. We introduce a particular type of tensor, called metric tensor which has a great importance in the theory of tensor analysis. We also consider two types of expressions due to Christoffel involving the derivatives of the components of metric tensor of fundamental tensor g_{ij} and g^{ij}. These expressions will be called Christoffel symbols of first and second kind.

12.1 Introduction

We know that in Euclidean space of three dimensional rectangular cartesian coordinates the distance ds between two neighbouring points (x^1, x^2, x^3) and $(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$ is given by

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = dx^i\, dx^i,\ i = 1, 2, 3 \qquad .....(12.1.1)$$

The distance ds is also called the line element. If we take the coordinates of points in any of the curvilinear coordinates (*e.g.* cylindrical or spherical polar coordinates) such as (x^{*1}, x^{*2}, x^{*3}) then x^i are functions of x^{*i} and dx^i are linear homogeneous functions of dx^{*i} given by

$$dx^i = \frac{\partial x^i}{\partial x^{*m}} dx^{*m} \quad (i, m = 1, 2, 3) \qquad .....(12.1.2)$$

when we substitute dx^i from (12.1.2) in (12.1.1), we get a homogeneous quadratic function in dx^{*i} *viz.*,

$$ds^2 \left(\frac{\partial x^i}{\partial x^{*m}} \frac{\partial x^i}{\partial x^{*n}} \right) dx^{*m} dx^{*n} \qquad \text{(summation over } i\text{)} \qquad .....(12.1.3)$$

which can be written as $\quad ds^2 = \overset{*}{g}_{mn}\, dx^{*m}\, dx^{*n} \quad (m, n = 1, 2, 3) \qquad .....(12.1.4)$

where $\quad \overset{*}{g}_{mn} = \dfrac{\partial x^i}{\partial x^{*m}} \dfrac{\partial x^i}{\partial x^{*n}} \quad$ (summing over i) $\qquad .....(12.1.5)$

The differential expression of R.H.S. of (12.1.3) which represents ds^2 is called the metric form or fundamental form of the space under consideration.

Motivated by the above fact, the idea of distance was extended by Riemann, originator of tensor calculus, to a space of N-dimensions.

12.2 Metric tensor and Riemannian space

The quadratic differential form

$$ds^2 = g_{ij}\, dx^i\, dx^j, \qquad .....(12.2.1)$$

which expresses the distance between two neighbouring points, whose coordinates in a V_N are x^i and $x^i + dx^i$, is called a Riemannian metric or line element, g_{ij} in called metric tensor or fundamental tensor.

The N-dimensional space characterised by a Riemannian metric is called a Riemannian space and is denoted by 'Riemannian – V_N'.

Here we postulate that the line element ds is independent of coordinate system *i.e.* ds^2 is an invariant. We will show that g_{ij} is a symmetric covariant tensor of order two, it is called the **fundamental covariant tensor or metric tensor** of Riemannian space.

Theorem 1. *The fundamental tensor g_{ij} is a covariant symmetric tensor of the order two.*

Proof : The line element or metric is given by

$$ds^2 = g_{ij}\, dx^i\, dx^j \qquad .....(12.2.2)$$

Consider a coordinate transformation from the system x^i to $\bar{x}^i \left(i = 1, 2, 3, ..., N\right)$ as

$$x^i = x^i\left(\bar{x}_1, \bar{x}_2, \bar{x}_3, ..., \bar{x}_N\right),$$

so that the metric $g_{ij}\, dx^i\, dx^j$ transforms to $\bar{g}_{ij}\, d\bar{x}^i\, d\bar{x}^j$. But we have ds^2 is invariant.

$\therefore$ $$ds^2 = g_{ij}\, dx^i\, dx^j = \bar{g}_{ij}\, d\bar{x}^i\, d\bar{x}^j \qquad .....(12.2.3)$$

$$= \bar{g}_{pq}\, d\bar{x}^p\, d\bar{x}^q$$

$$= \bar{g}_{pq} \frac{\partial \bar{x}^p}{\partial x^i} dx^i \frac{\partial \bar{x}^q}{\partial x^j} dx^j$$

or $$\left(\bar{g}_{ij} - \bar{g}_{pq} \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} \right) dx^i dx^j = 0 \qquad .....(12.2.4)$$

As explained in Example 11 of Unit 11, we deduce from (12.2.4)

that $$(g_{ij} + g_{ji}) = \left(\bar{g}_{pq} + \bar{g}_{qp} \right) \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial \bar{x}^q}{\partial x^j} \qquad .....(12.2.5)$$

which shows that $(g_{ij} + g_{ji})$ is covariant tensor of the second order.

Now we can write $$g_{ij} = \frac{1}{2}\left(g_{ij} + g_{ji} \right) + \frac{1}{2}\left(g_{ij} - g_{ji} \right) \qquad .....(12.2.6)$$

Then $$g_{ij}\, dx^i\, dx^j = \frac{1}{2}\left(g_{ij} + g_{ji} \right) dx^i dx^j + \frac{1}{2}\left(g_{ij} - g_{ji} \right) dx^i dx^j \qquad .....(12.2.7)$$

On interchanging the dummy indices in R.H.S., we get

$$g_{ij}\, dx^i\, dx^j = \frac{1}{2}\left(g_{ji} + g_{ij} \right) dx^j dx^i + \frac{1}{2}\left(g_{ji} - g_{ij} \right) dx^j dx^i \qquad .....(12.2.8)$$

Adding (12.2.7) and (12.2.8), we get

$$2g_{ij}\, dx^i\, dx^j = (g_{ij} + g_{ji})\, dx^i\, dx^j \qquad .....(12.2.9)$$

which show that g_{ij} is symmetric. Thus combining the two conclusions that $(g_{ij} + g_{ji})$ is a covariant tensor of the second order and g_{ij} is symmetric, we conclude that $2g_{ij}$ or g_{ij} is a symmetric covariant tensor of the second order.

Note : We call a N-dimensional space as Euclidean space of N-dimensions if its metric is

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + ... + (dx^N)^2$$

i.e. $g_{ij} = 0,\ i \neq j$ and $g_{ii} = 1$ (no summation).

12.3 Conjugate metric tensor

We know that g_{ij} is a symmetric covariant tensor of the second order and $g = | g_{ij} | \neq 0$, we can define

$$g^{ij} = \frac{G(i, j)}{g} \qquad .....(12.3.1)$$

where $G\,(i, j)$ is the cofactor of g_{ij} in the determinant g.

It follows from Theorem 13 of Unit 11 that g^{ij} is a symmetric contravariant tensor of the second order and is said to be conjugate of g_{ij} *i.e.* **conjugate metric tensor.** It is also called the **fundamental contravariant tensor.**

Hence the fundamental covariant tensor g_{ij} and fundamental contravariant tensor g^{ij}, being conjugate, are related to each other by the equation

$$g_{ij} g^{ik} = \delta^k_j. \qquad .....(12.3.2)$$

12.4 Indicator

It in implied that the metric in Euclidean space is positive definite *i.e.* $ds^2 > 0$.

But in the theory of relativity, the metric of the four dimensional space (space-time) is given by

$$ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + c^2 (dx^4)^2, \qquad .....(12.4.1)$$

where c is the velocity of light and x^4 is the time coordinate. This metric is not positive definite. We see that $ds^2 > 0$ when x^1, x^2, x^3 are constants along the curves, it is zero when, say, x^2 and x^3 are constants and $x^1 = cx^4$ and negative when x^4 in constant.

Thus, in general, for some displacement dx^i, the form ds^2 may be positive and for others it may be zero or negative. If $ds^2 = 0$, for dx^i not all zero, *i.e.* the two points are not coincident the displacement is called a **null** displacement. A curve along which the displacement $g_{ij}\, dx^i\, dx^j$ is null despite of the fact that the two points are not coincident, is called a null curve. For any displacement dx^i which is not null, we introduce an indicator e, which is $+1$ or -1, so as to make ds^2 always positive, *i.e.*

$$ds^2 = e g_{ij}\, dx^i\, dx^j, \qquad .....(12.4.2)$$

where e is called an indicator.

12.5 Illustrative examples

***Ex.*1.** *If a metric of a V_3 is given by*

$$ds^2 = 5\,(dx^1)^2 + 3\,(dx^2)^2 + 4\,(dx^3)^2 - 6\,(dx^1)\,(dx^2) + 4\,(dx^2)\,(dx^3)$$

find (i) g and (ii) g^{ij}.

Sol. When we compare the given metric with the metric

$$ds^2 = g_{ij}\, dx^i\, dx^j \ (i, j = 1, 2, 3)$$

we find that

$$g_{11} = 5, g_{22} = 3, g_{33} = 4, g_{12} = g_{21} = -3, g_{23} = g_{32} = 2, g_{13} = g_{31} = 0. \qquad .....(1)$$

$$\therefore \qquad g_{ij} = \begin{bmatrix} 5 & -3 & 0 \\ -3 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\therefore \qquad g = |g_{ij}| = 4. \qquad .....(2)$$

To get conjugate of g_{ij}, we find

$$G(1, 1) = 8, G(1, 2) = G(2, 1) = 12, G(2, 3) = G(3, 2) = -10,$$

$$G(2, 2) = 20, G(3, 1) = G(1, 3) = -6, G(3, 3) = 6. \qquad .....(3)$$

Since $g^{ij} = \dfrac{G(i,j)}{g}$, we obtain

$$g^{11} = 2, g^{22} = 5, g^{33} = \frac{3}{2}, g^{12} = g^{21} = 3\,,\ g^{23} = g^{32} = -\frac{5}{2}, g^{31} = g^{13} = -\frac{3}{2}. \qquad .....(4)$$

$$\therefore \qquad g^{ij} = \begin{bmatrix} 2 & 3 & -\frac{3}{2} \\ 3 & 5 & -\frac{5}{2} \\ -\frac{3}{2} & -\frac{5}{2} & \frac{3}{2} \end{bmatrix}. \qquad .....(5)$$

Ex.2. *Show that the metric of a Euclidean space, referred to cylindrical coordinates is given by*

$$ds^2 = (dr)^2 + (r\,d\theta)^2 + (dz)^2.$$

Determine its metric tensor and conjugate metric tensor.

Sol. We have

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{pq}d\bar{x}^p d\bar{x}^q, \qquad .....(1)$$

where in rectangular coordinates

$$x^1 = x, x^2 = y, x^3 = z$$

$$g_{12} = g_{21} = g_{13} = g_{31} = g_{23} = g_{32} = 0, g_{11} = g_{22} = g_{33} = 1, \qquad .....(2)$$

and in cylindrical coordinates

$$\bar{x}^1 = r, \bar{x}^2 = \theta, \bar{x}^3 = z\,;\ x = \cos\theta, y = r\sin\theta, z = z \text{ and } \bar{g}_{pq} = ? \qquad .....(3)$$

By covariant transformation law

$$\bar{g}_{pq} = \frac{\partial x^i}{\partial \bar{x}^p}\frac{\partial x^j}{\partial \bar{x}^q}g_{ij}. \qquad .....(4)$$

Therefore

$$\bar{g}_{11} = \frac{\partial x^i}{\partial \bar{x}^1}\frac{\partial x^j}{\partial \bar{x}^1}g_{ij}$$

$$= \left(\frac{\partial x^1}{\partial \bar{x}^1}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^1}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^1}\right)^2 g_{33}$$

$$= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2$$

$$= \cos^2\theta + \sin^2\theta + 0 = 1. \qquad .....(5)$$

$$\bar{g}_{22} = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2$$

$$= r^2\sin^2\theta + r^2\cos^2\theta + 0 = r^2 \qquad .....(6)$$

$$\bar{g}_{33} = \left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2$$

$$= 0 + 0 + 1 = 1 \qquad .....(7)$$

$$\bar{g}_{12} = \frac{\partial x^i}{\partial \bar{x}^1}\frac{\partial x^j}{\partial \bar{x}^2} g_{ij} = \left(\frac{\partial x^1}{\partial \bar{x}^1}\right)\left(\frac{\partial x^1}{\partial \bar{x}^2}\right) g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^1}\right)\left(\frac{\partial x^2}{\partial \bar{x}^2}\right) g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^1}\right)\left(\frac{\partial x^3}{\partial \bar{x}^2}\right) g_{33}$$

$$= \left(\frac{\partial x}{\partial r}\right)\left(\frac{\partial x}{\partial \theta}\right) + \left(\frac{\partial y}{\partial r}\right)\left(\frac{\partial y}{\partial \theta}\right) + \left(\frac{\partial z}{\partial r}\right)\left(\frac{\partial z}{\partial \theta}\right)$$

$$= - r \cos\theta \sin\theta + r \sin\theta \cos\theta + 0. \qquad .....(8)$$

Similarly $\bar{g}_{13} = 0 = \bar{g}_{23}$ due to symmetric property $\bar{g}_{21} = 0, \bar{g}_{31} = 0, \bar{g}_{32} = 0.$

Hence $$ds^2 = \bar{g}_{11}(d\bar{x}^1)^2 + \bar{g}_{22}(d\bar{x}^2)^2 + \bar{g}_{33}(d\bar{x}^3)^2$$

$$= (dr)^2 + (rd\theta)^2 + (dz)^2. \qquad .....(9)$$

The metric tensor $\bar{g}_{pq}$ in cylindrical coordinates is given by

$$\bar{g}_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Clearly $$g = |\bar{g}_{pq}| = r^2, \qquad .....(10)$$

and conjugate metric tensor $\bar{g}_{pq}$, which is the inverse of the matrix (10), is

$$\bar{g}_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

***Ex.* 3.** *Show that the metric of a Euclidean space, referred to spherical coordinates is given by*

$$ds^2 = (dr)^2 + (rd\theta)^2 + (r \sin\theta\, d\phi)^2.$$

Determine its metric tensor and conjugate metric tensor.

Sol. We have $$ds^2 = g_{ij}\, dx^i\, dx^j = \bar{g}_{pq} dx^p dx^q, \qquad .....(1)$$

where in rectangular coordinates

$$x^1 = x, x^2 = y, x^3 = z$$

$$g_{11} = g_{22} = g_{33} = 1, g_{ij} = 0, i \neq j \qquad .....(2)$$

and in spherical polar coordinates

$$\bar{x}^1 = r, \bar{x}^2 = \theta, \bar{x}^3 = \phi$$

$$x = r \sin\theta \cos\phi, y = r \sin\theta \sin\phi, z = r \cos\theta \qquad .....(3)$$

we have to find $\bar{g}_{pq}$.

By covariant law $$\bar{g}_{pq} = \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q} g_{ij.} \qquad .....(4)$$

Therefore $$\bar{g}_{11} = \frac{\partial x^i}{\partial \bar{x}^1} \frac{\partial x^j}{\partial \bar{x}^1} g_{ij} = \left(\frac{\partial x^1}{\partial \bar{x}^1}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^1}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^1}\right)^2 g_{33}$$

$$= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2$$

$$= \sin^2\theta \cos^2\phi + \sin^2\theta \sin^2\phi + \cos^2\theta = 1 \qquad .....(5)$$

$$\bar{g}_{22} = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2$$

$$= r^2 \cos^2\theta \cos^2\phi + r^2 \cos^2\theta \sin^2\phi + r^2 \sin^2\theta = r^2 \qquad .....(6)$$

$$\bar{g}_{33} = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 + \left(\frac{\partial z}{\partial \phi}\right)^2$$

$$= r^2 \sin^2\theta \sin^2\phi + r^2 \sin^2\theta \cos^2\phi + 0 \qquad .....(7)$$

$$= r^2 \sin^2\theta$$

$$\bar{g}_{12} = \frac{\partial x^i}{\partial \bar{x}^1} \frac{\partial x^j}{\partial \bar{x}^2} g_{ij}$$

$$= \left(\frac{\partial x^1}{\partial \bar{x}^1}\right)\left(\frac{\partial x^1}{\partial \bar{x}^1}\right) g_{11} + \left(\frac{\partial x^2}{\partial \bar{x}^1}\right)\left(\frac{\partial x^2}{\partial \bar{x}^2}\right) g_{22} + \left(\frac{\partial x^3}{\partial \bar{x}^1}\right)\left(\frac{\partial x^3}{\partial \bar{x}^2}\right) g_{33}$$

$$= \left(\frac{\partial x}{\partial r}\right)\left(\frac{\partial x}{\partial \theta}\right) + \left(\frac{\partial y}{\partial r}\right)\left(\frac{\partial y}{\partial \theta}\right) + \left(\frac{\partial z}{\partial r}\right)\left(\frac{\partial z}{\partial \theta}\right)$$

$$= \sin\theta \cos\phi (r\cos\theta \cos\phi) + (\sin\theta \sin\phi)(r\cos\theta \sin\phi) + \cos\theta(-r\sin\theta)$$

$$= r\sin\theta\cos\theta(\cos^2\phi + \sin^2\phi) - r\sin\theta\cos\theta \qquad .....(8)$$

$$= 0.$$

Similarly $\bar{g}_{13} = 0, \bar{g}_{23} = 0.$ .....(9)

By the symmetric property $\bar{g}_{21} = 0 = \bar{g}_{31} = \bar{g}_{32} = 0.$ .....(10)

Hence $$ds^2 = \bar{g}_{11}(d\bar{x}^1)^2 + \bar{g}_{22}(d\bar{x}^2)^2 + \bar{g}_{33}(d\bar{x}^3)^2$$

$$= (dr)^2 + (rd\theta)^2 + (r\sin\theta\, d\phi)^2. \qquad .....(11)$$

The metric tensor $\bar{g}_{pq}$ in spherical polar coordinates is given by

$$\bar{g}_{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin\theta \end{bmatrix}. \qquad .....(12)$$

Clearly $$g = |\bar{g}_{pq}| = r^4 \sin^2\theta. \qquad .....(13)$$

The conjugate metric tensor $\bar{g}^{pq}$, which is the inverse of matrix $\bar{g}_{pq}$, in given by

$$\bar{g}^{pq} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{r^2} & 0 \\ 0 & 0 & \dfrac{1}{r^2 \sin\theta} \end{bmatrix}.$$

***Ex.* 4.** *Show that*

(i) $(g_{hj}\, g_{ik} - g_{hk}\, g_{ij})\, g^{hj} = (N-1)\, g_{ik}$

(ii) $\dfrac{\partial k}{\partial x^j}(g_{hk} g_{il} - g_{hl} g_{ik}) g^{hj} = \dfrac{\partial k}{\partial x^k} g_{il} - \dfrac{\partial k}{\partial x^l} g_{ik}$

Sol. We have $\qquad g_{ij}\, g^{ik} = \delta^k_j.$ .....(1)

(i)

$$(g_{hj}\, g_{ik} - g_{hk}\, g_{ij}) g^{hj} = g_{hj}\, g^{hj}\, g_{ik} - g^{hj}\, g_{hk}\, g_{ij}$$
$$= N g_{ik} - \delta^j_k g_{ij}$$
$$= N g_{ik} - g_{ik}$$
$$= (N-1)\, g_{ik}.$$

(ii)

$$\frac{\partial k}{\partial x^j}(g_{hk} - g_{il} - g_{hl} g_{ik}) g^{hj} = \frac{\partial k}{\partial x^j} g_{hk} g^{hj} g_{il} - \frac{\partial k}{\partial x^j} g_{hl} g^{hj} g_{ik}$$
$$= \frac{\partial k}{\partial x^j} \delta^j_k g_{il} - \frac{\partial k}{\partial x^j} \delta^j_l g_{ik}$$
$$= \frac{\partial k}{\partial x^k} g_{il} - \frac{\partial k}{\partial x^l} g_{ik}.$$

***Ex.* 5.** *Show that*

(i) $g^{ij}\, g^{kl}\, dg_{ik} = -\, dg^{jl}$

(ii) $g_{ij}\, g_{kl}\, dg^{ik} = -\, dg_{jl}$

Sol. We have

(i) $\qquad g^{ij}\, g_{ik} = \delta^j_k$ .....(1)

On differentiation $\qquad g^{ij}\, dg_{ik} + g_{ik}\, dg^{ij} = 0$

or $\qquad g^{ij}\, dg_{ik} = -\, g_{ik}\, dg^{ij}.$ .....(2)

Taking inner product on both sides of (2) by g^{kl}, we get

$$g^{ij}\, g^{kl}\, dg_{ik} = -\, g^{kl}\, g_{ik}\, dg^{ij}$$
$$= -\delta^l_i dg^{ij}$$
$$= -\, dg^{lj}$$
$$= -\, dg^{jl}.$$ [by symmetric property]

(ii) Relation *(i)* may be written as

$$g^{ik} g_{ij} = \delta^k_j.$$

On differentiation $\qquad g_{ij}\, dg^{ik} + g^{ik}\, dg_{ij} = 0$

or $\qquad g_{ij}\, dg^{ik} = -\, g^{ik}\, dg_{ij}$

$$g_{ij}\, g_{kl}\, dg^{ik} = -\, g^{ik}\, g_{kl}\, dg_{ij}$$

$$= -\delta^i_l dg_{ij}$$

$$= -\, dg_{ij} = -\, dg_{jl}.$$

Hence proved.

12.6 Permutation symbols and tensors

The permutation symbol is written as e_{ijk} and in the Euclidean three dimensional space V_3 is defined as

$$e_{ijk} = \begin{cases} 0 \quad , & \text{if any two of } i, j, k \text{ are equal} \\ +1 \quad , & \text{if } i, j, k \text{ is a cyclic permutation} \\ -1 \quad , & \text{if } i, j, k \text{ is anticyclic permutation} \end{cases} \qquad \text{.....(12.6.1)}$$

Thus $\qquad e_{112} = e_{113} = e_{221} = e_{223} = e_{331} = e_{332} = e_{111} = e_{222} = e_{333} = 0$

$$e_{123} = e_{231} = e_{312} = +\, 1$$

$$e_{132} = e_{321} = e_{213} = -\, 1 \qquad \text{.....(12.6.2)}$$

We now introduce an entity defined by

$$\in_{ijk} = \sqrt{g} e_{ijk} ; \in^{ijk} = \frac{1}{\sqrt{g}} e_{ijk}, \qquad \text{.....(12.6.3)}$$

where g is the determinant of metric tensor g_{ij} of the space referred, which may not necessarily be rectangular. We shall now prove that although e_{ijk} is not a tensor, in general, both $\in_{ijk}$ and $\in^{ijk}$ are tensors covariant and contravariant respectively. These are called **permutation tensors** in three dimensional space, It is clear from the definitions of e_{ijk}, $\in_{ijk}$ and $\in^{ijk}$ that they are skew-symmetric in all three indices.

Theorem 2. *The entities defined by (permutation tensor)*

$$\in_{ijk} = \sqrt{g}\ e_{ijk}\ ; \in^{ijk} = \frac{1}{\sqrt{g}} e_{ijk},$$

are respectively covariant and contravariant tensors, where e_{ijk} is a permutation symbol and g is the determinant of the metric tensor g_{ij}.

Proof : We have

$$e_{ijk} \frac{\partial x^i}{\partial \bar{x}^l} \frac{\partial x^j}{\partial \bar{x}^m} \frac{\partial x^k}{\partial \bar{x}^n} = e_{jik} \frac{\partial x^j}{\partial \bar{x}^l} \frac{\partial x^i}{\partial \bar{x}^m} \frac{\partial x^k}{\partial \bar{x}^n} \quad \text{(interchanging the dummy indices } i \text{ and } j)$$

$$= -e_{ijk} \frac{\partial x^i}{\partial \bar{x}^m} \frac{\partial x^j}{\partial \bar{x}^l} \frac{\partial x^k}{\partial \bar{x}^n} \quad \text{(using skew-symmetric property of } e_{ijk})$$

This shows that $e_{ijk}\dfrac{\partial x^i}{\partial \overline{x}^l}\dfrac{\partial x^j}{\partial \overline{x}^m}\dfrac{\partial x^k}{\partial \overline{x}^n}$ is skew-symmetric in l and m.

Similarly, it can be shown that it can be shown that it is skew-symmetric in all l, m and n. But this expression, apart from the sign, is the Jacobian determinant $\left|\dfrac{\partial x^r}{\partial \overline{x}^s}\right|\cdot$ From the theory of determinants, it therefore follows that

$$e_{ijk}\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial x^k}{\partial \overline{x}^n}=e_{lmn}\left|\frac{\partial x^i}{\partial \overline{x}^j}\right| \qquad .....(12.6.4)$$

Now by covariant law we know that

$$\overline{g}_{pq}=\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q}g_{ij}. \qquad .....(12.6.5)$$

Therefore $$\left|\overline{g}_{pq}\right|=\left|\frac{\partial x^i}{\partial \overline{x}^p}\right|\left|\frac{\partial x^j}{\partial \overline{x}^q}\right|\left|g_{ij}\right|$$

or $$\overline{g}=\left|\frac{\partial x^r}{\partial \overline{x}^s}\right|^2\cdot g. \qquad .....(12.6.6)$$

Suppose in the coordinate system $\overline{x}^i$, the entity $\in_{ijk}$ be denoted by $\overline{\in}_{lmn}$, where

$$\overline{\in}_{lmn}=\sqrt{\overline{g}}\;e_{lmn}. \qquad .....(12.6.7)$$

Now, using (12.6.4) and (12.6.6) in (12.6.7), we find

$$\overline{\in}_{lmn}=\sqrt{g}\;e_{ijk}\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial x^k}{\partial \overline{x}^n}$$

$$=\in_{ijk}\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial x^k}{\partial \overline{x}^n}, \qquad .....(12.6.8)$$

which shows that $\in_{ijk}$ is a third order covariant tensor.

Also writing e^{lmn} for e_{lmn} and $\overline{e}^{ijk}$ for e_{ijk} we have

$$\in^{lmn}=\frac{1}{\sqrt{\overline{g}}}e^{lmn}=\left|\frac{\partial x^r}{\partial \overline{x}^s}\right|\frac{e^{lmn}}{\sqrt{g}} \qquad \text{[using (12.6.6)]}$$

$$=\frac{1}{\sqrt{g}}\overline{e}^{ijk}\frac{\partial x^l}{\partial \overline{x}^i}\frac{\partial x^m}{\partial \overline{x}^j}\frac{\partial x^n}{\partial \overline{x}^k} \qquad \text{[using (12.6.4)]}$$

$$=\in^{ijk}\frac{\partial x^l}{\partial \overline{x}^i}\frac{\partial x^m}{\partial \overline{x}^j}\frac{\partial x^n}{\partial \overline{x}^k}, \qquad .....(12.6.9)$$

which shows that $\in^{lmn}$ in a contravariant tensor of third order.

12.7 Christoffel symbols

These are the two expressions due to Christoffel involving contravariant fundamental tensor g^{ij} and the partial derivations of the components of the fundament tensor g_{ij}.

The Christoffel symbols of the **first and second** kind are denoted by $[i\,j,\,n]$ and $\left[\begin{smallmatrix} k \\ ij \end{smallmatrix}\right]$, respectively and are defined as

$$[ij,\,k] = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) \qquad \text{.....(12.7.1)}$$

$$\left\{\begin{smallmatrix} k \\ ij \end{smallmatrix}\right\} = g^{kh}\left[ij, h\right] \qquad \text{.....(12.7.2)}$$

Notes :

(i) The symbols $[ij,\,k]$ and $\left\{\begin{smallmatrix} l \\ ij \end{smallmatrix}\right\}$ may also be represented by $T_{k,ij}$ and T_{ij}^{l} respectively.

However we shall use only the brackets type representation.

(ii) These symbols, in general, are not tensors.

(iii) All, but one of the indices of the Christoffel symbols are regarded as subscripts. The exception in the index l which is treated as the superscript in the symbol of second kind.

(iv) Both the symbols are symmetric with respect to the indices i and j.

(v) In Euclidean space of N-dimensions $g_{11} = g_{22} = \ldots = g_{NN} = 1$ and $g_{ii} = 0,\ i \neq j$ in this case all the Christoffel symbols are zero.

(vi) Since g_{ij} is a symmetric tensor and has $\dfrac{N(N+1)}{2}$ independent components in the space V_N, then $\dfrac{\partial g_{ij}}{\partial x^k}$ will have $N\dfrac{N(N+1)}{2}$ independent components. Therefore the number of independent components of Christoffel symbols of a kind are $\dfrac{N^2(N+1)}{2}$.

12.8 Properties of Christoffel symbols

Property I : $[ij,\,m] = g_{lm}\left[\begin{smallmatrix} l \\ ij \end{smallmatrix}\right]$

Proof : By definition we have

$$\left[\begin{smallmatrix} l \\ ij \end{smallmatrix}\right] = g^{lk}\,[ij,\,k] \qquad \text{.....(12.8.1)}$$

Taking inner product by g_{lm}, we get

$$\therefore \qquad g_{lm}\left[\begin{smallmatrix} l \\ ij \end{smallmatrix}\right] = g_{lm}\,g^{lk}\,[ij,\,k]$$

$$= \delta_m^k\left[ij, k\right] = [ij,\,m].$$

Hence proved.

Property II : $\dfrac{\partial g_{ik}}{\partial x^j} = [ij, k] + [kj, i]$

Proof : We know by definition

$$[ij,\ k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) \qquad .....(12.8.2)$$

and $$[kj,\ i] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i}\right) \qquad .....(12.8.3)$$

keeping in the mind the symmetric property of g_{ij}, adding above two expressions, we get

$$[ij,k]+[kj,i] = \frac{\partial g_{ik}}{\partial x^j}. \quad \text{Hence proved}$$

Property III : $\dfrac{\partial g^{mk}}{\partial x^l} = -g^{mi}\left\{{k \atop il}\right\} - g^{ki}\left\{{m \atop il}\right\}$

Proof : We know that $g_{ij}\, g^{ik} = \delta^k_j.$ .....(12.8.4)

Differentiating with respect to x^l, we get

$$g_{ij}\frac{\partial g^{ik}}{\partial x^l} + \frac{\partial g_{ij}}{\partial x^l} g^{ik} = 0$$

or $$g_{ij}\frac{\partial g^{ik}}{\partial x^l} = -g^{ik}\frac{\partial g_{ij}}{\partial x^l} \qquad .....(12.8.5)$$

Taking inner product with g^{jm}, we obtain

$$\delta^m_i \frac{\partial g^{ik}}{\partial x^l} = -g^{ik} g^{jm}\frac{\partial g_{ij}}{\partial x^l} \qquad .....(12.8.6)$$

Now using property 2, we finally get

$$\begin{aligned}\frac{\partial g^{mk}}{\partial x^l} &= -g^{ik} g^{jm}\left([jl,i]+[il,j]\right)\\ &= -g^{ik} g^{jm}[jl,\ i] - g^{ik} g^{jk}[il, j]\\ &= -g^{jm}\left\{{k \atop ij}\right\} - g^{ik}\left\{{m \atop il}\right\}\\ &= -g^{mj}\left\{{k \atop ij}\right\} - g^{ki}\left\{{m \atop il}\right\},\end{aligned} \qquad .....(12.8.7)$$

where the dummy index in the first term has been replaced by i.

Property IV :

$$\begin{aligned}\left\{{i \atop ij}\right\} &= \frac{1}{2g}\frac{\partial g}{\partial x^j}\\ &= \frac{\partial}{\partial x^j}\left\{\log\sqrt{g}\right\}, \text{if } g \text{ is positive}\\ &= \frac{\partial}{\partial x^j}\left\{\log\sqrt{(-g)}\right\}, \text{if } g \text{ is negative}\end{aligned}$$

Proof : We have the matrix of g_{ij} as

$$g_{ij} = \begin{bmatrix} g_{11} & g_{12}\cdots & g_{1N} \\ g_{21} & g_{22}\cdots & g_{2N} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ g_{N1} & g_{N2} & g_{NN} \end{bmatrix}. \qquad \text{.....(12.8.8)}$$

Since g denotes the determinant of g_{ij}, we have

$$g = \begin{bmatrix} g_{11} & g_{12}\cdots & g_{1N} \\ g_{21} & g_{22}\cdots & g_{2N} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ g_{N1} & g_{N2} & g_{NN} \end{bmatrix}. \qquad \text{.....(12.8.9)}$$

Differentiating it with respect to x^j, we get

$$\frac{\partial g}{\partial x^j} = \begin{bmatrix} \frac{\partial g_{11}}{\partial x^j} & \frac{\partial g_{12}}{\partial x^j}\cdots & \frac{\partial g_{1N}}{\partial x^j} \\ g_{21} & g_{22}\cdots & g_{2N} \\ \cdots & \cdots & \cdots \\ g_{N1} & g_{N2} & g_{NN} \end{bmatrix} + \dots + \begin{bmatrix} g_{11} & g_{12}\cdots & g_{1N} \\ g_{21} & g_{22}\cdots & g_{2N} \\ \cdots & \cdots & \cdots \\ \frac{\partial g_{N1}}{\partial x^j} & \frac{\partial g_{N2}}{\partial x^j}\cdots & \frac{\partial g_{NN}}{\partial x^j} \end{bmatrix} \qquad \text{.....(12.8.10)}$$

Clearly, cofactor of $\dfrac{\partial g_{11}}{\partial x^j}$ = cofactor of g_{11} in $g = gg^{11}$ etc. .....(12.8.11)

Thus

$$\frac{\partial g}{\partial x^j} = \left(\frac{\partial g_{11}}{\partial x^j} gg^{11} + \frac{\partial g_{12}}{\partial x^j} gg^{12} + \dots + \frac{\partial g_{1N}}{\partial x^j} gg^{1N}\right) + \dots$$

$$+ \left(\frac{\partial g_{N1}}{\partial x^j} gg^{N1} + \frac{\partial g_{N2}}{\partial x^j} gg^{N2} + \dots + \frac{\partial g_{NN}}{\partial x^j} gg^{NN}\right)$$

$$= gg^{ik} \frac{\partial g_{ik}}{\partial x^j}. \qquad \text{.....(12.8.12)}$$

Now using property II, we get

$$\frac{\partial g}{\partial x^j} = gg^{ik}\,([ij,\, k] + [kj,\, i])$$

$$= g\begin{Bmatrix} i \\ ij \end{Bmatrix} + g\begin{Bmatrix} k \\ kj \end{Bmatrix}$$

$$= g\begin{Bmatrix} i \\ ij \end{Bmatrix} + g\begin{Bmatrix} i \\ ij \end{Bmatrix} \qquad \text{(as } k \text{ is dummy index so it is replaced by } i\text{)}$$

$$= 2g\begin{Bmatrix} i \\ ij \end{Bmatrix}. \qquad \text{.....(12.8.13)}$$

Hence $$\begin{Bmatrix} i \\ ij \end{Bmatrix} = \frac{1}{2g}\frac{\partial g}{\partial x^j} = \frac{\partial}{\partial x^j}\left\{\log\sqrt{g}\right\}, \text{ if } g \text{ is positive}$$

$$= \frac{\partial}{\partial x^j}\left\{\log\sqrt{-g}\right\}, \text{ if } g \text{ is negative}$$

Hence proved.

12.9 Illustrative examples

Ex.5. *Calculate the* Christoffel *symbols corresponding to the metric*

$$ds^2 = (dx^1)^2 + G(x^1, x^2)\,(dx^2)^2$$

where G is a function of x^1 and x^2.

Sol. Here $N = 2$ *i.e.* it is a two dimensional space, where

$$g_{11} = 1,\ g_{22} = G\,(x^1, x^2),\ g_{12} = 0,\ g_{21} = 0.$$

The number of Christoffel symbols of a kind will be

$$\frac{N^2(N+1)}{2} = \frac{2^2\cdot(2+1)}{2} = 6.$$

I. First kind :

Case I : $i = j = k$, then $\qquad [ii, i] = \frac{1}{2}\frac{\partial g_{ii}}{\partial x^i}$ (No summations)

Hence $\qquad [11, 1] = 0,\ [22, 2] = \frac{1}{2}\frac{\partial G}{\partial x^2}.$ .....(1)

Case II : $i = j \neq k$, then $\qquad [ii, k] = -\frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}.$ .....(2)

Therefore $\qquad [11, 2] = 0,\ [22, 1] = -\frac{1}{2}\frac{\partial G}{\partial x^1}.$ .....(3)

Case III : $i = k \neq j$, then $\qquad [ij, i] = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^i}.$ .....(4)

Therefore $\qquad [12, 1] = 0,\ [21, 2] = \frac{1}{2}\frac{\partial G}{\partial x^1}.$ .....(5)

Case IV : $i \neq j \neq k$

It is redundant in two dimensional space

II. Second kind : $$g^{ii} = \frac{1}{g_{ii}},\ g^{ij} = 0, i \neq j$$

$$\begin{Bmatrix} l \\ ij \end{Bmatrix} = g^{ll}[ij, l] \qquad .....(6)$$

Hence $$\begin{Bmatrix}1\\12\end{Bmatrix}=0,\ \begin{Bmatrix}2\\12\end{Bmatrix}=g^{22}[12,2]=\frac{1}{2G}\frac{\partial G}{\partial x^1}, \qquad .....(7)$$

$$\begin{Bmatrix}1\\11\end{Bmatrix}=0,\ \begin{Bmatrix}1\\22\end{Bmatrix}=g^{11}[22,1]=-\frac{1}{2}\frac{\partial G}{\partial x^1}, \qquad .....(8)$$

$$\begin{Bmatrix}2\\11\end{Bmatrix}=0,\ \begin{Bmatrix}2\\22\end{Bmatrix}=g^{22}[22,2]=\frac{1}{2G}\frac{\partial G}{\partial x^2}, \qquad .....(9)$$

Ex.6. *Surface of sphere is a two dimensional Riemannian space. Compute the* Christoffel *symbols*

Sol. For a sphere, r is constant, the metric of the surface of a sphere is given by

$$ds^2 = r^2\,(d\theta)^2 + r^2 \sin^2\theta\,(d\phi)^2.$$

Here $g_{11} = r^2,\ g_{22} = r^2 \sin^2\theta,\ g_{12} = g_{21} = 0$

and $$g^{11}=\frac{1}{r^2},\ g^{22}=\frac{1}{r^2\sin^2\theta},\ g^{12}=g^{21}=0 \qquad .....(1)$$

***(i)* First kind :**

(a) $i = j = k$, then $[ii, i] = \dfrac{1}{2}\dfrac{\partial g_{ii}}{\partial x^i}$. [No summation]

Therefore $[11, 1] = 0,\ [22, 2] = 0.$ .....(2)

(b) $i = j \neq k$, then $[ii, k] = -\dfrac{1}{2}\dfrac{\partial g_{ii}}{\partial x^k}$

Therefore $[11, 2] = 0,\ [22, 1] = -\, r^2 \sin\theta \cos\theta$ .....(3)

(c) $i = k \neq j$, then $[ij, i] = \dfrac{1}{2}\dfrac{\partial g_{ii}}{\partial x^j}$

Therefore $[12, 1] = 0,\ [21, 2] = r^2 \sin\theta \cos\theta.$ .....(4)

(d) $i \neq j \neq k$,

Redundant in two dimensional space

***(ii)* Second kind :** $\begin{Bmatrix}l\\ij\end{Bmatrix} = g^{ll}\,[ij, l]$ (No summation)

The non-zero components are

$$\begin{Bmatrix}1\\22\end{Bmatrix} = g^{11}\,[22, 1] = -\sin\theta\cos\theta$$

and $$\begin{Bmatrix}2\\21\end{Bmatrix} = g^{22}\,[21, 2] = \cot\theta \qquad .....(5)$$

The remaining four will be zero.

Ex.7. *Calculate the Christoffel symbols corresponding to the metric*

$$ds^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2 + (x^1)^2 (\sin x^2)^2 (dx^3)^2$$

Sol. For the given metric

$$g_{11} = 1, g_{22} = (x^1)^2, g_{33} = (x^1)^2 (\sin x^2)^2 \qquad .....(1)$$

and $g_{ij} = 0, i \neq j.$

(I) Christoffel symbols of the first kind :

Since $N = 3$, the total number of independent components to be determined are $\dfrac{N^2(N+1)}{2} = 18.$

We know that $$[ij, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right). \qquad .....(2)$$

Here, there are four cases.

Case I : $i = j = k$, then (1) becomes

$$[ii, i] = \frac{1}{2}\frac{\partial g_{ii}}{\partial x^i} \qquad \text{(No summations)}$$

$\therefore$ We find $$[11, 1] = 0, [22, 2] = 0, [33, 3] = 0 \qquad .....(3)$$

Case II : $i = j \neq k$, then (1) becomes

$$[ii, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ik}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) = -\frac{1}{2}\frac{\partial g_{ii}}{\partial x^k}$$

Therefore we find

$$[11, 2] = 0, [11, 3] = 0, [22, 1] = -x^1$$
$$[22, 3] = 0, [33, 1] = -x^1 (\sin x^2)^2,$$
$$[33, 2] = -(x^1)^2 \sin x^2 \cos x^2 \qquad .....(4)$$

Case III : $i = k \neq j$, then (1) gives

$$[ij, i] = \frac{1}{2}\left(\frac{\partial g_{ii}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^i}\right) = \frac{1}{2}\frac{\partial g_{ii}}{\partial x^j}$$

Therefore we find

$$[12, 1] = 0, [13, 1] = 0, [21, 2] = x^1, [23, 2] = 0$$
$$[31, 3] = x^1 (\sin x^2)^2, [32, 3] = (x^1)^2 \sin x^2 \cos x^2$$

Case IV : $i \neq j \neq k$, then (2) by virtue of (1), becomes $[ij, k] = 0$.

Hence, $$[12, 3] = 0, [23, 1] = 0, [31, 2] = 0 \qquad .....(6)$$

(II) Christoffel symbols of second kind :

We have $$g^{11} = \frac{1}{g_{11}} = 1, g^{22} = \frac{1}{(x^1)^2}, g^{33} = \frac{1}{(x^1)^2(\sin x^2)^2}$$

and $$g_{ij} = 0, i \neq j \qquad .....(7)$$

We know that $\begin{Bmatrix} l \\ ij \end{Bmatrix} = g^{lk}\,[ij,\ k]$ .....(8)

Here too, we have to find out 18 independent components

Since $g^{ij} = 0$ when $i \neq j$.

We have from (2) $\begin{Bmatrix} l \\ ij \end{Bmatrix} = g^{ll}\,[ij,\ l]$ (No summation over i) .....(9)

The non-zero components are

$$\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = g^{11}[22,1] = -x^1, \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = g^{11}[33,1] = -x^1(\sin x^2)^2$$

$$\begin{Bmatrix} 2 \\ 33 \end{Bmatrix} = g^{22}[33,2] = -\sin x^2 \cos x^2$$

$$\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = g^{22}[12,2] = \frac{1}{x^1}$$

$$\begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = g^{33}[13,3] = \frac{1}{x^1},$$

$$\begin{Bmatrix} 3 \\ 32 \end{Bmatrix} = g^{33}[32,3] = \cot x^2 \qquad .....(10)$$

The remaining twelve components will be zero.

Ex.8. *If the metric of a V_N is such that $g_{ij} = 0$ for $i \neq j$, show that*

$$\begin{Bmatrix} i \\ jk \end{Bmatrix} = 0, \quad \begin{Bmatrix} i \\ jj \end{Bmatrix} = -\frac{1}{2g_{ii}}\frac{\partial g_{ii}}{\partial x^i}$$

$$\begin{Bmatrix} i \\ ij \end{Bmatrix} = \frac{\partial}{\partial x^j}\left\{\log\sqrt{g_{ii}}\right\}; \begin{Bmatrix} i \\ ii \end{Bmatrix} = \frac{\partial}{\partial x^i}\left\{\log\sqrt{g_{ii}}\right\}$$

where i, j and k are not equal, and the summation convention does not apply.

Sol. Here $g^{ii} = \dfrac{1}{g_{ii}}$ and $g^{ij} = 0, i \neq j$

(i) $\begin{Bmatrix} i \\ jk \end{Bmatrix} = g^{il}[jk,l] = g^{ii}[jk,i], \qquad (\because g^{il} = 0,\ \ i \neq l)$

$= 0$ $\qquad$ ($\therefore$ fundamental tensors are zero when $i \neq j \neq k$) .....(1)

(ii) $\begin{Bmatrix} i \\ jj \end{Bmatrix} = g^{il}[jj,l] = -\dfrac{1}{2}g^{il}\dfrac{\partial g_{jj}}{\partial x^l} = -\dfrac{1}{2}g^{ii}\dfrac{\partial g_{jj}}{\partial x^i}$

$$= -\frac{1}{2g_{ii}}\frac{\partial g_{jj}}{\partial x^i}. \qquad .....(2)$$

(iii) $$\begin{Bmatrix} i \\ ij \end{Bmatrix} = g^{il}\,[ij,\, l] = g^{ii}\,[ij,\, l]$$

$$= \frac{1}{2} g^{ii} \frac{\partial g_{ii}}{\partial x^j} = \frac{1}{2g_{ii}} \frac{\partial g_{ii}}{\partial x^j}$$

$$= \frac{\partial}{\partial x^j} \left\{ \log \sqrt{g_{ii}} \right\}. \qquad .....(3)$$

(iv) $$\begin{Bmatrix} i \\ ii \end{Bmatrix} = g^{il}\,[ii,\, l]$$

$$= g^{ii}[ii, i] = \frac{1}{2} g^{ii} \frac{\partial g^{ii}}{\partial x^i}$$

$$= \frac{1}{2g^{ii}} \frac{\partial g_{ii}}{\partial x^i} = \frac{\partial}{\partial x^i} \left\{ \log \sqrt{g_{ii}} \right\}. \qquad .....(4)$$

12.10 Laws of transformation of Christoffel symbols

Theorem 3. *The Christoffel symbols are not tensor quantities.*

Proof : Let us consider the transformation of the Christoffel symbols from the coordinate system x^i to $\overline{x}^k$.

(i) By definition we know that

$$\overline{[lm, n]} = \frac{1}{2} \left[\frac{\partial \overline{g}_{in}}{\partial \overline{x}^m} + \frac{\partial \overline{g}_{mn}}{\partial \overline{x}^l} - \frac{\partial \overline{g}_{lm}}{\partial \overline{x}^n} \right] \qquad .....(12.10.1)$$

and

$$[ij,\, k] = \frac{1}{2} \left[\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right] \qquad .....(12.10.2)$$

We know that by covariant law

$$\overline{g}_{lm} = \frac{\partial x^i}{\partial \overline{x}^l} \frac{\partial x^j}{\partial \overline{x}^m} g_{ij} \qquad .....(12.10.3)$$

Differentiating with respect to $\overline{x}^n$, we get

$$\frac{\partial \overline{g}_{lm}}{\partial \overline{x}^n} = \frac{\partial x^i}{\partial \overline{x}^l} \frac{\partial x^i}{\partial \overline{x}^m} \frac{\partial g_{ij}}{\partial x^k} \frac{\partial x^k}{\partial \overline{x}^n} + \frac{\partial^2 x^i}{\partial \overline{x}^n \partial \overline{x}^l} \frac{\partial x^j}{\partial \overline{x}^m} g_{ij} + \frac{\partial x^i}{\partial \overline{x}^l} \frac{\partial^2 x^j}{\partial \overline{x}^m \partial \overline{x}^n} g_{ij} \qquad .....(12.10.4)$$

Similarly (by cyclic order)

$$\frac{\partial \overline{g}_{mn}}{\partial \overline{x}^l} = \frac{\partial x^j}{\partial \overline{x}^m} \frac{\partial x^k}{\partial \overline{x}^n} \frac{\partial x^i}{\partial \overline{x}^l} \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial^2 x^j}{\partial \overline{x}^l \partial \overline{x}^m} \frac{\partial x^k}{\partial \overline{x}^n} g_{jk} + \frac{\partial x^j}{\partial \overline{x}^m} \frac{\partial^2 x^k}{\partial \overline{x}^n \partial \overline{x}^l} g_{jk} \qquad .....(12.10.5)$$

$$\frac{\partial \overline{g}_{nl}}{\partial \overline{x}^m} = \frac{\partial x^k}{\partial \overline{x}^n} \frac{\partial x^i}{\partial \overline{x}^l} \frac{\partial x^j}{\partial \overline{x}^m} \frac{\partial g_{ki}}{\partial \overline{x}^j} + \frac{\partial^2 x^k}{\partial \overline{x}^m \partial \overline{x}^n} \frac{\partial x^i}{\partial \overline{x}^l} g_{ki} + \frac{\partial x^k}{\partial \overline{x}^n} \frac{\partial^2 x^i}{\partial \overline{x}^l \partial \overline{x}^m} g_{ki} \qquad .....(12.10.6)$$

Multiplying (12.10.5) and (12.10.6) by $\left(\frac{1}{2}\right)$ and (12.10.4) by $\left(-\frac{1}{2}\right)$ then adding keeping in view (12.10.1) and (12.10.2) we get on changing the dummy indices appropriately, we get

$$\left[\overline{lm,n}\right] = \frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial x^k}{\partial \overline{x}^n}[ij,k] + \frac{\partial x^k}{\partial \overline{x}^n}\frac{\partial^2 x^i}{\partial \overline{x}^l \partial \overline{x}^m} g_{ik}$$

$$= [ij,k]\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial x^k}{\partial \overline{x}^n} + \frac{\partial x^i}{\partial \overline{x}^n}\frac{\partial^2 x^j}{\partial \overline{x}^l \partial \overline{x}^m} \quad .....(12.10.7)$$

which shows that Christoffel symbols of first kind do not behave like tensors.

(ii) By contravariant law, we have

$$\overline{g}^{np} = \frac{\partial \overline{x}^n}{\partial x^r}\frac{\partial \overline{x}^p}{\partial x^s} g^{rs}. \quad .....(12.10.8)$$

Taking inner multiplication of (12.10.7) by $\overline{g}^{np}$ and its corresponding equivalent from (12.10.8), we get

$$\overline{g}^{np}\left[\overline{lm,n}\right] = [ij,k]\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\delta_r^k \frac{\partial \overline{x}^p}{\partial x^s} g^{rs} + g_{ij} g^{rs} \delta_r^i \frac{\partial \overline{x}^p}{\partial x^s}\frac{\partial^2 x^j}{\partial \overline{x}^l \partial \overline{x}^m}$$

or

$$\left\{\overline{\begin{matrix} p \\ lm \end{matrix}}\right\} = g^{ks}[ij,k]\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial \overline{x}^p}{\partial x^s} + g_{ij} g^{is}\frac{\partial \overline{x}^p}{\partial x^s}\frac{\partial^2 x^j}{\partial \overline{x}^l \partial \overline{x}^m}$$

$$= \left\{\begin{matrix} s \\ ij \end{matrix}\right\}\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial \overline{x}^p}{\partial x^s} + \frac{\partial \overline{x}^p}{\partial x^j}\frac{\partial^2 x^j}{\partial \overline{x}^l \partial \overline{x}^m} \quad .....(12.10.9)$$

which shows that Christoffel symbols of second kind also do not behave like tensors.

Remark : We have proved that Christoffel symbols are not tensor quantities. But in some very special case of linear transformation of coordinates, viz.

$$x^j = a_m^j \overline{x}^m + b^j,$$

where a_m^j and b^j are constants, we have

$$\frac{\partial^2 x^j}{\partial \overline{x}^l \partial \overline{x}^m} = 0.$$

and the equations (12.10.8) and (12.10.9) become

$$\left[\overline{lm,n}\right] = [ij,k]\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial x^k}{\partial \overline{x}^n}, \quad .....(12.10.10)$$

$$\left\{\overline{\begin{matrix} p \\ lm \end{matrix}}\right\} = \left\{\begin{matrix} s \\ ij \end{matrix}\right\}\frac{\partial \overline{x}^p}{\partial x^s}\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}, \quad .....(12.10.11)$$

which shows that in the case of linear transformation Christoffel symbols transform like a tensor.

Note : Taking inner multiplication of (12.10.9) by $\dfrac{\partial x^r}{\partial \overline{x}^p}$, we get

$$\left\{\begin{matrix}\overline{p}\\ lm\end{matrix}\right\}\frac{\partial x^r}{\partial \overline{x}^p} = \left\{\begin{matrix}s\\ ij\end{matrix}\right\}\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\delta_s^r + \frac{\partial^2 x^j}{\partial \overline{x}^l \partial \overline{x}^m}\delta_j^r$$

$$\therefore \qquad \frac{\partial^2 x^r}{\partial \overline{x}^l \partial \overline{x}^m} = \left\{\begin{matrix}\overline{p}\\ lm\end{matrix}\right\}\frac{\partial x^r}{\partial \overline{x}^p} - \left\{\begin{matrix}r\\ ij\end{matrix}\right\}\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m} \qquad .....(12.10.12)$$

It is an important result and should be remembered. It expresses the second order partial derivative of x^r with respect to $\overline{x}^i$ in terms of the first derivatives and Christoffel symbols of the second kind.

12.11 Self-learning exercises

1. Define metric tensor.
2. Define permutation tensors.
3. What do you mean by indicator.
4. Show that $g\, dx^i\, dx^j$ in invariant.
5. How many independent components of Christoffel symbols in V_3.

12.12 Summary

In this unit we have generalised the concept of distance to the Riemannian space by metric and defined metric tensor, a covariant symmetric tensor of rank two, some examples are given to calculate components of metric tensor in different Riemannian spaces. We have defined Christoffel symbols of first and second kinds, which are the expressions of partial derivatives of fundamental tensor g_{ij}. Some properties of these symbols are given and some examples are given to calculate these symbols. In the end we have shown that Christoffel symbols are not tensor quantities.

12.13 Answers to self-learning exercises

1. § 12.2
2. § 12.6
3. § 12.4
4. § 12.2
5. 18

12.14 Exercises

1. Show that g_{ij} is a covariant tensor of order two.

2. Prove that g^{ij} is a symmetric contravariant tensor of rank two.

3. What are the fundamental tensors and show that :

$$A^{ij}\, dg_{ij} = -A_{ij}\, dg^{ij}.$$

4. Prove that :

(i) $g^{ij}\, g^{kl}\, dg_{ik} = -\, dg^{il}$,

(ii) $g_{ij}\, g_{kl}\, dg^{ik} = -\, dg_{jl}$.

5. Prove that the permutation tensors are tensor of third order and also show that

$$\in_{ijk} = g_{il}\, g_{jm}\, g_{kn} \in^{lmn},$$

where the symbols have their usual meanings.

6. Show that the transformation of Christoffel symbols form a group.

7. Evaluate Christoffel symbols in spherical coordinates.

8. Prove that the Christoffel symbols are not tensor.

□□□

Unit 13 : Covariant Differentiation of Tensors, Ricci Theorem, Intrinsic Derivative

Structure of the Unit

13.0 Objective

The objective of this unit is to study the behavior of partial derivatives of vectors and tensors and consequently covariant differentiation. The properties of covariant differentiation and its uses are also the points of study.

13.1 Introduction

The transformation laws of partial derivatives of covariant, contravariant vectors and tensors are not like tensor quantities. So we investigate a particular form of partial derivative which behaves like tensors and it will be called covariant derivative.

13.2 Covariant differentiation of vectors

Here we study the transformation laws of the partial derivatives of contravariant and covariant vectors. We also investigate that these partial derivatives behave like tensors or not.

***(i)* Covariant derivative of contravariant vector :**

We have from contravariant law of vector

$$A^k = \frac{\partial x^k}{\partial \overline{x}^i} \overline{A}^i, \qquad \text{......(13.2.1)}$$

Differentiating partially with respect to x^j, we get

$$\frac{\partial A^k}{\partial x^j} = \frac{\partial x^k}{\partial \overline{x}^i} \frac{\partial \overline{x}^n}{\partial x^j} \cdot \frac{\partial \overline{A}^i}{\partial \overline{x}^n} + \frac{\partial^2 x^k}{\partial \overline{x}^n \partial \overline{x}^i} \frac{\partial \overline{x}^n}{\partial x^j} \overline{A}^i. \qquad \text{.....(13.2.2)}$$

The presence of second form on the R.H.S. of above equation shows that the partial derivative $\frac{\partial A^k}{\partial x^j}\cdot$ does not behave like a tensor.

Putting the value of

$$\frac{\partial^2 x^k}{\partial \overline{x}^n\, \overline{x}^i} = \left\{ \begin{matrix} \overline{p} \\ in \end{matrix} \right\} \frac{\partial x^k}{\partial \overline{x}^p} - \left\{ \begin{matrix} k \\ r_s \end{matrix} \right\} \frac{\partial x^r}{\partial \overline{x}^i} \frac{\partial x^s}{\partial \overline{x}^n}$$

in the above equation, we have

$$\frac{\partial x^k}{\partial x^j} = \left[\left\{ \begin{matrix} \overline{p} \\ in \end{matrix} \right\} \frac{\partial x^k}{\partial \overline{x}^p} - \left\{ \begin{matrix} k \\ r_s \end{matrix} \right\} \frac{\partial x^r}{\partial \overline{x}^i} \frac{\partial x^s}{\partial \overline{x}^n} \right] \frac{\partial \overline{x}^n}{\partial x^j} \overline{A}^i + \frac{\partial x^k}{\partial \overline{x}^i} \cdot \frac{\partial \overline{x}^n}{\partial x^j} \cdot \frac{\partial \overline{A}^i}{\partial \overline{x}^n}$$

or

$$\frac{\partial A^k}{\partial x^j} + \left\{ \begin{matrix} k \\ rs \end{matrix} \right\} \frac{\partial x^r}{\partial \overline{x}^i} \delta^s_j \overline{A}^i = \left\{ \begin{matrix} \overline{p} \\ im \end{matrix} \right\} \frac{\partial x^k}{\partial \overline{x}^p} \frac{\partial \overline{x}^n}{\partial x^j} \cdot \overline{A}^i + \frac{\partial x^k}{\partial \overline{x}^i} \frac{\partial \overline{x}^n}{\partial x^j} \cdot \frac{\partial \overline{A}^i}{\partial \overline{x}^n} \qquad \text{...(13.2.3)}$$

Using (13.2.1) and making suitable changes of dummy indices, we may write

$$\frac{\partial A^k}{\partial x^j} + \left\{ \begin{matrix} k \\ rj \end{matrix} \right\} A^r = \left[\frac{\partial \overline{A}^p}{\partial \overline{x}^n} + \left\{ \begin{matrix} \overline{p} \\ in \end{matrix} \right\} \overline{A}^i \right] \frac{\partial x^k}{\partial \overline{x}^p} \frac{\partial \overline{x}^n}{\partial x^j}. \qquad \text{......(13.2.4)}$$

Introducing following comma notation, viz.

$$A^k_{,j} \equiv \frac{\partial A^k}{\partial x^j} + \left\{ \begin{matrix} k \\ rj \end{matrix} \right\} A^r \qquad \text{.....(13.2.5)}$$

the above equation may be written as

$$A^k_{,j} = \overline{A}^p_{,n} \frac{\partial x^k}{\partial \overline{x}^p} \frac{\partial \overline{x}^n}{\partial x^j}, \qquad \text{......(13.2.6)}$$

which shows that $A^k_{,j}$ behaves like a mixed tensor of second order. It is called covariant derivative of a contravariant vector A^k with respect to x^j.

***(ii)* Covariant derivative of covariant vector :**

We have from covariant transformation law of vector

$$A^k = \frac{\partial \overline{x}^i}{\partial x^k} \overline{A}_i. \qquad .....(13.2.7)$$

Differentiating partially with respect to x^j, we get

$$\frac{\partial A_k}{\partial x^j} = \frac{\partial \overline{x}^i}{\partial x^k} \frac{\partial \overline{x}^n}{\partial x^j} \frac{\partial \overline{A}_i}{\partial \overline{x}^n} + \frac{\partial^2 \overline{x}^i}{\partial x^j \partial x^k} \overline{A}_{i,} \qquad ......(13.2.8)$$

Substituting
$$= \frac{\partial^2 \overline{x}^i}{\partial x^j \partial x^k} = \begin{Bmatrix} p \\ jk \end{Bmatrix} \frac{\partial \overline{x}^i}{\partial x^p} - \begin{Bmatrix} \overline{i} \\ lm \end{Bmatrix} \frac{\partial \overline{x}^l}{\partial x^j} \frac{\partial \overline{x}^m}{\partial x^k}$$

and making suitable changes in dummy indices.

$$\frac{\partial A_k}{\partial x^j} = \frac{\partial \overline{x}^i}{\partial x^k} \cdot \frac{\partial \overline{x}^n}{\partial x^j} \frac{\partial \overline{A}_i}{\partial \overline{x}^n} + \left[\begin{Bmatrix} p \\ jk \end{Bmatrix} \frac{\partial \overline{x}^i}{\partial x^p} - \begin{Bmatrix} \overline{i} \\ lm \end{Bmatrix} \frac{\partial \overline{x}^l}{\partial x^j} \frac{\partial \overline{x}^m}{\partial x^k} \right] \overline{A}_i$$

or
$$\frac{\partial A_k}{\partial x^j} - \begin{Bmatrix} p \\ jk \end{Bmatrix} A_p = \left[\frac{\partial \overline{A}_i}{\partial \overline{x}^n} - \begin{Bmatrix} \overline{l} \\ ni \end{Bmatrix} \overline{A}_l \right] \frac{\partial \overline{x}^i}{\partial x^k} \cdot \frac{\partial \overline{x}^n}{\partial x^j}. \qquad ......(13.2.9)$$

Introducing the comma notation, viz.

$$A_{k,j} = \frac{\partial A_k}{\partial x^j} - \begin{Bmatrix} p \\ jk \end{Bmatrix} A_p, \qquad ......(13.2.10)$$

we get above relation as
$$A_{k,j} = \overline{A}_{i,n} \frac{\partial \overline{x}^i}{\partial x^k} \frac{\partial \overline{x}^n}{\partial x^j}, \qquad .....(13.2.11)$$

which shows that $A_{k,j}$ behave like a covariant tensor of second order. If is called covariant derivative of covariant vector A_k with respect to x^j.

Note : Expression (13.2.6) and (13.2.11) are very important and should be remembered. These may be taken as definition of covariant derivatives of contravariant and covariant derivatives respectively.

13.3 Covariant differentiation of second order tensors

In order to extend the process of covariant differentiation to tensors of order more than one, we choose, without loss of generality, a mixed tensor A^i_j.

We have from transformation law

$$A^i_j = \frac{\partial x^i}{\partial \overline{x}^m} \cdot \frac{\partial \overline{x}^l}{\partial x^j} \cdot \overline{A}^m_l. \qquad ......(13.3.1)$$

Taking inner product with $\frac{\partial \overline{x}^m}{\partial x^j}$, we get

$$\frac{\partial \overline{x}^m}{\partial x^i} A^i_j = \frac{\partial \overline{x}^l}{\partial x^j} \overline{A}^m_l. \qquad ......(13.3.2)$$

Differentiating with respect to x^k, we find

$$\frac{\partial^2 \bar{x}^m}{\partial x^k \partial x^i} A^i_j + \frac{\partial \bar{x}^m}{\partial x^i}\frac{\partial \bar{A}^i_j}{\partial x^k} = \frac{\partial^2 \bar{x}^l}{\partial x^k \partial x^j} \cdot \bar{A}^m_l + \frac{\partial \bar{x}^l}{\partial x^j}\frac{\partial \bar{x}^r}{\partial x^k} \cdot \frac{\partial \bar{A}^m}{\partial \bar{x}^r} \qquad \text{......(13.3.3)}$$

Using value of second order derivative, we get

$$\left[\begin{Bmatrix} p \\ ki \end{Bmatrix}\frac{\partial \bar{x}^m}{\partial x^p} - \begin{Bmatrix} \bar{m} \\ st \end{Bmatrix}\frac{\partial \bar{x}^s}{\partial x^k}\frac{\partial \bar{x}^t}{\partial x^i}\right] A^i_j + \frac{\partial \bar{x}^m}{\partial x^i}\frac{\partial A^i_j}{\partial x^k}$$

$$= \left[\begin{Bmatrix} p \\ kj \end{Bmatrix}\frac{\partial \bar{x}^l}{\partial x^p} - \begin{Bmatrix} \bar{l} \\ st \end{Bmatrix}\frac{\partial \bar{x}^s}{\partial x^k}\frac{\partial \bar{x}^t}{\partial x^j}\right] \bar{A}^m_l + \frac{\partial \bar{x}^l}{\partial x^j}\frac{\partial \bar{x}^r}{\partial x^k}\frac{\partial \bar{A}^m_l}{\partial \bar{x}^r}. \qquad \text{.......(13.3.4)}$$

Now using (13.2.2) and changing appropriate dummy indices

$$\frac{\partial \bar{x}^m}{\partial x^i}\left[\frac{\partial A^i}{\partial x^k} + \begin{Bmatrix} i \\ kn \end{Bmatrix} A^n_j - \begin{Bmatrix} p \\ kj \end{Bmatrix} A^i_p\right] = \frac{\partial \bar{x}^l}{\partial x^j}\frac{\partial \bar{x}^r}{\partial x^k}\left[\frac{\partial \bar{A}^m_l}{\partial \bar{x}^r} + \bar{A}^t_l \begin{Bmatrix} \bar{m} \\ rt \end{Bmatrix} - \bar{A}^m_k \begin{Bmatrix} \bar{t} \\ rl \end{Bmatrix}\right]. \qquad \text{.....(13.3.5)}$$

Introducing the comma notation

$$A^i_{j,k} \equiv \frac{\partial A^i_j}{\partial x^k} + \begin{Bmatrix} i \\ kn \end{Bmatrix} A^n_j - \begin{Bmatrix} p \\ kj \end{Bmatrix} A^i_p, \qquad \text{.....(13.3.6)}$$

the above relation may be written as

$$\frac{\partial \bar{x}^m}{\partial x^i} A^i_{j,k} = \frac{\partial \bar{x}^l}{\partial x^j}\frac{\partial \bar{x}^r}{\partial x^k} \bar{A}^m_{l,r}. \qquad \text{.....(13.3.7)}$$

Taking inner multiplication by $\frac{\partial x^n}{\partial \bar{x}^m}$, we get

$$\delta^n_i A^i_{j,k} = \frac{\partial x^n}{\partial \bar{x}^m}\frac{\partial \bar{x}^l}{\partial x^j} \cdot \frac{\partial \bar{x}^r}{\partial x^k} \bar{A}^m_{l,r}$$

or

$$A^n_{j,k} = \frac{\partial x^n}{\partial \bar{x}^m}\frac{\partial \bar{x}^l}{\partial x^j} \cdot \frac{\partial \bar{x}^r}{\partial x^k} \bar{A}^m_{l,r} \qquad \text{......(13.3.8)}$$

This shows that $A^n_{j,k}$ is a mixed tensor of third order, contravariant of rank one and covariant of rank two. It is called the covariant derivative of A^m_l with respect to x^r.

Note : The covariant derivative $A^i_{j,k}$ defined by (13.3.6) contains three terms :

(i) The partial derivative of A_j^i with respect of x^k.

(ii) A positive sign term similar to that which occurs in the covariant derivative of a contravariant vector.

(iii) A negative sign term similar to that which occurs in the covariant derivative of a covariant vector.

13.4 Ricci's theorem

The covariant derivatives of the tensors g_{ij}, g^{ij} and δ^i_j all vanish identically.

Proof : ***(i)*** Covariant derivative of g_{ij} with respect to x^k

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} - g_{rj}\begin{Bmatrix} r \\ ik \end{Bmatrix} - g_{ir}\begin{Bmatrix} r \\ jk \end{Bmatrix}$$

$$= \frac{\partial g_{ij}}{\partial x^k} - [ik, j] - [jk, i] \quad \text{[Using property (1) of Christoffel symbols]}$$

$$= \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ij}}{\partial x^k} \quad \text{[Using property (2) of Christoffel symbols]}$$

$$= 0. \qquad .....(13.4.1)$$

(ii) Covariant derivative of g^{ij} with respect to x^k

$$g^{ij}_{,k} = \frac{\partial g^{ij}}{\partial x^k} + g^{rj}\begin{Bmatrix} i \\ rk \end{Bmatrix} + g^{ir}\begin{Bmatrix} j \\ rk \end{Bmatrix}$$

$$= \frac{\partial g^{ij}}{\partial x^k} - \frac{\partial g^{ij}}{\partial x^k}$$

$$= 0, \quad \text{[Using property (3) of Christoffel symbols]} \qquad ......(13.4.2)$$

(iii) Covariant derivative of δ^i_j with respect to x^k

$$\delta^i_{j,k} = \frac{\partial \delta^i_j}{\partial x^k} + \delta^l_j\begin{Bmatrix} i \\ lk \end{Bmatrix} - \delta^i_i\begin{Bmatrix} l \\ jk \end{Bmatrix}$$

$$= \frac{\partial \delta^i_j}{\partial x^k} + \begin{Bmatrix} i \\ jk \end{Bmatrix} - \begin{Bmatrix} l \\ jk \end{Bmatrix} \quad \text{[Using property of Kronecker delta]}$$

$$= \frac{\partial \delta^i_j}{\partial x^k}$$

$$= 0 \quad [\because \ \delta^i_j \text{ is a constant either 1 or 0}] \qquad ......(13.4.3)$$

Hence $\quad g_{ij,k} = 0,\ g^{ij}_{,k} = 0,\ \delta^i_{j,k} = 0$

which shows that the tensors g_{ij}, g^{ij} and δ^i_j may be treated as constants in covariant differentiation.

13.5 Illustrative examples

***Ex.*1.** *Prove that*

$$A^j_{i,j} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(A^j_i\sqrt{g}\right) - A^j_k\begin{Bmatrix} k \\ ij \end{Bmatrix}.$$

Show that if associate tensor A^{ij} is symmetric, then

$$A^j_{i,j} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(A^j_i\sqrt{g}\right) - \frac{1}{2}A^{jk}\frac{\partial g_{jk}}{\partial x^i}.$$

Sol. We have from process of covariant differentiation

$$A^j_{i,k} = \frac{\partial A^j_i}{\partial x^k} + A^r_i \begin{Bmatrix} j \\ rk \end{Bmatrix} - A^j_r \begin{Bmatrix} r \\ ik \end{Bmatrix} \qquad ......(1)$$

Putting $k = j$, we get

$$A^j_{i,j} = \frac{\partial A^j_i}{\partial x^j} + A^r_i \begin{Bmatrix} j \\ rj \end{Bmatrix} - A^j_r \begin{Bmatrix} r \\ ij \end{Bmatrix}$$

$$= \frac{\partial A^j_i}{\partial x^j} + A^r_i \frac{\partial}{\partial x^r}\left(\log\sqrt{g}\right) - A^j_r \begin{Bmatrix} r \\ ij \end{Bmatrix}$$

$$= \frac{\partial A^j_i}{\partial x^i} + A^j_i \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}\left(\sqrt{g}\right) - A^j_k \begin{Bmatrix} k \\ ij \end{Bmatrix}$$

[on making suitable changes in dummy indices]

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}\left(A^j_i \sqrt{g}\right) - A^j_k \begin{Bmatrix} k \\ ij \end{Bmatrix} \qquad ......(2)$$

This proves the first result.

Now $$A^j_k \begin{Bmatrix} k \\ ij \end{Bmatrix} = A^j_k g^{lk} [ij, l]$$

$$= A^{jl} [ij, l]$$

$$= \frac{1}{2} A^{jl} \left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

$$= \frac{1}{2} A^{jl} \frac{\partial g_{il}}{\partial x^j} + \frac{1}{2} \frac{\partial g_{jl}}{\partial x^i} - \frac{1}{2} A^{jl} \frac{\partial g_{ij}}{\partial x^l}$$

$$= \frac{1}{2} A^{lj} \frac{\partial g_{ij}}{\partial x^l} + \frac{1}{2} A^{jl} \frac{\partial g_{jl}}{\partial x^i} - \frac{1}{2} A^{jl} \frac{\partial g_{ij}}{\partial x^l}, \qquad ......(3)$$

where in the first term the dummy indices suitably have been changed. Since A^{ij} is symmetric the first and the last term will cancel out and therefore

$$A^j_k \begin{Bmatrix} k \\ ij \end{Bmatrix} = \frac{1}{2} A^{jk} \frac{\partial g_{jk}}{\partial x^i}. \qquad ......(4)$$

Substituting this result in (1), we get the required result as

$$A^j_{i,j} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}\left(A^j_i \sqrt{g}\right) - \frac{1}{2} A^{jk} \frac{\partial g_{ik}}{\partial x^i}.$$

Ex.2. *Prove that*

$$A^{ij}_{,j} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}\left(\sqrt{g}\, A^{ij}\right) + A^{jk} \begin{Bmatrix} i \\ jk \end{Bmatrix}$$

Show that the last term vanishes if A^{ij} is skew-symmetric.

Sol. We have from covariant differentiation

$$A^{ij}_{,l} = \frac{\partial A^{ij}}{\partial x^l} + A^{ir}\begin{Bmatrix} j \\ rl \end{Bmatrix} + A^{rj}\begin{Bmatrix} i \\ rl \end{Bmatrix}. \qquad \text{......(1)}$$

Putting $l = j$, we get

$$A^{ij}_{,j} = \frac{\partial A^{ij}}{\partial x^j} + A^{ir}\begin{Bmatrix} j \\ rj \end{Bmatrix} + A^{rj}\begin{Bmatrix} i \\ rj \end{Bmatrix}$$

$$= \frac{\partial A^{ij}}{\partial x^j} + A^{ir}\frac{1}{2g}\frac{\partial g}{\partial x^r} + A^{rj}\begin{Bmatrix} i \\ rj \end{Bmatrix}$$

[using property (4) of Christoffel symbols]

$$= \frac{\partial A^{ij}}{\partial x^j} + A^{ir}\frac{1}{\sqrt{g}}\frac{\partial \sqrt{g}}{\partial x^r} + A^{kj}\begin{Bmatrix} i \\ kj \end{Bmatrix}$$

$$= \frac{1}{\sqrt{g}}\left[\sqrt{g}\,\frac{\partial A^{ij}}{\partial x^j} + A^{ij}\frac{\partial \sqrt{g}}{\partial x^j}\right] + A^{kj}\begin{Bmatrix} i \\ kj \end{Bmatrix}$$

[making suitable changes in dummy indices]

$$= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}A^{ij}\right) + A^{jk}\begin{Bmatrix} i \\ jk \end{Bmatrix}. \qquad \text{.......(2)}$$

This proves the first result.

If A^{ij} is skew-symmetric, then

$$A^{kj}\begin{Bmatrix} i \\ kj \end{Bmatrix} = -A^{jk}\begin{Bmatrix} i \\ kj \end{Bmatrix}$$

$$= -A^{kj}\begin{Bmatrix} i \\ jk \end{Bmatrix} \qquad \text{[interchanging the dummy indices]}$$

$$= -A^{kj}\begin{Bmatrix} i \\ kj \end{Bmatrix} \qquad \text{[using symmetric prop. of Christoffel symbols]}$$

$\therefore$
$$2A^{kj}\begin{Bmatrix} i \\ kj \end{Bmatrix} = 0$$

$\Rightarrow$
$$A^{kj}\begin{Bmatrix} i \\ kj \end{Bmatrix} = 0.$$

Hence, if A^{ij} is skew-symmetric, then

$$A^{ij}_{,j} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}\,A^{ij}\right).$$

Hence Proved.

***Ex.*3.** *If A^{ijk} is a skew-symmetric tensor, show that*

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}\,A^{ijk}\right),$$

is a tensor.

Sol. Since A^{ijk} is a skew-symmetric tensor, we have

$$A^{ijk} = -A^{jik},\ A^{ijk} = -A^{ikj},\ A^{ijk} = -A^{kji}. \qquad \text{......(1)}$$

We know that

$$A^{ijk}_{,l} = \frac{\partial A^{ijk}}{\partial x^l} + A^{rjk}\begin{Bmatrix} i \\ rl \end{Bmatrix} + A^{irk}\begin{Bmatrix} j \\ rl \end{Bmatrix} + A^{ijr}\begin{Bmatrix} k \\ rl \end{Bmatrix} \qquad \text{......(2)}$$

Contracting over i and l ($l = i$), we get

$$A^{ijk}_{,i} = \frac{\partial A^{ijk}}{\partial x^i} + A^{rjk}\begin{Bmatrix} i \\ ri \end{Bmatrix} + A^{irk}\begin{Bmatrix} j \\ ri \end{Bmatrix} + A^{ijr}\begin{Bmatrix} k \\ ri \end{Bmatrix} \qquad \text{.....(3)}$$

But $$A^{irk}\begin{Bmatrix} j \\ ri \end{Bmatrix} = -A^{rik}\begin{Bmatrix} j \\ ri \end{Bmatrix} = -A^{irk}\begin{Bmatrix} j \\ ir \end{Bmatrix},$$

similarly, $$2A^{ijr}\begin{Bmatrix} k \\ ri \end{Bmatrix} = 0. \qquad \text{......(4)}$$

Hence, $$A^{ijk}_{,i} = \frac{\partial A^{ijk}}{\partial x^i} + A^{rjk}\begin{Bmatrix} i \\ ri \end{Bmatrix}$$

$$= \frac{\partial A^{ijk}}{\partial x^i} + A^{rjk}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^r}\left(\sqrt{g}\right) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(A^{ijk}\sqrt{g}\right). \qquad \text{.....(5)}$$

Since left hand side, which is a covariant derivative of a tensor is a tensor, the right hand side will also be a tensor.

13.6 Divergence of a vector

(i) Divergence of a contravariant vector A^i is defined as the contraction of its covariant derivative. It is denoted by *div* A^i and is an invariant. Thus

$$div\ A^i = A^i_{\ ,i}. \qquad \text{.....(13.6.1)}$$

(ii) The divergence of a covariant vector A_i is denoted by *div* A_i and is defined as

$$div\ A_i = g^{jk}A_{j,k}. \qquad \text{.....(13.6.2)}$$

It is also an invariant.

Note : The concept of divergence may be extended to the contravariant tensors of higher order or to mixed tensors. The divergence of a tensor may be obtained first by taking a covariant derivative of it and contracting over a superscript and the subscript of covariant derivative.

Thus, $$div\left(A^i_j\right) = A^i_{j,i}. \qquad \text{.....(13.6.3)}$$

The order is reduced by one in taking its divergence.

Theorem 1. *If A^i is a contravariant vector, then*

$$div\, A^i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(A^i \sqrt{g}\right).$$

Proof. The covariant derivative of A^i is given by

$$A^i_{,j} = \frac{\partial A^i}{\partial x^j} + A^r \begin{Bmatrix} i \\ rj \end{Bmatrix}. \qquad \text{......(13.6.4)}$$

Contracting over i and j ($j = i$), we get

$$\begin{aligned} div\, A^i &= A^i_{,i} \\ &= \frac{\partial A^i}{\partial x^i} + A^r \begin{Bmatrix} i \\ ri \end{Bmatrix} \\ &= \frac{\partial A^i}{\partial x^i} + A^r \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r}\left(\sqrt{g}\right) && \text{[changing dummy index } r \text{ to } i\text{]} \\ &= \frac{\partial A^i}{\partial x^i} + A^i \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\right) \\ &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(A^i \sqrt{g}\right). && \text{......(13.6.5)} \end{aligned}$$

Theorem 2. *To prove that*

$$div\, A_i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r}\left\{\sqrt{g}\, g^{rk} A_k\right\} = div\, A^i$$

where A^i and A_i are the contravariant and covariant components of the same vector A.

Proof. We have by definition

$$\begin{aligned} div\, A_i &= g^{jk} A_{j,k} \\ &= g^{jk}\left[\frac{\partial A_j}{\partial x^k} - A_r \begin{Bmatrix} r \\ jk \end{Bmatrix}\right] \\ &= g^{jk} \frac{\partial A_j}{\partial x^k} - A_r\left[-g^{rk}\begin{Bmatrix} j \\ kj \end{Bmatrix} - \frac{\partial g^{jr}}{\partial x^j}\right] && \text{[using property (3) of Christoffel symbols]} \\ &= g^{jk} \frac{\partial A_j}{\partial x^k} + g^{rk} A_r \begin{Bmatrix} j \\ jk \end{Bmatrix} + A_r \frac{\partial g^{jr}}{\partial x^j} \\ &= g^{rk} \frac{\partial A_r}{\partial x^k} + g^{rk} A_r \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k}\left(\sqrt{g}\right) + A_r \frac{\partial g^{jr}}{\partial x^j} \\ &= g^{kr} \frac{\partial A_k}{\partial x^r} + g^{kr} A_k \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r}\left(\sqrt{g}\right) + A_k \frac{\partial g^{rk}}{\partial x^r} && \text{[on changing the dummy indices]} \\ &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r}\left(\sqrt{g}\, g^{kr} A_k\right). && \text{......(13.6.6)} \end{aligned}$$

But $g^{rk} A_k = A^r$ (associate vector) ......(13.6.7)

Therefore above relation may be written as

$$div\ A_i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r}\left(\sqrt{g}\ A^r\right)$$

$$= div\ A^r$$

$$= div\ A^i.$$

Hence proved.

Remark : According to Ricci's theorem g^{ij} be have like a constant in covariant differentiation, we may write

$$div\ A_i = g^{jk} A_{j,k} = \left(g^{jk} A_j\right)_{,k}$$

$$= (A^k)_{,k} = div\ A^k$$

$$= div\ A^i.$$

13.7 Gradient of a scalar

If I is a scalar function of coordinates x^i, the gradient of I is defined by

$$grad\ I = I_{,i} = \frac{\partial I}{\partial x^i}, \qquad \text{......(13.7.1)}$$

where $I_{,i}$ is a covariant vector.

Theorem 3. *The covariant differentiation of invariants is commutative, that is*

$$\left(I_{,i}\right)_{,j} = \left(I_{,j}\right)_{,i}.$$

Proof. We have $I_{,i} = \dfrac{\partial I}{\partial x^i}$. .....(13.7.2)

Therefore,

$$\left(I_{,i}\right)_{,j} = \frac{\partial}{\partial x^j}\left(\frac{\partial I}{\partial x^i}\right) - I_{,r}\begin{Bmatrix} r \\ ij \end{Bmatrix}$$

$$= \frac{\partial^2 I}{\partial x^j \partial x^i} - \frac{\partial I}{\partial x^r}\begin{Bmatrix} r \\ ij \end{Bmatrix}$$

$$= \frac{\partial^2 I}{\partial x^i \partial x^j} - \frac{\partial I}{\partial x^r}\begin{Bmatrix} r \\ ji \end{Bmatrix}$$

$$= (I_{,j})_{,i}. \qquad \text{......(13.7.3)}$$

13.8 Laplacian of a scalar

If I is a scalar functions of coordinate x^i, then the divergence of *grad I* is defined as the Laplacian of I and it is denoted by $\nabla^2 I$.

Thus $\nabla^2 I = div\ grad\ \mathrm{I} = div\ I_{,i}$. ......(13.8.1)

Theorem 4. *Prove that*

(i) *div grad* $I = \nabla^2 I = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r}\left\{\sqrt{g}\, g^{rk} I_{,k}\right\}$,

(ii) *div grad* $I = \nabla^2 I = g^{jk}\left(\frac{\partial^2 I}{\partial x^j \partial x^k} - \frac{\partial I}{\partial x^r}\begin{Bmatrix} r \\ jk \end{Bmatrix}\right)$.

Proof. ***(i)*** From theorem 2 we have

$$div\ A_i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r}\left(\sqrt{g}\, g^{rk} A_k\right), \qquad .....(13.8.2)$$

where A_i is a covariant vector.

Since gradient of a scalar I is a covariant vector, setting $A_i = I_{,i}$ in the above equation, we get

$$div\ A_{,i} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r}\left(\sqrt{g}\, g^{rk} A_{,k}\right)$$

$$\Rightarrow \qquad \nabla^2 I = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r}\left(\sqrt{g}\, g^{rk} I_{,k}\right). \qquad .....(13.8.3)$$

(ii)

$$\nabla^2 I = div\ I_{,i} = g^{jk}\,(I_{,j})_{,k}$$

$$= g^{jk}\left(\frac{\partial I_{,i}}{\partial x^k} - I_{,r}\begin{Bmatrix} r \\ jk \end{Bmatrix}\right)$$

$$= g^{jk}\left(\frac{\partial^2 I_{,i}}{\partial x^j \partial x^k} - \frac{\partial I}{\partial x^r}\begin{Bmatrix} r \\ jk \end{Bmatrix}\right). \qquad ......(13.8.4)$$

13.9 Curl of a covariant vector

From a covariant vector A_i in V_3, we can form the contravariant vector

$$B^k = \in^{jik} A_{i,j} \qquad ......(13.9.1)$$

and call B^k the *curl* of vector A_i and written as *curl* A_i.

Thus $\qquad curl\ A_i = B^k = \in^{jik} A_{i,j} \qquad .....(13.9.2)$

We may also write on interchanging the dummy indices i and j as

$$B^k = \in^{ijk} A_{j,i} = - \in^{jik} A_{i,j} \qquad .....(13.9.3)$$

Hence $\qquad 2B^k = \in^{jik} (A_{i,j} - A_{j,i})$

$$\Rightarrow \qquad B^k = \frac{1}{2} \in^{jik} \left(A_{i,j} - A_{j,i}\right) \qquad .....(13.9.4)$$

Thus *curl* A_i may also be defined as

$$curl\ A_i = B^k = \frac{1}{2} e^{jik} \left(A_{i,j} - A_{j,i}\right). \qquad .....(13.9.5)$$

Ex.4. *If* A_{ij} *is the curl of a covariant vector, prove that*

$$A_{ij,k} + A_{jk,i} + A_{ki,j} = 0.$$

Show further that this expression is equivalent to

$$\frac{\partial A_{ij}}{\partial x^k} + \frac{\partial A_{ik}}{\partial x^k} + \frac{\partial A_{ki}}{\partial x^j} = 0.$$

If $\quad A_{ij} = B_{i,j} - B_{j,i},$

prove that $\quad A_{ij,k} + A_{jk,i} + A_{ki,j} = 0.$

Sol. Let B_i is a covariant vector and let its curl be A_{ij}.

Thus $\quad curl\, B_i = A_{ij}$

$$\Rightarrow \quad B_{i,j} - B_{j,i} = A_{ij}. \qquad \text{.....(1)}$$

we have
$$B_{i,j} = \frac{\partial B_i}{\partial x^j} - B_p \begin{Bmatrix} p \\ ij \end{Bmatrix}. \qquad \text{......(2)}$$

Inter changing i and j, we get

$$B_{j,i} = \frac{\partial B_j}{\partial x^i} - B_p \begin{Bmatrix} p \\ ji \end{Bmatrix} = \frac{\partial B_j}{\partial x^i} - B_p \begin{Bmatrix} p \\ ij \end{Bmatrix} \qquad \text{.....(3)}$$

$$\therefore \quad A_{ij} = B_{i,j} - B_{j,i} = \frac{\partial B_i}{\partial x^j} - \frac{\partial B_j}{\partial x^i} \qquad \text{.....(4)}$$

Now, we have

$$A_{ij,k} = \frac{\partial A_{ij}}{\partial x^k} - A_{pj} \begin{Bmatrix} p \\ ik \end{Bmatrix} - A_{ip} \begin{Bmatrix} p \\ jk \end{Bmatrix} \qquad \text{.....(5)}$$

$$= \frac{\partial^2 B_i}{\partial x^j \partial x^k} - \frac{\partial^2 B_j}{\partial x^i \partial x^k} - \left(\frac{\partial B_p}{\partial x^j} - \frac{\partial B_j}{\partial x^p} \right) \begin{Bmatrix} p \\ ik \end{Bmatrix} - \left(\frac{\partial B_i}{\partial x^p} - \frac{\partial B_p}{\partial x^i} \right) \begin{Bmatrix} p \\ jk \end{Bmatrix} \qquad \text{......(6)}$$

Similarly
$$A_{jk,i} = \frac{\partial A_{ik}}{\partial x^i} - A_{pk} \begin{Bmatrix} p \\ ji \end{Bmatrix} - A_{ip} \begin{Bmatrix} p \\ ki \end{Bmatrix} \qquad \text{.....(7)}$$

or
$$A_{jk,i} = \frac{\partial^2 B_j}{\partial x^k \partial x^i} - \frac{\partial^2 B_k}{\partial x^i \partial x^j} - \left(\frac{\partial B_p}{\partial x^k} - \frac{\partial B_k}{\partial x^p} \right) \begin{Bmatrix} p \\ ji \end{Bmatrix} - \left(\frac{\partial B_j}{\partial x^p} - \frac{\partial B_p}{\partial x^j} \right) \begin{Bmatrix} p \\ ki \end{Bmatrix} \qquad \text{.....(8)}$$

Similarly,
$$A_{ki,j} = \frac{\partial A_{ij}}{\partial x^j} - A_{pi} \begin{Bmatrix} p \\ kj \end{Bmatrix} - A_{ip} \begin{Bmatrix} p \\ ij \end{Bmatrix} \qquad \text{.....(9)}$$

$$A_{ki,j} = \frac{\partial^2 B_k}{\partial x^i \partial x^j} - \frac{\partial^2 B_i}{\partial x^j \partial x^k} - \left(\frac{\partial B_p}{\partial x^i} - \frac{\partial B_i}{\partial x^p} \right) \begin{Bmatrix} p \\ kj \end{Bmatrix} - \left(\frac{\partial B_k}{\partial x^p} - \frac{\partial B_p}{\partial x^k} \right) \begin{Bmatrix} p \\ ij \end{Bmatrix}. \qquad \text{......(10)}$$

Adding (5), (7) and (9), we get

$$A_{jk,i} + A_{ki,j} + A_{ij,k} = 0.$$

Proved I part.

Again, we are given $A_{ij} = B_{i,j} - B_{j,i}$

$\therefore$ $A_{ji} = B_{j,i} - B_{i,j}$

$= -(B_{i,j} - B_{j,i})$

$= -A_{ij}$

$$\Rightarrow \quad A_{ij} + A_{ji} = 0. \qquad .....(11)$$

Equation (11) shows that A_{ij} is antisymmetric.

Adding (5), (7) and (9) and using (11), we get

$$A_{ij,k} + A_{jk,i} + A_{ki,j} = \frac{\partial A_{ij}}{\partial x^k} + \frac{\partial A_{jk}}{\partial x^i} + \frac{\partial A_{ki}}{\partial x^j}. \qquad .....(12)$$

Hence from (12) if follows that

$$A_{ij,k} + A_{jk,i} + A_{ki,j} = 0$$

is equivalent to $\dfrac{\partial A_{ij}}{\partial x^k} + \dfrac{\partial A_{jk}}{\partial x^i} + \dfrac{\partial A_{ki}}{\partial x^j} = 0.$

Proved II part.

Ex.5. *Evaluate div A^j in (i) cylindrical polar coordinates, and (ii) spherical polar coordinates.*

Sol. ***(i)*** For cylindrical polar coordinates

$$x^1 = r, \quad x^2 = \theta, \quad x^3 = z$$

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = 1, \quad g_{ij} = 0, \, i \neq j$$

$$g = |g_{ij}| = r^2 \qquad ......(1)$$

The physical components in cylindrical polar coordinates of A^r, A^θ and A^z.

Therefore $\quad (A)^2 = (A^r)^2 + (A^\theta)^2 + (A^z)^2 \qquad .....(2)$

Also $\quad (A)^2 = g_{11}\,(A^1)^2 + g_{22}\,(A^2)^2 + g_{33}\,(A^3)^2 \qquad .....(3)$

Hence $\quad A^r = \sqrt{g_{11}}\,A^1 = A^1, \; A^\theta = \sqrt{g_{22}}\,A^2 = rA^2, \; A^z = \sqrt{g_{33}}\,A^3 = A^3 \qquad .....(4)$

Now, by definition $\quad div\,A^j = \dfrac{1}{\sqrt{g}}\dfrac{\partial}{\partial x^i}\left(\sqrt{g}\,A^i\right)$

$$= \frac{1}{r}\frac{\partial}{\partial x^i}\left(rA^i\right)$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial x^i}\left(rA^1\right) + \frac{\partial}{\partial x^2}\left(rA^2\right) + \frac{\partial}{\partial x^3}\left(rA^3\right)\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA^r\right) + \frac{\partial}{\partial \theta}\left(A^\theta\right) + \frac{\partial}{\partial z}\left(rA^z\right)\right]$$

$$= \frac{\partial A^r}{\partial r} + \frac{1}{r}\frac{\partial A^\theta}{\partial \theta} + \frac{\partial A^z}{\partial z} + \frac{A^r}{r}. \qquad .....(5)$$

(ii) For spherical polar coordinates

$$x' = r, \quad x^2 = \theta, \quad x^3 = \phi$$

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2\theta, \quad g_{ij} = 0, \quad i \neq j$$

$$g = |g_{ij}| = r^4 \sin^2\theta. \qquad .....(6)$$

The physical components are denoted by A^r, A^θ, A^ϕ and are given by

$$A^r = \sqrt{g_{11}}\, A^1 = A^1, \quad A^\theta = rA^2, \quad A^\phi = r\sin\theta\, A^3 \qquad .....(7)$$

By definition
$$div\, A^j = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, A^i\right)$$

$$= \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial x^i}\left(r^2 \sin\theta\, A^i\right)$$

$$= \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial x^i}\left(r^2 \sin\theta\, A^1\right) + \frac{\partial}{\partial x^2}\left(r^2 \sin\theta\, A^2\right) + \frac{\partial}{\partial x^3}\left(r^2 \sin\theta\, A^3\right)\right]$$

$$= \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial x}\left(r^2 \sin\theta\, A^r\right) + \frac{\partial}{\partial\theta}\left(r \sin\theta\, A^\theta\right) + \frac{\partial}{\partial\phi}\left(rA^\phi\right)\right]$$

$$= \frac{\partial A^r}{\partial r} + \frac{1}{r}\frac{\partial A^\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A^\phi}{\partial\phi} + \frac{2A^r}{r} + \frac{\cot\theta}{r} A^\theta. \qquad .....(8)$$

13.11 Intrinsic derivative (Absolute derivative)

The intrinsic derivative or absolute derivative of a covariant vector A_i along a curve $x^j = x^j(t)$ is defined as the inner product of the covariant derivative of A_i and $\frac{dx^j}{dt}$ *i.e.* $A_{i,j}\frac{dx^j}{dt}$ and is denoted by $\frac{\delta A_i}{\delta t}$.

Thus
$$\frac{\delta A_i}{\delta t} = A_{i,j}\frac{dx^j}{dt} = \frac{dA_i}{dt} = \frac{dA_i}{dt} - \begin{Bmatrix} r \\ ij \end{Bmatrix} A_r \frac{dx^j}{dt}. \qquad .....(13.11.1)$$

Similarly the intrinsic derivative of contravariant vector A^i.

$$\frac{\delta A^i}{\delta t} = A^i_{,j}\frac{dx^j}{dt} = \frac{dA^i}{dt} + \begin{Bmatrix} i \\ rj \end{Bmatrix} A^r \frac{dx^j}{dt}. \qquad .....(13.11.2)$$

The vectors A_i or A^i are said to move parallelly along a curve if their intrinsic derivatives along that curve are zero, respectively.

Similarly we can define the intrinsic derivative of higher order tensor $A^{i_1, i_2 .....i_m}_{j_1, j_2 ....j_n}$ along a curve $x^k = x^k(t)$ defined by

$$\frac{\delta A^{i_1, i_2 .....i_m}_{j_1, j_2 ....j_n}}{\delta t} = A^{i_1, i_2 .....i_m}_{j_1, j_2 ....j_{n,k}} \cdot \frac{dx^k}{dt}, \qquad .....(13.11.3)$$

where summation is taken over the index k.

Thus the intrinsic derivative is a tensor of the same order and type as the original tensor.

The intrinsic derivative of an invariant I is defined as

$$\frac{\delta I}{\delta t} = I_{,i}\frac{dx^i}{dt} = \frac{\partial I}{\partial x^i}\frac{dx^i}{dt} = \frac{dI}{dt}, \qquad \text{.....(13.11.4)}$$

which is same as its total derivative.

The intrinsic derivatives of higher order can easily be defined as

$$\frac{\delta^2}{\delta t^2}\left(A^i_j\right) = \frac{\delta}{\delta t}\left(\frac{\delta A^i_j}{\delta t}\right) = \frac{\delta}{\delta t}\left(A^i_{j,k}\frac{dx^k}{dt}\right)$$

$$= \left(A^i_{j,k}\frac{dx^k}{dt}\right)_{,p}\frac{dx^p}{dt}. \qquad \text{......(13.11.5)}$$

Theorem 8. *The intrinsic derivatives of* g_{ij}, g^{ij} *and* δ^i_j *are zero.*

Proof.
$$\frac{\delta}{\delta t}\left(g_{ij}\right) = \left(g_{ij}\right)_{,k}\frac{dx^k}{dt} = 0 \qquad \text{[using Ricci theorem]}$$

$$\frac{\delta}{\delta t}\left(g^{ij}\right) = \left(g^{ij}\right)_{,k}\frac{dx^k}{dt} = 0$$

$$\frac{\delta}{\delta t}\left(\delta^i_j\right) = \left(\delta^i_j\right)_{,k}\frac{dx^k}{dt} = 0.$$

13.12 Illustrative examples

***Ex.*6.** *Show that*

$$\frac{\delta}{\delta t}\left(\frac{dx^i}{dt}\right) = \frac{d^2x^i}{dt^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^j}{dt}\frac{dx^k}{dt}.$$

Sol. Let $\frac{dx^i}{dt} = A^i$ (contravariant vector)

Now
$$\frac{\delta}{\delta t}\left(\frac{dx^i}{dt}\right) = \frac{\delta}{\delta t}\left(A^i\right) = A^i_{,j}\frac{dx^j}{dt}$$

$$= \left[\frac{\partial A^i}{\partial x^j} + \begin{Bmatrix} i \\ rj \end{Bmatrix} A^r\right]\frac{dx^j}{dt}$$

$$= \frac{dA^i}{dt} + \begin{Bmatrix} i \\ rj \end{Bmatrix} A^r\frac{dx^j}{dt}$$

$$= \frac{d}{dt}\left(\frac{dx^i}{dt}\right) + \begin{Bmatrix} i \\ rj \end{Bmatrix}\frac{dx^r}{dt}\frac{dx^j}{dt}$$

$$= \frac{d^2x^i}{dt^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^j}{dt}\frac{dx^k}{dt}.$$

Ex.7. *If two unit vectors A^i and B^i are defined along a curve C such that their intrinsic derivatives along C are zer*o, show that the angle between them is constant.

Sol. It is given, that $$A^i_{,j}\frac{dx^j}{ds}=0,\ B^i_{,j}\frac{dx^j}{ds}=0, \qquad .....(1)$$

at every point of C.

Therefore $$A_{i,j}\frac{dx^j}{ds}=g_{ik}\ A^k_{,j}\frac{dx^j}{ds}=0, \qquad .....(2)$$

Also $$B_{i,j}\frac{dx^i}{ds}=g_{ik}\ B^k_{,j}\frac{dx^j}{ds}=0, \qquad .....(3)$$

at every point of C.

Let θ be the angle between unit vectors A^i and B^i, then

$$\cos\theta = A_i B^i$$

$$\frac{d}{ds}(\cos\theta) = \left(A_i B^i\right)_{,j}\frac{dx^j}{ds}$$

$$\Rightarrow \quad -\sin\theta\frac{d\theta}{ds} = \left(A_i B^i_{,j}+A_{i,j}B^i\right)\frac{dx^j}{ds} \qquad \text{[using (1) \& (2)]}$$

$$\sin\theta\frac{d\theta}{ds} = 0. \qquad .....(4)$$

From (4) it follows that either θ = 0 or θ = constant.

But 0 being included in constant, we conclude

$$\theta = \text{constant}.$$

Hence Proved.

Ex.8. *If the intrinsic derivative of a vector A^i along a curve C vanishes at every point of the curve, then show that the magnitude of the vector A^i is constant along the curve.*

Sol. Let the equation of the curve C be

$$x^i = x^i\,(s). \qquad .....(1)$$

It is given that $$A^i_{,j}\frac{dx^j}{ds}=0, \text{ at every point of } C. \qquad .....(2)$$

We know that $$A_i = g_{ik}A^k \quad \text{and} \quad (g_{ik})_{,j}=0. \qquad .....(3)$$

Therefore, $$A_{i,j}\frac{dx^j}{ds} = g_{ik}A^k_{,j}\frac{dx^j}{ds}=0 \text{ at every point of } C. \qquad .....(4)$$

Since, $$A^2 = A_i A^i$$

$$\frac{dA^2}{ds} = \frac{\delta}{\delta s}\left(A^2\right)=\left(A_i A^i\right)_{,j}\frac{dx^j}{ds}$$

$$=\left(A_{i,j}A^i + A_i A^i_{,j}\right)\frac{dx^j}{ds}$$

$$= \left(A_{i,j} \frac{dx^j}{ds} \right) A^i + A_i \left(A^i_{,j} \frac{dx^j}{ds} \right)$$

$$= 0. \qquad \text{[using (1) and (2)]} \qquad .....(5)$$

Hence A^2 = constant.

i.e. magnitude of vector A^i is constant.

Hence proved.

13.12 Self-learning exercises

1. What do you understand by the covariant derivative of a covariant vector ?
2. Define covariant derivative of a contravariant vector.
3. Show that the covariant derivative of a covariant tensor of second order is a covariant tensor of third order.
4. Define intrinsic derivatives of a tensor.
5. Show that $\dfrac{\partial A_r}{\partial x^s}$ is not a tensor even though A_p is a covariant tensor of rank one.

13.13 Summary

In this unit we have studied the partial differentiation of tensors. We have defined a particular process of it, called covariant differentiation. The properties of covariant differentiation like Ricci's theorem have also been studied. The use of covariant differentiation to define gradient, divergence and *curl* have also been discussed.

13.14 Answers to self-learning exercises

1. § 13.2*(ii)* **2.** § 13.2*(i)*

3. § 13.3 **4.** § 13.11

5. § 13.2*(i)*

13.15 Exercise

1. State and prove Ricci's theorem on fundamental tensors.
2. Prove that the covariant derivative of the tensor a^{ik} with respect to x^k, that is $a^{ik}_{,k}$ has the expression

$$a^{ik}_{,k} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} \left(\sqrt{g}\, a^{ik} \right) + a^{im} \begin{Bmatrix} i \\ km \end{Bmatrix}.$$

□ □ □

Unit 14 : Geodesics, Differential Equation of Geodesic, Geodesic Coordinates, Field of Parallel Vectors

Structure of the Unit

14.0 Objectives

The geodesic, a curve of stationary length on the surface, Riemannian coordinates, geodesic co-ordinates are the points of study of this unit. Parallelism of vectors and fundamental theorem on Rieman-nian geometry have also been given.

14.1 Introduction

In the calculus, we study the process of finding stationary values of a function. While in calculus of variation we find a path on which an integral has stationary value. This gives a process to find shortest path joining any two points on a surface, which we call the geodesic curve.

14.2 Geodesic

"A geodesic, in a Riemannian space V_N, is a curve whose length has stationary value with re-spect to arbitrary small variations of the curve, the end points being held fixed."

Geodesic on a surface in Euclidean three dimensional space may be defined as the curve along which the shortest distance measured on the surface between any two points in its plane.

The differential equations of a geodesic can be obtained with the help of Euler's equations, which are derived by the technique of calculus of variations.

14.3 Euler's condition

Theorem. The integral $\int_{t_0}^{t_1} f\left(x^i, \dot{x}^i\right) dt$ has stationary value on the curve whose differential equations are

$$\frac{\partial f}{\partial x^i} = -\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}^i}\right) = 0,$$

where $\dot{x}^i = \dfrac{dx^i}{dt}$.

Proof. Let $x^i = x^i\ (t)$ be a parametric equation of a curve C in V_N joining two fixed points $A\ (t = t_0)$ and $B\ (t = t_1)$ on it. Let the integral $I = \int_{t_0}^{t_1} f\left(x^i, \dot{x}^i\right) dt$ has stationary value on the curve C and C' be a neighbouring curve whose equation is given by

$$x^i = x^i\ (t) + \in \eta^i\ (t),$$

where $\in$ is small and $\eta^i\ (t)$ are arbitrary continuous differentiable functions of t, satisfying $\eta^i\ (t_0) = 0$, $\eta^i\ (t_1) = 0$ to ensure that the curve passes through A and B. The value of I taken along the curve C' is thus a function of $\in$ of the form

$$I(\in) = \int_{t_0}^{t_1} f\left(x^i + \in \eta^i,\ \dot{x}^i + \in \dot{\eta}^i\right) dt \qquad .....(14.3.1)$$

Since the integral I is stationary on C for which $\in = 0$, we have the condition $I'(0) = 0$.

Differentiating (14.3.1) with respect to $\in$

$$I'(\in) = \frac{\partial}{\partial \in}\int_{t_0}^{t_1} f\left(x^i + \in \eta^i,\ \dot{x}^i + \in \dot{\eta}^i\right) dt$$

$$= \int_{t_0}^{t_1} \frac{\partial}{\partial \in} f\left(x^i + \in \eta^i,\ \dot{x}^i + \in \dot{\eta}^i\right) dt$$

$$= \int_{t_0}^{t_1} \left\{\eta^i \frac{\partial}{\partial x^i} f\left(x^i + \in \eta^i,\ \dot{x}^i + \in \dot{\eta}^i\right) + \dot{\eta}^i f\left(x^i + \in \eta^i,\ \dot{x}^i + \in \dot{\eta}^i\right)\right\} dt$$

The stationary requirement $I'(0) = 0$, now gives

$$\int_{t_0}^{t_1} \left\{\eta^i \frac{\partial}{\partial x^i} f\left(x^i,\ \dot{x}^i\right) + \dot{\eta}^i \frac{\partial}{\partial \dot{x}^i} f\left(x^i,\ \dot{x}^i\right)\right\} dt = 0. \qquad .....(14.3.2)$$

The second term becomes as

$$\int_{t_0}^{t_1} \frac{\partial}{\partial \dot{x}_i}\dot{\eta}_i\ dt = \left[\frac{\partial f}{\partial \dot{x}^i}\eta^i\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}^i}\right)\eta^i\ dt$$

$$= -\int_{t_0}^{t_1} \eta^i \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}^i}\right) dt. \qquad .....(14.3.3)$$

Since $\eta^i(t_0) = 0 = \eta^i(t_1)$, thus equation (14.3.2) becomes

$$\int_{t_0}^{t_1} \left\{\frac{\partial f}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}^i}\right)\right\} \eta^i \, dt = 0. \qquad .....(14.3.4)$$

Since η^i is arbitrary, subject to its being differentiable and vanishing at A and B, equation (14.3.4) implies that

$$\frac{\partial f}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}^i}\right) = 0. \qquad .....(14.3.5)$$

Hence (14.3.5) are necessary conditions for the integral I to be stationary. These are called Euler's conditions or Euler's equations.

14.4 Differential equation of geodesics in a V_N

Using the property that geodesic curve is a path of stationary length joining two points A and B in it, we shall now find the differential equations of it in the space V_N.

In the Riemannian space V_N, we have

$$ds^2 = e g_{ij}\, dx^i\, dx^j$$

$$\Rightarrow \qquad \left(\frac{ds}{dt}\right)^2 = e g_{ij} \frac{dx^i}{dt} \cdot \frac{dx^j}{dt}$$

$$\Rightarrow \qquad \dot{s} = \left(e g_{ij}\, \dot{x}^i \dot{x}^j\right)^{1/2} = f\left(x^i, \dot{x}^i\right), \text{ (say)} \qquad .....(14.4.1)$$

Now, $\qquad s = \int_{t_0}^{t_1} \frac{ds}{dt}\, dt = \int_{t_0}^{t_1} \left(e g_{ij}\, \dot{x}^i \dot{x}^j\right)^{1/2} dt$

or $\qquad = \int_{t_0}^{t_1} f\left(x^i, \dot{x}^i\right) dt. \qquad .....(14.4.2)$

In order that s is stationary, the function f must satisfy the Euler's equations viz.,

$$\frac{\partial f}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}^i}\right) = 0. \qquad .....(14.4.3)$$

Now we have $\qquad f = \left(e g_{ij}\, \dot{x}^i \dot{x}^j\right)^{1/2} = \dot{s} \qquad .....(14.4.4)$

and g_{ij} being a function of x^i, therefore

$$\frac{\partial f}{\partial x^l} = \frac{1}{2\dot{s}} e \frac{\partial g_{ij}}{\partial x^l} \dot{x}^i \dot{x}^j. \qquad .....(14.4.5)$$

Also, $\qquad \frac{\partial f}{\partial \dot{x}^l} = \frac{1}{2\dot{s}} e g_{ij} \left(\dot{x}^i \delta_l^j + \dot{x}^j \delta_l^i\right)$

$$= \frac{e}{2\dot{s}} \left(\dot{x}^i g_{il} + \dot{x}^j g_{lj}\right)$$

$$= \frac{e}{2\dot{s}} \left(\dot{x}^i \, g_{il} + \dot{x}^i \, g_{il} \right)$$

$$= \frac{e}{\dot{s}} \, \dot{x}^i \, g_{il} \, . \qquad \text{.....(14.4.6)}$$

Now $$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}^l}\right) = \frac{e}{\dot{s}} \left(\dot{x}^i \, \frac{\partial g_{il}}{\partial x^j} \cdot \frac{dx^j}{dt} + g_{il} \, \ddot{x}^i \right) - \frac{e}{\dot{s}^2} \, \ddot{s} \, \dot{x}^i \, g_{il} \qquad \text{.....(14.4.7)}$$

Substituting (14.4.5) and (14.4.6) in (14.4.3), we get

$$\frac{1}{2} \dot{x}^i \, \dot{x}^j \, \frac{\partial g_{ij}}{\partial x^l} - \left(\dot{x}^i \, \dot{x}^j \, \frac{\partial g_{il}}{\partial x^j} + g_{il} \, \ddot{x}^i \right) + \frac{\ddot{s}}{\dot{s}} \dot{x}^i \, g_{il} = 0$$

$$\Rightarrow \qquad g_{il} \, \ddot{x}^i - \frac{\ddot{s}}{\dot{s}} \dot{x}^i \, g_{il} + \frac{1}{2} \left(\dot{x}^i \, \dot{x}^j \, \frac{\partial g_{il}}{\partial x^j} + \dot{x}^i \, \dot{x}^j \, \frac{\partial g_{jl}}{\partial x^i} - \dot{x}^i \, \dot{x}^j \, \frac{\partial g_{ij}}{\partial x^l} \right) = 0$$

$$\Rightarrow \qquad g_{il} \, \ddot{x}^i - \frac{\ddot{s}}{\dot{s}} \dot{x}^i \, g_{il} + \dot{x}^i \, \dot{x}^j \left[ij, \, l \right] = 0 \, . \qquad \text{.....(14.4.8)}$$

Taking inner product by g^{lm}, we get

$$\delta_i^m \, \ddot{x}^i - \frac{\ddot{s}}{\dot{s}} \delta_i^m \, \dot{x}^i + \dot{x}^i \, \dot{x}^j \begin{Bmatrix} m \\ ij \end{Bmatrix} = 0$$

$$\Rightarrow \qquad \frac{d^2 x^m}{dt^2} - \frac{\ddot{s}}{\dot{s}} \frac{dx^m}{dt} + \begin{Bmatrix} m \\ ij \end{Bmatrix} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0. \qquad \text{.....(14.4.9)}$$

These are the differential equations of a geodesic in parameter t. These may further be simplified, if we choose the arc distance s alone C as a parameter, *i.e.,* $s = t$. Then,

$$\dot{s} = 1, \quad \ddot{s} = 0. \qquad \text{.....(14.4.10)}$$

Hence (14.4.9) reduces to $$\frac{d^2 x^m}{ds^2} + \begin{Bmatrix} m \\ ij \end{Bmatrix} \frac{dx^i}{ds} \cdot \frac{dx^j}{ds} = 0. \qquad \text{.....(14.4.11)}$$

These are the required differential equations of geodesic. These constitute N-differential equations of the second order, and in terms of the intrinsic derivative may be written as

$$\frac{\delta}{\delta s} \left(\frac{dx^m}{ds} \right) = 0. \qquad \text{.....(14.4.12)}$$

Theorem 1. *In general, one and only one geodesic passes through two specified points lying in a small neighbourhood of a point O of a* V_N.

Proof. The differential equations of a geodesic curve in a V_N are

$$\frac{d^2 x^i}{ds^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{dx^j}{ds} \cdot \frac{dx^k}{ds} = 0. \qquad \text{......(14.4.13)}$$

These are N differential equations of second order, therefore, their general solution will involve $2N$ constants. The theory of differential equations states that these constants will be uniquely determined if the initial values of x^i and $\dfrac{dx^i}{ds}$ are given at a point. It means geometrically, that at any point of the space there is a unique geodesic with given direction. Since the geodesic is defined in terms of the curve passing through two points, it will be unique when the points are sufficiently close to one another.

Notes :

1. The geodesic may be unique unless the points are sufficiently close to one another. On the surface of the sphere there is a unique geodesic passing through any two points, except when the two points are at the ends of a diameter. In the latter case, all great circles passing through the two points are geodesics.

2. In Euclidean space V_N, using orthogonal coordinates, all Christoffel symbols vanish. Therefore, the differential equation of geodesic become

$$\frac{d^2x^i}{ds^2} = 0,$$

whose solution is $\quad x^i = A^i s + B^i,$

when A^i and B^i are constant vectors. These represent straight lines. Hence in Euclidean space V_N, the geodesic are straight lines.

14.5 Curvature of a curve

Let $x^i = x^i\,(s)$ be the equation of a curve C in the space V_N. The unit tangent vector to C is defined as (dx^i/ds) and it is denoted by $\hat{t}$ with t^i as its coutravariant component, thus

$$t^i = \frac{dx^i}{ds}. \qquad .....(14.5.1)$$

First Curvature :

The intrinsic derivative of t^i along the curve C is called the first curvature vector or principal normal of curve C relative to V_N and is denoted by p^i,

Thus, $$p^i = t^i_{,j}\frac{dx^j}{ds} \qquad .....(14.5.2)$$

or $$p^i = \frac{d^2x^i}{ds^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^j}{ds}\cdot\frac{dx^k}{ds}. \qquad .....(14.5.3)$$

The magnitude of a first curvature vector p^i is called **first curvature** of C relative to V_N and is denoted by κ. Therefore

$$\kappa^2 = p^i\,p_i = g_{ij}\,p^i\,p^j. \qquad .....(14.5.4)$$

If η^i denotes the components of the unit principal normal, then

$$\eta^i = \frac{p^i}{\kappa}, \Rightarrow \; p^i = \kappa\eta^i. \qquad .....(14.5.5)$$

We have the differential equation of geodesic as

$$\frac{d^2x^i}{ds^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{dx^i}{ds} \cdot \frac{dx^k}{ds} = 0$$

$$\Rightarrow \qquad p^i = \vec{0}$$

$$\Rightarrow \qquad \kappa\eta^i = \vec{0} \Rightarrow \kappa = 0. \qquad .....(14.5.6)$$

Thus we conclude that a geodesic in Riemannian space V_N is the curve whose first curvature relative to V_N is zero at all points. It gives the alternative definition of geodesic as :

"A geodesic in the Riemannian space V_N is a curve whose first curvature relative V_N vanishes at all points."

14.6 Null-geodesics

Along any portion of a curve which is not null, we have

$$g_{ij} \frac{dx^i}{ds} \cdot \frac{dx^j}{ds} = e. \qquad \left[\because \; e = \frac{1}{e}\right] \qquad .....(14.6.1)$$

Differentiating with respect to s, we get

$$\frac{d}{ds}\left(g_{ij} \frac{dx^i}{ds} \cdot \frac{dx^j}{ds}\right) = \frac{\delta}{\delta s}\left(g_{ij} \frac{dx^i}{ds} \cdot \frac{dx^j}{ds}\right)$$

$$= 2g_{ij} \frac{dx^i}{ds} \frac{\delta}{\delta s}\left(\frac{dx^j}{ds}\right) = 2g_{ij} \frac{dx^i}{ds} \cdot 0 = 0.$$

$$\Rightarrow \qquad \frac{de}{ds} = 0, \qquad .....(14.6.2)$$

which shows that the indicator e cannot change along a geodesic. Therefore, the unit tangent vector $\frac{dx^i}{ds}$ which is not null at any point, cannot be null at any other point on the geodesic.

Contrary to it, if the initial direction is null, then the curve is null and we cannot introduce the arc distance s (which is zero) as parameter.

Thus null geodesic is a null curve $x^i = x^i\,(t)$ which is the solution of the equation

$$\frac{d^2x^i}{dt^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{dx^i}{dt} \frac{dx^k}{dt} = 0. \qquad .....(14.6.3)$$

14.7 Illustrative examples

Ex.1. *Assume that we live in a space for which the line element is*

$$ds^2 = (dx^1)^2 + [(x^1)^2 + c^2]\,(dx^2)^2,$$

which is the surface of a right helocid immersed in a Euclidean three dimensional space. Determine the differential equation of geodesic.

Sol. Here we have $g_{11} = 1, \quad g_{12} = 0, \quad g_{21} = 0, g_{22} = (x^1)^2 + c^2.$

Therefore the Christoffel symbols of second kind are

$$\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = 0, \quad \begin{Bmatrix} 2 \\ 11 \end{Bmatrix} = 0, \quad \begin{Bmatrix} 1 \\ 21 \end{Bmatrix} = 0, \quad \begin{Bmatrix} 1 \\ 12 \end{Bmatrix} = 0,$$

$$\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} = \frac{\partial}{\partial x^1}\left(\log \sqrt{g_{22}}\right) = \frac{x^1}{\left(x^1\right)^2 + c^2},$$

$$\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -\frac{1}{2g_{11}}\frac{\partial g_{22}}{\partial x^1} = -x^1, \quad \begin{Bmatrix} 2 \\ 22 \end{Bmatrix} = 0.$$

Thus the differential equations of the geodesics on the surface are

$$\frac{d^2x^1}{ds^2} - x^1\left(\frac{dx^2}{ds}\right) = 0,$$

and
$$\frac{d^2x^2}{ds^2} + \frac{2x^1}{\left(x^1\right)^2 + c^2}\frac{dx^1}{ds}\cdot\frac{dx^2}{ds} = 0.$$

***Ex.*2.** *Obtain the differential equations of geodesics for the metric*

$$ds^2 = f\left(x\right)dx^2 + dy^2 + dz^2 + \frac{1}{f\left(x\right)}dt^2.$$

Sol. Here $\quad x^1 = x, \quad x^2 = y, \quad x^3 = z, \quad x^4 = t.$

$$g_{11} = f\,(x^1), \quad g_{22} = 1, \quad g_{33} = 1, \quad g_{44} = \frac{1}{f\,(x^1)}$$

$$g_{ij} = 0, \quad i \neq j. \qquad .....(1)$$

Therefore
$$g^{11} = \frac{1}{f\,(x^1)}, \quad g^{22} = 1, \quad g^{33} = 1, \quad g^{44} = f\,(x^1)$$

$$g^{ij} = 0, \quad i \neq j. \qquad .....(2)$$

Thus, the non-zero symbols of second kind are

$$\begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = -\frac{1}{2g_{11}}\frac{\partial g_{44}}{\partial x^1} = -\frac{1}{2f}\frac{\partial}{\partial x^1}\left(\frac{1}{f}\right) = \frac{1}{2f^3}\frac{\partial f}{\partial x^1}$$

$$\begin{Bmatrix} 4 \\ 14 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 41 \end{Bmatrix} = \frac{\partial}{\partial x^1}\left\{\log\sqrt{g_{44}}\right\} = -\frac{1}{2}\frac{d}{dx^1}\left(\log f\right)$$

$$\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \frac{\partial}{\partial x^1}\left\{\log \sqrt{g_{11}}\right\} = \frac{1}{2}\frac{d}{dx^1}\left(\log\, f\right). \qquad .....(3)$$

The differential equations of geodesics are

$$\frac{d^2x^i}{ds^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \qquad .....(4)$$

Hence, taking

(i) $i = 1$,
$$\frac{d^2x^1}{ds^2} + \begin{Bmatrix} 1 \\ 11 \end{Bmatrix} \frac{dx^1}{dx} \cdot \frac{dx^1}{ds} + \begin{Bmatrix} 1 \\ 44 \end{Bmatrix} \frac{dx^4}{ds} \cdot \frac{dx^4}{ds} = 0$$

$$\frac{d^2x}{ds^2} + \frac{1}{2}\frac{d}{dx}(\log f)\left(\frac{dx}{ds}\right)^2 + \frac{1}{2f^2}\frac{d}{dx}(\log f)\left(\frac{dt}{ds}\right)^2 = 0 \qquad .....(5)$$

(ii) $i = 2$,
$$\frac{d^2x^2}{ds^2} = 0, \quad i.e., \quad \frac{d^2y}{ds^2} = 0 \qquad .....(6)$$

(iii) $i = 3$,
$$\frac{d^2x^3}{ds^2} = 0, \quad i.e., \quad \frac{d^2z}{ds^2} = 0 \qquad .....(7)$$

(iv) $i = 4$,
$$\frac{d^2x^4}{ds^2} + 2\begin{Bmatrix} 4 \\ 41 \end{Bmatrix} \frac{dx^4}{ds} \cdot \frac{dx^1}{ds} = 0$$

or
$$\frac{d^2t}{ds^2} - \frac{d}{dx}(\log f)\frac{dt}{ds} \cdot \frac{dx}{ds} = 0. \qquad .....(8)$$

These are the required differential equation of the geodesics.

***Ex*3.** *Show that the curve given by*

$$x^1 = C \int r\cos\theta \cos\phi \, dr$$

$$x^2 = C \int r\cos\theta \sin\phi \, dr$$

$$x^3 = C \int r\sin\theta \, dr$$

$$x^4 = C \int r \, dr$$

where r, θ, ϕ are functions of t, is a real null curve in the V_4 space whose metric is

$$ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + C^2 (dx^4)^2,$$

but not a null geodesic, in general.

Sol. For the given curve, we have

$$\left(\frac{ds}{dt}\right)^2 = -\left(\frac{dx^1}{dt}\right)^2 - \left(\frac{dx^2}{dt}\right)^2 - \left(\frac{dx^3}{dt}\right)^2 + C^2\left(\frac{dx^4}{dt}\right)^2$$

$$= -C^2r^2\cos^2\theta\cos^2\phi - C^2r^2\cos^2\theta\sin^2\phi - C^2r^2\sin^2\theta + C^2r^2$$

$$= 0. \qquad .....(1)$$

Therefore,
$$S = \int_{t_1}^{t_2} \frac{ds}{dt} \cdot dt = 0, \qquad .....(2)$$

along the given curve. Hence it is a null curve.

In a V_4 space, whose metric is (1) the Christoffel symbols vanish and therefore the equations of a geodesic are

$$\frac{d^2x^i}{dt^2} = 0. \qquad .....(3)$$

These equation are not satisfied in general, by the given curve. Hence it is not a null geodesic in general. However if we take θ, ϕ and r as constants equations (3) will be satisfied and the null curve will become a null geodesic.

***Ex.*4.** *Show that on the surface of a sphere, all great circles are geodesics while no other circle is a geodesic.*

Sol. The metric on the surface of a sphere of radius a is given by

$$(ds)^2 = a^2\,(d\theta)^2 + a^2 \sin^2\theta\,(d\phi)^2. \qquad .....(1)$$

Here $$g_{11} = a^2, \quad g_{22} = a^2 \sin^2\theta, \quad g_{12} = g_{21} = 0. \qquad .....(2)$$

The non-zero Christoffel symbols of second kind are

$$\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 21 \end{Bmatrix} = \cot\theta, \quad \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -\sin\theta\cos\theta. \qquad .....(3)$$

The geodesic equation is

$$\frac{d^2x^i}{ds^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{dx^j}{ds} \cdot \frac{dx^k}{ds} = 0. \qquad .....(4)$$

Therefore the geodesic equations reduce to $(x^1 = \theta, \; x^2 = \phi)$

$$\frac{d^2\theta}{ds^2} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0 \quad (i = 1, \; j = k = 2) \qquad .....(5)$$

and $$\frac{d^2\phi}{ds^2} + 2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} = 0 \;\; (i = 2, \; j = 1, \; k = 2) \text{ or } (i = 2, \; j = 2, \; k = 1) \qquad .....(6)$$

(i) We consider a great circle on the surface of the sphere and choose the normal to the plane of the circle as the z-axis ($\theta = 0$), so that this great circle is the equator. Its parametric equation is

$$\theta = \frac{\pi}{2}, \quad \phi = C_1 s + C_2, \quad C_1 \neq 0 \qquad .....(7)$$

where C_1, C_2 are independent of s, θ and ϕ.

Clearly equation (7) satisfies equations (5) and (6).

Therefore the great circle is geodesic, since the choice of the polar axis $\theta = 0$ is arbitrary, it follows that any great circle is a geodesic.

(ii) Consider, a circle on the sphere, whose plane does not pass through the centre of the sphere. Taking the normal to the plane of the circle as $\theta = 0$, the parametric equation of the circle is

$$\theta = \theta_0, \; \theta_0 \neq \frac{\pi}{2}, \quad 0 < \theta_0 < \pi$$

$$\phi = k_1 s + k_2, \quad k_1 \neq 0, \qquad .....(8)$$

where k_1, k_2 are independent of s, θ and ϕ.

It may be noted that on substitution of equations (8) in (5) and (6), the equation (6) is satisfied but equation (5) reduces to

$$k_1{}^2 \sin\theta_0 \cos\theta_0 = 0,$$

which is not true under the given conditions on θ and ϕ. Hence any circle, not being a great circle, on the surface of a sphere is not a geodesic.

14.8 Geodesic coordinates

In Euclidean space V_N the components of the fundamental tensor g_{ij} are constants and therefore all the Christoffel symbols are zero at every point of the Euclidean space. However it is not possible to have such a coordinate system for an arbitrary V_N. Although it is always possible to choose a coordinates system, so that all Chritoffel symbols are zero at a particular point P_0, *i.e.,* in which g_{ij} are locally constants, such a coordinates system is known as **geodesic coordinate system** with the pole at P_0. Thus we can define it as :

"A coordinates system is said to be a geodesic coordinate system with the pole at a point P_0 if relative to this coordinate system the components of the fundamental tensor g_{ij} are locally constants in the neighbourhood of the point P_0, *i.e.,*

$$\frac{\partial g_{ij}}{\partial x^k} = 0$$

at P_0 for all values of i, j and k."

It is to be noted that in this case the first covariant derivative at P_0 reduces to the corresponding partial derivative. Hence

$$\left[A^i_{,j}\right]_{\text{at } P_0} = \left[\frac{\partial A^i}{\partial x^j} + A^r \begin{Bmatrix} i \\ rj \end{Bmatrix}\right]_{\text{at } P_0} = \frac{\partial A^i}{\partial x^j} \text{ at } P_0.$$

Theorem 2. *It is always possible to choose a coordinate system so that all the Christoffel symbols vanish at a particular point P_0 (Geodesic coordinate system).*

Proof. Let x^i be any coordinate system and at a particular point P_0 the value of x^i is $x^i_{(0)}$. We introduce a new coordinate system $\overline{x}^i$ defined by the equation

$$\overline{x}^i = x^i - x^i_{(0)} + \frac{1}{2}\begin{Bmatrix} i \\ mn \end{Bmatrix}_{(0)} \left(x^m - x^m_{(0)}\right)\left(x^n - x^n_{(0)}\right) \qquad .....(14.8.1)$$

Here the index (0) is used for the values at P_0. Differentiating (14.8.1) with respect to x^j

$$\frac{\partial \overline{x}^i}{\partial x^j} = \frac{\partial x^i}{\partial x^j} + \frac{1}{2}\begin{Bmatrix} i \\ mn \end{Bmatrix}_{(0)} \left(x^m - x^m_{(0)}\right)\frac{\partial x^n}{\partial x^j} + \frac{1}{2}\begin{Bmatrix} i \\ mn \end{Bmatrix}_{(0)} \frac{\partial x^m}{\partial x^j}\left(x^n - x^n_{(0)}\right)$$

$$= \delta^i_j + \frac{1}{2}\begin{Bmatrix} i \\ mn \end{Bmatrix}_{(0)} \left(x^m - x^m_{(0)}\right)\delta^n_j + \frac{1}{2}\begin{Bmatrix} i \\ mn \end{Bmatrix}_{(0)} \delta^m_j \left(x^n - x^n_{(0)}\right)$$

$$= \delta^i_j + \begin{Bmatrix} i \\ jn \end{Bmatrix}_{(0)} \left(x^n - x^n_{(0)}\right). \qquad .....(14.8.2)$$

Hence $$\left(\frac{\partial \overline{x}^i}{\partial x^j}\right)_{(0)} = \delta^i_j. \qquad .....(14.8.3)$$

This shows that the Jacobian determinant $\left|\left(\frac{\partial \overline{x}^i}{\partial x^j}\right)_{(0)}\right|$ is not zero and therefore the transformation (14.8.1) is permissible in the neighbourhood of P_0.

Taking inner multiplication of (14.8.2) with $\frac{\partial x^j}{\partial \overline{x}^k}$, we get

$$\delta^i_k = \frac{\partial x^i}{\partial \overline{x}^k} + \left\{\begin{matrix} i \\ jn \end{matrix}\right\}_{(0)} \left(x^n - x^n_{(0)}\right) \frac{\partial x^j}{\partial \overline{x}^k}, \qquad .....(14.8.4)$$

which implies that at P_0

$$\delta^i_k = \left(\frac{\partial x^i}{\partial \overline{x}^k}\right)_{(0)}. \qquad .....(14.8.5)$$

Differentiating (14.8.4) with respect to $\overline{x}^h$, we find

$$0 = \frac{\partial^2 x^i}{\partial \overline{x}^h \partial \overline{x}^k} + \left\{\begin{matrix} i \\ jn \end{matrix}\right\} \frac{\partial x^n}{\partial \overline{x}^h} \frac{\partial x^j}{\partial \overline{x}^k} + \left\{\begin{matrix} i \\ jn \end{matrix}\right\}_{(0)} \left(x^n - x^n_{(0)}\right) \frac{\partial^2 x^i}{\partial \overline{x}^h \partial \overline{x}^k}. \qquad .....(14.8.6)$$

Hence at P_0

$$0 = \left(\frac{\partial^2 x^i}{\partial \overline{x}^h \partial \overline{x}^k}\right)_{(0)} + \left\{\begin{matrix} i \\ jn \end{matrix}\right\}_{(0)} \left(\frac{\partial x^n}{\partial \overline{x}^h}\right)_{(0)} \left(\frac{\partial x^j}{\partial \overline{x}^k}\right)_{(0)}. \qquad .....(14.8.7)$$

Using (14.8.5), we get

$$0 = \left(\frac{\partial^2 x^i}{\partial \overline{x}^h \partial \overline{x}^k}\right)_{(0)} = -\left\{\begin{matrix} i \\ jn \end{matrix}\right\}_{(0)} \delta^n_h \, \delta^j_k = -\left\{\begin{matrix} i \\ hk \end{matrix}\right\}_{(0)}, \qquad .....(14.8.8)$$

therefore at P_0 $\left(\frac{\partial \overline{x}^i}{\partial x^j}\right)_{(0)} = \delta^i_j$, $\left(\frac{\partial x^i}{\partial \overline{x}^k}\right)_{(0)} = \delta^i_k$.

and $$\left(\frac{\partial^2 x^i}{\partial \overline{x}^h \partial \overline{x}^k}\right)_{(0)} = -\left\{\begin{matrix} i \\ hk \end{matrix}\right\}_{(0)}. \qquad .....(14.8.9)$$

We know that

$$\left\{\begin{matrix} \overline{p} \\ lm \end{matrix}\right\} = \left\{\begin{matrix} s \\ ij \end{matrix}\right\} \frac{\partial \overline{x}^p}{\partial x^s} \frac{\partial x^i}{\partial \overline{x}^l} \frac{\partial x^j}{\partial \overline{x}^m} + \frac{\partial \overline{x}^p}{\partial x^j} \frac{\partial^2 x^j}{\partial \overline{x}^l \partial \overline{x}^m},$$

therefore $$\left\{\begin{matrix} \overline{p} \\ lm \end{matrix}\right\}_{(0)} = \left\{\begin{matrix} s \\ ij \end{matrix}\right\}_{(0)} \left(\frac{\partial \overline{x}^p}{\partial x^s}\right)_{(0)} \left(\frac{\partial x^i}{\partial \overline{x}^l}\right)_{(0)} \left(\frac{\partial x^j}{\partial \overline{x}^m}\right)_{(0)} + \left(\frac{\partial \overline{x}^p}{\partial x^j}\right)_{(0)} \left(\frac{\partial^2 x^j}{\partial \overline{x}^l \partial \overline{x}^m}\right)_{(0)}$$

$$= \begin{Bmatrix} s \\ ij \end{Bmatrix}_{(0)} \delta_s^p \, \delta_j^i \, \delta_m^j - \delta_j^p \begin{Bmatrix} j \\ lm \end{Bmatrix}_{(0)}$$

$$= \begin{Bmatrix} p \\ lm \end{Bmatrix}_{(0)} - \begin{Bmatrix} p \\ lm \end{Bmatrix}_{(0)} = 0. \qquad .....(14.8.10)$$

This prove that theorem.

Theorem 3. *The necessary and sufficient conditions that a system of coordinates be geodesic with the pole P_0 are that their second covariant derivatives, with respect to the metric of the space, all vanish at P_0.*

Proof. We know that

$$\frac{\partial^2 x^r}{\partial \overline{x}^l \, \partial \overline{x}^m} = \begin{Bmatrix} \overline{p} \\ lm \end{Bmatrix} \frac{\partial x^r}{\partial \overline{x}^p} - \begin{Bmatrix} r \\ ij \end{Bmatrix} \frac{\partial x^i}{\partial \overline{x}^l} \frac{\partial x^j}{\partial \overline{x}^m}. \qquad .....(14.8.11)$$

Interchanging the coordinate system x^i and $\overline{x}^i$ the equation (14.8.11) can be written as

$$\frac{\partial^2 \overline{x}^r}{\partial x^l \, \partial x^m} = \begin{Bmatrix} p \\ lm \end{Bmatrix} \frac{\partial \overline{x}^r}{\partial x^p} - \begin{Bmatrix} \overline{r} \\ ij \end{Bmatrix} \frac{\partial \overline{x}^i}{\partial x^l} \frac{\partial \overline{x}^j}{\partial x^m}$$

$$\Rightarrow \qquad \begin{Bmatrix} \overline{r} \\ ij \end{Bmatrix} \frac{\partial \overline{x}^i}{\partial x^l} \frac{\partial \overline{x}^j}{\partial x^m} = -\left[\frac{\partial}{\partial x^m} \left(\frac{\partial \overline{x}^r}{\partial x^l} \right) - \begin{Bmatrix} p \\ lm \end{Bmatrix} \frac{\partial \overline{x}^r}{\partial x^p} \right]$$

$$= -\left[\frac{\partial}{\partial x^m} \left(\overline{x}^r_{,l} \right) - \begin{Bmatrix} p \\ lm \end{Bmatrix} x^r_{,p} \right].$$

Hence $$\begin{Bmatrix} \overline{r} \\ ij \end{Bmatrix} \frac{\partial \overline{x}^i}{\partial x^l} \frac{\partial \overline{x}^j}{\partial x^m} = -\left(\overline{x}^r_{,l} \right)_{,m} = -\overline{x}^r_{,lm}. \qquad .....(14.8.12)$$

Necessary condition : If the coordinate system $\overline{x}^i$ be a geodesic coordinate system with pole at P_0, then

$$\begin{Bmatrix} \overline{r} \\ ij \end{Bmatrix} = 0 \ \text{ at } P_0,$$

and therefore from (14.8.12) it follows that $\overline{x}^r_{,lm} = 0$ at P_0.

Sufficient condition : Courversely suppose that

$$\overline{x}^r_{,lm} = 0 \ \text{ at } P_0.$$

then equation (14.8.12) implies $$\begin{Bmatrix} \overline{r} \\ ij \end{Bmatrix} \frac{\partial \overline{x}^i}{\partial x^l} \frac{\partial \overline{x}^j}{\partial x^m} = 0 \ \text{ at } P_0.$$

Thus, $$\begin{Bmatrix} \overline{r} \\ ij \end{Bmatrix} = 0 \ \text{ at } P_0, \quad \text{as} \quad \left| \frac{\partial \overline{x}^i}{\partial x^l} \right| \neq 0, \quad \left| \frac{\partial \overline{x}^j}{\partial x^m} \right| \neq 0.$$

Hence the proposition.

***Ex.*5.** *Show that the coordinate system* $\overline{x}^i$ *defined by*

$$\overline{x}^i = x^i + \frac{1}{2}\begin{Bmatrix} i \\ mn \end{Bmatrix}_{(0)} x^m \cdot x^n,$$

is a geodesic coordinate system with the pole at the origin.

Sol. Since the pole is at the origin $(x^m)_{(0)} = 0$, then $\overline{x}^i$ becomes a particular case of theorem 2. We can prove it independently as follows :

$$\frac{\partial \overline{x}^i}{\partial x^j} = \delta^i_j + \frac{1}{2}\begin{Bmatrix} i \\ mn \end{Bmatrix}_{(0)} \delta^m_j x^n + \frac{1}{2}\begin{Bmatrix} i \\ mn \end{Bmatrix}_{(0)} x^m \delta^n_j$$

$$= \delta^i_j + \begin{Bmatrix} i \\ jn \end{Bmatrix}_{(0)} x^n. \qquad \text{.....(1)}$$

Also, we have

$$\frac{\partial^2 \overline{x}^i}{\partial x^j\,\partial x^k} = \begin{Bmatrix} i \\ jn \end{Bmatrix}_{(0)} \delta^n_k = \begin{Bmatrix} i \\ jk \end{Bmatrix}_{(0)}. \qquad \text{.....(2)}$$

Hence $$\left(\frac{\partial \overline{x}^i}{\partial x^j}\right)_{(0)} = \delta^i_j \text{ and } \left(\frac{\partial^2 \overline{x}^i}{\partial x^j\,\partial x^k}\right)_{(0)} = \begin{Bmatrix} i \\ jk \end{Bmatrix}_{(0)}. \qquad \text{.....(3)}$$

Now $$\overline{x}^i_{,jk} = \frac{\partial}{\partial x^k}\left(\frac{\partial \overline{x}^i}{\partial x^j}\right) - \frac{\partial \overline{x}^i}{\partial x^r}\begin{Bmatrix} i \\ jk \end{Bmatrix}. \qquad \text{.....(4)}$$

at the pole $$\left(\overline{x}^i_{,jk}\right)_{(0)} = \left(\frac{\partial^2 \overline{x}^i}{\partial x^j\,\partial x^k}\right)_{(0)} - \left(\frac{\partial \overline{x}^i}{\partial x^r}\right)_{(0)} \begin{Bmatrix} r \\ jk \end{Bmatrix}_{(0)}$$

$$= \begin{Bmatrix} i \\ jk \end{Bmatrix}_{(0)} - \delta^i_r \begin{Bmatrix} r \\ jk \end{Bmatrix}_{(0)} = 0. \qquad \text{.....(5)}$$

Hence proved the result.

***Ex.*6.** *Show that at the pole* P_0 *of a geodesic coordinate system*

$$A_{i,jk} = \frac{\partial^2 A_i}{\partial x^j \partial x^k} - A_l \frac{\partial}{\partial x^k}\begin{Bmatrix} l \\ ij \end{Bmatrix}.$$

Sol. Since we have $$A_{i,j} = \frac{\partial A_i}{\partial x^j} - A_r \begin{Bmatrix} r \\ ij \end{Bmatrix}. \qquad \text{.....(1)}$$

Taking covariant derivative of (1), we get

$$\left(A_{i,j}\right)_{,k} = \frac{\partial}{\partial x^k}\left[\frac{\partial A_i}{\partial x^j} - A_r \begin{Bmatrix} r \\ ij \end{Bmatrix}\right] - A_{i,r}\begin{Bmatrix} r \\ jk \end{Bmatrix} - A_{r,j}\begin{Bmatrix} r \\ ik \end{Bmatrix}$$

$$\Rightarrow \qquad A_{i,jk} = \frac{\partial^2 A_i}{\partial x^j \partial x^k} - A_r \frac{\partial}{\partial x^k}\begin{Bmatrix} r \\ ij \end{Bmatrix} - \frac{\partial A_r}{\partial x^k}\begin{Bmatrix} r \\ ij \end{Bmatrix} - A_{i,r}\begin{Bmatrix} r \\ jk \end{Bmatrix} - A_{r,j}\begin{Bmatrix} r \\ ik \end{Bmatrix}. \qquad .....(2)$$

Now, we know that in geodesic coordinate system at the pole P_0 the Christoffel symbols vanish and therefore (2) reduces to

$$A_{i,jk} = \frac{\partial^2 A_i}{\partial x^j \partial x^k} - A_r \frac{\partial}{\partial x^k}\begin{Bmatrix} r \\ ij \end{Bmatrix}. \qquad .....(3)$$

14.9 Riemannian coordinates

The Riemannian coordinate system is a particular case of geodesic coordinate system. Hence every Riemannian coordinate system is necessarily a geodesic coordinate system but converse is not always true. To define Riemannian coordinate system we consider an arbitrary fixed point P_0 in V_N. We define the quantity ξ^i such that

$$\xi^i = \left(\frac{dx^i}{ds}\right)_{(0)}, \qquad ......(14.9.1)$$

where suffix (0) indicates the value related to P_0. Let c be the geodesic through P_0 in V_N. Since one and one geodesic c will pass through P_0 in the direction of ξ^i, such that

$$y^i = s\xi^i, \qquad .....(14.9.2)$$

defines the Riemannian coordinate system. Here $P(y^i)$ is a point on the geodesic c and s is the arc length along the curve form P_0 to P.

Theorem 4. *The Riemannian coordinates are geodesic coordinates with the pole at* P_0.

Proof. The differential equation of geodesic c in terms of Riemannian coordinates y^i relative to V_N are given by

$$\frac{d^2 y^i}{ds^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{dy^j}{ds}\frac{dy^k}{ds} = 0, \qquad .....(14.9.3)$$

where $\begin{Bmatrix} i \\ jk \end{Bmatrix}$ is the Christoffel symbol relative to the coordinates y^i. Since the $P(y^i)$, defined by (14.9.2) is on the geodesic c given by (14.9.1). This must satisfy it, therefore

$$0 + \begin{Bmatrix} i \\ jk \end{Bmatrix} \xi^j \xi^k = 0, \qquad .....(14.9.4)$$

Q $y^i = s\xi^i$, then (14.9.4) becomes

$$\begin{Bmatrix} i \\ jk \end{Bmatrix} y^i y^k = 0. \qquad .....(14.9.5)$$

Equation (14.9.5) holds throughout the space V_N. It also implies that

$$\begin{Bmatrix} i \\ jk \end{Bmatrix} = 0 \quad \text{at } P_0. \qquad .....(14.9.6)$$

Since at P_0, $y^i \neq 0,\ y^k \neq 0.$

Hence the Riemannian coordinates are geodesic coordinates with the pole at P_0.

14.10 Field of parallel vectors (Parallelism of vectors)

The vectors A_i constitute a field of parallel vectors along the curve $x^i = x^i(t)$ in a V_N, if A_i is a solution of the differential equation

$$\frac{\delta A_i}{\delta t} \equiv A_{i,j}\frac{dx^i}{dt} = \frac{dA_i}{dt} - \begin{Bmatrix} l \\ ij \end{Bmatrix} A_l \frac{dx^i}{dt} = 0. \qquad .....(14.10.1)$$

The concept of parallelism is given by **Levi-Civita**. These are N-differential equations of first order, therefore the general solution will involve N-constants. According the theory of differential equation, if the initial values of A_i are given at a point of the curve these constants will uniquely be determined, *i.e.,* A_i will be determined uniquely at all other points if it is given at one point of the curve. Thus we can say that a field of parallel vectors is obtained from a given vector by parallel propagation (displacement) along the curve.

The condition (14.10.1) is in covariant form, we can write it in the contravariant form as

$$\frac{\delta A^i}{\delta t} \equiv \frac{\delta}{\delta t}\left(g^{ij} A_j\right) = g^{ij}\frac{\delta A_j}{\delta t} = 0$$

$$\Rightarrow \qquad \frac{\delta A^i}{\delta t} \equiv A^i_{,j}\frac{dx^j}{dt} = \frac{dA^i}{dt} + \begin{Bmatrix} i \\ jk \end{Bmatrix} A^k \frac{dx^j}{dt} = 0. \qquad .....(14.10.2)$$

Theorem 5. *The magnitude of all vectors of a field of parallel vectors is constant.*

Proof. If A be the magnitude of vector A^i, then

$$(A)^2 = e_{(A)}\, g_{ij}\, A^i A^j.$$

Differentiating with respect to parameter t, we find

$$2A\frac{dA}{dt} = \frac{d}{dt}\left(e_{(A)} g_{ij} A^i A^j\right) = \frac{\delta}{\delta t}\left(e_{(A)} g_{ij} A^i A^j\right)$$

$$= 2e_{(A)} g_{ij} A^j \frac{\delta A^i}{\delta t},$$

as the total derivative becomes the intrinsic derivative in the case of scalars ($e_{(A)}\, g_{ij}\, A^i A^j$ is a scalar).

Using (14.10.2), we find that

$$2A\frac{dA}{dt} = 0$$

$$\Rightarrow \qquad \frac{dA}{dt}\left(A^2\right) = 0$$

$$\Rightarrow \qquad A^2 = \text{constant}$$

$$\Rightarrow \qquad A = \text{constant}.$$

Hence the theorem.

Theorem 6. *Prove that the geodesic is an auto parallel curve.*

Or

Prove that the unit tangent vectors form a field of parallel vectors along a geodesic.

Proof. The differential equation of a geodesic are

$$\frac{d^2x^i}{ds^2}+\begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^j}{ds}\cdot\frac{dx^k}{ds}=0$$

$$\Rightarrow \qquad \frac{d}{ds}\left(\frac{dx^i}{ds}\right)+\begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^j}{ds}\cdot\frac{dx^k}{ds}=0$$

$$\Rightarrow \qquad \frac{\partial}{\partial x^j}\left(\frac{dx^i}{ds}\right)\frac{dx^j}{ds}+\begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^j}{ds}\cdot\frac{dx^k}{ds}=0$$

$$\Rightarrow \qquad \left[\frac{\partial}{\partial x^j}\left(\frac{dx^i}{ds}\right)+\begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^k}{ds}\right]\cdot\frac{dx^j}{ds}=0$$

$$\Rightarrow \qquad \left(\frac{dx^i}{ds}\right)_{,j}\frac{dx^j}{ds}=0$$

$$\Rightarrow \qquad t^i_{,j}\frac{dx^j}{ds}=0,$$

which shows that the unit tangent vectors $t^i=\dfrac{dx^i}{ds}$ form a field of parallel vectors along a geodesic.

Hence proved.

14.11 Parallelism of vector of variable magnitude along a curve

We know that A_i or A^i constitute a field of parallel vectors along the curve $x^i = x^i\ (t)$ if their intrinsic derivative with respect to t is zero. Further from Theorem 5, we find that the magnitude of all vectors of a field of parallel vectors is constant.

We shall now define the parallelism of two vectors whose magnitude need not to be constant.

'Two vectors at a point are said to be parallel, if their corresponding components are proportional.'

Clearly, a vector B^i is called parallel to a vector A^i at each point of a curve c if

$$B^i = \phi A^i,$$

where ϕ is an arbitrary scalar function of arc length s.

Theorem 7. *The necessary and sufficient condition for a vector B^i of variable magnitude to suffer a parallel displacement along a curve c is that*

$$B^i_{,j}\frac{dx^j}{ds}=B^i f\ (s).$$

Proof. Necessary condition : First we suppose that A^i constitute a field of parallel vectors along the curve c, then the magnitude of A^i is constant and

$$A^i_{,j}\frac{dx^j}{ds}=0. \qquad .....(14.11.1)$$

Now, we know that if B^i is parallel to A^i at each point of c, then

$$B^i=\phi A^i. \qquad .....(14.11.2)$$

Therefore,
$$B^i_{,j}\frac{dx^j}{ds}=\left(\phi A^i\right)_{,j}\frac{dx^j}{ds}$$

$$=\phi A^i_{,j}\frac{dx^j}{ds}+\frac{\partial\phi}{\partial x^j}A^i\frac{dx^j}{ds}$$

$$=0+\frac{d\phi}{ds}\cdot\frac{B^i}{\phi} \qquad \text{[using (14.11.1) \& (14.11.2)]}$$

$$=B^i\frac{d}{ds}(\log\phi)$$

$$\Rightarrow \qquad B^i_{,j}\frac{dx^j}{ds}=B^i f(s), \qquad .....(14.11.3)$$

where
$$\frac{d}{ds}\left[\log\phi(s)\right]=f(x). \qquad .....(14.11.4)$$

This shows that equation (14.11.3) is necessary condition for the vector B^i of variable magnitude to suffer a parallel displacement along c.

Sufficiently condition : Conversely suppose that B^i is a vector of variable magnitude, such that

$$B^i_{,j}\frac{dx^j}{ds}=B^i f(s). \qquad .....(14.11.5)$$

Taking
$$A^i=B^i F(s), \qquad .....(14.11.6)$$

we have
$$A^i_{,j}\frac{dx^j}{ds}=\left(FB^i\right)_{,j}\frac{dx^j}{ds}$$

$$=FB^i_{,j}\frac{dx^j}{ds}+\frac{\partial F}{\partial x^j}B^i\frac{dx^i}{ds}$$

$$=FB^i f(s)+B^i\frac{dF}{ds}$$

$$=B^i\left[Ff(s)+\frac{dF}{ds}\right]. \qquad .....(14.11.7)$$

Choosing F such that

$$Ff(s)+\frac{dF}{ds}=0, \qquad .....(14.11.8)$$

because of arbitrary nature of $F(s)$, we find

$$A^i_{,j}\frac{dx^i}{ds}=0,$$

which shows that A^i form a field of parallel vectors along c and is of constant magnitude, then (14.11.6) implies that B^i is parallel along c.

14.12 Illustrative examples

Ex.7. *Show that the vector B^i of variable magnitude suffers a parallel displacement along a curve c if and only if*

$$\left(B^l B^i_{,j}-B^i B^l_{,j}\right)\frac{dx^j}{ds}=0.$$

Sol. We know from Theorem 7, that B^i suffers a parallel displacement along c if and only if

$$B^i_{,j}\frac{dx^i}{ds}=B^i f(s)\cdot$$

Taking outer multiplication by B^l, we get

$$B^l B^i_{,j}\frac{dx^j}{ds}=B^l B^i f(s). \qquad .....(1)$$

Interchanging the suffixes i and l, we find

$$B^i B^l_{,j}\frac{dx^j}{ds}=B^i B^l f(s). \qquad .....(2)$$

Subtracting (2) from (1), the required result is obtained as

$$\left(B^l B^i_{,j}-B^i B^l_{,j}\right)\frac{dx^j}{ds}=0. \qquad .....(3)$$

Theorem 8. ***(Fundamental theorem of Riemannian geometry).***

With a given fundamental tensor of a Riemannian manifold (Riemannian space V_N), there exists exactly one symmetric connection with respect to which the parallel displacement preserves scalar product.

Proof. Let A^i and B^i be two unit vectors defined along a curve c in space V_N and these vectors suffer parallel displacement along c, then

$$A^i_{,j}\frac{dx^j}{ds}=0, \qquad .....(1)$$

$$B^i_{,j}\frac{dx^j}{ds}=0, \qquad .....(2)$$

The scalar product of A^i and B^i is $g_{ij}A^i B^j$, where g_{ij} is the given fundamental tensor.

The parallel displacement preserves scalar product if the intrinsic derivative of $g_{ij}A^i B^j$ is zero.

We have $$\left(g_{ij}A^iB^j\right)_{,k}\frac{dx^k}{ds}=\left(g_{ij,k}\frac{dx^k}{ds}\right)A^iB^j+g_{ij}\left(A^i_{,k}\frac{dx^k}{ds}\right)B^j+g_{ij}A^i\left(B^j_{,k}\frac{dx^k}{ds}\right)$$

Using (1), (2) and Ricci theorem *i.e.*, $g_{ij,k}=0$.

So the R.H.S. is zero and this proves the theorem.

14.13 Self-learning exercises

1. Define Geodesic.

2. Write Euler's condition of calculus of variation.

3. Define first curvature ?

4. What is null geodesic ?

5. What is field of parallel vectors ?

6. Write fundamental theorem of Riemannian geometry.

14.14 Summary

Geodesic in a surface is the curve of stationary length on a surface between any two points in its plane. It is the main point of the study in this chapter. We obtained a differential equation whose solution will give the geodesic curve. The geodesic coordinates and Riemannian coordinates have also been studied.

14.15 Answers to self-learning exercises

1. § 14.2	**2.** § 14.3	**3.** § 14.5
4. § 14.6	**5.** § 14.10	**6.** Theorem 8.

14.16 Exercises

1. Show that it is always possible to choose a geodesic coordinate system for any V_N with an arbitrary point P_0.

2. Obtain the equations of geodesic for the metric

$$ds^2=e^{-2kt}\left(dx^2+dy^2+dz^2\right)+\left(dt\right)^2.$$

3. Show that the great circles on sphere are geodesic.

4. Obtain the differential equation of geodesic for the metric

$$ds^2=f\left(x\right)dx^2+dy^2+dz^2+\frac{1}{f\left(x\right)}dt^2.$$

5. Give an example of a geodesic coordinate system.

□□□

Unit 15 : Riemannian-Christoffel Tensor and its Properties, Covariant Curvature Tensor, Einstein Space, Bianchi's Identity, Einstein Tensor, Flat Space, Isotropic Point, Schur's Theorem

Structure of the Unit

15.0 Objectives

The objective of this unit is to study the commutativity of the process of covariant differentiation of vectors and hence we define Riemann's symbols of first and second kinds. The contraction of Riemann-Christoffel tensor and Ricci tensor are the points of study. In the end Einstein space, Einstein tensor and flat space are the points of study.

15.1 Introduction

We have studies that the covariant differentiation of invariant is commutative. Now we shall investigate this property of commutative nature for covariant differentiation of vectors. For the study let us take the covariant derivative of an arbitrary covariant vector A_i

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - A_\alpha \begin{Bmatrix} \alpha \\ ij \end{Bmatrix}. \qquad \text{.....(15.1.1)}$$

Differentiating again covariantly with respect to x^k, we get

$$\left(A_{i,j}\right)_{,k} = \frac{\partial}{\partial x^k}\left(\frac{\partial A_i}{\partial x^j} - A_\alpha \begin{Bmatrix} \alpha \\ ij \end{Bmatrix}\right) - A_{\alpha,j}\begin{Bmatrix} \alpha \\ ik \end{Bmatrix} - A_{i,\alpha}\begin{Bmatrix} \alpha \\ jk \end{Bmatrix}.$$

$$\Rightarrow \qquad A_{i,jk} = \frac{\partial^2 A_i}{\partial x^k \partial x^j} - \frac{\partial A_\alpha}{\partial x^k}\begin{Bmatrix} \alpha \\ ij \end{Bmatrix} - A_\alpha \frac{\partial}{\partial x^k}\begin{Bmatrix} \alpha \\ ij \end{Bmatrix}$$

$$-\left[\frac{\partial A_\alpha}{\partial x^j} - A_\beta \begin{Bmatrix} \beta \\ \alpha j \end{Bmatrix}\right]\begin{Bmatrix} \alpha \\ ik \end{Bmatrix} - \left[\frac{\partial A_i}{\partial x^\alpha} - A_\beta \begin{Bmatrix} \beta \\ i\alpha \end{Bmatrix}\right]\begin{Bmatrix} \alpha \\ jk \end{Bmatrix} \qquad \text{.....(15.1.2)}$$

Now interchanging the suffices j and k, we find that

$$A_{i,jk} = \frac{\partial^2 A_i}{\partial x^j \partial x^k} - \frac{\partial A_\alpha}{\partial x^j}\begin{Bmatrix} \alpha \\ ik \end{Bmatrix} - A_\alpha \frac{\partial}{\partial x^j}\begin{Bmatrix} \alpha \\ ik \end{Bmatrix}$$

$$-\left[\frac{\partial A_\alpha}{\partial x^k} - A_\beta \begin{Bmatrix} \beta \\ \alpha k \end{Bmatrix}\right]\begin{Bmatrix} \alpha \\ ij \end{Bmatrix} - \left[\frac{\partial A_i}{\partial x^\alpha} - A_\beta \begin{Bmatrix} \beta \\ i\alpha \end{Bmatrix}\right]\begin{Bmatrix} \alpha \\ kj \end{Bmatrix} \qquad \text{.....(15.1.3)}$$

Subtracting (15.1.3) from (15.1.2) and interchanging α, β, we get

$$\left(A_{i,jk} - A_{i,kj}\right) = \left[\frac{\partial}{\partial x^j}\begin{Bmatrix} \alpha \\ ik \end{Bmatrix} - \frac{\partial}{\partial x^k}\begin{Bmatrix} \alpha \\ ij \end{Bmatrix} + \begin{Bmatrix} \alpha \\ \beta j \end{Bmatrix}\begin{Bmatrix} \beta \\ ik \end{Bmatrix} - \begin{Bmatrix} \alpha \\ \beta k \end{Bmatrix}\begin{Bmatrix} \beta \\ \alpha j \end{Bmatrix}\right] A^\alpha \qquad \text{.....(15.1.4)}$$

Since left hand side of (15.1.4) is a covariant tensor of third order and A_α be the arbitrary covariant vector in the right hand side, therefore from quotient law it follows that the coefficient of A_α in R.H.S. must be a mixed tensor of fourth order, contavariant of rank one and covariant of rank three. Let us denote this quantity by $R^\alpha_{.ijk}$ *i.e.,*

$$R^\alpha_{.ijk} = \frac{\partial}{\partial x^j}\begin{Bmatrix} \alpha \\ ik \end{Bmatrix} - \frac{\partial}{\partial x^k}\begin{Bmatrix} \alpha \\ ij \end{Bmatrix} + \begin{Bmatrix} \alpha \\ \beta j \end{Bmatrix}\begin{Bmatrix} \beta \\ ik \end{Bmatrix} - \begin{Bmatrix} \alpha \\ \beta k \end{Bmatrix}\begin{Bmatrix} \beta \\ ij \end{Bmatrix}. \qquad \text{.....(15.1.5)}$$

The tensor $R^\alpha_{.ijk}$ defined by (15.1.5) is known as **Riemann-Christoffel tensor** or **mixed curvature tensor** and thus, we have

$$A_{i,jk} - A_{i,kj} = R^\alpha_{.ijk} A_\alpha. \qquad \text{.....(15.1.6)}$$

The symbol $R^\alpha_{.ijk}$ is also called Riemann's symbol of second kind.*

* Riemann's symbol of the first kind is introduced later on.

Now it is clear that the necessary and sufficient condition for covariant differentiation of a vector A_i to be commutative is that Riemann-Christoffel tensor be identically zero or $R^{\alpha}_{.ijk} = 0$.

Note : $R^{\alpha}_{.ijk}$ is formed exclusively from the fundamental tensor g_{ij} and its derivatives upto second order. It does not depend on the choice of A_i.

15.2 Properties of Riemann-Christoffel tensor

Property–I : The Riemann-Christoffel tensor is skew-symmetric in the last two subscripts, *i.e.,*

$$R^{\alpha}_{.ijk} = -R^{\alpha}_{.ikj}. \quad .....(15.2.1)$$

Proof. The result immediately follows from (15.1.5).

Property–II : $R^{\alpha}_{.ijk}$ has cyclic property in its subscripts, *i.e.,*

$$R^{\alpha}_{.ijk} + R^{\alpha}_{.jki} + R^{\alpha}_{.kij} = 0. \quad .....(15.2.2)$$

Proof. By definition (15.1.5), we have

$$R^{\alpha}_{.ijk} = \frac{\partial}{\partial x^j}\begin{Bmatrix}\alpha \\ ik\end{Bmatrix} - \frac{\partial}{\partial x^k}\begin{Bmatrix}\alpha \\ ij\end{Bmatrix} + \begin{Bmatrix}\beta \\ ik\end{Bmatrix}\begin{Bmatrix}\alpha \\ \beta j\end{Bmatrix} - \begin{Bmatrix}\beta \\ ij\end{Bmatrix}\begin{Bmatrix}\alpha \\ \beta k\end{Bmatrix} \quad .....(15.2.3)$$

$$R^{\alpha}_{.jki} = \frac{\partial}{\partial x^k}\begin{Bmatrix}\alpha \\ ji\end{Bmatrix} - \frac{\partial}{\partial x^i}\begin{Bmatrix}\alpha \\ jk\end{Bmatrix} + \begin{Bmatrix}\beta \\ ji\end{Bmatrix}\begin{Bmatrix}\alpha \\ \beta k\end{Bmatrix} - \begin{Bmatrix}\beta \\ jk\end{Bmatrix}\begin{Bmatrix}\alpha \\ \beta i\end{Bmatrix} \quad .....(15.2.4)$$

$$R^{\alpha}_{.kij} = \frac{\partial}{\partial x^i}\begin{Bmatrix}\alpha \\ kj\end{Bmatrix} - \frac{\partial}{\partial x^j}\begin{Bmatrix}\alpha \\ ki\end{Bmatrix} + \begin{Bmatrix}\beta \\ kj\end{Bmatrix}\begin{Bmatrix}\alpha \\ \beta i\end{Bmatrix} - \begin{Bmatrix}\beta \\ ki\end{Bmatrix}\begin{Bmatrix}\alpha \\ \beta j\end{Bmatrix} \quad .....(15.2.5)$$

Adding these, we get

$$R^{\alpha}_{.ijk} + R^{\alpha}_{.jki} + R^{\alpha}_{.kij} = 0. \quad .....(15.2.6)$$

Hence the result follows.

Property–III : $R^{\alpha}_{.ijk}$ vanishes on contraction in α and *i, i.e.,*

$$R^{\alpha}_{.\alpha jk} = 0. \quad .....(15.2.7)$$

Proof. In equation (15.2.5) of §15.2 contracting over α and i, we get

$$R^{\alpha}_{.\alpha jk} = \frac{\partial}{\partial x^j}\begin{Bmatrix}\alpha \\ \alpha k\end{Bmatrix} - \frac{\partial}{\partial x^k}\begin{Bmatrix}\alpha \\ \alpha j\end{Bmatrix} + \begin{Bmatrix}\beta \\ \alpha k\end{Bmatrix}\begin{Bmatrix}\alpha \\ \beta j\end{Bmatrix} - \begin{Bmatrix}\beta \\ \alpha j\end{Bmatrix}\begin{Bmatrix}\alpha \\ \beta k\end{Bmatrix} \quad .....(15.2.8)$$

The last two terms cancel out as α and β are dummy indices, therefore using Property–4 of Christoffel symbols, we get

$$R^{\alpha}_{.\alpha jk} = \frac{\partial}{\partial x^j}\left(\frac{\partial}{\partial x^k}\log\sqrt{g}\right) - \frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^j}\log\sqrt{g}\right) = 0. \quad .....(15.2.9)$$

15.3 Covariant curvature tensor

The covariant curvature tensor is defined as

$$R_{rijk} = g_{r\alpha} R^{\alpha}_{.ijk}. \quad .....(15.3.1)$$

It is fourth order covariant tensor and is also called as **Riemann tensor**. The symbol R_{rijk} is also called **Riemann's symbol** of **first kind**. It is an associate tensor of Riemann-Christoffel tensor.

The properties of R_{rijk} can easily be studies if we express it in a more suitable form. Substituting the value of $R^{\alpha}_{.ijk}$ in the above definition, we get

$$R_{rijk} = g_{r\alpha}\left[\frac{\partial}{\partial x^j}\begin{Bmatrix}\alpha\\ ik\end{Bmatrix} - \frac{\partial}{\partial x^k}\begin{Bmatrix}\alpha\\ ij\end{Bmatrix} + \begin{Bmatrix}\beta\\ ik\end{Bmatrix}\begin{Bmatrix}\alpha\\ j\beta\end{Bmatrix} - \begin{Bmatrix}\beta\\ ij\end{Bmatrix}\begin{Bmatrix}\alpha\\ k\beta\end{Bmatrix}\right]$$

$$= \frac{\partial}{\partial x^j}\left(g_{r\alpha}\begin{Bmatrix}\alpha\\ ik\end{Bmatrix}\right) - \frac{\partial g_{r\alpha}}{\partial x^j}\begin{Bmatrix}\alpha\\ ik\end{Bmatrix} - \frac{\partial}{\partial x^k}\left(g_{r\alpha}\begin{Bmatrix}\alpha\\ ij\end{Bmatrix}\right)$$

$$+\frac{\partial g_{r\alpha}}{\partial x^k}\begin{Bmatrix}\alpha\\ ij\end{Bmatrix} + g_{r\alpha}\begin{Bmatrix}\beta\\ ik\end{Bmatrix}\begin{Bmatrix}\alpha\\ j\beta\end{Bmatrix} - g_{r\alpha}\begin{Bmatrix}\beta\\ ij\end{Bmatrix}\begin{Bmatrix}\alpha\\ k\beta\end{Bmatrix} \qquad .....(15.3.2)$$

Using properties (15.3.1) and (15.3.2) of Christoffel symbols, the above expression reduces to

$$R_{rijk} = \frac{\partial}{\partial x^j}[ik,r] - \begin{Bmatrix}\alpha\\ ik\end{Bmatrix}\left([rj,\alpha]+[\alpha j,r]\right) - \frac{\partial}{\partial x^k}[ij,r] - \begin{Bmatrix}\alpha\\ ij\end{Bmatrix}\left([rk,\alpha]+[\alpha k,r]\right)$$

$$+\begin{Bmatrix}\beta\\ ik\end{Bmatrix}[j\beta,r] - \begin{Bmatrix}\beta\\ ij\end{Bmatrix}[k\beta,r]$$

$$= \frac{\partial}{\partial x^j}[ik,r] - \frac{\partial}{\partial x^k}[ij,r] + \begin{Bmatrix}\alpha\\ ij\end{Bmatrix}[rk,r] - \begin{Bmatrix}\alpha\\ ik\end{Bmatrix}[rj,r] \qquad .....(15.3.3)$$

the remaining terms cancel out by suitable changes of dummy indices. It can further be simplified as

$$R_{rijk} = \frac{1}{2}\frac{\partial}{\partial x^j}\left(\frac{\partial g_{ir}}{\partial x^k} + \frac{\partial g_{kr}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^r}\right) - \frac{1}{2}\frac{\partial}{\partial x^k}\left(\frac{\partial g_{ir}}{\partial x^j} + \frac{\partial g_{ir}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^r}\right)$$

$$+g^{\alpha\beta}[ij,\beta][rk,\alpha] - g^{\alpha\beta}[ik,\beta][rj,\alpha]. \qquad .....(15.3.4)$$

Finally, we have

$$R_{rijk} = \frac{1}{2}\left(\frac{\partial^2 g_{rk}}{\partial x^i\,\partial x^j} + \frac{\partial^2 g_{ij}}{\partial x^r\,\partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^r\,\partial x^j} - \frac{\partial^2 g_{rj}}{\partial x^i\,\partial x^k}\right)$$

$$+g^{\alpha\beta}\left([rk,\alpha][ij,\beta] - [rj,\alpha][ik,\beta]\right), \qquad .....(15.3.5)$$

which is an important formula from the point of view of studying the properties.

15.4 Properties of covariant curvature tensor

Property–I : R_{rijk} is skew-symmetric in the pair of first two indices, *i.e.,*

$$R_{rijk} = -R_{irjk} \quad \text{(skew-symmetric property)} \qquad .....(15.4.1)$$

The above result can easily be proved by interchanging r and i in (15.3.2) of §15.3.

Property–II : R_{rijk} is skew-symmetric in the pair of last two indices, *i.e.,*

$$R_{rijk} = -R_{rikj} \quad \text{(skew-symmetric property)} \qquad .....(15.4.2)$$

It can also be proved by interchanging j and k in (15.3.2) of §15.3.

Property–III : R_{rijk} is symmetric in two pairs (first and last) of indices, *i.e.,*

$$R_{rijk} = R_{jkri} \quad \text{(symmetric property)} \qquad .....(15.4.3)$$

If may be easily seen from (15.3.2) of §15.3 by interchanging r and j, then i and k.

Property–IV : R_{rijk} has cyclic property in last three indices, *i.e.,*

$$R_{rijk} + R_{rjki} + R_{rkij} = 0 \quad \text{(cyclic property)} \qquad .....(15.4.4)$$

Proof : We have $\qquad R_{rijk} = g_{r\alpha}\, R^{\alpha}_{.ijk}. \qquad .....(15.4.5)$

Giving cyclic rotation to i, j, k and adding, we get

$$\begin{aligned} R_{rijk} + R_{rjki} + R_{rkij} &= g_{r\alpha}\, R^{\alpha}_{.ijk} + g_{r\alpha}\, R^{\alpha}_{.jki} + g_{r\alpha}\, R^{\alpha}_{.kij} \\ &= g_{r\alpha}\left(R^{\alpha}_{.ijk} + R^{\alpha}_{.jki} + R^{\alpha}_{.kij}\right) \\ &= 0. \quad \text{[using property–II of §15.2]} \end{aligned} \qquad .....(15.4.6)$$

Property–V : (Bianchi's identity).

The differential property satisfied by covariant derivative states

$$R_{rijk,p} + R_{rikp,j} + R_{ripj,i} = 0$$

or equivalently

$$R^{\alpha}_{.ijk,p} + R^{\alpha}_{.ikp,j} + R^{\alpha}_{.ipj,k} = 0. \qquad .....(15.4.7)$$

Proof : The identity is proved conveniently, by choosing geodesic coordinate system with the pole at P_0, so that all the Christoffel symbols vanish at P_0. We recollect that by choosing so, the first covariant derivative of any tensor at P_0 reduces to the corresponding partial derivative. For example

$$\left[A^i_{,j}\right]_{\text{at } P_0} = \left(\frac{\partial A^i}{\partial x^j}\right)_{\text{at } P_0}. \qquad .....(15.4.8)$$

Now $\qquad R^{\alpha}_{.ijk} = \frac{\partial}{\partial x^j}\begin{Bmatrix}\alpha\\ ik\end{Bmatrix} - \frac{\partial}{\partial x^k}\begin{Bmatrix}\alpha\\ ij\end{Bmatrix} + \begin{Bmatrix}\beta\\ ik\end{Bmatrix}\begin{Bmatrix}\alpha\\ \beta j\end{Bmatrix} - \begin{Bmatrix}\beta\\ ij\end{Bmatrix}\begin{Bmatrix}\alpha\\ \beta k\end{Bmatrix},$

at P_0 becomes $\qquad R^{\alpha}_{.ijk} = \frac{\partial}{\partial x^j}\begin{Bmatrix}\alpha\\ ik\end{Bmatrix} - \frac{\partial}{\partial x^k}\begin{Bmatrix}\alpha\\ ij\end{Bmatrix}$

and $\qquad \left(R^{\alpha}_{.ijk}\right)_{,p} = \frac{\partial^2}{\partial x^p\,\partial x^j}\begin{Bmatrix}\alpha\\ ik\end{Bmatrix} - \frac{\partial^2}{\partial x^p\,\partial x^k}\begin{Bmatrix}\alpha\\ ij\end{Bmatrix}. \qquad .....(15.4.9)$

Similarly, after cyclic rotation to $j, k, p,$ we get

$$\left(R^{\alpha}_{.ikp}\right)_{,j} = \frac{\partial^2}{\partial x^j\,\partial x^k}\begin{Bmatrix}\alpha\\ ip\end{Bmatrix} - \frac{\partial^2}{\partial x^j\,\partial x^p}\begin{Bmatrix}\alpha\\ ik\end{Bmatrix} \qquad .....(15.4.10)$$

and $\qquad \left(R^{\alpha}_{.ipj}\right)_{,k} = \frac{\partial^2}{\partial x^k\,\partial x^p}\begin{Bmatrix}\alpha\\ ij\end{Bmatrix} - \frac{\partial^2}{\partial x^k\,\partial x^j}\begin{Bmatrix}\alpha\\ ip\end{Bmatrix}. \qquad .....(15.4.11)$

Adding these three expressions (15.4.9) to (15.4.11), we get

$$\left(R^{\alpha}_{.ijk}\right)_{,p} + \left(R^{\alpha}_{.jkp}\right)_{,j} + \left(R^{\alpha}_{.kpj}\right)_{,i} = 0. \qquad .....(15.4.12)$$

Taking inner multiplication by $g_{\alpha r}$ and remembering that the fundamental tensor behave like a constant is covariant differentiation, we obtain

$$\left(g_{\alpha r}\, R^{\alpha}_{.ijk}\right)_{,p} + \left(g_{\alpha r}\, R^{\alpha}_{.jkp}\right)_{,j} + \left(g_{\alpha r}\, R^{\alpha}_{.ipj}\right)_{,k} = 0.$$

or $$R_{rijk,p} + R_{rikp,j} + R_{ripj,k} = 0. \qquad .....(15.4.13)$$

This is Bianchy identity.

15.5 Illustrative examples

***Ex.*1.** *Prove that* $R_{1212} = -G\dfrac{\partial^2 G}{\partial u^2}$ *for the* V_2 *whose line element is* $ds^2 = du^2 + G^2\, dv^2$, *where G is a function of u and v.*

Or

Show that the component R_{1212} *of the curvature tensor for a* V_2 *with metric*

$$ds^2 = dx^2 + f(x, y)\, dy^2 \quad \text{equals} \quad -\frac{1}{2}\frac{\partial^2 f}{\partial x^2} + \frac{1}{4f}\left(\frac{\partial f}{\partial x}\right)^2.$$

Sol. For the metric

$$ds^2 = du^2 + G^2\, dv^2,$$

we have $x^1 = u, \quad x^2 = v, g_{11} = 1, \quad g_{22} = G^2, \quad g_{12} = 0, \quad g_{21} = 0. \qquad .....(1)$

Since $$R_{rijk} = \frac{1}{2}\left(\frac{\partial^2 g_{rk}}{\partial x^i\, \partial x^j} + \frac{\partial^2 g_{ij}}{\partial x^r\, \partial x^k} - \frac{\partial^2 g_{rj}}{\partial x^i\, \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^r\, \partial x^j}\right)$$
$$+ g^{\alpha\beta}\left([rk,\alpha][ij,\beta] - [rj,\alpha][ik,\beta]\right)$$

the value of R_{1212} is given by

$$R_{1212} = \frac{1}{2}\left(0 + 0 - \frac{\partial^2 g_{11}}{\partial x^2\, \partial x^2} - \frac{\partial^2 g_{12}}{\partial x^1\, \partial x^1}\right) + g^{\alpha\beta}\left([12,\alpha][21,\beta] - [11,\alpha][22,\beta]\right)$$

$$= -\frac{1}{2}\frac{\partial^2}{\partial u^2}\left(G^2\right) + g^{11}\left([12,1][21,1] - [11,1][22,1]\right)$$
$$+ g^{22}\left([12,2][21,2] - [11,2][22,2]\right)$$

$$= -\frac{1}{2}\frac{\partial}{\partial u}\left(2G\frac{\partial G}{\partial u}\right) + g^{11}(0) + g^{22}\left(\frac{1}{2}\frac{\partial g_{22}}{\partial x^1}\,\frac{1}{2}\frac{\partial g_{22}}{\partial x^1}\right)$$

$$= -\frac{1}{2}\frac{\partial}{\partial u}\left(G\frac{\partial G}{\partial u}\right) + \frac{1}{4G^2}\left[\frac{\partial}{\partial u}\left(G^2\right)\right]^2. \qquad .....(2)$$

$\therefore$ $$R_{1212} = -G\frac{\partial^2 G}{\partial u^2}. \qquad .....(3)$$

Taking $G = \sqrt{f}$ and $u = x$ the alternative form may easily be obtained.

15.6 Contraction of Riemann-Christoffel tensor-Ricci tensor

Theorem 1. *The Riemann-Christoffel tensor* $R^{\alpha}_{.ijk}$ *(or mixed curvature tensor) can be contracted in two different ways-one of these leads to a zero tensor and other to a symmetric tensor* R_{ij}*, known as Ricci-tensor.*

Proof. We have by definition

$$R^{\alpha}_{.ijk} = \frac{\partial}{\partial x^j}\begin{Bmatrix}\alpha\\ ik\end{Bmatrix} - \frac{\partial}{\partial x^k}\begin{Bmatrix}\alpha\\ ij\end{Bmatrix} + \begin{Bmatrix}\beta\\ ik\end{Bmatrix}\begin{Bmatrix}\alpha\\ \beta j\end{Bmatrix} - \begin{Bmatrix}\beta\\ ij\end{Bmatrix}\begin{Bmatrix}\alpha\\ \beta k\end{Bmatrix} \quad .....(15.6.1)$$

(i) Contracting over i and α, *i.e.,* setting $i = \alpha$, we get

$$R^{\alpha}_{.\alpha jk} = \frac{\partial}{\partial x^j}\begin{Bmatrix}\alpha\\ \alpha k\end{Bmatrix} - \frac{\partial}{\partial x^k}\begin{Bmatrix}\alpha\\ \alpha j\end{Bmatrix} + \begin{Bmatrix}\beta\\ \alpha k\end{Bmatrix}\begin{Bmatrix}\alpha\\ \beta j\end{Bmatrix} - \begin{Bmatrix}\beta\\ \alpha j\end{Bmatrix}\begin{Bmatrix}\alpha\\ \beta k\end{Bmatrix} \quad .....(15.6.2)$$

Interchanging the dummy indices α and β, the last two terms cancel out. Further, using property of Christoffel symbols

$$R^{\alpha}_{.\alpha jk} = \frac{\partial}{\partial x^j}\left(\frac{\partial}{\partial x^k}\left(\log\sqrt{g}\right)\right) - \frac{\partial}{\partial x^k}\left(\frac{\partial}{\partial x^j}\left(\log\sqrt{g}\right)\right)$$

$$= 0. \quad .....(15.6.3)$$

(ii) Contracting over k and α, *i.e.,* setting $k = \alpha$, we get

$$R^{\alpha}_{.ij\alpha} = \frac{\partial}{\partial x^j}\begin{Bmatrix}\alpha\\ i\alpha\end{Bmatrix} - \frac{\partial}{\partial x^\alpha}\begin{Bmatrix}\alpha\\ ij\end{Bmatrix} + \begin{Bmatrix}\beta\\ i\alpha\end{Bmatrix}\begin{Bmatrix}\alpha\\ \beta j\end{Bmatrix} - \begin{Bmatrix}\beta\\ ij\end{Bmatrix}\begin{Bmatrix}\alpha\\ \beta\alpha\end{Bmatrix} \quad .....(15.6.4)$$

Writing, **(Ricci tensor)**

$$R_{ij} = R^{\alpha}_{.ij\alpha} = g^{\alpha r}R_{rij\alpha}. \quad .....(15.6.5)$$

The above expression becomes

$$R_{ij} = \frac{\partial}{\partial x^j}\left(\frac{\partial}{\partial x^i}\left(\log\sqrt{g}\right)\right) - \frac{\partial}{\partial x^\alpha}\begin{Bmatrix}\alpha\\ ij\end{Bmatrix} + \begin{Bmatrix}\beta\\ i\alpha\end{Bmatrix}\begin{Bmatrix}\alpha\\ \beta j\end{Bmatrix} - \begin{Bmatrix}\beta\\ ij\end{Bmatrix}\left(\frac{\partial}{\partial x^\beta}\left(\log\sqrt{g}\right)\right)$$

$$.....(15.6.6)$$

It may easily be observed that, by interchanging i and j in (15.6.6), that

$$R_{ij} = R_{ji} \qquad \text{(symmetric property)} \quad .....(15.6.7)$$

Thus R_{ij} is a symmetric tensor and is called Ricci tensor. It is defined by (15.6.5) and (15.6.6).

Notes :

1. The contraction over α and j ($j = \alpha$) does not yield any new tensor, because

$$R^{\alpha}_{.i\alpha k} = -R^{\alpha}_{ik\alpha} = -R_{ik}$$

[using skew-symmetric property of Riemann-Christoffel tensor.]

where is Ricci tensor with negative sign.

2. If g is negative, replace $\log\sqrt{g}$ by $\log\sqrt{-g}$ in (15.6.6).

15.7 Curvature invariant-Einstein space

The curvature invariant R is defined as

$$R = g^{ij} R_{ij} = R_i^i. \qquad .....(15.7.1)$$

A space for which at every point of it

$$R_{ij} = I g_{ij}, \qquad .....(15.7.2)$$

where I is an invariant and it is called on Einstein space. The inner multiplication of (15.7.2) by g^{ij} and using (15.7.1), we get

$$R = NI. \qquad .....(15.7.3)$$

Hence for an Einstein space

$$R_{ij} = \frac{R}{N} g_{ij}. \qquad .....(15.7.4)$$

15.8 Einstein tensor

It is defined as

$$G_j^i = g^{il} R_{jl} - \frac{1}{2} R\delta_j^i, \qquad .....(15.8.1)$$

and it has a considerable importance in the theory of relativity.

Theorem 2. *The divergence of Einstein tensor vanishes, i.e.,*

$$G_{j,i}^i = 0.$$

Proof. We have from Bianchy identity

$$R_{rijk,p} + R_{rikp,j} + R_{ripj,k} = 0. \qquad ......(15.8.2)$$

Taking inner multiplication of the above relation by $g^{ij} g^{rk}$, we get

$$\left(g^{ij} g^{rk} R_{rijk}\right)_{,p} + \left(g^{ij} g^{rk} R_{rikp}\right)_{,j} + \left(g^{ij} g^{rk} R_{ripj}\right)_{,k} = 0. \qquad .....(15.8.3)$$

Using the definition of Ricci tensor and the skew-symmetric property of covariant curvature tensor, we find

$$\left(g^{ij} R_{ij}\right)_{,p} - \left(g^{ij} R_{ip}\right)_{,j} - \left(g^{rk} R_{rp}\right)_{,k} = 0.$$

or
$$\left(R\right)_{,p} - \left(R_p^j\right)_{,j} - \left(r_p^k\right)_{,k} = 0$$

or
$$\left(R\right)_{,p} = \left(R_p^j\right)_{,j} + \left(R_p^j\right)_{,j}$$

or
$$\left(R\right)_{,p} = 2R_{p,j}^j$$

or
$$R_{p,j}^j = \frac{1}{2} R_{,p} = \frac{1}{2} \frac{\partial R}{\partial x^p}, \qquad .....(15.8.4)$$

where $R_p^j = g^{ij} R_{ip}$ is the associate tensor of R_{ip}.

Now
$$G_j^i = R_j^i - \frac{1}{2} R\delta_j^i. \qquad .....(15.8.5)$$

Taking its divergence and using (15.8.4), we get

$$div\left(G^i_j\right) = G^i_{j,i}$$

$$= \left(R^i_j\right)_{,j} - \frac{1}{2}\delta^i_j R_{,i}$$

$$= \frac{1}{2}R_{,j} - \frac{1}{2}R_{,j} = 0. \qquad .....(15.8.6)$$

Hence Proved.

Theorem 3. *An Einstein space V_N ($N > 2$) has constant curvature (Curvature invariant R).*

Proof. Taking inner multiplication of Bianchy identity by $g^{ij}\, g^{rk}$, we get

$$R^j_{p,j} = \frac{1}{2}\frac{\partial R}{\partial x^p}, \qquad .....(15.8.7)$$

where $\qquad R^j_p = g^{ij} R_{ip}.$

In an Einstein space V_N, we have

$$R_{ip} = \frac{R}{N} g_{ip}. \qquad .....(15.8.8)$$

Taking inner multiplication of it by g^{ij}, we get

$$g^{ij} R_{ip} = \frac{R}{N} g^{ij} g_{ip}$$

or $$R^j_p = \frac{R}{N}\delta^j_p. \qquad .....(15.8.9)$$

Now taking covariant derivative with respect to x^j, we get

$$R^i_{p,j} = \frac{1}{N}\delta^j_p R_{,j} = \frac{1}{N}R_{,p}. \qquad .....(15.8.10)$$

Hence from (15.8.7) & (15.8.8), we conclude that

$$\left(\frac{1}{2} - \frac{1}{N}\right)\frac{\partial R}{\partial x^p} = 0, \ i.e., \ \frac{\partial R}{\partial x^p} = 0, \text{ since } N > 2 \cdot$$

or we can say that R is constant.

Thus Einstein space V_N ($N > 2$) has constant curvature.

15.9 Riemannian curvature of a V_N at a point

If A^i and B^i be any two contravariant vectors at a point of a V_N, then

$$\kappa = \frac{R_{rijk} A^r A^j B^i B^k}{\left(g_{rj} g_{ik} - g_{rk} g_{ij}\right) A^r A^j B^i B^k},$$

is called the Riemannian curvature of the space V_N associated with the vectors A^i and B^i.

It is an invariant, which is unaltered at a point, when the two vectors determining it are replaced by any linear combination of them.

***Ex.*2.** *The metric of V_2 formed by the surface of a sphere of radius a is*

$$ds^2 = a^2\, d\theta^2 + a^2 \sin^2\theta\, d\phi^2,$$

in spherical polar coordinates. Show that the curvature of the surface of the sphere is $\frac{1}{a^2}$, *which is constant.*

Sol. For the metric $\quad ds^2 = a^2\, d\theta^2 + a^2 \sin^2\theta\, d\phi^2,$

we have $\quad x^1 = \theta, \quad x^2 = \phi, \; g_{11} = a^2, \quad g_{22} = a^2 \sin^2\theta, \quad g_{12} = 0, \quad g_{21} = 0.$ .....(1)

Therefore,

$$g \equiv \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = a^4 \sin^2\theta. \qquad .....(2)$$

Now

$$R_{1212} = -\frac{1}{2}\frac{\partial^2 g_{22}}{\partial x^1\, \partial x^1} + g^{22}\left(\frac{1}{2}\frac{\partial g_{22}}{\partial x^1}\right)^2$$

$$= -\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\left(a^2\sin^2\theta\right) + \frac{1}{4a^2\sin^2\theta}\left(\frac{\partial}{\partial\theta}\left(a^2\sin^2\theta\right)\right)^2$$

$\Rightarrow$

$$R_{1212} = a^2\sin^2\theta. \qquad .....(3)$$

$\therefore$

$$\kappa = \frac{R_{1212}}{g} = \frac{a^2\sin^2\theta}{a^4\sin^2\theta} = \frac{1}{a^2}, \qquad .....(4)$$

which is curvature of the surface of the sphere.

***Ex.*3.** *Show that the Riemannian of a V_2 is uniquely determined at each point, and its value is given by*

$$\kappa = \frac{R_{1212}}{g}$$

Sol. In a two-dimensional space, at any point of it, there exists only two independent vectors. Therefore the Riemannian curvature of a V_2 is uniquely determined at each point. In a V_2 the number of independent components of R_{rijk} is 1. The value of κ can easily be found by choosing the two vectors whose components are (1, 0) and (0, 1) respectively. Then

$$\kappa = \frac{R_{1212}A^1B^2A^1B^2}{\left(g_{11}g_{22} - g_{12}g_{21}\right)A^1B^2A^1B^2} = \frac{R_{1212}}{g},$$

as $A^1 = 1$, $B^2 = 1$.

15.11 Flat space

A space for which the Riemannian curvature is identically zero at every point of it ($\kappa = 0$), is called a flat space.

Theorem 4. *The necessary and sufficient condition for a space V_N to be flat is that the Riemann-Christoffel tensor be identically zero, i.e.,* $R^{\alpha}_{.ijk} = 0$.

Proof : Necessary condition :

Let the space V_N be flat, them $\kappa = 0$ at every point of V_N, *i.e.,*

$$R_{rijk} A^r B^j B^i B^k = 0,$$

for all vectors A^i and B^i. From it we cannot jump to the conclusion that $R_{rijk} = 0$. It must be remembered that in the form $R_{rijk} A^r B^j B^i B^k$ the coefficient of the product $A^r A^j B^i B^k$ is mixed up with the coefficients of $A^j A^r B^i B^k$, $A^r A^j B^k B^i$ and $A^j A^r B^k B^i$, it is fact $R_{rijk} + R_{jirk} + R_{rkji} + R_{jkri}$ which may be obtained by interchanging the dummy indices as follows :

$$R_{rijk} A^r A^j B^i B^k = 0 \quad .....(15.11.1)$$

Also

$$R_{jirk} A^j A^r B^i B^k = 0 \quad .....(15.11.2)$$

$$R_{rkji} A^r A^j B^k B^i = 0 \quad .....(15.11.3)$$

$$R_{jkri} A^j A^r B^k B^i = 0 \quad .....(15.11.4)$$

On addition,

$$(R_{rijk} + R_{jirk} + R_{rkji} + R_{jkri}) A^r A^j B^i B^k = 0. \quad .....(15.11.5)$$

This implies that for arbitrary A^i, B^i

$$R_{rijk} + R_{jirk} + R_{rkji} + R_{jkri} = 0.$$

$$\Rightarrow \quad R_{rijk} + R_{rkji} + R_{rkji} + R_{rijk} = 0 \quad \text{[using symmetric property]}$$

$$\Rightarrow \quad 2(R_{rijk} + R_{rkji}) = 0,$$

$$\Rightarrow \quad R_{rijk} = R_{rkji}. \quad \text{[using skew-symmetric property]} \quad .....(15.11.6)$$

Interchanging *i, j* and *k* cyclically in (15.11.6), we get

$$R_{rjki} = R_{rijk}. \quad .....(15.11.7)$$

From (15.11.6) and (15.11.7), we have

$$R_{rijk} = R_{rjki} = R_{rkij}. \quad .....(15.11.8)$$

Now substituting (15.11.8) in the cyclic property–IV of §15.4

$$R_{rijk} + R_{rjki} + R_{rkij} = 0. \quad .....(15.11.9)$$

we find

$$3R_{rijk} = 0$$

$$\Rightarrow \quad R_{rijk} = 0. \quad .....(15.11.10)$$

Since,

$$R_{rijk} = g_{r\alpha} R^{\alpha}_{.ijk},$$

and $g_{r\alpha}$ is an arbitrary, it implies that

$$R^{\alpha}_{.ijk} = 0. \quad .....(15.11.11)$$

Sufficeint condition : Conversely, if $R^{\alpha}_{.ijk} = 0$. *i.e.,* $R_{rijk} = 0$, then it in clear $\kappa = 0$.

Hence the theorem.

Note : Taking inner multiplication of $A_{i,j} = 0$ with $\frac{dx^j}{dt}$, we get

$$A_{i,j} \frac{dx^j}{dt} = 0$$

$\Rightarrow \qquad \dfrac{\delta A_i}{\delta t} = 0.$

This shows that in a flat space the property of parallelism is independent of the choice of a curve. Thus parallelism is an absolute property of a flat space.

15.12 Isotropic point

An isotropic point in Riemannian space is a point at which the Riemannian curvature is independent of the vectors A^i and B^i associated to it.

It implies that $\qquad [R_{rijk} - \kappa\,(g_{rj}\, g_{ik} - g_{ij}\, g_{rk})]\, A^r\, A^j\, B^i\, B^k = 0,$ .....(15.12.1)

for all vectors A^i and B^i at the isotropic point.

Now we define a tensor T_{rijk} by the equation

$$T_{rijk} = R_{rijk} - \kappa\,(g_{rj}\, g_{ik} - g_{ij}\, g_{rk}). \qquad .....(15.2.2)$$

then (15.2.1) reduces to $\qquad T_{rijk}\, A^r\, A^j\, B^i\, B^k = 0.$ .....(15.12.3)

Proceeding parallel to Theorem 4, we conclude from (15.12.3) that will be true for any vectors A^i and B^i if

$$T_{rijk} + T_{jirk} + T_{rkji} + T_{jkri} = 0. \qquad ......(15.12.4)$$

According to (15.12.2) we see that the tensor T_{rijk} satisfies the same four properties as by R_{rijk} viz.

$$T_{rijk} = -\,T_{irjk}\,; \quad T_{rijk} = -\,T_{rikj}\,; \quad T_{rijk} = T_{jkri}$$

and $\qquad T_{rijk} + T_{rjki} + T_{rkij} = 0.$ .....(15.12.5)

Hence repeating the same steps as in (15.12.3), replacing R_{rijk} by T_{rijk}, we finally get

$$T_{rijk} = 0. \qquad .....(15.12.6)$$

Thus from (15.12.2) to (15.12.6), it follows that at an isotropic point the Riemannian curvature satisfies the condition

$$R_{rijk} = \kappa\,(g_{rj}\, g_{ik} - g_{ij}\, g_{rk}). \qquad .....(15.12.7)$$

Now we state an important theorem due to Schur.

Theorem 5. (Schur's theorem).

If a Riemannian space V_N $(N > 2)$ is isotropic at each point in a region, then the Riemannian curvature is constant throughout that region.

(Such a space V_N is called a space of constant curvature).

Proof. We know that if the Riemannian space V_N is isotropic at each point in a region, then

$$R_{rijk} = \kappa\,(g_{rj}\, g_{ik} - g_{ij}\, g_{rk}), \qquad .....(1)$$

where κ is the function of coordinates x^i.

Taking covariant differentiation of (1) with respect to x^i, we get

$$R_{rijk,t} = (g_{rj}\, g_{ik} - g_{ij}\, g_{rk})\, \kappa_{,t}. \qquad .....(2)$$

since the covariant derivative of the metric tensor vanishes.

Permuting the subscripts j, k, t cyclically, we find

$$R_{rikt,j} = (g_{rk}\, g_{it} - g_{ik}\, g_{rt})\, \kappa_{,j} \qquad .....(3)$$

and

$$R_{ritj,k} = (g_{rt}\, g_{ij} - g_{it}\, g_{rj})\, \kappa_{,k}\,. \qquad .....(4)$$

Adding equations (2), (3) and (4) and using Bianchi identity, we get

$$(g_{rj}\, g_{ik} - g_{ij}\, g_{rk})\, \kappa_{,t} + (g_{rk}\, g_{it} - g_{ik}\, g_{rt})\, \kappa_{,j} + (g_{rt}\, g_{ij} - g_{it}\, g_{rj})\, \kappa_{,k} = 0. \qquad .....(5)$$

Taking inner multiplication by $g^{rj}\, g^{ik}$, we find

$$\left(N^2 - \delta_k^j \delta_j^k\right)\kappa_{,t} + \left(\delta_k^j \delta_t^k - N\delta_t^i\right)\kappa_{,j} + \left(\delta_r^j \delta_j^k - N\delta_t^k\right)\kappa_{,k} = 0$$

$$\Rightarrow \qquad \left(N^2 - N\right)\kappa_{,t} + (1-N)\delta_t^j \kappa_{,j} + (1-N)\delta_t^k \kappa_{,k} = 0$$

$$\Rightarrow \qquad (1 - N)\,[\kappa_{,t} + \kappa_{,t} - N\kappa_{,t}] = 0$$

$$\Rightarrow \qquad (1 - N)\,(2 - N)\,\kappa_{,t} = 0. \qquad .....(6)$$

But $N > 2$, therefore $\kappa_{,t} = 0$, but this is simply $\dfrac{\partial \kappa}{\partial x^t} = 0$. Hence it follows that κ is a constant. Such a V_N is called **space of constant curvature**. Hence the theorem is proved.

15.13 Illustrative examples

Ex.4. *If the metric of a two dimensional flat space is* $f(r)\,[(dx^1)^2 + (dx^2)^2]$, *where* $(r)^2 = (x^1)^2 + (x^2)^2$, *show that* $f(r) = c\,(r)^k$, *where c and k are constants.*

Sol. We have the metric

$$ds^2 = f(r)\,\{(dx^1)^2 + (dx^2)^2\},$$

therefore

$$g_{11} = f(r), \quad g_{22} = f(r), \quad g_{12} = 0, \quad g_{21} = 0. \qquad .....(1)$$

The only non-zero component of R_{rijk} in a V_2 is R_{1212} which in the present case is given by

$$R_{1212} = -\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^1\, \partial x^1} + \frac{\partial^2 f}{\partial x^2\, \partial x^2}\right) + \frac{1}{2f}\left\{\left(\frac{\partial f}{\partial x^1}\right)^2 + \left(\frac{\partial f}{\partial x^2}\right)^2\right\} \qquad .....(2)$$

In a V_2,

$$\kappa = \frac{R_{1212}}{g}.$$

For a flat space $\kappa = 0$, *i.e.*

$$R_{1212} = 0.$$

Hence

$$\frac{\partial^2 f}{\partial x^1\, \partial x^1} + \frac{\partial^2 f}{\partial x^2\, \partial x^2} = \frac{1}{f}\left[\left(\frac{\partial f}{\partial x^1}\right)^2 + \left(\frac{\partial f}{\partial x^2}\right)^2\right]. \qquad .....(3)$$

But

$$f = f(r)\; (r)^2 = (x^1)^2 + (x^2)^2. \qquad .....(4)$$

Therefore, changing (3) to polar coordinates, we get

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} = \frac{1}{f}\left(\frac{\partial f}{\partial r}\right)^2. \qquad .....(5)$$

Let, $$F = \log f(r). \qquad .....(6)$$

Then (5) reduces to

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} = 0$$

$$\Rightarrow \qquad \frac{d}{dr}\left(r\frac{dF}{dr}\right) = 0, \ \textit{i.e.,}\ \ r\frac{dF}{dr} = \kappa \text{ (constant)}$$

or $$\frac{dF}{dr} = \frac{\kappa}{r}, \ \textit{i.e.,}\ \ F = \kappa \log\frac{r}{A}. \qquad .....(7)$$

From (6) and (7), we conclude that $f(r) = c\,(r)^k$, where c and k are constants.

***Ex.*5.** *Prove that, in space V_N of constant curvature κ,*

(i) $R_{ij} = -(N-1)\,\kappa g_{ij}$, *and*

(ii) $R = -N(N-1)\,\kappa$.

Deduce that a space of constant curvature is an Einstein space.

Sol. In the space of constant curvature

(i) $$R_{rijk} = \kappa\,(g_{rj}\,g_{ik} - g_{ij}\,g_{rk}). \qquad .....(1)$$

Taking inner product with g^{rk}, we get

$$g^{rk}\,R_{rijk} = \kappa\left(\delta^k_j g_{ik} - g_{ij}N\right), \qquad .....(2)$$

Using the definition of Ricci tensor, we have

$$R_{ij} = \kappa\,(1-N)\,g_{ij}. \qquad .....(3)$$

(ii) Again taking the inner product of (1) by g^{ij}, we get

$$g^{ij}\,R_{ij} = \kappa\,(1-N)\,g^{ij}\,g_{ij}$$

$$\Rightarrow \qquad R = \kappa\,(1-N)\,N. \qquad .....(4)$$

From (3) and (4), it follows that

$$R_{ij} = \frac{R}{N}\,g_{ij}. \qquad .....(5)$$

This shows that a space V_N $(N > 2)$ of constant curvature is an Einstein space.

***Ex.*6.** *In a V_2, prove that*

$$R\,(g_{ij}\,g_{rj} - g_{ij}\,g_{rk}) = -2\,R_{rijk}$$

and hence that $$R_g = -2\,R_{1212}.$$

In this case, prove also that the components of Ricci tensor are proportional to the components of metric tensor that is

$$g\,R_{ik} = -R_{1212}\,g_{ik}.$$

Sol. Since a V_2 is isotropic, the equation

$$R_{rijk} = \kappa\,(g_{rj}\,g_{ik} - g_{ij}\,g_{rk}), \qquad .....(1)$$

holds throughout any V_2.

Using the method of example 5, we get

$$R_{ij} = -\kappa\, g_{ij}, \quad N = 2 \qquad .....(2)$$

and
$$R = -2\kappa. \qquad .....(3)$$

From (1) and (3), $\quad -2R_{rijk} = R\,(g_{rj}\, g_{ik} - g_{ij}\, g_{rk})$

$\Rightarrow \quad -2R_{1212} = R\,(g_{11}\, g_{22} - g_{12}{}^2)$

$\Rightarrow \quad -2R_{1212} = R.g. \qquad .....(4)$

Now from equation (2), (3) and (4),

$$gR_{ij} = \frac{R}{2} gg_{ij} = -R_{1212} g_{ij}. \qquad .....(5)$$

This gives the required result.

15.14 Self-learning exercises

1. Define Riemann-Christoffel tensor.
2. Write the necessary and sufficeint conditions for covariant differentiation of a vector A_i to be commutative.
3. Define covariant curvature tensor.
4. What is Ricci tensor ?
5. What is Bianchy identity ?
6. Define Einstein tensor.
7. What is the divergence of Einstein tensor ?
8. What do you mean by flat space ?
9. Write the statement of Schur's theorem.

15.14 Summary

In this unit we have studied the commutativity of covariant differentiation of vectors and defined Riemann-Christoffel tensor. On the basis of this we have defined covariant curvature tensor. The properties of covariant curvature tensor are also given. The contraction in Riemann-Christoffel tensor gives Ricci tensor. Then the Einstein space has been defined. Bianchy identity and Einstein tensor have also been studied. The divergence of Einstein tensor vanishes. A space for which Riemann curvature is identically zero at every point of it, is called flat space.

15.15 Answers to self-learning exercises

1. § 15.1	**2.** § 15.1	**3.** § 15.3
4. § 15.6	**5.** § 15.4	**6.** § 15.8
7. § 15.8	**8.** § 15.11	**9.** § 15.12

15.16 Exercises

1. Prove that

$$B_{l,mn} - B_{l,nm} = R^{p}_{lmn} B_p,$$

where B_p is an arbitrary covariant tensor of rank 1 and deduce that R^{p}_{lmn} is a tensor.

2. Define Riemann's symbols of first and second kind. If B_i are components of a vector, prove that

$$B_{i,jk} - B_{i,kj} = B_{\alpha} R^{\alpha}_{ijk}.$$

3. Show that the space of constant curvature is Einstein space.

4. For a V_2 space, prove that

$$gR_{ij} = -g_{ij} R_{1212} \quad \text{and} \quad gR = -2R_{1212}.$$

Hence deduce that every V_2 is an Einstein space.

5. Show that the number of independent components of the covariant curvature tensor in a space of N-dimension is

$$\frac{1}{12} N^2 \left(N^2 - 1\right).$$

□ □ □

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