MA/MSc MT-03

Vardhaman Mahaveer Open University, Kota

Differential Equations, Calculus of Variations and Special Functions

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Editor
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Writers

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MNIT, Jaipur
2. **Dr. Atul Garg**
Lecturer, Deptt. of Mathematics
Seth Moti Lal PG College, Jhunjhunu
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Differential Equations, Calculus of Variations and Special Functions

PREFACE

The Present book entitled ***"Differential Equations, Calculus of Variations and Special Functions"*** *has been designed so as to cover the unit-wise syllabus of Mathematics-Third paper for M.A./M.Sc. (Previous) students of Vardhaman Mahaveer Open University, Kota. It can also be used for competitive examinations. The basic principles and theory have been explained in a simple, concise and lucid manner. Adequate number of illustrative examples and exercises have also been included to enable the students to grasp the subject easily. The units have been written by various experts in the field. The unit writers have consulted various standard books on the subject and they are thankful to the authors of these reference books.*

Unit 1 : Non-Linear Ordinary Differential Equations of Particular Forms and Riccati's Equation

Structure of the Unit

1.0 Objective

The purpose of this unit is to discuss various methods for solving some particular forms of second and higher order non-linear differential equations. The methods for solving exact non-linear differential equations and Riccati's equation are also discussed.

1.1 Introduction

In earlier classes we studied a great deal about linear differential equations of second and higher orders when coefficient may or may not be constant. It is a known fact that due to superimposition of linearly independent solutions, it is easy to solve linear differential equation and we have well established theories for such types of equations.

On the other hand, the non-linear differential equations are difficult to handle. In the case of some first order equations, we have well established methods. However, there is no known general method for solving second and higher order non linear differential equations. It is only some particular forms that may be reduced to linear equations by suitable transformation and integrated to yield compact results. The aim of this unit is to study those easily integrable non-linear equations.

Next we shall discuss the general solution of Riccati's equation. The solution of this equation when one, two or three particular solutions are known will also be discussed.

1.2 Exact Non-linear Differential Equations

There is no simple method for testing the exactness of non-linear differential equations as in the case of linear equations. One possible method is that if the terms of the equation be grouped, by inspection, in such a way that they become perfect differential and their integrals may be written directly. The other method of obtaining the integral of an exact differential equation, which is applicable both for linear and non-linear equations is explained below.

Let $s = f(x)$ be a differential equation of n^{th} order. If it is an exact deferential equation it should be derived merely by differentiation, so as to contain $\frac{d^n y}{dx^n}$ in the first degree. Now we write the equation in the form $sdx = f(x)\, dx$ and will integrates assuming that as if $\frac{d^{n-1} y}{dx^{n-1}}$ were the only variable in the differential equation and $\frac{d^n y}{dx^n}$ is its differential coefficient.

Denoting the result by s_1 then $sdx - ds_1$ will contain differential coefficients at the most upto $(n-1)^{\text{th}}$ order. Restriction of taking $\frac{d^{n-1} y}{dx^{n-1}}$ as the only variable should be removed while finding ds_1.

Repeating the above process as many times as necessary, we shall finally get

$$sdx - ds_1 - ds_2 - \ldots = 0$$

or $$ds_1 + ds_2 + \ldots = sdx$$

On integration, we get

$$s_1 + s_2 \ldots = \int sdx = \int f(x)\, dx$$

Ex.1. Show that the differential equation

$$\boldsymbol{y+3x\frac{dy}{dx}+2y\left(\frac{dy}{dx}\right)^3+\left(x^2+2y^2\frac{dy}{dx}\right)\frac{d^2y}{dx}=0}$$

is an exact equation and find its first integral.

Sol. The given equation can be written as

$$sdx \equiv \left[x^2\frac{d^2y}{dx^2}+2y^2\frac{dy}{dx}\frac{d^2y}{dx^2}+2y\left(\frac{dy}{dx}\right)^3+3x\frac{dy}{dx}+y\right]dx=0$$

Now here the first three terms are the differentiation of

$$x^2\frac{dy}{dx}+y^2\left(\frac{dy}{dx}\right)^2$$

So putting $$s_1=\left\{x^2\frac{dy}{dx}+y^2\left(\frac{dy}{dx}\right)^2\right\}$$

On differentiation, we get

$$ds_1=\left\{x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+2y\left(\frac{dy}{dx}\right)^3+2y^2\frac{dy}{dx}\frac{d^2y}{dx^2}\right\}dx$$

Thus $$sdx-ds_1=\left[y+x\frac{dy}{dx}\right]dx \qquad .....(1)$$

Again the terms on R.H.S. are the differentiation of xy, so putting

$$s_2=xy$$

On differentiation, we get

$$ds_2=\left[x\frac{dy}{dx}+y\right]dx \qquad .....(2)$$

From (1) and (2), we finally get

$$sdx-ds_1-ds_2=0$$

which on integration gives

$$s_1+s_2=\text{constant}$$

This relation shows that the given equation is exact and the first integral will be given by

$$x^2\frac{dy}{dx}+y^2\left(\frac{dy}{dx}\right)^2+xy=c.$$

Ex.2. Solve the following differential equation :

$$\boldsymbol{2\sin x\frac{d^2y}{dx^2}+2\cos x\frac{dy}{dx}+2\sin x\frac{dy}{dx}+2y\cos x=\cos x}$$

Sol. We can writte the given equation as

$$sdx \equiv \left[2\sin x\frac{d^2y}{dx^2}+2\cos x\frac{dy}{dx}+2\sin x\frac{dy}{dx}+2y\cos x\right]dx=\cos x\,dx$$

Here first term of above equation will arise from the differentiation of $2\sin x\frac{dy}{dx}$, so putting

$$s_1 = 2\sin x\frac{dy}{dx}$$

which implies that $$ds_1 = \left[2\sin x\frac{d^2y}{dx^2} + 2\cos x\frac{dy}{dx}\right]dx$$

Thus $$sdx - ds_1 = \left[2\sin x\frac{dy}{dx} + 2y\cos x\right]dx$$

Again putting

$$s_2 = 2y\sin x$$

On differentiation, we get

$$ds_2 = \left[2\sin x\frac{dy}{dx} + 2y\cos x\right]dx$$

$\therefore$ $$sdx - ds_1 - ds_2 = 0$$

This shows that the given equation is exact and on integrating, we get

$$s_1 + s_2 = \int sdx = \int \cos x\, dx$$

or $$2\sin x\frac{dy}{dx} + 2y\sin x = \sin x + 2c_1$$

or $$\frac{dy}{dx} + y = \frac{1}{2} + c_1\text{cosec } x$$

This is a linear differential equation of first order whose integrating factor (*I.F.*) is e^x

Thus its solution is

$$y.(I.F.) = \int\left(\frac{1}{2} + c_1\text{cosec } x\right)(I.F.)dx + c_2$$

or $$y\,e^x = \frac{1}{2}e^x + c_1\int e^x\text{cosec } x\,dx + c_2$$

Ex.3. Solve $$2x^2\cos y\frac{d^2y}{dx^2} - 2x^2\sin y\left(\frac{dy}{dx}\right)^2 + x\cos y\frac{dy}{dx} - \sin y = \log x$$

Sol. The given equation is

$$sdx \equiv \left[2x^2\cos y\frac{d^2y}{dx^2} - 2x^2\sin y\left(\frac{dy}{dx}\right)^2 + x\cos y\frac{dy}{dx} - \sin y\right]dx = \log x\,dx \qquad .....(3)$$

Let $$s_1 = 2x^2\cos y\frac{dy}{dx}$$

So that $$ds_1 = \left[2x^2\cos y\frac{d^2y}{dx^2} - 2x^2\sin y\left(\frac{dy}{dx}\right) + 4x\cos y\frac{dy}{dx}\right]dx$$

$\therefore \qquad sdx - ds_1 = \left[-3x\cos y\frac{dy}{dx} - \sin y\right]dx$

Again let $\qquad s_2 = -\,3x\sin y$

So that $\qquad ds_2 = \left[-3x\cos y\frac{dy}{dx} - 3\sin y\right]dx$

$\therefore \qquad s\,dx - ds_1 - ds_2 = 2\sin y\,dx$

Hence the equation is not exact.

So dividing the given equation (3) by x^2, we get

$$sdx \equiv \left[2\cos y\frac{d^2y}{dx^2} - 2\sin y\left(\frac{dy}{dx}\right)^2 + \frac{1}{x}\cos y\frac{dy}{dx} - \frac{1}{x^2}\sin y\right]dx = \frac{\log x}{x^2}dx$$

Now let $\qquad s_1 = 2\cos y\frac{dy}{dx}$

so that $\qquad ds_1 = \left[2\cos y\frac{d^2y}{dx} - 2\sin y\left(\frac{dy}{dx}\right)^2\right]dx$

$\therefore \qquad sdx - ds_1 = \left[\frac{1}{x}\cos y\frac{dy}{dx} - \frac{1}{x^2}\sin y\right]dx$

Again let $\qquad s_2 = \frac{1}{x}\sin y$

So that $\qquad ds_2 = \left[\frac{1}{x}\cos y\frac{dy}{dx} - \frac{1}{x^2}\sin y\right]dx$

$\therefore \qquad sdx - ds_1 - ds_2 = 0$

Hence the equation is exact, and

$$ds_1 + ds_2 = sdx = \frac{\log x}{x^2}dx$$

Integrating we get

$$s_1 + s_2 = \int \frac{1}{x^2}\log x\,dx + c_1$$

$$2\cos y\frac{dy}{dx} + \frac{1}{x}\sin y = -\frac{1}{x}(\log x + 1) + c_1 \qquad .....(4)$$

Let $\sin y = u$. Then

$$\cos y\frac{dy}{dx} = \frac{du}{dx}$$

$\therefore$ (4) reduces to

$$\frac{du}{dx} + \frac{u}{2x} = -\frac{1}{2x}(\log x + 1) + \frac{c_1}{2} \qquad .....(5)$$

which is linear with

$$I.F. = e^{\frac{1}{2}\int \frac{1}{x}dx} = \sqrt{x}$$

Hence the solution of (5) is

$$u\sqrt{x} = -\frac{1}{2}\int \frac{(\log x + 1)\sqrt{x}}{x}\,dx + \frac{c_1}{2}\int \sqrt{x}\,dx + c_2$$

or $$\sqrt{x}\sin y = -\frac{1}{2}\int (w+1)e^{w/2}dw + \frac{c_1}{3}x^{3/2} + c_2\text{, where } w = \log x$$

$$= -(w+1)e^{w/2} + 2e^{w/2} + \frac{c_1}{3}x^{3/2} + c_2$$

$$= -(\log x + 1)\sqrt{x} + 2\sqrt{x} + \frac{c_1}{3}x^{3/2} + c_2$$

or $$\sin y = -\log x + 1 + \frac{c_1}{3}x + c_2 x^{-1/2}$$

which is the required solution.

Ex.4. Solve $$x^2 y\frac{d^2y}{dx^2} + \left(x\frac{dy}{dx} - y\right)^2 = 0$$

Sol. The given equation is

$$sdx \equiv \left[x^2 y\frac{d^2y}{dx^2} + x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + y^2\right]dx = 0 \qquad .....(6)$$

Let $$s_1 = x^2 y\frac{dy}{dx}$$

$\Rightarrow$ $$ds_1 = \left[x^2 y\frac{d^2y}{dx^2} + x^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx}\right]dx$$

So that $$sdx - ds_1 = \left[-4xy\frac{dy}{dx} + y^2\right]dx$$

Again let $$s_2 = -2xy^2$$

So that $$ds_2 = \left[-4xy\frac{dy}{dx} - 2y^2\right]dx$$

$\therefore$ $$sdx - ds_1 - ds_2 = 3y^2\,dx$$

Hence the equation is not exact.

Therefore dividing the given equation (6) by x^2, we get

$$sdx = \left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - \frac{2y}{x}\frac{dy}{dx} + \frac{y^2}{x^2}\right]dx = 0$$

Now let $$s_1 = y\frac{dy}{dx}$$

Then $$ds_1 = \left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 \right] dx$$

So that $$sdx - ds_1 = \left[-\frac{2y}{x}\frac{dy}{dx} + \frac{y^2}{x^2} \right] dx$$

Let $$s_2 = -\frac{y^2}{x} \quad \text{so that} \quad ds_2 = \left(-\frac{2y}{x}\frac{dy}{dx} + \frac{y^2}{x^2} \right) dx$$

Hence $sdx - ds_1 - ds_2 = 0$

or $ds_1 + ds_2 = sdx = 0$

or $s_1 + s_2 = c_1$

or $$y\frac{dy}{dx} - \frac{y^2}{x} = c_1 \qquad .....(7)$$

Let $$u = \frac{y^2}{2} \quad \text{so that} \quad y\frac{dy}{dx} = \frac{du}{dx}$$

Hence equation (7) becomes

$$\frac{du}{dx} - \frac{2}{x}u = c_1 \qquad .....(8)$$

which is linear with $I.F. = e^{\int(-2/x)dx} = \frac{1}{x^2}$

Thus solution of (8) is

$$\frac{u}{x^2} = -\frac{c_1}{x} + c_2 \quad \text{or} \quad \frac{y^2}{2} = x(-c_1 + c_2 x)$$

or $y^2 = x(Ax - B)$,

where A and B are arlitrary corstants.

1.3 Riccati's Equation

Originally, the name Riccati's equation was given to the differential equation

$$\frac{dy}{dx} + by^2 = cx^m \qquad .....(1)$$

where b are c are constants. Equation (1) can be written in the form

$$y_1 + by^2 = cx^m \qquad .....(2)$$

where suffixes denotes differentiation w.r.t. x

The more general form of (2) is

$$xy_1 - ay + by^2 = cx^m \qquad .....(3)$$

which can be easily reduced to the form

$$\frac{du}{dt} + \frac{b}{a}u^2 = \frac{c}{a}z^{(m/a)-2} \qquad .....(4)$$

by using the substitution $t = x^a$ and then changing the variable y to u by substitution $y = ut$.

The Equation (4) can be easily written in the form

$$y_1 = P + Qy + Ry^2 \qquad .....(5)$$

where P, Q and R are function of x.

The equation (5) is known as the **generalised Riccati's equation.**

French Mathematician Liouville, in 1841, proved that equation (5) is one of the simplest differential equation of the first order and first degree that can not, in general be integrated by quadratures. Due to historical and theoretical importance and its usefulness in Differential Geometry, the study of Riccati's equation becomes quite useful.

1.3.1 General solution of Riccati's equation

Equation (5) can be reduced to a second order linear differential equation by introducing another dependent variable S such that

$$y = \frac{S_1}{RS} = -S_1 (RS)^{-1} \qquad .....(6)$$

On differentiation, we get

$$y_1 = -S_2(RS)^{-1} + S_1(RS)^{-2}[R_1S + RS_1] \qquad .....(7)$$

where a subscript denote differentiation with respect to x.

Substituting (6) and (7) in (5), we get

$$-\frac{S_2}{RS} + \frac{R_1S_1}{R^2S} + \frac{S_1^2}{RS^2} = P + Q\left[-\frac{S_1}{RS}\right] + R\left[\frac{S_1^2}{R^2S^2}\right]$$

or $\qquad -RS_2 + R_1S_1 = PR^2S - QS_1R$

or $\qquad RS_2 - (QR + R_1)S_1 + PR^2S = 0 \qquad .....(8)$

This is linear differential equation of second order. We know that the general solution of (8) is of the form

$$S = Af(x) + Bg(x) \qquad .....(9)$$

where A and B are arbitrary constants and $f(x)$, $g(x)$ are two linearly independent integrals.

Now, from (6) and (9), we get

$$y = -\frac{[Af_1 + Bg_1]}{R[Af + Bg]} = -\frac{(A/B)f_1 + g_1}{R[(A/B)f + g]}$$

which is of the form

$$y = -\frac{cf_1(x) + g_1(x)}{R[cf(x) + g(x)]} \qquad .....(10)$$

where $c = A/B$ is an arbitrary constant. Hence the general solution of (5) is (10).

1.3.2 Theorem : The cross ratio of any four particular integrals of a Riccati's equation is independent of *x*

Proof : We know that the general solution of Riccati's equation

$$y_1 = P + Qy + Ry^2 \qquad .....(11)$$

is of the form $$y = -\frac{cf_1 + g_1}{R[cf + g]} \qquad .....(12)$$

where f_1, g_1, f, g are appropriate functions of x and c is an arbitrary constant.

Let p(x), q(x), r(x) and s(x) are four particular solutions of (11) obtained from (12) by giving four different values of c, say α, β, γ, δ.

Then $$p(x) = -\frac{[\alpha f_1 + g_1]}{R[\alpha f + g]}$$

$$q(x) = -\frac{[\beta f_1 + g_1]}{R[\beta f + g]}$$

$$r(x) = -\frac{[\gamma f_1 + g_1]}{R[\gamma f + g]}$$

$$s(x) = -\frac{[\delta f_1 + g_1]}{R[\delta f + g]}$$

Then $$p - q = \frac{(\alpha - \beta)[fg_1 - f_1 g]}{R[\alpha f + g][\beta f + g]}$$

$$r - s = \frac{(\gamma - \delta)[fg_1 - f_1 g]}{R[\gamma f + g][\delta f + g]}$$

$$p - s = \frac{(\alpha - \delta)[fg_1 - f_1 g]}{R[\alpha f + g][\delta f + g]}$$

$$r - q = \frac{(\gamma - \beta)[fg_1 - f_1 g]}{R[\gamma f + g][\beta f + g]}$$

Thus $$\frac{(p-q)(r-s)}{(p-s)(r-q)} = \frac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\delta)(\gamma-\beta)} = k(\text{say})$$

when k is independent of x. This shows that the cross-ratio of any four particular solutions of a Riccati's equation is independent of x.

1.3.3 Method of solution of Riccati's equation when one particular solutions is known

Let $p(x)$ be the known particular solution of Riccati's's equation

$$y_1 = P + Qy + Ry^2 \qquad .....(13)$$

So that $$p_1 = P + Qp + Rp_2$$

Let u be the another dependent variable such that

$$y = p(x) + \frac{1}{u} \qquad .....(14)$$

then equation (13) reduces

$$p_1 - \frac{u_1}{u^2} = P + Q\left(p + \frac{1}{u}\right) + R\left(p^2 + \frac{2p}{u} + \frac{1}{u^2}\right) \qquad .....(15)$$

Using (14) and (15) in (13), we get

$$\frac{u_1}{u^2} = -\frac{Q}{u} - R\left[\frac{2p}{u} + \frac{1}{u^2}\right]$$

or $$u_1 + (Q + 2pR)\, u = -R$$

which is a linear differential equation of first order and first degree in u and x. Its integrating factor is given by

$$I.F. = e^{\int (Q+2Rq)\, dx}$$

and hence the required general solution is

$$ue^{\int (Q+2Rq)\, dx} = \int Re^{\int (Q+2Rq)\, dx} dx + c$$

where c is an arbitrary constant.

1.3.4 Method of solution of Riccati's equation when two particular solutions are known

Let $p(x)$ and $q(x)$ be the two know particular solutions of Riccati's equation

$$y_1 = P + Qy + Ry^2 \qquad .....(16)$$

so that $$p_1 = P + Qp + Rp^2 \qquad .....(17)$$

$$q_1 = P + Qq + Rq^2 \qquad .....(18)$$

From (16) and (17), we get

$$y_1 - p_1 = (y - p)\, Q + (y^2 - p^2)R$$

or $$y_1 - p_1 = (y - p)\, [Q + (y + p)R]$$

or $$\frac{y_1 - p_1}{y - p} = Q + (y + p)R \qquad .....(19)$$

Similarly from (16) and (18), we get

$$\frac{y_1 - q_1}{y - q} = Q + (y + q)R \qquad .....(20)$$

From (19) and (20), we get

$$\frac{y_1 - p_1}{y - p} - \frac{y_1 - q_1}{y - q} = (p - q)R$$

On integration, we get

$$\log (y - p) - \log (y - q) = c + \int(p - q)\, Rdx$$

which is the required general solution.

1.3.5 Method of solution of Riccati's equation when three particular solutions are known

Let $p(x)$. $q(x)$ and $r(x)$ be the three known particular solutions of Riccali's equation

$$y_1 = P + Qy + Ry^2$$

and the corresponding values of c be α, β and γ. Then by Theorem 1.3.2, we can write

$$p = -\frac{[\alpha f_1 + g_1]}{R[\alpha f + g]}$$

$$q = -\frac{[\beta f_1 + g_1]}{R[\beta f + g]}$$

$$r = -\frac{[\gamma f_1 + g_1]}{R[\beta f + g]}$$

then, we have $$\frac{(p-q)(r-y)}{(r-q)(p-y)} = k(\text{constant})$$

where k is independent of x. This is the required solution of Riccati's equation when three particular solutions are known.

Ex.1. solve $y_1 = \cos x - y \sin x + y^2$

Sol. Taking $y = \sin x$ so that $y_1 = \cos x$. Substituting these in the given equation, we get

$$\cos x = \cos x - \sin^2 x + \sin^2 x$$

This shows that $y = \sin x$ is a particular solution of given equation.

Now taking $$y = \sin x + \frac{1}{u} \quad \text{so that} \quad y = \cos x - \frac{u_1}{u^2}$$

Using these in given equation, we get

$$\cos x - \frac{u_1}{u^2} = \cos x - \sin x\left(\sin x + \frac{1}{u}\right) + \left(\sin x + \frac{1}{u}\right)^2$$

or $$-\frac{u_1}{u^2} = \frac{\sin x}{u} + \frac{1}{u^2}$$

or $$\frac{du}{dx} + u \sin x = -1 \quad .....(21)$$

Equation (21) is a linear equation of first order whose integrating factor is

$I.F. = e^{\int \sin x\, dx} = e^{-\cos x}$ and hence the solution of (21) is

$$u.\, e^{-\cos x} = c - \int e^{-\cos x}\, dx \quad .....(22)$$

Now putting the value of

$$u = \frac{1}{(y - \sin x)}$$

in equation (22), we get

$$\frac{-e^{\cos x}}{y - \sin x} = c - \int e^{-\cos x} dx$$

which is the required solution of given equation.

Ex.2. Find the general solution of the Riccati's equation

$$\boldsymbol{\frac{dy}{dx} = 2 - 2y + y^2}$$

whose one particular solution is (1 + tan x).

Sol. The given equation is

$$\frac{dy}{dx} = 2 - 2y + y^2 \quad .....(23)$$

Since $(1 + \tan x)$ is a given particular solution then taking

$$y = (1 + \tan x) + \frac{1}{u} \quad \text{so that} \quad y_1 = \sec^2 x - \frac{1}{u^2}\frac{du}{dx} \quad .....(24)$$

Putting (24) in (23), we get

$$-\frac{1}{u^2}\frac{du}{dx} = \frac{1}{u^2} + \frac{2\tan x}{u}$$

or $$\frac{du}{dx}+\left(2\tan x\right)u=-1$$

It is a linear differential equation of first order having integrating factor

$$I.F.=e^{\int(2\tan x)dx}=e^{2\log\sec x}=\sec^2 x$$

Hence the solution is

$$u\sec^2 x=c-\int\sec^2 x\,dx=c-\tan x \qquad .....(25)$$

From (24) and (25), the required general solution is

$$y=1+\tan x+\frac{\sec^2 x}{c-\tan x}$$

Ex.3. Show that there are two values of the constant for which $\frac{k}{x}$ is an integral of $x^2(y_1+y^2)=2$, and hence obtain the general solution.

Sol. Rewriting the given equation in the standard Riccati's form as

$$y_1=P+Qy+Ry^2 \qquad .....(26)$$

$$y_1=\left(\frac{2}{x^2}\right)-y^2 \qquad .....(27)$$

Let $p(x)$ and $q(x)$ are two particular integrals of (26), than by §1.3.4, we have

$$\log\left[\frac{(y-p)}{(y-q)}\right]=c+\int\left(p-q\right)Rdx \qquad .....(28)$$

Now let $\qquad y=\frac{k}{x}$ so that $y_1=-\frac{k}{x^2}$

Substituting these in (27), we get

$$-\frac{k}{x^2}=\frac{2}{x^2}-\frac{k^2}{x^2} \quad \text{or } k^2-k-2=0 \text{ so that. } k=2,-1$$

Hence $\frac{2}{x}$ and $-\frac{1}{x}$ are two particular solutions of (27)

Now taking

$$p\left(x\right)=\frac{2}{x} \text{ and } q\left(x\right)=-\frac{1}{x} \qquad .....(29)$$

On comparing (26) and (27), we get $R=-1$ .....(30)

Using (29) and (30) in (28), we get

$$\log\frac{xy-2}{xy+1}=\log k+\int\left(\frac{2}{x}+\frac{1}{x}\right)\left(-1\right)dx, \text{ taking } c=\log k$$

or $$\log\frac{xy-2}{xy+1}=\log k-3\log x$$

or $$\left(\frac{xy-2}{xy+1}\right)x^3=k$$

or $\qquad x^3(xy-2)=k\,(xy+1)$, where k is an arbitrary constant.

***Ex.4. Show that* 1, *x*, x^2 *are three particular integrals of* $x(x^2-1)y_1+x^2-(x^2-1)y-y^2=0$*, and hence obtain the general solution* $y(x+k)=x+kx^2$*, k* being an arbitrary constant.**

Sol. Re writing the given equation in the standard Riccati's form as

$$y_1=-\frac{x}{x^2-1}+\frac{1}{x}y+\frac{1}{x\left(x^2-1\right)}y^2 \qquad .....(31)$$

Now putting $y=1$ (one of the three given integrals) so that $y_1=0$, and we get

$$0=-\frac{x}{x^2-1}+\frac{1}{x}+\frac{1}{x\left(x^2-1\right)}=0$$

This show that $y=1$ is an particular integral of (1). Similarly we can prove that $y=x$ and $y=x^2$ are also particular integrals of (31).

Now taking $p(x)=1$, $q(x)=x$, $r(x)=x^2$ and using § 1.3.5, we get

$$\frac{(1-x)\left(x^2-y\right)}{\left(x^2-x\right)(1-y)}=\frac{1}{k} \text{ (say)}$$

or
$$\frac{(1-x)\left(x^2-y\right)}{-x(1-x)(1-y)}=\frac{1}{k}$$

or
$$k(x^2-y)=-x(1-y)$$

or
$$y(k+x)=x+kx^2$$

which is the required solution.

1.4 Equation of the Form $\dfrac{d^2y}{dx^2}=f(y)$

To find the solution of above equation, we multiply both side by $2\dfrac{dy}{dx}$, then we get

$$2\frac{dy}{dx}\frac{d^2y}{dx^2}=2f(y)\frac{dy}{dx}$$

On integration we obtain

$$\left(\frac{dy}{dx}\right)^2=2\int f(y)\,dy+a$$

or
$$\frac{dy}{\sqrt{2}\left[\int f(y)\,dy+a\right]^{1/2}}=dx$$

Again integrating, we finally obtain

$$\int\frac{dy}{\sqrt{2}\left[\int f(y)\,dy+a\right]^{1/2}}=x+b$$

Ex.*1. *Solve $\quad sin^3 y \dfrac{d^2 y}{dx^2} = cos\ y$

Sol. We can write the given equation as

$$\frac{d^2 y}{dx^2} = \text{cosec}^2\, y \cot\, y$$

Now multiplying both sides by $2\dfrac{dy}{dx}$ and integrating, we get

$$\left(\frac{dy}{dx}\right)^2 = a - \cot^2 y = \frac{a \sin^2 y - \cos^2 y}{\sin^2 y}$$

or
$$\frac{\sin y\, dy}{\sqrt{a-(1+a)\ cos^2\ y}} = dx$$

Again integrating, we get the required solution as

$$-\frac{1}{\sqrt{1+a}} \sin^{-1}\left\{\sqrt{\frac{1+a}{a}} \cos\, y\right\} = x + c$$

Ex.*2. *Solve $\quad y^3 \dfrac{d^2 y}{dx^2} = c$

Sol. We can write the given equation as

$$\frac{d^2 y}{dx^2} = \frac{c}{y^3}$$

Now multiplying both side by $2\dfrac{dy}{dx}$ and integrating, we get

$$\left(\frac{dy}{dx}\right)^2 = -\frac{c}{y^2} + a$$

or
$$\frac{y\, dy}{\sqrt{ay^2 - c}} = dx$$

Again integrating, we get the required solution as

$$ay^2 = c + (ax + b)^2$$

where a and b are two constants.

1.5 Equation not Containing *y* Directly

In this case general equation is given in the form

$$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1} y}{dx^{n-1}}, \ldots\ldots, \frac{dy}{dx}, x\right) = 0 \qquad \ldots..(1)$$

To solve it, the order of equation is depressed by assuming the lowest differential coefficient present in the equation as a dependent variable. So let

$$\frac{dy}{dx} = p,\ \frac{d^2y}{dx^2} = \frac{dp}{dx},\ ....,\ \frac{d^ny}{dx^n} = \frac{d^{n-1}p}{dx^{n-1}}$$

therefore equation (1) reduces to

$$f\left(\frac{d^{n-1}p}{dx^{n-1}}, \frac{d^{n-2}p}{dx^{n-2}}, ..., p, x\right) = 0$$

which may be possibly solved for p.

Let $$p = \frac{dy}{dx} = \phi(x)$$

then the solution is

$$y = \int \phi(x)\, dx + c\,.$$

Ex.*1. *Solve $$\left(\frac{d^3y}{dx^3}\right)^2 + x\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = 0$$

Sol. The given equation does not contain y directly. Here the lowest differential coefficient is $\frac{d^2y}{dx^2}$. So putting

$$\frac{d^2y}{dx^2} = p \quad \text{and} \quad \frac{d^3y}{dx^3} = \frac{dp}{dx}.$$

We get from the given equation

$$\left(\frac{dp}{dx}\right)^2 + x\,\frac{dp}{dx} - p = 0$$

or $$p = x\frac{dp}{dx} + \left(\frac{dp}{dx}\right)^2$$ [Clairaut`s form $y = px + f(p)$]

So its solution is

$$p = cx + c^2$$

or $$\frac{d^2y}{dx^2} = cx + c^2$$

on integration, $$\frac{dy}{dx} = c\frac{x^2}{2} + c^2x + c_1$$

Again integrating, we get the general solution as

$$y = c\frac{x^3}{6} + c^2\,\frac{x^2}{2} + c_1x + c_2$$

Ex.2. Solve $$2\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2+4=0$$

Sol. The given equation does not contain y directly. Here the lowest differential coefficient is $\frac{dy}{dx}$. So putting

$$\frac{dy}{dx}=p \ \ and \ \ \frac{d^2y}{dx^2}=\frac{dp}{dx}.$$

We get from the given equation

$$2\frac{dp}{dx}-p^2+4=0$$

or $$\frac{2dp}{p^2-4}=dx$$

Integrating

$$\frac{1}{2}\log\frac{p-2}{p+2}=x+a$$

or $(p-2)=(p-2)\,be^{2x}$, where $b=e^{2a}$.

or $$p=\frac{dy}{dx}=2\left(1+\frac{2be^{2x}}{1-be^{2x}}\right)$$

On integration, we get the general solution as

$$y=2x-2\log(1-be^{2x})+c.$$

1.6 Equation not Containing *x* Directly

In this case general equation is given in the form

$$f\left(\frac{d^ny}{dx^n},\frac{d^{n-1}y}{dx^{n-1}},\ldots,\frac{dy}{dx},y\right)=0$$

Now putting $$\frac{dy}{dx}=p,\ \frac{d^2y}{dx^2}=\frac{dp}{dx}=\frac{dp}{dy}\cdot\frac{dy}{dx}=p\frac{dp}{dy}$$

Similarly $$\frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d}{dy}\left(p\frac{dp}{dy}\right)\cdot\frac{dy}{dx}$$

$$=\left[p\frac{d^2p}{dy^2}+\left(\frac{dp}{dy}\right)^2\right]p$$

$$=p^2\frac{d^2p}{dy^2}+p\left(\frac{dp}{dy}\right)^2$$

Hence the given equation reduces to

$$f\left(\frac{d^{n-1}p}{dy^{n-1}}, \ldots, p, y\right)=0$$

which may be possibly solved for p.

Let $$p=\frac{dy}{dx}=\phi(y).$$

Then the solution is

$$\int\frac{dy}{\phi(y)}=x+c$$

Ex.*1. *Solve $$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+4\left(\frac{dy}{dx}\right)^3=0$$

Sol. The given equation does not contain x directly, so substituting

$$\frac{dy}{dx}=p, \frac{d^2y}{dx^2}=p\frac{dp}{dy}, \text{ we get}$$

$$p\frac{dp}{dy}+2p+4p^3=0$$

or $$\frac{dp}{1+2p^2}=-2\,dy$$

On integration, we get

$$\frac{1}{\sqrt{2}}\tan^{-1}\left(p\sqrt{2}\right)=-2y+a$$

or $$\tan^{-1}\left(p\sqrt{2}\right)=b-2\sqrt{2}\,y, \text{ where } b=\sqrt{2a}$$

or $$\sqrt{2}\cot\left(b-2\sqrt{2}\;y\right)dy=dx.$$

Again integrating, we get the general solution as

$$\log\sin\left(b-2\sqrt{2}\;y\right)=-2x+\log c$$

or $$\sin\left(b-2\sqrt{2}\;y\right)=ce^{-2x}$$

Ex.*2. *Solve $$y(1-\log y)\frac{d^2y}{dx^2}+(1+\log y)\left(\frac{dy}{dx}\right)^2=0$$

Sol. The given equation does not contain x directly, so substituting

$$\frac{dy}{dx}=p, \frac{d^2y}{dx^2}=p\frac{dp}{dy}, \text{ we get}$$

$$y(1-\log y)\,p\frac{dp}{dy}+(1+\log y)\,p^2=0$$

or $$\frac{dp}{p}+\frac{(1+\log y)}{y(1-\log y)}dy=0.$$

On integration, we get by substituting $\log y = t$

$$\log\ p=\log y+2\log(\log y-1)+\text{constant}$$

or $$p=\frac{dy}{dx}=ay(\log y-1)^2$$

or $$\frac{dy}{y(\log y-1)^2}=a\ dx$$

Again integrating , we get the general solution as

$$-\frac{1}{(\log y-1)}=ax+b$$

or $$(1-\log y)=\frac{1}{ax+b}$$

1.7 Equation in which *y* Appears in only Two Derivatives Whose Orders Differ by Two.

In this case general equation is given in the form

$$f\left(\frac{d^n y}{dx^n},\frac{d^{n-2}y}{dx^{n-2}},\ x\right)=0$$

Now putting $$\frac{d^{n-2}y}{dx^{n-2}}=p$$

so that $$\frac{d^n y}{dx^n}=\frac{d^2 p}{dx^2}$$

then the given equation becomes

$$f\left(\frac{d^2 p}{dx^2},p,x\right)=0$$

which gives $$p=\frac{d^{n-2}y}{dx^{n-2}}=\phi(x).$$

By successive integration, we can find the value of y.

Ex.*1. *Solve $$\frac{d^5 y}{dx^5}-n^2\frac{d^3 y}{dx^3}=e^{ax}$$

Sol. In the given equation y appears in two derivatives whose order differs by two. Now substituting $\frac{d^3 y}{dx^3}=p$. So the given equation transforms to

$$\frac{d^2 p}{dx^2} - n^2\ p = e^{ax}$$

whose solution will be

$$p = \frac{d^3 y}{dx^3} = c_1\, e^{nx} + c_2\, e^{-nx} + \frac{e^{ax}}{\left(a^2 - n^2\right)}$$

On integration, we get

$$\frac{d^2 y}{dx^2} = \frac{c_1}{n} e^{nx} - \frac{c_2}{n} e^{-nx} + \frac{e^{ax}}{a\left(a^2 - n^2\right)} + c_3$$

Again integrating

$$\frac{dy}{dx} = \frac{c_1}{n^2} e^{nx} + \frac{c_2}{n^2} e^{-nx} + \frac{e^{ax}}{a^2\left(a^2 - n^2\right)} + c_3\, x + c_4$$

which on integration gives the general solution as

$$y = \frac{c_1}{n^3} e^{nx} - \frac{c_2}{n^3} e^{-nx} + \frac{e^{ax}}{a^3\left(a^2 - x^2\right)} + c_3\, \frac{x^2}{2} + c_4 x + c_5$$

1.8 Equation in which *y* Appears in only Two Derivatives Whose Orders Differ by Unity

In this case general equation is given in the form

$$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1} y}{dx^{n-1}}, x\right) = 0$$

Now putting $\quad \dfrac{d^{n-1} y}{dx^{n-1}} = p$

so that $\quad \dfrac{d^n y}{dx^n} = \dfrac{dp}{dx}.$

Hence the given equation reduces to

$$f\left(\frac{dp}{dx}, p, x\right) = 0$$

This is an equation of first order. We can here easily find the value of p in terms of x as

$$p = \frac{d^{n-1} y}{dx^{n-1}} = \phi(x).$$

By successive integration, we get the general solution.

Ex.*1. *Solve $\quad a\dfrac{d^2y}{dx^2}=\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{1}{2}}$

Sol. In the given equation y appears in two derivatives whose order differs by unity. Now substituting

$$\frac{dy}{dx}=p,\ \frac{d^2y}{dx^2}=\frac{dp}{dx}$$

so the given equation transforms to

$$a\frac{dp}{dx}=\left(1+p^2\right)^{\frac{1}{2}}$$

or $$\frac{dp}{\sqrt{1+p^2}}=\frac{1}{a}dx$$

Integrating $$\sin h^{-1}p=\frac{x}{a}+c_1$$

$$p=\frac{dy}{dx}=\sin h\left(\frac{x}{a}+c_1\right)$$

Again integrating, we get the general solution as

$$y=a\ \cos\ h\left(\frac{x}{a}+c_1\right)+c_2$$

1.9 Homogeneous Equation

We mean by homogeneous equation that an equation in which all the terms will be of the same dimensions.

Dimention of a differential equation is calculated as given under

$$x\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+\frac{dy}{dx}=2$$

Now $$\text{Dim}\left(x\frac{d^2y}{dx^2}\right)=\text{Dim}\left(x\frac{y}{x^2}\right)=\text{Dim}\left(y^1x^{-1}\right)$$

$$\text{Dim}\left(\left(\frac{dy}{dx}\right)^2\right)=\text{Dim}\left(\left(\frac{y}{x}\right)^2\right)=\text{Dim}\left(y^2x^{-2}\right)$$

$$\text{Dim}\left(\frac{dy}{dx}\right)=\text{Dim}\left(y^1x^{-1}\right)$$

$$\text{Dim}(2)=0$$

Hence the given equation has the 0 dimension

Note :

(a) Derivative in a differential equation does not alter the dimension of the variables x and y.

(b) The dimension of x is invariably taken as unity.

In such cases suitable transformations are made to lower the order of the equation

Ex.*1. *Solve $$nx^3\frac{d^2y}{dx^2}=\left(y-x\frac{dy}{dx}\right)^2$$

Sol. Here x and y both of dimension unity. There for the given equation is homogeneous of dimension 2. Substituting $y = zx$ and $x = e^{\theta}$, we get

$$ne^{2\theta}\left(\frac{dz}{d\theta}+\frac{d^2z}{d\theta^2}\right)=\left\{xz-x\left(z+\frac{dz}{d\theta}\right)\right\}^2$$

or $$n\left(\frac{dz}{d\theta}+\frac{d^2z}{d\theta^2}\right)=\left(\frac{dz}{d\theta}\right)^2$$

Now if we put $\frac{dz}{d\theta}=\alpha$, then above equation becomes

$$n\left(\alpha+\frac{d\alpha}{d\theta}\right)=\alpha^2$$

or $$\left[\frac{1}{\alpha-n}-\frac{1}{\alpha}\right]d\alpha=d\theta$$

on integrating $$\frac{1}{n}\log\frac{\alpha-n}{\alpha}=\theta+\text{constant}$$

Now substituting $\alpha=\frac{dz}{d\theta}$ and then integrating , we get the general solution as

$$y=n\,x\log\left(c_1+\frac{c_2}{x}\right)$$

1.10 Summary

In this unit, you studied the exactness of differential equation and the method by which we can solve exact equations. Methods for solution of the standard Riccati's equation of first order, with one, two or three known particular solutions were discussed. The methods have been illustrated with the help of examples.

Self-Learning exercise

1. What do you mean by exact equation ?
2. Write down the Riccati's equation of first order.
3. Riccati's equation is a non-linear differential equation. Is it true ?

1.11 Answers of Self-Learning Exercise

1. A differential equation which is integrable directly.
2. $\frac{dy}{dx} = P + Qy + Ry^2$, where P, Q, R are functions of x or constants.
3. True

1.12 Exercise

1. Solve the following differential equations :

(a) $x^2 y \frac{d^2y}{dx^2} + \left(x\frac{dy}{dx} - y \right)^2 - 3y^2 = 0$ **[Ans.** $xy^2 = c_2 x^5 - \frac{2c_1}{5}$]

(b) $(2y + x)\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\left(1 + \frac{dy}{dx}\right) = 0$ **[Ans.** $y^2 + xy = c_1 x + c_2$]

(c) $\cos y \frac{d^2y}{dx^2} - \sin y \left(\frac{dy}{dx}\right)^2 + \cos y \frac{dy}{dx} = x + 1$ **[Ans.** $\sin y = \frac{(x+1)^2}{2} - x + c_1 + c_2 e^{-x}$]

2. Solve the following differential equations :

(a) $x(1 - x^3) y_1 = x^2 + y - 2xy^2$, x^2 is an integral **[Ans.** $\frac{x^4 - x}{y - x^2} = c - \frac{2x^3}{3}$]

(b) $\frac{dy}{dx} = 1 + y^2$, $\tan x$ is an integral **[Ans.** $y(c - \tan x) = c \tan x + 1$]

(c) $x^3 y_1 = x^2 y + y^2 - x^2$ **[Ans.** $y\,(ce^{2/x} - 1) = x + cxe^{2/x}$]

(d) $x(x-1) y_1 - (2x+1) y + yh^2 + 2x = 0$, x is a solution **[Ans.** $y = \frac{(x^2 + c)}{(x + c)}$]

3. Solve :

(a) $\frac{d^2y}{dx^2} = \frac{1}{\sqrt{ay}}$ **[Ans.** $3x = 2a^{1/4}\left(\sqrt{y} - 2c_1\right)\left(\sqrt{y} + c_1\right)^{1/2} + c_2$]

(b) $\frac{d^2y}{dx^2} + \frac{a^2}{y^2} = 0$ **[Ans.** $\sqrt{c_1 y^2 + y} - \frac{1}{\sqrt{c_1}} \log\left(\sqrt{c_1 y} + \sqrt{1 + c_1 y}\right) = ac_1\sqrt{2x} + c_2$]

4. Solve :

(a) $\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ **[Ans.** $y = \cos h(x + c_1) + c_2$]

(b) $\left(1+x^2\right)\dfrac{d^2y}{dx^2}+1+\left(\dfrac{dy}{dx}\right)^2=0$ **[Ans.** $y=cx+\left(1+c^2\right)\log\left(x-c\right)+c_1$]

(c) $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+\left(\dfrac{dy}{dx}\right)^3=0$ **[Ans.** $y=-\sin^{-1}\left(c_1e^{-x}+c_2\right)$]

(d) $2x\dfrac{d^3y}{dx^3}\cdot\dfrac{d^2y}{dx^2}=\left(\dfrac{d^2y}{dx^2}\right)^2-a^2$ **[Ans.** $y=\dfrac{4\left(c_1x+a\right)^{5/2}}{15c_1^2}+c_2x+c_3$]

5. Solve :

(a) $y\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=y^2$ **[Ans.** $y^2=c_1\sin h\sqrt{2x+c_2}$]

(b) $\dfrac{d^2y}{dx^2}+a\left(\dfrac{dy}{dx}\right)^2=0$ **[Ans.** $e^{ay}=c_1x+c_2$]

(c) $y\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2+1=0$ **[Ans.** $y^2+x^2+c_1x+c_2=0$]

(d) $y\dfrac{d^2y}{dx^2}-\left(\dfrac{dy}{dx}\right)^2=y^2\log y$ **[Ans.** $\log y=c_1e^x+c_2e^{-x}$]

(e) $\left(\dfrac{dy}{dx}\right)^2-y\dfrac{d^2y}{dx^2}=n\left\{\left(\dfrac{dy}{dx}\right)^2+a^2\left(\dfrac{d^2y}{dx^2}\right)^2\right\}^{1/2}$ **[Ans.** $cy+n\left(1+a^2c^2\right)^{1/2}=c_2e^{cx}$]

6. Solve :

(a) $\dfrac{d^4y}{dx^4}-a^2\dfrac{d^2y}{dx^2}=0$ **[Ans.** $y=c_1e^{ax}+c_2e^{-ax}+c_3x+c_4$]

(b) $x^2\dfrac{d^4y}{dx^4}+a^2\dfrac{d^2y}{dx^2}=0$ **[Ans.** $y=c_1+c_2x+x^{5/2}\left[c_3\sqrt{x}\sqrt{1-4a^2}+\dfrac{c_4}{\sqrt{x}}\sqrt{1-4a^2}\right]$

when $a<\dfrac{1}{2}$ and $y=c_1+c_2x+c_3x^{5/2}\cos\left(\dfrac{1}{2}\sqrt{4a^2-1}\log\dfrac{x}{c_4}\right)$ when $a>\dfrac{1}{2}$]

7. Solve :

(a) $x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=0$ **[Ans.** $y=c_1\log x+c_2$]

(b) $\dfrac{d^3y}{dx^3}\cdot\dfrac{d^2y}{dx^2}=2$ **[Ans.** $15y=8\left(x+c_1\right)^{5/2}+c_2x+c_3$]

8. Solve :

(a) $xy\dfrac{d^2y}{dx^2}+x\left(\dfrac{dy}{dx}\right)^2=3y\dfrac{dy}{dx}$ **[Ans.** $y^2+\sqrt{y^4+c_1x^4}=c_2x^4$]

(b) $(2y+x)\dfrac{d^2y}{dx^2}+2\left(\dfrac{dy}{dx}\right)^2+2\dfrac{dy}{dx}=2$ **[Ans.** $xy+y^2-x^2=c_1x+c_2$]

9. By reduction to a linear equation show that the solution of the Riccati's equation

$$x^2\frac{dy}{dx}+2-2xy+x^2y^2=0 \quad \text{is}$$

$$y\left(x^2+c_1x\right)=2x+c_1$$

10. Show that tan x is one integral of the equation

$$y_1=1+y^2$$

and hence obtain the general solution in the form

$$y(c_1-\tan x)=c_1\tan x+1$$

where c_1 is a constant.

11. Determine the curve whose radius of curvature varies as the cube of the length of the normal intercepted the curve and x-axis. **[Ans.** $c_3+c_1y^2=(c_1x+4)$]

□□□

Unit 2 : Total Differential Equations

Structure of the Unit

2.0 Objective

In this unit, you will learn various methods for solving different types of total differential equations. Some of the methods are : Method of inspection, method for homogeneous equations, method of Auxiliary equations and general method. You will also study the geometrical meaning and method for solving total differential equations involving three or four variables.

2.1 Introduction

In this unit, we propose to discuss differential equations with one independent variable and more than one dependent variables.

The expression $\sum_{i=1}^{n} u_i dx_i$, where u_i, $i = 1, 2 .... n$ are, in general, functions of some or all of n independent variables $x_1, x_2 .... x_n$ is called a **total differential forms** in n variables and the equation

$$\sum_{i=1}^{n} u_i dx_i = 0 \qquad .....(1)$$

is called a **total differential equation** in n variables $x_1, x_2 .... x_n$. It is also known as **Pfaffian differential equation.**

In the case of two variables, equation (1) may be written as

$$M(x, y)\, dx + N(x, y)\, dy = 0 \qquad .....(2)$$

It is a differential equation of first order and first degree. The necessary and sufficient condition for its exactness (integrability) is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad .....(3)$$

In the case of three variables x, y, z the total differential equation (1) may be written as

$$Pdx + Qdy + Rdz = 0 \qquad .....(4)$$

where P, Q and R are functions of x, y and z. In vector notations, equation (4) may be written as

$$\mathbf{X} \cdot d\mathbf{r} = 0 \quad \text{where} \quad \mathbf{X} = (P, Q, R) \quad \text{and} \quad d\mathbf{r} = (dx, dy, dz).$$

It is not always possible to integrate equation (4) directly. If however, the equation is such that there exist a function $u(x, y, z)$ whose total differential du is equal to the left hand side of (4), then only it is integrated directly. In other cases equations (4) may or may not be integrable.

Now we proceed to find the condition which P, Q, R must satisfy, so that equation (4) is integrable. This is also known as condition of integrability.

2.2 Necessary and Sufficient Condition for integrability of the Total Differential Equation *Pdx* + *Qdy* + *Rdz* = 0.

2.2.1. Theorem :

The necessary and sufficient condition for the total differential equation Pdx + Qdy + Rdz = 0 to be integrable is

$$P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0$$

or $\quad \mathbf{X} \cdot \mathbf{curl}\, \mathbf{X} = \mathbf{0}, \quad$ ***where*** $\quad \mathbf{X} = (P, Q, R)$

or

$$\begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = 0$$

Proof : Condition is necessary :

Let $u(x, y, z) = C$ .....(1)

be an integral of total differential equation

$$Pdx + Qdy + Rdz = 0 \quad .....(2)$$

Then total differential du of (1), must be equal to $Pdx + Qdy + Rdz,$ or it multiplied by a factor. But we know the differentiation of (1) is

$$du = \left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z}\right)dz \quad .....(3)$$

Since (1) is an integral of (2), therefore P, Q, R must be proportional to $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$.

So, $$\frac{\partial u/\partial x}{P} = \frac{\partial u/\partial y}{Q} = \frac{\partial u/\partial z}{R} = \mu(x, y, z) \qquad (\text{say})$$

$$\therefore \quad \mu P = \frac{\partial u}{\partial x},\ \mu Q = \frac{\partial u}{\partial y},\ \mu R = \frac{\partial u}{\partial z} \quad .....(4)$$

From the first two parts of (4), we get

$$\frac{\partial}{\partial y}(\mu P) = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(\mu Q)$$

or $$\mu\frac{\partial P}{\partial y} + P\frac{\partial \mu}{\partial y} = \mu\frac{\partial Q}{\partial x} + Q\frac{\partial \mu}{\partial x}$$

or $$\mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial y} \quad .....(5)$$

Similarly, we can write

$$\mu\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = R\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial z} \quad .....(6)$$

and $$\mu\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = P\frac{\partial \mu}{\partial z} - R\frac{\partial \mu}{\partial x} \quad .....(7)$$

Multiplying (5), (6) and (7) by R, P and Q respectively and adding, we get

$$P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0 \quad .....(8)$$

This is the condition for the integrability of total differential equation (2).

Sufficient Condition :

Now we prove that if the condition (8) is satisfied, then the equation (2) will have a solution of the form (1).

Now if the condition (8) is satisfied for P, Q, R of the equation (2) then it can be easily verified that the same condition will hold for the coefficients of

$$\mu Pdx + \mu Qdy + \mu Rdz = 0$$

where μ is any function of x, y, z and replacing P, Q, R by μP, μQ, μR respectively.

Here, if we treat variable z as constant then the differential equation (2) becomes $Pdx+Qdy = 0$.

Now $Pdx+ Qdy$ may be regarded as an exact differential. For if it not so, then an integrating factor μ can be found to make it exact. Thus there is no loss of generality in regarding $Pdx + Qdy$ as an exact differential. Therefore

$$\int (Pdx + Qdy) = V \quad \text{(say).} \qquad .....(9)$$

It follows that

$$P = \frac{\partial V}{\partial x} \quad \text{and} \quad Q = \frac{\partial V}{\partial y}$$

Differentiating (9), we get

$$Pdx + Qdy = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy \qquad .....(10)$$

Substituting these values in the given condition (8), we find that

$$\frac{\partial V}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial^2 V}{\partial z \partial y}\right) - \frac{\partial V}{\partial y}\left(\frac{\partial R}{\partial x} - \frac{\partial^2 V}{\partial z \partial x}\right) = 0$$

or

$$\frac{\partial V}{\partial x} \cdot \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z} - R\right) - \frac{\partial V}{\partial y} \cdot \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial z} - R\right) = 0$$

or

$$\begin{vmatrix} \dfrac{\partial V}{\partial x} & \dfrac{\partial}{\partial x}\left(\dfrac{\partial V}{\partial z} - R\right) \\ \dfrac{\partial V}{\partial y} & \dfrac{\partial}{\partial y}\left(\dfrac{\partial V}{\partial z} - R\right) \end{vmatrix} = 0$$

This shows that a relation independent of x and y exists between V and $\left(\frac{\partial V}{\partial z} - R\right)$. Consequently $\frac{\partial V}{\partial z} - R$ can be expressed as a function of z and V. That is we can take

$$\frac{\partial V}{\partial z} - R = \phi(z, V) \qquad .....(11)$$

Hence

$$\begin{aligned} Pdx + Qdy + Rdz &= \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \left(\frac{\partial V}{\partial z} - \phi\right)dz \\ &= \left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right) - \phi\, dz \\ &= dV - \phi dz \end{aligned}$$

Thus (2) may be written as $dV - \phi\, dz = 0$ which is a first order equation in two variables hence integrable will give equation in two variables.

Suppose the integral is $U(z, V) = c$, then subestituting the value of V from (9), we get the solution in the form given by (1).

Thus the condition is sufficient.

2.1.1 Theorem : ***Prove that the necessary condition for integrability of the total differential equation*** $\boldsymbol{X} \cdot d\boldsymbol{r} = Pdx + Qdy + Qdz = 0$ ***is*** $\boldsymbol{X} \cdot \text{curl}\, \boldsymbol{X} = 0$**.**

Proof : Let $\quad \boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, so that

$$d\boldsymbol{r} = dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k}$$

and $\quad \boldsymbol{X} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$

Then we have

$$\boldsymbol{X} \cdot d\boldsymbol{r} = Pdx + Qdy + Rdz \qquad \text{.....(12)}$$

Then we see that (12) is satisfied by usual rule of dot product of two vectors $\boldsymbol{X}$ and $d\boldsymbol{r}$.

Now, we know that

$$\text{Curl } \boldsymbol{X} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\boldsymbol{j} + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k}.$$

Now by usual rule of dot product of two vectors, we get

$$\boldsymbol{X} \cdot \text{Curl } \boldsymbol{X} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

which is equal to zero. So the necessary condition is $\boldsymbol{X} \cdot \text{curl}\, \boldsymbol{X} = 0$

2.3 Methods of Solving Total Differential Equation $Pdx + Qdy + Rdz = 0$

If the following condition of integrability

$$P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0$$

is satisfied, then the total differential equation may be solved by several methods as given below.

2.3.1 Method of Inspection

If the condition of integrability is satisfied, then sometimes it will be possible to rearrange the terms of the given equation, by dividing or multiplying by a suitable function, so that it can be integrated directly.

The following list will help to rewrite the given equation in the form of exact differential.

(i) $x\,dy + y\,dx = d\,(xy)$

(ii) $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$

(iii) $\dfrac{x\,dy - y\,dx}{xy} = d\left(\log\dfrac{y}{x}\right)$

(iv) $\dfrac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$

(v) $\dfrac{x\,dy + y\,dx}{xy} = d\left(\log(xy)\right)$

(vi) $\dfrac{x\,dy + y\,dx}{x^2 + y^2} = d\left[\dfrac{1}{2}\log\left(x^2 + y^2\right)\right]$

(vii) $\dfrac{2xy\,dy - y^2\,dx}{xy} = d\left(\dfrac{y^2}{x}\right)$

(viii) $\dfrac{ye^x\,dx - e^x\,dy}{y^2} = d\left(\dfrac{e^x}{y}\right)$

Ex.*1.** Show that **$(2x + y^2 + 2xz)\,dx + 2xy\,dy + x^2dz = 0$ *is integrable (i.e., condition of integrability is satisfied).

Sol. Comparing the given equation with $Pdx + Qdy + Rdz = 0$

We get, $P = 2x + y^2 + 2xz$; $Q = 2xy$; $R = x^2$

Now the condition of integrability is

$$P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0$$

Substituting the values of P, Q, R in it, we get

$$(2x + y^2 + 2xz)(0 - 0) - 2xy(2x - 2x) + x^2(2y - 2y) = 0$$

Showing that the condition of integrability is satisfied and hence the given equation is integrable.

Ex.*2. *Solve **$(yz + xyz)\,dx + (zx + xyz)\,dy + (xy + xyz)\,dz = 0$**

Sol. Comparing the given equation with $Pdx + Qdy + Rdz = 0$

We get $P = yz + xyz$; $Q = zx + xyz$; $R = xy + xyz$

Now the condition of integrability is

$$P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0$$

$$= yz(1+x)\{(x+xz)-(x+xy)\} - zx(1+y)\{(y+yz)-(y+xy)\}$$
$$+ xy(1+z)\{(z+yz)-(z+xz)\}$$

$$= yz(1+x)x(z-y) - zx(1+y)y(z-x) + xy(1+z)z(y-x)$$

$$= xyz\left[\{(z-y)-(z-x)+(y-x)\} + \{x(z-y) - y(z-x) + z(y-x)\}\right]$$

$$= xyz[0+0] = 0$$

This shows that the given equation is integrable.

Now dividing the whole equation by xyz, then given equation becomes

$$\left(\frac{1}{x}+1\right)dx + \left(\frac{1}{y}+1\right)dy + \left(\frac{1}{z}+1\right)dz = 0$$

On integration, we get

$$\log x + x + \log y + y + \log z + z = C$$

or $$\log(xyz) + x + y + z = C$$

which is the required general solution, C being an arbitrary constant.

Ex.*3. *Solve **$(y^2 + z^2 - x^2)\,dx - 2xy\,dy - 2xz\,dz = 0$**

Sol. As usual, we see that the condition of integrability is satisfied. Now rearranging the terms of the given equation as

$$\left(x^2 + y^2 + z^2\right)dx = 2x^2dx + 2xy\,dy + 2x\,zdz$$

or $$\left(x^2 + y^2 + z^2\right)dx = 2x\left(xdx + ydy + zdz\right)$$

or $$\frac{dx}{x} = \frac{2xdx + 2ydy + 2zdz}{x^2 + y^2 + z^2}$$

On integration, we get

$$\log x + \log c = \log\left(x^2 + y^2 + z^2\right)$$

or $$x^2 + y^2 + z^2 = cx$$

is the required general solution.

Ex.4. Solve $\boldsymbol{(2x^2y + 2xy^2 + 2xyz + 1)\, dx + (x^3 + x^2y + x^2z + 2xyz + 2y^2z + 2yz^2 + 1)\, dy + (xy^2 + y^3 + y^2z + 1)\, dz = 0}$

Sol. As usual, it may be verified that the condition of integrability is satisfied. Now rearranging the terms of the given equation as

$$\left\{2xy(x+y+z)+1\right\}dx + \left\{x^2(x+y+z) + 2yz(x+y+z)+1\right\} dy + \left\{y^2(x+y+z)+1\right\}dz = 0$$

or $$(x+y+z)\left(2xy\,dx + x^2dy + 2y\,zdy + y^2dz\right) + dx + dy + dz = 0$$

or $$\left(2xy\,dx + x^2dy\right) + \left(2yzdy + y^2dz\right) + \left(\frac{dx+dy+dz}{x+y+z}\right) = 0$$

On integration, we get

$$x^2y + y^2z + \log(x + y + z) = C$$

This is the required general solution.

Ex.5. Solve $\boldsymbol{\frac{yz}{x^2+y^2}dx - \frac{xz}{x^2+y^2}dy - \tan^{-1}\left(\frac{y}{x}\right)dz = 0}$

Sol. It can be easily verified that the condition of integrability is satisfied. Arranging the terms of the given equation as

$$\frac{ydx - xdy}{\left(x^2+y^2\right)\tan^{-1}\left(\frac{y}{x}\right)} = \frac{dz}{z} \qquad .....(13)$$

Taking $\tan^{-1}\left(\frac{y}{x}\right) = s$, so that $\frac{xdy - ydx}{x^2\left(1+\frac{y^2}{x^2}\right)} = ds$. Then equation (13) becomes

or $$-\frac{ds}{s} = \frac{dz}{z}$$

Integrating $$-\log s = \log z + \log c$$

or $$s = \frac{1}{cz}$$

i.e. $$\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{cz}$$

which gives $\left(\frac{y}{x}\right)=\tan\left(\frac{1}{cz}\right)$

This is the required general solution.

2.3.2 Method for Homogeneous Equations

The equation $Pdx + Qdy + Rdz = 0$ is called a homogeneous equation if P, Q, R are homogeneous functions of x, y, z of the same degree. In such a case one variable is separated from the other two by the substitution

$$x = uz, \qquad y = vz \qquad .....(14)$$

then $$dx = udz + zdu, \quad dy = vdz + zdv \qquad .....(15)$$

Further, let

$$P = z^n f_1(u,v), \quad Q = z^n f_2(u,v) \text{ and } R = z^n f_3(u,v) \qquad .....(16)$$

Hence the given equation $Pdx + Qdy + Rdz = 0$ becomes

$$z^{n+1}\{f_1(u,v)du + f_2(u,v)dv\} + z^n\{uf_1(u,v) + vf_2(u,v) + f_3(u,v)\}dz = 0$$

On multiplying by z, we get

$$z^{n+2}\{f_1(u,v)du + f_2(u,v)dv\} + z^{n+1}\{uf_1(u,v) + vf_2(u,v) + f_3(u,v)\}dz = 0 \qquad .....(17)$$

Now following two cases arise :

Case I : $Px + Qy + Rz = 0$

If $Px + Qy + Rz = 0$ that is by substituting the values of x, y from (14) and P, Q, R from (16) in it, we find

$$z^{n+1}\{uf_1(u,v) + vf_2(u,v) + f_3(u,v)\} = 0$$

then the coefficient of dz in equation (17) will become zero and hence it reduces to

$$f_1(u,v)du + f_2(u,v)dv = 0 \qquad .....(18)$$

which can be integrated easily.

Case II : $Px + Qy + Rz \neq 0$

In this case the coefficient of dz will not be zero and therefore equation (17) may be written as.

$$\frac{f_1(u,v)du + f_2(u,v)dv}{\{uf_1(u,v) + vf_2(u,v) + f_3(u,v)\}} + \frac{dz}{z} = 0 \qquad ...(19)$$

Now since the given equation $Pdx + Qdy + Rdz = 0$ is integrable so equation (19) will be an exact differential and hence this equation may be integrated easily.

2.3.3 Working Rule for Solving Homogeneous Equations

(i) First of all verify the condition of integrability.

(ii) If $Px + Qy + Rz = 0$, then substitute $x = uz$, $y = vz$ and solve

(iii) If $Px + Qy + Rz \neq 0$ then $\frac{1}{Px + Qy + Rz}$ will be an integrating factor of the homogeneous equation $Pdx + Qdy + Rdz = 0$. After multiplying this equation by this integrating factor and rearranging the terms we can integrate the equation by inspection.

Ex.6. Solve $\boldsymbol{z^2dx+\left(z^2-2yz\right)dy+\left(2y^2-yz-xz\right)dz=0}$

Sol. Comparing the given equation with the standard equation $Pdx + Qdy + Rdz = 0$, we get

$$P = z^2,\ Q = z^2 - 2yz,\ R = 2y^2 - yz - xz$$

The given equation is homogeneous of degree 2. Now first of all we test the condition of integrability

$$P\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)-Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)$$

$$=z^2\left(4y-z-2z+2y\right)-\left(z^2-2yz\right)\left(-z-2z\right)+\left(2y^2-yz-xz\right)\left(0-0\right)$$

$$=6yz^2-3z^3+3z^3-6yz^2=0$$

Hence the condition of integrability is satisfied

Further, $\quad Px+Qy+Rz=xz^2+yz^2-2y^2z+2y^2z-yz^2-xz^2=0$

Therefore, we substitute

$$x = uz,\ \ y = vz$$

Hence $\quad dx = udz + zdu, \quad dy = vdz + zdv$

and the given equation reduces to

$$z^2\left(udz+zdu\right)+z^2\left(1-2v\right)\left(vdz+zdv\right)+z^2\left(2v^2-v-u\right)dz=0$$

or $\quad du + (1 - 2v)\, dv = 0$

Integrating, we get

$$u + v - v^2 = C$$

or $\quad xz + yz - y^2 = cz^2$

This is the required general solution.

Ex.7. Solve $\quad \boldsymbol{\left(yz+z^2\right)dx-xzdy+xydz=0}$

Sol. On comparing the given equation with $Pdx + Qdy + Rdz = 0$,

we have $P = yz + z^2, Q = -xz,\ R = xy$

Here the given equation is homogeneous of degree 2 and the condition of integrability is satisfied (do your self)

Now Let $\quad D = Px + Qy + Rz$

$$= x\,(yz + z^2) - xyz + xyz = xz\,(y + z) \neq 0$$

Multiplying the given equation by integrating factor 1/D, we get

$$\frac{\left(yz+z^2\right)dx-xz\,dy+xy\,dz}{D}=0 \qquad .....(17)$$

Now $\quad d\left(D\right)=d\left[xz\left(y+z\right)\right]=\left(z\,dx+xdz\right)\left(y+z\right)+xz\left(dy+dz\right)$

or $\quad d\left(D\right)=z\left(y+z\right)dx+x\left(y+2z\right)dz+xz\,dy$

Now rewriting the numerator of (17) as

$$d(D)-d(D)+\left(yz+z^2\right)dx-xzdy+xydz=d(D)-2xz(dy+dz)$$

$\therefore$ Equation (17) becomes

$$\frac{d(D)}{D}-\frac{2xz(dy+dz)}{D}=0$$

or
$$\frac{d(D)}{D}-\frac{2xz(dy+dz)}{xz(y+z)}=0$$

Integrating, $\log D-2\log(y+z)=\log C$

or $D=C(y+z)^2$

or $xz(y+z)=C(y+z)^2$

or $xz=C(y+z)$

which is the required general solution, C being an arbitrary constant.

Ex.8. Solve $\boldsymbol{(2xz-yz)dx+(2yz-xz)dy-\left(x^2-xy+y^2\right)dz=0}$

Sol. First of we verify the condition of integrability (do yourself). Since the given equation is homogeneous, so putting

$$x=uz,\ y=vz \quad \text{so that} \quad dx=udz+zdu,\ \ dy=zdv+vdz \qquad .....(18)$$

Now using these values in given equation, we get

$$\left(2uz^2-vz^2\right)(udz+zdu)+\left(2vz^2-uz^2\right)(vdz+zdv)-\left(u^2z^2-4vz^2+v^2z^2\right)dz=0$$

or
$$(2u-v)(udz+zdu)+(2v-u)(vdz+zdv)-\left(u^2-uv+v^2\right)dz=0$$

or
$$z\left[(2u-v)du+(2v-u)dv\right]+\left[u(2u-v)+v(2v-u)-\left(u^2-uv+v^2\right)\right]dz=0$$

or
$$z\left[2udu-(udv+vdu)+2vdv\right]+\left(u^2-uv+v^2\right)dz=0$$

or
$$z\left[du^2-d(uv)+dv^2\right]+\left(u^2-uv+v^2\right)dz=0$$

or
$$\frac{d\left(u^2-uv+v^2\right)}{u^2-uv+v^2}+\frac{dz}{z}=0$$

On integration, we get

$$\log(u^2-uv+v^2)+\log z=\log C$$

or
$$z\left(u^2-uv+v^2\right)=C$$

or
$$z\left(\frac{x^2}{z^2}-\frac{x}{z}\cdot\frac{y}{z}+\frac{y^2}{z^2}\right)=C$$

or
$$x^2-xy+y^2=cz$$

which is the required general solution.

Ex.9. Solve $\quad yz(y+z)dx + zx(x+z)dy + xy(x+y)dz = 0$

Sol. First of all verify the condition of integrability (do your self). Since the given equation is homogeneous, we put

$x = uz,\ y = vz$ so that $dx = zdu + udz,\ dy = zdv + vdz$ .....(19)

Substituting these in the given equation, we get

$$v(v+1)z^3(zdu + udz) + u(u+1)z^3(zdv + vdz) + uv(u+v)z^3dz = 0$$

or $\quad [v(v+1)du + u(u+1)dv]z^4 + [uv(v+1) + uv(u+1) + uv(u+v)]z^3dz = 0$

or $\quad [v(v+1)du + u(u+1)dv]z^4 + 2uv(u+v+1)z^3dz = 0$

Dividing above equation by $uv(u+v+1)z^4$, we get

$$\frac{(v+1)du}{u(u+v+1)} + \frac{(u+1)dv}{v(u+v+1)} + 2\frac{dz}{z} = 0$$

or $$\left(\frac{1}{u} - \frac{1}{u+v+1}\right)du + \left(\frac{1}{v} - \frac{1}{u+v+1}\right)dv + 2\frac{dz}{z} = 0$$

or $$\frac{du}{u} + \frac{dv}{v} - \frac{du+dv}{u+v+1} + 2\frac{dz}{z} = 0$$

On integration, we get

$$\log u + \log v - \log(u+v+1) + 2\log z = \log C$$

or $\quad uvz^2 = C(u+v+1)$

or $$\left(\frac{x}{z}\right)\left(\frac{y}{z}\right)z^2 = C\left(\frac{x}{z} + \frac{y}{z} + 1\right) \qquad \text{by using (9)}$$

or $\quad xyz = C(x+y+z)$

this is the required general solution.

2.3.4 Method of Auxiliary Equations

Let $\quad Pdx + Qdy + Rdz = 0$ .....(20)

by the given equation. Its condition of integrability is

$$P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0. \qquad \text{.....(21)}$$

On comparing (20) and (21), we obtain simultaneous equations, known as auxiliary equations.

$$\frac{dx}{\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}} = \frac{dy}{-\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)} = \frac{dz}{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)} \qquad \text{.....(22)}$$

For solving (22) let $u = c_1$ and $v = c_2$ be their two integrals. After finding the value of $Adu + Bdv = 0$ and comparing it with the given equation, the values of A and B will be obtained. Integration of $Adn + Bdv = 0$, will give the required solution.

This method will fail if $\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$ and $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

Ex.10. Solve $\qquad$ $\boldsymbol{xz^3dx - zdy + 2ydz = 0}$

Sol. Here the condition of integrability is satisfied (do your self) now given equation is

$$xz^3dx - zdy + 2ydz = 0 \qquad .....(23)$$

Comparing it with $Pdx + Qdy + Rdz = 0$, we have

$$P = xz^3,\ Q = -z,\ R = 2y$$

The auxiliary equations of the given equation are

$$\frac{dx}{\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)} = \frac{dy}{\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)} = \frac{dz}{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}$$

or $$\frac{dx}{2+1} = \frac{dy}{3xz^2} = \frac{dz}{0}$$

or $$\frac{dx}{1} = \frac{dy}{xz^2} = \frac{dz}{0}$$

Taking last two terms, we get

$$dz = 0 \quad \text{so that} \quad z = c_1 = u \text{ (say)} \qquad ......(24)$$

Taking first two terms, we get

$$xz^2dx - dy = 0$$

or $\qquad 2xu^2dx - 2dy = 0 \qquad$ [by using (23)]

Integrating, $\qquad x^2u^2 - 2y = c_2 = v \qquad$ (say)

or $\qquad x^2z^2 - 2y = v \qquad$ [by using (23)] $\qquad .....(25)$

Substituting the values of u and v from (24) and (25) in $Adu + Bdv = 0$, we get

$$Adz + Bd\,(x^2z^2 - 2y) = 0$$

or $$Adz + B\,(2xz^2dx + 2x^2zdz - 2dy) = 0$$

or $$2Bxz^2dx - 2Bdy + (A + 2Bx^2z)\,dz = 0 \qquad .....(26)$$

Comparing (23) and (26), we have

$$xz^3 = 2Bxz^2, -z = -2B$$

and $$2y = A + 2Bx^2z \Rightarrow B = \left(\frac{1}{2}\right)z \quad \text{and} \quad A = 2y - 2Bx^2z = 2y - x^2z^2$$

or $B = \left(\frac{1}{2}\right)u$ and $A = -v$, [by using (24) and (25)]. Substituting these values of A and B in $\qquad Adu + Bdv = 0$, we get

$$-vdu + \left(\frac{1}{2}\right)udv = 0$$

or $$\frac{1}{v}dv = 2\left(\frac{1}{u}\right)du$$

On integration, we get

$$\log v = 2\log u + \log c$$

$$v = cu^2 \qquad .....(27)$$

Putting the values of u and v from (24) and (25) in (27), we get

$$x^2z^2 - 2y = cz^2$$

which is the required general solution.

2.3.5 General Method

Step I : Let the condition of integrability is satisfied for the given equation

$$Pdx + Qdy + Rdz = 0 \qquad .....(28)$$

Step II : Treating one of the variables of (28), say z, as a constant then dz = 0 and the given equation is reduced to

$$Pdx + Qdy = 0$$

Integrating it, keeping z as constant. If necessary the help of an integrating factor may be taken. Let the result so obtained be

$$u\,(x, y, z) = f(z) \qquad .....(29)$$

where $f(z)$ is a function of z alone. This is possible because the arbitrary function $f(z)$ is constant with respect to x and y.

Step III : Now we differentiate (29) totally with respect to x, y, z and then compare the result with the given equation (28). We will get a relation between df and dz. If the of df and dz involve functions of x and y, it would be possible to eliminate them with the help of (22). Thus we shall get an equation in df and dz which will be independent of x and y.

Step IV : The values of $f(z)$ will be obtained by integrating the above equation. After sustituting it in (32), we get the complete solution.

Remark : General method, for solving the total differential equation of the type

$$Pdx + Qdy + Rdz = 0$$

should be adopted only when the equations are non-homogeneous and the method of inspection fails.

Ex.*11. *Solve $\mathbf{3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})\, dz = 0}$

Sol. Here, the condition of integrability is satisfied. Let us treat z as constant, so that $dz = 0$. Then the given equation become

$$3x^2dx + 3y^2dy = 0$$

On integration, we get

$$x^3 + y^3 = f(z) \qquad \text{(say)} \qquad .....(30)$$

where the constant of integration has been taken as a function $f(z)$ as we have treated z as constant.

Now differentiating (30), we have

$$3x^2dx + 3y^2dy - f'(z)\,dz = 0 \qquad .....(31)$$

Comparing (31) with the given equation, we get

$$f'(z) = x^3 + y^3 + e^{2z}$$

or $$f'(z) = f(z) + e^{2z} \qquad \text{[by using (30)]}$$

or $\quad \dfrac{df}{dz} - f = e^{2z}$, which is a linear equation having integrating factor as

$IF = e^{\int(-1)dz} = e^{-z}$. Hence the solution is

$$f(z)e^{-z} = \int\left(e^{2z}e^{-z}\right)dz + c = e^{z} + c$$

or $\quad f(z) = e^{2z} + ce^{z}$

or $\quad x^3 + y^3 = e^{2z} + ce^{z}$ [by using (30)]

Which is the required general, C being an arbitrary constant.

***Ex.*12.** $\quad \left(e^{x}y + e^{z}\right)dx + \left(e^{y}z + e^{x}\right)dy + \left(e^{y} - e^{x}y - e^{y}z\right)dz = 0$

Sol. Here, the condition of integrability is satisfied. Let us treat z as constant so that dz = 0. Then the given equation becomes

$$\left(e^{x}y\,dx + e^{z}dy\right) + \left(e^{y}zdy + e^{z}dx\right) = 0$$

On integration. we get

$$e^{x}y + e^{y}z + e^{z}x = f(z) \qquad .....(32)$$

Now differentiating equation (32), we obtain

$$\left(e^{x}y + e^{z}\right)dx + \left(e^{y}z + e^{x}\right)dy + \left(e^{y} + e^{z}x\right)dz = f'(z)dz \qquad .....(33)$$

Comparing (33) with the given equation, we get

$$e^{y} + e^{z}x - f'(z) = e^{y} - e^{x}y - e^{y}z$$

which gives

$$f'(z) = e^{x}y + e^{y}z + e^{z}x = f(z) \qquad \text{(by 32)}$$

or $\quad \dfrac{df}{dz} = f$

Integrating, we get

$$f(z) = ce^{z}$$

Putting the value of f (z) from equation (32), we get the required general solution as

$$e^{x}y + e^{y}z + e^{z}x = ce^{z}$$

Ex.*13. *Solve $\quad y^2z(x\cos x - \sin x)dx + x^2z(y\cos y - \sin y)dy$

$$+ xy(y\sin x + x\sin y + xy\cos z)dz = 0$$

Sol. Here, the condition of integrability, is satisfied. Let us treat z as constant so that $dz = 0$. Then the given equation becomes

$$y^2z(x\cos x - \sin x)dx + x^2z(y\cos y - \sin y)dy = 0$$

or $\quad \dfrac{x\cos x - \sin x}{x^2}dx + \dfrac{y\cos y - \sin y}{y^2}dy = 0$

or $$d\left(\frac{\sin x}{x}\right)+d\left(\frac{\sin y}{y}\right)=0$$

On integration, we get

$$\frac{\sin x}{x}\times\frac{\sin y}{y}=f(z) \qquad .....(34)$$

where the constant of integration has been taken as a function *f* (*z*) as we have treated *z* as constant.

Now differentiating (34), we get

$$\frac{x\cos x-\sin x}{x^2}dx+\frac{y\cos y-\sin y}{y^2}dy=f'(z)\,dz$$

or $$zy^2(x\cos x-\sin x)dx+zx^2(y\cos y-\sin y)dy-x^2y^2z\ f'(z)dz=0 \qquad .....(35)$$

Comparing (35) with the given equation, we have

$$-x^2y^2z\ f'(z)=xy(y\sin x+x\sin y+xy\cos z)$$

or $$-z\,f'(z)=\frac{\sin x}{x}+\frac{\sin y}{y}+\cos z=f(z)+\cos z \qquad \text{[by using (34)]}$$

or $$\frac{df}{dz}+\frac{1}{z}f=-\frac{\cos z}{z}$$, which is a linear equation having integrating factor (IF) as

$$IF=e^{\int(1/z)dz}=e^{\log z}=z$$ and the solution is

$$z\,f(z)=\int z\left(\frac{-\cos z}{z}\right)dz+c=-\sin z+c$$

or $$z\left(\frac{\sin x}{x}+\frac{\sin y}{y}\right)=c-\sin z \qquad \text{[by using (34)]}$$

which is the required general solution, *c* being an arbitrary constant.

Self Learning Exercise-I

1. Write down pfaffian differential equation in *n* variables.
2. Write the condition when an equation of the type $Mdx + Ndy = 0$ become exact.
3. What is the condition of integrability for the equation $Pdx + Qdy + Rdz = 0$?
4. Which equations are called homogeneous ?

2.4 Geometrical Meaning of $Pdx + Qdy + Rdz = 0$

We know that direction cosines of the tangent at a point (*x, y, z*) on a curve are proportional to *dx, dy, dz*. Therefore, the differential equation $Pdx + Qdy + Rdz = 0$ .....(1) signifies that the tangent to a curve at the point (*x, y, z*) is perpendicular to a line, whose direction cosines are proportional to *P, Q, R*.

Whereas the simultaneous equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad .....(2)$$

express that the tangent to a curve at a point (*x, y, z*) is parallel to a line with direction cosines proportional to *P, Q, R*.

We thus have two sets of curve, and if they intersect, they intersect at right angle. Now we discuss two cases.

Case I : If the equation $Pdx + Qdy + Rdz = 0$ is integrable, it means that family of surfaces can be obtained such that all curves on it are perpendicular to the curves represented by the equation (2) at all points where curves cut the surface. Since the solution of equation (1) will be of the form $\phi(x, y, z) = C$ and that of (2) will be of the form $f_1(x, y, z) = C_1$ and $f_2(x, y, z) = C_2$, it means that in this case an infinite number of surfaces can be drawn to cut orthogonally a doubly infinite set of curves.

Case II : If equation (1) is not integrable than the curves represented by $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ may not admit of such a family of orthogonal surfaces.

Ex.*1.** ***Solve ***Find the system of curves satisfying the differential equating.***

$$xdx + ydy + c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dz = 0 \quad ....(3)$$

which lie on the surface

$$1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z^2}{c^2} \quad ....(4)$$

Sol. Equation of the given surface can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad ....(5)$$

with the help of (3), the given equation can be written as

$$xdx+ydy \ +zdz=0$$

on Integration, we get

$$x^2 + y^2 + z^2 = k \quad ....(6)$$

Hence the required system of curves will be given by the intersection of (5) and (6).

Ex.*2.** ***Find the differential equation of the family of twisted cubic curves $y = ax^2$, $y^2 = bzx$. Show that all these curves cut orthogonally the family of ellipsoids $x^2 + 2y^2 + 3z^2 = c^2$.

Sol. Family of twisted cubic curves as given in question is

$$y = ax^2 \quad .....(7)$$

$$y^2 = bzx \quad ....(8)$$

On differentiating (7), we get

$$dy = 2ax\, dx$$

or $$dy = 2\frac{y}{x}dx$$ [by using (7)]

or $$2ydx - xdy = 0 \qquad ....(9)$$

Now similarly, differentiating (8), we obtain

$$2ydy = b(zdx + xdz)$$

or $$2ydy = \frac{y^2}{zx}(zdx + xdz)$$ [by using (8)]

or $$yz\,dx - 2zx\,dy + xy\,dz = 0 \qquad ....(10)$$

From (9) and (10), we get

$$\frac{dx}{-x^2y} = \frac{dy}{-2xy^2} = \frac{dz}{(-2zx)2y-(-x)yz}$$

or $$\frac{dx}{-x^2y} = \frac{dy}{-2xy^2} = \frac{dz}{-3xyz}$$

or $$\frac{dx}{x} = \frac{dy}{2y} = \frac{dz}{3z}$$

which are the required differential equations of the family of curves.

The differential equations of the surfaces which are cut orthogonally by the given curves is

$$x\,dx + 2y\,dy + 3z\,dz = 0$$

Integrating, we get

$$x^2 + 2y^2 + 3z^2 = k = c^2 \text{ (say)}$$

2.5 Equations Containing More than Three Variables

Let us consider an equation of the form

$$Pdx + Qdy + Rdz + Tdt = 0 \qquad ....(1)$$

Treating t as constant, so that $dt = 0$, then equation (1) becomes

$$Pdx + Qdy + Rdz = 0 \qquad ....(2)$$

Condition of integrability for equation (2) will be

$$P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0 \qquad ....(3)$$

Similarly If we take z, x and y as constant, then we get dz = 0, dx = 0, dy = 0. The condition of integrability in these cases will be

$$P\left(\frac{\partial T}{\partial y} - \frac{\partial Q}{\partial t}\right) - Q\left(\frac{\partial T}{\partial x} - \frac{\partial P}{\partial t}\right) + T\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0 \qquad ....(4)$$

$$Q\left(\frac{\partial T}{\partial z} - \frac{\partial R}{\partial t}\right) - R\left(\frac{\partial T}{\partial y} - \frac{\partial Q}{\partial t}\right) + T\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) = 0 \qquad ....(5)$$

and $$R\left(\frac{\partial T}{\partial x} - \frac{\partial P}{\partial t}\right) - P\left(\frac{\partial T}{\partial z} - \frac{\partial R}{\partial t}\right) + T\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) = 0 \qquad ....(6)$$

Hence we see that in the case of more than three variables, the condition of integrability must be satisfied for the coefficients of all the terms taken three at a time.

Here we note that only three of the relations (3), (4), (5) and (6) are independent and the fourth one can be derived from the remaining three.

2.6 Method for Obtaing Solution Involving Four Variables

If the condition of integrability is satisfied, then the solution the total differential equation can be obtained by two methods.

Method 1. By Inspection : In this method we can arrange the coefficients in such way that the given equation is directly integrable.

Method 2. In this method, we take any two of the four variables constant. The equation is integrated and the constant of integration is taken as the function of those variables which were kept constant. The result is compared with the given equation after obtaining its differential and in such a way the values of constants of integration are obtained. This will give the complete solution.

Ex.*1. *Solve $\quad (2x + y^2 + 2xz)\, dx + 2xy\, dy + x^2 dz = dt.$

Sol. We can write the given equation as

$$(2x + y^2 + 2xz)\, dx + 2xy\, dy + x^2 dz - dt = 0$$

we can easily verify the condition of integrability as given by equations (3), (4), (5) and (6) of §2.5.

Now the given equation can be written as $2xdx + (y^2dx + 2xy\, dy) + (2xzdx + x^2dz) - dt = 0$. Which on integration gives the complete solution as $x^2 + xy^2 + x^2z - t = c$.

Ex.*2. *Solve $\quad z\,(y + z)\, dx + z\,(t - x)\, dy + y\,(x - t)\, dz + y\,(y + z)\, dt = 0$

Sol. On comparing the given question by the standard equation $Pdx + Qdy + Rdz + Tdt = 0$, we get

$$P = z(y + z),\ Q = z(t - x),\ R = y(x - t),\ T = y(y + z)$$

Here we can easily show that the conditions of integrability (equations (3), (4), (5) and (6) of §2.5) are satisfied.

Now we solve the given question by treating two variables as constant. Treating y and z as constants so that $dy = 0$ and $dz = 0$. Then the given equation reduces to

$$z(y + z)\, dx + y(y + z)\, dt = 0$$

or $\quad zdx + ydt = 0$

On integration, we get

$$zx + yt = f\,(x, z) \quad (\text{ say}) \qquad .....(7)$$

Now on differentiation (7), we get

$$zdx + tdy + xdz + ydt = df$$

or $\quad (y + z)\,(zdx + tdy + xdz + ydt) = (y + z)\, df$

or $\quad z(y+z)\,dx + t(y+z)\,dy + x(y+z)\,dz + y(y+z)\,dt = (y+z)\,df$ .....(8)

Comparing (8) with the given equation, we have

$$t(y+z)\,dy + x(y+z)\,dz - (y+z)\,df = z(t-x)\,dy + y(x-t)\,dz$$

or $\quad (ty+xz)\,dy + (ty+xz)\,dz = (y+z)\,df$

or $\quad (ty+xz)(dy+dz) = (y+z)\,df$

or $\quad f(dy+dz) = (y+z)\,df$ [by using (7)]

or $\quad \dfrac{dt}{f} = \dfrac{dy+dz}{y+z}$ .....(9)

Integration of (9) yields

$$\log f = \log(y+z) + \log c$$

or $\quad f = c(y+z)$

or $\quad zx + yt = c(y+z)$ [by using (7)]

2.7 Total Differential Equation of Second Degree

It the given equation be of the form

$$Adx^2 + Bdy^2 + Cdz^2 + 2Ddydz + 2Edzdx + 2Fdxdy = 0$$

where A, B, C, D, E and F are functions of x, y, and z then it can be easily resolved into factors, if

$$ABC + 2DEF - AD^2 - BE^2 - CF^2 = 0$$

Let the two factors be

$$Pdx + Qdy + Rdz = 0$$

and $\quad P'dx + Q'dy + R'dz = 0$

The solutions of either of these may be obtained by the methods discussed earlier. The two general solutions taken together constitute the complete solution.

Ex.*1. *Solve $\quad \mathbf{(xdx + ydy + zdz)^2 z = \{(z^2x^2y^2)(xdx + ydy + zdz)\,dz\}}$

Sol. We can factorize the given equation as

$$(xdx + ydy + zdz)\{z(xdx + ydy + zdz) - (z^2 - x^2 - y^2)\,dz\} = 0$$

i.e., $\quad xdx + ydy + zdz = 0$ .....(1)

and $\quad z(xdx + ydy + zdz) - z^2dz + (x^2+y^2)\,dz = 0$ .....(2)

On integration of (1), we get

$$x^2 + y^2 + z^2 = c_1 \quad .....(3)$$

To obtain the integral of (2), the equation may be written as

$$z(xdx + ydy) + (x^2+y^2)\,dz = 0$$

or $\quad z^2(2xdx + 2ydy) + (x^2+y^2)\,2zdz = 0$

On integration, we get

$$z^2(x^2+y^2) = c_2 \quad .....(4)$$

Hence the required solution is

$$(x^2 + y^2 + z^2 - c_1)\,(z^2x^2 + z^2y^2 - c_2) = 0$$

Self Learning Exercise-II

1. The direction cosines of the tangent at a point (x, y, z) on a curve are proportional to _, _, _.

2. What is the equation of family of twisted cubic curves ?

2.8 Summary

In this unit, you studied about the condition of integrability of total differential equation and various methods for solving it. Now you must be knowing about the geometrical meaning of $Pdx + Qdy + Rdz = 0$ and methods of finding solution of total differential equation containing three or more than three variables

2.9 Answers of Self Learning Exercises

Exercise I

1. $\sum_{i=1}^{n} u_i dx_i = 0$, where u_i $(i = 1, 2 \ldots\ldots\ldots n)$ are n functions of some or all of n independent variables $x_1, x_2, \ldots\ldots, x_n$.

2. $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

3. $P\left(\dfrac{\partial R}{\partial y} - \dfrac{\partial Q}{\partial z}\right) - Q\left(\dfrac{\partial R}{\partial x} - \dfrac{\partial P}{\partial z}\right) + R\left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right) = 0$

4. Equation $Pdx + Qdy + Rdz = 0$ is called homogeneous if P, Q, R are homogenous functions of x, y, z of the same degree.

Exercise II

1. dx, dy, dz

2. $y = ax^2,\ \ y^2 = bzx$

2.10 Exercise

Solve the following differential equations

1. $(2x^2 + 2xy + 2xz^2 + 1)\, dx + dy + 2zdz = 0$ **[Ans.** $e^{x^2}\left(x + y + z^2\right) = c$ **]**

2. $xdy - ydx + 2x^2z\, dz = 0$ **[Ans.** $\dfrac{y}{x} + z^2 = c$ **]**

3. $(y + a)^2 dx + zdy - (y + a)dz = 0$ **[Ans.** $z = (x + c)\,(y + a)$**]**

4. $yzdx + zxdy + xy\,dz = 0$ **[Ans.** $xyz = c$]

5. $(ydx + xdy)\,(a - z) + xydz = 0$ **[Ans.** $xy = c(a - z)$]

6. $zdz + (x - a)\,dx = \{h^2 - z^2 - (x - a)^2\}^{1/2}\,dy$ **[Ans.** $h^2 - z^2 - (x - a)^2 = (y - c)^2$]

7. $zydx = zxdy + y^2dz$ **[Ans.** $x - cy - y\log z = 0$]

8. $yz^2(x^2 - yz)\,dx + x^2z(y^2 - xz)\,dy + xy^2(z^2 - xy)\,dz = 0$ **[Ans.** $x^2z + yz^2 + xy^2 = cxyz$]

9. $(y^2 + yz + z^2)\,dx + (x^2 + xz + z^2)\,dy + (x^2 + xy + y^2)\,dz = 0$

[Ans. $xy + yz + zx = c(x + y + z)$]

10. $(x^2 - y^2 - z^2 + 2xy + 2xz)\,dx + (y^2 - z^2 - x^2 + 2yz + 2yx)\,dy + (z^2 - x^2 - y^2 + 2zx + 2zy)\,dz = 0$ **[Ans.** $x^2 + y^2 + z^2 = c\,(x + y + z)$]

11. $2(y + z)\,dx - (x + z)\,dy + (2y - x + z) = 0$ **[Ans.** $(x + z)^2 = c(y + z)$]

12. $z(z - y)\,dx + (z + x)zdy + x(x + y)dz = 0$ **[Ans.** $z(x + y) = c(x + z)$]

13. $(x^2y - y^3 - y^2z)\,dx + (xy^2 - x^2z - x^3)\,dy + (xy^2 + x^2y)\,dz = 0$ **[Ans.** $\frac{x}{y} + \frac{y}{x} + \frac{z}{x} + \frac{z}{y} = c$]

14. $(y^2 + yz)\,dx + (xz + z^2)\,dy + (y^2 - xy)\,dz = 0$ **[Ans.** $y(x + z) = c(y + z)$]

15. $(y^2 + z^2 + 2xy + 2xz)\,dx + (x^2 + z^2 + 2xy + 2yz)\,dy + (x^2 + y^2 + 2xz + 2yz)\,dz = 0$

[Ans. $x(y^2 + z^2) + y(z^2 + x^2) + z(x^2 + y^2) = c$]

16. $(2xy + z^2)\,dx + (x^2 + 2yz)\,dy + (y^2 + 2xz)\,dz = 0$ **[Ans.** $x^2y + y^2z + z^2x = c$]

17. $(mz - ny)\,dx + (nx - lz)\,dy + (ly - mx)\,dz = 0$ **[Ans.** $\frac{nx - lz}{mz - ny} = c$]

18. $(\cos x + e^xy)\,dx + (e^x + e^yz)\,dy + e^y\,dz = 0$ **[Ans.** $e^yy + e^yz + \sin x = c$]

19. $2xz(y - z)\,dx + z(x^2 + 2z)\,dy + y(x^2 + 2y)\,dz = 0$ **[Ans.** $\frac{x^2 + 2z}{y - z} = \frac{c}{z} - 2$]

20. $xdy - ydx - 2x^2zdz = 0$ **[Ans.** $y = x\,(c - z^2)$]

21. $(z + z^2)\cos x\,dx - (z + z^2)\,dy + (1 - z^2)\,(y - \sin x)\,dz = 0$ **[Ans.** $y = \sin x - cze^{-z}$]

22. $y\sin\alpha\,dx + x\sin\alpha\,dy - xy\sin\alpha\,dz - xy\cos\alpha\,d\alpha = 0$ **[Ans.** $xy = c\sin\alpha\,e^z$]

23. $yzdx + 2xzdy - 3xydz = 0$ **[Ans.** $xy^2 = cz^3$]

24. $(2y^2 + 4az^2x^2)\,xdx + [3y + 2x^2 + (y^2 + z^2)^{-1/2}]\,ydy + [4z^2 + 2ax^4 + (y^2 + z^2)^{-1/2}]\,zdz = 0$

[Ans. $x^2y^2 + ax^4z^2 + y^2 + z^2 + \sqrt{(y^2 + z^2)} = c.$]

25. Find the equation of the curve that passes through the point (3, 2, 1) and cut orthogonally the family of surfaces $x + yz = c$

[Ans. $y^2 - z^2 = 3,\ y + z = 3e^{x-3}$]

□ □ □

Unit 3 : Partial Differential Equations of Second order, Monge's Method

Structure of the Unit

3.0 Objective

The purpose of this unit is to discuss partial differential equations of order two with variable co-efficients. Here you will learn how a large class of second order partial differential equations may be solved by using the methods applicable for solving ordinary differential equations ? You will also study Monge's method for solution of some special type of second order partial differential equations.

3.1 Introduction

A partial differential equation (P.D.E) is said to be of order two, if it involves at least one of the differential coefficients r, s, t and none of order higher than two. The general form of a second order partial differential equation in two independent variables x, y is given as

as $$F(x, y, z, p, q, r, s, t) = 0\ ;$$

where $$p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y},\ r = \frac{\partial^2 z}{\partial x^2},\ s = \frac{\partial^2 z}{\partial x \partial y},\ t = \frac{\partial^2 z}{\partial y^2}$$

The most general linear partial differential equation of second order in two independent variable x and y with variable coefficient is given as

$$Rr + Ss + Tt + Pp + Qq + Zz = F$$

where R, S, T, P, Q, Z, F are functions of x and y only and not all R, S, T are zero.

3.2 Solution of P.D.E. of Second Order by Inspection

Before taking up the general equation of second degree P.D.E., we discuss the solution of simple problems which can be integrated merely by inspection. On two successive integral of given P.D.E., we get the general solution which is a relation in x, y, z. To understand this, we discuss the following problems.

Ex.1. Solve t + s + q = 0

Sol. We can write the given problem as

$$\frac{\partial^2 z}{\partial y^2}+\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial z}{\partial y}=0$$

Integrating with respect to y, treating x as constant, we get

$$\frac{\partial z}{\partial y}+\frac{\partial z}{\partial x}+z=f(x) \qquad \text{or} \qquad p+q=f(x)-z$$

which is the form of standard Lagrange's linear equation $Pp + Qq = R$, so the auxiliary equation will be

$$\frac{dx}{1}=\frac{dy}{1}=\frac{dz}{f(x)-z}$$

from first two terms, we obtain

$$x-y=c_1 \text{ (constant)} \qquad .....(1)$$

and from first and last terms, we have

$$\frac{dz}{dx}+z=f(x) \qquad .....(2)$$

which is linear differential equation of first order having integrative factor e^x.

Hence the solution of (2) will be

$$z \cdot e^x = \int f(x)\, e^x dx + c_2 \text{ (constant)}$$

Therefore the required solution of given equation will be (by using (1)]

$$ze^x - \phi(x) = \psi(x-y)$$

where c_2 is a function of c_1 or of $(x-y)$.

Ex.2. Solve t – qx = x²

Sol. We can write the given problem as

$$\frac{\partial q}{\partial y}-qx=x^2 \qquad .....(3)$$

which is linear in q and y having integrating factor $e^{-x\int dy} = e^{-xy}$. Therefore the solution of (3) is

$$q \cdot e^{-xy} = \int x^2 e^{-xy}\, dy + f(x) \quad \text{(as } x \text{ is constant)}$$

or $\quad q \cdot e^{-xy} = -xe^{-xy} + f(x)$

or $\quad \dfrac{\partial z}{\partial y}=-x+f(x)e^{xy}$

Again integrating with respect to y (treating x as constant), we get.

$$z=-xy+\frac{1}{x}f(x)e^{xy}+\phi(x).$$

Ex.3. Solve $x - t = \frac{x}{y^2}$

Sol. We can write the given problem as

$$\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y^2} = \frac{x}{y^2}$$

Integrating with respect to y (treating x as constant), we get

$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = -\frac{x}{y} + f(x)$$

or $$p - q = -\frac{x}{y} + f(x)$$

which is the form of standard Lagrange's linear equation $Pp + Qq = R$, so the auxiliary equation will be

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{-x/y + f(x)}$$

From first two terms, we obtain

$$x + y = c_1 \text{ (constant)} \qquad .....(4)$$

and from first and last terms, we have

$$dz = \frac{-x}{y} dx + f(x) dx$$

or $$dz = \frac{-x}{c_1 - x} dx + f(x) dx \qquad \text{[by using (4)]}$$

or $$dz = \left\{1 - \frac{c}{c_1 - x}\right\} dx + f(x) dx$$

On integrating, we get

$$z = x + c_1 \log(c_1 - x) + \int f(x) dx + c_2$$

or $$z = x + c_1 \log y + \phi(x) + F(x + y)$$

where c_2 is a function of c_1 or of $(x + y)$.

Ex.4. Solve rx = (n – 1) p

Sol. We can write the given problem as

$$x \frac{\partial^2 z}{\partial x^2} = (n-1) \frac{\partial z}{\partial x}$$

or $$\frac{\frac{\partial^2 z}{\partial x^2}}{\frac{\partial z}{\partial x}} = \frac{n-1}{x}$$

Now integrating both sides with respect to x treating y as constant, we get

$$\log\left(\frac{\partial z}{\partial x}\right) = (n-1) \log x + \log f_1(y)$$

or $$\frac{\partial z}{\partial x} = x^{n-1} f_1(y)$$

Again integrating w.r.t. x treating y as constant, we obtain

$$z = \frac{x^n}{n} f_1(y) + f_2(y)$$

Ex.5. Solve 2yq + y^2t = 1

Sol. We can write the given problem as

$$2yq + y^2 \frac{\partial q}{\partial y} = 1$$

or $$\frac{\partial}{\partial y}\left(y^2 q\right) = 1$$

Now integrating both side with respect to y treating x as constant, we get

$$y^2 q = f_1(x)$$

or $$q = \frac{\partial z}{\partial y} = \frac{1}{y^2} f_1(x)$$

Again integrating with respect to y, we obtain

$$z = -\left(\frac{1}{y}\right) f_1(x) + f_2(x).$$

Ex.6. Show that a surface passing through the circle z = 0, $x^2 + y^2 = 1$ and satisfying the differential equation s = 8xy is z = $(x^2 + y^2)^2 - 1$

Sol. We can write the given differential equation as

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 8xy$$

Integrating with respect to x, we get

$$\frac{\partial z}{\partial y} = 4x^2 y + f(y)$$

Again integrating with respect to y, we obtain

$$z = 2x^2y^2 + \int f(y)\,dy + \phi_1(x)$$

or $$z = 2x^2y^2 + \phi_2(y) + \phi_1(x) \qquad .....(5)$$

where $$\phi_2(y) = \int f(y)\,dy$$

where ϕ_1 and ϕ_2 are two arbitrary functions.

Now given circle is

$$x^2 + y^2 = 1, \; z = 0$$

Putting $z = 0$ in (5), we get

$$2x^2y^2 + \phi_2(y) + \phi_1(x) = 0 \qquad .....(6)$$

Now, $$x^2 + y^2 = 1 \Rightarrow (x^2 + y^2)^2 = 1^2$$

or $$2x^2y^2 + x^4 + y^4 = 1 \qquad .....(7)$$

On comparing (6) with (7), we get

$$\phi_2(y) + \phi_1(x) = x^4 + y^4 - 1$$

Substituting this in (5), we obtain

$$z = 2x^2y^2 + x^4 + y^4 - 1$$

or $$z = \left(x^2 + y^2\right)^2 - 1$$

Hence the result.

Self-Learning Exercise-I

1. What is the general form of a second order p.d.e. in two independent variables x and y ?
2. The most general linear p.d.e. of second order in two independent variables x and y is ....
3. The solution of $r = 6x$ is ....

3.3 Exercise-1

Solve the following partial differential equations :

1. $ar = xy$ **[Ans.** $az = \frac{1}{6}x^3y + x\,f(y) + F(y)$]
2. $r = 2y^2$ **[Ans.** $z = x^2y^2 + x\,f(y) + F(y)$]
3. $s - t = x/y^2$ **[Ans.** $z = (x+y)\log y + f(x) + F(x+y)$]
4. $xr + p = 9x^2y^2$ **[Ans.** $z = x^3y^3 + \log a\ f(y) + F(y)$]
5. $yt - q = xy$ **[Ans.** $z = \frac{1}{2}xy^2\log y - \frac{1}{4}xy^2\frac{y^2}{2}f(x) + F(x)$]
6. $\log s = x + y$ **[Ans.** $z = e^{x+y} + f(y) + F(x)$]
7. $p + r + s = 1$ **[Ans.** $z = x + e^{-y}\left(-y\ e^y + F(y)\right) + e^{-y}f(x-y)$]
8. $ys + p = \cos(x+y) - y\sin(x+y)$ **[Ans.** $yz = y\sin(x+y) + f(x) + F(y)$]
9. $s = x/y + a$ **[Ans.** $z = \frac{x^2}{2}\log y + axy + f(x) + F(y)$]

It may he noted here that a *p.d.e.* $f(x, y, z, p, q, r, s, t) = 0$ can he integrated only in special cases. The most important method of solution, due to Monge, is applicable to a wide class of such equations but not to all equations.

3.4 Monge's Method for Solving Equation of the Type *Rr* + *Ss* +*Tt* = *V*

Monge's gives a method for solving p.d.e. of second order of the type

$$Rr + Ss + Tt = V \qquad .....(1)$$

where R, S, T and V are, in general, functions of x, y, z, p and q. Indeed this a equation of first degree in r, s and t. To solve such type of equations, first we determine the intermediate integrals. For this we have

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

or $$dp = rdx + sdy \qquad ....(2)$$

hence $$r = \frac{dp - sdy}{dx} \qquad ....(3)$$

Similarly $$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy$$

or $$dq = sdx + tdy \qquad ....(4)$$

hence $$t = \frac{dq - sdx}{dy} \qquad ....(5)$$

Now, r and t are eliminated from equation (1) with the help of (3) and (5). Thus we get an equation in s as

$$R\left(\frac{dp - sdy}{dx}\right) + Ss + T\left(\frac{dq - sdx}{dy}\right) = V$$

or $$(Rdpdy + Tdqdx - Vdydx) - s\left(Rdy^2 - Sdydx + Tdx^2\right) = 0 \qquad ....(6)$$

Equation (6) will be identically satisfied if we take

$$Rdpdy + Tdqdx - Vdydx = 0 \qquad ....(7)$$

and $$Rdy^2 - Sdydx + Tdx^2 = 0 \qquad ....(8)$$

which are called **Monge's subsidiary equations** and will provide us the intermediate integrals. Here we note that the equation (8) is quadratic for the ratio $dy : dx$ and therefore can be decomposed into two linear equations in dx and dy of the form

$$dy - m_1 dx = 0 \quad \text{and} \quad dy - m_2 dx = 0$$

Now combining equations $dy - m_1 dx = 0$ and (7) with $dz = pdx + qdy$, two integrals $u_1 = u_1(x, y, z, p, q)$ and $v_1 = v_1(x, y, z, p, q)$ can be obtained. Then we get $u_1 = f_1(v_1)$ as the first intermediate integral. Similarly on combining equations $dy - m_2 dx = 0$ and (7) with $dz = pdx + qdy$, and following the above procedure, the second intermediate integral $u_2 = f_2(v_2)$ can be obtained.

From these two intermediate integrals, the values of p and q may be obtained in terms of x and y and then substituting them in $dz = pdx + qdy$ and integrating it, the complete integral of (1) is obtained.

Ex.1. Solve $r = a^2 t$ by Monge's method.

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we get $R = 1, S = 0, T = -a^2, V = 0$. The Monge's subsidiary equations are given by

$$Rdpdy + Tdqdx - Vdydx = 0$$

and $$Rdy^2 + Sdydx + Tdx^2 = 0$$

Substituting the values of R, S, T and V, the subsidiary equations will be

$$dpdy - a^2 dqdx = 0 \qquad ....(9)$$

$$dy^2 - a^2 dx^2 = 0 \qquad ....(10)$$

Equation (10) may be factorised as

$$(dy - adx) = 0 \qquad ....(11)$$

and $$(dy + adx) = 0 \qquad ....(12)$$

Combining equation (11) with subsidiary equation (9), we get

$$dp(adx) - a^2 dqdx = 0$$

or $$dp - adq = 0 \qquad (\because dx = 0, \text{ gives trivial solution}) \qquad ....(13)$$

Now from (11) and (13) we obtain

$$y - ax = c_1,\ p - aq = c_2$$

therefore the first intermediate integral is

$$(p - aq) = f_1(y - ax) \qquad ....(14)$$

Similarly combining $(dy + adx) = 0$ with subsidiary equation (9), we get the second intermediate integral as

$$(p + aq) = f_2(y + ax) \qquad ....(15)$$

Now from above two intermediate integrals (14) and (15) we deduce the value of p and q as.

$$p = \frac{1}{2}\left[f_1(y - ax) + f_2(y + ax)\right]$$

$$q = \frac{1}{2a}\left[f_2(y + ax) - f_1(y - ax)\right]$$

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = \left(\frac{dy + adx}{2a}\right) f_2(y + ax) - \left(\frac{dy - adx}{2a}\right) f_1(y - ax)$$

On integration, we have

$$z = \frac{1}{2a}\phi_2(y + ax) - \frac{1}{2a}\phi_1(y - ax)$$

Hence the required solution is

$$z = F_1(y + ax) + F_2(y - ax)$$

Ex.2. Solve $r + (a + b)s + abt = xy$ by Monge's method.

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we have $R = 1, S = a + b, T = ab, V = xy$. Here Monge's subsidiary equations

$$Rdpdy + Tdqdx - Vdydx = 0$$

$$Rdy^2 - Sdxdy + Tdx^2 = 0$$

become $$dpdy + abdqdx - xydxdy = 0 \qquad .....(16)$$

and $$dy^2 - (a + b)\,dxdy + ab\,dx^2 = 0 \qquad .....(17)$$

Equation (17) may be factorised as

$$(dy - bdx) = 0 \qquad .....(18)$$

and $$(dy - adx) = 0 \qquad .....(19)$$

On integration $$y - bx = c_1 \qquad .....(20)$$

$$y - ax = c_2 \qquad .....(21)$$

Combining equation (18) with subsidiary equation (16), we get

$$dp\,(bdx) + abdqdx - xydx\,(bdx) = 0$$

or $$dp + adq - xy\,dx = 0$$

or $$dp + adq - x\,(c_1 + bx)\,dx = 0 \qquad \text{[by using (20)]}$$

On integration, we get

$$p + aq - \left(\frac{c_1}{2}\right)x^2 - \left(\frac{b}{3}\right)x^3 = c_3$$

or $$p + aq - \frac{x^2}{2}(y - bx) - \left(\frac{b}{3}\right)x^3 = c_3 \qquad \text{[by using (20)]}$$

or $$p + aq - \left(\frac{1}{2}\right)yx^2 + \frac{1}{6}bx^3 = c_3$$

Therefore the first intermediate integral is

$$p + aq - \frac{1}{2}yx^2 + \frac{1}{6}bx^3 = f_1(y - bx) \qquad .....(22)$$

Similarly, the second intermediate integral corresponding to equation (19) is

$$p + bq - \frac{1}{2}yx^2 + \frac{1}{6}ax^3 = f_2(y - ax) \qquad .....(23)$$

Now from above two intermediate integrals (22) and (23), we deduce the values of p and q as

$$p = \frac{1}{2}x^2y - \frac{1}{6}(a+b)x^3 + \frac{1}{a-b}\left[a\,f_2(y - ax) - b\,f_1(y - bx)\right]$$

and $$q = \frac{1}{6}x^3 + \left(\frac{1}{a-b}\right)\left[f_1(y - bx) - f_2(y - ax)\right]$$

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = \frac{1}{2}x^2ydx - \frac{1}{6}(a+b)x^3dx + \frac{1}{(a-b)}\left[af_2(y - ax\;dx - bf_1(y - bx)dx)\right]$$
$$+\frac{1}{6}x^3dy + \frac{1}{(a-b)}\left[f_1(y - bx)dy - f_2(y - ax)dy\right]$$

or $$dz = \frac{1}{6}\left(3x^2ydx + x^3dy\right) - \frac{1}{6}(a+b)x^3dx - \frac{1}{(b-a)}\left[af_2(y - ax)dx - bf_1(y - bx)dx\right]$$
$$-\frac{1}{(b-a)}\left[f_1(y - bx)dy - f_2(y - ax)dy\right]$$

or $$dz = \frac{1}{6} d\left(x^3 y\right) - \frac{1}{6}(a+b)x^3 dx + \frac{1}{(b-a)} f_2(y-ax)(dy-adx)$$
$$- \frac{1}{(b-a)} f_1(y-bx)(dy-bdx)$$

Integrating, we get the required solution as

$$z = \frac{1}{6}x^3 y - \frac{1}{24}(a+b)x^4 + \phi(y-ax) + \phi_1(y-bx)$$

Ex.3. Solve **$x^2 r + 2xy\, s + y^2 t = 0$ by Monge's method.**

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we have $R = x^2$, $S = 2xy$, $T = y^2$, and $V = 0$. Hence Monge's subsidiary equations

$$Rdpdy + Tdqdx - Vdydx = 0$$

$$Rdy^2 - Sdx\,dy + Tdx^2 = 0$$

become

$$x^2 dpdy + y^2 dqdx = 0 \qquad .....(24)$$

and $$x^2 dy^2 - 2xy\,dy + y^2 dx^2 = 0 \qquad .....(25)$$

Equation (25) may be factorised as

$$(xdy - ydx)^2 = 0$$

or $$(xdy - ydx) = 0 \qquad .....(26)$$

Combining it with the equation (24), we get

$$xdp\ (ydx) + y^2 dq\ dx = 0$$

or $$xdp + ydq = 0$$

or $$xdp + pdx + qdy + ydq = pdx + qdy$$

or $$d\,(xp) + d\,(yq) = dz$$

On integration, we get

$$px + qy = z + c_1$$

Now equation (26) gives

$$\frac{y}{x} = c_2$$

Thus the intermediate integral will be

$$px + qy = z + f(c_2)$$

which is of Lagrange's form having the subsidiary equations

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z + f(c_2)}$$

First two terms gives

$$\frac{y}{x} = c_2$$

and the last two terms gives $z + f(c_2) = cy$

Hence required solution is

$$z = yf_1\left(\frac{y}{z}\right) + f_2\left(\frac{y}{x}\right)$$

Ex.4. Solve ***(x – y) (xr – xs – ys – yt) = (x + y) (p – q) by Monge's method.***

Sol. Monge's subsidiary equations in this case will be

$$x(x-y)dpdy + y(x-y)dqdx - (x+y)(p-q)dxdy = 0 \quad .....(27)$$

and $$x(dy)^2 + (x+y)dxdy + y(dx)^2 = 0 \quad .....(28)$$

Factors of equation (28) are

$$xdy + ydx = 0,$$

which on integration gives $xy = c_1$

and $dx + dy = 0,$

which on integration gives $x + y = c_2$. Combining equation (27) with $(xdy + ydx) = 0$, we get

$$(x-y)(dp-dq) = (p-q)(dx-dy)$$

On integration, we obtain

$$\frac{p-q}{x-y} = \text{constant.}$$

Therefore the intermediate integral is

$$(p - q) = (x - y) f (xy)$$

for which the Lagrange's subsidiary equation will be

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{(x-y)f(xy)} = \frac{f(xy)(ydx+xdy)+dz}{0}$$

From first two terms, we get

From the last two relations, we get $x + y = c_2$

$$dz + f(xy)\, d(xy) = 0$$

On integration

$$z = F_1(xy) + \text{constant}$$

Hence required solution is

$$z = F_1(xy) + F_2(x + y)$$

Ex.5. Solve ***$q^2r - 2pqs + p^2t = 0$ by Monge's method.***

Sol. Monge's subsidiary equations in this case will be

$$q^2 dpdy + p^2 dqdx = 0 \quad .....(29)$$

and $$q^2dy^2 + 2pq\,dx\,dy + p^2dx^2 = 0 \quad .....(30)$$

Factors of equation (30) are

$$(qdy + pdx)^2 = 0$$

or $$qdy + pdx = 0$$

which on integration gives (after putting in $dz = pdx + qdy$)

$$\text{dz} = 0 \Rightarrow z = c_1 \qquad (\text{constant})$$

Now substituting $qdy = -pdx$ in (29), we get

$$qdp(-pdx) + p^2 dqdx = 0$$

or $$qdp - pdq = 0$$ [$dx = 0$ will give the trivial solution]

On integration, we get

$$\frac{p}{q} = b \qquad (\text{constant})$$

Therefore the intermediate integral is

$$\frac{p}{q} = f(z)$$

or $$p - q\ f(z) = 0$$

For which the Lagrange's subsidiary equation will be

$$\frac{dx}{1} = \frac{dy}{-f(z)} = \frac{dz}{0}$$

from first two terms, we get

$$y + x\ f(z) = c$$

and from last two terms, we get

$$z = c_1$$

Hence the required solution is

$$y + x\ f(z) = F(z) \quad \text{as} \quad c = F(z)$$

Ex.6. Solve ***$t - r\sec^4 y = 2q\tan y$ by Monge's method.***

Sol. Monge's subsidiary equations in this case will be

$$-\sec^4 y dpdy + dqdx - 2q\tan y\, dxdy = 0 \qquad .....(31)$$

and $$-\sec^4 y\, dy^2 + dx^2 = 0 \qquad .....(32)$$

Factors of equation (32) are

$$dx - \sec^2 y\ dy = 0, \qquad .....(33)$$

which on integration gives $x - \tan y = \text{constant}$

and $$dx + \sec^2 y\ dy = 0 \qquad .....(34)$$

which on integration gives

$$x + \tan y = \text{constant}$$

Now combining (34) with equation (31), we get

$$\sec^2 y\, dp + dq - 2q\tan y\ dy = 0$$

On integration, we get

$$p + q\cos^2 y = \text{constant} \ = f_1(x + \tan y) \qquad .....(35)$$

Similarly, when (33) is combined with (31), and integrated gives

$$p - q\cos^2 y = f_2(x - \tan y) \qquad .....(36)$$

On solving (35) and (36), we get the values of p and q as

$$p = \frac{1}{2}\left[f_1(x+\tan y) + f_2(x-\tan y)\right]$$

$$q = \frac{1}{2}\sec^2 y\left[f_1(x+\tan y) + f_2(x-\tan y)\right]$$

Substituting these values in $dz = pdx + qdy$, we obtain

$$dz = \frac{1}{2}\left[f_1(x+\tan y) + f_2(x-\tan y)\right]dx + \frac{1}{2}\left[f_1(x+\tan y) + f_2(x-\tan y)\sec^2 y\, dy\right]$$

or $$2dz = f_1(x+\tan y)\left(dx + \sec^2 y\, dy\right) + f_2(x-\tan y)\left(dx - \sec^2 y\, dy\right)$$

which on integration gives the required solution as

$$2z = F_1(x+\tan y) + F_2(x-\tan y)$$

3.5 Monge's Method for Solving Equation of the Type Rr + Ss + Tt + U (rt – s²) = V

Prof G. Monge gave a method for solving equation

$$Rr + Ss + Tt + U(rt - s^2) = V \qquad .....(1)$$

where R, S, T, U and V are, in general, functions of x, y, z, p and q.

We know that

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

or $$dp = rdx + sdy$$

or $$r = \frac{dp - sdy}{dx} \qquad .....(2)$$

Similarly

$$dq = \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy$$

therefore $$t = \frac{dq - Sdx}{dy} \qquad ......(3)$$

Putting the values of r and t from (2) and (3) in (1), we get

$$R\left(\frac{dp - sdy}{dx}\right) + Ss + T\left(\frac{dq - sdx}{dy}\right) + U\left\{\frac{dp - sdy}{dx}\cdot\frac{dq - sdx}{dy} - s^2\right\} = V$$

or $$(Rdpdy + Tdqdx + Udpdq - Vdxdy) - s\,(Rdy^2 - Sdxdy + Tdx^2 + Udpdx + Udqdy) = 0 \qquad .....(4)$$

Equation (4) will be identically satisfied if we take

$$Rdpdy + Tdqdx + Udpdq - Vdxdy = 0 \qquad .....(5)$$

and $$Rdy^2 - S\,dxdy + Tdx^2 + Udpdx + Udqdy = 0 \qquad .....(6)$$

These simultaneous equations (5) and (6) are known as Monge's subsidiary equations.

Here the equation (6) can not be factorized. So we will try to factorize

$$\left(Rdy^2 - S\,dxdy + Tdx^2 + Udpdx + Udqdy\right) + \lambda\left(Rdpdy + Tdq\,dx + Udpdq - Vdxdy\right) = 0 \qquad .....(7)$$

where λ is some multiple and is determined later.

Let us suppose that the factors of (7) are

$$\left(Rdy + m_1 Tdx + m_2 Udp\right)\left(dy + \frac{1}{m_1}dx + \frac{\lambda}{m_2}dq\right) = 0 \qquad .....(8)$$

On comparing (7) with (8), we obtain

$$\frac{R}{m_1} + m_1 T = -\left(S + \lambda V\right), \quad m_2 = m_1, \quad \frac{R\lambda}{m_2} = U \qquad .....(9)$$

The last two relations gives $m_1 = \frac{R\lambda}{U}$. Putting this in the first relation of (9), we obtain

$$\lambda^2\left(UV + RT\right) + \lambda SU + U^2 = 0 \qquad .....(10)$$

This equation is called **λ–equation**, where λ, in general, is a function of *x, y, z, p* and *q*.

Now since equation (10) is quadratic in λ so suppose that it is satisfied by two values of λ say λ_1 and λ_2 then the factors corresponding to these values will be

$$\left(Rdy + \frac{R\lambda_1}{U}Tdx + R\lambda_1 dp\right)\left(dy + \frac{U}{R\lambda_1}dx + \frac{U}{R}dq\right) = 0$$

as $$m_1 = m_2 = \frac{R\lambda_1}{U}$$

or $$\left(Udy + \lambda_1 Tdx + \lambda_1 Udp\right)\left(Udx + \lambda_1 Rdy + \lambda_1 Udq\right) = 0 \qquad .....(11)$$

Similarly corresponding to λ_2, we can obtain

$$\left(Udy + \lambda_2 Tdx + \lambda_2 Udp\right)\left(Udx + \lambda_2 Rdy + \lambda_2 Udq\right) = 0 \qquad .....(12)$$

Now one factor from (11) and one from (12) will be combined in pairs to get intermediate integrals in the form $u = f(v)$. We can combine factors as

$$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$$

$$Udx + \lambda_2 Rdy + \lambda_2 Udp = 0$$

and

$$Udx + \lambda_1 Rdy + \lambda_1 Udp = 0$$

$$Udy + \lambda_2 Tdx + \lambda_2 Udp = 0$$

These two pairs will give intermediate integrals provided these total differential equations are integrable, from which the values of *p* and *q* can be determined. Substituting these values of *p* and *q* in $dz = pdx + qdy$, we get the general solution on integration.

Ex.1. Solve $\quad 3r + 4s + t + (rt - s^2) = 1$

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - S^2) = V$, we have $R = 3$, $S = 4$, $T = 1$, $U = 1$, $V = 1$. Then λ – quadratic equation

$$\lambda^2(UV+RT)+\lambda SU+U^2=0$$

becomes $$4\lambda^2+4\lambda+1=0$$

or $$(2\lambda+1)^2=0 \Rightarrow \lambda_1=\lambda_2=-\frac{1}{2}$$

Hence there is only one intermediate integral given by the equations

$$Udy+\lambda_1 Tdx+\lambda_1 Udp=0$$

and $$Udx+\lambda_2 Rdy+\lambda_2 Udq=0$$

On putting above values, we get

$$dy+\left(-\frac{1}{2}\right)dx+\left(-\frac{1}{2}\right)dp=0$$

and $$dx+\left(-\frac{1}{2}\right)3dy+\left(-\frac{1}{2}\right)dq=0$$

or $$-2dy+dx+dp=0$$

and $$3dy-2dx+dq=0$$

On integration, we obtain

$$-2y+x+p=c_1 \qquad .....(13)$$

and $$3y-2x+q=c_2 \qquad .....(14)$$

Hence the only intermediate integral is

$$-2y+x+p=f(3y-2x+q)$$

where f is any arbitrary function

Now solving (13) and (14) for p and q, we get

$$p=2y-x+c_1$$

$$q=-3y+2x+c_2$$

Putting these values of p and q in $dz=pdx+qdy$, we get

$$dz=(2y-x+c_1)dx+(-3y+2x+c_2)dy$$

or $$dz=2(ydx+xdy)-xdx-3ydy+c_1dx+c_2dy$$

On integrating, we obtain the general solution as

$$z=2xy-\frac{1}{2}x^2-\frac{3}{2}y^2+c_1x+c_2y+c_3$$

where c_1, c_2, c_3 are arbitrary constants.

Ex.2. Solve **$2s+(rt-s^2)=1$**

Sol. Comparing the given equation with $Rr+Ss+Tt+U(rt-S^2)=V$, we have $R=0$, $S=2$, $T=0$, $U=1$, $V=1$.

Then the λ–quadratic equation

$$\lambda^2(UV+RT)+\lambda SU+U^2=0$$

becomes $$\lambda^2+2\lambda+1=0$$

giving $\lambda_1 = \lambda_2 = -1$

Hence there is only one intermediate integral given by the equations

$$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$$

and $$Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$$

On putting above values, we get

$$(dy - dp) = 0 \text{ and } dx - dq = 0$$

Integrating, we obtain

$$y - p = c_1 \text{ and } x - q = c_2 \qquad .....(15)$$

Hence the only intermediate integral is

$$(x - q) = f(y - p)$$

where f is any arbitrary function.

Now putting the values of p and q from (15) in $dz = pdx + qdy$, we get

$$dz = (y - c_1)dx + (x - c_2)dy$$

or $$dz = (y\,dx + x\,dy) - c_1 dx - c_2 dy$$

On integrating, we get the general solution as

$$z = xy - c_1 x - c_2 y + c_3.$$

Ex.3. Solve $\quad$ **$2r + (p + x) S + yt + y (rt - s^2) + q = 0$**

Sol. Comparing the given equation with standard equation we have $R = q$, $S = (p + x)$, $T = y$, $U = y$ and $V = -q$. Then λ–equation

$$\lambda^2(UV + RT) + \lambda SU + U^2 = 0$$

becomes $$\lambda^2(-yq + yq) + \lambda(p + x)y + y^2 = 0$$

Which gives $$\lambda_1 = -\left(\frac{y}{p+x}\right) = 0 \text{ and } \lambda_2 = \infty$$

Hence the two intermediate integrals are given as

$$y\,dy - \frac{y^2}{(p+x)}dx - \frac{y^2}{(p+x)}dp = 0$$

and $$0 + q\,dy + y\,dq = 0 \qquad \left(\text{as } \frac{1}{\lambda_2} = 0\right)$$

which gives

$$\left(\frac{p+x}{y}\right) = c_1 \text{ and } qy = c_2$$

Hence the intermediate integral will be given by

$$qy = f\left(\frac{p+x}{y}\right) \qquad .....(16)$$

Similarly, the second intermediate integral obtained as

$$p + x = c_3 \qquad .....(17)$$

Substituting the values of p and q from (16) and (17) in $dz = pdx + qdy$, we get

$$dz = (c_3 - x)\,dx + \frac{1}{y} f\left(\frac{p+x}{y}\right) dy$$

or
$$dz = (c_3 - x)\,dx + \frac{1}{y} f\left(\frac{c_3}{y}\right) dy$$

On integration, we get the general solution as

$$z = c_3 x - \frac{1}{2}x^2 + F\left(\frac{c_3}{y}\right) + G(c_3)$$

Ex.4. Solve $\quad$ **$(rt - s^2) - s\,(\sin x + \sin y) = \sin x \sin y$**

Sol. Comparing the giving equation with standard equation we have $R = 0$, $S = -(\sin x + \sin y)$, $T = 0$, $U = 1$, and $V = \sin x \sin y$. Then λ-equation is

$$\lambda^2 (UV + RT) + \lambda U + U^2 = 0$$

becomes
$$\lambda^2 (\sin x \sin y) - \lambda(\sin x + \sin y) + 1 = 0$$

which gives
$$\lambda_1 = \operatorname{cosce} x \text{ and } \lambda_2 = \operatorname{cosce} y$$

The first intermediate integral is given by

$$\sin x\, dy + dp = 0, \quad \sin y\, dx + dq = 0$$

which are not integrable.The other intermediate integrable is given by

$$\sin y dy + dp = 0, \quad \sin x dx + dq = 0$$

On integration, we get

$$p - \cos y = c_1 \quad \text{and } q - \cos x = c_2$$

Hence the intermediate integral will be given by

$$(p - \cos y) = f(q - \cos x)$$

This can not be integrated further unless we know f. Therefore, let us suppose that the arbitrary function f is linear, *i.e.,*

$$(p - \cos y) = \alpha(q - \cos x) + \beta \qquad .....(18)$$

where α and β are constants.

Lagrange's subsidiary equations for (18) will be

$$\frac{dx}{1} = \frac{dy}{-\alpha} = \frac{dz}{\cos y - \alpha \cos x + \beta}$$

From first two terms, we get

$$y + \alpha x = c_3$$

and from the first and last term, we obtain

$$dz = \left[\cos(c_3 - \alpha x) - \alpha \cos x + \beta\right] dx$$

On integration, we get the general solution as

$$\alpha z + \sin y + \alpha^2 \sin x - \alpha\beta x = \alpha c_4$$

Ex.5. Solve $\quad z(1+q^2)r - 2pqzs(1+p^2)t + z^2(rt - s^2) + 1 + p^2 + q^2 = 0$

Sol. Comparing the given equation with the standard equation, we have $R = z(1+q^2)$,

$S = -2pqz$, $T = (1+p^2)$, $U = z^2$ and $V = (1+p^2+q^2)$. Then the λ-equation is

$$\lambda^2 p^2 q^2 - 2\lambda pqz + z^2 = 0$$

or
$$(\lambda pq - z)^2 = 0$$

which gives
$$\lambda = \frac{z}{pq}$$

Putting the value of λ in

$$U dy + \lambda T\, dx + \lambda U\, dp = 0$$

and
$$U\, dx + \lambda R\, dy + \lambda U\, dq = 0$$

we get

and
$$pq dy + (1+p^2)\, dx + z dp = 0 \qquad .....(19)$$

$$pq dx + (1+q^2)\, dy + z dp = 0 \qquad .....(20)$$

$$dz = p\, dx + q\, dy \qquad .....(21)$$

Combining (19) and (21), and on integration, we obtain

$$x + zp = c_1 \qquad .....(22)$$

Similarly by combining (20) and (21), and on integration, we obtain

$$y + zq = c_2 \qquad .....(23)$$

Putting the values of p and q obtained from (22) and (23) in $dz = p\, dx + q dy$, we get

$$dz = \left(\frac{c_1 - x}{z}\right) dx + \left(\frac{c_2 - y}{z}\right) dy$$

Integrating
$$z^2 + (c_1 - x)^2 + (y - c_2)^2 = c_3$$

which is the required solution.

Ex.6. Solve $\quad \mathbf{5r + 6s + 3t + 2(rt - s^2) + 3 = 0}$

Sol. Comparing the given equation with the standard equation, we have $R = 5$, $S = 6$, $U = 2$, and $V = -3$. Then the λ–equation will be

$$9\lambda^2 + 12\lambda + 4 = 0$$

or
$$(3\lambda + 2)^2 = 0$$

which gives
$$\lambda_1 = \lambda_2 = -\frac{2}{3}$$

There is only one intermediate integral given by the equations

$$2dy + \left(-\frac{2}{3}\right)\cdot 3\, dx + \left(-\frac{2}{3}\right)\cdot 2dp = 0$$

and
$$2dx + \left(-\frac{2}{3}\right)\cdot 5\, dy + \left(-\frac{2}{3}\right)\cdot 2dq = 0$$

or
$$3\, dy - 3\, dx - 2dp = 0$$

and $\quad 3\,dx - 5\,dy - 2dq = 0$

Integrating,we get $\quad 3y - 3x - 2p = c_1$ .....(24)

and $\quad 3x - 5y - 2q = c_2$ .....(25)

Hence the only intermediate integral is

$$3y - 3x - 2p = f(3x - 5y - 2q) \quad .....(26)$$

where f is an arbitrary function.

Solving (24) and (25) for p and q, we get

$$p = \frac{1}{2}(3y - 3x - c_1) \quad \text{and} \quad q = \frac{1}{2}(3x - 5y - c_2)$$

Putting p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{1}{2}(3y - 3x - c_1)\,dx + \frac{1}{2}(3x - 5y - c_2)\,dy$$

or $\quad 2dz = 3\,(ydx + xdy) - 3xdx - 5ydy - c_1dx - c_2dy$

Integrating, we get

$$2z = 3xy - \left(\frac{3}{2}\right)x^2 - \left(\frac{5}{2}\right)y^2 - c_1x - c_2y + c_3$$

which is the required solution. c_1, c_2 and c_3 are arbitrary constants.

Self-Learning Exercise-II

1. For p.d.e. $R\,r + Ss + Tt = V_1$ the Monge's subsidiary equations are .... and ....
.

2. The Monge's subsidiary equations for p.d.e. $r = kt$ are .... and ....

3. The λ-equation in Monge's method for solving p.d.e. $r + 3s + t + (rt - s^2) = 1$ is ....

3.6 Summary

In this unit, you learn about partial differential equations of second order and their solution. You also studied the solution of two types of P.D.E. by Monge's method.

3.7 Answers of Self–Learning Exercise

Exercise-I

1. $F(x, y, z, p, q, r, s, t) = 0$

2. $Rr + Ss + Tt + Pp + Qq + Zz = F$

3. $z = x^3 + x\;f(y) + \phi(y)$

Exercise-II

1. $R\,dpdy + T\,dq\,dx - V\,dy\,dx = 0$

$R\,dy^2 - S\,dy\,dx + Tdx^2 = 0$

2. $dp\,dy - R\,dq\,dx = 0$ and $dy^2 - r\,dx^2 = 0$

3. $2\lambda^2 + 3\lambda + 1 = 0$

3.8 Exercise-II

Solve the following P.D.E by Monge's method :

1. $pt - qs = q^3$ **[Ans.** $y = xz + f(z) + F(x)$]
2. $y^2r - 2ys + t = p + 6y$ **[Ans.** $z = y^3 - y\,f(y^2 + 2x) + F(y^2 + 2x)$]
3. $2x^2r - 5xys + 2y^2t + 2(px + qy) = 0$ **[Ans.** $z = f(x^2 y) + F(xy^2)$]
4. $(1+q)^2 r - 2(1+p+q+pq)s + (1+p)^2 t = 0$ **[Ans.** $y = f(x + y + z) + x\,F(x + y + z)$]
5. $(q + 1)\,s = (p + 1)\,t$ **[Ans.** $z = f(x) + \phi(x + y + z)$ or $y - \psi(x + y + z = \phi(x)$]
6. $r - t\cos^2 x + p\tan x = 0$ **[Ans.** $2z = f(y + \sin x) - F(y - \sin x)$]
7. $s^2 - rt = a^2$ **[Ans.** $z = x\,f_1(q - ax) + qy + \phi(q - ax)$]
8. $ar + bs + ct + e(rt - s^2) = b$, where $a, b, c, e,$ and h are constants

$$\left[\textbf{Ans.}\quad ez = xf_1(ay + eq - m_2x) - \frac{x^2}{2} - \frac{ay^2}{2} + y(ay + eq) + \text{constant}\right]$$

9. $2\,pr + 2qt - 4pq\,(rt - s^2) = 1$ $\left[\textbf{Ans.}\quad 3z = \pm\,2(c_1 + x)^{3/2} \pm 2(c_2 + y)^{3/2} + c_3\right]$

Solve the following partial differential equations :

10. $2r + te^x - (rt - s^2) = 2e^x$ **[Ans.** $z = e^x + bx + y^2 - ay + c$]
11. $3r + s + t + (rt - s^2) + 9 = 0$ $\left[\textbf{Ans.}\quad z = cy - 2xy - \frac{x^2}{2} - \frac{3y^2}{2} + f(c - 5x) + F(c)\right]$
12. $r + 3s + t + (rt - s^2) = 1$ $\left[\textbf{Ans.}\quad z = -\frac{1}{2}(x - y)^2 + F_1(\alpha) + F_2(\beta) - \beta\,f_2(\beta) + \beta\,f_1(\beta)\right]$
13. $(rt - s^2) + 3s = 2$ $\left[\textbf{Ans.}\quad x = \frac{1}{2}(\beta - \alpha);\ y = f(\alpha) - g(\beta);\ z = \{xy - \phi(\alpha) + \Psi(\beta) + \beta y\}\right]$
14. $qxr + (x + y)s + pyt + xy(rt - s^2) = 1 - pq$ $\left[\textbf{Ans.}\quad z + \frac{1}{m}y + mx - n\log x = f(x^m y)\right]$

□ □ □

Unit 4 : Classification of Linear PDE of Second Order, Cauchy Problem and Method of Separation of Variables

Structure of the Unit

4.0 Objective

Partial differential equations generally occur in the problems of physics and engineering. After studying this unit, you should be able to identify and classify partial differential equations (PDE). You will have an idea of Cauchy problem. At last you will get knowledge of how to solve the partial differential equations by method of separation of variables.

4.1 Introduction

The importance of partial differential equations among the topis of applied mathematics has been recognized for many years. However, the increasing complexity of today's technology is demanding of the mathematician, the engineer and the scientists, an understanding of the subject previously attained only by specialists. This unit of partial differential equations (PDE) comprises identification and classification of PDE. It also presents the principal technique method of separation of variables for constructing solution to partial differential equation problems. The solved and supplementary problems have the vital role of applying reinforcing and sometimes expanding the theoretical concepts.

4.2 Classification of PDE of Second Order

Consider the second order partial differential equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad ....(1)$$

where R, S and T are continuous functions of x and y only possessing continuous partial derivatives. The PDE can be classified into three categories depending on nature of values of the discriminant $S^2 - 4RT$. Thus (1) is known as

(i) Hyperbolic if $S^2 - 4RT > 0$

(ii) Parabolic if $S^2 - 4RT = 0$

(iii) Elliptic if $S^2 - 4RT < 0$

***Ex.* 1 :** Consider the one dimensional Laplace's equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ *i.e.* $r + t = 0$. Comparing it with equation (1), we have $R = 1$, $S = T = 0$. Hence $S^2 - 4RT = 0$ and so given equation is parabolic.

***Ex.* 2 :** Consider the one dimensional diffusion equation $\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}$ *i.e.* $r - q = 0$. Comparing it with equation (1), we have $R = 1$, $S = 0$ and $T = -1$. Hence $S^2 - 4RT = 4 > 0$ and so given equation is hyperbolic.

4.3 Classification of a Second Order PDE in More Than Two Independent Variables

A linear second order partial differential equation having more than two independent variables can suitably be reduced, in general, to a canonical form only when the coefficients are constants. Let x, x_2,..., x_n be n independent variables and u be the dependent variable, then such a second order PDE may be written as

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = 0 \qquad \text{....(1)}$$

where a_{ij}, b_i and c are constants and $a_{ij} = a_{ji}$. Now we consider a one-one transformation

$$\xi_i = \xi_i (x_1, x_2, ..., x_n),\ i = 1, 2, ..., n \qquad \text{....(2)}$$

Then the equation (1) transforms to

$$\sum_{k=1}^{n} \sum_{l=1}^{n} A_{kl}\, u_{\xi_k \xi_l} + F\left(\xi_1, \xi_2, ..., \xi_n ; u, u_{\xi_1}, u_{\xi_2}, ..., u_{\xi_n}\right) = 0 \qquad \text{.....(3)}$$

where $$A_{kl} = a_{ij} \left(\xi_k\right)_{x_i} \left(\xi_i\right)_{x_j} \qquad \text{...(4)}$$

The characteristic quadratic $Q(\alpha)$ associated with equation (1) in this case is

$$Q(\alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, \alpha_i \alpha_j \qquad \text{...(5)}$$

The associated real symmetric matrix in this case will be

$$M = \begin{bmatrix} a_{11} & a_{12} & ..... & a_{1n} \\ a_{21} & a_{22} & ..... & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & ..... & a_{nn} \end{bmatrix} \qquad \text{....(6)}$$

and the characteristic roots "eigenvalues" will be given by

$$|M - \alpha I| = 0 \qquad ...(7)$$

and their nature and signs will determine the type of the given PDE.

Case I : Elliptic PDE : If all the eigenvalues are nonzero and of the same sign, the given PDE is of elliptic type.

Case II : Hyperbolic PDE **:** If all the eigenvalues are nonzero and have the same sign except precisely one of them, the given PDE is of hyperbolic type.

Case III : Ultra Hyperbolic PDE ($n \geq 4$) : If all the eigenvalues are nonzero and atleast two of them are positive and two negative then the given PDE is of ultra hyperbolic type.

Case IV : Parabolic PDE : If any of the eigenvalues is zero, the given PDE is of parabolic type.

Note : As an alternative of finding the eigenvalues of matrix M, which sometimes may be cumber -some, the classification can be made with the help of by expressing the quadratic form (5) as a sum of squares. The number of positive and negative squares will be the same as the number of positive and negative eigenvalues of the associated matrix. Either of the methods, as per convenience, may be chosen for the classification of partial differential equation.

Ex. 1. Determine the nature of following PDE

$$\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$$

Sol.
$$\frac{\partial^2 z}{\partial x^2} - x^2 \frac{\partial^2 z}{\partial y^2} = 0$$

Comparing with standard second order PDE, we have

$$R = 1, S = 0, T = -x^2$$

$$S^2 - 4RT = 0 - 4(-x^2) = 4x^2$$

Since $x^2 > 0$, therefore given PDE is hyperbolic.

Ex. 2. Classify the following PDE as hyperbolic, parabolic or elliptic :

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

Sol. On comparing it with equation (1), we have

$$R = 1, S = 2, T = 1$$

Hence the value of discriminant $S^2 - 4RT = 0$

Therefore given PDE is parabolic in nature.

Ex. 3. Find the nature of following PDE

$$3\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + 5\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial y} = 0$$

Sol. On comparing given equation with standard PDE, we have

$$R = 3, S = 2, T = 5$$

So $S^2 - 4RT = 1 - 15 = -14 < 0$

then given PDE is elliptic in nature.

Ex. 4. Show that the equation $\frac{\partial^2 z}{\partial x^2} + 2x\frac{\partial^2 z}{\partial x^2 y} + \left(1 - y^2\right)\frac{\partial^2 z}{\partial y^2} = 0$

is elliptic for values of x and in the region $x^2 + y^2 < 1$, parabolic on the boundary and hyperbolic outside this region.

Sol. Given equation is

$$\frac{\partial^2 z}{\partial x^2} + 2x\frac{\partial^2 z}{\partial x \partial y} + \left(1 - y^2\right)\frac{\partial^2 z}{\partial y^2} = 0$$

Obviously $R = 1,\ S = 2x,\ T = 1 - y^2$

Now discriminant is

$$S^2 - 4RT = 4x^2 - 4(1 - y^2) = 4(x^2 + y^2 - 1)$$

Given equation is elliptic in nature if

$$S^2 - 4RT < 0$$

or $4(x^2 + y^2 - 1) < 0 \Rightarrow x^2 + y^2 < 1$ (inside boundary)

Given equation is parabolic in nature if

$$S^2 - 4RT = 0$$

or $4(x^2 + y^2 - 1) = 0 \Rightarrow x^2 + y^2 = 1$ (on boundary)

Given equation is hyperbolic in nature if

$$S^2 - 4RT > 0$$

or $4(x^2 + y^2 - 1) > 0 \Rightarrow x^2 + y^2 > 1$ (outside the boundary)

Ex. 5. Classify the following differential equation as to type in the second quadrant of xy-plane

$$\sqrt{y^2 + x^2}\frac{\partial^2 u}{\partial x^2} + 2(x - y)\frac{\partial^2 u}{\partial x \partial y} + \sqrt{y^2 + x^2}\frac{\partial^2 u}{\partial y^2} = 0$$

Sol. : Here $R = \sqrt{y^2 + x^2},\ S = 2(x - y),\ T = \sqrt{y^2 + x^2}$

Now $S^2 - 4RT = 4(x - y)^2 - 4(x^2 - y^2)$

$= 4(x^2 + y^2 - 2xy - y^2 - x^2)$

$= -8xy$

In second quadrant, y is positive while x is – negative,therefore

$$S^2 - 4RT = +\text{ve} > 0$$

Hence given PDE is **hyperbolic** in nature.

Ex. 6. Classify the equations :

(a) $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 2\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y \partial z}$

(b) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

Sol. ***(a)*** The given PDE can be written as

$$\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}-2\frac{\partial^2 u}{\partial x\partial y}-2\frac{\partial^2 u}{\partial y\partial z}=0$$

Here $a_{11}=1,\ a_{22}=2,\ a_{33}=1,$

$a_{12}=a_{21}=-1,\ a_{23}=a_{32}=-1,\ a_{13}=a_{31}=0,$

therefore the quadratic form

$$Q(\alpha)=a_{ij}\,\alpha_i\,\alpha_j$$

becomes

$$Q(\alpha)=\alpha_1^2+2\alpha_2^2+\alpha_3^2-2\alpha_1\alpha_2-2\alpha_2\alpha_3$$

$$=(\alpha_1-\alpha_2)^2+(\alpha_2-\alpha_3)^2+(0)^2$$

here the two shares are positive and one is zero therefore the given PDE is of parabolic type.

Aliter : The associated matrix is

$$M=\begin{bmatrix}-1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}$$

The eigenvalues of the matrix are given by

$$|M-\alpha I|=0$$

$$\Rightarrow (1-\alpha)(\alpha^2-3\alpha)=0 \ \text{ i.e. } \alpha=0,\ \alpha=1,\ \alpha=3$$

Since one of the eigenvalues is zero, the given PDE is a parabolic type

(b) The given equation can be written as

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=0$$

Here $a_{11}=1,\ a_{22}=1,\ a_{33}=1,\ a_{44}=-\frac{1}{c^2}$ and $a_{ij}=a_{ji}=0,\ i\neq j$

Hence the quadratic form

$$Q(\alpha)=a_{ij}\,\alpha_i\,\alpha_j$$

becomes

$$Q(\alpha)=\alpha_1^2+\alpha_2^2+\alpha_3^2-\left(\frac{1}{c}\alpha_4\right)^2$$

This shows that the three shares are positive and only one is negative and therefore the given PDE is of hyperbolic type.

Ex. 7. Classify the equations

$$\frac{\partial^2 u}{\partial x^2}+3\frac{\partial^2 u}{\partial y^2}+84\frac{\partial^2 u}{\partial z^2}+28\frac{\partial^2 u}{\partial y\partial z}+16\frac{\partial^2 u}{\partial z\partial x}+2\frac{\partial^2 u}{\partial x\partial y}=0$$

Sol. Here, $a_{11}=1,\ a_{22}=3,\ a_{33}=84$

$a_{12}=a_{22}=1,\ a_{23}=a_{32}=14,\ a_{31}=a_{13}=8.$

The associated matrix is

$$M=\begin{bmatrix}1 & 1 & 8\\ 1 & 3 & 14\\ 8 & 14 & 84\end{bmatrix}$$

The eigenvalues of the matrix are given by

$$|M - \alpha I| = 0$$

$$\Rightarrow \qquad \alpha^3 - 98\alpha^2 + 78\alpha - 4 = 0$$

By Descarte's rule of signs, The given equation has all the three positive roots and therefore the given PDE is of elliptic type.

Aliter : The quadratic form

$$Q(\alpha) = a_{ij}\alpha_i\,\alpha_j$$

becomes $\quad Q(\alpha) = \alpha_1^2 + 3\alpha_2^2 + 84\alpha_3^2 + 2\alpha_1\alpha_2 + 16\alpha_1\alpha_3 + 28\alpha_2\alpha_3$

$$= \left(\alpha_1 + \alpha_2 + 8\alpha_3\right)^2 + \left\{\sqrt{2}\left(\alpha_2 + 3\alpha_3\right)^2\right\} + \left(\sqrt{2}\,\alpha_3\right)^2$$

Here all the three squares are positive the given PDE is of elliptic type.

Self -Learning Exercise–1

1. Mark the correct alternative :

   (i) The second order PDE $Rr + Ss + Tt + f(x, y, z, p, q) = 0$ is parabolic if

   (a) $S^2 - 4RT > 0$ *(b)* $S^2 - 4RT = 0$ *(c)* $S^2 - 4RT < 0$ *(d)* none of these

   (ii) The PDE $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$ is

   (a) hyperbolic *(b)* parabolic *(c)* elliptic *(d)* none of these

   (iii) In the region $x^2 > 4y$ the PDE $\dfrac{\partial^2 u}{\partial x^2} + x\dfrac{\partial^2 u}{\partial x \partial y} + y\dfrac{\partial^2 u}{\partial y^2} = 0$ is

   (a) hyperbolic *(b)* parabolic *(c)* elliptic *(d)* none of these

   (iv) The differential equation $4\dfrac{\partial^2 u}{\partial x^2} - 16\dfrac{\partial^2 u}{\partial x\,\partial y} + 9\dfrac{\partial^2 u}{\partial y^2} = 0$ is

   (a) hyperbolic *(b)* parabolic *(c)* elliptic *(d)* none of these

2. Write the condition under which *a* second order PDE in more than two independent variables is elliptic.
3. The region in which the equation $(x \log y)\, r + 4yt = 0$ is hyperbolic is...
4. Classify the following PDE $4\dfrac{\partial^2 u}{\partial x^2} + 4\dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$
5. Classify the PDE $5\dfrac{\partial^2 u}{\partial x^2} - 9\dfrac{\partial^2 u}{\partial x \partial y} + 4\dfrac{\partial^2 u}{\partial y^2} = 0$
6. Classify the PDE $\dfrac{\partial^2 u}{\partial t^2} + t\dfrac{\partial^2 u}{\partial x \partial t} + x\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} + 6u = 0$

4.4 Cauchy Problem

The Cauchy problem is a boundary value problem dealing with the unique solution of a second order quasi–linear PDE when its initial value and slope are specified.

Statement : Determine the solution $z = z(x, y)$ of the PDE

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad ...(1)$$

where R, S and T are in general functions of x, y, z, p and q such that the solution takes on a given space curve C, having the parametric equation

$$x = x(t),\ y = y(t),\ z = z(t) \quad ...(2)$$

prescribed value of z and $\frac{\partial z}{\partial n}$, where n is the distance measured along the normal to the curve.

The latter set of boundary conditions is equivalent to assuming that the values of x, y, z, p, q are determined on the curve, but it be noted that the values of p and q can not be assigned arbitrarily along the curve.

Method of solution : The solution of eq. (1) will be some surface, called **integral surface**, passing through C. Hence at each point of C, by relations (2) we have

$$\dot{z}_0 = p\dot{x} + q\dot{y}_0 \quad .....(3)$$

which shows that p_0 and q_0 are not independent.

Thus, the Cauchy problem finds the solution of (1) passing through the integral strip of the first order formed by the planar elements $(x_0, y_0, z_0, p_0, q_0)$ of the curve C. At every point of the integral strip $p_0 = p_0(t),\ q_0 = q_0(t)$, so that of we differentiate these equation w.r.t. 't' we find that

$$\dot{p}_0 = r\dot{x}_0 + s\dot{y}_0,\ \ \dot{q}_0 = s\dot{x}_0 + t\dot{y}_0 \quad ....(4)$$

Knowing $R, S, T, f,\ \dot{x}_0, \dot{y}_0, p_0, q_0, \dot{p}_0, \dot{q}_0$ at each point of C, we may regard equations (1) and (4) as linear simultaneous equations for the unknowns r, s, t at each point of C. Solving by Cramer's's rule, we get

$$\frac{r}{\Delta_1} = \frac{-s}{\Delta_2} = \frac{t}{\Delta_3} = -\frac{1}{\Delta} \quad .....(5)$$

where

$$\Delta_1 = \begin{vmatrix} S & T & f \\ \dot{y}_0 & 0 & -\dot{p}_0 \\ \dot{x}_0 & \dot{y}_0 & -\dot{q}_0 \end{vmatrix},\quad \Delta_2 = \begin{vmatrix} R & T & f \\ \dot{x}_0 & 0 & -\dot{p}_0 \\ 0 & \dot{y}_0 & -\dot{q}_0 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} R & S & f \\ \dot{x}_0 & \dot{y}_0 & -\dot{p}_0 \\ 0 & \dot{x}_0 & -\dot{q}_0 \end{vmatrix} \quad .....(6)$$

$$\Delta = \begin{vmatrix} R & S & T \\ \dot{x}_0 & \dot{y}_0 & 0 \\ 0 & \dot{x}_0 & \dot{y}_0 \end{vmatrix} \quad .....(7)$$

If $\Delta \neq 0$, we can easily calculate the expressions for second order derivatives r_0, s_0 and t_0 along C.

The third order partial differential coefficient of z can similarly be calculated at every point of C by differentiating (1) w.r.t. x and y respectively, making use of

$$\dot{r}_0 = z_{xxx}\dot{x}_0 + z_{xxy}\dot{y}_0 \qquad .....(8)$$

etc. and solving as above.

Proceeding in this manner, we can calculate partial derivatives of every order of the points of C. The values of the function z at neighbouring points, can be obtained by using Taylor's Theorem for functions of two independent variables. Thus the Cauchy problem possesses a solution as long as $\Delta \neq 0$. In the elliptic case $4RT - S^2 > 0$, so that $\Delta \neq 0$ always holds and the derivatives, of all orders, of z are uniquely determined.

If $\Delta = 0$, then the Cauchy's method of solution breaks down. This critical case leads to the condition

$$R\dot{y}^2 - S\dot{x}\dot{y} + T\dot{x}^2 = 0$$

or

$$Rdy^2 - Sdydx + Tdx^2 = 0 \qquad ....(9)$$

At each point $(x, y, 0)$ of Γ (which is orthogonal projection of the curve C on the plane $z = 0$) the eq. (9) would give a pair of directions along which $\Delta = 0$. These directions are called as **characteristics.**

Thus curves known as **characteristic base curves,** may be drawn through every point $(x, y, 0)$ of the base curve Γ simply by approximating them by straight line segments whose directions are taken to coincide with those of the tangents given by the roots of (9), viz.

$$\frac{dy}{dx} = \frac{S \pm \sqrt{S^2 - 4RT}}{2R} \qquad .....(10)$$

Thus a curve Γ in the xy plane satisfying (10) is called a characteristic base curve of the PDE (1), and the curve C of which it is the projection is called a **characteristics curve** of the same equation.

Note that **characteristics** are :

(i) Real and distinct if $S^2 - 4RT > 0$

(ii) Coincident if $S^2 - 4RT = 0$ and

(iii) Imaginary if $S^2 - 4RT < 0$

Hence these are two families of characteristics if the given PDE is hyperbolic, one family if it is parabolic and none if it is elliptic. Thus the Cauchy problem will fail to have unique solution if an arc element of the base curve Γ coincides with the characteristics. Consequently, the condition $\Delta \neq 0$ is both necessary and sufficient to solve the Cauchy problem.7

Characteristic equations :

Corresponding to (1), consider λ–quadratic

$$R\lambda^2 + S\lambda + T = 0 \qquad ....(11)$$

when $S^2 - 4RT \geq 0$, eq. (11) has real roots. Then, the ordinary differential equation

$$\frac{dy}{dx} + \lambda(x, y) = 0$$

are called the characteristic equations.

Again the solution of (11) will be characteristic curves or simply the characteristic of the second order PDE (1).

4.5 Method of Separation of Variables

For given linear second order partial differential equation

$$Rr + Ss + Tt + Pp + Qq + Zz = F(x,y) \quad ....(1)$$

where R, S, T, P, Q, Z and F are functions of independent variables x and y only. Let $Z(x,y)$ be solution of (1).

The method of separation of variables for this problem is a powerful tool and begins with assumption that $Z(x,y)$ is of the form $X(x) . Y(y)$ i.e.

$$Z(x,y) = X(x) . Y(y) \quad ....(2)$$

where X is function of independent variables x only and Y is function of independent variables y only.

On substituting (2) in (1) we have

$$\frac{1}{X} f(D) X = \frac{1}{Y} g(D') Y \quad ....(3)$$

where f(D) and g(D′) are quadratic functions of $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$ respectively. This has the effect of replacing the single PDE with two second order linear ordinary differential equations since LHS of (3) is function of x alone and the RHS is function of y alone. Since x and y are independent variables, the two sides of (3) will be equal only if each side is a constant (say λ) be

$$\frac{1}{X} f(D) X = \frac{1}{Y} g(D') Y = \lambda$$

or $$f(D) X = \lambda X \quad \text{and} \quad g(D') Y = \lambda Y \quad .....(4)$$

which can be solved by the methods of ordinary differential equation.

The theory of eigenfunction expansions enters into the treatment of any in homogenous aspect of the problem. The general solution of equation (4) will depend on the choice of λ positive or negative or zero. In practical problems, the nature of the boundary conditions determine the nature of λ and it becomes an eigenvalue problem.

The method of separation of variables can be employed in a similar manner for more than two independent variables also.

In the application of ordinary linear differential equation, we first find the general solution and then determine the arbitrary constant from the initial values, But the same method is not applicable to problem involving PDE In method of separation of variables right from the beginning we try to find the particular solution of PDE which satisfy all or some of the boundary conditions and then the remaining conditions are also satisfied. The combination of these particular solutions gives the solution of the problem.

Ex. 1. Find the characteristics of

$$\boldsymbol{y^2 r - x^2 t = 0.}$$

Sol. Given $$y^2 r - x^2 t = 0 \quad ....(5)$$

Comparing (5) with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \text{ we have} \quad .....(6)$$

$$R = y^2,\ S = 0 \text{ and } T = -x^2$$

Hence $\quad S^2 - 4RT = 0 - 4y^2(-x^2) = 4x^2y^2 > 0$

and thus (1) is hyperbolic everywhere except on the coordinate axes $x = 0$ and $y = 0$. The λ quadratic is

$$R\lambda^2 + S\lambda + T = 0 \text{ or } y^2\lambda^2 - x^2 = 0 \qquad ....(7)$$

Solving (7), we get $\lambda = \frac{x}{y}, -\frac{x}{y}$ (two district real roots)

Corresponding characteristic equations are

$$\frac{dy}{dx} + \frac{x}{y} = 0 \text{ and } \frac{dy}{dx} - \frac{x}{y} = 0$$

or $\quad xdx + ydy = 0$ and $xdx - ydy = 0$

Integrating, we get

$$x^2 + y^2 = C_1 \quad \text{and} \quad x^2 - y^2 = C_1$$

which are required families of characteristics.

Here these are families of circles and hyperbolas respectively.

Ex. 2. Find the characteristics of

$$\boldsymbol{x^2r + 2xys + y^2t = 0} \qquad .....(8)$$

Sol. Comparing (8) with (6) we have

$$R = x^2,\ S = 2xy \text{ and } T = y^2$$

Hence $\quad S^2 - 4RT = 0$

and hence (3) is parabolic everywhere. The λ quadratic is

$$\lambda^2x^2 + 2\lambda xy + y^2 = 0$$

Solving it we get

$$(\lambda x + y)^2 = 0 \quad \text{or } \lambda = -\frac{y}{x}, -\frac{y}{x} \text{ (two equal roots)}$$

The characteristic equations is

$$\frac{dy}{dx} - \frac{x}{y} = 0 \text{ or } \frac{1}{y}dy - \frac{1}{x}dx = 0$$

Integrating, we get

$$\frac{y}{x} = c_1 \quad \text{and} \quad y = c_1x \qquad ....(9)$$

which is the required family of characteristics. (9) represents a family of straight lines passing through the origin.

Ex. 3. Solve the followings P.D.E.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}, \quad \boldsymbol{0 < x < \pi, \quad y > 0}$$

satisfying the boundary conditions

(i) $\boldsymbol{z = 0}$ ***when*** $\boldsymbol{x = 0}$

(ii) $\boldsymbol{z = 0}$ ***when*** $\boldsymbol{x = \pi}$

(iii) $\boldsymbol{z = \sin 3x}$ ***when*** $\boldsymbol{y = 0}$

Sol. Let $z(x, y)$ be solution of given PDE Assume that

$$z(x, y) = X(x)Y(y)$$

where X and Y be function of only x and y respectively.

On substituting the value of $z(x, y) = X(x)Y(y)$ in given PDE, we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{dY}{dy} = -n^2 \text{ (say)}$$

then $$\frac{d^2X}{dx^2} + n^2X = 0, \quad \frac{dY}{dy} + n^2Y = 0$$

Hence $$X = a\sin(nx + \alpha), \quad \text{and } y = be^{-n^2y}$$

where a, b, α are arbitrary constants

Thus $$z = X(x)Y(y) = A\sin(nx + \alpha)e^{-n^2y}, A = ab \qquad .....(10)$$

According to conditions *(i)* and *(ii)* given with the problem, from (10), we get

$$0 = A\sin\alpha\, e^{-n^2y} \text{ and } 0 = A(-1)^n \sin\alpha\, e^{-n^2y}. \text{ Thus } \alpha = 0 \text{ as } A \neq 0$$

Hence $$z = A\sin nx\, e^{-n^2y} \qquad .....(11)$$

Also by condition *(iii)*, from (11), we get

$$\sin 3x = A\sin nx \Rightarrow A = 1, n = 3$$

Hence $$z = \sin 3xe^{-9y}$$

be required solution of given PDE under specified boundary conditions.

Ex. 4. Use the method of separation of variables to solve the equation

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \text{ given that } u(x,0) = 6e^{-3x}$$

Sol. Let $u(x, t) = X(x)T(t)$ be solution of given PDE where X is a function of x only and T is a function of t only.

Now $$\frac{\partial u}{\partial x} = T\frac{dX}{dx} \text{ and } \frac{\partial u}{\partial t} = X\frac{dT}{dt}$$

On substituting these values in given PDE, we get

$$T\frac{dX}{dx} = 2X\frac{dT}{dt} + XT$$

Dividing by XT, we have

$$\frac{X'}{X} = \frac{2T'}{T} + 1 = -n^2 \text{ (say)}$$

Now we have two ordinary differential equations.

$$\frac{X'}{X} = -n^2 \text{ and } \frac{2T'}{T} + 1 = -n^2$$

or $$\frac{dX}{dx} + n^2X = 0, \text{ and } \frac{T'}{T} = -\left(\frac{n^2+1}{2}\right)$$

Sloving these equations, we find that

$$X = c_1 e^{-n^2 x} \text{ and } T = c_2 e^{-\left(\frac{n^2+1}{2}\right)t}$$

Hence $$u(x,t) = X(x)T(t) = c_1 c_2 e^{-n^2 x - \left(\frac{n^2+1}{2}\right)t}$$

Under given condition we get $6e^{-3x} = c_1 c_2 e^{-n^2 x}$

$\Rightarrow$ $c_1 c_2 = 6$ and $n^2 = 3$

Thus the required solution of the problem is $u(x, t) = 6\, e^{-3x-2t}$

Ex. 5. Use the method of separation of variables to solve the PDE

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

Sol. : Let $u(x, y)$ be solution of given PDE. For method of separation of variables, we assume

$$u(x, y) = X(x)\, Y(y) \qquad .....(12)$$

where X is function of x only and Y is function of y only.

Now we have $$\frac{\partial u}{\partial x} = Y\frac{dX}{dx},\ \frac{du}{dt} = X\frac{dY}{dy},\ \frac{\partial^2 u}{\partial x^2} = Y\frac{d^2 X}{dx^2}$$

On substituting these values in given problem, we get

$$Y\frac{d^2 X}{dx^2} - 2Y\frac{dX}{dx} + X\frac{dY}{dy} = 0$$

On dividing by XY, we have

$$\frac{X''}{X} - \frac{2X'}{X} + \frac{Y'}{Y} = 0$$

or $$\frac{X'' - 2X'}{X} + \frac{Y'}{Y} = 0$$

or $$\frac{X'' - 2X'}{X} = -\frac{Y'}{Y} = -p^2 \text{ (say)}$$

From above equalities, we have two ordinary differential equation.

$$X'' - 2X' + p^2 X = 0 \text{ and } Y' - p^2 Y = 0$$

Now consider first differential equation from the above pair of equations i.e.

$$X'' - 2X' + p^2 X = 0 \qquad .....(13)$$

Now auxiliary equation for (13) is

$$m^2 - 2m + p^2 = 0$$

$$m = \frac{2 \pm \sqrt{4 - 4p^2}}{2} = 1 \pm \sqrt{1 - p^2}$$

Therefore $$CF = c_1 e^{\left(1+\sqrt{1-p^2}\right)x} + c_2 e^{\left(1-\sqrt{1-p^2}\right)x}$$

i.e. $$X = c_1 e^{\left(1+\sqrt{1-p^2}\right)x} + c_2 e^{\left(1-\sqrt{1-p^2}\right)x} \qquad .....(14)$$

Again $$\frac{dY}{dy} = p^2 Y \Rightarrow \frac{dY}{Y} = p^2 dy$$

$$\log Y = p^2 y + \log c_3$$

$$Y = c_3 e^{p^2 y} \qquad .....(15)$$

Substituting the values of X and Y from equation (14) and (15) respectively in (12), we get

$$u(x,y) = X(x)Y(y) = \left[c_1 e^{\left(1+\sqrt{1-P^2}\right)x} + c_2 e^{\left(1-\sqrt{1-p^2}\right)x} \right] c_3 e^{p^2 y}$$

Thus $$u\,(x,y) = \left[A e^{\left(1+\sqrt{1-p^2}\right)x} + B e^{\left(1-\sqrt{1-p^2}\right)x} \right] e^{p^2 y}$$

where $A = c_1 c_3$ and $B = c_2 c_3$.

Ex. 6. Solve by the method of separation of variables the PDE

$$4\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 3u, \text{ given that } u = 3e^{-x} - e^{-5x} \text{ when } t = 0$$

Sol. Let $u(x, t) = X(x)T(t)$ be solution of given PDE where X is a function of x only and T is a function of only t.

On substituting the value of $u(x, t)$ in the given PDE and dividing by XT, we get

$$\frac{4T'}{T} + \frac{X'}{X} = 3$$

$$\frac{4T'}{T} - 3 = \frac{-X'}{X} = p^2 \text{ (say)}$$

So we have $$\frac{4T'}{T} = p^2 + 3 \text{ and } -\frac{X'}{X} = p^2$$

Now $$\frac{4T'}{T} = p^2 + 3 \Rightarrow \frac{dT}{T} = \left(\frac{3+p^2}{4}\right)dt$$

$\Rightarrow$ $$\log T = \left(\frac{3+p^2}{4}\right)t + \log c_1$$

or $$T = c_1 e^{\left(p^2+3\right)t/4}$$

Again $$\frac{-X'}{X} = p^2 \Rightarrow \frac{dX}{X} = -p^2 dx$$

or $$\log X = -p^2 x + \log c_2$$

or $$X = c_2 e^{-p^2 x}$$

Hence $$u(x,t) = XT = c_1 c_2 e^{-p^2 x + (p^2+3)t/4} = b_n e^{-p^2 x + (p^2+3)t/4}$$

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-p^2 x + (p^2+3)t/4}$$

By the condition given in the problem, for $t = 0$ we have,

$$u(x,0) = 3e^{-x} - e^{-5x} = \sum_{n=1}^{\infty} b_n e^{-p^2 x}$$

So we have, $p^2 = 1, b_1 = 3$ or $p^2 = 5, b_2 = -1$

Hence the general solution is

$$u(x,t) = 3e^{-x+t} - e^{-5x+2t}$$

which required solution of given PDE under specified condition.

Self Learning Exercise–II

1. The equation $4r + 5s + t + p + q - 2 = 0$
has .... real characteristic family of curves.

2. For one family of characteristic of PDE

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

$S^2 - 4RT$ should be ....

3. If $S^2 - 4RT < 0$ for PDE

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

then it has .... real characteristics.

4. If PDE $Rr + Ss + Tt + f(x, y, z, p, q) = 0$
is hyperbolic the number of real characteristics will be ....

5. By the method of separation of variables to solve the one dimensional wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \; z(x, t) = ...$$

4.6 Summary

In this unit, we get an idea and importance of partial differential equation for physical and practical problems. We have learnt how we can classify the nature of different equations. Cauchy problem is physical roblem arise in analysis of physical and mathematical problem . A very powerful tool 'The method of separation of variables' is also introduced in this unit. At last for concrete depth in PDE, we have included the self- learning exercises, illustrative Ex.s and questions for practice.

4.7 Answers to Self-Learning Exercises

Exercise – I

1. *(i)* *b* *(ii)* *c* *(iii)* *a* *(iv)* *a*

2. See § 4 **3.** $xy \log y < 0$

4. Parabolic **5.** Hyperbolic

6. Elliptic if $t^2 - 4x < 0$, Hyperbolic if $t^2 - 4x > 0$ and Parabolic if $t^2 - 4x = 0$

Exercise – II

1. 2 **2.** $S^2 - 4RT = 0$ **3.** Zero

4. 2 **5.** $X(x)\,T(t)$

4.8 Exercise

1. Find the characteristics of

(i) $4r + 5s + t + p + q - 2 = 0$ **[Ans.** $y - x = c_1$, and $y - \frac{x}{y} = c_2$ **]**

(ii) $(\sin^2 x)r + (2\cos x)s - t = 0$ **[Ans.** $y + \operatorname{cosec} x - \cot x = c_1$; $y + \operatorname{cosec} x + \cot x = c_2$**]**

2. Show that the equation $u_{xx} + xu_{yy} + uy = 0$ is elliptic for $x > 0$ and hyperbolic for $x < 0$.

3. Find whther the following PDE are parabolic or elliptic

(i) $x^2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = 0$ **[Ans.** Hyperbolic if $x > 0$, parabolic if $x = 0$ and elliptic if $x < 0$ **]**

(ii) $t \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x\,\partial t} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} = 0$

[Ans. $tx < 1$ for hyperbolic, $tx > 1$ for elliptic and $tx = 1$ for parabolic**]**

4. Solve by the method of separation of variables :

$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$; $u(0, y) = 8e^{-3y}$ **[Ans.** $u(x, y) = 8e^{-3y-12x}$**]**

5. Solve by the method of separation of variables :

$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} - 2u$; $u(x, 0) = 10e^{-x} - 6e^{-4x}$ **[Ans.** $u(x, t) = 10e^{-(x+3t)} - 6e^{-2(2x+3t)}$**]**

6. Solve $2u_{xx} - u_y = 0$ by separation of variables. **[Ans.** $u(x, y) = \left(Ae^{\sqrt{kx}} + Be^{-\sqrt{kx}}\right)e^{2ky}$ **]**

7. Solve the following PDE by the method of separation of variables,

(i) $4u_x + u_y = 3u$ and $u(0, y) = e^{-5y}$ **[Ans.** $u(x, y) = e^{2x-5y}$**]**

(ii) $\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial u}{\partial t}$; $u(\mathrm{x}, 0) = x(\mathrm{a} - \mathrm{x})$ **[Ans.** $u(x, y) = x(a - x)\; e^{-(p^2 t/2)}$ **]**

(iii) $y^3 u_x + x^2 u_y = 0$ **[Ans.** $u(x, y) = ce^{k\{(x^3/3) - (y^4/4)\}}$ **]**

8. Use the method of separation of variables to solve the equation,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$$ **[Ans.** $u(x, y) = (A\cos px + B\sin px)e^{-(p2+2)y}$ **]**

9. Solve the method of separation of variables,

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0 \;;\; u(x, 0) = 4e^{-x}$$ **[Ans.** $u(x, y) = 4e^{-x+(3/2)y}$ **]**

10. Solve by method of separation of variables,

$$4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u \;;\; u(0, y) = 4e^{-y} - e^{-5y}$$ **[Ans.** $u(x, y) = 4e^{x-y} - e^{2x-5y}$ **]**

11. Solve by method of separation of variables,

$$\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial t} + u = 0 \text{ when } u\,(x, 0) = 6e^{-3x}$$ **[Ans.** $u(x, y) = 6e^{-3x-2t}$ **]**

□ □ □

Unit 5 : Laplace, Wave and Diffusion Equations And Canonical Forms

Structure of the Unit

5.0 Objective

After studying this unit, you should be able to know application of partial differential equations. You will get an idea of wave, diffusion and Laplace equations in different coordinate system and their solutions. You will also study the reduction of the second order P.D.E's to canonical forms.

5.1 Introduction

In physical and engineering application, PDE's of second order are of utmost significance. These equations arise in the modelling of vibration of string and membranes, theory of hydraulics, gravitational and potential problems and so on. Since a comprehensive treatment of the subject is not possible in this unit, we restrict our study to a consideration of some special types of equations.

5.2 Laplace, Wave and Diffusion Equations

In applied mathematics and theoretical physics three types of equations occur frequently. These are

(i) Laplace Equation

(ii) Wave Equation and

(iii) Diffusion Equation.

In many practical problems the solution of these equations may be obtained with the help of separation of variables.

5.2.1 Laplace Equation

One of the most important PDE appearing in theoretical physics is **Laplace's equation.** It is usually written as

$$\nabla^2 u = 0 \qquad .....(1)$$

where the operator ∇^2, known as **Laplacian** depends on the coordinate system chosen. It is an elliptic PDE.

(i) in three dimensions, this equation in Cartesian system of coordinates (x, y, z) is written as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad .....(2)$$

(ii) in cylindrical polar coordinates (r, θ, z), eq. (2) becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad .....(3)$$

(iii) in antisymmetric case *i.e.* u is independent of θ, therefore equation (3) reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad .....(4)$$

(iv) in spherical polar coordinates (r, θ, ϕ), eq. (2) reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0 \qquad .....(5)$$

(v) when u is independent of the azimuthal angle ϕ, (5) reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0$$

or

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) = 0 \qquad .....(6)$$

(vi) in two dimensions, Laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad .....(7)$$

in Cartesian coordinates (x, y) and

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad .....(8)$$

in polar coordinates (r, θ).

Equation (7) is also known as **Harmonic equation.**

5.2.2 Wave Equation

The wave equation is

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad .....(9)$$

It is **hyperbolic** PDE. ∇^2 is a Laplacian operator which depends on the coordinate system chosen.

(i) Three dimensional wave equation (sound waves in space) in Cartesian coordinates is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad .....(10)$$

(ii) Transverse vibrations of a membrane are governed by two dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad .....(11)$$

(iii) Transverse vibrations of a string are governed by the one dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad .....(12)$$

5.2.3 Diffusion Equation or Heat Conduction Equation

The diffusion equation or heat conduction equation in general, is written as

$$\nabla^2 u = \frac{1}{k}\frac{\partial u}{\partial t} \qquad .....(13)$$

where u is interpreted as temperature. It is **parabolic** PDE.

The one dimensional diffusion equation, which is very much used, may be written as

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \qquad .....(14)$$

Ex. 1. Find the most general functions X(x) and T(t), each of one is variable, such that u(x, y) = XT satisfies the PDE.

$$\boldsymbol{\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}}$$

Also obtain a solution of the above equation for k =1 and which satisfies the boundary conditions $\quad u = 0 \quad$ ***when*** $x = 0$ ***or*** π

$u = \sin 3x$ ***when*** $t = 0$ ***and*** $0 < x < \pi$

Sol. The given differential equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \qquad .....(15)$$

Let the solution of eq. (15) by method of separation of variables is of the form

$$u(x, t) = X(x)\, T(t) \qquad .....(16)$$

Substituting (16) in (15), we get

$$\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{kT}\frac{dT}{dt} \quad ....(17)$$

The expression on LHS of eq. (17) is a function of independent variable x while on RHS, it is function of independent variable t only. Both are equal if both are constant and equal to either $-n^2$, 0 or n^2. Hence three cases arise as follows :

Case I : $\frac{d^2X}{dx^2}=0$ and $\frac{dT}{dt}=0$

The solution will be $X=Ax+B$ and $T=C$

Case II : $\frac{d^2X}{dx^2}-n^2x=0$ and $\frac{dT}{dt}=n^2kt$

The solution will be $X=Ae^{nx}+Be^{-nx}$ and $T=Ce^{n^2kt}$

Case III : $\frac{d^2X}{dx^2}+n^2x=0$ and $\frac{dT}{dt}=-n^2kt$

The solution is $X=A\sin(nx+\alpha)$ and $T=Be^{-n^2kt}$

where A, B, C and α are arbitrary constants. Since when $t\to\infty$, $u(x, t)\to 0$, hence case III is most appropriate solution of eq. (15). Hence

$$u(x,t)=Ae^{-n^2kt}\sin(nx+\alpha)$$

is the most general solution of given problem

Special case : $u(x, t)=0$ when $x=0$ or π gives $\alpha=0$

Further $u(x, t)=\sin 3x$ when $t=0$ gives

$\sin 3x=A\sin nx \Rightarrow A=1$ and $n=3$

Also $k=1$

Hence solution of $\frac{\partial^2u}{\partial x^2}=\frac{\partial u}{\partial t}$ is given by

$u(x. t)=e^{-9t}\sin 3x$

Example 2 : Solve the two dimensional heat conduction equation

$$\frac{\partial^2u}{\partial x^2}=\frac{\partial^2u}{\partial y^2}=\frac{1}{k}\frac{\partial u}{\partial t} \quad .....(18)$$

by the method of separation of variables.

Sol. : Let the solution of (18) is

$$u(x, y, t)=X(x)\,Y(y)\,T(t) \quad ....(19)$$

Substituting (19) in (18), we get

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=\frac{1}{kT}\frac{dT}{dt} \quad ....(20)$$

The RHS of (20) is a function of independent variable 't' only whereas LHS is a function of two independent variables x and y. They are equal if both are constant only. If RHS of (20) is a constant and sum of two functions of two independent variables then both are constants also. Now three cases arise.

Case I : $$\frac{1}{X}\frac{d^2X}{dx^2} = 0, \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = 0 \quad \text{and} \quad \frac{1}{kT}\frac{dT}{dt} = 0$$

The solution of these relations will give

$$X = ax + b, \qquad Y = cy + d \quad \text{and} \quad T = e$$

where a, b, c, d and e are arbitrary constants.

Case II : $$\frac{1}{X}\frac{d^2X}{dx^2} = m^2, \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = n^2 \quad \text{and} \quad \frac{1}{kT}\frac{dT}{dt} = p^2$$

or $$\frac{d^2X}{dx^2} - m^2X = 0, \quad \frac{d^2Y}{dy^2} - n^2Y = 0 \quad \text{and} \quad \frac{dT}{dt} = p^2kT$$

where $m^2 + n^2 = p^2$

On solving these equations, we get

$$X = a_1e^{mx} + b_1e^{-mx}, \ Y = a_2ex^{nx} + b_2e^{-nx} \quad \text{and} \quad T = a_3e^{p^2kt}$$

Case III : $$\frac{1}{X}\frac{d^2X}{dx^2} = -m^2, \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = -n^2 \quad \text{and} \quad \frac{1}{kT}\frac{dT}{dt} = -p^2$$

or $$\frac{d^2X}{dx^2} + m^2X = 0, \quad \frac{d^2Y}{dy^2} + n^2Y = 0 \quad \text{and} \quad \frac{dT}{dt} = -p^2kT$$

where $m^2 + n^2 = p^2$

Solving these equations, we get

$$X = c_1\cos(mx + c_m), \quad Y = c_3\cos(ny + c_n) \quad \text{and} \quad T = c_5e^{-(m^2+n^2)kt}$$

Since $u(x, y, t) \to 0$ as $t \to \infty$, therefore case III is most appropriate. Hence solution of (18) which is linear can be written as

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}\cos(mx + c_m)\cos(ny + c_n)e^{-k(m^2+n^2)t}$$

Ex. 3. A thin rectangular plate whose surface is impervious to heat flows has at t* = 0 *an arbitrary function f (x, y). Its four edges x* = 0, *x* = *a*, *y* = 0, *y* = *b are kept at zero temperature. Determine the temperature at a point of the plate as 't' increases.

Sol. Here the temperature $U(x, y, t)$ in the plate is governed by the two dimensional heat equation

$$\frac{\partial^2U}{\partial x^2} + \frac{\partial^2U}{\partial y^2} = \frac{1}{k}\frac{\partial U}{\partial t} \qquad \text{....(21)}$$

with boundary conditions

$$U(0,y,t)=0,\ U(a,y,t)=0,\ U(x,0,t)=0,\ U(x,b,t)=0 \qquad .....(22)$$

and initial condition is

$$U(x, y, t) = f(x, y) \qquad .....(23)$$

Proceeding similarily to Ex.2, we find that if solution of (21) may be assumed as

$$U(x, y, 0) = X(x)\, Y(y) T(t)$$

then $X = c_1 \cos(mx + c_m) = A_1 \cos mx + B_1 \sin mx,$

$$Y = A_2 \cos nx + B_2 \sin nx$$

and $T = A_3 e^{-k\left(m^2+n^2\right)t}$

Using boundary conditions (22), we find that

$$A_1 = 0,\ B_1 \sin ma = 0,\ A_2 = 0,\ B_2 \sin nb = 0$$

$\therefore$ $A_1 = 0 = A_2$, $\sin ma = \sin u\pi$ and $\sin nb = \sin v\pi$ $(u,\ v = 1, 2, 3 ....)$ as $B_1 \neq 0$ and $B_2 \neq 0$

Thus $A_1 = 0 = A_2$, $m = \dfrac{u\pi}{a}$ and $n = \dfrac{v\pi}{b}$

Hence the general solution of (21) will be

$$U(x,y,t) = \sum_{u=1}^{\infty}\sum_{v=1}^{\infty} F_{uv} \sin\frac{u\pi x}{a} \sin\frac{v\pi y}{b} e^{-k\left(\frac{u^2}{a^2}+\frac{v^2}{b^2}\right)\pi^2 t}$$

Now under initial condition (23), we have

$$U(x,y,o) = f(x,y) = \sum_{u=1}^{\infty}\sum_{v=1}^{\infty} F_{uv} \sin\frac{u\pi x}{a} \sin\frac{v\pi y}{b} \qquad .....(24)$$

which is a double Fourier series of $f(x, y)$.

Hence
$$F_{uv} = \frac{4}{ab}\int_{x=0}^{a}\int_{y=0}^{b} f(x,y) \sin\frac{u\pi x}{a} \sin\frac{v\pi y}{b}\, dxdy \qquad .....(25)$$

Thus (24) is a general solution of (21) under boundary and initial condition (22) and (23) where constant F_{uv} as given by (25).

Ex. 4. By separating the variables, show that the one dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad .....(26)$$

has solution of the form $A \exp(\pm\, inx \pm inct)$ ***where A and n are constants.***

Sol. Let the solution of (26) is

$$u(x, t) = X(x)\, T(t) \qquad ....(27)$$

Substituting (27) in (26), we get

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2 T}\frac{d^2T}{dt^2} = -n^2 \qquad \text{(say)}$$

$$\Rightarrow \qquad \frac{d^2X}{dx^2} + n^2X = 0 \text{ and } \frac{d^2T}{dt^2} + n^2c^2T = 0$$

Solving these we get

$$X = c_1 e^{\pm inx} \text{ and } T = c_2 e^{\pm\, in\, ct} \qquad .....(28)$$

Hence from (27) and (28), we get the solution of (26) as

$$u(x, t) = A \exp(\pm\, in\, x \pm in\, ct)$$

Ex. 5. A tightly stretched sting which has fixed end points x = 0 and x = l is initially in a position given by y = k sin³ (πx/l). It is released from rest from this position. Find the displacement y (x, t).

Sol. Since the string is tightly stretched initially between two fixed points and released from rest, it will make transverse vibrations in (x, y) plane. The displacement $y(x, t)$ of any point on it will be a governed by the following wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \qquad .....(29)$$

with the boundary conditions

$$t > 0 \ : \ y(0,t) = 0 = y(l, t) \qquad .....(30)$$

and the initial condition

$$t = 0 : \ y(x,0) = k \sin^3(\pi x/l) \qquad .....(31)$$

which also implies $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

Applying the method of separation of variables if solution of (29) is of the form $X(x)T(t)$ we find that

$$X = A \cos \lambda x + B \sin \lambda x$$

and $\quad T = C \cos \lambda ct + D \sin \lambda ct \qquad ....(32)$

Using boundary condition (30), we get

$$A = 0 \qquad \text{and} \qquad B \sin \lambda\ell = 0 \Rightarrow \lambda = \frac{n\pi}{\ell} (\because B \neq 0)(n = 1, 2, 3....)$$

Hence $\quad X_n(x) = A_n \sin(n\pi/l) \qquad ....(33)$

Under initial condition $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$, we get $D = 0$ from (32). Therefore

$$T_n(t) = B_n \cos(n\pi ct/l) \qquad ....(34)$$

Hence (33) and (34), we get the general solution of (29) as

$$y_n(x,t) = C_n \sin\frac{n\pi x}{\ell} \cos\frac{n\pi ct}{\ell}, n \in N$$

where $C_n = A_n B_n$ is an arbitrary constant

Hence $\quad y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{\ell} \cos\frac{n\pi ct}{\ell} \qquad .....(35)$

To determine the constant C_n we apply the condition (31) on (35), we get

$$k\sin^3\left(\frac{\pi x}{\ell}\right)=\sum_{n=1}^{\infty}C_n\sin\left(\frac{n\pi x}{\ell}\right)$$

or $$\frac{k}{4}\left[3\sin\left(\frac{\pi x}{\ell}\right)-\sin\left(\frac{3\pi x}{\ell}\right)\right]=\sum_{n=1}^{\infty}C_n\sin\left(\frac{n\pi x}{\ell}\right)$$

$\Rightarrow$ $$C_1=\frac{3}{4}k,\ C_3=-\frac{k}{4}\ \text{ and }\ c_2=c_4=c_5=c_6=....0$$

Hence the required solution is

$$y(x,t)=\frac{k}{4}\left[3\sin\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{\pi ct}{\ell}\right)-\sin\left(\frac{3\pi x}{\ell}\right)\cos\left(\frac{3\pi ct}{\ell}\right)\right]$$

Ex. 6. Solve the harmonic equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad .....(36)$$

satisfying the conditions

$$\left.\begin{array}{l} u(x,0)=0,\ u(x,a)=\sin\left(\frac{\pi x}{\ell}\right) \\ u(0,y)=u(l,y)=0 \end{array}\right\} \qquad .....(37)$$

Sol. Let the solution of (36) is

$$u(x, y) = X(x)\ Y(y)$$

Substituting in (36) we get

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=0$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}=-\frac{1}{Y}\frac{d^2Y}{dy^2}=-\lambda^2\ (\text{say})$$

Now $$\frac{d^2X}{dx^2}+\lambda^2X=0\Rightarrow X(x)=A\cos\lambda x+B\sin\lambda x$$

Applying $u(0, y) = X(0) = 0$ and $u(l, y) = X(l) = 0$

we get A = 0 and $\lambda\ell = n\pi$ *or* $\lambda=\frac{n\pi}{\ell}, n\in N$

thus $$X_n(x)=B_n\sin\left(\frac{n\pi x}{l}\right)$$

Again $$\frac{d^2Y}{dy^2}-\lambda^2Y=0\Rightarrow Y(y)=C\cos h\lambda y+D\sin h\lambda y$$

Now $u(x, 0) = y(0) = 0$ gives $C = 0$

thus $$Y_n(y) = D_n \sin\left(\frac{n\pi y}{l}\right)$$

Hence we have $$u_n(x,y) = X_n(x)Y_n(y) = F_n \sin\left(\frac{n\pi x}{l}\right)\sin h\left(\frac{n\pi y}{l}\right)$$

where F_n is arbitrary constant. Therefore

$$u(x,y) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi x}{l}\right)\sin h\left(\frac{n\pi y}{l}\right)$$

Now applying the boundary condition

$$u(x, a) = \sin(\pi x/l)$$

We find that $$\frac{\sin \pi x}{l} = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi x}{l}\right)\sin h\left(\frac{n\pi a}{l}\right) \forall x$$

Equating coefficients of like terms, we get

$$F_1 \sinh(\pi a/l) = 1 \quad \text{and} \quad F_2 = F_3 = ... = 0$$

Hence, the required solution is

$$u(x,y) = \text{cosec}h\left(\frac{\pi a}{l}\right)\sin\left(\frac{\pi x}{l}\right)\sin h\left(\frac{\pi y}{l}\right)$$

5.3 Canonical Forms

Let us consider the equations of the type

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad ....(1)$$

where R, S, T are continuous functions of x and y possessing continuous partial derivatives of as high as order as necessary. It is a typical class of semi-linear equations of the type of

$$Rr + Ss + Tt = V$$

Changing the independent variables x, y to ξ, η such that

$$\xi = \xi(x, y), \quad \eta = \eta(x, y) \qquad .....(2)$$

$$z = z(\xi, \eta) \qquad ....(3)$$

Here it is assumed that ξ, η are doubly differentiable and the transformation from (x, y)–plane to (ξ, η)–plane is locally one to one. This requires that the Jacobian of the transformation is nonzero, *i.e.*

$$J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x} \neq 0$$

Now from (2) and (3), we get

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial z}{\partial\eta}\frac{\partial\eta}{\partial x} = \left(\frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi} + \frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta}\right)z$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial\xi}\frac{\partial\xi}{\partial y} + \frac{\partial z}{\partial\eta}\frac{\partial\eta}{\partial y} = \left(\frac{\partial\xi}{\partial y}\frac{\partial}{\partial\xi} + \frac{\partial\eta}{\partial y}\frac{\partial}{\partial\eta}\right)z$$

$$r=\frac{\partial^2 z}{\partial x^2}=\left(\frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi}+\frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta}\right)=\left(\frac{\partial\xi}{\partial x}\frac{\partial z}{\partial\xi}+\frac{\partial\eta}{\partial x}\frac{\partial z}{\partial\eta}\right)$$

$$=\left(\frac{\partial\xi}{\partial x}\right)^2\frac{\partial^2 z}{\partial\xi^2}+\left(2\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}\right)\frac{\partial^2 z}{\partial\xi\partial\eta}+\left(\frac{\partial\eta}{\partial x}\right)^2\frac{\partial^2 z}{\partial\eta^2}+2\left(\frac{\partial^2\xi}{\partial x^2}\right)\frac{\partial z}{\partial\xi}+2\left(\frac{\partial^2\eta}{\partial x^2}\right)\frac{\partial z}{\partial\eta}$$

$$s=\frac{\partial^2 z}{\partial x\partial y}=\left(\frac{\partial\xi}{\partial x}\cdot\frac{\partial}{\partial\xi}+\frac{\partial\eta}{\partial x}\cdot\frac{\partial}{\partial\eta}\right)\left(\frac{\partial\xi}{\partial y}\cdot\frac{\partial z}{\partial\xi}+\frac{\partial\eta}{\partial y}\cdot\frac{\partial z}{\partial\eta}\right)=\left(\frac{\partial\xi}{\partial x}\cdot\frac{\partial\xi}{\partial y}\right)\frac{\partial^2 z}{\partial\xi^2}$$

$$+\left(\frac{\partial\xi}{\partial x}\cdot\frac{\partial\eta}{\partial y}+\frac{\partial\eta}{\partial x}\cdot\frac{\partial\xi}{\partial y}\right)\frac{\partial^2 z}{\partial\xi\partial\eta}+\left(\frac{\partial\eta}{\partial x}\cdot\frac{\partial\eta}{\partial y}\right)\frac{\partial^2 z}{\partial\eta^2}+\frac{\partial^2\xi}{\partial x\partial y}\cdot\frac{\partial z}{\partial\xi}+\frac{\partial^2\eta}{\partial x\partial y}\cdot\frac{\partial z}{\partial\eta}$$

and $$t=\frac{\partial^2 z}{\partial y^2}=\left(\frac{\partial\xi}{\partial y}\cdot\frac{\partial}{\partial\xi}+\frac{\partial\eta}{\partial y}\cdot\frac{\partial}{\partial\eta}\right)\left(\frac{\partial\xi}{\partial y}\cdot\frac{\partial z}{\partial\xi}+\frac{\partial\eta}{\partial y}\cdot\frac{\partial z}{\partial\eta}\right)=\left(\frac{\partial\xi}{\partial y}\right)^2\frac{\partial^2 z}{\partial\xi^2}$$

$$+\left(2\frac{\partial\xi}{\partial y}\cdot\frac{\partial\eta}{\partial y}\right)\frac{\partial^2 z}{\partial\xi\partial\eta}+\left(\frac{\partial\eta}{\partial y}\right)^2\frac{\partial^2 z}{\partial\eta^2}+\left(\frac{\partial^2\xi}{\partial y^2}\right)\frac{\partial z}{\partial\xi}+\left(\frac{\partial^2\eta}{\partial y^2}\right)\frac{\partial z}{\partial\eta} \qquad .....(4)$$

Now substituting these values in (1), it takes the form

$$A+\frac{\partial^2 z}{\partial\xi^2}+2B\frac{\partial^2 z}{\partial\xi\partial\eta}+C\frac{\partial^2 z}{\partial\eta^2}+F\left(\xi,\eta,z,\frac{\partial z}{\partial\xi},\frac{\partial z}{\partial\eta}\right)=0 \qquad .....(5)$$

where $$A=R\left(\frac{\partial\xi}{\partial x}\right)^2+S\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+T\left(\frac{\partial\xi}{\partial y}\right)^2 \qquad .....(6)$$

$$B=R\frac{\partial\xi}{\partial x}\cdot\frac{\partial\eta}{\partial x}+\frac{1}{2}S\left(\frac{\partial\xi}{\partial x}\cdot\frac{\partial\eta}{\partial y}+\frac{\partial\eta}{\partial x}\cdot\frac{\partial\xi}{\partial y}\right)+T\frac{\partial\xi}{\partial y}\cdot\frac{\partial\eta}{\partial y} \qquad .....(7)$$

$$C=R\left(\frac{\partial\eta}{\partial x}\right)^2+S\frac{\partial\eta}{\partial x}\cdot\frac{\partial\eta}{\partial y}+T\left(\frac{\partial\eta}{\partial y}\right)^2 \qquad .....(8)$$

and $$F\left(\xi,\eta,z,\frac{\partial z}{\partial\xi},\frac{\partial z}{\partial\eta}\right)$$

is obtained from the transformed form of $f(x, y, z, p, q)$ and the remaining terms containing first order partial derivatives of transformed Rr, Ss, and Tt.

One of the relations satisfied by A, B, C and R, S, T which can be easily seen, is

$$AC-B^2=\frac{1}{4}\left(4RT-S^2\right)J^2 \qquad .....(9)$$

We shall now determine the functional relationship [equations (2)] of ξ,η with x and y so that the transformed equation (5) takes the simplest possible form.

The procedure is simple when the discriminant $S^2 - 4RT$ of the quadratic equation (called α equation)

$$Q(\alpha) = R\alpha^2 + S\alpha + T = 0 \qquad .....(10)$$

is either positive, negative or zero everywhere. We shall discuss these cases separately. It may be noted that $Q(\alpha)$ is called the 'characteristic quadratic form' and the discriminant of the quadratic will determine the nature of P.D.E. This will depend on the characteristic roots of the associated real symmetric metric.

$$M = \begin{bmatrix} R & S/2 \\ S/2 & T \end{bmatrix} \qquad .....(11)$$

Case I : $S^2 - 4RT > 0$.

In this case the roots α_1 and α_2 of equation (10), which are in general functions of x and y, would be real and distinct.

Let us take $$\frac{\partial \xi}{\partial x} = \alpha_1 \frac{\partial \xi}{\partial y} \qquad .....(12)$$

and $$\frac{\partial \eta}{\partial x} = \alpha_2 \frac{\partial \eta}{\partial y} \qquad ....(13)$$

then from (6) and (8), we find that

$$A = \left(R\alpha_1^2 + S\alpha_1 + T\right)\left(\frac{\partial \xi}{\partial y}\right)^2 = 0 \qquad .....(14)$$

and $$C = \left(R\alpha_2^2 + S\alpha_2 + T\right)\left(\frac{\partial \eta}{\partial y}\right)^2 = 0 \qquad .....(15)$$

where α_1 and α_2 are roots of (10).

The equation (5) reduces to

$$2B\frac{\partial^2 z}{\partial \xi \partial \eta} + F\left(\xi, \eta, z, \frac{\partial z}{\partial \xi}, \frac{\partial z}{\partial \eta}\right) = 0 \qquad .....(16)$$

Equation (12) is a Lagrange's linear equation of first order, whose subsidary equations are

$$\frac{dx}{1} = \frac{dy}{-\alpha_1} = \frac{d\xi}{0}$$

which gives ξ = constant,

and $$\frac{dy}{dx} + \alpha_1 = 0 \qquad ....(17)$$

Let $f_1(x, y)$ = constant be the solution of equation (17) then the general solution of equation (12) will be

$$\xi = f_1(x, y) \qquad ....(18)$$

In a similar manner the general solution of equation (13) will be

$$\eta = f_2(x, y) \qquad .....(19)$$

where f_1 = constant and f_2 = constant are the solution of differential equations

$$\frac{dy}{dx} + \alpha_1 = 0, \quad \frac{dy}{dx} + \alpha_2 = 0 \qquad .....(20)$$

respectively. Relations (18) and (19) are the desired transformations for independent variables which reduce the given equations (1) to the form (16).

Now from (9), we have

$$AC - B^2 = \frac{1}{4}\left(4RT - S^2\right)\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right)^2 \qquad .....(21)$$

This shows that the sign of $(AC - B^2)$ is the same as of $(4RT - S^2)$ *i.e.* it is invariant under transformation.

Therefore, when $A = C = 0$, from (21), we have

$$4B^2 = \left(S^2 - 4RT\right)\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right)^2 \qquad .....(22)$$

Since we have assumed that $S^2 > 4RT$, it implies from (22) that $B^2 > 0$ *i.e.* $B \neq 0$ and therefore we may devide both sides of equation (16) by it and write it finally as

$$\frac{\partial^2 z}{\partial \xi \partial \eta} = \phi_1\left(\xi, \eta, z, \frac{\partial z}{\partial \xi}, \frac{\partial z}{\partial \eta}\right) \qquad .....(23)$$

which is the canonical form of equation (1) when $S^2 - 4RT > 0$.

Case II : $S^2 - 4RT = 0$.

In this case the two roots of the quadratic equation (10) are equal *i.e.* $\alpha_1 = \alpha_2$ Therefore one of the functions, say ξ will be defined by equation (18) of case I. We may now take η to be any suitable function of x and y which should be independent of ξ . Therefore, as before, $A = 0$ but $C \neq 0$. Further, from (21), since $S^2 - 4RT = 0$ we have

$$B = 0$$

Hence equation (5) reduces to

$$C\frac{\partial^2 z}{\partial \eta^2} + F\left(\xi, \eta, z, \frac{\partial z}{\partial \xi}, \frac{\partial z}{\partial \eta}\right) = 0$$

or

$$\frac{\partial^2 z}{\partial \eta^2} = \phi_2\left(\xi, \eta, z, \frac{\partial z}{\partial \xi}, \frac{\partial z}{\partial \eta}\right) \qquad .....(24)$$

which is the canonical form of the equation (1) when $S^2 - 4RT = 0$.

Case III : $S^2 - 4RT < 0$.

This is particularly the same as case I except that the roots of the quadratic equation (10) in this case are complex. If we proceed in the same manner as we did in case I, we shall arrive at equation (21) but in this case the variables are not real and in fact complex conjugates. To get a real cononical form we transform the independent varialdes ξ and η again be the following relations.

$$\lambda = \frac{1}{2}(\xi + \eta), \quad \mu = \frac{1}{2}i(\eta - \xi) \qquad ....(25)$$

then $$\frac{\partial z}{\partial \xi}=\frac{\partial z}{\partial \lambda}\cdot\frac{\partial \lambda}{\partial \xi}+\frac{\partial z}{\partial \mu}\cdot\frac{\partial \mu}{\partial \xi}=\frac{1}{2}\left(\frac{\partial}{\partial \lambda}-i\frac{\partial}{\partial \mu}\right)z$$

$$\frac{\partial z}{\partial \eta}=\frac{\partial z}{\partial \lambda}\cdot\frac{\partial \lambda}{\partial \eta}+\frac{\partial z}{\partial \mu}\cdot\frac{\partial \mu}{\partial \eta}=\frac{1}{2}\left(\frac{\partial}{\partial \lambda}+i\frac{\partial}{\partial \mu}\right)z$$

Hence $$\frac{\partial^2 z}{\partial \xi \partial \eta}=\frac{1}{4}\left(\frac{\partial}{\partial \lambda}-i\frac{\partial}{\partial \mu}\right)\left(\frac{\partial z}{\partial \lambda}+i\frac{\partial z}{\partial \mu}\right)=\frac{1}{4}\left(\frac{\partial^2 z}{\partial \lambda^2}+\frac{\partial^2 z}{\partial \mu^2}\right) \quad .....(26)$$

and therefore the relation (23) reduces to

$$\frac{\partial^2 z}{\partial \lambda^2}+\frac{\partial^2 z}{\partial \mu^2}=\phi_3\left(\lambda,\mu,z,\frac{\partial z}{\partial \lambda},\frac{\partial z}{\partial \mu}\right) \quad ....(27)$$

which is the Canonical form of equation (1) when $S^2-4RT<0$.

Ex. 1. Reduce the equation

$$(n-1)^2\frac{\partial^2 z}{\partial x^2}-y^{2n}\frac{\partial^2 z}{\partial y^2}=ny^{2n-1}\frac{\partial z}{\partial y}$$

to canonical form and find its general solution.

Sol. Comparing the given equation with the standard form

$Rr+Ss+Tt+f(x,y,z,p,q)=0$, we get

$$R=(n-1)^2,\ S=0,\ T=-y^{2n},\ f=-ny^{2n-1}\frac{\partial z}{\partial y}$$

Here, $S^2-4RT=4(n-1)^2y^{2n}>0$ provided $n\neq 1$.

Hence the given differential equation is hyperbolic differential equation. The roots of the α-equation $R\alpha^2+S\alpha+T=0$

or $(n-1)^2\alpha^2-y^{2n}=0$

are $$\alpha_1=\frac{y^n}{n-1} \text{ and } \alpha_2=\frac{-y^n}{n-1}$$

Changing the independent variables from x, y to ξ,η such that $\xi=f_1(x,y)$, $\eta=f_2(x,y)$ where $f_1=$ constant and $f_2=$ constant are the solution of the differential equations

$$\frac{dy}{dx}+\alpha_1=0 \text{ and } \frac{dy}{dx}+\alpha_2=0 \quad \text{respectively.}$$

These gives $f_1(x,y)=y^{1-n}-x=$ constant

and $f_2(x,y)=y^{1-n}+x=$ constant

Hence $\xi=y^{1-n}-x$ and $\eta=y^{1-n}+x$

Now, $$p=\frac{\partial z}{\partial x}=-\frac{\partial z}{\partial \xi}+\frac{\partial z}{\partial \eta}$$

$$q=\frac{\partial z}{\partial y}=(1-n)y^{-n}\left\{\frac{\partial z}{\partial \xi}+\frac{\partial z}{\partial \eta}\right\}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial \xi^2} - 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2}$$

$$t = \frac{\partial^2 z}{\partial y^2} = (n-1)^2 y^{-2n} \left\{ \frac{\partial^2 z}{\partial \xi^2} + 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2} \right\} + n(n-1) y^{-n-1} \left\{ \frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} \right\}$$

Therefore, the given equation reduces to

$$(n-1)^2 \left\{ \frac{\partial^2 z}{\partial \xi^2} - 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2} \right\} - (n-1)^2 \left\{ \frac{\partial^2 z}{\partial \xi^2} + 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2} \right\}$$

$$-n(n-1) y^{n-1} \left\{ \frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} \right\} = n(n-1) y^{n-1} \left\{ \frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} \right\}$$

or $$-4(n-1)^2 \frac{\partial^2 z}{\partial \xi \partial \eta} = 0$$

or $$\frac{\partial^2 z}{\partial \xi \partial \eta} = 0$$

which is the required canonical form if $n \neq 1$.

The general solution of the above equation may be easily obtained as

$$z = \phi_1(\xi) + \phi_1(\eta)$$

where ϕ_1 and ϕ_2 arbitrary functions of ξ and η respectively. Changing to original variables we get finally

$$z = \phi_1\left(y^{1-n} - x\right) + \phi_2\left(y^{1-n} + x\right)$$

Note : If $n = 1$, the character of the given differential equation changes. It becomes a parabolic equation, viz.

$$y\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} = 0$$

whose general solution is

$$z = \phi_1(x)\log y + \phi_2(x).$$

Ex.2. Reduce the equation

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

to canonical form and hence solve it.

Sol. Comparing the given equation with the standard form $Rr + Ss + Tt + f(x, y, z, p, q) = 0$,

we get, $R = 1,\ S = 2,\ T = 1, f = 0$

Here $S^2 - 4RT = 4 - 4 = 0$

Hence the given equation is a parabolic differential equation.

The roots of the α-equation

$$R\alpha^2 + S\alpha + T = 0$$

or $\qquad \alpha^2 + 2\alpha + 1 = 0$

are $\qquad \alpha = -1, -1$

Changing the independent variables x, y to ξ, η where $\xi = f_1(x, y)$, such that f_i = const. is the solution of the differential equation

$$\frac{dy}{dx} + \alpha_1 = 0$$

or $\qquad \dfrac{dy}{dx} - 1 = 0$ which gives $x - y$ = const.

Hence $\qquad \xi = x - y$

We may now take η to be any suitable function of x and y which should be independent of ξ. Let

$$\eta = x + y$$

Now, $\qquad \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial \xi}\cdot\dfrac{\partial \xi}{\partial x} + \dfrac{\partial z}{\partial \eta}\cdot\dfrac{\partial \eta}{\partial x} = \dfrac{\partial z}{\partial \xi} + \dfrac{\partial z}{\partial \eta}$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = -\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}$$

$$\frac{\partial^2 z}{\partial x^2} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}\right) = \frac{\partial^2 z}{\partial \xi^2} + 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(-\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}\right) = -\frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \eta^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \left(-\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(-\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}\right)$$

$$= \frac{\partial^2 z}{\partial \xi^2} - 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2}$$

Therefore the given equation reduces to

$$\frac{\partial^2 z}{\partial \eta^2} = 0$$

which is the required canonical form.

The general solution of equation may be easily obtained as

$$z = \eta\phi(\xi) + \phi_2(\xi)$$

where ϕ_1 and ϕ_2 are arbitrary functions of ξ.

Changing to the original variables, we get finally

$$z = (x + y)\phi_1(x - y) + \phi_2(x - y)$$

Ex. 3. Reduce $\dfrac{\partial^2 z}{\partial x^2} = x^2 \dfrac{\partial^2 z}{\partial y^2}$ ***to canonical form.***

Sol. Comparing the given equation with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0,$$

We get $\quad R = 1,\ S = 0,\ T = -x^2$

Now the roots of the α-equation

$$R\alpha^2 + S\alpha + T = 0$$

or $\quad \alpha^2 - x^2 = 0$

are $\quad \alpha = \pm x$

Changing the independent variables x, y to ξ, η where

$$\xi = f_1(x, y) \quad \text{and} \quad \eta = f_2(x, y)$$

such that f_1 = const. and f_2 = const. are the solutions of the differential equations.

Hence $\quad \dfrac{dy}{dx} + \alpha_1 = 0 \quad \text{and} \quad \dfrac{dy}{dx} + \alpha_2 = 0$

becomes $\quad \dfrac{dy}{dx} + x = 0 \quad \text{and} \quad \dfrac{dy}{dx} - x = 0$

Integrating $\quad y + \dfrac{x^2}{2} = \text{const. and } y - \dfrac{x^2}{2} = \text{const.}$

Hence $\quad \xi = y + \dfrac{x^2}{2} \quad \text{and} \quad \eta = y - \dfrac{x^2}{2}$

Now, $\quad \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial \xi} \cdot \dfrac{\partial \xi}{\partial x} + \dfrac{\partial z}{\partial \eta} \cdot \dfrac{\partial \eta}{\partial x} = x\dfrac{\partial z}{\partial \xi} - x\dfrac{\partial z}{\partial \eta}$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}$$

$$\frac{\partial^2 z}{\partial x^2} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}\right) = x^2\left[\frac{\partial^2 z}{\partial \xi^2} - 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2}\right] + \frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 y}{\partial \xi^2} + 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2}$$

Therefore the given equation reduces to

$$x^2\left[\frac{\partial^2 z}{\partial \xi^2} - 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2}\right] + \frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta} - x^2\left[\frac{\partial^2 z}{\partial \xi^2} + 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2}\right] = 0$$

or $\quad \dfrac{\partial^2 z}{\partial \xi \partial \eta} = \dfrac{1}{4x^2}\left(\dfrac{\partial z}{\partial \xi} - \dfrac{\partial z}{\partial \eta}\right)$

or $\quad \dfrac{\partial^2 z}{\partial \xi \partial \eta} = \dfrac{1}{4(\xi - \eta)}\left(\dfrac{\partial z}{\partial \xi} - \dfrac{\partial z}{\partial \eta}\right)$

which is the required canonical form.

Ex.4. Reduce the equation

$$\boldsymbol{xyr-\left(x^2-y^2\right)s-xyt+py-qx=2\left(x^2-y^2\right)}$$

to canonical form and hence solve it.

Sol. Comparing the given equation with standard form $Rr+Ss+Tt+f(x,y,z,p,q)=0$, we get

$R = xy,\ S = -(x^2 - y^2),\ T = -xy$

So α-equation $R\alpha^2 + S\alpha + T = 0$

becomes $xy\,\alpha^2 - (x^2 - y^2)\,\alpha - xy = 0$

or $\alpha = -\dfrac{y}{x}, \dfrac{x}{y}$

Hence $\dfrac{dy}{dx}+a_1=0$ and $\dfrac{dy}{dx}+a_2=0$

becomes $\dfrac{dy}{dx}-\dfrac{y}{x}=0$ and $\dfrac{dy}{dx}+\dfrac{x}{y}=0$

Integrating, $\dfrac{y}{x}=c_1,\ x^2+y^2=c_2$

Now, we take $\xi=\dfrac{y}{x},\ \eta=x^2+y^2$

Then $\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial \xi}\cdot\dfrac{\partial \xi}{\partial x}+\dfrac{\partial z}{\partial \eta}\cdot\dfrac{\partial \eta}{\partial x}=-\dfrac{y}{x^2}\dfrac{\partial z}{\partial \xi}+2x\dfrac{\partial z}{\partial \eta}$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial z}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}=\frac{1}{x}\frac{\partial z}{\partial \xi}+2y\frac{\partial z}{\partial \eta}$$

$$\frac{\partial^2 z}{\partial x^2}=\left(-\frac{y}{x^2}\right)^2\frac{\partial^2 z}{\partial \xi^2}+2(2x)\left(-\frac{y}{x^2}\right)\frac{\partial^2 z}{\partial \xi\partial \eta}+4x^2\frac{\partial^2 z}{\partial \eta^2}+\frac{2y}{x^3}\frac{\partial z}{\partial \xi}+2\frac{\partial z}{\partial \eta}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\left(-\frac{y}{x^2}\right)\left(\frac{1}{x}\right)\frac{\partial^2 z}{\partial \xi^2}+\left\{2y\left(-\frac{y}{x^2}\right)+2x\cdot\frac{1}{x}\right\}\frac{\partial^2 z}{\partial \xi\partial \eta}+4xy\frac{\partial^2 z}{\partial \eta^2}-\frac{1}{x^2}\frac{\partial z}{\partial \xi}$$

$$\frac{\partial^2 z}{\partial y^2}=\left(\frac{1}{x}\right)^2\frac{\partial^2 z}{\partial \xi^2}+2\cdot\frac{1}{x}\cdot(2y)\frac{\partial^2 z}{\partial \xi\partial \eta}+4y^2\frac{\partial^2 z}{\partial \eta^2}+2\frac{\partial z}{\partial \eta}$$

Therefore the given equation reduces to

$$\left(x^2+y^2\right)^2\frac{\partial^2 y}{\partial \xi\partial \eta}=\left(y^2-x^2\right)x^2$$

or $$\frac{\partial^2 y}{\partial \xi\partial \eta}=\frac{\left(y^2-x^2\right)x^2}{\left(x^2+y^2\right)^2}=\frac{\xi^2-1}{\left(\xi^2+1\right)^2} \qquad .....(28)$$

Integrating (28) w.r.t. ξ, we get

$$\frac{\partial z}{\partial \eta} = \int \frac{\xi^2 - 1}{\left(\xi^2 + 1\right)^2} d\xi + \phi(\eta)$$

$$= \int \frac{d\xi}{\xi^2 + 1} - 2\int \frac{d\xi}{\left(\xi^2 + 1\right)^2} + \phi(\eta)$$

$$= \int \frac{d\xi}{\xi^2 + 1} - 2\left[\frac{\xi}{2\cdot 1\cdot 1\left(\xi^2 + 1\right)} + \frac{1}{2\cdot 1\cdot 1}\int \frac{d\xi}{\xi^2 + 1} + \right] + \phi(\eta)$$

$\therefore$
$$\frac{\partial z}{\partial \eta} = -\frac{\xi}{\left(\xi^2 + 1\right)} + \phi(\eta)$$

Integrating it, we get

$$z = -\frac{\xi\eta}{\left(\xi^2 + 1\right)} + \phi_1(\eta) + \phi_2(\xi)$$

or
$$z = -xy + \phi_1\left(x^2 + y^2\right) + \phi_2\left(\frac{y}{x}\right)$$

Self Learning Exercise

1. The Harmonic equation is ....

2. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{k}\frac{\partial u}{\partial t}$ is two-dimensional .... equation.

3. Write general Laplace's equation.

4. Write wave equation.

5. Give a common method for solving Laplace, wave and diffusion equations.

5.4 Summary

In this unit, we have covered nature and types of Laplace, wave and diffusion equations and their solutions under different boundary and initial conditions, with illustrative examples. We have also presented the canonical form of PDE and its general solution also for hyperbolic, parabolic and elliptic equations.

5.5 Answers of Self-Learning Exercises

1. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

2. Diffusion

3. $\nabla^2 u = 0$

4. $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

5. Separation of variables

5.6 Exercise

1. A string is stretched between the fixed point O ($x = 0$) and A ($x = 1$) and released at rest from the position $U(x, 0) = A \sin \pi x$. Find the formula for its subsequent displacement $U(x, t)$

[Ans : $U(x, t) = A \cos \pi ct \cos \pi x$]

2. A string is stretched between the fixed points (0, 0) and (l, 0). If it is released at rest from the initial deflection

$$f(x) = \begin{cases} \frac{2k}{l} x & ; \quad 0 < x < \frac{l}{2} \\ \frac{2k}{l}(l - x) & ; \quad \frac{l}{2} < x < l \end{cases}$$

where 'k' is arbitrary constant. Find the expression of deflection of the string at any instant 't'.

[Ans : $U(x,t) = \frac{8k}{\pi^2}\left[\frac{\sin \pi x}{l}\frac{\cos \pi ct}{l} - \frac{1}{9}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l} + ...\right]$

3. A tightly string stretched string with fixed end points $x = 0$ and $x = \pi$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points an initial velocity

$$\left(\frac{\partial U}{\partial t}\right)_{t=0} = 0.03 \sin x - 0.04 \sin 3x$$

then find the displacement $U(x, t)$ at any point x and at any instant t.

[Ans. $U(x,t) = \frac{1}{c}\left[0.03 \sin x \sin ct - 0.01333 \sin 3x \sin 3ct\right]$]

4. Solve $y_{tt} = 4y_{xx}$, $y(5, t) = 0 = y(5, t)$, $y(x, 0) = 0$ and $\left(\frac{\partial y}{\partial t}\right)_{t=0} = f(x) = 5 \sin \pi x$

[Ans. $y(x, t) = \frac{5}{2\pi} \sin \pi x \sin 2\pi t$ **]**

5. Solve diffusion equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < l, t > 0$

$u(x, 0) = 3 \sin n\pi x$, $u(0, t) = 0$, $u(l, t) = 0$.

[Ans. $u(x,t) = 3\sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \sin n\pi x$ **]**

6. The temperature distribution in a bar of length π which is perfectly insulated at ends $x = 0$ and $x = \pi$ is governed by PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Assuming the initial temperature distribution as $u(x, 0) = \cos 2x$. Find the temperature distribution at any instant of time. **[Ans.** $u(x, t) = e^{-4t} \cos 2x$]

7. A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the temperature initially is

$$u(x,0)=\begin{cases} x, & 0 \le x \le 50 \\ 100-x, & 50 \le x \le 100 \end{cases}$$

Find the temperature $u(x,t)$ at any time.

[Ans. $u(x,t)=\dfrac{400}{\pi^2}\sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n+1)^2}\sin\dfrac{(2n+1)\pi x}{100}e^{-\left(\frac{(2n+1)c\pi}{100}\right)^2 t}$ **]**

8. Solve $u_t = a^2 u_{xx}$ under the conditions $u_x(0, t) = 0 = u_x(\pi, t)$, $u(x, 0) = x^2$, $0 < x < \pi$, $t > 0$.

[Ans. $u(x,t)=\dfrac{\pi^3}{3}+4\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}\cos n\, x e^{-a^2n^2t}$ **]**

9. Solve $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$; $0<x<\pi, 0<y<\pi$

which satisfies the conditions $u(0,y)=u(\pi,y)=u(x,\pi)=0$ and $u(x,0)=\sin^2 x$.

[Ans. $u(x,t)=\dfrac{-8}{\pi}\sum_{n=1}^{\infty}\dfrac{\sin(2n-1)x\sinh(2n-1)(\pi-y)}{(2n-1)\left[(2n-1)^2-y\right]\sinh(2n-1)\pi}$ **]**

10. Reduce the equation $y\dfrac{\partial^2 z}{\partial y^2}+\dfrac{\partial z}{\partial y}=0$ to canonical form and find its general solution.

[Ans. $z=\phi_1(x)\log y+\phi_2(x)$**]**

11. Reduce the equation

$$y^2\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x\partial y}+x^2\frac{\partial^2 z}{\partial y^2}=\frac{y^2}{x}\frac{\partial z}{\partial x}+\frac{x^2}{y}\frac{\partial z}{\partial y}$$

to canonical form and hence solve it. **[Ans.** $z=\left(x^2-y^2\right)\phi_1\left(x^2+y^2\right)+\phi_2\left(x^2+y^2\right)$**]**

12. Reduce the equation $\dfrac{\partial^2 z}{\partial x^2}=x^2\dfrac{\partial^2 z}{\partial y^2}$ to canonical form .

Also state the nature of the equation.

[Ans. $\dfrac{\partial^2 z}{\partial\xi\partial\eta}=\dfrac{1}{4(\xi-\eta)}\left(\dfrac{\partial z}{\partial\xi}-\dfrac{\partial z}{\partial\eta}\right)$; $\xi=y+\dfrac{x^2}{2}, \eta=y-\dfrac{x^2}{2}$, hyperbolic.**]**

13. Reduce the equation $\dfrac{\partial^2 z}{\partial x^2}+x^2\dfrac{\partial^2 z}{\partial y^2}=0$ to canonical form.

Also find its nature. **[Ans.** $\dfrac{\partial^2 z}{\partial\lambda^2}+\dfrac{\partial^2 z}{\partial\mu^2}=-\dfrac{1}{2\lambda}\dfrac{\partial z}{\partial\lambda}, \mu=y, \lambda=\dfrac{x^2}{2}$, elliptic**]**

□ □ □

Unit 6 : Eigenvalues, Eigenfunctions and Sturm-Liouville Boundary Value Probleon

Structure of the Unit

6.0 Objective

After completing the present unit, you will get a basic knowledge about eigenvalue and eigenfunction of boundary value problems. You will study special boundary value problem known as Sturm-Liouvelle problem and properties of eigenfunctions in later part of unit. The knowledge which you gain here, can be used to study various special functions that arise in physical and engineering problems.

6.1 Introduction

In the eighteenth century much attention was given to the problem of determing the mathematical laws governing the notion of a vibrating string with fixed end points. We wish to motivate the physics of vibrating string. In the last unit, we dealt the wave equation in detail with some other physical problems where we had derived boundary value problems for seeking non-trivial solution of partial differental equations involved in formulating physical problems. In this unit we study the condition of parameter involved in boundary value problem and corresponding non-trivial solution. We will also see special boundary value problem, known as Sturm-Liouville problem in detail which helps in studying regular boundary value problem and special functions in future.

6.2 Linear Homogeneous Boundary Value Problems

In previous unit, we have noticed that most important application of the idea is in boundary value problems of any type. For second order linear differential equation, boundary value problem is defined as

$$Ly = h \qquad .....(1)$$

where L is a second order linear differential operator defined on a finite interval $[a, b]$ and h is a function in $[a, b]$ and pair of homogeneous boundary conditions of the form

$$\alpha_1 y(a) + \alpha_2 y(b) + \alpha_3 y'(a) + \alpha_4 y'(b) = \gamma_1 \qquad .....(2)$$

$$\beta_1 y(a) + \beta_2 y(b) + \beta_3 y(a) + \beta_4 y(b) = \gamma_2 \qquad .....(3)$$

where α_i, β_i and γ_i for $i = 1, 2$ are constants. The problem (1) with boundary conditions (2) and (3) is known as linear homogeneous boundary value problem. In this problem, we seek all non-trivial functions of $y(x)$ in $[a, b]$ which simultaneously satisfy differential equation (1) and boundary conditions (2) to (3).

For example, $$y'' + \lambda y = 0 \qquad .....(4)$$

with boundary conditions

$$y(0) = 0 \text{ and } y(\pi) = 0 \qquad .....(5)$$

is a boundary value problem of above type on the interval $[a, b]$. The parameter 'λ' in (4) is free to assume any real value.

The situation with boundary conditions is quite different from that for initial condition. The initial value problem is a sophisticated variation of the fundamental theorem of calcalus. The boundary value problem is rather more subtle.

6.3 Eigenvalues and Eigenfunctions

In previous study, we have considered initial value problem, in which the solution of second order differential equation is sought that satisfies two conditions at a single value of the independent variable. Here we have absolutely different situation for we wish to satsfy one condition at each of two distinct values of independent variable x. The part of our task is to discover the values of $\lambda's$ for which problem can be solved for getting non-trivial solution. The solution of given problem in (4) with boundary conditions (5) is not difficult to find. We simply apply the boundary conditions to the general solution. But we have to analyse the solution for all possible values of $\lambda's$. So, three cases arise as follows.

Case I : λ is negative or $\lambda < 0$

Let $\lambda = -m^2$

The given problem (4) with (5) becomes

$$y'' - m^2 y = 0 \qquad .....(1)$$

and $\qquad y(0) = 0$ and $y(\pi) = 0$

so, the general solution is

$$y(x) = c_1 e^{mx} + c_2 e^{-mx}$$

Now $\qquad y(0) = 0 \Rightarrow c_1 + c_2 = 0 \qquad .....(2)$

and $\qquad y(\pi) = 0 \Rightarrow c_1 e^{m\pi} + c_2 e^{-m\pi} = 0 \qquad .....(3)$

Equations (2) and (3) give

$c_1 \sinh m\pi = 0 \Rightarrow c_1 = 0$ as $\sinh m\pi \neq 0$ for $m \neq 0$

Hence $c_1 = c_2 = 0$. Thus we get only one trivial solution exists.

Case II : λ = 0

The given problem (4) with (5) becomes

$$y'' = 0$$

and $\qquad y(0) = 0$ and $y(\pi) = 0$

Hence the general solution is

$$y(x) = c_1 x + c_2$$

When $y(0) = 0$, we have $c_2 = 0$

So $\qquad y(x) = c_1 x$

When $y(\pi) = 0$, we have $c_1 = 0$

Under given boundary conditions, $c_1 = c_2 = 0$

i.e. we have trivial solution for given problem for this value of λ or $y \equiv 0$

Thus, we are restricted to the case in which λ is postive for seeking non-trivial solution.

Case III : λ > 0

Let $\lambda = m^2$

The given problem (4) with (5) reduces to

$$y'' + m^2 y = 0 \qquad .....(9)$$

and $\qquad y(0)= 0$ and $y(\pi) = 0$

so, the general solution is

$$y(x)= c_1 \sin mx + c_2 \cos mx$$

for $y(0) = 0$, we have $c_2 = 0$

Hence $\qquad y(x) = c_1 \sin mx$

and for $y(\pi) = 0$, $0 = c_1 = \sin m\pi$

Since $c_1 \neq 0$ for seeking non-trivial solution, we must have

$\qquad \sin m\pi = 0 \Rightarrow \sin m\pi = n\pi$; for some positive integer

or $\qquad m\pi = n\pi;\ n = 1, 2, 3, ...$

or $\qquad m = n$

Hence $\lambda_n = n^2;\ n = 1, 2, 3,$ .... which is known as eigenvalues and corresponding solution is

$$y_n(x) = c_1 \sin nx;\ n = 1, 2, 3, ...$$

which is called as eigenfunction.

6.3.1 Eigenvalue or Characteristic Value

The values $\lambda's$, for which given boundary value problem has non-trivial solutions, are called eigen-values of given problem.

For example $\lambda = 1, 4, 9, ...n^2$ are eigenvalues of problem (4)

6.3.2 Eigenfunction or Characteristic Function

The non-trivial solution of given boundary value problem corresponding to particular eigenvalues is termed as eigenfunction.

For example $y_n(x) = \sin x, \sin 2x$, ..., $\sin nx$, ... are eigenfunctions for eigenvalues in = 1, 4, 9, ... n^2, ... respectivoly for problem in (4)

It is to be noted here that the eigenvalues are uniqely determined by the problem but the eigenfunctions are not. Any non-zero sealar multiple of eigenfunction is also a eigenfunction.

From the above study, we have three important conclusions for eigenvalues and eigenfunctions as follows

(i) The eigenvalues form an increasing seauence of positive numbers that approaches ∞ i.e.

$$\lambda_1 < \lambda_2 < \lambda_3 \ldots\ldots\ \lambda_n \ldots\ldots$$

and $$\lambda_n \to \infty \text{ as } n \to \infty$$

For example, $1 < 4 < 9 \ldots\ldots < n^2 < \ldots\ldots$ in above problem

(ii) The n^{th} eigenfunction vanishes at the end points of the interval and has exactly $n - 1$ zeros inside this interval.

For example, for $\lambda_n = n^2$, $y_n = \sin nx$ vanisheos at the end points of the interval $[0, \pi]$ and has exactly $n - 1$ zeros inside this interval $(0, \pi)$ in above problem in (4).

(iii) If $y_n(x)$ is an eigenfunction for eigenvalue λ for given problem, then $cy_n(x)$ is also eigenfunction where c is arbitrary constant for same eigenvalue. Hence the eigenfunction corresponding to each eigenvalue is unique except for a multiple of an arbitrary constant factor.

The problems of heat, wave and Laplace in previous unit or many other physical or applied mathematical problems are boundary value problems. In solution procedure by seperation of variables for any problem, notice that we have calculated eigenvalues and corresponding eigenfunctions also.

Ex.1. Find the eigenvalues λ's and corresponding eigenfunctions $y_n(x)$ for the equation $y'' + \lambda y = 0$ under the boundary condition $y(0) = 0$ and $y(\pi/2) = 0$

Sol. We have three cases.

Case I : λ is negatve or $\lambda < 0$

Let $$\lambda = -m^2$$

The given differential equation becomes

$$y'' - m^2 y = 0$$

whose general solution is

$$y(x) = c_1 e^{mx} + c_2 e^{-mx}$$

Now $$y(0) = 0 \Rightarrow c_1 + c_2 = 0$$

and $$y(\pi/2) = 0 \Rightarrow c_1 e^{m\pi/2} + c_2 e^{-m\pi/2} = 0$$

The above thwo equations give us

$c_1 \sinh (m\pi/2) = 0 \Rightarrow c_1 = 0 \qquad (\because\ m\pi \neq 0)$

Thus we get only one trivial solution i.e. $y(x) = 0$

Case II : when $\lambda = 0$

The given problem reduces to

$$y'' = 0$$

Hence the general solution is

$$y(x) = c_1x + c_2$$

So, under given boundary conditions, $c_1 = c_2 = 0$

which gives trivial solution only i.e. $y \equiv 0$ for $\lambda = 0$

Thus $\lambda \leq 0$ are not eigenvalues for given problem.

Case III : when λ is positive or $\lambda > 0$

Let $\quad \lambda = m^2$

Then problem becomes

$$y'' + m^2y = 0 \qquad .....(6)$$

and $\quad y(0) = 0$ and $y(\pi/2) = 0$

The general solution is $\quad y(x) = c_1 \sin mx + c_2 \cos mx$

When $y(0) = 0$, $c_2 = 0$ and hence $y(x) = c_1 \sin mx$

When $\quad y(\pi/2) = 0,\ 0 = c_1 \sin n\, \pi/2$

For seeking non-trivial solution, we should have $c_1 \neq 0$ then $\sin n\, \pi/2 = 0$

or $\quad \sin m\pi/2 = n\pi$; for some positive integer n

$\Rightarrow \quad m\pi/2 = n\pi;\ n = 1, 2, 3, ...$

$\Rightarrow \quad m = 2n;\ n = 1, 2, 3, ...$

Therefore $\quad \lambda_n = m^2 = 4n^2\ ; \quad n = 1, 2, 3, ...$

Hence $\lambda_n = 4, 16, 36, ..., 4n^2$ ...... are the increasing sequence of eigenvalues. The corresponding eigenfunctions are

$$y_n(x) = \sin 2nx;\ n = 1, 2, 3, ...$$

Ex.2. Find the eigenvalues and eigenfunctions for the boundary value problem $y'' + \lambda y = 0$ under the boundary condition $y(a) = 0$ and $y(b) = 0$, $0 < a < b$; a, b are arbitrary real constants.

Case I : $\lambda < 0$ or $\lambda = -\alpha^2$

Given problem reduces to

$$y'' - \alpha^2y = 0$$

with $\quad y(a) = 0$ and $y(b) = 0$

The general solution is

$$y(x) = c_1e^{\alpha x} + c_2e^{-\alpha x}$$

When $\quad y(a) = 0,\ c_1e^{\alpha a} + c_2e^{-\alpha a} = 0 \Rightarrow -c_1e^{2\alpha a} = c_2$

$\quad y(b) = 0,\ c_1e^{\alpha b} + c_2e^{-\alpha b} = 0 \Rightarrow -c_1e^{2\alpha b} = c_2$

Hence, $\quad -c_1e^{2\alpha a} = -c_1e^{2\alpha b}$

$\Rightarrow \quad c_1(e^{2\alpha a} - e^{2\alpha b}) = 0$

Since $\quad a \neq b,\ c_1 = 0$

and hence $\quad c_2 = 0$

which implies $y \equiv 0$ i.e. only trivial solution exists.

Case II : If $\lambda = 0$

Given problem reduces to

$$y'' = 0$$

with $y(a) = 0$ and $y(b) = 0$

The general solution is $y(x) = c_1 x + c_2$

For $y(a) = 0$, $c_1 a + c_2 = 0$ and for $y(b) = 0$, $c_1 b + c_2 = 0$

On subtracting we have $c_1(a-b) = 0$

Since $a \neq b,\ c_1 = 0$

and hence $c_2 = 0$ and $y \equiv 0$ i.e. we get only trivial solution.

Case III : When $\lambda > 0$ or $\lambda = \alpha^2$

Given problem becomes, $y'' + \alpha^2 y = 0$

With $y(a) = 0$ and $y(b) = 0$

The general solution is $y(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$

For $y(a) = 0,\ 0 = c_1 \cos \alpha a + c_2 \sin \alpha a$

For $y(b) = 0,\ 0 = c_1 \cos \alpha b + c_2 \sin \alpha b$

Non-trivial solution for c_1 and c_2 in above system of equation may exist only when we have

$$\begin{vmatrix} \cos \alpha a & \sin \alpha a \\ \cos \alpha b & \sin \alpha b \end{vmatrix} = 0$$

i.e. $\sin \alpha(b-a) = 0$

or $\sin \alpha(b-a) = \sin n\pi$; for $n = 1, 2, 3, \ldots..$

or $\alpha(b-a) = n\pi$

or $$\alpha = \frac{n\pi}{b-a};\ n = 1, 2, 3, \ldots...$$

Hence the eigenvalues are

$$\lambda_n = \alpha^2 = \frac{n^2\pi^2}{(b-a)^2};\ n = 1, 2, 3, \ldots...$$

and correspondingeigenfunctions are

$$y_n(x) = c_1 \cos \frac{n\pi}{b-a} x + c_2 \sin \frac{n\pi}{b-a} x$$

If we suppose that $$c_1 = \sin \frac{n\pi b}{b-a} \text{ and } c_2 = \cos \frac{n\pi b}{b-a}$$

then eigenfurctions are $$y_n(x) = \sin \frac{n\pi}{b-a}(b-x)$$

Ex.3. Find the eigenvalues and eigenfunction for the boundary value problem

$$y'' - 2y + \lambda y = 0;\ y(0) = 0,\ y(\pi) = 0$$

Sol. Put $y = e^{mx}$

Auxillary equation is $m^2 - 2m + \lambda = 0$

$$m = 1 \pm \sqrt{1-\lambda}$$

Case I : If $1 - \lambda > 0$ or $\lambda < 1$

The general solution is

$$y(x) = c_1 e^{\left(1+\sqrt{1-\lambda}\right)x} + c_2 e^{\left(1-\sqrt{1-\lambda}\right)x}$$

Under given boundary conditions, $y(0) = y(\pi) = 0$, we have

$$c_1 = c_2 = 0 \text{ or } y \equiv 0$$

So, only trival solution exist i.e. $\lambda < 1$ does not give any eigenvalue.

Case II : If $1 - \lambda = 0$ or $\lambda = 1$

The general solution of given problem is

$$y(x) = (c_1 x + c_2)\, e^x$$

On applying boundary conditions,

$y(0) = 0$ and $y(\pi) = 0$ we have $c_1 + c_2 = 0$

Hence, only trivial solution exists and therefore $\lambda = 1$ is not an eigenvalue.

Case III : If $1 - \lambda < 0$ or $\lambda > 1$

The general solution is

$$y = \left[A\cos\sqrt{\lambda-1}x + B\sin\sqrt{\lambda-1}x\right]e^x$$

When $y(0) = 0$, we have $A = 0$ or $y(x) = B\sin\sqrt{\lambda-1}xe^x$

For $\quad y(\pi) = 0, \ \sin\sqrt{\lambda-1}\pi = 0$

since $e^\pi \neq 0$ and $B \neq 0$ for seeking non-zero solutions.

Hence $\sin\sqrt{\lambda-1}\ \pi = 0 = \sin n\pi$, $n = 1, 2, 3, ...$

$\Rightarrow \quad \lambda - 1 = n^2$

or $\quad \lambda_n = n^2 + 1;\ n = 1, 2, 3, ...$

are required eigenvalues and corresponding eigenfunctions are

$$y_n(x) = e^x \sin nx \quad n = 1, 2, 3, ...$$

Ex.4. Find the eigenvalues and eigenfunctions for the following boundary value problem

$$y'' - 4y' + (4 - 9\lambda)y = 0,\ y(0) = 0,\ y(a) = 0,$$

where 'a' is a positive real constant.

Sol. The auxillary equation of a given problem is

$$m^2 + 4m + (4 - 9\lambda) = 0$$

$$m = -4 \pm \sqrt{16 - 4(4 - 9\lambda)} = 2 \pm 3\sqrt{\lambda}$$

Case I : when $\lambda = 0$

The general solution of given problem is

$$y(x) = e^{-2x}(c_1 + c_2 x)$$

When $\quad y(0) = 0,\ c_1 = 0$

or $\quad y(x) = c_2 x e^{-2x}$

Also when $\quad y(a) = 0,\ c_2 a e^{-2a} = 0$

Since $\quad a > 0$, therefore $c_2 = 0$

Hence, $y \equiv 0$ i.e. only trivial solution exists.

Case II : When λ > 0

The general solution is

$$y(x) = e^{-2x}\left(c_1 e^{3\sqrt{\lambda x}} + c_2 e^{-3\sqrt{\lambda x}}\right)$$

On applying boundary condition

$$y(0) = 0,\ c_1 + c_2 = 0 \text{ or } c_2 = -c_1$$

$\therefore$ $$y(x) = c_1 e^{-2x}\left(e^{3\sqrt{\lambda x}} - e^{-3\sqrt{\lambda x}}\right)$$

Again $$y(a) = 0, \text{ gives } c_1 e^{-2a}\left(e^{3\sqrt{\lambda a}} - e^{-3\sqrt{\lambda a}}\right) = 0$$

$\Rightarrow$ $c_1 = 0 \quad \therefore c_2 = 0, y \equiv 0$, only trival solution exists.

For $\lambda \geq 0$, the given problem has no non-zero eigenfunction.

Case III : When λ < 0

The general solution of given differential equation is

$$y(x) = e^{-2x}\left(c_1 \sin\left(3\sqrt{-\lambda x}\right) + c_2 \cos\left(3\sqrt{-\lambda x}\right)\right)$$

Now $y(0) = 0$ gives $c_2 = 0$

$\therefore$ $$y(x) = e^{-2x}\sin\left(3\sqrt{-\lambda x}\right)$$

Also $$y(a) = 0 \text{ gives } c_1 e^{-2a}\sin\left(3\sqrt{-\lambda a}\right) = 0$$

For non-trivial solution, we have $c_1 \neq 0$, then

$$\sin\left(3\sqrt{-\lambda a}\right) = 0$$

or $$\sin\left(3\sqrt{-\lambda a}\right) = \sin n\pi;\ n = 1, 2, 3, ...$$

$\therefore$ $$\sqrt{-\lambda} = \frac{n\pi}{3a}$$

or $$-\lambda = \frac{n^2\pi^2}{9a^2}$$

Hence $$\lambda_n = \frac{-n^2\pi^2}{9a^2};\ n = 1, 2, 3.....$$

are the required eigenvalues for given problem. Hence the corresponding eigenfunctions are

$$y_n(x) = e^{-2x}\sin\left(\frac{n\pi x}{a}\right);\ n = 1, 2, 3.....$$

Ex.5. Find the eigenvalues and eigenfunctions for the following boundary value problem

$$y'' - 3y' + 2(1 + \lambda)y = 0,\ y(0) = 0,\ y(1) = 0$$

Sol. Auxillary equation for given differential equation is

$$m^2 - 3m + 2(1 + \lambda) = 0$$

Solving, we get $$m=\frac{3\pm\sqrt{9-4\cdot 2\cdot(1+\lambda)}}{2}$$

$$=\frac{3}{2}\pm\frac{\sqrt{1-8\lambda}}{2}$$

Thus, three cases arise

Case I : When 1 – 8λ = 0 or $\lambda=\frac{1}{8}$

The general solution of equation is

$$y(x) = e^{(3/2)x}(c_1 + c_2x)$$

Now $y(0) = 0$ gives $c_1 = 0$.

Therefore $$y(x) = c_2x^{(3/2)x}$$

Again $y(1) = 0$ gives $c_2 = 0$

Hence, $y \equiv 0$ is the only trivial solution of the given problem.

Case II : when 1 – 8λ > 0 or $\lambda<\frac{1}{8}$

The solution of given equation is

$$y(x)=e^{(3/2)x}\left(c_1e^{(1/2)\sqrt{1-8\lambda}x}+c_2e^{(-1/2)\sqrt{1-8\lambda}x}\right)$$

when, $y(0) = 0 \Rightarrow c_1 + c_2 = 0$ or $c_2 = -c_1$

$\therefore$ $$y(x)=c_1e^{(3/2)x}\left(e^{(1/2)\sqrt{1-8\lambda}x}-e^{(-1/2)\sqrt{1-8\lambda}x}\right)$$

or $$y(x)=2c_1e^{(3/2)x}\sin h\left(\frac{\sqrt{1-8\lambda}}{2}\right)x$$

Again $$y(1)=0 \Rightarrow y(1)=2c_1e^{(3/2)}\sin h\left(\frac{\sqrt{1-8\lambda}}{2}\right)=0$$

$\therefore$ $c_1 = 0$

Therefore $c_2 = 0$. Hence $y(x) \equiv 0$,

Thus for $\lambda\le\frac{1}{8}$, only trivial solution exists.

Case III : when 1 – 8 λ < 0 or $\lambda>\frac{1}{8}$

The solution is

$$y(x)=e^{(3/2)x}\left[c_1\sin\frac{\sqrt{8\lambda-1}}{2}x+c_2\cos\frac{\sqrt{8\lambda-1}}{2}x\right]$$

Now for $y(0) = 0$, we have $c_2 = 0$

$$y(x) = c_1 e^{(3/2)x} \sin \frac{\sqrt{8\lambda - 1}}{2} x$$

Also $\quad y(1) = 0 \Rightarrow \; c_1 e^{(3/2)x} \sin \frac{\sqrt{8\lambda - 1}}{2} = 0$

For seeking non-trivial solution, we have $c_1 \neq 0$

therefore $\quad \sin \frac{\sqrt{8\lambda - 1}}{2} = 0$

or $\quad \sin \frac{\sqrt{8\lambda - 1}}{2} = \sin n\pi$; for positive integral n

$\therefore \quad \frac{\sqrt{8\lambda - 1}}{2} = n\pi \Rightarrow \sqrt{8\lambda - 1} = 2n\pi \Rightarrow \lambda_n = \frac{4n^2\pi^2 + 1}{8}; n = 1, 2, 3, \ldots.$

are required eigenvalues and corresponding eigenfunctions are $y_n(x) = e^{(3/2)x} \sin n\pi x \; (n \in N)$

6.4 Sturm-Liouville Problem

A boundary value problem consisting of second order homogeneous linear differential equation of the form

$$\frac{d}{dx}\left(p(x) \frac{dy}{dx} \right) + \left[\lambda q(x) + r(x) \right] y = 0 \qquad .....(1)$$

where p, q and r are continuous real valued functions defined on $a \leq x \leq b$ such that p has a continuous derivative, $p(x) > 0$ and $q(x) > 0$ and λ is a parameher independent of x and two homogenous boundary conditions

$$A_1 y(a) + A_2 y'(a) = 0 \qquad .....(2)$$

$$B_1 y(b) + A_2 y'(b) = 0 \qquad .....(3)$$

where A_1, A_2, B_1 and B_2 are real constants such that A_1 and A_2 are not both zero and B_1 and B_2 are not both zero simultaneously, is called Sturm-Liouville problem. All the problems we have discussed in previous section are Sturm-Liouville problems.

Ex.1. Check whether the boundary value problem

$$y'' - \lambda y = 0 \quad \textit{with} \quad y(0) = 0 = y(\pi)$$

is Sturm-Liouville problem or not

Sol. On comparing with stanard form of Sturm-Liouville problem, we have

$$p(x) = 1, q(x) = 1, r(x) = 0, a = 0 \text{ and } b = \pi;$$

$$A_1 = B_1 = 1 \text{ and } A_2 = B_2 = 0$$

Hence given problem is Sturm-Liouville problem.

Ex.2. Check whether the following boundary value problem

$$xy'' + y' + (x^2 + 1 + \lambda)\, y = 0$$

$$y(0) = 0 \; \textit{and} \; y'(L) = 0, L \text{ is constant such that } L > 1$$

is Sturm-Liouville problem or not.

Sol. $xy'' + y' + (x^2 + 1 + \lambda)\, y = 0$

$(xy')' + (x^2 + 1 + \lambda)\, y = 0$

$\therefore$ $p(x) = x$, $q(x) = 1$, $r(x) = 1 + x^2$, $a = 0$ and $B = L$;

$A_1 = 1$, $B_1 = 0$, $A_2 = 0$ and $B_2 = 1$

Since $p(x) > 0$ for $0 \le x \le L$

Given boundary value problem is Sturm-Liouville problem

Ex.3. Find the eigenvalues and eigenfunctions of the following Sturm-Liouville problem

$$\frac{d}{dx}\left(e^{2x}\frac{dy}{dx}\right)+(\lambda+1)e^{2x}y=0;$$

$$y(0) = 0 = y(\pi)$$

Sol. Transform dependent variable from y to u by using transformation

$$y = e^{-x}u$$

$$\therefore \quad \frac{dy}{dx} = e^{-x}\frac{du}{dx} - e^{-x}u$$

Therefore given differential equation reduces to

$$\frac{d}{dx}\left(e^{2x}\left(e^{-x}\frac{du}{dx}-e^{-x}u\right)\right)+(\lambda+1)e^{2x}e^{-x}u=0$$

$$=2e^{2x}\left(e^{-x}\frac{du}{dx}-e^{-x}u\right)$$

$$+e^{2x}\left(-e^{-x}\frac{du}{dx}+e^{-x}\frac{d^2u}{dx^2}+e^{-x}u-e^{-x}\frac{du}{dx}\right)$$

$$+\lambda e^{2x}\cdot e^{-x}u + e^{2x}e^{-x}u = 0$$

or $$e^{x}\left[\frac{d^2u}{dx^2}+\lambda u\right]=0$$

i.e. $$u'' + \lambda u = 0$$

and boundary conditions reduce to

$$u(0) = 0 = u(\pi) \text{ since } e^{-x} \neq 0 \quad \forall\, x \in R$$

we know that $\lambda_n = n^2$; $n = 1, 2, 3, ...$

are the eigenvalues for reduced problem and corresponding eigenfunctions are $u_n(x) = \sin nx$ (see §6.3)

Hence $\lambda_n = n^2$; $n = 1, 2, 3, ...$ are the eigenvalues for given problem and corresponding eigenfunctions are

$$y_n(x)\, e^{-x} \sin nx\, ;\ n \in \mathbf{N}$$

Ex.4. Solve the following Sturm-Liouville problem

$$\boldsymbol{y'' + \lambda y = 0;\ y'(-\pi) = 0,\ y'(\pi) = 0}$$

Sol. Let $\lambda < 0$ i.e. $\lambda = -\alpha^2$

Then given problem becomes

$$y'' - \alpha^2 y = 0;\ y'(-\pi) = 0,\ y'(\pi) = 0$$

The general solution is $y(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$

$\therefore$ $y'(x) = c_1 \alpha e^{\alpha x} - \alpha c_2 e^{-\alpha x}$

Now $y'(-\pi) = 0 \Rightarrow c_1 \alpha e^{-\alpha\pi} - c_2 \alpha e^{\alpha\pi} = 0$

and $y'(\pi) = 0 \Rightarrow c_1 \alpha e^{\alpha\pi} - c_2 \alpha e^{-\alpha\pi} = 0$

For non-trivial solution for c_1 and c_2 for above system of equations, the coffecient determinant must vanish. Hence

$$\begin{vmatrix} \alpha e^{-\alpha\pi} & -\alpha e^{\alpha\pi} \\ \alpha e^{\alpha\pi} & -\alpha e^{-\alpha\pi} \end{vmatrix} = 0$$

$\Rightarrow \quad -e^{-2\alpha\pi} + e^{-2\alpha\pi} = 0$

which is not possible. Hence $c_1 = c_2 = 0$

Therefore only trivial solution exists that is $y = 0$

When $\lambda = 0$.

The general solution is $y = c_1 x + c_2$

So, $y' = c_1$

For boundary condition $y'(-\pi) = 0$ and $y'(\pi) = 0$, $c_1 = 0$

Hence $y(x) = c_2$ is solution

When $\lambda > 0$. Let $\lambda = \alpha^2$

Then given problem becomes

$$y'' + \alpha^2 y = 0$$

The general solution of the differential equation is

$$y(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$y'(x) = -c_1 \alpha \sin \alpha x + c_2 \alpha \cos \alpha x$$

An appling boundary condition $y'(-\pi) = 0$, we have $c_2 = 0$

$\therefore$ $y'(x) = -c_1 \alpha \sin \alpha x$

Again for $y'(\pi) = 0; -c_1 \alpha \sin \alpha\pi = 0$

Since $c_1 \neq 0$; therefore $\sin \alpha\pi = 0$, *i.e.* $\sin \alpha\pi = \sin n\pi$; $n = 1,2,3, ...$

or $\alpha = n$; $n = 1,2,3, ...$

$\therefore$ $\lambda_n = n^2$; $n = 1,2,3, ...$

are the required eigenvalues and corresponding eigenfunctions are $y_n(x) = \cos nx$

Hence from Case II and Case III, the eigenvalues for given problem are $\lambda_n = 0,1,4,9, ...n^2 .....$ and corresponding eigenfunctions are $y_n(x) = 1, \cos x, \cos 2x, \cos 3x, ...\cos nx, ...$

6.5 Orthogonality of Eigenfunctions

From previous section, it is very much clear that the Sturm-Liouville problem is advanced boundary value problem and have non-trivial solution if function $p(x)$ and $q(x)$ are restricted for $p(x) > 0$ and $q(x) > 0$ on $[a, b]$ and iff the parameter λ takes a certain specific value. These are termed as eigenvalues of boundary value problem. They are real numbers that can be arranged in an increasing sequence:

$$\lambda_1 < \lambda_2 < .... < \lambda_n < \lambda_{n+1} < ...... \quad ....(1)$$

and furthurmore $\lambda_n \to \infty$ as $n \to \infty$

This ordering is desirable to arrange corresponding eigenfunctions

$$y_1(x), y_2(x), ..., y_n(x), y_{n+1}(x), ... \quad .....(2)$$

in their own natural order. The eigenfunctions are not unique, but with the boundary conditions, they are determined up to a non-zero constant factor.

Now, we introduce a new concept in broader context that will assist to understand the property of various special functions that generally arise in various physical and engineering modelling.

A sequence of eigenfunctions $y_n(x)$ in (2) having the property

$$\int_a^b y_m(x) y_n(x) dx = \begin{cases} 0 & \text{if } m \neq n \\ \alpha_n \neq 0 & \text{if } m = n \end{cases}$$

is said to be orthogonal on the interval $[a, b]$.

If $\alpha_n = 1$, $\forall n$, the function $y_n(x)$ are said to be normalized and sequence of eigenfunctions is known as orthonormal sequence.

If sequence of eigenfunctions $y_n(x)$ have the following general property

$$\int_a^b q(x) y_m(x) y_n(x) dx = \begin{cases} 0, & m \neq n \\ \alpha_n \neq 0, & m = n \end{cases}$$

then, this sequence is said to be orthogonal with respect to a weight function $q(x)$.

6.6 Important Theorems of Sturm-Liouville Systems

6.6.1 Theorem 1. The eigenvalues of Sturm-Liouville system are real

Proof. We have

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda r(x)\right] = 0 \quad .....(1)$$

where $a_1 y(a) + a_2 y'(a) = 0$, and $b_1 y(b) + b_2 y'(b) = 0$ ......(2)

Suppose the $p(x)$, $q(x)$, $r(x)$, a_1, a_2, b_1 and b_2 are real, while λ and y may be complex. Let $\bar{\lambda}$ and $\bar{y}$ denote complex conjugates of λ and y respectively. Now we have from (1) and (2)

$$\frac{d}{dx}\left[p(x)\frac{d\bar{y}}{dx}\right] + \left[q(x) + \bar{\lambda} r(x)\right] = 0 \quad .....(3)$$

where $a_1\bar{y}(a) + a_2\bar{y}'(a) = 0$, and $b_1\bar{y}(b) + b_2\bar{y}'(b) = 0$. .....(4)

Multiplying (1) by $\bar{y}$ and (3) by y and then subtracting we find that

$$\frac{d}{dx}\left[p(x)\{y\bar{y}' - \bar{y}y'\}\right] = \left(\lambda - \bar{\lambda}\right) r(x) y\bar{y} \quad .....(5)$$

Integrating it from a to b and using boundary conditions (2) and (4), we find that

$$\left(\lambda - \bar{\lambda}\right)\int_a^b r(x) y\bar{y}\, dx = 0 \quad .....(6)$$

Since $r(x)$ is a non-negative and $r(x) \neq 0$ for $a \leq x \leq b$, therefore (6) gives

$$\lambda - \bar{\lambda} = 0 \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \text{ is real.}$$

6.6.2 Theorem 2 : Let λ_m and λ_n be two distinct eigenvalues of the Sturm-Liouville problem

$$\frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\}+\left[\lambda q(x)+r(x)\right]y=0 \qquad .....(6)$$

and $y_m(x)$ and $y_n(x)$ be their corresponding eigenfunctions. Then $y_m(x)$ and $y_n(x)$ are orthogonal with respect to the weight function $q(x)$ on the interval $a \le x \le b$.

Proof : If λ_m and λ_n are eigenvalues of given Sturm-Liouville problem

$$\left[p(x)y'(x)\right]'+\left[\lambda q(x)+r(x)\right]y(x)=0 \qquad .....(7)$$

then we have

$$\left[p(x)y_m'(x)\right]'+\left[\lambda_m q(x)+r(x)\right]y_m(x)=0 \qquad .....(8)$$

and

$$\left[p(x)y_n'(x)\right]'+\left[\lambda_n q(x)+r(x)\right]y_n(x)=0 \qquad .....(9)$$

On multiplying by (8) by y_n and (9) y_m respectively and on subtracting we get.

$$y_n(x)\left[p(x)y_m'(x)\right]'-y_m(x)\left[p(x)y_n'(x)\right]'+(\lambda_m-\lambda_n)q(x)y_m(x)y_n(x)=0$$

$$\Rightarrow \quad (\lambda_m-\lambda_n)q(x)y_m(x)y_n(x)=y_m(x)\left[p(x)y_n'(x)\right]'-y_n(x)\left[p(x)y_n'(x)\right]'$$

On integrating writh respect to x between a and b, we have

$$(\lambda_m-\lambda_n)\int_a^b q(x)y_m(x)y_n(x)\,dx=\int_a^b y_m(x)\left[p(x)y_n'(x)'\right]'dx-\int_a^b y_n(x)\left[p(x)y_m'(x)\right]'dx$$

$$\Rightarrow \quad (\lambda_m-\lambda_n)\int_a^b q(x)y_m(x)y_n(x)\,dx=\left[y_m(x)p(x)y_n'(x)\right]_a^b-\int_a^b y_m'(x)p(x)y_n'(x)\,dx$$

$$-\left[y_n(x)p(x)y_m'(x)\right]_a^b+\int_a^b y_n'(x)p(x)y_m'(x)\,dx$$

$$\Rightarrow \quad (\lambda_m-\lambda_n)\int_a^b q(x)y_m(x)y_n(x)\,dx=y_m(b)p(b)y_n'(b)-y_m(a)p(a)y_n'(a)$$

$$-y_n(b)p(b)y_m'(b)+y_n(a)p(a)y_m'(a)$$

$$\Rightarrow \quad (\lambda_m-\lambda_n)\int_a^b q(x)y_m(x)y_n(x)\,dx=p(b)\left[y_m(b)y_n'(b)-y_n(b)y_m'(b)\right]$$

$$-p(a)\left[y_m(a)y_n'(a)-y_n(a)y_m'(a)\right] \qquad .....(10)$$

Now define $w(x)$, a Wronskian determinant of the solution or eigenfunctions $y_m(x)$ and $y_n(x)$ as

$$w(x)=\begin{vmatrix} y_m(x) & y_m'(x) \\ y_n(x) & y_n'(x)\end{vmatrix}=y_m(x)y_n'(x)-y_n(x)y_m'(x)$$

So, expression (10) can be written as

$$(\lambda_m-\lambda_n)\int_a^b q(x)y_m(x)y_n(x)\,dx=p(b)w(b)-p(a)w(a) \qquad .....(11)$$

For obtaining the orthogonality property

$$\int_a^b q(x)\, y_m(x)\, y_n(x)\, dx = 0 \text{ for } m \# n$$

We seek right hand side of (10) or (11) to vanish, that is

$$p(b)\, w(b) - p(a)\, w(a) = 0$$

This will certainly happen if the boundary conditions required for a non-trival solution of (7) are

$$\left.\begin{array}{c} y(a)=0 \text{ and } y(b)=0 \\ \text{or} \\ y'(a)=0 \text{ and } y'(b)=0 \end{array}\right\} \quad .....(12)$$

Above boundary conditions are special cases of more general boundary conditions.

$$c_1 y(a) + c_2 y'(a) = 0 \text{ and } d_1 y(b) + d_2 y'(b) = 0 \quad .....(13)$$

where c_1 and c_2 donot vanish simultaneously and simillary d_1 and d_2 do not vanish simultaneously. To verify that the general boundary condition in (13) really vanishes the right hand side of (11), Let eigenfunction $y_m(x)$ and $y_n(x)$ also satify boundary condition (13) i.e.

$$c_1 y_m(a) + c_2 y'_m(a) = 0$$

$$c_1 y_n(a) + c_2 y'_n(a) = 0$$

For non-trivial solution of c_1 and c_2 in above system of equations, the determinant

$$\begin{vmatrix} y_m(a) & y'_m(a) \\ y_n(a) & y'_n(a) \end{vmatrix} = w(a)$$

must vanish. Hence $w(a) = 0$. Simillarly $w(b) = 0$.

So right hand side of (11) definitely vanishes and orthogonality of eigenfunctions is validated under suitable boundary condition (13) which are homogeneous in nature. The problem (7) with boundary condition (13) is known as Sturm-Liouville problem.

The significance of orthogonality property of eigenfunctions of Sturm-Liouville problem is to represent series expansions of function $f(x)$ in terms of eigenfunctions $y_n(x)$ as

$$f(x) = a_1 y_1(x) + a_2 y_2(x) + \ldots\ldots + a_n y_n(x) + \ldots\ldots .$$

where the cofficient $a_1, a_2 \ldots. a_n, \ldots..$ can be derived using orthogonality property of eigenfunctions.

6.6.3 Theorem 3 :To every eigenvalue of a Sturm-Liouville system there corresponds only one linearly independent eigenfunction.

Proof. Let if possible, $y_1(x)$ and $y_2(x)$ be two distinct eigenfunctions of the systems, corresponding to same eigenvalue λ. In order to prove the linear independence of $y_1(x)$ and $y_2(x)$, it is sufficient to prove that the wronskian

$$w(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix} \text{ is identically zero.}$$

By definition,

$$w(x) = y_1 y'_2 - y_2 y'_1$$

We have $\qquad w'(x) = y_1 y_2'' - y_2 y_1''$

and from the given boundary conditions

$$w(a) = w(b) = 0 \qquad .....(14)$$

Since $y_1(x)$ and $y_2(x)$ are solutions of Sturm-Liouville's problem, therefore

$$(p\, y_1)' + (q + \lambda r)\, y_1 = 0$$

and $\qquad (p\, y_2')' + (q + \lambda r)\, y_2 = 0$

Eliminating $(q + \lambda\, r)$, we get

$$(y_2'' y_1 - y_1'' y_2)\, p(x) + (y_2' y_1 - y_1' y_2)\, p'(x) = 0$$

or $\qquad p(x)\, w'(x) + p'(x)\, w(x) = 0$

or $\qquad d\left[p(x)\, w(x)\right] = 0 \Rightarrow w(x) = \dfrac{C}{p(x)}$

Since $p(x) \neq 0$, the boundary condition (14) gives $C = 0$ for all x. Hence $w(x) \equiv 0$ in $[a, b]$, which means that, the eigenfunction $y_1(x)$ and $y_2(x)$ corresponding to same eigenvalue λ are linearly independent.

6.6.4 Theorem 4 : (Expansion of a function in terms of eigenfunctions of Sturm-Liouville system). If $\{\phi_n(x)\}$ be a set of eigenfunctions of Sturm-Liouville system, then $\sum_{n=1}^{\infty} A_n \phi_n(x)$ converges uniformly to a function $f(x)$ in $[a, b]$ such that

$$f(x) = \sum_{n=1}^{\infty} A_n \phi_n(x), \quad a \le x \le b \qquad .....(15)$$

where $$A_m = \frac{\int_a^b r(x) f(x) \phi_m(x)\, dx}{\int_a^b r(x)\, \phi_m^2(x)\, dx}, \quad m \in N \qquad .....(16)$$

Proof. Without taking the proof of convergence, let $f(x)$ is given by (15). Multiplying both sides of (15) by $r(x)\, \phi_m(x)$, integrating from a to b and changing the order of integration and summation (which is justified due to uniform convengence of the series) we find that

$$\int_a^b r(x) f(x) \phi_m(x)\, dx = \sum_{n=1}^{\infty} A_n \int_a^b r(x) \phi_n(x) \phi_m(x)\, dx \qquad .....(17)$$

Since the set of eigenfunctions of Sturm-Liouville system are orthogenal in $[a, b]$ w.r.t weight function $r(x)$, therefore relation (17) reduces to

$$\int_a^b r(x) f(x) \phi_m(x)\, dx = A_m \int_a^b r(x) \phi_m^2(x)\, dx$$

which gives A_m given by (16).

Ex.*1. *Compute the eigenvalues and eigenfunctions for boundary value problem

$$y'' + 2y' + (1-\lambda)y = 0; \quad y = (0) = 0 \text{ and } y(1) = 0$$

Also prove that the set of eigenfunctions for the given problem is an orthogonal set.

Sol. The auxillary equation is $m^2 + 2m + (1 - \lambda) = 0$

or $$m = \frac{-2 + \sqrt{4 - 4(1-\lambda)}}{2} = -1 \pm \sqrt{\lambda}$$

Now, three cases arise

Case I : When $\lambda > 0$ or $\lambda = \alpha^2$

The general solution of the given differential equation in this case will be

Now $$y(x) = c_1 e^{(-1+\sqrt{\lambda})x} + c_2 e^{(-1-\sqrt{\lambda})x}$$

For $y(0) = 0 \Rightarrow c_1 + c_2 = 0$ or $c_2 = -c_1$

$\therefore$ $$y(x) = c_1\left[e^{(-1+\sqrt{\lambda})x} - e^{(-1-\sqrt{\lambda})x}\right]$$

Now $y(1) = 0$ gives $c_1\left[e^{(-1+\sqrt{\lambda})} - e^{(-1-\sqrt{\lambda})}\right] = 0$

$\Rightarrow$ $c_1 = 0$

Hence $c_2 = 0 = c_1 \Rightarrow y(x) \equiv 0$ i.e. only trivial solution exists.

Case II : When $\lambda = 0$:

The general solution is $y(x) = e^{-x}(c_1 + c_2 x)$

For $y(0) = 0$, we get $c_1 = 0$. Hence $y(x) = c_2 x e^{-x}$

When $y(1) = 0$, we get $c_2 e^{-1} = 0 \Rightarrow c_2 = 0$.

Thus $c_1 = c_2 = 0$, which gives $y \equiv 0$ i.e. only trivial solution exists.

Case III : When $\lambda < 0$ or $\lambda = -\alpha^2$

Then general solution is

$$y(x) = e^{-x}\left[c_1 \cos\sqrt{-\lambda}x + c_2 \sin\sqrt{-\lambda}x\right]$$

For $y(0) = 0$, we have $c_1 = 0$

So $$y(x) = c_2 e^{-x} \sin\sqrt{-\lambda}x$$

Now, for $y(1) = 0$, we have $c_2 e^{-1} \sin\sqrt{-\lambda} = 0$

For seeking non-trivial solution of given problem, we have $c_2 \neq 0$, so $\sin\sqrt{-\lambda} = 0$

or $\sin\sqrt{-\lambda} = \sin n\pi$; n is positive integer

$\Rightarrow$ $\sqrt{-\lambda} = n\pi$

$\Rightarrow$ $-\lambda = n^2\pi^2$

$\therefore$ $\lambda_n = -n^2\pi^2$; $n = 1, 2, 3, \ldots\ldots$.

Hence, corresponding eigenfunctions are

$$y_n(x) = e^{-x} \sin n\pi x$$

Let $y_m(x) = e^{-x} \sin m\pi x$ and $y_n(x) = e^{-x} \sin n\pi x$ are two eigenfunctions corresponding to eigen-values $\lambda_m = -m^2\pi^2$ and $\lambda_n = -n^2 \pi^2$ resectively. Then the integral

$$\begin{aligned}
\int_0^1 e^{2x} y_m(x) y_n(x) &= \int_0^1 e^{2x} e^{-x} \sin m\pi x \; e^{-x} \sin n\pi x \, dx \\
&= \int_0^1 \sin m\pi x \sin n\pi x \, dx \\
&= \int_0^1 \frac{1}{2}\left[\cos(m-n)\pi x - \cos(m+n)\pi x\right] dx \\
&= \frac{1}{2}\left[\frac{\sin(m-n)\pi x}{(m-n)} - \frac{\sin(m+n)\pi x}{(m+n)}\right]_0^1 \\
&= 0
\end{aligned}$$

prompts that $y_m(x)$ and $y_n(x)$ are orthogonal in [0, 1] with respect to weight function e^{2x}.

Self-Learning Exercise

1. Classify the following problem as boundary value problem or initial value problem

(*a*) $y'' - \lambda y = 0, y(0) = 0$ and $y(1) = 0$

(*b*) $y'' + 2y' + 2y = 0, y(0) = 1$

(*c*) $(xy')' + (9\lambda + 4) y = 0, y(a) = 0,$ and $y(b) = 0$, a, b are constonts

(*d*) $3y'' + 4y' + 2y = 0, y(2) = 5, y'(2) = 6$

2. Find the eigenvalues λ_n and eigenfunctions $y_n(x)$ for $y'' + \lambda y = 0$ in each of the following boundary conditions

(*a*) $y(0) = 0, y(1) = 0$

(*b*) $y(-2) = 0, y(2) = 0$

(*c*) $y(-3) = 0, y(0) = 0$

(*d*) $y(1) = 0, y(4) = 0$

3. Check whether following boundary value problems are Sturm-Liouville problem or not

(*a*) $e^x y'' + e^x y' + \lambda y = 0; y(0) = 0, y'(1) = 0$

(*b*) $y'' + \lambda(1 + x)y = 0; y'(0) = 0, y(2) + y'(2) = 0$

(*c*) $\left(\frac{1}{x} y'\right)' + (x + \lambda) y = 0; y(0) + 3y'(0) = 0, y(1) = 0$

(*d*) $(xy')' + (x^2 + 1 - \lambda x^2)y = 0; y(0) = 0; y(0) + 3y'(0) = 0, y(1) + y'(1) = 0$

(*e*) $(xy')' + (x^2 + 1 + \lambda e^x)y = 0; y(1) = 0; y(1) + 2y'(1) = 0; y(2) - 3y'(2) = 0$

4. Find eigeavalues and corresponding eigenfunction of the following Sturm-Liouville problems.

(a) $y'' + \lambda y = 0; y(0) = 0$ and $y'(\pi) = 0$

(b) $y'' + \lambda y = 0; y'(0) = 0$ and $y'(L) = 0$

(c) $y'' + \lambda y = 0; y'(-\pi) = 0$ and $y'(\pi) = 0$

6.7 Summary

In this unit, we introduced a special type of boundary value problem known as Sturm-Liouville problem which gives fundamental basics for important concepts like eigenvalue, eigenfunction, orthogonality and Fourier series. These concepts directly involved in solving practical problems arise in physical and engineering challenges.

6.8 Answer to Self-Learning Exercise

1. *(a)* Boundary value problem

(b) Initial value problem

(c) Boundary value problem

(d) Initial value problem

2. *(a)* $\lambda_n = n^2\pi^2\,; n = 1, 2, 3 ......y_n(x) = \sin n\pi x$

(b) $\lambda_n = \dfrac{n^2\pi^2}{16};\ n = 1, 2, 3 ......y_n(x) = \sin\dfrac{n\pi}{4}(x+L)$

(c) $\lambda_n = \dfrac{n^2\pi^2}{9};\ n = 1, 2, 3 ......y_n(x) = \sin\dfrac{n\pi x}{3}$

(d) $\lambda_n = \dfrac{n^2\pi^2}{9};\ n = 1, 2, 3 ......y_n(x) = \sin\dfrac{n\pi}{3}(4-x)$

3. *(a)* Yes *(b)* Yes

(c) No, since $p(x)$ is not continuous in [0, 1]

(d) No, since $q(x) < 0$ in [0, 1]

(d) Yes

4. *(a)* $\lambda_n = \dfrac{(2n+1)^2}{4};\ n = ,1, 2, 3 ....., \ y_n(x) = \sin\dfrac{2n+1}{2}x$

(b) $\lambda_n = \dfrac{n^2\pi^2}{L^2};\ n = 0, 1, 2, 3 ....., \ y_n(x) = \dfrac{\cos n\pi x}{L}$

(c) $\lambda_n = n^2;\ n = 1, 2, 3 ....., y_n(x) = \cos nx$

6.9 Exercise

1. Find the eigenvalues λ_n and eigenfunction $y_n(x)$ for the following boundary value problem $y'' + \lambda y = 0$ in each of the following boundary conditions :

(a) $y(0) = 0, y(2\pi) = 0$ [**Ans.** $\lambda_n = \dfrac{n^2}{4};\ n = 1, 2, 3, ..., y_n(x) = \sin\dfrac{nx}{2}$]

(b) $y(0) = 0, y(L) = 0\,; L > 0$, L is positive constant

[**Ans.** $\lambda_n = \dfrac{n^2\pi^2}{L^2};\ n = 1, 2, 3, ..., y_n(x) = \sin\dfrac{n\pi x}{L}$]

(c) $y(-L) = 0, y(L) = 0$; $L > 0$, L is positive constant

[**Ans.** $\lambda_n = \dfrac{n^2\pi^2}{4L^2}$; $n = 1, 2, 3, ..., y_n(x) = \sin\dfrac{n\pi(x+L)}{2L}$]

2. Solve the following Sturm-Liouville problem

(a) $\dfrac{d}{dx}\left(x\dfrac{dy}{dx}\right) + \dfrac{\lambda}{x}y = 0;\ y(1) = 0, y\left(e^{\pi}\right) = 0$

[**Ans.** $\lambda_n = n^2$; $n = 1, 2, 3, ..., y_n(x) = \sin(n \ln |x|)$]

(b) $\dfrac{d}{dx}\left(\left(x^2+1\right)\dfrac{dy}{dx}\right) + \dfrac{\lambda}{x^2+1}y = 0$; $y(0) = 0$ and $y(1) = 0$ (Hint put $x = \tan t$)

[**Ans.** $\lambda_n = 16n^2$; $n = 1, 2, 3, ..., y_n(x) = \sin\left(4n \tan^{-1} x\right)$]

3. Compute the eigenvalues and eigenfunctions for boundary value problem and determine Euclidean space in which a complete set of eigenfunctions for the given problem is an orthogonal set

(a) $y'' + (1+\lambda)y = 0;\ y(0) = 0, y(\pi) = 0$

[**Ans.** $\lambda_n = n^2 - 1$; $n = 1, 2, 3, ..., y_n(x) = \sin nx$ orthogonal in $[0,\pi]$]

(b) $4y'' - 4y' + (1+\lambda)y = 0, y(-1) = 0, y(1) = 0$

[**Ans.** $\lambda_n = n^2\pi^2$, $y_n(x) = \begin{cases} e^{x/2}\sin\dfrac{n\pi x}{2};\ n = 2, 4, 6...... \\ e^{x/2}\cos\dfrac{n\pi x}{2};\ n = 1, 3, 5...... \end{cases}$

Orthogonal in $(-1, 1)$ with resepect to function e^{-x}]

(c) $y'' + 2y' + (1-\lambda)y = 0;\ y'(0) = 0$ and $y'(\pi) = 0$

[**Ans.** $\lambda_n = -n^2$; n = 1, 2, 3, ..., $y_n = e^{-x}(n \cos nx + \sin nx)$, $\lambda_n = 1$, $y_n = 1$. Orthogonal in $[0, \pi]$ with respect to weight function e^{2x}]

4. Find the real eigenvalues and eigenfunctions for the boundary value problem $y'' + \lambda y = 0$; $y(0) = 0,\ y'(1) = 0$ [**Ans.** $\lambda_0 = 0; y_0(x) = 1$; $\lambda_n = n^2\pi^2, y_n(x) = \cos n\pi x, n \in \mathbf{N}$]

5. Find the solution of Sturm-Liouville problem $y'' + \dfrac{1}{x}y' + \dfrac{\lambda}{x^2}y = 0,\quad 1 \le x \le 2$

with boundary conditions $y(1) = 0 = y(2)$ [**Ans.** $y = \sum\limits_{n=1}^{\infty} B_n \sin\left(n\pi\dfrac{\log x}{\log 2}\right)$]

7. Determine the normalized eigenfunctions of the problem $y'' + \lambda y = 0$, $y(0) = 0$, $y'(1) + y(1) = 0$. Hence expand the function $f(x) = x$, $0 \le x \le 1$, in terms of these normalized eigenfunctions.

[**Ans.** $y_n(x) = \left\{\dfrac{2}{1+\cos^2\sqrt{\lambda_n}}\right\}^{1/2} \sin\left(x\sqrt{\lambda_n}\right), n \in N, x = \sum\limits_{n=1}^{\infty} \dfrac{4\sin\sqrt{\lambda_n}}{\lambda_n\left(1+\cos^2\sqrt{\lambda_n}\right)}\sin\left(x\sqrt{\lambda_n}\right)$]

□ □ □

Unit 7 : Variational Problems with Fixed Boundaries and Euler-Lagrange Equation

Structure of the Unit

7.0 Objective

In this unit you will study the methods of finding curves connecting two given points which either maximizes or minimizes some given integral. You will also know about Euler-Lagrange equation for an extremal. Variational problems involving several independent variables will also be discussed.

7.1 Introduction

Calculus of variations is a field of mathematics that deals with extremizing functionals as opposed to ordinary calculus which deals with functions. The origin of calculus of variations was based on famous *"Brachistochrone problem or quickest path problem."* In calculus of variation, we generally encoumter with the problems where one has to find the maximal and minimal value that is extreme value of special quantities called functionals.

7.2 Definitions and Fundamental Problems

7.2.1 Functionals : Functionals are variable quantities whose values are determined by choice of one or several functions. In short, we may say that functionals are *functions of functions.*

***Ex.*1.** Let the parametric equations of the plane curve be $x = x(t)$, $y = y(t)$, t being the parameter. The arc length of the plane curve from $P(t_0)$ to $Q(t)$ is given by

$$s\,[x(t),y(t)] = \int_{t_0}^{t} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

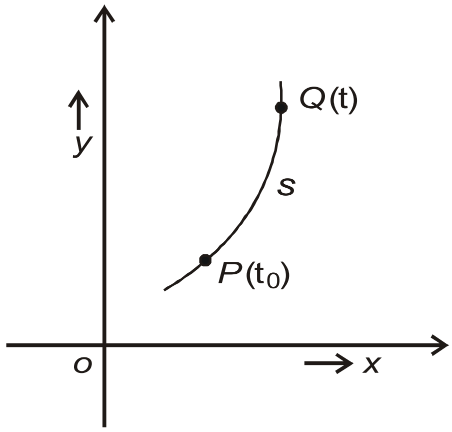

Fig. 7.1

where $\dot{x}$ and $\dot{y}$ represent the differentiation of x and y with respect to 't' respectively.
Here s is a **functional** which is function of functions $x(t)$ and $y(t)$.

7.2.2 Linear Functionals : A functional $L\,[y(x)]$ satisfying the conditions.

(i) $L\,[cy(x)] = cL\,[y(x)]$

(ii) $L\,[y_1(x) + y_2(x)] = L\,[y_1(x)] + L[y_2(x)]$

where c is a arbitrary constant is known as *linear functional*.

*Ex.*2. $L\left[y(x)\right]=\int\limits_{x_0}^{x}\left\{a(x)\frac{dy}{dx}+b(x)y\right\}dx$, ***is a linear functional.***

The calculus of variations provides a method for determining maximal and minimal values of functionals. Such problems are known as *variational problems.*

Now we deal with three problems of historical importance which influenced the development of this subject.

7.2.3 Brachistochrone Problem

Suppose P and Q are two points in the plane but not in the same vertical line. Imagine, there is a thin flexible wire connecting those two points. Suppose P is above Q, and we let a frictionless bead travel under gravity from P to Q. The Bachistochrone problem (or quickest discent problem) is concerned with determining the path of the bead when it reaches the point Q in the least possible time. This problem was first introduced by *J.* Bernaulli in the mid of 17[th] century and was first solved by Sir Isaac Newton.

7.2.4 Problem of Geodesics

In general relativity, a geodesic generalizes the concept of *straight line* to curve spacetime. For example : Find the curve of shortest length connecting two points in space. If there is no constraints the solution obviously is a straight line joining the points. However, if the curve is constrained and is to lie on a surface, then in space, the solution is less obvious and possibly many solutions may exist.

The solutions are called geodesics. In other words a geodesics on a surface is a curve along which the distance between two points on the surface is a minimum. To find the geodesics on a surface is a variational involving conditional extremum.

7.2.5 Isoperimetric Problem

In this problem, we required to find a closed plane curve of a given length l bounding a maximal area S. Let the parametric equation of the plane curve be $x = x(t)$, $y = y(t)$, and the curve is traversed once in anti-clockwise as t increases from t_0 to t_1, then length l of given curve is

$$l=\int\limits_{t_0}^{t_1}\sqrt{\left(\dot{x}(t)\right)^2+\left(\dot{y}(t)\right)^2}\;dt \qquad ......(1)$$

which is a constant, and enclosed area is given by

$$S=\frac{1}{2}\int\limits_{t_0}^{t}\sqrt{\left[x\dot{y}-y\dot{x}\right]}\;dt \qquad .....(2)$$

The problem is to maximize the functional S, given by (2) subject to the condition that the length l of the curve given by (1) must have a constant value.

7.3 Euler-Lagrange Equation

7.3.1 Basic Lemma : Let $M(x)$ be a continuous function on the internal $[a, b]$. Suppose that for any continuous function $h(x)$, we have $\int\limits_a^b M(x)h(x)\,dx=0$ ***then*** $M(x)\equiv 0$ ***on the interval*** $[a,b]$.

Proof : Let $M(x) \neq 0$ (say positive) at a point $\bar{x}$ where $a \leq \bar{x} \leq b$. Since $M(x)$ is continuous on $[a, b]$, it follows that if $M(x) \neq 0$. Then $M(x)$ maintains its sign in a certain neighbourhood $x_0 \leq x \leq x_1$ of the point $\bar{x}$.

Since $h(x)$ is arbitrary continuous function, we may choose $h(x)$ s.t. $h(x)$ remains positive in $x_0 \leq x \leq x_1$ while it vanishes outside the interval. Hence, we obtain.

$$\int_a^b h(x) M(x) dx = \int_{x_0}^{x_1} h(x) M(x) dx > 0 \qquad .....(1)$$

Since the product $h(x)M(x)$ remains positive in $[x_0, x_1]$ and vanishes outside this interval.

By the hypothesis $\int_a^b h(x) M(x) dx = 0$ .....(2)

which contradicts (1). This contradiction shows that our assumption $M(x) \neq 0$ at some point $\bar{x}$ must be wrong and so $M(x) \equiv 0$ on $[a, b]$.

7.3.2 Euler-Lagrange Equation : If $y(x)$ is a curve in interval $[a, b]$ which is a twice differentiable and satisfying the conditions $y(a) = y_1$ and $y(b) = y_2$ and minimizes the functional.

$$F[y(x)] = \int_a^b f(x, y, y') dx \qquad .....(3)$$

where $y' \equiv \dfrac{dy}{dx}$.

Then the following differential equation must be satisfied

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad .....(4)$$

Proof. Suppose $y \equiv y(x)$ is a curve which minimizes the functional F. That is, for any permissible curve $y = g(x)$, $F[y(x)] \leq F[g(x)]$. We have to construct a function of one real variable satisfying following properties.

1. $H(\in)$ is a differentiable near $\in = 0$

2. $H(0)$ is a local minimum for H.

We begin by constructing a variation of $y(x)$. Let $\in$ be a small real number (positive or negative). s.t.

$$y_\in(x) = y(x) + \in h(x)$$

where $h(x)$ is a continuous function in $[a, b]$ and $h(a) = h(b) = 0$.

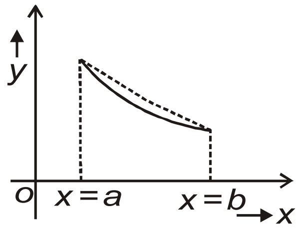

Fig. 7.2

We can define a function H to be

$$H_{\in} = F\left[y_{\in}(x)\right]$$

Since $y(x)$ minimizes $F(y(x))$, it follows that it minimizes $H(\in)$. Since $H(0)$ is minimum value of H, we know that from ordinary calculus that $H'(0) = 0$.

The function H can be differentiated by using Leibnitz rule, that is

$$\frac{d}{d\in}H(\in) = \frac{d}{d\in}\left[\int_a^b f(x, y_{\in}, y'_{\in})\,dx\right]$$

$$= \int_a^b \frac{\partial}{\partial\in}\left[f(x, y_{\in}, y'_{\in})\,dx\right] \qquad .....(5)$$

Now applying chain rule within the integral, we abtain

$$\frac{\partial f}{\partial\in} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial\in} + \frac{\partial f}{\partial y_{\in}}\cdot\frac{\partial y_{\in}}{\partial\in} + \frac{\partial f}{\partial y'_{\in}}\cdot\frac{\partial y'_{\in}}{\partial\in}$$

$$= \frac{\partial f}{\partial y_{\in}}\cdot\frac{\partial y_{\in}}{\partial\in} + \frac{\partial f}{\partial y'_{\in}}\cdot\frac{\partial y'_{\in}}{\partial\in}$$

$$= \frac{\partial f}{\partial y_{\in}}h(x) + \frac{\partial f}{\partial y'_{\in}}h'(x)$$

Substituting the value of $\frac{\partial f}{\partial\in}$ in the equation (5), we have

$$\frac{dH(\in)}{d\in} = \int_a^b\left[\frac{\partial f}{\partial y_{\in}}h(x) + \frac{\partial f}{\partial y'_{\in}}h'(x)\right]dx$$

Using $H'(0) = 0$, we find that

$$H'(0) = \int_a^b\left[\frac{\partial f}{\partial y}h(x) + \frac{\partial f}{\partial y'}h'(x)\right]dx = 0$$

Integrating by parts, we get

$$H'(0) = \int_a^b \frac{\partial f}{\partial y}h(x)\,dx + \left[\frac{\partial f}{\partial y'}h(x) - \int_a^b \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]h(x)\,dx\right]_a^b = 0$$

$$= \int_a^b \left\{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right\}h(x)\,dx + \left[\frac{\partial f}{\partial y'}h(x)\right]_a^b = 0$$

$$= \int_a^b\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]h(x)\,dx = 0 \qquad \text{[Using } h(a) = h(b) = 0\text{]}$$

By using lemma, we conclude that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad .....(6)$$

This equation is called *Euler-Lagrange equation*.

7.3.5 Remark : The statement of the lemma and its proof donot change if restriction $h(a) = h(b) = 0$ is imposed on the function $h(x)$.

7.4 Some Elementary Cases of the Integrability of the Euler-Lagrange Equation

7.4.1. *f* is independent of *y′* : If f is independent of y', then f is function of (x, y) only. Therefore $\frac{\partial f}{\partial y'} = 0$. Thus the Euler-Lagrange equation reduces to following form :

$$\frac{\partial f}{\partial y} = 0 \qquad .....(1)$$

Now integrating (1), with respect to y, we obtain a arbitrary curve $f = g(x)$, without any constant and in general, does not satisfy boundary conditions $y(a) = y_1$ and $y(b) = y_2$. Thus this type of equation does not posses a solution.

7.4.2. *f* is indpendent of *x* and *y* : In this case,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial y'} = 0 \qquad .....(2)$$

From Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0, \text{ we get}$$

$$\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - y'\frac{\partial^2 f}{\partial y \partial y'} - y''\frac{\partial^2 f}{\partial y'^2} = 0 \qquad \left[\because f \equiv f(x, y, y')\right]$$

From equation (2), we have

$$-y''\frac{\partial^2 f}{\partial y'^2} = 0 \qquad .....(3)$$

This implies that either $y'' = 0$ or $\frac{\partial^2 f}{\partial y'^2} = 0$

Now $\qquad y'' = 0$

$\Rightarrow \qquad y = Ax + B \qquad .....(4)$

where A and B are arbitrary constants, which is a two parameter family of straight lines. But if $\frac{\partial^2 f}{\partial y'^2} = 0$ has one or several real roots $y' = K_n$, then $y = K_n x + c$

which is one parameter family of straight line contained in two parameter family of straight lines. Thus extremals are all possible straight lines.

7.4.3. *f* is indpendent of only *y* : Here $f \equiv f(x, y')$, therefore Euler-Lagrange equation can be written as

$$-\frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]=0 \text{ as } \frac{\partial f}{\partial y}=0$$

$$\Rightarrow \qquad \frac{\partial f}{\partial y'}=c \qquad .....(5)$$

where c is a constant. Since this relation is independent of y it can be solved for y' as a function of x. Another integration leads to a solution involving two arblitraray constants which can be obtained by using given boundary conditions.

7.4.4. *f* is a linear function of *y′* or f is binearly dependent on y' *such that* $f(x, y, y') = p(x, y) + q(x,y)y'$

Forming the Euler-Lagrange equation for this particular f, we have

$$\frac{\partial p}{\partial y}+\frac{\partial q}{\partial y}y'-\frac{dq}{dx}=\frac{\partial p}{\partial y}+\frac{\partial q}{\partial y}y'-\left(\frac{\partial q}{\partial x}+\frac{\partial q}{\partial y}\cdot y'\right)=\frac{\partial p}{\partial y}-\frac{\partial q}{\partial x}=0 \qquad .....(7)$$

for all x and y.

Solution of this problem, in general, not possible because solution does not satisfy given boundary conditions. But if we consider $\frac{\partial p}{\partial y}-\frac{\partial q}{\partial x}=0$, then the expression $pdx + qdy$ becomes exact differential equation whose solution does not depend on path of extremal and therefore variational becomes meaningless.

***Ex.*1. Test for an extremum of the functional**

$$F\left[y(x)\right]=\int_0^1\left[x^2y^2+x^2y'\right]dx,\quad y(0)=0,\quad y(1)=1 \qquad .....(8)$$

Sol. Clearly we see that

$$f(x,y,y')=x^2y^2+x^2y'$$

is a linear function of y'. Now from case 7.4.4, we have $p(x,y) = x^2y^2$, $q(x,y) = x^2$

Hence from equation (7), we find that

$$\frac{\partial p}{\partial y}-\frac{\partial q}{\partial x}=0$$

$$\Rightarrow \qquad 2x^2y - 2x = 0$$

$$\Rightarrow \qquad 2x\,(xy-1) = 0$$

$$\Rightarrow \qquad xy=1 \quad \text{or} \quad x=0$$

Obviously first boundary condition is satisfied by only $x = 0$, by and second boundary condition is satisfied by only $xy = 1$. Both boundary conditions are not satisfied by the curves $x = 0$ and $xy = 1$.Thus no solution exist for this problem.

***Ex.*2. Test for extremum of the functional**

$$\boldsymbol{F[y(x)] = \int_a^b [\cos y - xy' \sin y]\,dx} \quad .....(9)$$

with boundary conditions $y(a) = y_0$, $y(b) = y_1$

Sol. For this problem, Euler-Lagrange equation is given by

$$-\sin y - xy' \cos y - \frac{d}{dx}[-x \sin y] = 0$$

or $$-\sin y - xy' \cos y + \sin y + xy' \cos y = 0$$

Thus, integrand being an exact differential equation. Therefore variational problem becomes meaningless

7.4.5. *f* is independent of *x* : In this case, $\frac{\partial f}{\partial x} = 0$, therefore Euler-Lagrange equation reduces to

$$\Rightarrow \qquad \frac{d}{dx}\left[f - y' \frac{\partial f}{\partial y'}\right] = 0$$

Hence Euler-Lagrange equation has its integral as $f - y' \frac{\partial f}{\partial y'} = c$

where c is arbitrary constant

Ex.*1. *Test for extremum of the functional

$$\boldsymbol{F(y(x)) = \int_0^1 \sqrt{1+y'^2}\,dy}, \qquad \boldsymbol{y(0) = 0, y(1) = 2}$$

Sol. Using Euler-Lagrange equation, we get

$$-\frac{d}{dx}\left[\frac{2y'}{\sqrt{1+y'^2}}\right] = 0$$

Integrating with respect to 'x', we get

$$\frac{y'}{\sqrt{1+y'^2}} = c \text{, where c is arbitray constant}$$

$$\Rightarrow \qquad y' = \pm\sqrt{\frac{c^2}{1-c^2}} = A(\text{say})$$

Again integrating with respect to 'x'

$$y = Ax + B$$

$y(0) = 0$ and $y(1) = 2$, implies that $B = 0$, $A = 2$

Thus $y = 2x$ which is a straight line.

Ex.2. Test for extremum of the functional

$$F\left[y(x)\right]=\int_0^1\left[y'^2+x^2\right]dx,\ \ y(0)=1,\ \ y(1)=2$$

Sol. Using Euler-Lagrange equation, we get

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$$

$\Rightarrow$ $$-\frac{d}{dx}\left[2y'\right]=0$$

$\Rightarrow$ $$y''=0$$

Integrating two times we get $y=Ax+B$

Using $y(0)=1, y(1)=2$, we get $A=B=1$.

Thus solution is $y=x+1$.

Ex.5. (Brachistochrone problem or quickest descent problem)

Find the shape of the curve on which a bead is sliding from rest and accelerated by gravity will ship (without friction) in least time from one point to another.

Sol. Let us consider a particle P descending from $A(0,0)$ to $B(a,b)$ under gravity along some curve. We have to determine shape of the curve which gives minimum possible time to descent. Let $P(x,y)$ be the position of the particle at any time t and having actual arc length s from a point A.

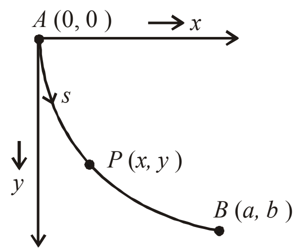

Fig. 7.3

Under the gravity, the motion of particle is given by

$$v=\frac{ds}{dt}=\sqrt{2gy}$$

$\Rightarrow$ $$dt=\frac{ds}{\sqrt{2yg}}$$

Hence time T of descent is (from A to B).

$\Rightarrow$ $$T=\int_0^a\frac{ds}{\sqrt{2yg}} \qquad .....(10)$$

But we know that

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \qquad ds = \sqrt{1+y'^2}\,dx \qquad \text{where } y' = \frac{dy}{dx}$$

Putting the value of *ds* in equation (10), we obtain

$$T = \int_0^a \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx$$

Here $f(x, y, y') = \sqrt{\dfrac{1+y'^2}{2gy}} \neq f(x)$

Now from case (7.4.5), we have

$$y'\frac{\partial f}{\partial y'} - f = c_1$$

$$\Rightarrow \qquad y' \times \frac{1}{\sqrt{2gy}} \times \frac{y'}{\sqrt{1+y'^2}} - \sqrt{\frac{1+y'^2}{2gy}} = c_1$$

$$\Rightarrow \qquad \frac{1}{\sqrt{y\left(1+y'^2\right)}} = c_2 \qquad (\text{where } c_2 = -\sqrt{2g}\,c_1)$$

$$\text{or} \qquad y\left(1+y'^2\right) = c_3 \qquad \left(\text{where } c_3 = 1/c_2^2\right)$$

Now putting $y' = \cot\theta \;\Rightarrow\; y = c_3 \sin^2\theta = \frac{c_3}{2}(1-\cos 2\theta)$

Since $\dfrac{dy}{dx} = y' \quad \Rightarrow \quad dx = \dfrac{dy}{y'}$

$$\Rightarrow \qquad dx = \frac{2c_3 \cos\theta \sin\theta d\theta}{\cot\theta}$$

$$\Rightarrow \qquad dx = 2c_3 \sin^2\theta d\theta = c_3\left(1-\cos 2\theta\right)$$

Integrating we get

$$x = c_3\left(\theta - \frac{\sin 2\theta}{2}\right) + c_4 = \frac{c_3}{2}\left(2\theta - \sin 2\theta\right) + c_4$$

$$\text{and} \qquad y = \frac{c_3}{2}(1-\cos 2\theta)$$

If we substitute $2\theta = \phi$, and using intial condition (that is at $A(0,0)$), we have

$$c_4 = 0\text{ ; and } x = \frac{c_3}{2}(\phi - \sin\phi)\text{, and } y = \frac{c_3}{2}(1-\cos\phi)$$

which is equation of the cycloid with radius $\frac{c_3}{2}$ of rolling circle and c_3 can be obtained by using appropriate boundary condition.

***Ex.4.* (*The minimal surface of revolution problem*)**

Find the curve with fixed boundary revolves such that its rotation about x-axis generate mininal surface area.

Sol. We know that, surface area of the relvolution is given by

$$S\left[y(x)\right]=\int_a^b 2\pi y\ ds$$

$$=\int_a^b 2\pi y\sqrt{1+y'^2}\ dx$$

Here $f(x,y,y')=2\pi y\ \sqrt{1+y'^2}\ \neq f(x)$

From case (7.4.5), the first integral of Euler's equation is

$$f-y'\frac{\partial f}{\partial y'}=c_1$$

$$\Rightarrow \qquad 2\pi\ y\sqrt{1+y'^2}-\frac{2\pi yy'^2}{\sqrt{1+y'^2}}=c_1$$

$$\Rightarrow \qquad \frac{y}{\sqrt{1+y'^2}}=c_2 \qquad\qquad \left(\text{where } c_2=\frac{c_1}{2\pi}\right)$$

$$\Rightarrow \qquad dx=\frac{dy}{\sqrt{y^2-c_2^2}}$$

Integrating with respect to 'y' we get

$$x=c_2\cosh^{-1}\left(\frac{y}{c_2}\right)+c_3$$

$$y=c_2\cosh\left(\frac{x-c_3}{c_2}\right)$$

where c_2 and c_3 are arbitrary contestants, which is a equation of the "catenary" and the corresponding surface of revolution is called "centroid" of revolution.

7.5 Functionals Involving Several Dependent Variables and Their First Order Derivatives.

We now proceed to derive the differential equations that must be satisfied by the twice differentiable functions $x_1(t), x_2(t), ..., x_n(t)$ that extremize the integral

$$I=\int_{t_1}^{t_2} f\left(x_1,x_2,.....x_n,\dot{x}_1,\dot{x}_2,.....\dot{x}_n,t\right)dt \qquad .....(1)$$

with respect to those functions of $x_1, x_2, ..., x_n$ which achieve prescribed values at the fixed limits of integration t_1 and t_2, where $t_1 < t_2$. The superior dot represents ordinary differentiation with respect to the independent variable t.

We denote the set of actual extremizing functions by $x_1(t), x_2(t), ..., x_n(t)$ and proceed to form the one-parameter family of comparison functions

$$X_1(t) = x_1(t) + \in\xi_1(t), X_2(t) = x_2(t) + \in\xi_2(t) ,....., X_n(t) = x_n(t) + \in\xi_n(t) \qquad .....(2)$$

where $\xi_1, \xi_2, ..., \xi_n$ are arbitrary differentiable functions for which

$$\xi_1(t_1) = \xi_1(t_2) = \xi_2(t_1) = \xi_2(t_2) = ...= \xi_n(t_1) = \xi_n(t_2) = 0 \qquad .....(3)$$

and $\in$ is the parameter of the family. The condition (3) assures us that every member of each comparison family satisfies the required prescribed end point conditions. We see, moreover, that no matter what the choice of $\xi_1, \xi_2, ..., \xi_n$, the set of extremizing functions $x_1(t), x_2(t), ..., x_n(t)$ is a member of each comparison family for the penameter value $\in = 0$. Thus if we form the integral.

$$I(\in) = \int_{t_1}^{t_2} f\left(X_1, X_2, ..., X_n, \dot{X}_1, \dot{X}_2, ..., \dot{X}_n, t\right) dt \qquad .....(4)$$

by replacing $x_1, x_2, ..., x_n$ etc, in (4) by $X_1, X_2, ..., X_n$ etc., respectively, we have that $I(0)$ is the extremum value sought. We therefore conclude that

$$I'(0) = 0 \qquad .....(5)$$

It follows from (2) that

$$\dot{X}_1 = \dot{x}_1 + \in \dot{\xi}_1, \dot{X}_2 = \dot{X}_n + \in \dot{\xi}_2, ..., \dot{X}_n = \dot{x}_n + \in \dot{\xi}_n \qquad .....(6)$$

Now differentiate (4) with respect to '$\in$', we have

$$\frac{dI}{d\in} = \int_{t_1}^{t_2} \left[\frac{\partial f}{\partial X_1}\xi_1 + \frac{\partial f}{\partial \dot{X}_1}\dot{\xi}_1 + \frac{\partial f}{\partial X_2}\xi_2 + \frac{\partial f}{\partial \dot{X}_2}\dot{\xi}_2 + ... + \frac{\partial f}{\partial X_n}\xi_n + \frac{\partial f}{\partial \dot{X}_n}\dot{\xi}_n \right] dt, \qquad .....(7)$$

where we use (2) and (6) to derive the sequence of substitution $\left(\frac{\partial \dot{X}_1}{\partial \in}\right) = \dot{\xi}_1, ..., \left(\frac{\partial \dot{X}_n}{\partial \in}\right) = \dot{\xi}_n$.

It is clear from (2) and (6) that setting $\in = 0$ is equivalent to replaeing $X_1, X_2, ..., X_n, \dot{X}_1, \dot{X}_2, ..., \dot{X}_n$ by $x_2, x_2, ..., x_n, \dot{x}_1, \dot{x}_2, ...\dot{x}_n$ respectively. Thus because of (5), we abtain from (7) on setting $\in = 0$

$$I'(0) = \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial x_1}\xi_1 + \frac{\partial f}{\partial \dot{x}_1}\dot{\xi}_1 + \frac{\partial f}{\partial x_2}\xi_2 + \frac{\partial f}{\partial \dot{x}_2}\dot{\xi}_2 + ... + \frac{\partial f}{\partial x_n}\xi_n + \frac{\partial f}{\partial \dot{x}_n}\dot{\xi}_n \right) dt = 0 \qquad .....(8)$$

This last relation holds for all choices of the functions $\xi_1(t), \xi_2(t), ..., \xi_n(t)$. In particular, it holds for the special choice in which $\xi_2, ..., \xi_n$ are identically zero, but for which $\xi_1(t)$ is still arbitrary, consistent with (3). With this selection of $\xi_1, \xi_2, ..., \xi_n$, we integrate by parts the second term of the second member of (8) to obtain, since $\xi_1(t_1) = \xi_1(t_2) = 0$,

$$\int_{t_1}^{t_2} \left[\frac{\partial f}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_1}\right) \right] \xi_1 dt = 0 \qquad .....(9)$$

Since (9) holds for all, ξ_1 we conclude by applying the basic Lemma that

$$\frac{\partial f}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_1}\right) = 0 \quad .....(10)$$

Through similar treatment of the successive pairs of terms of the second member of (8) we derive like equations, with x_1 replaced by x_2, ..., x_n, Joining these equations with (10), we have

$$\frac{\partial f}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_1}\right) = 0, \frac{\partial f}{\partial x_2} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_2}\right) = 0, ..., \frac{\partial f}{\partial x_n} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_n}\right) = 0 \quad .....(11)$$

for the system of simultaneaes Euler-Lagrange equations which must be satisfied by the functions $x_1(t)$, $x_2(t)$, ..., $x_n(t)$ which render the integral (1) an extremum.

Ex.*1. *Find the extremals of the functional

$$I[y,z] = \int_0^{\pi/2} \left[\dot{y}^2 + \dot{z}^2 + 2yz\right] dt$$

with the boundary conditions $y(0) = 0, y(\pi/2) = -1$; $z(0) = 0, z(\pi/2) = 1$

Sol. Here $f(y, z, \dot{y}, \dot{z}, t) = \dot{y}^2 + \dot{z}^2 + 2yz$

Then from equation (11), we can see that

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\left[\frac{\partial f}{\partial \dot{y}}\right] = 0$$

$$\frac{\partial f}{\partial z} - \frac{d}{dt}\left[\frac{\partial f}{\partial \dot{z}}\right] = 0$$

or $\quad \ddot{y} - z = 0$ and $\ddot{z} - y = 0$ .....(12)

Eliminating 'z' from this system, we get

$$y^{(iv)} - y = 0 \text{ or } \frac{d^4 y}{dt^4} - y = 0$$

Its solution is given by

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t \quad .....(13)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. Now from equation (12) we have

$$z = \ddot{y} = c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t \quad .....(14)$$

Applying the given boundary conditions

$y(0) = 0, y(\pi/2) = -1, \ z(0) = 0, z(\pi/2) = 1$, we find that

$c_1 = c_2 = c_3 = 0, \ c_4 = -1.$

Hence the extremal curve is the intersection of the surfaces

$y = -\sin t, z = \sin t.$

***Ex.*2. *(a)* Find the extremum of the function**

$$F[y(x)] = \int_{x_1}^{x_2} \frac{\left(1 + y'^2\right)^{1/2}}{x} dx$$

***(b)* Show that the curve through (1,0) and (2,1) which minimize**

$$\int_1^2 \frac{\left(1+y'^2\right)^{1/2}}{x}\,dx \text{ is a circle.}$$

Sol. ***(a)*** Comparing the given functional with $\int_{x_1}^{x_2} f\left(x, y, y'\right)dx,$ we get

where $$f\left(x,y,y'\right)=\frac{\left(1+y'^2\right)^{1/2}}{x} \quad .....(15)$$

Euler-Lagrange equation is

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0 \quad .....(16)$$

From (15), we have

$$\frac{\partial f}{\partial y}=0 \text{ and } \frac{\partial f}{\partial y'}=\frac{y'}{x\left(1+y'^2\right)^{1/2}} \quad .....(17)$$

Since $\frac{\partial f}{\partial y}=0,$ (16) reduces to $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)=0$

Integrating it, we get $\frac{\partial f}{\partial y'}=c$

or $$\frac{y'}{x\sqrt{1+y'^2}}=c$$

Thus, $$y'=cx\left(1+y'^2\right)^{1/2} \quad .....(18)$$

Now let $$\frac{dy}{dx}=y'=\tan\theta$$

Then (18) yields $\tan\theta = cx\sec\theta$

$\Rightarrow$ $\quad x = c_1 \sin\theta$ where $c_1 = 1/c$

Now $dy = \tan\theta\, dx = c_1 \tan\theta \cos\theta d\theta = c_1 \sin\theta d\theta$

Integrating it, we get $y = -c_1 \cos\theta + c_2$

Thus $x = c_1 \sin\theta$ and $y - c_2 = -c_1\cos\theta$ or $x^2 + (y - c_2)^2 = c_1^2$ .....(19)

which is a family of circle with center at axis.

(b) Proceed exactly as in part *(a)* upto (19). In the present problem, using the boundary conditions $x = 1, y = 0$ and $x = 2, y = 1$, (19) yields

$1+c_2^2=c_1^2$ and $4+\left(1-c_2\right)^2=c_1^2$ giving $c_1=\sqrt{5},\ c_2=2$.

Hence from (19) the required curve is the circle $x^2 + (y - 2)^2 = 5$.

Ex.3. ***Obtain the Euler-Lagrange equation for the extremals of the functional***

$$\int_{x_1}^{x_2}\left[y^2 - yy' + y'^2\right]dx$$

Sol. Comparing the given functional with $\int_{x_1}^{x_2} f(x, y, y')\,dx$, we get

$$f(x, y, y') = y^2 - yy' + y'^2 \qquad .....(20)$$

Euler-Lagramge's equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad .....(21)$$

From (10), we get $\dfrac{\partial f}{\partial y} = 2y - y', \dfrac{\partial f}{\partial y'} = -y + 2y'$

and $\dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = -y' + 2y''$

Using these values, the required Euler-Lagrange equation (21), becomes

$$2y - y' - (-y' + 2y'') = 0 \text{ or } y'' - y = 0$$

Ex.4. ***Test for an extremal of the functional***

$$F[y(x)] = \int_0^{\pi/2}\left(y'^2 - y^2\right)dx, \quad y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 1$$

Sol. Comparing the given functional with $\int_0^{\pi/2} f(x, y, y')\,dx$, we get

$$f(x, y, y') = y'^2 - y^2 \qquad .....(22)$$

Euler-Lagrange's equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad .....(23)$$

From (22), we have $\dfrac{\partial f}{\partial y} = -2y, \dfrac{\partial f}{\partial y'} = 2y'$ and $\dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 2y''.$

Using these values, (23) reduces to

$$-2y - (2y'') = 0 \quad \text{or} \quad y'' + y = 0$$

or

$$(D^2 + 1)\,y = 0 \text{ where } D \equiv \frac{d}{dx} \qquad .....(24)$$

$$y = c_1 \cos x + c_2 \sin x \qquad .....(25)$$

Using boundary condition, we get

$$c_1 = 0 \text{ and } c_2 = 1.$$

Hence, from (25), an extremun can be attained only on the curve $y = \sin x$

Self-Learning Exercise

1. Is $L\left[y(x)\right]=\int_{x_0}^{x} y^2 dx$ is linear ? (Yes / No.)

2. Is $L\left[y(x)\right]=\int_{x_0}^{x}\left[\frac{d^2y}{dx^2}+c(x)y\right]dx$ is linear ? (Yes / No.)

3. As extremal of the functional

   $F\left[y(x)\right]=\int_{a}^{b} f(x,y,y')dx,\ y(a)=y_1,\ y(b)=y_2$ satisfies Euler-Lagrange equation, which in general is a

   (a) linear second order ODE
   (b) admits a unique solution
   (c) non-linear ODE of order greater than two.
   (d) may not admit a solution.

4. The curve of shortest distance between two fixed points is
   (a) straight line
   (b) circle
   (c) parabola
   (d) none of these

5. The Euler-Lagrange equation for a functional of the form $\int_{a}^{b} f(x,y)dx$ is

   (a) $f_{y'}=c_1$
   (b) $f_y-y'f_{y'}=c_1$
   (c) $f_y=c_1$
   (d) none of these

6. The extremizing curve of the brachistochrone problem is a
   (a) circle
   (b) catenary
   (c) cycloid
   (d) straight line.

7.6 Summary

The caluclus of variation, which plays an important role in both pure and applied mathematics, dates from the time of Newton. Development of the subject started mainly with the work of Euler and Lagrange. In this unit we have solved a number of problem of engineering and physics with the help of Euler-Lagrange equations.

7.7 Answers to Self-Learning Exercises

(1) No (2) Yes

(3) *(d)* (4) *(a)*

(5) *(d)* (6) *(a)*

7.8 Exercise

1. Find the extemal of the function $I\left[y(x)\right]=\int_0^1 \frac{1+y^2}{y'}\,dx,$ throught the origin and the point (1, 1).

[Ans. y = tan($\pi x/4$)]

2. *(a)* Show that if y satisfies the Euler-Lagrange's equation associated with the integral

$$I=\int_{x_1}^{x_2}\left(p^2y'^2+q^2y^2\right)dx$$

where $p(x)$ and $q(x)$ are known functions, then I has the value $\left[\left(p^2yy'\right)\right]_{x_1}^{x_2}$

(b) Show that, it y satisfies the Euler-Lagrange's equation associated with part (*a*) and if $z(x)$ is an arbitrary differentiable function for which $z(x_1) = z(x_2) = 0$

then $$I=\int_{x_1}^{x_2}\left(p^2y'z'+q^2yz\right)dx=0$$

3. Prove that the extremal of $\int_a^b y\left(1+y'^2\right)^{1/2}$ is the catenary $y = a \cos h(ax + b)$

4. Prove that the extremal of $\int_0^2 \frac{y'^2}{x}dx$ with $y(0) = 0$ and $y(2) = 1$ is a parabola.

5. Prove that the extremals of

$$I=\int_{x_1}^{x_2}\left[u(x)y'^2-v(x)y^2\right]dx$$

subject to the condition that

$$J=\int_{x_1}^{x_2}\omega(x)y^2dx=k \qquad \text{(a constant)}$$

are the solution of Sturm-Liouville equation

$$\frac{d}{dx}\left[u(x)\frac{dy}{dx}\right]+\left[v(x)+\lambda\omega(x)\right]y=0, \quad \text{with} \quad y(x_1)=y(x_2)=0$$

6. Show that the extremum of the functional

$$I = \int_{x_1}^{x_2} \left[y^2 + y'^2 - 2y \sin x \right] dx,$$

is given by $$y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} \sin x$$

7. Show that the Euler's equation for the functional

$$I = \int_a^b f(x, y) \sqrt{1 + y'^2}\, dx \text{ has the form } f_{y'} - f_{x',y'} - \frac{f_{y''}}{1 + y'^2} = 0$$

8. Find an extremal to

$$I = \int_1^2 \frac{\sqrt{1 + y'^2}}{x} dx, \quad y(1) = 0, \quad y(2) = 1$$

[Ans. $x^2 + (y - 2)^2 = 5$**]**

9. Find the curve $y = \phi(x)$ which corresponds to the extreme value of

$$F[y(x)] = \int_a^b x^n \left(\frac{dy}{dx} \right)^2 dx$$

[Ans. $y = \frac{c_1 x^{1-n}}{1-n} + c_2, \; n \neq 1 = c_1 \log x + c_2, \; n = 1$**]**

10. Show that the curve of shortest distance (geodesic) on a right circular cylinder is a Helix or a generator.

11. Find the extremals of the functional $F[y(x), z(x)] = \int_a^b \left(2yz - 2y^2 + y'^2 + z'^2 \right) dx$

Deduce the extremals if $a = 0$, $b = \pi$; $y(0) = 0, \; y(\pi) = 1, \; z(0) = 0, \; z(\pi) = -1$.

[Ans. $y = (c_1 x + c_2)\cos x + (c_3 + c_4)\sin x \;\; z = (c_1 x + c_2 + 2c_3)\cos x + (c_3 + c_4 - 2c_1)\sin x$**]**

□□□

Unit 8 : Functionals Dependent on Higher Order Derivatives and Variational Problems in Parametric Form

Structure of the Unit

8.0 Objective

This unit deals with the functionals dependent on higher order derivatives and functions of more than one independent variable. The variational problems in parametric form are also included in the present unit.

8.1 Introduction

In the previous unit, we have discussed the Euler-Lagrange's equation and various variational problems having their first order derivatives. In this unit, we will disuss the variational problem with functional dependent on higher order derivatives, several independent variables and variational problem in parametric form.

8.2 Varitional Problems Involving Several Higher Order Derivatives

Theorem : If the function *f* contains higher order derivatives, say upto any order *n*, then

$$f \equiv f\left(x, y, y', ..., y^{(n)}\right) \qquad .....(1)$$

and we need to extremize the integral

$$I = \int_{x_1}^{x_2} f\left(x, y, y', ...y^{(n)}\right) dx \qquad .....(2)$$

where we consider the funciton f is differentiable (n + 2)- times with respect to 'x'. and also assume that the boundary conditions are given by

$$y(x_1)=y_1,\ y'(x_1)=y_1',\ y''(x_1)=y_1'',\ldots\ldots,y^{(n)}(x_1)=y_1^{(n)}$$

$$y(x_2)=y_2,\ y'(x_2)=y_2',\ y''(x_2)=y_2'',\ldots\ldots,y^{(n)}(x_2)=y_2^{(n)} \quad \ldots..(3)$$

Then I extremized by $\dfrac{\partial f}{\partial y}-\dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right)+\ldots\ldots+(-1)^n\dfrac{d^n}{dx^n}\left(\dfrac{\partial f}{\partial y^n}\right)=0$ .....(4)

Proof : Let the extremum is attained on the curve $y = y(x)$ and $y = \bar{y}(x)$ be comparison curve to extremizing curve $y = y(x)$, and let both of these be $2n$ times differentiable.

Now we consider

$$\bar{y}(x)=y(x)+\in\eta(x), \quad \ldots..(4)$$

where $\eta(x_1)=\eta(x_2)=\eta'(x_1)=\eta'(x_2)=\ldots..=\eta^{(n)}(x_1)=\eta^{(n)}(x_2)=0$

Obviously $y(x,0) = y(x)$, the extremizing curve.

Now substituting it in equation (1), we get

$$I(\in)=\int_{x_1}^{x_2} f\left(x,\bar{y},\bar{y}',\ldots\ldots,\bar{y}^{(n)}\right)dx \quad \ldots..(5)$$

Since setting $\in=0$ has the effect of replacing $\bar{y},\bar{y}',\ldots\ldots,\bar{y}^{(n)}$ in (5) by the $y,y',y'',\ldots\ldots,y^{(n)}, I(\in)$ must take extreme value when $\in=0$. This happens no matter what particular value function $\eta(x)$ is involved in (4) and (5). But by elementary calculus, a necessary condition of extremum is given by $\quad I(\in)=0 \quad$ ....(6)

Using Leibniz's rule of differentiation under integral sign, (6) gives.

$$I'(\in)=\int_{x_1}^{x_2}\frac{d}{d\in}f\left(x,\bar{y},\bar{y}',\ldots\ldots,\bar{y}^{(n)}\right)dx$$

Now using the chain rule for differentiating functions of several variables, we get

$$\frac{d}{d\in}f\left(x,\bar{y},\bar{y}',\ldots\ldots,\bar{y}^{(n)}\right)=\frac{\partial f}{\partial x}\frac{\partial x}{\partial\in}+\frac{\partial f}{\partial\bar{y}}\frac{\partial\bar{y}}{\partial\in}+\ldots..+\frac{\partial f}{\partial\bar{y}^{(n)}}\cdot\frac{\partial\bar{y}^{(n)}}{\partial\in} \quad \ldots..(7)$$

By using (4), we have

$$\text{L.H.S. of (7)} =\frac{\partial f}{\partial\bar{y}}\eta(x)+\frac{\partial f}{\partial\bar{y}}\eta'(x)+\ldots\ldots..+\frac{\partial f}{\partial\bar{y}^{(n)}}\eta^{(n)}(x) \quad \ldots..(8)$$

From (8), we get

$$I'(\in)=\int_{x_1}^{x_2}\left\{\frac{\partial f}{\partial\bar{y}}\eta(x)+\frac{\partial f}{\partial\bar{y}}\eta'(x)+\ldots\ldots..+\frac{\partial f}{\partial\bar{y}^{(n)}}\eta^{(n)}(x)\right\}dx=0$$

which, upon setting $\in=0$ and making use of (6), gives

$$\int_{x_1}^{x_2}\left\{\frac{\partial f}{\partial y}\eta(x)+\frac{\partial f}{\partial y'}\eta'(x)+\ldots\ldots..+\frac{\partial f}{\partial y^{(n)}}\eta^{(n)}(x)\right\}dx=0 \quad \ldots..(9)$$

where we have used the fact that when $\in = 0,\ \bar{y} = y, \bar{y}' = y', ..., \bar{y}^{(n)} = y^{(n)}$.

Now integrating by parts, we have

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\eta'(x) = \left[\frac{\partial f}{\partial y'}\eta(x)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)dx$$

$$= -\int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)dx$$

and $$\int_{x_1}^{x_2} \frac{\partial f}{\partial y''}\eta''(x)dx = \left[\frac{\partial f}{\partial y''}\eta'(x)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right]\eta'(x)$$

$$= -\int_{x_1}^{x_2} \frac{d}{dx}\left[\frac{\partial f}{\partial y''}\right]\eta'(x) \qquad .....(10)$$

Again integrating with respect to 'x' we get

$$\int_{x_1}^{x_2} \frac{\partial y}{\partial y''}\eta''(x)dx = -\left\{\left[\frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)\eta(x)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2}\left[\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)\eta(x)dx\right]\right\}$$

Thus, $$\int_{x_1}^{x_2} \frac{\partial f}{\partial y''}\eta''(x)dx = \int_{x_1}^{x_2} \frac{d^2}{dx^2}\left[\frac{\partial f}{\partial y''}\right]\eta(x)dx$$

Similarly $$\int_{x_1}^{x_2} \frac{\partial f}{\partial y^{(n)}}\eta^{(n)}dx = (-1)^n \int_{x_1}^{x_2} \frac{d^n}{dx^n}\left[\frac{\partial f}{\partial y^{(n)}}\right]\eta(x)dx \qquad ...(11)$$

Using it in equation (9), we obtain

$$\int_{x_1}^{x_2}\left\{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) + ...+(-1)^n \frac{d^n}{dx^n}\left(\frac{\partial f}{\partial y^{(n)}}\right)\right\}\eta(x) = 0$$

which gives

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) + ... + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial f}{\partial y^{(n)}}\right) = 0 \qquad .....(12)$$

8.3 Variational Problem Involving Functionals Dependent on the Functions of Several Independent Variables and Dependent Variables

In this section, we will discuss the variational problems which is dependent on several dependent and independant variables.

Theorem : If *z* is a curve which is dependent on *x*, *y* and is twice differentiable in its domain *D*, and extremize the functional

$$i\left[z(x,y)\right] = \iint_D F(x,y,p,q)\,dx\,dy \qquad .....(1)$$

Then following differential equaiton must be satisfied

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right) = 0 \qquad .....(2)$$

where $\quad p = \frac{\partial z}{\partial x}$, **and** $q = \frac{\partial z}{\partial y}$

Proof : Take some admissible surface $z = \overline{z}(x,y)$ close to $z = z(x,y)$ and include the surfaces $z = z(x,y)$ and $z = \overline{z}(x,y)$ in a one-parameter family of surfaces

$$z(x,y,\alpha) = z(x,y) + \alpha\delta z$$

where $\delta z = \overline{z}(x,y) - z(x,y)$

For $\alpha = 0$, we get the surface $z = z(x,y)$, for $\alpha = 1$, we have $z = \overline{z}(x,y)$.δz is called the variation of the fucntion $z(x,y)$.

On fucntions of the family $z = z(x,y,\alpha)$, the functional I reduces to the fucntion of α, which has an extremum for $\alpha = 0$. Hence, we have

$$\left[\frac{\partial}{\partial\alpha} I\left(z\,(x\,,y\,,\alpha)\right)\right]_{\alpha=0} = 0$$

The derivative of $I\,[z(x,y,\alpha)]$ with respect to α, for $\alpha = 0$ is known as the variation of the function and is denoted by δI. Accordingly, we have

$$\delta I = \left[\frac{\partial}{\partial\alpha}\iint_D F\left(x,y,z(x,y,\alpha),p(x,y,\alpha),q(x,y,\alpha)\right)dx\,dy\right]_{\alpha=0}$$

or
$$\delta \text{I} = \iint_D \left[F_z\delta z + F_p\delta p + F_q\delta q\right]dx\,dy \qquad .....(3)$$

where $z(x,y,\alpha) = z(x,y) + \alpha\delta z$

$$p(x,y,\alpha) = \frac{\partial z(x,y,\alpha)}{\partial x} = p(x,y) + \alpha\delta p$$

and
$$q(x,y,\alpha) = \frac{\partial z(x,y,\alpha)}{\partial y} = q(x,y) + \alpha\delta q$$

Now, we have

$$\frac{\partial(F_p\delta z)}{\partial x} = \frac{\partial F_p}{\partial x}\delta z + F_p\delta p \quad\Rightarrow\quad F_p\delta p = \frac{\partial(F_p\delta z)}{\partial x} - \frac{\partial F_p}{\partial x}\delta z$$

and
$$\frac{\partial(F_q\delta z)}{\partial y} = \frac{\partial F_q}{\partial y}\delta z + F_q\delta q \quad\Rightarrow\quad F_q\delta q = \frac{\partial(F_q\delta z)}{\partial y} - \frac{\partial F_q}{\partial y}\delta z$$

Using above two results, we have

$$\iint_D \left[F_p\,\delta p + F_q \delta q +\right] dx\,dy = \iint_D \left\{\frac{\partial}{\partial x}\left(F_p \delta z\right) + \frac{\partial}{\partial y}\left(F_q \delta z\right)\right\} dx\,dy$$

$$-\iint_D \left(\frac{\partial F_p}{\partial x} + \frac{\partial F_q}{\partial y}\right)\delta z dx\,dy \qquad .....(4)$$

where $\partial F_p/\partial x$ is known as total partial derivative with respect to 'x'. While computing it, y is assumed to be fixed, but depenence of z, p and q upon x is taken account. Therefore, we have

$$\frac{\partial F_p}{\partial x} = F_{px} + F_{pz}\frac{\partial z}{\partial x} + F_{pp}\frac{\partial p}{\partial x} + F_{pq}\frac{\partial q}{\partial x}$$

Similarly $$\frac{\partial F_q}{\partial y} = F_{qy} + F_{qz}\frac{\partial z}{\partial x} + F_{qp}\frac{\partial p}{\partial y} + F_{qq}\frac{\partial q}{\partial y}$$

Using the well-known Green's theorem. We have

$$\iint_D \left\{\frac{\partial}{\partial x}\left(F_p \delta z\right) + \frac{\partial}{\partial y}\left(F_q \delta z\right)\right\} dx\,dy = \int_C \left(F_p dy - F_q dx\right)\delta z = 0 \qquad .....(5)$$

The last imtegral is equal to zero, since on the contour C the variation $\delta z = 0$ because all permissible surfaces pass through one and same spatial cantour C. Using (5), (4) reduces to

$$\iint_D \left(F_p \delta p + F_q \delta q\right) dx\,dy = -\iint_D \left(\frac{\partial}{\partial x}F_p + \frac{\partial}{\partial y}F_q\right)\delta z\,dx\,dy \qquad .....(6)$$

Using (6) in (3), it gives

$$\delta I = \iint_D F_z \delta z\,dx\,dy - \iint_D \left(\frac{\partial}{\partial x}F_p + \frac{\partial}{\partial y}F_q\right)\delta z\,dx\,dy$$

Hecne the neccessary condtion for $\delta I = 0$ for an extremum of the functional (2) takes from

$$\iint_D \left(F_z - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q\right)\delta z\,dx\,dy = 0$$

Since the variation δz is arbitrary and the factor is continuous, it follows from the fundamental lemma of the calculus of variation that on extemizing surface $z = z\,(x,y)$, we must have

$$F_z - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q = 0 \text{ that is } \frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right) = 0 \qquad .....(7)$$

Remark . For the functional

$$I\left[z\left(x_1, x_2, ..., x_n\right)\right] = \int\iint_D F\left(x_1, x_2, ...x_n, z, p_1, p_2, ..., p_n\right)\,dx_1 dx_2 .....dx_n$$

where $p_i = \dfrac{\partial z}{\partial x_i}$, in exactly similar way, we get from the basic necessery conditon for extremum $\delta I = 0$, the following equation

$$F_z - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}F_{pi} = 0$$

which the function $z = z(x_1, x_2, ..., x_n)$ extremizing the functional I must satisfy.

8.4 Variational Problems in Parameteric Ferm

In some problems, the requirement of single valuedness is excessively restrictive; for it turns out that Euler-Lagrange's equation-derived under assumption that the extremizing function is single valued- may have for the solution which satisfies the given end point conditions, a relationship in which dependent variable is not a single valued function of the independent variable. One cannot, without further justificaiton, accept such a solution as valid.

We proceed to show, that the extremizing relationship between a pair of variables x and y is the same, whether the solution is derived under the assumption that y is a single valued function of x or that a more general parametric representation is required to express the relation between x and y. We do this by showing that the solution of Euler-Lagrange equation derived on the basis of the assumption of the single valuedness of y as a funciton x satisfies also the system of Euler-Lagrange's equations derived on the basis of the parametric relationship between x and y.

Under the assumption that y is a single valued funciton of x, the integral to be extremized is given as

$$\mathrm{I} = \int_{x_1}^{x_2} f\left(x, y, y'\right) dx \qquad .....(1)$$

where y is required to have values y_1 and y_2 at $x = x_1$ and $x = x_2$. If instead, we use the parametric representation $\mathrm{x} = x(t)$, $y = y(t)$ where $x(t_j) = x_j$ and $y(t_j) = y_j$ for $j = 1,2$, the integral (1) transformed to through the relationships

$$y' = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \text{ and } dx = \dot{x}\, dt \qquad .....(2)$$

where the supirior dot represents differntiation with respect ot 't' .

Therefore
$$I = \int_{t_1}^{t_2} f\left(x, y, \frac{\dot{y}}{\dot{x}}\right) \dot{x}\, dt \qquad .....(3)$$

The Euler-Lagrange's equation for (1) is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad .....(4)$$

According to § 7.5, the system of Euler-Lagrange's equation assiociated with (3) can be written as

$$\therefore \qquad \frac{\partial g}{\partial x} - \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right) = 0, \; \frac{\partial g}{\partial y} - \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{y}}\right) = 0$$

where
$$g\left(x, y, \dot{x}, \dot{y}\right) = f\left(x, y, y'\right)\dot{x} \qquad .....(5)$$

From (5), we obtain

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial x} \cdot \dot{x}, \; \frac{\partial g}{\partial \dot{x}} = f - \dot{x}\frac{\partial f}{\partial y'} \cdot \frac{\dot{y}}{\dot{x}^2} = f - y'\frac{\partial f}{\partial y'} \qquad .....(6)$$

With the aid of second relation of (2), we obtain

$$\frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right)=\dot{x}\frac{d}{dx}\left(f-y'\frac{\partial f}{\partial y'}\right)=\dot{x}\left\{y'\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]+\frac{\partial f}{\partial x}\right\} \qquad .....(7)$$

Further, we obtain from (5)

$$\frac{\partial g}{\partial y}=\frac{\partial f}{\partial y}\dot{x},\ \frac{\partial g}{\partial y}=\dot{x}\frac{\partial f}{\partial y'}\frac{1}{\dot{x}}=\frac{\partial f}{\partial y'} \qquad .....(8)$$

According to the second relation of (5), we have

$$\frac{d}{dt}\left(\frac{\partial g}{\partial \dot{y}}\right)=\dot{x}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \qquad .....(9)$$

Combining this last result with the first of (8), we obtain the pair of equations

$$\frac{\partial g}{\partial x}-\frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right)=-\dot{y}\left[\frac{\partial g}{\partial y}-\frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right)\right],$$

$$\frac{\partial g}{\partial y}-\frac{d}{dt}\left(\frac{\partial g}{\partial \dot{y}}\right)=-\dot{x}\left[\frac{\partial g}{\partial y}-\frac{d}{dx}\left(\frac{\partial g}{\partial y'}\right)\right] \qquad .....(10)$$

From this result, we conclude that any relationship, single-valued or not, that satisfies the Euler-Lagrange's equaiton (4), derived on the basis of an assumed single valued solution $y = y(x)$, satisfies also the system (5), whose derivation requires no assumption of single-valuedness of y as function of x.

8.5 Isoperimetric Problem

In this section, we seek to derive the differentiable equation which must be satisfied by the funciton which renders the integral

$$I=\int_a^b f(x,y,y')\,dx \qquad .....(1)$$

an extremum with respect to continuously differentiable functions $y = y(x)$ for which the second integral.

$$J=\int_a^b g(x,y,y')\,dx \qquad .....(2)$$

possesses a given prescribed value, and with $y(a) = y_1$, $y(b) = y_2$ both prescribed boundary conditions The given functions f and g are twice differentiable with respect to x.

To solve this type of problem, we will use the method of Lagrange's multiplier. But first of all, we need to choose suitable extremizing function for this problem. If we choose $Y(x) = y(x) + \in_1\eta(x)$ which is a function of one perameter family. Then it yields the problem, because any change of the value of the single perameter would in general alter the value of J, whose constancy must be maintained as prescribed. For this reason we introduce the two perameter family

$$Y(x) = y(x) + \in_1\eta_1(x) + \in_2\eta_2(x) \qquad .....(3)$$

in which , η_1 and η_2 are arbitrary differentiable function for which $\eta_1(a) = \eta_2(a) = 0$ and $\eta_1(b) = \eta_2(b) = 0$. These conditions ensures that $Y(a) = y(a) = y_1$ and $Y(b) = y(b) = y_2$ as prescribed, for all values of parameters $\in_1$ and $\in_2$.

We replace y by $Y(x)$, given by (3), in both equations (1) and (2) so as to form respectively

$$I(\in_1,\in_2) = \int_a^b f(x,Y,Y')\,dx \qquad .....(4)$$

and

$$J(\in_1,\in_2) = \int_a^b g(x,Y,Y')\,dx \qquad .....(5)$$

Clearly, the parameters $\in_1$ and $\in_2$ are not independent, because J is to be maintained at a constant value, it is clear from (5) that there is a functional relation between them-namely,

$$J(\in_1,\in_2) = \text{constant (prescribed)} \qquad .....(6)$$

Now using, method of Lagranges multipliers, we introduce the function for $\in_1$, $\in_2$,

$$I^* = I(\in_1,\in_2) + \lambda J(\in_1,\in_2) = \int_a^b f^*(x,Y,Y')\,dx \qquad .....(7)$$

where, according to (1) and (2),

$$f^* = f + \lambda g \qquad .....(8)$$

The constant λ is the undetermined multiplier whose value remains to be determined by conditions of each individual problem to which the method is applied. Thus for extemizing the value of I^*, we have

$$\frac{\partial I^*}{\partial \in_1} = \frac{\partial I^*}{\partial \in_2} = 0, \text{ when } \in_1 = \in_2 = 0 \qquad .....(9)$$

From (7), with the help of (3), it follows that

$$\frac{\partial I^*}{\partial \in_j} = \int_a^b \left\{ \frac{\partial f^*}{\partial Y}\frac{\partial Y}{\partial \in_j} + \frac{\partial f^*}{\partial Y'}\frac{\partial Y'}{\partial \in_j} \right\} dx$$

$$= \int_a^b \left\{ \frac{\partial f^*}{\partial Y}\eta_j + \frac{\partial f^*}{\partial Y'}\eta_j' \right\} dx \qquad .....(10)$$

(j = 1,2)

Setting $\in_1 = \in_2 = 0$, so that according to (3), (Y,Y') is replaced by (y,y'), we thus have that

$$\left.\frac{\partial I^*}{\partial \in_j}\right|_0 = \int_a^b \left\{ \frac{\partial f^*}{\partial y}\eta_j + \frac{\partial f^*}{\partial y'}\eta_j' \right\} dx = 0 \qquad (j = 1,2), \qquad .....(11)$$

Note that the symbol $|_0$ indicates that the setting of $\in_1 = \in_2 = 0$. Integrating by parts the second term of the integrand of (11), we obtain with aid of boundary conditions that

$$\int_{a_1}^b \eta_j \left[\frac{\partial f^*}{\partial y} - \frac{d}{dx}\left(\frac{\partial f^*}{\partial y'} \right) \right] dx = 0 \qquad (j = 1,2) \qquad .....(12)$$

Now using basic lemma, we obtain the differential equation

$$\frac{\partial f^*}{\partial y} - \frac{d}{dx}\left(\frac{\partial f^*}{\partial y'}\right) = 0 \qquad .....(13)$$

as the Euler-Lagrange's equation which must be satisfied by the function $y(x)$ which extrimizes (1) under the restriction that (2) be maintained at a prescribed value.

Ex.1. Find the extremal of the functional

$$I = \int_0^1 \left(1 + y''^2\right) dx,$$

under the conditions $y(0) = 0, y'(0) = 1, y(1) = 1, \; y'(1) = 1$

Sol. In this problem,

$$f(x, y, y', y'') = 1 + y''^2$$

Therefore, the extremal function is given by solution of the following differential equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0$$

$$\Rightarrow \qquad 0 - 0 + \frac{d^2}{dx^2}[2y''] = 0$$

$$\Rightarrow \qquad \frac{d^4 y}{dx^4} = 0 \qquad .....(14)$$

The solution of differential equation (14) is

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

Using the given conditions we easily obtain $y = x$

Thus extremal curve is a straight line.

Ex.2 . Find the extremal of the functional.

$$I[y(x)] = \int_0^{\pi/2} \left[y''^2 - y^2 + x^2\right] dx,$$

$$y(0) = 1, y'(0) = 0, y\left(\pi/2\right) = 0, y'\left(\pi/2\right) = -1.$$

Sol. Comparing the given functional with

$$I[y(x)] = \int_0^{\pi/2} f(x, y, y', y'')\, dx$$

we get

$$f(x, y, y', y'') = y''^2 - y^2 + x^2 \qquad .....(15)$$

Using equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] + \frac{d^2}{dx^2}\left[\frac{\partial f}{\partial y''}\right] = 0 \qquad .....(16)$$

From (15) we get

$$\frac{\partial f}{\partial y} = -2y, \frac{\partial f}{\partial y'} = 0, \frac{\partial f}{\partial y''} = 2y''$$

So (16) reduces to

$$-2y + \frac{d^2}{dx^2}[2y''] = 0 \quad \text{or} \quad \frac{d^2 y}{dx^4} - y = 0$$

or $$(D^4 - 1)y = 0 \text{ where } D \equiv \frac{d}{dx} \qquad .....(17)$$

The auxilliay equation of (17) is

$$m^4 - 1 = 0 \Rightarrow m = \pm 1, \pm i$$

Thus solution of (17) is $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$ .....(18)

Using boundary conditions $y(0) = 1$, $y(\pi/2) = 0$, we get

$$c_1 + c_2 + c_3 = 1 \qquad .....(19)$$

and $$c_1 e^{\pi/2} + c_2 e^{-\pi/2} + c_4 = 0 \qquad .....(20)$$

Since $y'(0) = 0$ and $y'(\pi/2) = -1$ therefore we find that

$$c_1 - c_2 + c_4 = 0 \qquad .....(21)$$

$$c_1 e^{\pi/2} - c_2 e^{-\pi/2} - c_3 = -1 \qquad .....(22)$$

Adding (19) and (22), we get

$$c_1\left(1 + e^{\pi/2}\right) + c_2\left[1 - e^{-\pi/2}\right] = 0$$

and subtracting (20) from (21), we get

$$c_1\left(1 - e^{+\pi/2}\right) - c_2\left(1 + e^{-\pi/2}\right) = 0$$

Above two relations give $c_1 = c_2 = 0$ and using it in (19) and (21), we get

$$c_4 = 0, \ c_3 = 1$$

Hence extremum can be attained only on the curve $y = \cos x$

Ex.3 . Find the extremal equation for the following functional

$$\boldsymbol{I}\left[z(x_1, x_2)\right] = \iint_D \left\{\left(\frac{\partial z}{\partial x_1}\right)^2 + \left(\frac{\partial z}{\partial x_2}\right)^2\right\} dx_1\, dx_2$$

Sol. Here the integrand f is a function of two independent variables x_1 and x_2 , i.e.

$$F\left[z, \frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, x_1, x_2\right] = \left(\frac{\partial z}{\partial x_1}\right)^2 + \left(\frac{\partial z}{\partial x_2}\right)^2 \qquad .....(23)$$

Therefore, using the result

$$-\frac{\partial}{\partial x_1}\left(\frac{\partial F}{\partial p_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial F}{\partial p_2}\right) + \frac{\partial F}{\partial z} = 0 \qquad .....(24)$$

where $p_1 = \frac{\partial z}{\partial x_1}$, and $p_2 = \frac{\partial z}{\partial x_2}$,

From (23) and (24), we obtain $\frac{\partial}{\partial x_1}\left[2\frac{\partial z}{\partial x_1}\right] + \frac{\partial}{\partial x_2}\left[2\frac{\partial z}{\partial x_2}\right] = 0$

$$\Rightarrow \qquad \frac{\partial^2 z}{\partial x_1^2} + \frac{\partial^2 z}{\partial x_2^2} = 0,$$

which is the familiar Laplace equation.

Ex.4 . Obtain the surface of minimum area, stretched over a given closed curve C, enclosing the domain D in the xy plane.

Sol. From calculus, we know that the required given problem reduces to find the extremal of the functional

$$I\left[z(x,y)\right] = \iint_D \left\{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\}^{1/2} dx\,dy$$

Now we have $F(x, y, z, p_1, p_2) = \left(1 + p_1^2 + p_2^2\right)^{1\backslash 2}$ .....(25)

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x_1}\left(\frac{\partial F}{\partial p_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial F}{\partial p_2}\right) = 0, \qquad .....(26)$$

where $p_1 \equiv \frac{\partial z}{\partial x} = z_x$ and $p_2 = \frac{\partial z}{\partial y} = z_y$

(25) implies

$$\frac{\partial F}{\partial z} = 0, \frac{\partial F}{\partial p_1} = p_1\left(1 + p_1^2 + p_2^2\right)^{-1/2}, \frac{\partial F}{\partial p_2} = p_2\left(1 + p_1^2 + p_2^2\right)^{-1/2}$$

From (26), we have

$$-\frac{\partial}{\partial x}\left(\frac{p_1}{\left(1 + p_1^2 + p_2^2\right)^{1/2}}\right) - \frac{\partial}{\partial y}\left(\frac{p_2}{\left(1 + p_1^2 + p_2^2\right)^{1/2}}\right) = 0$$

or

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\left(1 + z_x^2 + z_y^2\right)^{1/2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\left(1 + z_x^2 + z_y^2\right)^{1/2}}\right) = 0 \qquad .....(27)$$

From (27), we get

$$z_{xx}\left(1 + z_x^2 + z_y^2\right)^{-1/2} - \frac{1}{2}z_x\left(1 + z_x^2 + z_y^2\right)^{-3/2} \times 2\left(z_x z_{xx} + z_y z_{yx}\right)$$

$$+z_{yy}\left(1 + z_x^2 + z_y^2\right)^{-1/2} - \frac{1}{2}z_y\left(1 + z_x^2 + z_y^2\right)^{-3/2} \times 2\left(z_x z_{xy} + 2z_y z_{yy}\right) = 0$$

or $$z_{xx}\left[\frac{1}{\left(1+z_x^2+z_y^2\right)^{1/2}}-\frac{z_x^2}{\left(1+z_x^2+z_y^2\right)^{3/2}}\right]+z_{yy}\left[\frac{1}{\left(1+z_x^2+z_y^2\right)^{1/2}}-\frac{z_y^2}{\left(1+z_x^2+z_y^2\right)^{3/2}}\right]$$

$$-\frac{2z_x z_y z_{xy}}{\left(1+z_x^2+z_y^2\right)^{3/2}}=0$$

or $$z_{xx}\left(1+z_y^2\right)+z_{yy}\left(1+z_x^2\right)-2z_x\, z_y\, z_{xy}=0$$

That is $$\frac{\partial^2 z}{\partial x^2}\left\{1+\left(\frac{\partial z}{\partial y}\right)^2\right\}+\frac{\partial^2 z}{\partial y^2}\left\{1+\left(\frac{\partial z}{\partial x}\right)^2\right\}-2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x\partial y}=0$$

whose solution will yield the desired minimal surface.

Ex.5 . Find the closed convex curve of length L that encloses greatest possible area.

Sol. We know that the area of the closed plane curve is given by the integral

$$I=\frac{1}{2}\int_a^b\left[x\dot{y}-y\dot{x}\right]dt \qquad .....(27)$$

where $$\dot{x}=\frac{dx}{dt},\ \dot{y}=\frac{dy}{dt}.$$

The total length of the curve is, given by

$$L=\int_a^b\left[\dot{x}^2+\dot{y}^2\right]^{1/2}dt \qquad .....(28)$$

has the same value L where L is the length of the plane curve. Now the question is to maximize (extremize) (27) under the restriction (28), We will use the equation (13) of (§ 8.5), which is given below :

$$\frac{\partial f^*}{\partial x}-\frac{d}{dt}\left(\frac{\partial f^*}{\partial \dot{x}}\right)=0,\ \frac{\partial f^*}{\partial y}-\frac{d}{dt}\left(\frac{\partial f^*}{\partial \dot{y}}\right)=0 \qquad .....(29)$$

where $$f^*=\frac{1}{2}\left(x\dot{y}-y\dot{x}\right)+\lambda\sqrt{\dot{x}^2+\dot{y}^2} \qquad .....(30)$$

From (29) and (30), we have

$$\frac{1}{2}\dot{y}-\frac{d}{dt}\left(-\frac{1}{2}y+\frac{\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)=0$$

$$-\frac{1}{2}\dot{x}-\frac{d}{dt}\left(\frac{1}{2}x+\frac{\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)=0$$

From which we obtain, by direct integration with respect to 't',

$$y-\frac{\lambda\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}=c_1,\quad x+\frac{\lambda\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}=c_2 \qquad ......(30)$$

From these, we have

$$(y-c_1)^2+(x-c_2)^2 = \lambda^2\left[\frac{\dot{x}^2+\dot{y}^2}{\dot{x}^2+\dot{y}^2}\right]=\lambda^2$$

Thus we have the well-known result *"that the closed curve of given perimeter for which the enclosed area is a maximum is a circle."*

Ex.6 . (Shape of hanging rope). Find the shape assumed by a uniform rope when suspended by its end from two points. at equal heights.

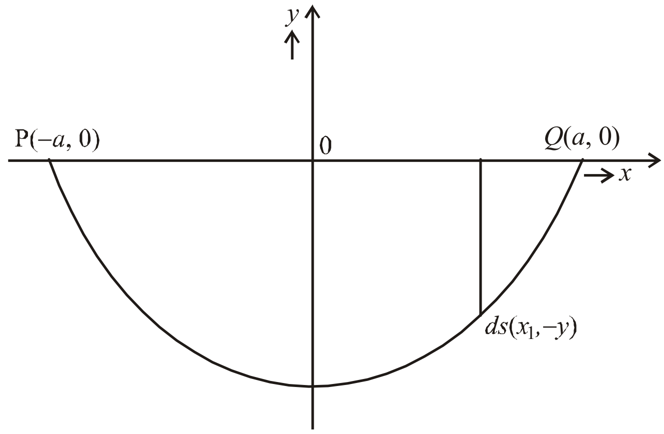

Fig. 8.1

Sol. Let the rope of lenght $2L$ be suspended between two points $P(-a,0)$ and $Q(a,0)$ in the same straight line, as points are at equal heights.

Thus if σ denotes the constant mass per unit length of rope, the potential energy of an element of length ds at $(x, -y)$ is given by $(-gy\,\sigma ds)$ where g is the constant acceleration due to gravrity. Accordingly, the total potential energy of the rope in the arbitrary configuration $y=y(x)$ is given by

$$I = \int_{-a}^{a}\sigma g\, y\, ds=\sigma g\int_{-a}^{a} y\sqrt{1+y'^2}\,dx \qquad .....(31)$$

where prime represents the differentiation with respect to 'x'. and taking absolute value.

According to minimum energy principle the equillibrium configuration is supplied by particular relation $y=y(x)$ for which (31) is a minimum with respect to functions $y(x)$ for which $y(a)=0$, $y(-a)=0$, and for which the total length of arc

$$J = \int_{-a}^{a}\sqrt{1+y'^2}\,dx=2L \qquad .....(32)$$

We may therefore apply the Euler-Lagrange equation to the integrand function

$$f^* = \sigma g y\sqrt{1+y'^2}+\lambda\sqrt{1+y'^2} \qquad .....(33)$$

formed from (31) and (32). Since f^* is explicitly independent of the variable x, however, we may use Euler-Lagrange equation and so substitute (30) into (13) (§ 8.5), we easily abtain.

$$(\sigma g y+\lambda)\left(\frac{y'^2}{\sqrt{1+y'^2}}-\sqrt{1+y'^2}\right)=c_1$$

$$\Rightarrow \qquad -\frac{1}{\sqrt{1+y'^2}} = \frac{c_1}{\sigma g y+\lambda}$$

$$\Rightarrow \qquad \left(1+y'^2\right) = \frac{(\sigma g y+\lambda)^2}{c_1^2}$$

$$\Rightarrow \qquad y'^2 = \frac{(\sigma g y+\lambda)^2}{c_1^2}-1$$

$$\Rightarrow \qquad y' = \sqrt{\frac{(\sigma g y+\lambda)^2-c_1^2}{c_1^2}}$$

$$\Rightarrow \qquad \frac{c_1\, dy}{\sqrt{(\sigma g y+\lambda)^2-c_1^2}} = dx$$

Putting $\sigma g y+\lambda = c_1 \cos ht$ and integrating, we find that $\frac{c_1}{\sigma g}\cos h^{-1}\left(\frac{\sigma g y+\lambda}{c_1}\right)+c_2 = x$

Solving we get

$$y = -\frac{\lambda}{\sigma g}+\frac{c_1}{\sigma g}\cosh\frac{\sigma g(x-c_2)}{c_1} \qquad .....(34)$$

where c_2 is an arbitrary constant of integration.

Thus, according to (34), the shape of a hanging rope is that of a catanary with vertical axis. By specifying that catenry passing through (–a,0) and (a,0) and that arc included between these points have length 2L, we may assign value to constants c_1, c_2, λ. appearing in (34).

Ex.7 . Determine the curve of prescribed lengh 2*l* which joins the points (–a,b) and (a,b) and has its centre of gravity as low as possible.

Sol. Let P_1 P_2 be an are joining the given paints (–a,b) and (a,b). The y-coordinate of the centre of gravity of the required curve is given by

$$I = \frac{\int_{-a}^{a} y\, ds}{\int_{-a}^{a} ds} = \frac{1}{2l}\int_{-a}^{a} y\left(1+y'^2\right)^{1/2} dx$$

where we have used the given constraint; namely

$$\int_{-a}^{a} ds = \int_{-a}^{a}\left(1+y'^2\right)^{1/2} dx = 2l, \text{ that is } \frac{1}{2l}\int_{-a}^{a}\left(1+y'^2\right)^{1/2} dx = 1 \qquad .....(35)$$

The boundary conditions are y(–a) = b, and y(a) = b

Let $$F(x,y,y') = \frac{y}{2l}\left(1+y'^2\right)^{1/2}+\frac{\lambda}{2l}\left(1+y'^2\right)^{1/2}$$

$$= \frac{(y+\lambda)}{2l}\left(1+y'^2\right)^{1/2}$$

where λ is the Lagrange's multiplier. Since F does not contain x, thus from Euler-Lagrange's equation

$$F - y'\left(\frac{\partial F}{\partial y'}\right) = c \quad \text{(a constant)}$$

or
$$\frac{(y+\lambda)\left(1+y'^2\right)^{1/2}}{2l} - \frac{(y+\lambda)}{2l}\times\frac{y'^2}{\left(1+y'^2\right)^{1/2}} = c \qquad .....(36)$$

or
$$\frac{(y+\lambda)}{\left(1+y'^2\right)^{1/2}} = c_1$$

where $c_1 = 2cl$. Re-writing the above equation we have

$$1+y'^2 = \frac{(y+\lambda)^2}{c_1^2} \text{ or } \frac{dy}{dx} = \left\{\frac{(y+\lambda)^2 - c_1^2}{c_1^2}\right\}^2$$

Separating variables and then integrating, we get

$$x = c_1\int\frac{dy}{\left\{(y+\lambda)^2 - c_1^2\right\}^{1/2}} + c_2 \text{ or } x = c_1\cosh^{-1}\frac{y+\lambda}{c_1} + c_2$$

So that
$$y = c_1\cosh\left(\frac{x-c_2}{c_1}\right) - \lambda \qquad .....(37)$$

which is a complete solution of equation (36) on [–a,a] and boundary condition will be satisfied by this solution if and only if

$$\frac{b+\lambda}{c_1} = \cosh\left[\frac{-a-c_2}{c_1}\right] \text{ and } \frac{b+\lambda}{c_1} = \cosh\left[\frac{a-c_2}{c_1}\right]$$

that is to say if and only if $(a + c_2)/c_1 = (a - c_2)/c_1$

Hence $c_2 = 0$. Thus equation (37) reduces to $y = c_1\cosh\left(\frac{x}{c_1}\right) - \lambda$ .....(38)

This shows curve must be symmetric with respect to y-axis. Thus, we get.

$$\lambda = c_1\cosh\ a/c_1 - b \qquad .....(39)$$

Using (38) in (35), we get

$$\frac{1}{2l}\int_{-a}^{a}\left\{1+\sinh^2(x/c_1)\right\}^{1/2}dx = 1$$

or
$$\int_{-a}^{a}\cos h(x/c_1)\,dx = 2l$$

or
$$2c_1\sin h(a/c_1) = 2l \Rightarrow l = c_1\sin(a/c_1) \qquad .....(40)$$

From (39), we have

$$\lambda = c_1 \left\{1+\sinh^2\left(a/c_1\right)\right\}^{1/2} - b$$

$$= c_1 \left\{1+\frac{l^2}{c_1^2}\right\}^{1/2} - b \quad \text{(using (40))}$$

$$= \left\{c_1^2 + l^2\right\}^{1/2} - b$$

Thus equation of the curve is given by

$$y = \cosh\left(\frac{x}{c_1}\right) - \left\{c_1^2 + l^2\right\}^{1/2} + b$$

Self-Learning Exercise

1.The possible value of α for which the functional

$$I\left[y(x)\right] = \int_0^1 \left[3y^2 + 3y'''\right] dy, \; y(\alpha) = 1$$

can be extremized ?

(a) –1,0 *(b)* 0,1 *(c)* –1,1 *(d)* –1,0,1

2. Find Euler-Lagrange's equation for

$$I = \int_{x_1}^{x_2} F\left(x, y, z, y', z', y'', z'', \ldots\ldots, y^{(k)}, z^{(k)}\right) dx$$

8.6 Summary

In this chapter, we obtain solution of some variabtional problems involving higher order derivatives, some functional dependent on some dependent and independent variables. A number of problems are included to illustrate various concepts of calculus of variation.

8.7 Answer to of Self-Leanning Exercise

(i) *(b)*

(ii) $$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) \ldots\ldots + (-1)^k \frac{d^k}{dx^k}\left(\frac{\partial F}{\partial y^{(k)}}\right) = 0$$

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial z''}\right) \ldots\ldots + (-1)^k \frac{d^k}{dx^k}\left(\frac{\partial F}{\partial z^{(k)}}\right) = 0$$

8.8 Exercise

1. Show that the Euler's equation for the surface area functional

$$I(u) = \iint_{\lambda} \sqrt{1+u_x'^2+u_y'^2}\; dx\, dy$$

is $\left(1+u_y'^2\right)u_{xx}'' - 2u_x'\, u_y'\, u_{xy}'' + \left(1+u_x'^2\right)u_{yy}'' = 0$

2. Find the Euler's equation for the functional.

$$I = \iint_{\lambda} \left[u_x^2 + u_y^2 + 2f(x,y)u(x,y)\right] dx\, xy$$

where λ is a closed region in the *xy*-plane and *u* has continuous partial derivatives.

[**Ans :** $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = f(x,y)$]

3. Find the general solution of the extremals

(i) $\iint_D \left(\frac{p^2 xy}{2} - \frac{qx^2y^2}{2}\right) dx\, dy$

(ii) $\iint_D \left(xyz + ypq + xp^2\right) dx\, dy$

where $p = \partial z/\partial x,\ q = \partial z/\partial y$ [**Ans :** ***(i)*** $z = c_1(y) \log x + c_2(y) + (x^3/9)$

(ii) $z = c_1(y) - \{-c_2(y)/2x^2\} + (yx^2/15)$]

4. Find the extremal for the functional

$$I[x(t), y(t)] = \int_{t_1}^{t_2} \left\{\left(\dot{x}^2 + \dot{y}^2\right)^{1/2} + a^2\left(x\dot{y} - y\dot{x}\right)\right\} dt$$

where *a* being a constant. [**Ans :** circles]

5. Find the extremal of the functional

$$I[x(t), y(t)] = \int_0^{\pi/4} \left(\dot{x}\dot{y} + 2x^2 + 2y^2\right) dt,$$

subject to the initial conditions at $t = 0$, $x = y = 0$; at $t = \frac{\pi}{4}$ $x = y = 1$.

[**Ans.** $x = y = \frac{\sin h2t}{\sin h\pi/2}$]

6. Find the curve of length *L* that join the paints (0, 0) and (1, 0) lie above the *x*-axis, and encloses the maxinmm area between itself and *x*-axis.

[Ans. $(x-c_1)^2+(y-c_1)^2=\lambda^2$ where $c_1=\frac{1}{2}, c_2=\left(\lambda^2-\frac{1}{4}\right)^{1/2}$

and λ is the solution of $\frac{1}{2\lambda}=\sin\left(\frac{L}{2\lambda}\right)$**]**

7. Find the extremals of the isoperimetric problem

$$I\left[y(x)\right]=\int_0^1\left(y'^2+x^2\right)dx, \text{ given that } \int_0^1 y^2\,dx=2\ ;\ y(0)=0,\ y(1)=0.$$

[Ans. $y=\sin m\pi x$, m = 1,2,3**]**

8. Find the curve joining two points (x_1,y_1) and (x_2,y_2) that yields a surface of revolution of stationary area when revolved about the x-axis. **[Ans.** a circle**]**

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Unit 9 : Series Solution of Second Order Linear Differential Equation

Structure of the Unit

9.0 Objective

The main object of this unit is to find the solution of a linear differential equation of second order with variable coefficients in terms of a series near ordinary and singular points with special reference to Gauss hypergeometric equation and Legendre equation.

9.1 Introduction

We know about the methods of solving linear differential equations of second order with constant coefficients and in certain cases with variable coefficients. But sometimes, in case of variable coefficients the problem becomes intricate and we are not able to find the solution in a closed form. Under such situation, we can find a power series in terms of the independent variable x satisfying certain conditions. This method is called the method of **solution in series** or **integration in series.** Legendre's equation, Hypergeometric equation and Bessel's equation are the examples whose solutions have been expressed in the form of a infinite power series e·g. the general solution of $y'' + y = 0$ is $y = a \cos x + b \sin x$ and this may be rewritten as

$$y = a\left\{1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \ldots\right\} + b\left\{x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \ldots\right\}$$

This shows that the general solution of the linear differential equation may be expressed by the superposition of a pair of infinite series.

9.2 Power Series Method

The basic concept of power series method is simple and we will apply this technique to the solution of some second order differential equations.

Let us consider the differential equation

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0 \qquad .....(1)$$

where $P(x)$, $Q(x)$ and $R(x)$ are polynomial in x and $P(x) \neq 0$.

The above equation may be written as

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0 \qquad .....(2)$$

where $\qquad p_1(x) = \dfrac{Q(x)}{P(x)}$, and $p_2(x) = \dfrac{R(x)}{P(x)}$

To find the solution of the equation (1), we assume a series for y of the form

$$y = a_0 + a_1x + a_2x^2 + \ldots = \sum_{r=0}^{\infty} a_r x^r \qquad .....(3)$$

Now substituting the values of y, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in equation (2) and rearranging the terms of different powers of x, we get an algebraic equation of the type

$$\lambda_0 + \lambda_1x + \lambda_2x^2 + \ldots = 0 \qquad .....(4)$$

Since equation (4) holds good for all values of x, identically, we obtain

$$\lambda_0 = 0\ ,\ \lambda_1 = 0,\ \lambda_2 = 0\ ,\ldots,\ \lambda_n = 0\ \ldots$$

From these equations, we can determine the coefficients a_0, a_1, a_2 ... etc. Putting the values of a_0, a_1, a_2, ... in the equation (3), we get the required solution which will be clear from the following example.

Ex.*1. *Solve in series

$$\left(1 - x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

Sol. Let the solution of the equation be

$$y = a_0 + a_1x + a_2x^2 + \ldots \qquad .....(5)$$

$$\therefore \qquad \frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \ldots$$

and $$\frac{d^2y}{dx^2} = 2a_2 + 6a_3x + 12a_4x^2 + ...$$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation and simplifying, we get

$$2a_0 + 2a_2 + 6a_3x + (12a_4 - 4a_2)x^2 + (20a_5 - 10a_3)x^3 + ... = 0$$

Equating to zero, the coefficients of various powers of x, we obtain

$$a_2 = -a_0,\ a_3 = 0,\ a_4 = \frac{a_2}{3} = -\frac{a_0}{3},\ a_5 = 0$$

Substituting for $a's$ in equation (5), we get

$$y = a_1x + a_0\left(1 - x^2 - \frac{x^4}{3}\right) + ...$$

which is the required solution.

9.2.1 Validity of The Power Series Method

In general an infinite series of the form

$$\sum_{r=0}^{\infty} a_r (x - x_0)^r = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + ...$$

is called a power aseries

Let us consider a differential equation

$$x^2 \frac{d^2y}{dx^2} + (x^2 - x)\frac{dy}{dx} + 2y = 0$$

If we assume a solution of the form

$$y = a_0 + a_1x + a_2x^2 + ...$$

and solve the equation by the above method, we find that

$$a_0 = 0,\ a_1 = 0,\ a_2 = 0,\ ...$$

This shows that the above equation has no series solution and if it is not so then what should be the conditions under which the above equation admits of the series solution.

9.2.2 Definitions

The following definitions will help us in establishing the validity of the series methods.

***(a)* Ordinary and singular points**

If $P(x_0) \neq 0$, then $x = x_0$ is called an **ordinary point** of (1), otherwise a **singular point.** If $P(x_0) = 0$, then $P_1(x)$ and/or $P_2(x)$ become unbounbed as $x_0 \to 0$, such a point is called singular point of eq. (1). For example, in the Legendre equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0,$$

the point $x_0 = 0$ is an **ordinary point** because

$$P(x_0)=1-x_0^2 \neq 0 \text{ at } x_0 = 0,$$

while $x_0 = \pm 1$ are the **singular points** of the Legendre equation.

In Bessel's equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-n^2\right)y=0$ clearly, $x_0 = 0$ is a **singular point** and all other points are **ordinary points.**

It is found that every solution of the eq. (1) at the ordinary point is analytic.

***(b)* Regular singular point**

A singular point $x = x_0$ of (1) is called **regular** if the following conditions are satisfied

$$\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)} = \lim_{x\to x_0}(x-x_0)\,p_1(x) = \text{finite}$$

and $$\lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)} = \lim_{x\to x_0}(x-x_0)^2\,p_2(x) = \text{finite}$$

For more general functions than polynomials, x_0 is a regular singular point of equation (1) if the expressions $(x-x_0)\frac{Q(x)}{P(x)}$ and $(x-x_0)^2\frac{R(x)}{P(x)}$ are analytic at $x = x_0$, i.e., they have convergent Taylor's series expansion about x_0.

where $P(x)$, $Q(x)$ and $R(x)$ are polynomials in x and $p_1(x)$, $p_2(x)$ are defined by eq. (2).

***(c)* Irregular singular point**

Any singular point of the equation (1) which is not a regular singular point is called an **irregular singular** point. For example

(i) the differential equation

$x(x-1)^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+(x-1)y=0$ has the singular points $x_0 = 0$, $x_0 = 1$. It can be easily seen that $x_0 = 0$ is a regular singular point as

$$\lim_{x\to 0}(x-0)\,p_1(x) = \lim_{x\to 0}(x-0)\frac{2x}{x(x-1)^2}=0$$

and $$\lim_{x\to 0}(x-0)^2\,p_2(x) = \lim_{x\to 0}(x-0)^2\frac{(x-1)}{x(x-1)^2}=0$$

whereas $x_0 = 1$ is an irregular singular point, since

$$\lim_{x\to 1}(x-1)\,p_1(x) = \lim_{x\to 1}(x-1)\frac{2x}{x(x-1)^2}\cdot = \lim_{x\to 1}\left(\frac{2}{x-1}\right) \text{ does not exist.}$$

(ii) the point $x_0 = 1$ is a regular singular point of the Legendre equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0, \text{ since}$$

$$\lim_{x\to 1}(x-1)\,p_1(x) = \lim_{x\to 1}(x-1)\frac{(-2x)}{\left(1-x^2\right)}=1$$

and $\lim_{x\to1}(x-1)^2 p_2(x) = \lim_{x\to1}(x-1)^2 \frac{n(n+1)}{(1-x^2)} = 0$

In a similar manner, it can be shown that $x_0 = -1$ is also a regular singular point of the Legendre equation.

***(c)* Radius of convergence**

Whether x_0 is ordinary or singular point, the power series method for solving the differential equation (1) is based on the idea of expressing y as intinite series in powers of $(x - x_0)$. Here note that only convergent series will yield desired solutions, if it exist.

A power series $\sum_{r=0}^{\infty} a_r (x-x_0)^r$ is said to converge at a point x, if

$$\lim_{m\to\infty} \sum_{r=0}^{m} a_r (x-x_0)^r \text{ exists}$$

Obviously if the series converges for $x = x_0$ it may converge for all x or only for some values of x for which the convergence tests studied in Real analysis may be used.

If there exists a number $R \geq 0$, such that $\sum_{r=0}^{\infty} a_r (x-x_0)^r$ converges absolutely for $|x - x_0| < R$ and diverges for $|x - x_0| > R$, the number R is caled the **Radius of convergence** of the series.

For a series that converges no where except at x_0, the radius of convergence is said to be zero. If it converges for all x, we say that radius of convergence is infinite. Also note that $R = \lim_{r\to\infty} \left| \frac{a_r}{a_{r+1}} \right|$, provided the limit exists.

9.3 Series Solution Near an Ordinary Point

If $x = x_0$ is an ordinary point of the equation (1), then each solution can be expressed in the form

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 y_1(x) + a_1 y_2(x),$$

where a_0 and a_1 are arbitrary constants and y_1 and y_2 are linearly independent series solutions which are analytic at x_0.

Following examples will make the method more clear.

Ex.*1. *Solve in series $\left(2-x^2\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = 0.$

Sol. Since $x_0 = 0$ is an ordinary point $\left(i \cdot e\ P(x_0) = 2 - x_0^2 \neq 0 \text{ at } x_0 = 0\right)$, we assume the solution in the form

$$y = \sum_{r=0}^{\infty} a_r (x-0) x^r = \sum_{r=0}^{\infty} a_r\, x^r$$

Substituting for y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we have

$$\left(2-x^2\right)\left[\sum_{r=0}^{\infty} a_r r\left(r-1\right)x^{r-2}\right]+2x\left[\sum_{r=0}^{\infty} a_r r x^{r-1}\right]-2\left[\sum_{r=0}^{\infty} a_r x^r\right]=0$$

or
$$2\sum_{r=0}^{\infty} a_r r\left(r-1\right)x^{r-2}-\sum_{r=0}^{\infty} a_r\left(r-1\right)\left(r-2\right)x^r=0$$

Equating to zero, the coefficient of the smallest power of x *i.e.* x^{r-2}, we get

$$2\,a_r r(r-1)-a_{r-2}(r-3)\,(r-4)=0$$

or
$$a_r=\frac{\left(r-3\right)\left(r-4\right)}{2r\left(r-1\right)}a_{r-2},\ r\geq 2$$

This gives
$$a_2=\frac{a_0}{2},\ a_3=0;\ a_4=0;\ a_5=0;\ a_6=0,\ .....$$

This shows that all the coefficients beyond a_2 are zero.

Hence the solution of the given equation is given by

$$y=a_0+a_1x+a_2x^2$$

or
$$y=a_0\left(1+\frac{x^2}{2}\right)+a_1x.$$

Ex.3. Solve the Legendre's equation

$$\boldsymbol{\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0.}$$

Sol. Since $x_0 = 0$ is an ordinary point $\left(i.e.\ P\left(x_0\right)=1-x_0^2\neq 0 \text{ at } x_0=0\right)$, therefore we may assume the solution in the form

$$y=\sum_{r=0}^{\infty} a_r\left(x-0\right)^r=\sum_{r=0}^{\infty} a_r x^r \qquad .....(1)$$

so that
$$\frac{dy}{dx}=\sum_{r=0}^{\infty} a_r\ r\ x^{r-1} \text{ and } \frac{d^2y}{dx^2}=\sum_{r=0}^{\infty} a_r\ r\left(r-1\right)x^{r-2}$$

Putting these values, in the given equation, we get

$$\left(1-x^2\right)\left[\sum_{r=0}^{\infty} a_r\ r\left(r-1\right)x^{r-2}\right]-2x\left[\sum_{r=0}^{\infty} a_r\ r\ x^{r-1}\right]+n\left(n+1\right)\left[\sum_{r=0}^{\infty} a_r\ x^r\right]=0$$

or
$$\sum_{r=0}^{\infty} a_r\ r\left(r-1\right)x^{r-2}-\sum_{r=0}^{\infty} a_r\left(r-n\right)\left(r+n+1\right)x^r=0$$

Equating to zero, the coefficient of x^r the recurrence relation is given by

$$a_{r+2}(r+2)(r+1)-a_r(r-n)\,(r+n+1)=0$$

or $$a_{r+2} = \frac{(r-n)(r+n+1)}{(r+1)(r+2)} a_r, \text{ where } r = 0, 1, 2 \ldots.. \qquad \ldots..(2)$$

The relation (2) gives even and odd coefficients in terms of the one immediately preceding it, except for a_1 and a_2 which are arbitrary.

$\therefore$ From (1), we find that

$$a_2 = \frac{-n(n+1)}{2\cdot 1} a_0$$

$$a_4 = \frac{(2-n)(2+n+1)}{3\cdot 4} a_2$$

or $$a_4 = \frac{(n-2)n(n+1)(n+3)}{4\cdot 3\cdot 2\cdot 1} a_0$$

and $$a_3 = \frac{-(n-1)(n+2)}{3\cdot 2} a_1$$

$$a_5 = \frac{-(n-3)(n+4)}{5\cdot 4} a_3$$

or $$a_5 = \frac{(n-1)(n-3)(n+2)(n+4)}{5\cdot 4\cdot 3\cdot 2\cdot 1} a_1$$

and so on.

Putting these coefficients in (1), the solution of the given equation can be written as

$$y = a_0\left[1 - \frac{n(n+1)}{\lfloor 2}x^2 + \frac{(n-2)n(n+1)(n+3)}{\lfloor 4}x^4 + \ldots\right]$$

$$+a_1\left[x - \frac{(n-1)(n+2)}{\lfloor 3}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{\lfloor 5}x^5 + \ldots\right]$$

$\therefore$ $$y = a_0 y_1(x) + a_1 y_2(x).$$

9.4 Series Solution Near a Regular Singular Point

If $x = x_0$ is a regular singularity of the equation (1) (§9.2), then at least one of the solutions can be expressed as

$$y = (x - x_0)^m \sum_{r=0}^{\infty} a_r (x - x_0)^r = \sum_{r=0}^{\infty} a_r (x - x_0)^{m+r} \qquad \ldots..(1)$$

where 'm' may be a positive or negative integer or a fraction and is called the index of the series solution. This method of solution was suggested by **George Frobenius (1849–1917)** and is called **Frobenius method.** We now discuss the method of solving equation (1) in the neighbourhood of a regular singular point $x = x_0$. Without loss of generality, we can take $x_0 = 0$. If $x_0 \neq 0$, we can transform the equation by letting $x = x_0 = z$.

Since $x_0 = 0$ is a regular singular point of the equation (1), its solution can be expressed in the following form

$$y = x^m \sum_{r=0}^{\infty} a_r\, x^r = \sum_{r=0}^{\infty} a_r\, x^{m+r}, \quad \text{where } a_0 \neq 0 \qquad \text{.....(2)}$$

9.4.1 Working Rule :

(*i*) Substitute the value of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation.

(ii) Rearrange the terms in powers of x and equate to zero the coefficient of lowest power of x. This gives us a quadratic equation in m which is called the **indicial equation.**

(iii) Solve the indicial equation. The following cases arise :

(a) The roots of the indicial equation are different and not differing by an integer.

(b) The roots of the indicial equation are equal.

(c) The roots of the indicial equation are different, differing by an integer and also making a coefficient of y infinite.

(d) The roots of the indicial equation are different, differing by an integer and making a coefficient of y indeterminate.

(iv) We equate to zero the coefficient of general power of x ($e \cdot g.$ x^{m+r} or x^{m+r-1} whichever may be the lowest) in the equation obtained in step *(ii)*. The equation so obtained will be called the **recurrence relation,** because it connects together the coefficients a_m, a_{m-2} or a_m, a_{m-1} etc.

(v) If the recurrence relation connects a_m and a_{m-2}, then we, in general, determine a_1 by equating to zero the coefficient of the next higher power. On the other hand, if the recurrence relation connects a_m, a_{m-1}, this step may be omitted.

(vi) With the help of the recurrence relation all the a′s are determined in terms of a_0 and these a′s will be put in eq. (2). Then replacing m by m_1 and m_2 and a_0 by a and b respectively, we shall obtain two independent solutions, say au and bv. Therefore the complete solution of the given differential equation is given by

$y = au + bv$, where a and b are arbitrary constants.

The method is illustrated with the help of following examples %

Case I. When the roots m_1, m_2 of the indicial equation are different and not differing by an integer, the complete solution is

$$y = c_1 (y)_{m_1} + c_2 (y)_{m_2}$$

where c_1 and c_2 are arbitrary constants

Ex.*1. *Solve in series $2x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + \left(1 - x^2\right) y = 0.$

Sol. Here $x_0 = 0$ is a regular singular point as

$$\lim_{x\to 0}(x-0)\frac{Q(x)}{P(x)} = \lim_{x\to 0}(x-0)\,p_1(x) = \lim_{x\to 0}(x-0)\left(\frac{-x}{2x^2}\right) = \frac{-1}{2} = \text{finite}$$

and $$\lim_{x\to 0}(x-0)^2\frac{R(x)}{P(x)} = \lim_{x\to 0}(x-0)^2\,p_2(x) = \lim_{x\to 0}(x-0)^2\left(\frac{1-x^2}{2x^2}\right) = \frac{1}{2} = \text{finite}$$

therefore we assume the series solution in the form

$$y = \sum_{r=0}^{\infty} a_r x^{m+r},\ a_0 \neq 0 \qquad .....(3)$$

Substituting for y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we find that

$$2x^2\left[\sum_{r=0}^{\infty} a_r(m+r)(m+r-1)x^{m+r-2}\right] - x\left[\sum_{r=0}^{\infty} a_r(m+r)x^{m+r-1}\right] + \left(1-x^2\right)\left[\sum_{r=0}^{\infty} a_r x^{m+r}\right] = 0$$

or $$\sum_{r=0}^{\infty} a_r\left[(m+r-1)(2m+2r-1)\right]x^{m+r} - \sum_{r=0}^{\infty} a_r\, x^{m+r+2} = 0 \qquad .....(4)$$

which is an identity. Now equating to zero, the coefficient of smallest power x *i.e.* x^m (put $r = 0$ in the first summation) then the equation (4) gives the indicial equation or quadratic equation in m as

$$a_0(m-1)(2m-1) = 0$$

which implies that $m = 1, 1/2$ as $a_0 \neq 0$

so the roots of the indicial equal are different and not differing by an integer.

To obtain the recurrence relation, we equate to zero the coefficient of x^{m+r} and obtain

$$a_r = \frac{1}{(m+r-1)(2m+2r-1)}\,a_{r-2} \qquad .....(5)$$

This formula connects a_r with a_{r-2}. Now we proceed to find a_1 as explained in step (v) of § 9.4.1. For this purpose, we equate to zero, the coefficient of next higher power of x *i.e.* x^{m+1}

(put $r = 1$ in the first summation), we get

$$a_1[m(2m+1)] = 0$$

Since the quantity within the bracket is not zero for any above values of m $\left(1 \text{ or } \frac{1}{2}\right)$, this gives

$a_1 = 0$

Since $a_1 = 0$, then from (5), we have $a_3 = a_5 = ... = 0$.

Also taking $r = 2$, in (5), we get

$$a_2 = \frac{1}{(m+1)(2m+3)}\,a_0 \qquad .....(6)$$

Next taking $r = 4$, in (5) and using (6), we obtain

$$a_4 = \frac{1}{(m+1)(m+3)(2m+3)(2m+7)}\,a_0$$

and so on.

Putting these values in (3), i.e. $y = x^m[a_0 + a_1x + a_2x^2 + a_3x^3 + ...]$ gives

$$y = a_0x^m\left[1+\frac{x^2}{(m+1)(2m+3)}+\frac{x^4}{(m+1)(m+3)(2m+3)(2m+7)}+...\right] \quad .....(7)$$

Putting $m = 1$, and replacing a_0 by a in (7), we get

$$y = ax\left[1+\frac{1}{2\cdot 5}x^2+\frac{1}{2\cdot 4\cdot 5\cdot 9}x^4+...\right] = au \text{ (say)}$$

Next putting $m = 1/2$, and replacing a_0 by b, we obtain

$$y = bx^{1/2}\left[1+\frac{x^2}{2\cdot 3}+\frac{x^4}{2\cdot 3\cdot 4\cdot 7}+...\right] = bv \text{ (say)}$$

Therefore the complete solution is given by

$$y = au + bv,$$

where a and b are arbitrary constants.

Ex.2. Solve the Gauss hypergeometric equation

$$x(1-x)\frac{d^2y}{dx^2}+\{\gamma-(1+\alpha+\beta)x\}\frac{dy}{dx}-\alpha\beta y=0$$

in series in the neighbourhood of the regular singular point (i) x = 0 (ii) x = 1 and (iii) x = ∞.

Sol. Given

$$x(1-x)\frac{d^2y}{dx^2}+\{\gamma-(1+\alpha+\beta)x\}\frac{dy}{dx}-\alpha\beta y=0 \quad .....(8)$$

Dividing by $x(1-x)$, we get

$$\frac{d^2y}{dx^2}+\frac{\{\gamma-(1+\alpha+\beta)x\}}{x(1-x)}\frac{dy}{dx}-\frac{\alpha\beta}{x(1-x)}y=0$$

Comparing it with $y'' + p_1(x)\,y' + p_2(x)\,y = 0$, we have

$$p_1(x) = \frac{\{\gamma-(1+\alpha+\beta)x\}}{x(1-x)}$$

and
$$p_2(x) = \frac{\alpha\beta}{x(1-x)}$$

Since $x\,p_1(x)$ and $x^2\,p_2(x)$ both tends to a finite value at $x = 0$, so $x = 0$ is regular singular point of (8).

Case I. Solution in the neighbourhood of $x = 0$.

We assume that the given equation (8) has the solution of the form

$$y = \sum_{r=0}^{\infty} a_r x^{m+r},\ a_0 \neq 0 \quad .....(9)$$

Substituting the values of y, y' and y'' in the given equation (8), we get

$$\left(x-x^2\right)\left[\sum_{r=0}^{\infty} a_r(m+r)(m+r+1)x^{m+r-2}\right]+$$

$$\{\gamma-(1+\alpha+\beta)x\}\left[\sum_{r=0}^{\infty}a_r(m+r)x^{m+r-1}\right]-\alpha\beta\left[\sum a_r x^{m+r}\right]=0$$

or $$\sum_{r=0}^{\infty}a_r(m+r)(m+r-1+\gamma)x^{m+r-1}-\sum_{r=0}^{\infty}a_r(m+r+\alpha)(m+r+\beta)x^{m+r}=0 \quad ....(10)$$

which is an identity. Equating to zero, the coefficient of the smallest power of x *i.e.* x^{m-1} (put $r = 0$ in the first summation), we get the indicial equation as

$$a_0 m(m-1+\gamma)=0,\ a_0 \neq 0$$

This gives $m = 0,\ 1-\gamma$

To obtain the recurrence relation, we equate to zero the coefficient of x^{m+r-1}. Then we have

$$a_r(m+r)(m+r-1+\gamma)-a_{r-1}(m+r-1+\alpha)(m+r-1+\beta)=0$$

or $$a_r=\frac{(m+r-1+\alpha)(m+r-1+\beta)}{(m+r)(m+r-1+\gamma)}\ a_{r-1} \quad .....(11)$$

For the solution corresponding to $m = 0$, the recurrence relation (11) reduces to

$$a_r=\frac{(r-1+\alpha)(r-1+\beta)}{r(r-1+\gamma)}\ a_{r-1}$$

from which it follows that

$$a_1=\frac{\alpha\cdot\beta}{1\cdot\gamma}a_0,$$

$$a_2=\frac{(1+\alpha)(1+\beta)}{2\cdot(1+\gamma)}a_1=\frac{\alpha(1+\alpha)\beta(1+\beta)}{1\cdot 2\gamma(1+\gamma)}a_0$$

and so on.

Putting these values and $m = 0$ and replacing a_0 by a in (2) gives

$$y=a\left[1+\frac{\alpha\cdot\beta}{1\cdot\gamma}x+\frac{\alpha(1+\alpha)\beta(1+\beta)}{1\cdot 2\gamma(1+\gamma)}x^2+....\right] \quad .....(12)$$

If we take $a = 1$ in (12), the series on the right hand side of (12) is called the **hypergeometric series** and is represented by ${}_2F_1(\alpha, \beta, \gamma; x)$. Thus we see that ${}_2F_1(\alpha, \beta, \gamma; x)$ is a solution of (8).

For the solution corresponding to $m = 1-\gamma$, when $1-\gamma$ is neither zero nor an integer, the recurrence relation (11) reduces to.

$$a_r=\frac{(1-\gamma+r-1+\alpha)(1-\gamma+r-1+\beta)}{(1-\gamma+r)(1-\gamma+r-1+\gamma)}a_{r-1}$$

or $$a_r=\frac{(\alpha'+r-1)(\beta'+r-1)}{r(\gamma'+r-1)}a_{r-1} \quad .....(13)$$

where $\alpha' = 1-\gamma+\alpha,\ \beta' = 1-\gamma+\beta,\ \gamma' = 2-\gamma$ .....(14)

Replacing $r = 1, 2, 3, ...$ successively in (13), we have

$$a_1=\frac{\alpha'\cdot\beta'}{1\cdot\gamma'}a_0$$

$$a_2 = \frac{(\alpha'+1)(\beta'+1)}{2(\gamma'+1)} a_1 = \frac{\alpha'(\alpha'+1)\beta'(\beta'+1)}{1\cdot 2\cdot \gamma'(\gamma'+1)} a_0 \text{ etc.}$$

Hence putting $m = 1 - \gamma$ ..., using the above values of a_1, a_2 ... in (9) and replacing a_0 by b gives

$$y = bx^{1-\gamma}\left[1 + \frac{\alpha'\cdot\beta'}{1\cdot\gamma'}x + \frac{\alpha'(\alpha'+1)\beta'(\beta'+1)}{1\cdot 2\gamma'(\gamma'+1)}x^2 + ...\right] \quad .....(15)$$

If we take $b = 1$ in (15), the series on the right hand side of (15) would be

$$x^{1-\gamma}\,{}_2F_1\,(\alpha', \beta'; \gamma'; x) \text{ i.e. } x^{1-\gamma}\,{}_2F_1\,(1-\gamma+\alpha, 1-\gamma+\beta; 2-\gamma\,; x)$$

which is another independent solution of (8).

Hence the general solution of (8) is

$$y = a\,{}_2F_1\,(\alpha, \beta; \gamma; x) + bx^{1-\gamma}\,{}_2F_1\,(1-\gamma+\alpha, 1-\gamma+\beta; 2-\gamma\,; x) \quad ....(16)$$

which a and b are arbitrary conatants.

Case II. Solution in the neighbourhood of $x = 1$.

It can be easily see that

$$\lim_{x\to 1}(x-1)\,p_1(x) = \lim_{x\to 1}(x-1)\frac{\{\gamma - (1+\alpha+\beta)x\}}{x(1-x)} = \text{finite value}$$

and $$\lim_{x\to 1}(x-1)^2\,p_2(x) = \lim_{x\to 1}(x-1)^2\frac{\{-\alpha\beta\}}{x(1-x)} = 0 = \text{finite value}$$

so $x = 1$ is also a regular singular point of (8).

If we substitute $\xi = 1 - x$ in the equation (8), it reduces to

$$\xi(1-\xi)\frac{d^2y}{d\xi^2} + \{\alpha+\beta-\gamma+1-(\alpha+\beta+1)\xi\}\frac{dy}{d\xi} - \alpha\beta y = 0 \quad .....(17)$$

On comparing (8) and (17), we find that (17) is the same as (8) except that γ is replaced by $\alpha + \beta - \gamma + 1$ and x by ξ.

Hence the solution (16) of (8) near $x = 0$ will be valid for (17) near $\xi = 0$, *i.e.* near $x = 1$.

Hence in this case, the required solution will be

$$y = A\,{}_2F_1\,(\alpha, \beta; \alpha+\beta-\gamma+1; 1-x)$$
$$+ B(1-x)^{\gamma-\alpha+\beta}\,{}_2F_1\,(\gamma-\alpha, \gamma-\beta;\ \gamma-\alpha-\beta+1; 1-x) \quad .....(18)$$

where $\gamma - \alpha - \beta$ is neither zero nor an integer

Case III. Solution in the neighbourhood of $x = \infty$.

To find the solution of the given hypergeometric differential equation (8) for large values of the independent variable *i.e.* about $x = \infty$, we change the independent variable from x to t with the help of the following transformation $x = 1/t$ *i.e.*, $t = 1/x$ .....(19)

Clearly large values of x correspond to small values of t. Using the above equation (19), we rewrite (8) and obtain the transformed equation near $t = 0$, say

$$\frac{d^2y}{dx^2} + p_1(t)\frac{dy}{dx} + p_2(t)y = 0 \quad .....(20)$$

Then the given equation (8) is said to have a regular singular point at $x = \infty$ if the transformed equation (20) has regular singular point at $t = 0$.

For $$x = \frac{1}{t} \text{ or } t = \frac{1}{x}, \quad \frac{dt}{dx} = \frac{-1}{x^2} \qquad .....(21)$$

and $$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\left(\frac{-1}{x^2}\right) = -t^2\frac{dy}{dt} \qquad .....(22)$$

Also $$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt} \qquad .....(23)$$

Using (21), (22) and (23), the given equation (8) transforms to

$$t^2(t-1)\frac{d^2y}{dt^2} + \{2(t-1) - \gamma t + (\alpha + \beta + 1)\}t\frac{dy}{dt} - \alpha\beta y = 0 \qquad .....(24)$$

To solve (24), let its series solution be

$$y = \sum_{r=0}^{\infty} a_r\, t^{m+r}, \; a_0 \neq 0 \qquad .....(25)$$

so that $\frac{dy}{dt} = \sum_{r=0}^{\infty} a_r(m+r)\,t^{m+r-1}$ and $\frac{d^2y}{dt^2} = \sum_{r=0}^{\infty} a_r(m+r)(m+r+1)\,t^{m+r-2}$

Putting these values of y, $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ in (24), we get

$$\left(t^3 - t^2\right)\sum_{r=0}^{\infty} a_r(m+r)(m+r-1)t^{m+r-2}$$
$$+\{2(t-1) - \gamma t + \alpha + \beta + 1\}t\sum_{r=0}^{\infty} a_r(m+r)t^{m+r-1} - \alpha\beta\sum_{r=0}^{\infty} a_r t^{m+r} = 0$$

or $$\sum_{r=0}^{\infty} a_r(m+r-\alpha)(m+r-\beta)t^{m+r} - \sum_{r=0}^{\infty} a_r(m+r)(m+r+1-\gamma)t^{m+r+1} = 0 \qquad .....(26)$$

which is an identity. Equating to zero, the coefficient of the smallest power of t (put $r = 0$, in the first summation), we get

$a_0(m - \alpha)(m - \beta) = 0 \Rightarrow m = \alpha, \beta$ as $a_0 \neq 0$

Next equating to zero, the coefficient of t^{m+r+1} in (26), we find that

$$a_{r+1} = \frac{(m+r)(m+r+1-\gamma)}{(m+r+1-\alpha)(m+r+1-\beta)}a_r \qquad .....(27)$$

For the solution, corresponding to $m = \alpha$, the recurrence relation (27) reduces to

$$a_{r+1} = \frac{(\alpha+r)(\alpha+r+1-\gamma)}{(r+1)(\alpha+r+1-\beta)}a_r$$

from which it follows that $$a_1 = \frac{\alpha(\alpha+1-\gamma)}{1\cdot(\alpha+1-\beta)} a_0$$

$$a_2 = \frac{(\alpha+1)(\alpha+2-\gamma)}{2\cdot(\alpha+2-\beta)} a_1 = \frac{\alpha(\alpha+1)(\alpha+1-\gamma)(\alpha+2-\gamma)}{1\cdot 2(\alpha+1-\beta)(\alpha+2-\beta)} a_0$$

and so on.

Putting these values and replacing a_0 by A in (25), gives

$$y = At^{\alpha}\left[1 + \frac{\alpha(1+\alpha-\gamma)}{1\cdot(1+\alpha-\beta)}t + \frac{\alpha(\alpha+1)(1+\alpha-\gamma)(1+\alpha-\gamma+1)}{1\cdot 2(1+\alpha-\beta)(1+\alpha-\beta+1)}t^2 + \ldots\right]$$

$$= At^{\alpha}\sum_{k=0}^{\infty}\frac{(\alpha)_k(1+\alpha-\gamma)_k}{(1+\alpha-\beta)_k}\frac{t^k}{\lfloor k}$$

or $$y = A\left(\frac{1}{x}\right)^{\alpha}\sum_{k=0}^{\infty}\frac{(\alpha)_k(1+\alpha-\gamma)_k}{(1+\alpha+\beta)_k}\frac{1}{\lfloor k}\left(\frac{1}{x}\right)^k$$

or $$y = Ax^{-\alpha}{}_2F_1\left(\alpha, 1+\alpha-\gamma;\ 1+\alpha-\beta; \frac{1}{x}\right) \qquad \ldots\ldots(28)$$

By symmetry for $m = \beta$, we get

$$y = Bx^{-\beta}{}_2F_1\left(\beta, 1+\beta-\gamma; 1+\beta-\alpha; \frac{1}{x}\right) \qquad \ldots\ldots(29)$$

Therefore the complete solution of the Gauss hypergeometric equation when $\beta - \alpha$ is neither zero nor an integer, is given by

$$y = Ax^{-\alpha}{}_2F_1\left(\alpha, 1+\alpha-\gamma; 1+\alpha-\beta; \frac{1}{x}\right) + Bx^{-\beta}{}_2F_1\left(\beta, 1+\beta-\gamma; 1+\beta-\alpha; \frac{1}{x}\right)$$

Case II. When the roots m_1, m_2 of the indicial equation are equal, the complete solution is $y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$.

This case is illustrated in the following example :

Ex.3. Solve in series $x(1-x)\frac{d^2y}{dx^2} + (1-5x)\frac{dy}{dx} - 4y = 0$

Sol. Since $x_0 = 0$ is a regular singular point therefore we assume that the solution is of the form

$$y = \sum_{r=0}^{\infty} a_r x^{m+r}, \quad a_0 \neq 0 \qquad \ldots\ldots(30)$$

Putting the values for y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation and rearranging the terms, we get

$$\sum_{r=0}^{\infty} a_r (m+r)^2 x^{m+r-1} - \sum_{r=0}^{\infty} a_r (m+r+2)^2 x^{m+r} = 0 \qquad .....(31)$$

Equating to zero, the coefficients of lowest power of x, the indicial equation gives

$$a_0 m^2 = 0 \Rightarrow m = 0, 0 \text{ as } a_0 \neq 0.$$

Since both the values of m are equal so it gives us only one independent solution. Equating to zero, the coefficient of x^{m+r}, we find that

$$a_{r+1} = \left(\frac{m+r+2}{m+r+1}\right)^2 a_r \qquad .....(32)$$

Which gives
$$a_1 = \left(\frac{m+2}{m+1}\right)^2 a_0$$

$$a_2 = \left(\frac{m+3}{m+2}\right)^2 a_1 = \left(\frac{m+3}{m+1}\right)^2 a_0$$

and so on.

Hence the solution is given by

$$y = a_0 x^m \left[1 + \left(\frac{m+2}{m+1}\right)^2 x + \left(\frac{m+3}{m+1}\right)^2 x^2 + \left(\frac{m+4}{m+1}\right)^2 x^3 + \ldots\right] \qquad .....(33)$$

Putting $m = 0$ and replacing a_0 by a in (33) gives

$$y = a\,[1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \ldots] = au \text{ (say)} \qquad .....(34)$$

To get the second solution, we procecd as follows :

Rewriting (33)

$$y = a_0 \left[x^m + \left(\frac{m+2}{m+1}\right)^2 x^{m+1} + \left(\frac{m+3}{m+1}\right)^2 x^{m+2} + \ldots.\right]$$

which on differentiation with respect to x gives

$$\frac{dy}{dx} = a_0 \left[m x^{m-1} + \left(\frac{m+2}{m+1}\right)^2 (m+1) x^m + \left(\frac{m+3}{m+1}\right)^2 (m+2) x^{m+1} + \ldots.\right] \text{ and}$$

$$\frac{d^2y}{dx^2} = a_0 \left[m(m-1) x^{m-2} + \left(\frac{m+2}{m+1}\right)^2 (m+1) m x^{m-1} + \left(\frac{m+3}{m+1}\right)^2 (m+2)(m+1) x^m + \ldots.\right]$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the left hand side of the given equation, we get

$$\left(x - x^2\right) a_0 \left[m(m-1) x^{m-2} + \left(\frac{m+2}{m+1}\right)^2 m(m+1) x^{m-1} + \left(\frac{m+3}{m+1}\right)^2 (m+1)(m+2) x^m + \ldots\right]$$

$$+(1-5x)a_0\left[mx^{m-1}+\left(\frac{m+2}{m+1}\right)^2(m+1)x^m+\left(\frac{m+3}{m+1}\right)^2(m+2)x^{m+1}+...\right]$$

$$-4a_0\left[x^m+\left(\frac{m+2}{m+1}\right)^2x^{m+1}+\left(\frac{m+3}{m+1}\right)^2x^{m+2}+...\right]=a_0m^2x^{m-1}$$

The coefficient of remaining powers of x being zero, it can be easily verified by considering the coefficients one by one.

Thus we may write

$$\left(x-x^2\right)\frac{d^2y}{dx^2}+(1-5x)\frac{dy}{dx}-4y=a_0\,m^2x^{m-1}$$

which on partial differentiation with respect to m, gives

$$\frac{\partial}{\partial m}\left[\left(x-x^2\right)\frac{d^2}{dx^2}+(1-5x)\frac{d}{dx}-4\right]y=2a_0\,mx^{m-1}+a_0m^2x^{m-1}\log x$$

Since the operators are commutative, therefore the above relation may be rewritten as

$$\left[\left(x-x^2\right)\frac{d^2}{dx^2}+(1-5x)\frac{d}{dx}-4\right]\frac{\partial y}{\partial m}=2a_0\,mx^{m-1}+a_0m^2x^{m-1}\log x$$

Putting $m=0$, we get

$$\left[\left(x-x^2\right)\frac{d^2}{dx^2}+(1-5x)\frac{d}{dx}-4\right]\left(\frac{\partial y}{\partial m}\right)_{m=0}$$

which shows that $\left(\frac{\partial y}{\partial m}\right)_{m=0}$ is a second solution of the given differential equation.

Hence differentiating (33) partially with respect to m, we get

$$\frac{\partial y}{\partial m}=a_0x^m\log x\left[1+\left(\frac{m+2}{m+1}\right)^2x+\left(\frac{m+3}{m+1}\right)^2x^2+...\right]$$

$$+a_0x^m\left[2\left(\frac{m+2}{m+1}\right)\left\{\frac{1}{(m+1)}-\frac{(m+2)}{(m+1)^2}\right\}x+...\;+2\left(\frac{m+3}{m+1}\right)\left\{\frac{1}{(m+1)}-\frac{(m+3)}{(m+1)^2}\right\}x^2+...\right]$$

Putting $m=0$ and replacing a_0 by b gives

$$\left(\frac{\partial y}{\partial m}\right)_{m=0}=b\log x\left[1+2^2x+3^2x^2+...\right]+2b\left[2(1-2)x+3(1-3)x^2+....\right]$$

$$\therefore\quad\left(\frac{\partial y}{\partial m}\right)_{m=0}=b\left[u\log x-2\left(1\cdot 2x+2\cdot 3x^2+...\right)\right]=bv,\quad\text{(say)}$$

Thus the required solution is

$$y = au + bv,$$

where a and b are arbitrary constants.

Case III. When the roots m_1, m_2 ($m_1 > m_2$) of the indicial equation are different and differing by an integer and also making a coefficient of y infinite.

Working Rule. If the indicial equation has unequal roots, say m_1 and m_2 ($m_1 > m_2$) differing by an integer and if some of the coefficients of y become infinite when $m = m_2$, we modify the form of y by replacing a_0 by $d_0(m - m_2)$ where $d_0 \neq 0$. Then two independent solutions can be obtained by putting $m = m_2$ in the modified form of y and $\frac{\partial y}{\partial m}$. In this case the solution by putting $m = m_1$ in y is rejected because it only gives a numerical multiple of the solution obtained by putting $m = m_2$ in modified y. Thus the complete solution is

$$y = c_1 (y)_{m_2} + c_2 \left(\frac{\partial y}{\partial m} \right)_{m_2}$$

Ex.4. Solve $\quad \boldsymbol{x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \left(x^2 - 1\right) y = 0}$ ***in series.***

Sol. Given $\quad x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \left(x^2 - 1\right) y = 0 \qquad$ .....(35)

Since $x = 0$ is a regular singular point as $x\, p_1(x)$ and $x^2\, p_2(x)$ tends to a finite limit as $x \to 0$, therefore we assume the solution of the given equation (35) in the form

$$y = \sum_{r=0}^{\infty} a_r x^{m+r}, \quad a_0 \neq 0$$

then $\quad y' = \sum_{r=0}^{\infty} a_r (m+r) x^{m+r-1}, \quad y'' = \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2}$

Substituting for y, y' and y'' in (35), then it gives

$$x^2 \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2} + x \sum_{r=0}^{\infty} a_r (m+r) x^{m+r-1} + \left(x^2 - 1\right) \sum_{r=0}^{\infty} a_r x^{m+r} = 0$$

or $\quad \sum_{r=0}^{\infty} a_r \left[(m+r)(m+r-1) + (m+r) - 1\right] x^{m+r} + \sum_{r=0}^{\infty} a_r x^{m+r+2} = 0$

or $\quad \sum_{r=0}^{\infty} a_r (m+r+1)(m+r-1) x^{m+r} + \sum_{r=0}^{\infty} a_r x^{m+r+2} = 0 \qquad$ .....(36)

which is an identity. Equating to zero, the coefficients of the smallest power of x, namely x^m (put $r = 0$ in the first summation), gives the indicial equation

$$a_0(m + 1)(m - 1) = 0$$

so that $\quad m = 1, -1$ as $a_0 \neq 0 \qquad$ .....(37)

The roots given by (37) are different and differing by an integer.

To obtain the recurrence relation, we equate to zero, the coefficient of x^{m+r} and obtain

$$a_r(m+r+1)(m+r-1) + a_{r-2} = 0$$

or $$a_r = \frac{-1}{(m+r+1)(m+r-1)} a_{r-2} \qquad .....(38)$$

[Since (38) gives the relationship between a_r and a_{r-2}, we proceed to find a_1 as explained in step (v) of § 9.4.1]

Equating to zero, the coefficient of x^{m+1} in (36) (put $r = 1$ in the first summation), we find that

$$a_1(m+2)m = 0, \quad \text{giving } a_1 = 0$$

Since the quantity within the bracket is not zero for any above values of m.

From (38) and $a_1 = 0$, we have

$$a_1 = a_3 = a_5 = a_7 = ... = 0$$

Further, taking $r = 2$ in (38), we get

or $$a_2 = -\frac{1}{(m+3)(m+1)} a_0 \qquad .....(39)$$

For $r = 4$, in (38) and using (39), we find that

$$a_4 = -\frac{1}{(m+5)(m+3)} a_2 = \frac{1}{(m+1)(m+3)^2(m+5)} a_0$$

Putting these values in $y = \sum_{r=0}^{\infty} a_r x^{m+r}$, we get

$$y = a_0 x^m \left\{1 - \frac{1}{(m+1)(m+3)} x^2 + \frac{1}{(m+1)(m+3)^2(m+5)} x^4 - ...\right\} \qquad .....(40)$$

Since the factor $(m + 1)$ appears in the denominator, the coefficient of y will be infinite for $m = -1$.

To overcome this difficulty, we put $a_0 = d_0(m + 1)$, of course the condition $a_0 \neq 0$ is now violated, therefore we assume in its place $d_0 \neq 0$. The above equation (40) becomes

$$y = d_0 x^m \left[(m+1) - \frac{x^2}{(m+3)} + \frac{x^4}{(m+3)^2(m+5)} - ...\right] \qquad .....(41)$$

Putting $m = -1$ and replacing d_0 by a, we get

$$y = ax^{-1}\left[-\frac{1}{2}x^2 + \frac{1}{2^2 \cdot 4}x^4 - ...\right] = au \text{ (say)} \qquad .....(42)$$

The obtain another solution, $m = -1$ will be substituted in $\left(\frac{\partial y}{\partial m}\right)$ obtained from (41).

Now $$\frac{\partial y}{\partial m} = d_0 x^m \log x \left[(m+1) - \frac{x^2}{(m+3)} + \frac{x^4}{(m+3)^2 (m+5)} - \ldots \right]$$

$$+ d_0 x^m \left[1 + \frac{x^2}{(m+3)^2} - \left\{ \frac{2}{(m+3)^3 (m+5)} + \frac{1}{(m+3)^2 (m+5)^2} \right\} x^4 - \ldots \right]$$

Putting $m = -1$, replacing d_0 by b, the second solution will be obtained as

$$\left(\frac{\partial y}{\partial m} \right)_{m=-1} = bx^{-1} \log x \left[-\frac{1}{2} x^2 + \frac{1}{2^2 \cdot 4} x^4 - \ldots \right]$$

$$+ bx^{-1} \left[1 + \frac{1}{2^2} x^2 - \frac{1}{2^2 \cdot 4} \left(\frac{2}{2} + \frac{1}{4} \right) x^4 + \ldots \right]$$

$$= bu \log x + bx^{-1} \left[1 + \frac{x^2}{2^2} - \frac{5}{2^2 \cdot 4^2} x^4 + \ldots \right]$$

$$= b \left[u \log x + x^{-1} \left\{ 1 + \frac{x^2}{2^2} - \frac{5}{2^2 \cdot 4^2} x^4 + \ldots \right\} \right] \qquad \text{.....(43)}$$

$= b\, v$ (say)

Hence the complete solution of the given differential equation is

$$y = au + bv.$$

Note : If we substitute $m = 1$ and $d_0 = \frac{1}{2}$ in (41), we get

$$y = x \left\{ 1 - \frac{1}{2 \cdot 4} x^2 + \frac{x^4}{2 \cdot 4^2 \cdot 6} - \ldots \right\}$$

$$y = \left\{ x - \frac{x^3}{2 \cdot 4} + \frac{x^5}{2 \cdot 4^2 \cdot 6} - . \right\} = -2u$$

which gives no new independent solution.

Case IV. When the roots m_1, m_2 of the indicial equation are different and differing by an integer and also making a coefficient of y indeterminate.

Working Rule. If the indicial equation has two different roots say m_1, m_2 ($m_1 > m_2$) differing by an integer and if one of the coefficients of y become indeterminate when $m = m_2$, the complete solution is given by putting $m = m_2$ in y, which contains two arbitrary constants. In this case, the solution obtained by putting $m = m_1$ in y is rejected because it only gives a numerical multiple of one of the series contained in the first solution.

Ex.5. Solve $x^2\dfrac{d^2y}{dx^2}+\left(x+x^2\right)\dfrac{dy}{dx}+\left(x-9\right)y=0$ ***in series.***

Sol. Since $x_0 = 0$ is a regular singular point, we assume the solution in the form

$$y=\sum_{r=0}^{\infty} a_r x^{m+r}\ ,\quad a_0 \neq 0$$

then
$$x^2\left[\sum_{r=0}^{\infty} a_r\left(m+r\right)\left(m+r-1\right)x^{m+r-2}\right]+\left(x+x^2\right)\left[\sum_{r=0}^{\infty} a_r\left(m+r\right)x^{m+r-1}\right]$$
$$+\left(x-9\right)\left[\sum_{r=0}^{\infty} a_r x^{m+r}\right]=0$$

or
$$\sum_{r=0}^{\infty} a_r\left[\left(m+r\right)\left(m+r-1\right)+\left(m+r\right)-9\right]x^{m+r}+\sum_{r=0}^{\infty} a_r\left(m+r+1\right)x^{m+r+1}=0$$

or
$$\sum_{r=0}^{\infty} a_r\left[\left(m+r-3\right)\left(m+r+3\right)\right]x^{m+r}+\sum_{r=0}^{\infty} a_r\left(m+r+1\right)x^{m+r+1}=0$$

which is an identity. Equating to zero, the coefficient of the smallest power of x, namely x^m (putting $r = 0$ in the first summation), we get

$$a_0(m-3)(m+3)=0,\ \ m=3,\ -3\quad (\because\ \ a_0\neq 0)$$

The roots of the equation are different and differing by an integer. To obtain the recurrence relation, we equate to zero, the coefficient of the general term *i.e.* x^{m+r}, we get

$$a_r(m+r+3)(m+r-3)+a_{r-1}(m+r)=0$$

or
$$a_r=\frac{-\left(m+r\right)}{\left(m+r+3\right)\left(m+r-3\right)}a_{r-1} \qquad .....(44)$$

Taking $m = -3$, we get $\quad a_r=\dfrac{-\left(r-3\right)}{r\left(r-6\right)}a_{r-1}$

Thus for $r = 1$, we have $a_1=\dfrac{-2}{5}a_0$ and for $r = 2, 3, 4, 5$, and 6 we have

$$a_2=-\frac{1}{8}a_1=\frac{2}{5}\cdot\frac{1}{8}a_0$$

$$a_3=0,\ a_4=0,\ a_5=0 \text{ and}$$

$$a_6=\frac{-\left(6-3\right)}{6\left(6-6\right)}a_5=\frac{0}{0}\ \text{(inderminate)}$$

and may be taken as a free constant

Also $\quad a_7=\dfrac{-4}{7}a_6$ and $a_8=\dfrac{-5}{16}a_7=\dfrac{4\cdot 5}{7\cdot 16}a_6$

and so on.

$$\therefore \qquad y = \sum_{r=0}^{\infty} a_r x^{m+r} = x^m \left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \right]$$

$$y = x^{-3} \left[a_0 + a_1 x + a_2 x^2 + a_6 x^6 + a_7 x^7 + a_8 x^8 + \ldots \right]$$

$$= x^{-3} \left[a_0 - \frac{2}{5} a_0 x + \frac{1}{8} \cdot \frac{2}{5} a_0 x^2 \right] + x^{-3} \left[a_6 x^6 - \frac{4}{7} a_6 x^7 + \frac{4 \cdot 5}{7 \cdot 16} a_6 x^8 - \ldots \right]$$

$$\therefore \qquad y = a_0 x^{-3} \left[1 - \frac{2}{5} x + \frac{2}{5} \cdot \frac{1}{8} x^2 \right] + a_6 x^3 \left[1 - \frac{4}{7} x + \frac{4 \cdot 5}{7 \cdot 16} x^2 - \ldots \right]$$

This contains two arbitrary constants a_0 and a_6 and therefore may be taken as the complete solution

Note. If we put $m = 3$ in (44), we get a series solution

$$y = a_0 x^3 \left[1 - \frac{4}{7} x + \frac{4 \cdot 5}{7 \cdot 16} x^2 - \ldots \right]$$

which gives no new independent solution.

9.5 Series Solution in Descending Powers of the Independent Variable

Till now we have obtained series solutions in ascending powers of the independent variable. However, the following cases may arise.

(i) There exists no solution of the form $\sum_{r=0}^{\infty} a_r . x^{m+r}$.

(ii) The usual Frobenius method breaks down.

(iii) The series solution obtained by earlier methods does not converge.

In such cases we obtain the series solution in descending powers of the independent variable. Sometimes, the series solution in descending powers are desirable and are more useful in practice.

Working Rule

(i) We assume a solution of the form $y = \sum_{r=0}^{\infty} a_r . x^{m-r}, \quad a_0 \neq 0$

(ii) For indicial equation, we equate to zero the coefficient of the highest power of x in the identity.

(iii) For recurrence relation, the coefficient of the higher power, in general, in the identity is equated to zero.

To illustrate the method we consider following examples :

Ex.*1. *Integrate in descending series the Legendre's equation or determine the solution of Legendre's equation.

Sol. The differential equation of the form

$$(1 - x^2)\, y'' - 2x\, y' + n(n + 1)\, y = 0 \qquad .....(1)$$

is called the Legendre's equation, where $n \in \mathbf{N}$. Let the series solution of (1) be of the form

$$y = \sum_{r=0}^{\infty} a_r x^{m-r}, \; a_0 \neq 0 \qquad .....(2)$$

Substituting the values of y'', y' and y in the given equation, we get

$$\left(1-x^2\right)\sum_{r=0}^{\infty} a_r(m-r)(m-r-1)x^{m-r-2} - 2x\sum_{r=0}^{\infty} a_r(m-r)x^{m-r-1} + n(n+1)\sum_{r=0}^{\infty} a_r x^{m-r} = 0$$

or $$\sum_{r=0}^{\infty} a_r(m-r)(m-r-1)x^{m-r-2} - \sum_{r=0}^{\infty} a_r(m-r-n)(m-r+n+1)x^{m-r} = 0 \qquad .....(3)$$

which is an identity. Equating to zero, the coefficient of the highest power of x, namely x^m, (put $r = 0$ in the second summation), we get the indicial equation

$$a_0(m-n)(m+n+1) = 0$$

Since $a_0 \neq 0 \Rightarrow m = n, -(n+1)$

which shows that the roots are different.

To obtain the recurrence relation, we equate to zero the coefficient of x^{m-r} and obtain

$$a_{r-2}(m-r+2)(m-r+1) - a_r(m-r-n)(m-r+n+1) = 0$$

or $$a_r = \frac{(m-r+2)(m-r+1)}{(m-r-n)(m-r+n+1)}a_{r-2} \qquad .....(4)$$

Here we need to evaluate a_1. It can be done by equating to zero, the coefficient of the next lower power of x *i.e.* x^{m-1}, which gives

$$a_1(m-1-n)(m+n) = 0$$

$\Rightarrow$ $a_1 = 0$, since the quantity within the bracket is not zero for any above values of m

Since $a_1 = 0$, then from (4), we have $a_3 = a_5 = ... = 0$

Also $$a_2 = \frac{m(m-1)}{(m-n-2)(m+n-1)}a_0$$

$$a_4 = \frac{(m-2)(m-1)}{(m-n-4)(m+n-3)}a_2$$

$$a_4 = \frac{m(m-1)(m-2)(m-3)}{(m-n-2)(m-n-4)(m+n-1)(m+n-3)}a_0$$

Putting these values in (2), the solution is.

$$y = a_0x^m\left[1+\frac{m(m-1)}{(m-n-2)(m+n-1)}x^{-2}+\frac{m(m-1)(m-2)(m-3)}{(m-n-2)(m-n-4)(m+n-1)(m+n-3)}x^{-4}+...\right] \qquad .....(5)$$

When $m = n$, replacing a_0 by a, in (5) one of the solution is

$$y = ax^n\left[1-\frac{n(n-1)}{2(2n-1)}x^{-2}+\frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{-4}-...\right] = au \text{ (say)} \qquad .....(6)$$

When $y = -(n+1)$ and replacing a_0 by b, in (5) the other solution is

$$y = bx^{-n-1}\left[1+\frac{(n+1)(n+2)}{2\cdot(2n+3)}x^{-2}+\frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4(2n+3)(2n+5)}x^{-4}+\ldots\right]= bv \text{ (say)} \quad .....(7)$$

Hence the complete solution is

$$y = au + bv,$$

where a and b are arbitrary constants.

Self Learning Exercise

Fill up the blanks :

(1) The ordinary point of $(x^2 - 1)\, y'' + xy' - y = 0$ is ... ...

(2) For differential equation $2x^2y'' + 7x(x + 1)\, y' - 3y = 0$, $x = 0$ is a ... ... singular point.

(3) The regular and irregular singular points of the differential equation

$x^2(x + 1)^2\, y'' + (x^2 - 1)\, y' + 2y = 0$

are .... and .... respectively

(4) The nature of the point $x = 0$ for the equation $xy'' + y \sin x = 0$ is ..... .

9.7 Summary

In this unit you studied the Frobenius method for finding the solution of a linear differential equation of second order with variable coefficient near ordinary and regular singular points. Various cases of this important method were discussed and illustrated with the help of examples.

9.8 Answers to Self Learning Exercise

(1) $x = 0$ (2) Regular (3) $x = 0$ and $x = -1$ (4) Regular singular

9.9 Exercise

Solve the following differential equations in series :

1. $(1 - x^2)\, y_2 - xy_1 + 4y = 0$

[Ans. $y = a_0\left(1-2x^2\right)+a_1\left(x-\frac{1}{2}x^3-\frac{1}{8}x^5-\frac{1}{16}x^7\ldots\right)$**]**

2. $(1 - x^2)\, y_2 + 2xy_1 + y = 0$

[Ans. $y = a_0\left(1-\frac{1}{2}x^2+\frac{1}{8}x^4+\frac{1}{18}x^6\ldots\right)+a_1\left(x-\frac{1}{2}x^3+\frac{1}{40}x^5+\frac{3}{560}x^7+\ldots\right)$**]**

3. $y_2 + x^2y = 0$

[Ans. $y = a_0\left(1-\frac{1}{3\cdot 4}x^4+\frac{1}{3\cdot 4\cdot 7\cdot 8}x^8+\ldots\right)+a_1\left(x-\frac{1}{4\cdot 5}x^3+\frac{1}{4\cdot 5\cdot 8\cdot 9}x^5\ldots\right)$**]**

4. $(2 + x^2)\, y_2 + xy_1 + (1 + x)\, y = 0$

[Ans. $y = a_0\left(1-\frac{1}{4}x^2-\frac{1}{12}x^3+\frac{5}{96}x^4\ldots\right)+a_1\left(x-\frac{1}{6}x^3-\frac{1}{24}x^4+\frac{1}{24}x^5+\ldots\right)$**]**

5. $2x(1-x)\,y_2 + (1-x)\,y_1 + 3y = 0$

[Ans. $y = a\left(1 - 3x + \frac{3}{1\cdot 3}x^2 + \frac{3}{3.5}x^3 + \frac{3}{5\cdot 7}x^4 + \ldots\right) + bx^{1/2}(1-x)$ **]**

6. $x^2y_2 + xy_1 + (x^2 - n^2)\,y = 0$, when n is not an integer.

[Ans. $y = ax^n\left\{1 - \frac{1}{4(1+n)}x^2 + \frac{1}{4\cdot 8(1+n)(2+n)}x^4 - \ldots\right\}$

$$+bx^{-n}\left\{1 - \frac{1}{4(1-n)}x^2 + \frac{1}{4\cdot 8(1-n)(2-n)}x^4 - \ldots\right\}$$ **]**

7. $(2x + x^3)\,y_2 - y_1 - 6xy = 0$

[Ans. $y = a\left(1 + 3x^2 + \frac{3}{5}x^4 - \frac{3\cdot 1}{5\cdot 9}x^6 + \ldots\right) + bx^{3/2}\left(1 + \frac{3}{8}x^2 - \frac{3\cdot 1}{8\cdot 16}x^4 + \frac{3\cdot 1\cdot 5}{8\cdot 16\cdot 24}x^6 + \ldots\right)$ **]**

8. $9x(1-x)\,y_2 - 12y_1 + 4y = 0$

[Ans. $y = a\left(1 + \frac{1}{3}x + \frac{1\cdot 4}{3.6}x^2 + \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}x^3 + \ldots\right)$

$$+bx^{7/3}\left(1 + \frac{8}{10}x - \frac{8\cdot 11}{10\cdot 13}x^2 + \frac{8\cdot 11\cdot 14}{10\cdot 13\cdot 16}x^3 + \ldots\right)$$ **]**

9. $4xy_2 + 2y_1 + y = 0$

[Ans. $y = a\left(1 - \frac{x}{\lfloor 2} + \frac{x^2}{\lfloor 4}\ldots\right) + bx^{1/2}\left(1 - \frac{x}{\lfloor 3} + \frac{x^2}{\lfloor 5}\ldots\right)$ **]**

10. $x(1-x)\,y_2 + 3y_1 + 2y = 0$

[Ans. $y = a\left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots\right) + b\left(x + \frac{x^3}{6} + \frac{1}{12}x^4 + \ldots\right)$ **]**

11. $xy_2 + y_1 + xy = 0$

[Ans. $y = ay_1 + by_2$, where $y_1 = \left(1 - \frac{1}{2^2}x^2 + \frac{1}{2^2\cdot 4^2}x^4 - \ldots\right)$ and

$$y_2 = y_1\log x + \left(\frac{1}{2^2}x^2 - \frac{1}{2^2\cdot 4^2}\left(1 + \frac{1}{2}\right)x^4 + \frac{1}{2^2\cdot 4^2\cdot 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^6 + \ldots\right)$$ **]**

12. $(x - x^2)\,y_2 + (1-x)y_1 - y = 0$

[Ans. $y = (a + b\log x)\left(1 + x + \frac{2}{4}x^2 + \frac{2\cdot 5}{4\cdot 9}x^3 + \ldots\right) + b\left(-2x - x^2 - \ldots\right)$ **]**

13. $xy_2 + (1+x)y_1 + 2y = 0$

[Ans. $y = (a + b\log x)\left(1 - 2x + \frac{3}{\lfloor 2}x^2 - \frac{4}{\lfloor 3}x^3 + \ldots\right) + b\left\{2\left(2 - \frac{1}{2}\right)x - \frac{3}{\lfloor 2}\left(2 + \frac{1}{2} - \frac{1}{3}\right)x^2 + \ldots\right\}$ **]**

14. $x(1-x^2)\,y_2 + (1-3x^2)y_1 - xy = 0$

[Ans. $y = (a + b\log x)\left(1 + \frac{1^2}{2^2}x^2 + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}x^4 + \ldots\right) + b\left(\frac{1}{4}x - \frac{21}{128}x^4 + \ldots\right)$**]**

15. $x(1-x)\,y_2 - (1+3x)y_1 - y = 0$

[Ans. $y = (a + b\log x)\left(-1\cdot 2x^2 - 2\cdot 3x^3 \ldots\right) + b\left(1 - x - 5x^2 \ldots\right)$**]**

16. $xy_2 + xy_1 + (x^2 - 4)y = 0$

[Ans. $y = (a + b\log x)\,x^{-2}\left\{-\frac{1}{2^2\cdot 4}x^4 + \frac{1}{2^3\cdot 4\cdot 6}x^6 - \frac{1}{2^3\cdot 4^2\cdot 6\cdot 8}x^8 + \ldots\right\}$

$+bx^{-2}\left(1 + \frac{1}{2^2}x^2 + \frac{1}{2^2\cdot 4^2}x^4 - \frac{1}{2^2\cdot 4^2\cdot 6^2}x^6 - \ldots\right)$**]**

17. $x(1-x)\,y_2 - 3xy_1 - y = 0$

[Ans. $y = (a + b\log x)\left(x + 2x^2 + 3x^3 + \ldots\right) + b\left(1 + x + x^2 + \ldots\right)$**]**

18. $x^2y_2 + x(1+2x)y_1 - 4y = 0$

[Ans. $y = a_0x^{-2}\left(1 - \frac{4}{3}x + \frac{2}{3}x^2\right) + a_4x^2\left(1 - \frac{4}{5}x + \frac{4}{10}x^2 - \ldots\right)$**]**

19. $(1-x^2)y_2 + 2xy_1 + y = 0$

[Ans. $y = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \ldots\right) + a_1\left(x - \frac{x^3}{2} + \frac{1}{40}x^5 - \ldots\right)$**]**

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Unit 10 : Gauss Hypergeometric Function: its Properties And Integral Representation

Structure of the Unit

10.0 Objective

The aim of this unit is to study a special function known as Gauss hypergeometric function. Also its special cases, properties, convergence conditions and summation theorems such as Gauss's theorem, Kummer's theorem and Vandermonde's theorem are obtained.

10.1 Introduction

The series

$$1+\frac{ab}{c}z+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{\underline{|2}}+\frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{\underline{|3}}+\ldots \qquad .....(1)$$

is called the Gauss series or the Ordinary hypergeometric series. It is usually represented by the symbol ${}_2F_1(a, b ; c ; z)$, The three quantities a, b and c are called the parameters and z is the variable of the series. All these four quantities may be any number, real or complex. In the notation ${}_2F_1(.)$, the left suffix

2 and the right suffix 1 indicate the number of parameters in the numerator and denominator respectively. If either of the parameters a or b (or both) is a negative integer, the series terminates i.e. it has only a finite number of terms and becomes in fact a polynomial. Also when c is zero or a negative integer, the series is not defined.

C.F. Gauss carried out an exhaustive study of this function in a systematic way and Euler discovered many properties of the function.

The function has its importance because of its application in solving various problems arising in physical and engineering sciences. It is interesting to note that apart from the elementary functions such as exponential function, logarithmic function, sine and cosine functions etc., it is also possible to derive Bessel's functions, Kummer's confluent hypergeometric function, Bessel polynomials, Hermite polynomials, Jacobi polynomials etc. either as a limiting case or as a special case of this function.

If we introduce the conventional notation (Pochammer symbol)

$$(\alpha)_n = \frac{\Gamma(a+n)}{\Gamma(\alpha)} = \alpha(\alpha+1)(\alpha+2)...(\alpha+n-1),\ n \geq 1 \qquad .....(2)$$

and $(\alpha)_0 = 1, \alpha \neq 0$,

then the equation (1) can be written in the contracted form

$$_2F_1(a,b;c;z) \text{ or } {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{\lfloor n} \qquad .....(3)$$

As pointed out earlier, in general a, b, and c are complex parameters and z is a complex variable. If a or b is a negative integer then series terminates. Also c is neither zero nor a negative integer *i.e.* $c \neq 0, -1, -2,$ .....

From (1), it follows easily that

(i) $_2F_1(a, b; c; 0) = 1$

(ii) $_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z)$

The last property indicates that the hypergeometric function is **symmetric** in the upper parameters a and b.

10.2 Convergence of the Series in (3)

To test the convergence of the series in (3), let us apply the **D' Alembert's ratio test**. We see that

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}} \frac{z^{n+1}}{\lfloor n+1} \cdot \frac{(c)_n \lfloor n}{(a)_n (b)_n z^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(a+n)(b+n)}{(c+n)} \cdot \frac{z}{n+1}\right|$$

$$= |z|,$$

so long as non of a, b, c is zero or a negative interer.

Therefore, the series converges absolutely within the circle of convergence if $|z|<1$ and diverges outside the circle of convergence i.e. $|z|>1$, provided that c is neither zero nor a negative integer. If either or both of a and b is zero or a negative integer, the series terminates, and convergence does not enter the discussion.

For $|z|=1$, i.e. on the circle of convergence, the test fails. In this ease, let us compare this series with the series

$$\Sigma v_n = \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}},$$

where $2\delta = Re\,(c-a-b) > 0.$

Since $$\lim_{n\to\infty}\left|\frac{u_n}{v_n}\right| = \left|\frac{(a)_n (b)_n}{(c)_n \lfloor n} \cdot n^{1+\delta}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(a)_n}{\lfloor n-1 \cdot n^a} \cdot \frac{(b)_n}{\lfloor n-1 \cdot n^b} \frac{\lfloor n-1 \cdot n^c}{(c)_n} \frac{\lfloor n-1\, n^{1+\delta}}{\lfloor n\; n^{c-a-b}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\Gamma(a+n)}{\lfloor n-1 \cdot n^a} \cdot \frac{1}{\Gamma(a)} \cdot \frac{\Gamma(b+n)}{\lfloor n-1 \cdot n^b} \cdot \frac{1}{\Gamma(b)} \cdot \frac{\lfloor n-1\, n^c \Gamma(c)}{\Gamma(c+n)} \cdot \frac{\lfloor n-1}{\lfloor n} \cdot \frac{n^{1+\delta}}{n^{c-a-b}}\right|$$

But we know that $\lim_{n\to\infty} \frac{\lfloor n-1\, n^z}{\Gamma(z+n)} = 1$

therefore $$\lim_{n\to\infty}\left|\frac{u_n}{v_n}\right| = \left|\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\right| \cdot \lim_{n\to\infty}\left|\frac{1}{n^{(c-a-b-\delta)}}\right| = 0,$$

because Re $(c-a-b-\delta) = 2\delta - \delta > 0$, therefore the series in (3) is absolutely convergent on $|z| = 1$ when Re $(c-a-b) > 0$.

To summarise, we conclude that the hypergeometric series (3) or (1) is

(a) absolutely convergent within the circle of convergence $|z| < 1$

(b) divergent outside the circle of convergence $|z| > 1$.

(c) for $|z| = 1$ *i.e.* on the circle of convergence, it converges absolutely if Re $(c-a-b) > 0$. It also converges conditionally for $z = -1$ if $-1 < Re\,(c-a-\text{b}) \le 0$, and divergent if Re $(c-a-b) \le -1$.

10.3 Special cases of the Gauss function

When $a = 1$, $b = c$, the R.H.S. of (1) reduces to

$$1 + z + z^2 + \ldots\ldots = \frac{1}{1-z}, \quad |z| < 1$$

which is simply a geometric series. This is why (1) called the **hypergeometric series.**

Most of the elementary functions which occur in Mathematical Physics, can be expressed in terms of the Gauss function. For example,

(i) $${}_2F_1\,(a,b;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{\lfloor n} z^n = \sum_{n=0}^{\infty} (-a)(-a-1)\ldots.(-a-n+1)\frac{(-z)^n}{\lfloor n}$$

or $\quad {}_2F_1$ (a, b; b; z) = $(1-z)^{-a}$

This is simply a statement of the Binomial theorem for $|z| < 1$.

(ii) $${}_2F_1\,(1,1;2;-z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{1+n} = \frac{1}{z}\log(1+z)$$

(iii) For $|z| < 1$, $${}_2F_1\,(\frac{1}{2},1;\frac{3}{2};z^2) = \frac{1}{2z}\log\frac{(1+z)}{(1-z)}$$

(iv) Since $${}_2F_1\,(1;b;1;\frac{z}{b}) = 1 + \sum_{n=1}^{\infty}\left(1+\frac{1}{b}\right)\left(1+\frac{2}{b}\right)\ldots.\left(1+\left(\frac{n-1}{b}\right)\right)\frac{z^n}{\lfloor n},$$

therefore, $$\lim_{b\to 0}\left\{{}_2F_1\left(1,b;1;\frac{z}{b}\right)\right\} = \sum_{n=0}^{\infty}\frac{z^n}{\lfloor n} = e^z$$

(v) $${}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2\right) = \frac{1}{z}\sin^{-1} z$$

(vi) $${}_2F_1\left[\frac{1}{2},1;\frac{3}{2};-z^2\right] = \frac{1}{z}\tan^{-1} z$$

The Legendre polynomial $P_n(x)$ is defined as the coefficient of z^n in the expansion, in ascending powers of z, of $(1-2xz+z^2)^{-1/2}$. By direct expansion, we can prove that the coefficient is in fact ${}_2F_1\left[-n,1+n;1;\frac{1}{2}-\frac{1}{2}x\right] = P_n(x)$. This result is known as Murphy's formula.

Other elementary special cases are

$${}_2F_1\left[a,a+\frac{1}{2};\frac{1}{2};z\right] = \frac{1}{2}\left(1+\sqrt{z}\right)^{-2a} + \frac{1}{2}\left(1-\sqrt{z}\right)^{-2a}$$

$${}_2F_1\left[a-\frac{1}{2},a;2a;z\right] = \left[\frac{1}{2}+\frac{1}{2}\sqrt{1-z}\right]^{1-2a}$$

and $${}_2F_1\left[2a,a+1;a;z\right] = (1+z)/(1-z)^{2a+1}$$

10.4 Integral Representation

If $|z| < 1$ and if $Re(c) > Re(b) > 0$, then

$$B(b, c-b)\ {}_2F_1(a, b ; c ; z) = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt$$

or

$${}_2F_1(\text{a,b;c;z}) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt \qquad .....(1)$$

Proof. Let

$$I = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt$$

$$= \int_0^1 t^{b-1}(1-t)^{c-b-1}\sum_{r=0}^{\infty}\frac{(a)_r (zt)^r}{\lfloor r}dt$$

Now interchanging the order of integration and summation, we see that

$$I = \sum_{r=0}^{\infty}\frac{(a)_r z^r}{\lfloor r}\int_0^1 t^{b+r-1}(1-t)^{c-b-1}\, dt$$

$$= \sum_{r=0}^{\infty}\frac{\Gamma(b+r)\Gamma(c-b)}{\Gamma(c+r)}\cdot\frac{(a)_r z^r}{\lfloor r}$$

$$= \frac{\Gamma(b)}{\Gamma(c)}\Gamma(c-b)\sum_{r=0}^{\infty}\frac{(a)_r (b)_r}{\Gamma(c)_r}\cdot\frac{z^r}{\lfloor r}$$

$$= B(b, c-b)\ {}_2F_1(a, b ; c ; z)$$

10.4.1 Deductions from integral representation

As a consequence of equation (1), we derive the **Gauss's theorem** which gives rise to **Vandermonde's theorem** of the hypergeometric function. **Kummer's** theorem is also derived. These theorems are of great importance in the study of various special functions of mathematical physics.

***(a)* Gauss's theorem.** If ***Re* (*c* – *a* – *b*) > 0, *Re* (*c*) > 0, then**

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

Prof. Putting $z = 1$ in the equation (1), we get

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-a-b-1}\, dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)}$$

$$\therefore \quad {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad .....(2)$$

***(b)* Vandermonde's theorem**

$$ {}_2F_1(-n,b;c;1) = \frac{(c-b)_n}{(c)_n} $$

Prof. If we make $a = -n$ in eq. (2), where n is a positive integer, then we get

$$ {}_2F_1(-n,b;c;1) = \frac{\Gamma(c)\Gamma(c-b+n)_n}{\Gamma(c+n)\Gamma(c-b)} = \frac{(c-b)_n}{(c)_n} $$

***(c)* Kummer's Theorem**

$$ {}_2F_1(a,b;1-a+b;-1) = \frac{\Gamma(1-a+b)\Gamma\left(1+\frac{b}{2}\right)}{\Gamma(1+b)\Gamma\left(1+\frac{b}{2}-a\right)} \qquad .....(3) $$

Prof. To prove (3), we put $z-1$ and $c = 1-a+b$ in equation (1), we abtain

$$ {}_2F_1(a,b;1-a+b;-1) = \frac{\Gamma(1-a+b)}{\Gamma(b)\Gamma(1-a)}\int_0^1 t^{b-1}\left(1-t^2\right)^{-a} dt \qquad .....(4) $$

Putting $t^2 = u$ in the above equation (4), we get

$$ {}_2F_1(a,b;1-a+b;-1) = \frac{\Gamma(1-a+b)}{2\Gamma(b)\Gamma(1-a)}\int_0^1 u^{(b/2)-1}(1-u)^{1-a-1} du $$

$$ = \frac{\Gamma(1-a+b)}{2\Gamma(b)\Gamma(1-a)} \cdot \frac{\Gamma\left(\frac{b}{2}\right)\Gamma(1-a)}{\Gamma\left(\frac{b}{2}+1-a\right)} $$

$$ \therefore \quad {}_2F_1(a,b;1-a+b;-1) = \frac{\Gamma[1+(b/2)]}{\Gamma(1+b)} \frac{\Gamma(1-a+b)}{\Gamma(1-a+b/2)} $$

10.5 Gauss's Hypergeometric Differential Equation and its Solution

Let $\theta = z\frac{d}{dz}$. Then $\theta z^n = nz^n$

Therefore, $\theta(\theta + c - 1) z^n = n(n + c - 1) z^n$.

Now $$ y = {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \lfloor n} z^n $$

We have $$ \theta(\theta+c-1)y = \sum_{n=0}^{\infty} \frac{n(n+c-1)(a)_n (b)_n}{(c)_n \lfloor n} z^n $$

$$ = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_{n-1} \lfloor n-1} z^n $$

$$= \sum_{n=0}^{\infty} \frac{(a)_{n+1} (b)_{n+1}}{(c)_n \lfloor n} z^{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(a+n)(b+n)(a)_n (b)_n}{(c)_n \lfloor n} z^n$$

$$= z\,(\theta + a)\,(\theta + b)\,y$$

$$\left(\text{since } (\theta + a) y = \sum_{n=0}^{\infty} \frac{(a+n)(a)_n (b)_n}{(c)_n \lfloor n} z^n \right)$$

Hence $y = {}_2F_1(a, b; c; z)$ is a solution of differential equation

$$\left[\theta(\theta + c - 1) - z(\theta + a)(\theta + b)\right] y = 0, \quad \theta = z \cdot \frac{d}{dz}$$

The above equation can be easily written in the following form

$$\boldsymbol{z(1-z)\frac{d^2y}{dz^2} + \{c - (1+a+b)z\}\frac{dy}{dz} - ab\ y = 0} \qquad .....(1)$$

(by employing the relations $\theta y = zy'$ and $\theta(\theta - 1)\, y = z^2 y''$) is known as Gauss's hypergeometric differential equation.

From the theory of differential equation, it follows that the regular singular points of the above equation (1) are:

(i) $z = 0$ with exponents $0,\ 1 - c$

(ii) $z = 1$ with exponents $0,\ c - a - b$

(iii) $z = \infty$ with exponents a, b.

For details of the solution of the differential equation (1), students are advised to reter Ex. 2 in §9.4 of the last unit.

10.6 Two summation Theorems

In this section, we discuss two theorems concerning elementary series manipulations which are important techniques in establishing several transformation formulae, summation formulae and in investigating several other properties of hypergeometric functions, Bessel's functions and Orthogonal polynomials etc.

10.6.1 Theorem 1.

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha(n, m) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \alpha(n, m-n) \qquad .....(1)$$

and.

$$\sum_{m=0}^{\infty} \sum_{n=0}^{m} \beta(n, m) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \beta(n, m+n) \qquad .....(2)$$

Proof. Consider the L.H.S. of the equation (1) in which the term u^{m+n} has been inserted for convenience *i.e.*

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha(n, m)\, u^{m+n} \qquad .....(3)$$

Let us collect the powers of u in (3). We introduce new indices of summation s and r by

$$n = r,\ m = s - r \qquad \text{.....(4)}$$

so that $$n + m = s \qquad \text{.....(5)}$$

The indices n and m now satisfy the inequalities $m \geq 0,\ n \geq 0$.

From (4) and (5), it follows that $s - r \geq 0,\ r \geq 0$ or $0 \leq r \leq s$

provided that s is restricted to be a non-negative integer. Thus we have

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha(n,m)\, u^{m+n} = \sum_{s=0}^{\infty} \sum_{r=0}^{s} \alpha(r, s-r)\, u^{s}$$

Now putting $u = 1$ and replacing the dummy indices r and s on the right by n and m respectively, we get the required result.

In Theorem 1, equation (2) is merely written in reverse order; hence no separate proof is needed.

Theorem 2.

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha(n,m) = \sum_{m=0}^{\infty} \sum_{n=0}^{[m/2]} \alpha(n, m-2n) \qquad \text{.....(6)}$$

and $$\sum_{m=0}^{\infty} \sum_{n=0}^{[m/2]} \beta(n,m) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \beta(n, m+2n) \qquad \text{.....(7)}$$

where the symbol $\sum_{l=0}^{[m/2]}$ indicates that n runs from 0 to the greatest integer less than or equal to $m/2$.

Proof. If we consider

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha(n,m) u^{m+2n}$$

in which u^{m+2n} is inserted for convenience, i.e. $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha(n, m) u^{m+2n}$ and taking $n = r$ and $m = s - 2r$ so that $m + 2n = s$.

Since $m \geq 0,\ n \geq 0,\ s - 2r \geq 0,\ r \geq 0$ from which $0 \leq 2r \leq s$ and $s \geq 0$.

Since $0 \leq r \leq \frac{s}{2}$ and r is integral, the index r runs from 0 to the greatest integer $s/2$. Thus we obtain

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha(n, m) u^{m+2n} = \sum_{s=0}^{\infty} \sum_{r=0}^{[s/2]} \alpha(r, s-2r) u^{s}$$

Now putting $u = 1$ and replacing the dummy indices r and s on the right by n and m respectively, we get the required result (6). Equation (7) is written in reverse order. If we combine the above two theorems, we find that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{m} \gamma(n,m) = \sum_{m=0}^{\infty} \sum_{n=0}^{[m/2]} \gamma(n, m-n)$$

Ex.*1. *Prove that $\quad {}_2F_1\left[\frac{a}{2},\frac{a}{2}+\frac{1}{2};\frac{1}{2};z^2\right] = \frac{1}{2}\left[(1-z)^{-a}+(1-z)^{-a}\right]$

Sol. Taking $\quad$ R.H.S. $=\frac{1}{2}\left[(1-z)^{-a}+(1-z)^{-a}\right]$

$$=\frac{1}{2}\Bigg[\Bigg\{1+az+\frac{a(a+1)}{\underline{|2}}\cdot z^2+\frac{a(a+1)(a+2)}{\underline{|3}}\cdot z^3+\frac{a(a+1)(a+2)(a+3)}{\underline{|4}}z^4$$

$$+\frac{a(a+1)(a+2)(a+3)(a+4)}{\underline{|5}}z^5+\ldots\Bigg\}$$

$$+\Bigg\{1-az+\frac{a(a+1)}{\underline{|2}}\cdot z^2-\frac{a(a+1)(a+2)}{\underline{|3}}\cdot z^3+\frac{a(a+1)(a+2)(a+3)}{\underline{|4}}z^4$$

$$-\frac{a(a+1)(a+2)(a+3)(a+4)}{\underline{|5}}z^5+\ldots\Bigg\}\Bigg]$$

$$=\frac{1}{2}\left[2+a(a+1)z^2+\frac{a(a+1)(a+2)(a+3)}{12}z^4+\ldots.\infty\right]$$

$$=\left[1+\frac{a}{2}(a+1)z^2+\frac{a(a+1)(a+2)(a+3)}{2\cdot\ 2\cdot\ 2\cdot\ 3}z^4+\ldots.\infty\right]$$

$$=\left[1+\frac{\frac{a}{2}\left(\frac{a}{2}+\frac{1}{2}\right)}{\left(\frac{1}{2}\right)}z^2+\frac{\left(\frac{a}{2}\right)\left(\frac{a}{2}+\frac{1}{2}\right)\left(\frac{a}{2}+1\right)\left(\frac{a}{2}+\frac{3}{2}\right)}{\frac{1}{2}\cdot\frac{3}{2}\cdot 2\cdot 1}\left(z^2\right)^2+\ldots.\right]$$

$$={}_2F_1\left[\frac{a}{2},\frac{a}{2}+\frac{1}{2};\frac{1}{2};z^2\right]=\text{ L.H.S.}$$

Ex.*2. *Establish the result

$$\boldsymbol{{}_2F_1[-n,a+n\ ;\ c\ ;1]=\frac{(-1)^n(1+a-c)_n}{(c)_n}}$$

Sol. Here $\quad$ L.H.S. $={}_2F_1[-n,a+n;c;1]$

$$=\frac{\Gamma(c)\Gamma(c-a)}{\Gamma(c+n)\Gamma(c-a-n)} \qquad \text{(by Gauss's summation Theorem)}$$

$$=(-1)^n\frac{\Gamma(1-c+a+n)}{\Gamma(1-c+a)}$$

$$=\frac{(-1)^n(1+a-c)_n}{(c)_n}$$

Hence proved.

Ex.3. Prove that

$$\mathrm{B}(\lambda, c-\lambda)\, {}_2F_1(a, b; c; z) = \int_0^1 t^{\lambda-1}(1-t)^{c-\lambda-1}\, {}_2F_1(a, b; \lambda; zt)\, dt$$

***where** $|z| < 1$, $\lambda > 0$, $c - \lambda > 0$.*

Sol. Let

$$I = \int_0^1 t^{\lambda-1}(1-t)^{c-\lambda-1}\, {}_2F_1(a, b; \lambda; zt)\, dt$$

$$= \int_0^1 t^{\lambda-1}(1-t)^{c-\lambda-1} \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(\lambda)_r} \cdot \frac{(zt)^r}{\lfloor r} dt$$

$$= \sum_{r=0}^{\infty} \frac{z^r}{\lfloor r} \frac{(a)_r (b)_r}{(\lambda)_r} \int_0^1 t^{\lambda+r-1}(1-t)^{c-\lambda-1} dt$$

$$= \sum_{r=0}^{\infty} \frac{z^r}{\lfloor r} \frac{(a)_r (b)_r}{(\lambda)_r} \frac{\Gamma(\lambda+r)\Gamma(c-\lambda)}{\Gamma(c+r)}$$

$$= \frac{\Gamma(\lambda)\Gamma(c-\lambda)}{\Gamma(c)} \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c)_r} \cdot \frac{z^r}{\lfloor r}$$

$$= B(\lambda, c-\lambda)\, {}_2F_1(a, b; c; z)$$

***Ex.4. Show that if** $b > 0$,*

$${}_2F_1(a, b; 2b; z) = \frac{2\{1-(z/2)\}^{-a}}{2^{2b-1} B(b,b)} \int_0^{\pi/2} (\sin\phi)^{2b-1}\left[(1+\xi\cos\phi)^{-a} + (1-\xi\cos\phi)^{-a}\right] d\phi$$

***where** $\xi = \dfrac{z}{2-z}$.*

Deduce that

$${}_2F_1(a, b; 2b; z) = 2\left(1-\frac{1}{2}z\right)^{-a} {}_2F_1\left(\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; b+\frac{1}{2}; \xi^2\right)$$

Sol. We know that if $|z| < 1$ and if $Re(c) > Re(b) > 0$, then

$$B(b, c-b)\, {}_2F_1(a, b; c; z) = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt \quad .....(9)$$

For $c = 2b$, it reduces to

$${}_2F_1(a, b; 2b; z) = \frac{1}{B(b,b)} \int_0^1 t^{b-1}(1-t)^{b-1}(1-tz)^{-a} dt \quad .....(10)$$

Putting $t = \sin^2\theta$, we have

$${}_2F_1(a, b; 2b; z) = \frac{2}{B(b,b)} \int_0^{\pi/2} (\sin\theta)^{2b-1}(\cos\theta)^{2b-1}\left(1 - z\sin^2\theta\right)^{-a} d\theta$$

$$= \frac{2}{B(b,b)} \int_0^{\pi/2} (\sin\theta)^{2b-1} (\cos\theta)^{2b-1} \left[1 - z\left(\frac{1-\cos 2\theta}{2}\right)\right]^{-a} d\theta$$

$$= \frac{2}{B(b,b)} \int_0^{\pi/2} (\sin\theta)^{2b-1} (\cos\theta)^{2b-1} \left[\frac{2 - z + z\cos 2\theta}{2}\right]^{-a} d\theta \cdot$$

$$= \frac{2\left[1-(z/2)\right]^{-a}}{B(b,b)} \int_0^{\pi/2} (\sin\theta)^{2b-1} (\cos\theta)^{2b-1} \left[1 + \frac{z}{2-z}\cos 2\theta\right]^{-a} d\theta$$

$$= \frac{2\left[1-(z/2)\right]^{-a}}{2^{2b-1} B(b,b)} \int_0^{\pi/2} (\sin 2\theta)^{2b-1} (1 + \xi\cos 2\theta)^{-a} d\theta \qquad .....(11)$$

where $\xi = \frac{z}{2-z}$. If we put $2\theta = \phi$, then (11) becomes

$${}_2F_1(a, b; 2b; z) = \frac{\left[1-(z/2)\right]^{-a}}{2^{2b-1} B(b,b)} \int_0^{\pi} (\sin\phi)^{2b-1} (1 + \xi\cos\phi)^{-a} d\phi \qquad .....(12)$$

In the same way, if we substitute $t = \cos^2\theta$ in (10), we get

$${}_2F_1(a, b; 2b; z) = \frac{\left[1-(z/2)\right]^{-a}}{2^{2b-1} B(b,b)} \int_0^{\pi} (\sin\phi)^{2b-1} (1 - \xi\cos\phi)^{-a} d\phi \qquad .....(13)$$

Adding (12) and (13) and applying the property of the definite integral, viz.

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x), \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

we obtain the desired result

$${}_2F_1(a, b; 2b; z) = \frac{\left[1-(z/2)\right]^{-a}}{2^{2b-1} B(b,b)} \int_0^{\pi/2} (\sin\phi)^{2b-1} \left[(1 + \xi\cos\phi)^{-a} + (1 - \xi\cos\phi)^{-a}\right] d\phi$$

To deduce the second part, we find from example 1 that

$$\left[(1 + \xi\cos\phi)^{-a} + (1 - \xi\cos\phi)^{-a}\right] = 2 \cdot {}_2F_1\left(\frac{a}{2}, \frac{a}{2} + \frac{1}{2}; \frac{1}{2}; \xi^2\cos^2\phi\right)$$

Hence $${}_2F_1(a, b; 2b; z) = \frac{4\left[1-(z/2)\right]^{-a}}{2^{2b-1} B(b,b)} \int_0^{\pi/2} (\sin\phi)^{2b-1} \, {}_2F_1\left(\frac{a}{2}, \frac{a}{2} + \frac{1}{2}; \frac{1}{2}; \xi^2\cos^2\phi\right) d\phi$$

Expanding ${}_2F_1$ ($\xi^2 \cos^2\phi$) in terms of its series and integrating with the help of beta function formula, we have

$$ {}_2F_1\,(a,\,b;\,2b;\,z) = \frac{4\left[1-(z/2)\right]^{-a}}{2^{2b-1}B(b,b)} \sum_{r=0}^{\infty} \frac{(a/2)_r \{(a+1)/2\}_r}{(1/2)_r \lfloor r} \xi^{2r} \times \int_0^{\pi/2} \sin^{2b-1}\phi \cos^{2r}\phi \, d\phi $$

$$ = \frac{4\left[1-(z/2)\right]^{-a}}{2^{2b-1}B(b,b)} \sum_{r=0}^{\infty} \frac{(a/2)_r \{(a+1)/2\}_r}{(1/2)_r \lfloor r} \xi^{2r} \cdot \frac{\Gamma(b)\Gamma(r+1/2)}{2\Gamma(b+r+1/2)} $$

Applying Legendre's duplication formula, we get

$$ = 2\left(1-\frac{z}{2}\right)^{-a} \sum_{r=0}^{\infty} \frac{(a/2)_r \left(\frac{a+1}{2}\right)_r}{\left(b+\frac{1}{2}\right)_r} \cdot \frac{\xi^{2r}}{\lfloor r} $$

$$ \therefore \quad {}_2F_1\,(a,\,b;\,2b;\,z) = 2\left(1-\frac{z}{2}\right)^{-a} {}_2F_1\left(\frac{a}{2},\,\frac{a}{2}+\frac{1}{2}\,;\,b+\frac{1}{2}\,;\,\xi^2\right) $$

Ex.5. Show that if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$,

$$ \sin nx = n \sin x \; {}_2F_1\left(\frac{1}{2}+\frac{1}{2}n,\,\frac{1}{2}-\frac{1}{2}n\,;\,\frac{3}{2}\,;\,\sin^2 x\right) $$

and $\cos nx = {}_2F_1\left(\frac{n}{2},\frac{-n}{2}\,;\,\frac{1}{2}\,;\,\sin^2 x\right)$.

Sol. We know that sin nx and cos nx satisfy the following differential equation

$$ \frac{d^2y}{dx^2} + n^2 y = 0 \qquad .....(14) $$

Let us transform (14) by the substitution $u = \sin^2 x$. Then

$$ \frac{du}{dx} = \sin 2x \text{ and } \frac{d^2u}{dx^2} = 2\cos 2x $$

Now,
$$ \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \sin 2x \frac{dy}{du} $$

$$ \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\sin 2x \cdot \frac{dy}{du}\right) $$

$$ = 2\cos 2x \frac{dy}{du} + \sin 2x \cdot \frac{d^2y}{du^2}\cdot\frac{du}{dx} $$

$$ = 2\cos 2x \frac{dy}{du} + \sin^2 2x \frac{d^2y}{du^2} $$

$$ = 2\left(1-2\sin^2 x\right)\frac{dy}{du} + 4\sin^2 x \cos^2 x \cdot \frac{d^2y}{du^2} $$

$$\therefore \qquad \frac{d^2y}{dx^2} = 2(1-2u)\frac{dy}{du} + 4u(1-u)\frac{d^2y}{du^2}$$

Substituting the value of $\frac{d^2y}{dx^2}$ in (14), it becomes

$$u(1-u)\frac{d^2y}{du^2} + \left(\frac{1}{2} - u\right)\frac{dy}{du} + \frac{n^2}{4}y = 0$$

The above equation may be written as

$$u(1-u)\frac{d^2y}{du^2} + \left[\frac{1}{2} - \left(1 + \frac{n}{2} - \frac{n}{2}\right)u\right]\frac{dy}{du} - \left(\frac{n}{2}\right)\left(\frac{-n}{2}\right)y = 0$$

which is a Gauss's hypergeometric equation with $a = \frac{n}{2}, b = \frac{-n}{2}, c = \frac{1}{2}$. Hence the general solution of (14) is given by

$$y = A\,{}_2F_1\left(\frac{n}{2}, \frac{-n}{2}; \frac{1}{2}\ ;\ \sin^2 x\right) + B\sin x\ {}_2F_1\left(\frac{1}{2} + \frac{1}{2}n, \frac{1}{2} - \frac{1}{2}n\ ;\ \frac{3}{2}\ ;\ \sin^2 x\right)$$

Since $\sin nx$ is the solution

$$\therefore \qquad \sin nx = A\ {}_2F_1\left(\frac{n}{2}, \frac{-n}{2}\ ;\ \frac{1}{2}\ ;\ \sin^2 x\right) + B\sin x\ {}_2F_1\left(\frac{1}{2} + \frac{1}{2}n,\ \frac{1}{2} - \frac{1}{2}n\ ;\ \frac{3}{2}\ ;\ \sin^2 x\right) \qquad .....(15)$$

For $x = 0$, equation (15) gives $A = 0$

Further

$$\frac{\sin nx}{\sin x} = B\ {}_2F_1\left(\frac{1}{2} + \frac{1}{2}n, \frac{1}{2} - \frac{1}{2}n\ ;\ \frac{3}{2}\ ;\ \sin^2 x\right)$$

Now taking limit of both sides as $x \to 0$, and noting that $\lim_{\theta \to 0}\left(\frac{\sin\theta}{\theta}\right) = 1$,

we get $B = n$

$$\therefore \qquad \sin nx = n\sin x\ {}_2F_1\left(\frac{1}{2} + \frac{1}{2}n, \frac{1}{2} - \frac{1}{2}n\ ;\ \frac{3}{2}\ ;\ \sin^2 x\right)$$

Again, if $y = \cos nx$, then putting $x = 0$, we see that $A = 1$, and on differentiating and putting $x = 0$, we get $B = 0$, which establishes the second part.

Ex.6. Show that

$$\boldsymbol{\Gamma(a)\Gamma(b)\ {}_2F_1\left(a, b\ ;\ \frac{1}{2}\ ;\ z\right) = \int_0^\infty\int_0^\infty e^{-u-v}\cosh\left\{2\sqrt{(uv)}z\right\}u^{a-1}v^{b-1}du\,dv}$$

provided Re (*a*) > 0 and Re (*b*) > 0.

Sol. R.H.S $= \int_0^\infty\int_0^\infty e^{-u-v}\cosh\left\{2\sqrt{(uv)}z\right\}u^{a-1}v^{b-1}du\,dv \qquad .....(16)$

But we know that $\cosh\left\{2\sqrt{(uv)z}\right\} = \sum_{r=0}^{\infty} \frac{2^{2r} u^r v^r z^r}{\lfloor 2r}$

Putting this value in the above integral (16), then it breaks up into product of two integrals, and we have

$$= \sum_{r=0}^{\infty} \frac{2^{2r} z^r}{\lfloor 2r} \int_0^{\infty} e^{-u} u^{a+r-1} du \int_0^{\infty} e^{-v} v^{b+r-1} dv$$

$$= \sum_{r=0}^{\infty} \frac{2^{2r}}{\Gamma(2r+1)} \Gamma(a+r)\Gamma(b+r)\cdot z^r$$

$$= \sum_{r=0}^{\infty} \frac{2^{2r}}{2r\Gamma(2r)} \frac{(a)_r \Gamma(a)(b)_r \Gamma(b)}{\Gamma\left(\frac{1}{2}\right)} \cdot z^r$$

$$= \sum_{r=0}^{\infty} \frac{2^{2r-1}\Gamma\left(\frac{1}{2}\right)(a)_r \Gamma(a)(b)_r \Gamma(b) z^r}{r\, 2^{2r-1}\Gamma(r)\Gamma\left(r+\frac{1}{2}\right)}$$

(applying Legendre's duplication formula)

$$= \Gamma(a)\Gamma(b) \sum_{r=0}^{\infty} \frac{(a)_r (b)_r z^r}{\lfloor r \left(\frac{1}{2}\right)_r}$$

$$= \Gamma(a)\Gamma(b)\ {}_2F_1\left(a, b\ ;\ \frac{1}{2}\ ;\ z\right) = L.H.S$$

Ex.7. Prove that

$$\lim_{c \to -n} \frac{1}{\Gamma(c)}\ {}_2F_1(a,b\ ;\ c;\ z) = \frac{(a)_{n+1}(b)_{n+1}}{\lfloor n+1} z^{n+1}\ {}_2F_1(a+n+1,\ b+n+1\ ;\ n+2\ ;\ z)$$

Sol. L.H.S $= \lim_{c \to -n} \frac{1}{\Gamma(c)}\ {}_2F_1(a,b\ ;\ c\ ;\ z)$

$$= \lim_{c \to -n} \frac{1}{\Gamma(c)} \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c)_r} \cdot \frac{z^r}{\lfloor r}$$

$$= \lim_{c \to -n} \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{\Gamma(c+r)} \cdot \frac{z^r}{\lfloor r}$$

$$= \sum_{r=n+1}^{\infty} \frac{(a)_r (b)_r}{\Gamma(-n+r)} \cdot \frac{z^r}{\lfloor r}$$

$$= \sum_{s=0}^{\infty} \frac{(a)_{s+n+1}(b)_{s+n+1}}{\Gamma(s+1)} \cdot \frac{z^{s+n+1}}{\lfloor s+n+1} \qquad \text{(Putting } r-n-1=s\text{)}$$

$$= \sum_{s=0}^{\infty} \frac{(a+n+1)_s (a)_{n+1} (b+n+1)_s (b)_{n+1}}{\lfloor s \qquad (n+2)_s} \cdot \frac{z^{s+n+1}}{\lfloor n+1}$$

$$= \frac{(a)_{n+1}(b)_{n+1} z^{n+1}}{\lfloor (n+1)} \sum_{s=0}^{\infty} \frac{(a+n+1)_s (b+n+1)_s}{(n+2)_s} \cdot \frac{z^s}{\lfloor s}$$

$$= \frac{(a)_{n+1}(b)_{n+1}}{\lfloor (n+1)} \cdot z^{n+1} \; {}_2F_1(a+n+1, b+n+1\,;\, n+2\,;\, z)$$

$= \text{R.H.S}$

Ex.*8. *It the complete elliptic integral of first kind being

$$\boldsymbol{K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}}}$$

Show that $\quad \boldsymbol{K = \frac{\pi}{2} \, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}\,;\, 1\,;\, k^2\right)}$

Sol. We have $\quad K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}}$

Putting $\quad \sin \phi = \sqrt{t}$

then $\quad \cos\phi \, d\phi = \frac{1}{2} \frac{1}{\sqrt{t}} dt$

or $\quad d\phi = \frac{1}{2} \frac{1}{\sqrt{t}} \cdot \frac{1}{\sqrt{1-t}} dt$

$\therefore \quad K = \frac{1}{2} \int_0^1 t^{-1/2} (1-t)^{-1/2} (1-k^2 t)^{-1/2} \, dt$

$\therefore$ By integral representation of ${}_2F_1\,(a, b\,;\, c\,; z)$, we have

$$K = \frac{1}{2} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right)}{\Gamma(1)} \; {}_2F_1\left(\frac{1}{2}, \frac{1}{2}\,;\, 1\,;\, k^2\right)$$

$$= \frac{\pi}{2} \, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}\,;\, 1\,;\, k^2\right) = R.H.S$$

Ex.*9. *Prove that

$$\boldsymbol{{}_2F_1(a,b\,;\, c\,;\, z) = \frac{1}{B(b,c,b)} \int_0^1 u^{b-1} (1-u)^{c-b-1} (1-zu)^{-a} \, du,}$$

where c > b > 0. Hence prove that

$$\boldsymbol{{}_2F_1(1,2\,;\, 3\,;\, z) = \log\left\{e(1-z)^{1/z}\right\}^{-2/z}}$$

Sol. By integral representation of ${}_2F_1(a, b\,; c\,; z)$, we have, if $|z| < 1$ and if $Re\,(c) > Re\,(b) > 0$, then

$$F(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-zu)^{-a}\,du$$

Now, $$F(1, 2\,; 3\,; z) = \frac{1}{B(2,1)} \int_0^1 u(1-u)^0(1-zu)^{-1}\,du$$

$$= \frac{1}{B(2,1)} \int_0^1 \frac{u\,du}{1-zu}$$

$$= \frac{2}{z} \int_0^1 \left\{\frac{1}{1-zu} - 1\right\} du$$

$$= \frac{2}{z}\left[\left\{-\frac{1}{z}\log(1-zu)\right\}_0^1 - (u)_0^1\right]$$

$$= \frac{2}{z}\left[-\frac{1}{z}\{\log(1-z) - \log 1\} - 1\right]$$

$$= \frac{2}{z}\left[-\frac{1}{z}\log(1-z) - 1\right]$$

$$= -\frac{2}{z}\left[\log(1-z)^{1/z} + \log e\right] = -\frac{2}{z}\left[\log e(1-z)^{1/z}\right]$$

$$= \log\left\{e(1-z)^{1/z}\right\}^{-2/z} = \text{R.H.S.}$$

Ex.*10. *Show that if Re* (*b*) > 0 *and if n is a nonnegative integer, then

$$ {}_2F_1\left[-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}\,; b+\frac{1}{2}\,; 1\right] = \frac{2^n (b)_n}{(2b)_n}$$

Sol. L.H.S $= {}_2F_1\left[-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}\,; b+\frac{1}{2}\,; 1\right]$

$$= \frac{\Gamma(b)\Gamma\left(b+\frac{1}{2}\right)\ (b)_n}{\Gamma\left(b+\frac{n}{2}\right)\Gamma\left(b+\frac{n}{2}+\frac{1}{2}\right)} \qquad \text{(by Gauss's theorem)}$$

Using Legendre's duplication formula, we have

$$\text{L.H.S} = \frac{\Gamma(2b)\cdot 2^{2\left(b+\frac{n}{2}\right)-1}}{2^{2b-1}\,\Gamma\left[2\left(b+\frac{n}{2}\right)\right]}\cdot(b)_n$$

$$= 2^n \frac{(b)_n}{(2b)_n} = R.H.S$$

Ex.*11. *Show that

$$\int_0^t x^{1/2}(t-x)^{-1/2}\left[1-x^2(t-x)^2\right]^{-1/2} dx = \frac{1}{2}\pi t \;{}_2F_1\left[\frac{1}{4},\frac{3}{4};1;\frac{t^4}{16}\right]$$

Sol. Let
$$I = \int_0^t x^{1/2}(t-x)^{-1/2}\left\{1-x^2(t-x)^2\right\}^{-1/2} dx$$

$$= \int_0^t x^{1/2}(t-x)^{-1/2}\sum_{n=0}^{\infty}\frac{(1/2)_n\left(x^2(t-x)^2\right)^n}{\lfloor n}dx$$

$$= \sum_{n=0}^{\infty}\frac{(1/2)_n}{\lfloor n}\int_0^t x^{2n+\frac{1}{2}}(t-x)^{2n-\frac{1}{2}}dx$$

Putting $x = tu$, we have

$$I = \sum_{n=0}^{\infty}\frac{(1/2)_n}{\lfloor n}\; t^{4n+1}\int_0^1 u^{2n+\frac{1}{2}}(1-u)^{2n-\frac{1}{2}}du$$

$$= \sum_{n=0}^{\infty}\frac{(1/2)_n}{\lfloor n}\; t^{4n+1}\frac{\Gamma\left(\frac{3}{2}+2n\right)\Gamma\left(\frac{1}{2}+2n\right)}{\Gamma(2+4n)}$$

Applying Legendre's duplication formula for $\Gamma(2+4n)$, we find that

$$I = \sum_{n=0}^{\infty}\frac{(1/2)_n}{\lfloor n}\cdot t^{4n+1}\frac{\Gamma\left(\frac{1}{2}+2n\right)\Gamma\left(\frac{1}{2}\right)}{2^{4n+1}\Gamma(2n+1)}$$

Again applying the Legendre's duplication formula and simplifying, we have

$$I = \sum_{n=0}^{\infty}\frac{(1/2)_n}{\lfloor n}\cdot t^{4n+1}\cdot\frac{1}{2^{4n+(3/2)}}\frac{\Gamma\left(\frac{1}{4}+n\right)\Gamma\left(\frac{3}{4}+n\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+n\right)\Gamma(n+1)}$$

$$= \sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}\right)_n\Gamma\left(\frac{1}{4}\right)\cdot\left(\frac{3}{4}\right)_n\Gamma\left(\frac{3}{4}\right)}{(1)_n\lfloor n}\cdot\frac{t^{4n+1}}{2^{4n+(3/2)}}$$

$$= \frac{1}{2\sqrt{2}}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(1)_n\lfloor n}\left(\frac{t^4}{16}\right)^n\frac{\pi t}{\sin(\pi/4)}$$

$$= \frac{\pi t}{2}\;{}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{t^4}{16}\right)$$

which completes the solution.

Self-Learning Exercise

1. Define Gauss hypergeometric function in terms of a series.

2. What is circle of convergence for the series representing ${}_2F_1\,(a, b\,;c\,;z)$?

3. ${}_2F_1\,(-\,n,\,b\,;\,c\,;\,1) =$ ....

4. ${}_2F_1\,(a,\,b\,;\,1-a+b\,;-1) =$ ....

5. ${}_2F_1\,(a,\,b\,;\,c\,;\,1) =$ ....

6. ${}_2F_1\,(a,\,b\,;\,b\;\;;\;z) =$ ....

7. $\lim\limits_{b\to 0}\ {}_2F_1\left(1,b;1;\dfrac{z}{b}\right) = \ldots$

8. ${}_2F_1\,(-\,n,\,1-b-n\,;\,a\,;\,1) =$ ....

10.7 Summary

In this unit, the function introduced by C.F. Gauss was studied. The important special cases, properties and convergence conditions of this function were discussed in detail.

10.8 Answers of Self-Learning Exercise

1. $\sum\limits_{n=0}^{\infty} \dfrac{(a)_n (b)_n z^n}{(c)_n \lfloor n}$ **2.** $|\,z\,| < 1$ **3.** $\dfrac{(c-b)_n}{(c)_n}$

4. $\dfrac{\Gamma(1-a+b)\Gamma\left(1+\dfrac{b}{2}\right)}{\Gamma(1+b)\Gamma\left(1+\dfrac{b}{2}-a\right)}$ **5.** $\dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$ **6.** $(1-z)^{-a}$

7. e^z **8.** $\dfrac{(a+b-1)_{2n}}{(a)_n (a+b-1)_n}$

10.9 Exercise

1. Define hypergeometric function ${}_2F_1\,(a,\,b\,;\,c\,;\,z)$ and state the condition on its elements a, b and c for its convergence.

2. Find representation of following functions in terms of Gauss hypergeometric function :

(i) $(1+z)^n$ **[Ans.** ${}_2F_1\,(-\,n,\,1\,;\,1\,;-z)$**]**

(ii) $\dfrac{1}{2az}\left[(1-z)^{-a} - (1+z)^{-a}\right]$ **[Ans.** ${}_2F_1\left(\dfrac{a}{2}+\dfrac{1}{2},\dfrac{a}{2}+1\,;\,\dfrac{3}{2}\,;\,z^2\right)$**]**

(iii) $\dfrac{1}{z}\log(1+z)$ **[Ans.** ${}_2F_1\,(1,\,1\,;\,2\,;-\mathrm{z})$**]**

(iv) $\dfrac{1}{2z}\log\left(\dfrac{1+z}{1-z}\right)$ **[Ans.** ${}_2F_1\left(\dfrac{1}{2},1\,;\,\dfrac{3}{2}\,;\,z^2\right)$**]**

(v) $\dfrac{\sin^{-1} z}{z}$ **[Ans.** $_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right)$]

(vi) $\dfrac{1}{z}\tan^{-1} z$ **[Ans.** $_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; z^2\right)$]

(vii) $\sin z$ **[Ans.** $z\ _0F_1\left(-; \frac{3}{2}; -\frac{1}{4}z^2\right)$]

(viii) $\cos z$ **[Ans.** $_0F_1\left(-; \frac{1}{2}; -\frac{1}{4}z^2\right)$**]**

3. Express complete elliptic integral of the second kind in terms of Gauss's hypergeometric function

[Ans. $\frac{1}{2}\pi\ _2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; k^2\right)$**]**

4. By transforming the equation $\dfrac{d^2y}{dx^2} + n^2 y = 0$ to hypergeometric form by the substitution $\xi = \sin^2 z$, prove that if $0 \le z \le \pi$ then,

$$\cos nz = \cos\left(\frac{n\pi}{2}\right)\ _2F_1\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; \cos^2 z\right) + n\sin\left(\frac{n\pi}{2}\right)\cos z\ _2F_1\left(\frac{1}{2}-\frac{n}{2}, \frac{1}{2}+\frac{n}{2}; \frac{3}{2}; \cos^2 z\right)$$

and $$\sin nz = \sin\left(\frac{n\pi}{2}\right)\ _2F_1\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; \cos^2 z\right) - n\cos\left(\frac{n\pi}{2}\right)\cos z\ _2F_1\left(\frac{1}{2}-\frac{n}{2}, \frac{1}{2}+\frac{n}{2}; \frac{3}{2}; \cos^2 z\right)$$

5. Establish the transformation formula

$$_2F_1\left(2a, 2b; a+b+\frac{1}{2}; z\right) = {}_2F_1\left\{a, b; a+b+\frac{1}{2}; 4z(1-z)\right\}$$

provided that $a + b + \frac{1}{2}$ is not zero or a negative integer and if $|z| < 1$ and $|4z(1-z)| < 1$

6. Show that $$\lim_{z\to 1} \frac{_2F_1(a, b; a+b; z)}{-\log(1-z)} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

7. If the complete elliptic integral of the first kind is $K' = \int_0^{\pi/2} \left(1 - k^2 \sin^2\phi\right)^{-1/2} d\phi$,

then show that $$K' = \frac{\pi}{2}\ _2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \qquad |k| < 1$$

□ □ □

Unit 11 : Gauss and Confluent Hypergeometric Functions

Structure of the Unit

11.0 Objective

In the last unit the Gauss hypergeometric function was introduced and some properties, summation theorems and convergence conditions for this function were discussed. The aim of this unit is to study further the hypergeometric function. Precisely you will study the linear transformation formulas, contiguous function relations, differentiation formulas and a linear relation between the solutions of hypergeometric differential equation. You will also study the kummer's confluent hypergeometric function and important formulas concerned with this function.

11.1 Introduction

Here some more results for the Gauss hypergeometric function (introduced in the last unit) will be established. In fact linear transformation formulas, contiguous function relations, differentiation formulas etc. will be discussed in this unit.Next, the Kummer's confluent hypergeometric function will be introduced and important formulas for this function will also be established.

11.2 Linear Transformation Formulas

Result :

If $|z| < 1$ **and** $\left|\frac{z}{1-z}\right| < 1$**, then**

(i) $_2F_1(a, b, c; z) = (1-z)^{-a}\ _2F_1\left(a, c-b, c; \frac{z}{z-1}\right)$ .....(1)

(ii) $_2F_1(a, b, c; z) = (1-z)^{-b}\ _2F_1\left(c-a, b; c; \frac{z}{z-1}\right)$ .....(2)

(iii) $_2F_1(a, b, c; z) = (1-z)^{c-a-b}\ _2F_1(c-a, c-b; c; z)$ .....(3)

Proof. ***(i)*** We know that by integral representation of $_2F_1(a, b, c; z)$, if $|z| < 1$ and if $\text{Re}(c) > \text{Re}(b) > 0$

Then $B(b, c-b)\ _2F_1(a, b; c; z) = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt$

$$= \int_0^1 (1-t)^{b-1}\{1-(1-t)\}^{c-b-1}\left\{1-z(1-t)^{-a}\right\} dt$$

$$= \int_0^1 t^{c-b-1}(1-t)^{b-1}(1-z+tz)^{-a}\, dt$$

$$= (1-z)^{-a}\int_0^1 t^{c-b-1}(1-t)^{b-1}\left(1-\frac{tz}{z-1}\right)^{-a} dt$$

$$= (1-z)^{-a}\, B(c-b, b)\ _2F_1\left(a, c-b; c; \frac{z}{z-1}\right)$$

Thus $_2F_1(a, b; c; z) = (1-z)^{-a}\ _2F_1\left(a, c-b\ ;\ c\ ; \frac{z}{z-1}\right)$

(ii) Taking L.H.S $_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z)$

$$= (1-z)^{-b}\ _2F_1\left(b\ ; c-a\ ;\ c\ ; \frac{z}{z-1}\right)$$

(by first transformation formula)

$$= (1-z)^{-b}\ _2F_1\left(c-a\ ;\ b\ ; c\frac{z}{z-1}\right)$$

(by symmetric properly)

Hence $_2F_1(a, b; c; z) = (1-z)^{-b}\ _2F_1\left(c-a,\ b;\ c;\ \frac{z}{z-1}\right)$

(iii) From (1), we have $_2F_1(a, b; c; z) = (1-z)^{-a}\ _2F_1\left(c-b, a;\ c;\ \frac{z}{z-1}\right)$ .....(4)

Putting $\frac{z}{z-1} = y$ or 1 - y = $(1-z)^{-1}$, we have

Now $${}_2F_1\left(c-b,a;c;\frac{z}{z-1}\right) = {}_2F_1(c,b;a;c;y)$$

$$= (1-y)^{-(c-b)}\ {}_2F_1\left(c-b,\ c-a;c;\frac{y}{y-1}\right) \quad .....(5)$$

or $${}_2F_1\left(c-b,a\ ;\ c\ ;\frac{z}{z-1}\right) = (1-z)^{c-b}\ {}_2F_1(c-b,c-a\ ;\ c;z) \quad ......(6)$$

Using (6) in (4), we have

$${}_2F_1(a,b;\ c;z) = (1-z)^{c-a-b}\ {}_2F_1(c-a,\ c-b\ ;c\ ;\ z)$$

11.2.1 Applications

If we set $z = \frac{1}{2}$ in the first transformation formula, then

$${}_2F_1\left(a,b;c;\frac{1}{2}\right) = 2^a{}_2F_1(a,c-b;c;-1) \quad .....(7)$$

The series on the R.H.S. of (7) can be summed in terms of product of gamma functions with the help of Kummer's theorem in the following cases :

(i) $c = c-a-b+1$ that is $b = 1-a$

(ii) $c = a-(a-b)+1$ or $c = \frac{1+a+b}{2}$

From the first case, we get

$${}_2F_1\left(a,1-a;c;\frac{1}{2}\right) = 2^a{}_2F_1(a,c+a-1\ ;\ c\ ;\ -1)$$

$${}_2F_1\left(a,1-a;c;\frac{1}{2}\right) = \frac{2^a\Gamma(c)\Gamma\left(\frac{1+c+a}{2}\right)}{\Gamma(c+a)\Gamma\left(\frac{1+c-a}{2}\right)}$$

Further, applying the Legendre's duplication formula for $\Gamma(c)$ and $\Gamma(c+a)$, then we obtain

$$\therefore \quad {}_2F_1\left(a,1-a;c;\frac{1}{2}\right) = \frac{\Gamma\left(\frac{c}{2}\right)\Gamma\left(\frac{1+c}{2}\right)}{\Gamma\left(\frac{c+a}{2}\right)\Gamma\left(\frac{1+c-a}{2}\right)}$$

In the same way, in the second case, we can prove the following result.

$${}_2F_1\left(a,b\ ;\frac{1+a+b}{2}\ ;\ \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+a+b}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1+b}{2}\right)}$$

11.3 Differentiation of Hypergeometric Functions

Result : Show that

(i) $$\frac{d}{dx}\left[{}_2F_1(a,b\,;\,c\,;\,x)\right] = \frac{ab}{c}\,{}_2F_1(a+1,b+1;\,c+1\,;\,x) \qquad .....(1)$$

(ii) $$\frac{d^n}{dx^n}\left[{}_2F_1(a,b\,;\,c\,;\,x)\right] = \frac{(a)_n(b)_n}{(c)_n}\,{}_2F_1(a+n,\ b+n;\,c+n\,;\,x) \qquad .....(2)$$

Proof of (i), we have

$$\frac{d}{dx}\,{}_2F_1(a,b\,;\,c\,;\,x) = \frac{d}{dx}\left[\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r}\cdot\frac{x^r}{\lfloor r}\right]$$

$$= \sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r}\frac{r\,x^{r-1}}{\lfloor r}$$

$$= \sum_{r=1}^{\infty}\frac{(a)_r(b)_r}{(c)_r\lfloor r-1}\cdot x^{r-1}$$

$$= \sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}}\cdot\frac{x^n}{\lfloor n}$$

Since $\qquad (a)_{n+1} = a(a+1)_n,$

Therefore $$\frac{d}{dx}\left[{}_2F_1(a,b\,;\,c\,;\,x)\right] = \sum_{n=0}^{\infty}\frac{a(a+1)_n\,b(b+1)_n}{c(c+1)_n}\cdot\frac{x^n}{\lfloor n}$$

$\therefore$ $$\frac{d}{dx}\left[{}_2F_1(a,b\,;\,c\,;\,x)\right] = \frac{ab}{c}\,{}_2F_1(a+1,b+1\,;\,c+1\,;\,x)$$

(ii) We prove the result by the principle of mathematical induction

Since by (1), we have

$$\frac{d}{dx}\left[{}_2F_1(a,b\,;\,c\,;\,x)\right] = \frac{a\,b}{c}\,{}_2F_1(a+1,b+1\,;\,c+1\,;\,x)$$

Therefore the result (2) is true for $n = 1$

Suppose that (2) is true for $n = m$ (a fixed positive integer) i.e.

$$\frac{d^m}{dx^m}\left[{}_2F_1(a,b\,;\,c\,;\,x)\right] = \frac{(a)_m(b)_m}{(c)_m}\,{}_2F_1(a+m,b+m\,;\,c+m\,;\,x)$$

Now, $$\frac{d^{m+1}}{dx^{m+1}}\left[{}_2F_1(a,b\,;\,c\,;\,x)\right] = \frac{d}{dx}\left[\frac{d^m}{dx^m}\left\{{}_2F_1(a+m,b+m\,;\,c+m\,;\,x)\right\}\right]$$

$$= \frac{(a)_m(b)_m}{(c)_m}\frac{d}{dx}\left[{}_2F_1(a+m,b+m\,;\,c+m\,;\,x)\right]$$

[by equation (4)]

$$=\frac{(a)_m(b)_m}{(c)_m}\cdot\frac{(a+m)(b+m)}{c+m}\ {}_2F_1(a+m+1,b+m+1\ ;\ c+m+1\ ;\ x)$$

$$=\frac{(a)_{m+1}(b)_{m+1}}{(c)_{m+1}}\ {}_2F_1(a+m+1,b+m+1\ ;\ c+m+1\ ;\ x)$$

Thus result (2) holds for $n = m + 1$. Hence by P.M.I the result (2) is true for every positive integer n.

11.4 Linear Relation between the Solutions of Hypergeometric equations

In the unit 9, we have seen that the differential equation

$$z(1-z)\frac{d^2y}{dz^2}+\{c-(1+a+b)z\}\frac{du}{dz}-abu=0 \qquad .....(1)$$

has the solutions $A\ {}_2F_1(a,b\ ;\ c\ ;\ z)$ and $B\ z^{1-c}\ {}_2F_1(a+1-c,\ b+1-c\ ;\ 2-c\ ;\ z)$ which are convergent for $|\ z\ | < 1$ whereas the solutions $A\ {}_2F_1(a,\ b\ ;\ a+b+1-c\ ;1-z)$ and $B(1-z)^{c-a-b}$ ${}_2F_1(c-a\ ,\ c-b\ ;1+c-a-b\ ;1-z)$ of the hypergeometric differential equation are convergent for $|1-z| < 1$. (Refer Ex.2. §9.4)

Hence there exist an interval (0, 1) in which all the four solutions exist. Since only two solutions of the second order differential equation are linearly independent, which implies that there may exist a linear relation between the solutions.

Let the relation be

$$F(a,b\ ;\ c\ ;\ z) = A\ {}_2F_1(a,\ b,\ a+b-c+1\ ;1-z)+$$
$$B(1-z)^{c-a-b}\ {}_2F_1(c-a,\ c-b\ ;\ c-a-b+1\ ;1-z) \qquad .....(2)$$

where A and B are constants.

Putting $z = 1$ in the above equation (2) and applying the Gauss's theorem, we have

$${}_2F_1(a,b\ ;\ c\ ;\ 1)= A=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad .....(3)$$

where $\qquad R(c-a-b)>0$

Again, if we put $z = 0$ in (2), then it gives

$$1 = A\ {}_2F_1\ (a,\ b\ ;\ a+b-\ c+1\ ;\ 1) + B\ {}_2F_1\ (c-a\ ;\ c-b\ ;\ c-a-b+1\ ;\ 1)$$

or

$$1=A\frac{\Gamma(a+b-c+1)\Gamma(1-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}+B\frac{\Gamma(c-a-b+1)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)} \qquad .....(4)$$

Putting the value of A from the equation (3) in the equation (4) we obtain

$$1=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\frac{\Gamma(a+b-c+1)\Gamma(1-c)}{\Gamma(a-c+1)\Gamma(b-c+1)}+B\frac{\Gamma(c-a-b+1)\Gamma(1-c)}{\Gamma(1-c)\Gamma(1-b)}$$

or $\quad 1 = \dfrac{\Gamma(c)\Gamma(1-c)\Gamma(c-a-b)\Gamma(1-c+a+b)}{\Gamma(c-a)\Gamma(1-c+a)\Gamma(c-b)\Gamma(1-c+b)} + B\dfrac{\Gamma(c-a-b+1)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)}$

Since $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$, therefore

$$1 = \frac{\sin \pi(c-a)\sin \pi(c-b)}{\sin \pi c \sin \pi(c-a-b)} + B\frac{\Gamma(c-a-b+1)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)}$$

$$\therefore \quad \frac{\Gamma(c-a-b+1)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)}B = 1 - \frac{\sin \pi(c-a)\cdot \sin \pi(c-b)}{\sin \pi c \cdot \sin \pi(c-a-b)}$$

$$= \frac{\sin \pi c \cdot \sin \pi(c-a-b) - \sin \pi(c-a)\sin \pi(c-b)}{\sin \pi c \cdot \sin \pi(c-a-b)}$$

$$= \frac{\left[\{\cos \pi(a+b) - \cos \pi(2c-a-b)\} - \{\cos \pi(b-a) - \cos \pi(2c-a+b)\}\right]}{2\sin \pi c \sin \pi(c-a-b)}$$

$$= \frac{\cos \pi(a+b) - \cos \pi(b-a)}{2\sin \pi c \sin \pi(c-a-b)}$$

$$= -\frac{\sin \pi a \sin \pi b}{\sin \pi c \, \sin \pi(c-a-b)}$$

$$\therefore \quad B = \frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(1-c)\Gamma(c-a-b+1)} \cdot \frac{\sin \pi a \sin \pi b}{\sin \pi c \, \sin \pi(a+b-c)}$$

Applying $\sin \pi_{z} = \dfrac{\pi}{\Gamma(z)\Gamma(1-z)}$, we have

$$B = \frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(1-c)\Gamma(c-a-b+1)} \cdot \frac{\pi}{\Gamma(a)\Gamma(1-a)} \cdot \frac{\pi}{\Gamma(b)\Gamma(1-b)}$$

$$\cdot \frac{\Gamma(c)\Gamma(1-c)\Gamma(a+b-c)\Gamma(1-a-b+c)}{\pi^2}$$

$$\mathrm{B} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$$

Substituting these values of A and B in (2), we get the following linear relation :

$$\boldsymbol{F(a\,;b\,;c\,;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\ {}_2F_1(a, b\,;\ a+b-c+1\,;\ 1-z)}$$

$$\boldsymbol{+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}\ {}_2F_1(c-a,\ c-b\,; c-a-b+1\ ;\ 1-z)}$$

11.5 Relations of Contiguity

The functions obtained by increasing or decreasing any one of the parameters of the hypergeometric function ${}_2F_1\,(a\,;b\,;c\,;z)$ by unity, are called the functions contiguous to it. In this way, we obtain the following six functions contiguous to ${}_2F_1\,(a\,;b\,;c\,;z)$:

(i) $\boldsymbol{F(a+) = {}_2F_1(a+1\,;\ b;\ c;\ z)}$

(ii) $\boldsymbol{F(a-) = {}_2F_1(a-1,\ b;\ c;\ z)}$

(iii) $\boldsymbol{F(b+) = {}_2F_1(a, \mathrm{b}+1;\ c;\ z)}$

(iv) $\boldsymbol{F(b-) = {}_2F_1(a, \mathrm{b}-1;\ c;\ z)}$

(v) $\boldsymbol{F(c+) = {}_2F_1(a, \mathrm{b};\ c+1;\ z)}$

(vi) $\boldsymbol{F(c-) = {}_2F_1(a, \mathrm{b};\ c-1;\ z)}$

Now we shall see that the function ${}_2F_1$ can be connected with any two of its contiguous functions giving rise to fifteen (that is ${}^6C_2$) relations in this way. These relations were first obtained by Gauss and are called **contiguous function relations.**

If we write $\dfrac{(a)_n(b)_n}{(c)_n}\cdot\dfrac{z^n}{\lfloor n} = \delta_n$, then clearly $F = {}_2F_1 = \sum\limits_{n=0}^{\infty}\delta_n$ .....(1)

Now we have

$$F(a+) = {}_2F_1(a+1;\ b;\ c;z)$$
$$= \sum_{n=0}^{\infty}\frac{(a+1)_n(b)_n}{(c)_n}\cdot\frac{z^n}{\lfloor n}$$
$$= \sum_{n=0}^{\infty}\frac{a+n}{a}\cdot\frac{(a)_n(b)_n}{(c)_n}\cdot\frac{z^n}{\lfloor n}$$
$$= \sum_{n=0}^{\infty}\frac{a+n}{a}\cdot\delta_n \qquad \text{[using (1)]}$$

In this way, we obtain the following relations

$$F(a+) = \sum_{n=0}^{\infty}\frac{(a+n)}{a}\cdot\delta_n\ ,\ F(a-) = \sum_{n=0}^{\infty}\frac{(a-1)}{(a-1+n)}\delta_n$$

$$F(b+) = \sum_{n=0}^{\infty}\frac{(b+n)}{(b)}\delta_n\ ,\ F(b-) = \sum_{n=0}^{\infty}\frac{(b-1)}{(b-1+n)}\delta_n$$

$$F(c+) = \sum_{n=0}^{\infty}\frac{(c)}{(c+n)}\delta_n\ ,\ F(c-) = \sum_{n=0}^{\infty}\frac{(c-1+n)}{(c-1)}\delta_n$$

In proving these relations, the formulae

$\Gamma(z+1) = z\Gamma(z)$ and $(a-1)_n = \dfrac{\Gamma(a+n-1)}{\Gamma(a-1)} = \dfrac{(a-1)}{(a+n-1)}(a)_n$ were used.

There are **fifteen** contiguous function relations for the hypergeometric function, which are given below :

(i) $(a-b)\,F = a\,F\,(a+) - b\,F\,(b+)$

(ii) $(a-c+1)\,F = a\,F\,(a+) - (c-1)\,F\,(c-)$

(iii) $[(a+(b-c)\,z]\,F = a\,(1-z)\,F\,(a+) - c^{-1}\,(c-a)\,(c-b)\,z\,F\,(c+)$

(iv) $(1-z)\,F = F\,(a-) - c^{-1}\,(c-b)\,z\,F\,(c+)$

(v) $(1-z)\,F = F\,(b-) - c^{-1}\,(c-a)\,z\,F\,(c+)$

(vi) $[2a-c+(b-a)\,z]\,F = a\,(1-z)\,F\,(a+) - (c-a)\,F\,(a-)$

(vii) $(a+b-c)\,F = a\,(1-z)\,F\,(a+) - (c-b)\,F\,(b-)$

(viii) $(c-a-b)\,F = (c-a)\,F\,(a-) - b\,(1-z)\,F\,(b+)$

(ix) $(b-a)\,(1-z)\,F = (c-a)\,F\,(a-) - (c-b)\,F\,(b-)$

(x) $[1-a+(c-b-1)\,z]\,F = (c-a)\,F\,(a-) - (c-1)\,(1-z)\,F\,(c-)$

(xi) $[2b-c+(a-c)\,z]\,F = b\,(1-z)\,F\,(b+) - (c-b)\,F\,(b-)$

(xii) $[b+(a-c)\,z]\,F = b\,(1-z)\,F\,(b+) - c^{-1}\,(c-a)\,(c-b)\,z\,F\,(c+)$

(xiii) $(b-c+1)\,F = b\,F\,(b+) - (c-1)\,F\,(c-)$

(xiv) $[1-b+(c-a-1)\,z]\,F = (c-b)\,F\,(b-) - (c-1)\,(1-z)\,F\,(c-)$

(xv) $[c-1+(a+b+1-2c)\,z]\,F = (c-1)\,(1-z)\,F\,(c-) - c^{-1}\,(c-a)\,(c-b)\,z\,F\,(c+)$

Again since $z\dfrac{d}{dz}\left(z^n\right) = n\,z^n$, writing $\theta = z\dfrac{d}{dz}$, we have

$$\theta\left(z^n\right) = nz^n \text{ and } (\theta+a)\,z^n = (n+a)\,z^n \qquad .....(2)$$

Hence $$(\theta+a)\,F = \sum_{n=0}^{\infty}(n+a)\,\delta_n \qquad ......(3)$$

Using the relation $F\,(a+) = \sum_{n=0}^{\infty}\left(\dfrac{a+n}{a}\right)\delta_n$

$$(\theta+a)\,F = a\,F\,(a+) \qquad .....(4)$$

Similarly from $F\,(a+) = \sum_{n=0}^{\infty}\left(\dfrac{a+n}{a}\right)\delta_n$ and $F\left(b+\right) = \sum_{n=0}^{\infty}\left(\dfrac{b+n}{b}\right)\delta_n$

$$(\theta+b)\,F = b\,F\,(b+) \qquad .....(5)$$

and $$(\theta+c-1) = (c-1)\,F\,(c-) \qquad ......(6)$$

Proof. ***(i)*** Subtracting (5) from (4), we obtain *(i) i.e.,*

$$(\theta+a)\,F - (\theta+b)\,F = a\,F\,(a+) - b\,F(a+)$$

$$\Rightarrow \quad (a-b)\,F = a\,F\,(a+) - b\,F(b+)$$

(ii) Subtracting (6) from (4), we have

$$(\theta+a)\,F - (\theta+c-1)\,F = a\,F\,(a+) - (c-1)\,F\,(c-)$$

$$\Rightarrow \quad (a-c+1)\,F = a\,F\,(a+) - (c-1)\,F\,(c-)$$

(iii) We know that $\theta(z^n) = n\,z^n$,

$$\therefore \qquad \theta F = \sum_{n=0}^{\infty} \frac{n(a)_n (b)_n}{(c)_n \lfloor n} z^n = z \sum_{n=1}^{\infty} \frac{n(a)_n (b)_n}{(c)_n \lfloor n-1} z^n$$

$$= z \sum_{n=0}^{\infty} \frac{(a)_{n+1} (b)_{n+1}}{(c)_{n+1} \lfloor n} z^n = z \sum_{n=0}^{\infty} \frac{(a+n)(a)_n (b+n)(b)_n}{(c+n)(c)_n} \cdot \frac{z^n}{\lfloor n}$$

$$\therefore \qquad \theta F = z \sum_{n=0}^{\infty} \frac{(a+n)(b+n)}{(c+n)} \cdot \delta_n \qquad .....(7)$$

But $$\frac{(a+n)(b+n)}{(c+n)} = n + (a+b-c) + \frac{(c-a)(c-b)}{(c+n)} ... \qquad .....(8)$$

$\therefore$ The above equation (7) with the help of (8) is transformed to

$$\theta F = z \sum_{n=0}^{\infty} n\delta_n + (a+b-c) z \sum_{n=0}^{\infty} \delta_n + z \frac{(c-a)(c-b)}{c} \sum_{n=0}^{\infty} \frac{c\delta_n}{c+n}$$

$$= z\,\theta F + (a+b-c) zF + c^{-1}(c-a)(c-b) z\, F(c+),$$

or $$(1-z)\theta\; F = (a+b-c) zF + c^{-1}(c-a)(c-b) z\, F(c+) \qquad .....(9)$$

Also from (4), we have $\theta F = -\,a\,F + a\,F(a+)$
which implies that $(1-z)\,\theta F = -\,a\,(1-z)\,F + a\,(1-z)\,F\,(a+)$ .....(10)
From (9) and (10), we have.

$$\left[a(1-z) + (a+b-c) z\right] F = a(1-z) F(a+) - c^{-1}(c-a)(c-b) z\, F(c+)$$

or $$\left[a + (b-c) z\right] F = a(1-z) F(a+) - c^{-1}(c-a)(c-b) z\, F(c+). \qquad .....(11)$$

(iv) Consider $$\theta\, F(a-) = \sum_{n=1}^{\infty} \frac{(a-1)_n (b)_n}{(c)_n} \frac{z^n}{\lfloor n-1} = \sum_{n=0}^{\infty} \frac{(a-1)_{n+1} (b)_{n+1}}{(c)_{n+1} \lfloor n} z^{n+1} \qquad .....(12)$$

$$\because \qquad (a-1)_{n+1} = (a-1)(a)_n$$

$$\therefore \qquad \theta F(a-) = \sum_{n=0}^{\infty} \frac{(a-1)(a)_n (b+n)(b)_n}{(c+n)(c)_n \lfloor n} z^n$$

$$= (a-1) z \sum_{n=0}^{\infty} \frac{(b+n)}{(c+n)} \delta_n \qquad .....(13)$$

Since $$\frac{b+n}{c+n} = 1 - \frac{(c-b)}{c+n}$$

Putting this value in the above relation (13), we get

$$\theta F(a-) = (a-1) z \sum_{n=0}^{\infty} \left(1 - \frac{(c-b)}{(c+n)}\right) \delta_n$$

$$= (a-1)z\sum_{n=0}^{\infty}\delta_n - \frac{(a-1)(c-b)z}{c}\sum_{n=0}^{\infty}\left(\frac{c}{(c+n)}\right)\delta_n$$

$$\theta F(a-) = (a-1)z\,F - c^{-1}(a-1)(a-b)z\,F(c+) \qquad .....(14)$$

But in equation (4), if we write $(a-1)$ in place of a, we get

$$\theta F(a-) = (a-)F - (a-1)F(a-) \qquad .....(15)$$

Combining the equations (14) and (15), we get the required result *(iv)*.

(v) If we interchange a and b in *(iv)*, we obtain *(v)*.

The remaining ten relations can be deduced by making use of the above five relations.

11.6 Kummer's Confluent Hypergeometric Function

The hypergeometric differential equation is

$$z(1-z)\frac{d^2u}{dz^2} + \{c-(1+a+b)z\}\frac{du}{dz} - abu = 0 \qquad .....(1)$$

Replacing z by z/b in (1), we get

$$z\left(1-\frac{z}{b}\right)\frac{d^2u}{dz^2} + \left\{c-\left(1+\frac{1+a}{b}\right)z\right\}\frac{du}{dz} - au = 0 \qquad .....(2)$$

Now take the limit as $b \to \infty$, the equation (2) reduces to

$$z\frac{d^2u}{dz^2} + (c-z)\frac{du}{dz} - au = 0 \qquad .....(3)$$

whose solution is given by $\lim\limits_{b\to\infty} {}_2F_1\left(a,b;c;\frac{z}{b}\right)$ .....(4)

The equation (3) is known as the confluent hypergeometric differential equation or Kummer's equation.

Now,
$$\lim_{b\to\infty}\frac{(b)_r}{b^r} = \lim_{b\to\infty}\frac{b(b+1)(b+2)....(b+r-1)}{b.b.b....\ r \text{ times}}$$

$$= \lim_{b\to\infty}\left(1+\frac{1}{b}\right)\left(1+\frac{2}{b}\right)...\left(1+\frac{r-1}{b}\right) = 1$$

Hence the solution (4) may be written as

$$\lim_{b\to\infty} {}_2F_1\left(a,b;c;\frac{z}{b}\right) = \lim_{b\to\infty}\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r\lfloor r}\left(\frac{z}{b}\right)^r$$

$$= \lim_{b\to\infty}\sum_{r=0}^{\infty}\frac{(a)_r}{(c)_r}\frac{z^r}{\lfloor r}\frac{(b)_r}{b^r}$$

$$= \sum_{r=0}^{\infty}\frac{(a)_r}{(c)_r}\cdot\frac{z^r}{\lfloor r} = {}_1F_1(a;c;z)$$

The function ${}_1F_1(a;c;z)$ is called the **confluent hypergeometric function.**

Now considering the equation (3), we find that $z = 0$ is a regular singular point, so if c is neither zero nor a negative integer, two independent solutions in series of it can be easily found by Frobenius method described in unit 9

$$\therefore \qquad u_1 = {}_1F_1(a; c; z)$$

$$u_2 = z^{1-c}\, {}_1F_1(a - c + 1; 2 - c; z)$$

Hence the general solution of equation (1) is

$$u = A\, {}_1F_1(a;c;z) + Bz^{1-\gamma}\, {}_1F_1(a-c+1;2-c;z)$$

where A and B are arbitrary constants.

11.6.1 Convergency of the Confluent hypergeometric function.

If u_n and u_{n-1} are the n^{th} and $(n + 1)^{th}$ terms of the series representing confluent hypergeometric function, then

$$u_n = \frac{(a)_n}{(c)_n}\frac{x^n}{\lfloor n} \quad \text{and} \quad u_{n+1} = \frac{(a)_{n+1}}{(c)_{n+1}} \cdot \frac{x^{n+1}}{\lfloor n+1}$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(a)_{n+1}}{(a)_n}\frac{\lfloor n}{\lfloor n+1} \cdot \frac{(c)_n}{(c)_{n+1}} \cdot x\right|$$

$$= \left|\frac{(a+n)}{(c+n)(n+1)} \cdot x\right|$$

$$\therefore \qquad \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(a+n)}{(c+n)(n+1)} \cdot x\right| \to 0$$

Hence $\left|\frac{u_{n+1}}{u_n}\right| < 1$ for all z. Thus the series is always convergent.

11.6.2 Differentiation of Confluent hypergeometric function.

Results :

(i) $$\frac{d}{dx}\, {}_1F_1(a;c;x) = \frac{a}{c}\, {}_1F_1(a+1;c+1;x)$$

(ii) $$\frac{d^n}{dx^n}\, {}_1F_1(a;c;x) = \frac{(a)_n}{(c)_n}\, {}_1F_1(a+n;c+n;x)$$

The proofs of above formulas are similar to formulas given in §11.3 for Gauss hypergeometric function.

11.6.3 Integral representation for confluent hypergeometric function

If $|z| < 1$ and Re $(c)>$ Re $(a) > 0$, then

$$B(a, c-a)\, {}_1F_1(a; c; z) = \int_0^1 t^{a-1}(1-t)^{c-a-1}\, e^{zt}\, dt$$

or $${}_1F_1(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}e^{zt}\,dt$$

Proof we have R.H.S. $$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}\sum_{n=0}^{\infty}\frac{(zt)^n}{\lfloor n}dt$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\sum_{n=0}^{\infty}\frac{z^n}{\lfloor n}\frac{\Gamma(a+n)\Gamma(c-a)}{\Gamma(c+n)}$$

$$= {}_1F_1(a;c;z)$$

11.6.4 Kummer's first transformation

Result :

If c is neither zero nor a negative integer, then ${}_1F_1(a;c;z) = e^z\,{}_1F_1(c-a;c-z)$.

Proof : By integral representation of confluent hypergeometric function, we have.

$$B(a,c-c)\,{}_1F_1(a;c;z) = \int_0^1 t^{a-1}(1-t)^{c-a-1}e^{zt}\,dt$$

Using the property of definite integral, we get

$$= e^z\int_0^1 t^{c-a-1}(1-t)^{c-1}e^{-zt}\,dt$$

$$= e^z B\quad c-a,a\quad {}_1F_1(c-a;c;-z)$$

$$\therefore\qquad {}_1F_1(a;c;-z) = e^z\,{}_1F_1(a-c;a;-z)$$

Ex.*1. *If m is a positive integer, show that

$${}_2F_1(-m,a+m;c;x) = \frac{x^{1-c}(1-x)^{c-a}}{\Gamma(m+c)}\Gamma(c)\frac{d^m}{dx^m}\left\{x^{c+m-1}(1-x)^{a-c+m}\right\}$$

and deduce that

$${}_2F_1\left(-m,a+m;\frac{a}{2}+\frac{1}{2};\frac{1}{2}-\frac{1}{2}\mu\right) = \frac{(\mu^2-1)^{\frac{1}{2}-\frac{1}{4}a}\,\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)}{2^m\,\Gamma\left(\frac{1}{2}+\frac{a}{2}+m\right)}\frac{d^m}{d\mu^m}(\mu^2-1)^{m+\frac{a}{2}-\frac{1}{2}}$$

Sol. R.H.S. $$= \frac{x^{1-c}(1-x)^{c-a}}{\Gamma(m+c)}\Gamma(c)\frac{d^m}{dx^m}\left\{\sum_{r=0}^{\infty}x^{c+m-1}\frac{(c-a-m)_r\,x^r}{\lfloor r}\right\}$$

$$= \frac{x^{1-c}(1-x)^{c-a}}{(c)_m}\frac{d^m}{dx^m}\left\{\sum_{r=0}^{\infty}x^{c+m+r-1}\frac{(c-a-m)_r}{\lfloor r}\right\}$$

$$= \frac{x^{1-c}(1-x)^{c-a}}{(c)_m} \sum_{r=0}^{\infty} \frac{(c+r)_m (c-a-m)_r \, x^{c+r-1}}{\lfloor r}$$

$$= x^{1-c}(1-x)^{c-a} \sum_{r=0}^{\infty} \frac{(c+m)_r}{(c)_r} \frac{(c-a-m)_r}{\lfloor r} x^{c+r-1}$$

$$= (1-x)^{c-a} {}_2F_1(c+m, c-a-m; c; x)$$

But we know by transformation formula

$$ {}_2F_1(a, b; c; z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z)$$

$\therefore$ R.H.S. $= {}_2F_1(-m, a+m; c; x) =$ L.H.S.

Deduction. Putting $x = \dfrac{1-\mu}{2}$ and $c = \dfrac{1+a}{2}$, we obtain the second part of the question.

Ex.2. If m is a positive integer, and | x | > 1, show that

$$ {}_2F_1\left(\frac{m+1}{2}, \frac{m+2}{2}; 1; \frac{-1}{x^2}\right) = \frac{(-1)^m x^{m+1}}{\lfloor m} \frac{d^m}{dx^m}\left\{\frac{1}{\sqrt{x^2+1}}\right\}.$$

Sol. We know that

$$\frac{1}{(1+x^2)^{1/2}} = \frac{1}{x\left(1+\frac{1}{x^2}\right)^{1/2}} = \frac{1}{x}\left(1+\frac{1}{x^2}\right)^{-1/2}$$

$$\therefore \quad \frac{1}{(1+x^2)^{1/2}} = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{2}\right)_r x^{-2r-1}}{\lfloor r}$$

Hence $$\frac{d^m}{dx^m}(1+x^2)^{-1/2} = \frac{d^m}{dx^m}\left[\sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r}\left(\frac{1}{2}\right)_r x^{-2r-1}\right]$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r}\left(\frac{1}{2}\right)_r (-2r-1)(-2r-2)......(-2r-m)x^{-2r-m-1}$$

$$= \sum_{r=0}^{\infty} (-1)^{m+r} \frac{\left(\frac{1}{2}\right)_r (2r+1)_m x^{-2r-m-1}}{\lfloor r}$$

But $$(2r+1)_m = \frac{\Gamma(2r+m+1)}{\Gamma(2r+1)} = \frac{2^m \Gamma\left(r+\frac{m+1}{2}\right)\Gamma\left(r+\frac{m}{2}+1\right)}{\Gamma\left(r+\frac{1}{2}\right)\Gamma(r+1)}$$

Putting the value of $(2r+1)_m$ in the above relation

$$\frac{d^m}{dx^m}\left(1+x^2\right)^{-1/2} = \sum_{r=0}^{\infty} \frac{(-1)^{m+r}\left(\frac{1}{2}\right)_r 2^m \Gamma\left(r+\frac{m}{2}+\frac{1}{2}\right)\Gamma\left(r+\frac{m}{2}+1\right) x^{-2r-m-1}}{\lfloor r \, \Gamma\left(r+\frac{1}{2}\right)\Gamma(r+1)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^{m+r}\left(\frac{m+1}{2}\right)_r \left(\frac{m+2}{2}\right)_r \lfloor m \; x^{-2r-m-1}}{\lfloor r \quad \lfloor r}$$

(Again applying Legendre's duplication formula)

$$\therefore \quad \frac{(-1)^m x^{m+1}}{\lfloor m} \frac{d^m}{dx^m}\left(1+x^2\right)^{-1/2} = \sum_{r=0}^{\infty} \frac{(-1)^{2m}\left(\frac{m+1}{2}\right)_r \left(\frac{m+2}{2}\right)_r (-1)^r x^{-2r}}{(1)_r \lfloor r}$$

$$= {}_2F_1\left(\frac{m+1}{2}, \frac{m+2}{2}; 1; \frac{-1}{x^2}\right)$$

Ex.3. Prove that

$$\frac{d^m}{dx^m}\left[x^{\alpha-1+m} {}_2F_1(a, b; c; x)\right] = (a)_m \, x^{a-1} \; {}_2F_1(a+m, b; c; x)$$

Sol. L.H.S. $= \frac{d^m}{dx^m}\left[x^{a-1+m} {}_2F_1(a, b; c; x)\right]$

$$= \frac{d^m}{dx^m}\left[\sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c)_r \lfloor r} \cdot x^{a+m+r-1}\right]$$

But $\frac{d^m}{dx^m}\left(x^{a+m+r-1}\right) = (a+m+r-1)(a+m+r-2)....(a+r)x^{a+r-1}$

$$= (a+r)_m \, x^{a+r-1} = \frac{(a+m)_r (a)_m}{(a)_r} x^{a+r-1}$$

$\therefore$ L.H.S. $= \sum_{r=0}^{\infty} \frac{(b)_r}{(c)_r \lfloor r} \cdot (a+m)_r \, (a)_m \, x^{a+r-1}$

$$= (a)_m \, x^{a-1} \; {}_2F_1(a+m, b; c; x)$$

= R.H.S.

Ex4. Prove that If a + b + c > 0, then

$$\lim_{x\to 1}\left\{(1-x)^{a+b-c} {}_2F_1(a, b; c; x)\right\} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$$

Sol. L.H.S. $= \lim_{x\to 1}\left\{(1-x)^{a+b-c} \; {}_2F_1(a, b; c; x)\right\}$

Now applying the transformation formula of

$${}_2F_1(a,b;c;x) = (1-x)^{c-a-b}{}_2F_1(c-a,c-b;c;x)$$

$$\therefore \quad \text{L.H.S.} = \lim_{x\to 1}\left\{(1-x)^{a+b-c}(1-x)^{c-a-b}{}_2F_1(c-a,c-b;c;x)\right\}$$

$$= \lim_{x\to 1}\left\{{}_2F_1(c-a,c-b;c;x)\right\}$$

$$= {}_2F_1(c-a,c-b;c;L)$$

$$= \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \qquad \text{(applying Gauss's theorem)}$$

$$= \text{R.H.S.}$$

Ex.5. Prove that $\quad {}_1F_1(a,b;z) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-t}t^{a-1}{}_0F_1(-;b;zt)\,dt$

Sol. R.H.S.

$$= \frac{1}{\Gamma(a)}\int_0^\infty e^{-t}t^{a-1}\sum_{r=0}^{\infty}\frac{(z)^r(t)^r}{(b)_r \lfloor r}dt$$

$$= \frac{1}{\Gamma(a)}\sum_{r=0}^{\infty}\frac{(z)^r}{(b)_r \lfloor r}\int_0^\infty e^{-t}t^{a-r-1}dt$$

$$= \frac{1}{\Gamma(a)}\sum_{r=0}^{\infty}\frac{(z)^r}{(b)_r \lfloor r}\Gamma(a+r)\,dt$$

$$= {}_1F_1(a,b;z) = \text{L.H.S.}$$

Self-Learning Exercise

1. $\frac{d^2}{dx^2}\left[{}_2F_1(a,\,b;c;x)\right] = ...$

2. $\lim_{b\to\infty}{}_2F_1\left(a,\,b\,;\,c;\frac{x}{b}\right) = ...$

3. $\lim_{a\to\infty}{}_1F_1\left(a,c;-\frac{x}{c}\right) = ...$

4. Write the Kummer's first transformation for ${}_1F_1$

5. $aF(a+) - bF(b+) = ...$

6. $\lim_{x\to 1}\left\{(1-x)^{a+b-c}\,{}_2F_1(a,b;c;x)\right\} = ...$

11.7 Summary

In this unit we established some important formulae such as differentiation formulas, contiguous function relations, linear relations etc. for Gauss hypergeometric function introduced in the last unit. We also introduced and studied Kummer's confluent hypergeometric function.

11.8 Answers to self-Learning Exercise

1. $\dfrac{a(a+1)b(b+1)}{c(c+1)}\,{}_2F_1(a+2,b+2;c+2;x)$

2. ${}_1F_1(a;c;x)$

3. $x^{(1-c)/2}\Gamma(c)J_{c-1}\left(2\sqrt{z}\right)$

4. ${}_1F_1(a;c;z)=e^{z}{}_1F_1(c-a;c;-z)$

5. $(a-b)F$

6. $\dfrac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$

11.9 Exercise

1. Prove that $\quad (b)_n\dfrac{d^n}{dx^n}\left[e^{-x}{}_1F_1(a;b;x)\right]=(-1)^n(b-a)_n\,e^{-x}{}_1F_1(a;b+n;x)$

2. Show that $\quad {}_1F_1(a;c;x)=\lim\limits_{b\to\infty}{}_2F_1\left(a;b;c;\dfrac{x}{b}\right)$

3. Show that $\quad (c)_m\dfrac{d^m}{dx^m}\left[e^{-x}{}_1F_1(a;c;x)\right]=(-1)^m(c-a)_m\,e^{-x}{}_1F_1(a;c+m;x)$

(**Hint.** Use Kummer's first transformation)

4. If incomplete gamma function is defined by $\gamma(a,x)=\int_0^x e^{-t}t^{a-1}dt,\ \operatorname{Re}(a)>0.$

Show that $\quad \gamma(a,x)=a^{-1}x^{a}{}_1F_1(a;a+1;-x).$

5. State Confluent hypergeometric differential equation and explain its solution,

6. Prove that

$$\int_0^\infty t^{\lambda-1}e^{-zt}{}_1F_1\left(a;\frac{a+\lambda+1}{2};\frac{zt}{2}\right)dt=\frac{\sqrt{\pi}\,z^{-\lambda}\Gamma(\lambda)\Gamma\left(\frac{1+a+\lambda}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1+\lambda}{2}\right)},\operatorname{Re}(\lambda)>0,\operatorname{Re}(z)>0.$$

[Hint First replace ${}_1F_1$ by its integral representation, then change the order of integration, Evaluate the inner integral in terms of the gamma function. Write down the remaining integral in terms of

$$ {}_2F_1\left(\lambda, a; \frac{\lambda+a+1}{2}; 1\right) = \frac{\Gamma\left(\frac{\lambda+a+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\lambda+1}{2}\right)\Gamma\left(\frac{\alpha+1}{2}\right)} $$

7. Prove that $\quad F(a, b+1; c+1; x) - F(a, b; c; x) = \dfrac{a(c-b)x}{c(c+1)} F(a+1, b+1; c+2; x)$

8. Prove the following relations :

(i) $F(a-1, b-1; c; x) - F(a, b-1, c; x) = \dfrac{(1-b)x}{b} F(a, b; c+1; x)$

(ii) $aF(a+1, b; c; x) - (c-1)F(a, b; c-1; x) = (a+1-c)F(a, b; c; x)$

9. Show that

(i) $e^x - 1 = x\,F(1, 2; x)$

(ii) $\left(1 + \dfrac{x}{a}\right)e^x = F(a+1; a; x)$

10. Prove the following relations

(i) $bF(a; b; x) = bF(a-1; b; x) + x\,F(a; b+1; x)$

(ii) $aF(a+1; b; x) - (b-1)\,F(a; b-1; x) = (a-b+1)F(a; b; x)$

□ □ □

Unit 12 : Legendre's Polynomials and Functions $P_n(x)$ and $Q_n(x)$

Structure of the Unit

12.0 Objective

Our aim of this unit is to develop the Legendre Polynomials and to discuss its important properties.

12.1 Introduction

Legendre polynomials may be introduced either through solution of a **differential equation** or through a **generating funcition.** We shall discuss both the methods. Legendre polynomials have many applications to mathematical physics and these applications depend on a number of special properties which Legendre polynomials possess.

12.2 Legendre Equation and its Solution

The differential equation of the form

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0 \qquad .....(1)$$

is called Legendre's equation, where n is a positive integer. This equation has regular singular points at $x = \pm 1$ and $x = \infty$, whereas all other points are ordinary, one of which be chosen as $x = 0$ since all other ordinary points may be transferred at the origin.

The solution of equation (1) in series of descending powers of x can be referred to example 1§ 9.5 of unit 9.

However for sake of completeness we here reproduce the solution of (1).

Let the solution of (1) be

$$y=\sum_{r=0}^{\infty}a_r x^{k-r},\ a_0 \neq 0 \qquad .....(2)$$

then $$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r x^{k-r-1}\left(k-r\right)$$

and $$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r\left(k-r\right)\left(k-r-1\right)x^{k-r-2}$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$\left(1-x^2\right)\sum_{r=0}^{\infty}a_r\left(k-r\right)\left(k-r-1\right)x^{k-r-2}-2x\sum_{r=0}^{\infty}a_r x^{k-r-1}\left(k-r\right)+n\left(n+1\right)\sum_{r=0}^{\infty}a_r x^{k-r}=0$$

or $$\sum_{r=0}^{\infty}a_r\left(k-r\right)\left(k-r-1\right)x^{k-r-2}+\sum_{r=0}^{\infty}\left\{n\left(n+1\right)-2\left(k-r\right)-\left(k-r\right)\left(k-r-1\right)\right\}a_r\,x^{k-r}=0$$

or $$\sum_{r=0}^{\infty}a_r\left(k-r\right)\left(k-r-1\right)x^{k-r-2}+\sum_{r=0}^{\infty}a_r\left(n-k+r\right)\left(n+k-r+1\right)x^{k-r}=0 \qquad .....(3)$$

Equating to zero the coefficient of the highest power of x namely x^k in (3), we get

$$a_0\,(n-k)(n+k+1)=0$$

$\Rightarrow$ $$k=n,\ -(n+1) \qquad (\because\ a_0\neq 0) \qquad .....(4)$$

The next lower power of x is $k-1$, so we equate to zero the coefficient of x^{k-1} in (3) and obtain

$$(n-k+1)\,(n+k)\,a_1=0 \qquad .....(5)$$

For $k=n$ and $-(n+1)$, neither $(n-k+1)$ nor $(n+k)$ in zero. thesefore $a_1=0$

Next equating to zero the coefficient of x^{k-r} in (3), we have

$$(k-r+2)\,(k-r+1)\,a_{r-2}+(n-k+r)\,(n+k-r+1)\,a_r=0$$

$$a_r=-\frac{(k-r+2)(k-r+1)}{(n-k+r)(n+k-r+1)}\,a_{r-2} \qquad .....(6)$$

Putting $n=3, 5, 7...$ in (6) and noting that $a_1=0$, we have

$$a_1=a_3=a_5=a_7=.....=0 \qquad .....(7)$$

To obtain a_2, a_4, a_6.... etc, we consider following two cases

Case I. When $k=n$ then (6) becomes

$$a_r=-\frac{(n-r+2)(n-r+1)}{r(2n-r+1)}\,a_{r-2} \qquad .....(8)$$

Putting $r=2, 4, 6, ...$ in (8), we have

$$a_2=-\frac{n(n-1)}{2(2n-1)}\,a_0$$

$$a_4=-\frac{(n-2)(n-3)}{4(2n-3)}\,a_2=\frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}\,a_0$$

and so on

Re-writing (2), we have for $k=n$

$$y=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+a_3x^{n-3}+a_4x^{n-4}+... \qquad ....(9)$$

Using (7) and the above values of a_2, a_4, a_4, etc in (9) we get

$$y=a_0\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4}-....\right] \qquad .....(10)$$

Case II. When $k=-(n+1)$ then (6) becomes

$$a_r=\frac{(n+r-1)(n+r)}{r(2n+r+1)}\,a_{r-2} \qquad .....(11)$$

Putting $r=2, 4, 6, ...$ etc., we get

$$a_2=\frac{(n+1)(n+2)}{2(2n+3)}\,a_0$$

$$a_4=\frac{(n+3)(n+4)}{4(2n+5)}\,a_2=\frac{(n+1)(n+2)(n+3)(n++4)}{2\cdot 4(2n+3)(2n+5)}\,a_0$$

and so on.

For $k = -(n+1)$, (2) gives

$$y = a_0 x^{-n-1} + a_1 x^{-n-2} + a_2 x^{-n-3} + a_3 x^{-n-4} + a_4 x^{-n-5} + \ldots \qquad \ldots(12)$$

Using (7) and the above values of a_2, a_4, a_4, etc. in (12), we find that

$$y = a_0 \left[x^{-n-1} - \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4 (2n+3)(2n+5)} x^{-n-5} + \ldots \right] \qquad \ldots(13)$$

Thus two independent solutions of (1) are given by (10) and (13). If we take

$$a_0 = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{\lfloor n}$$

then solution (10) is denoted by $P_n(x)$ and is called **Legendre polynomial of first kind** and if we take $a_0 = \dfrac{\lfloor n}{1 \cdot 3 \cdot 5 \ldots (2n+1)}$ then solution (13) is denoted by $Q_n(x)$ and is called **Legender polynomial of second kind** so the general solution of (1) is

$$y = A\, P_n(x) + B\, Q_n(x)$$

where A and B are arbitrary constants

12.3 Definition

12.3.1 Legendre's polynomial of degree *n* or Legendre's function of first kind

Legendre's polynomial of degree n is denoted and defined by

$$P_n(x) = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{n!} \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 (2n-1)(2n-3)} x^{n-4} \ldots \right]$$

$$= \sum_{r=0}^{[n/2]} (-1)^r \frac{\lfloor (2n-2r)}{2^n \lfloor r \lfloor (n-r) \lfloor (n-2r)} x^{n-2r}, \qquad \ldots(1)$$

where

$$\left[\frac{n}{2}\right] = \begin{cases} n/2, & \text{if } n \text{ is even} \\ (n-1)/2, & \text{if } n \text{ is odd,} \end{cases} \qquad \ldots(2)$$

12.3.2 Legendre's Function of Second Kind

This is denoted and defined by

$$Q_n(x) = \frac{n!}{1 \cdot 3 \ldots (2n+1)}$$

$$\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-(n+3)} + \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4 (2n+3)(2n+5)} x^{-(n+5)} + \ldots \right] \qquad \ldots(3)$$

12.3.3 Values of $P_n(x)$ for n = 0, 1, 2, 3, 4 and 5

Putting $n = 0, 1, 2, 3, 4$, and 5 in (1), and simplying the expression thus obtained we easily find that

$$P_0(x) = 1,\ P_1(x) = x$$

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right),\ P_3(x) = \frac{1}{2}\left(5x^2 - 3x\right) \quad .....(4)$$

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right) \text{ and} \quad .....(5)$$

$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right) \quad .....(6)$$

12.4 Generating Function for $P_n(x)$

Result. Show that $\left(1 - 2xh + h^2\right)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x), |x| \le 1, |h| < 1$

or show that $P_n(x)$ is the coefficient of h^n in the expansion of the $(1 - 2xh + h^2)$ in ascending powers of h. $(1 - 2xh + h^2)^{-1/2}$ is called generating function for Legendre polynomial $P_n(x)$.

Proof. Since $|h| < 1$ and $|x| \le 1$, we have

$$(1 - 2xh + h^2)^{-1/2} = \left[1 - h(2x - h)\right]^{-1/2}$$

$$= 1 + \frac{1}{2}h(2x-h) + \frac{1\cdot 3}{2\cdot 4}h^2(2x-h)^2 + ...$$

$$+ \frac{1\cdot 3....(2n-3)}{2\cdot 4....(2n-2)}h^{n-1}(2x-h)^{n-1} + \frac{1\cdot 3....(2n-1)}{2\cdot 4....(2n)}h^n(2x-h)^n + ... \quad .....(1)$$

Now, the coefficient of h^n in

$$\frac{1\cdot 3....(2n-1)}{2\cdot 4....(2n)}h^n(2x-h)^n = \frac{1\cdot 3\cdot 5....(2n-1)}{2\cdot 4\cdot 6....(2n)}(2x)^n$$

$$= \frac{1\cdot 3\cdot 5\cdot(2n-1)}{\lfloor n}x^n \quad .....(2)$$

Again the coefficient of h^n in

$$\frac{1\cdot 3....(2n-3)}{2\cdot 4....(2n-2)}h^{n-1}(2x-h)^{n-1} = \frac{1\cdot 3....(2n-3)}{2^{n-1}\cdot 1\cdot 2\cdot 3....(n-1)}\left[-(n-1)2^{n-2}x^{n-2}\right]$$

$$= -\frac{1\cdot 3....(2n-1)}{\lfloor n} \times \frac{n(n-1)}{2(2n-1)}x^{n-2} \quad .....(3)$$

and so on. Using (2), (3), ..., we see that coefficient of h^n in expansion of $(1 - 2xh + h^2)^{-1/2}$, viz. (1) is given by

$$\frac{1\cdot 3\cdot 5....(2n-1)}{\lfloor n}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4} - ...\right] = P_n(x)$$

Thus we can say that $P_1(x)$, $P_2(x)$, ... Will be coefficientts of h, h^2, ... in the expansion of $(1 - 2xh + h^2)^{-1/2}$. Hence we have

$$(1 - 2x + h^2)^{-1/2} = 1 + h P_1(x) + h^2P_2(x) + h^3P_3(x) + ... + h^nP_n(x) + ...$$

or $$\mathbf{(1 - 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)}$$

12.5 Rodrigues Formula for $P_n(x)$

Result. Show that $$P_n(x) = \frac{1}{2^n \lfloor n} \frac{d^n}{dx^n}\left(x^2 - 1\right)^n$$

Proof. Let $$y = (x^2 - 1)^n$$

$$\therefore \quad \frac{dy}{dx} = n(x^2 - 1)^{n-1} \cdot 2x$$

Multiplying both sidees by $(x^2 - 1)$, we get

$$(x^2 - 1)\frac{dy}{dx} = n(x^2 - 1)^n \cdot 2x = 2nxy$$

Differentiating $(n + 1)$ times both sides of the above equation and using Leibnitz theorem, we get

$$\left(x^2 - 1\right)\frac{d^{n+2}y}{dx^{n+2}} + {}^{n+1}c_1 \cdot 2x \cdot \frac{d^{n+1}y}{dx^{n+1}} + {}^{n+1}c_2 \cdot 2\frac{d^n y}{dx^n}$$

$$= 2n\left[x\frac{d^{n+1}y}{dx^{n+1}} + {}^{n+1}c_1 \frac{d^n y}{dx^n} \cdot 1\right]$$

Simplifying the above equation, we find that

or $$\left(1 - x^2\right)\frac{d^{n+2}y}{dx^{n+2}} - 2x\frac{d^{n+1}y}{dx^{n+1}} + n\left(n+1\right)\frac{d^n y}{dx^n} = 0 \qquad .....(1)$$

Let $\dfrac{d^n y}{dx^n} = z$ in (4). Then

$$\left(1 - x^2\right)\frac{d^2 z}{dx^2} - 2x\frac{dz}{dx} + n\left(n+1\right)z = 0 \qquad .....(2)$$

Now (2) is Legendre's equation and shows that z is a solution to this equation. Hence one of its solution be

$$z = \frac{d^n y}{dx^n} = c\, P_n(x) \qquad .....(3)$$

where c is constant

To find c, put $x = 1$ in both sides of (3), therefore

$$c\,P_n(1) = \left[\frac{d^n y}{dx^n}\right]_{x=1}$$

$$\Rightarrow \qquad c = \left[\frac{d^n y}{dx^n}\right]_{x=1} \qquad [\because\ P_n(1) = 1] \qquad .....(4)$$

Again $\qquad y = (x^2 - 1)^n = (x-1)^n \cdot (x+1)^n$

Differentiating both sides n times by Leibnitz's theorem, we get

$$\frac{d^n y}{dx^n} = (x-1)^n \frac{d^n (x+1)^n}{dx^n} + n \frac{d^{n-1}(x+1)^n}{dx^{n-1}} \left\{ n(x-1)^{n-1} \right\} + \ldots + (x+1)^n \frac{d^n (x-1)^n}{dx^n}$$

Now putting $x = 1$ in both sides of above relation, we see that all the terms in RHS except the last term vanishes since each term contains the factor $(x - 1)$, and also

$$\frac{d^n (x-1)^n}{dx^n} \lfloor n$$

Thus $\qquad \left(\frac{d^n y}{dx^n}\right)_{x=1} = (1+1)^n \lfloor n = 2^n \lfloor n \qquad .....(5)$

Now using (5) in (4), we find that

$$c = 2^n \lfloor n$$

Substituting the values of y and c in (3), we easily arrive at the Rodrigue's formula.

12.5.1 Alternative form of Rodrigue's formula

We have

$$P_n(x) = \frac{1}{2^n \lfloor n} \cdot \frac{d^n}{dx^n} \left\{ (x-1)^n (x+1)^n \right\}$$

By Leibnitz's rule we have

$$P_n(x) = \frac{1}{2^n \lfloor n} \sum_{r=0}^{\infty} {}^n c_r D^{n-r} (x-1)^n D^r (x+1)^n$$

$$= \sum_{r=0}^{\infty} \left({}^n c_r\right)^2 \left(\frac{x-1}{2}\right)^r \left(\frac{x+1}{2}\right)^{n-r} \qquad .....(6)$$

12.5.2 Application

Multiplying (6) by $\frac{t^n}{(\lfloor n)^2}$ and summing from $n = 0$ to ∞, we get

$$\sum_{n=0}^{\infty} \frac{P_n(x) t^n}{(\lfloor n)^2} = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \left(\frac{1}{\lfloor n-r \lfloor r}\right)^2 \left(\frac{x-1}{2}\right)^r \left(\frac{x+1}{2}\right)^{n-r} t^n$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{1}{(\underline{|n})^2 (\underline{|r})^2} \left(\frac{x-1}{2} \right)^r \left(\frac{x+1}{2} \right)^n t^{n+r}$$

$$\left[\text{using} \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n+k) \right]$$

$$= {}_0F_1\left(-;1;\frac{x-1}{2}t \right) {}_0F_1\left(-;1;\frac{x+1}{2}t \right) \qquad .....(7)$$

12.6 Orthogonal Property for $P_n(x)$

Result : Prove that

(i) $$\int_{-1}^{+1} \boldsymbol{P_m(x) P_n(x) dx = 0} \quad \text{if } \boldsymbol{m \neq n}$$

and *(ii)* $$\int_{-1}^{+1} \boldsymbol{[P_n(x)]^2 dx = \frac{2}{2n+1}} \quad \text{if } \boldsymbol{m = n}$$

Proof. The Legendre equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

or
$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0 \qquad .....(1)$$

Now since $P_m(x)$ and $P_n(x)$ are solutions of (1), hence

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} + m(m+1)P_m = 0 \qquad .....(2)$$

and
$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} + n(n+1)P_n = 0 \qquad .....(3)$$

Multiplying (2) by P_n and (3) by P_m and substracting, we get

$$P_m(x)\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} - P_n\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} + \{n(n+1) - m(m+1)\}P_nP_m = 0$$

Integrating above w.r.t. x form –1to 1, we get

$$\int_{-1}^{+1} P_m(x)\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}dx - \int_{-1}^{+1} P_n\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}dx + \{n(n+1) - m(m+1)\}\int_{-1}^{+1} P_nP_m dx = 0$$

On integration by parts, we get

$$\left[P_m(x)(1-x^2)\frac{dP_n}{dx}\right]_{-1}^{+1} - \int_{-1}^{+1}\left[\frac{dP_m}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}\right]dx - \left[P_n(x)(1-x^2)\frac{dP_m}{dx}\right]_{-1}^{+1}$$

$$-\int_{-1}^{+1}\left[\frac{dP_n}{dx}\left\{\left(1-x^2\right)\frac{dP_m}{dx}\right\}\right]dx+\left\{n\left(n+1\right)-m\left(m+1\right)\right\}\int_{-1}^{+1}P_mP_n dx=0$$

or $$\left(n-m\right)\left(n+m+1\right)\int_{-1}^{+1}P_mP_n dx=0$$

$$\Rightarrow \quad \int_{-1}^{+1}P_mP_n dx=0,\ m\neq n \qquad .....(4)$$

Case II. **When $m = n$.** From generating function

$$(1-2xh+h^2)^{-1/2}=\sum_{n=0}^{\infty}h^nP_n\left(x\right) \qquad .....(5)$$

$$(1-2xh+h^2)^{-1/2}=\sum_{m=0}^{\infty}h^mP_m\left(x\right) \qquad .....(6)$$

Multiplying the corresponding sides of (5) and (6), we get

$$(1-2xh+h^2)^{-1}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}P_n\left(x\right)P_m\left(x\right)h^{m+n}$$

Integrating both sides of the above with respect to 'x' from -1 to 1,we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left\{\int_{-1}^{+1}P_m\left(x\right)P_n\left(x\right)dx\right\}h^{m+n}=\int_{-1}^{+1}\left(1-2xh+h^2\right)^{-1}dx \qquad ..(7)$$

Making use of (4), (7) reduces to

$$\sum_{n=0}^{\infty}\left[\int_{-1}^{+1}\left\{P_n\left(x\right)\right\}^2dx\right]h^{2n}=-\frac{1}{2h}\left[\log\left(1-2xh+h^2\right)\right]_{-1}^{+1}$$

$$=\frac{1}{h}\left[\log\left(1+h\right)-\log\left(1-h\right)\right]$$

$$=\frac{1}{h}\left[\left(h-\frac{h^2}{2}+\frac{h^3}{3}-...\right)-\left(-h-\frac{h^2}{2}-\frac{h^3}{3}-...\right)\right]$$

$$=\frac{2}{h}\left[h+\frac{h^3}{3}+\frac{h^5}{5}+...\right]=\sum_{n=0}^{\infty}\frac{2}{2n+1}h^{2n}$$

Equating coefficients of h^{2n} from both sides, we get

$$\int_{-1}^{+1}P_n^2\left(x\right)dx=\frac{2}{2n+1}$$

12.7 Recurrence Formulas for $P_n(x)$

12.7.1 $(n+1)P_{n+1}(x) = (2n+1)\,xP_n(x) - nP_{n-1}(x),\ n \geq 1,$

or $(2n+1)\,xP_n(x) = (n+1)\,P_{n+1}(x) + n\,P_{n-1}(x)$

Proof. We know that

$$(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x) \qquad .....(1)$$

Differentiating (1) both sides *w.r.t. h*, we get

$$-\frac{1}{2}\left(1-2xh+h^2\right)^{-3/2}(-2x+2h) = \sum_{n=0}^{\infty} nh^{n-1}P_n(x)$$

or
$$(x-h)(1-2xh+h^2)^{-1/2} = (1-2xh+h^2)\sum_{n=0}^{\infty} nh^{n-1}P_n(x)$$

or
$$(x-h)\sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)\sum_{n=0}^{\infty} nh^{n-1}P_n(x)$$

Equating coefficients of h^n from both sides, we get

or
$$xP_n(x) - P_{n-1}(x) = (n+1)\,P_{n+1}(x) - 2xn\,P_n(x) + (n-1)\,P_{n-1}(x)$$

or
$$(2n+1)xP_n(x) = (n+1)\,P_{n+1}(x) + nP_{n-1}(x)$$

12.7.2 $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

Proof.

Differentiating (1) w.r.t. *'h'*, we get

$$-\frac{1}{2}\left(1-2xh+h^2\right)^{-3/2}(-2x+2h) = \sum_{n=0}^{\infty} nh^{n-1}P_n(x)$$

or
$$(x-h)(1-2xh+h^2)^{-3/2} = \sum_{n=0}^{\infty} nh^{n-1}P_n(x) \qquad .....(2)$$

Again differentiating (1) *w.r.t.'x'*, we find that

$$-\frac{1}{2}\left(1-2xh+h^2\right)^{-3/2}\times(-2h) = \sum_{n=0}^{\infty} h^n P_n'(x)$$

or
$$h(1-2xh+h^2)^{-3/2} = \sum_{n=0}^{\infty} h^n P_n'(x)$$

Multiplying by $(x-h)$ on both sides, we get

$$h(x-h)\left(1-2xh+h^2\right)^{-3/2} = (x-h)\sum_{n=0}^{\infty} h^n P_n'(x)$$

Using (2), we get $\sum_{n=0}^{\infty} nh^n P_n(x) = (x-h)\sum_{n=0}^{\infty} h^n P_n'(x)$

Equating coefficients of h^n from both sides of the above equation, we get

$$nP_n(x) = xP_n'(x) - P_{n-1}'(x)$$

12.7.3 $\quad$ $\mathbf{(2n+1)\,P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)n}$

Proof. From recurrence formulas 12.7.1, we have

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

Differentiating it *w.r.t.'x'*, we get

$$(2n+1)P_n(x) + (2n+1)xP'_n(x) = (n+1)P'_{n+1}(x) + nP'_{n-1}(x) \quad .....(3)$$

From recurrence 12.7.2, we have

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x)$$

or $\quad$ $$\mathbf{xP_n'(x) = nP_n(x) - P_{n-1}'(x)} \quad .....(4)$$

Using (4) in (3), we get

$$(2n+1)P_n(x) + (2n+1)\,[nP_n(x) + P'_{n-1}(x)] = (n+1)P'_{n+1}(x) + nP'_{n-1}(x)$$

or $\quad$ $(2n+1)(n+1)P_n(x) = (n+1)P'_{n+1}(x) - (n+1)P'_{n-1}(x)$

or $\quad$ $\mathbf{(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)}$

12.7.4 $\quad$ $\mathbf{(n+1)\,P_n(x) = [P'_{n+1}(x) - xP'_n(x)]}$

Proof. From recurrence formulae 12.7.2 and 12.7.3, we have

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x) \quad .....(5)$$

$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x) \quad .....(6)$$

Substracting, we get

$$\mathbf{(n+1)P_n(x) = P'_{n+1}(x) - xP'_n(x)}$$

12.7.5 $\quad$ $\mathbf{(1-x^2)\,P'_n(x) = x\,[P_{n-1}(x) - xP_n(x)]}$

Proof. From recurrence formulae 12.7.2, we have

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x)$$

Multiplying by x, we get $\quad nxP_n(x) = x^2P'_n(x) - xP'_{n-1}(x) \quad .....(7)$

Replacing n by $(n-1)$ in formula 12.7.4, we have

$$nP_{n-1}(x) = P'_n(x) - xP'_{n-1}(x) \quad .....(8)$$

Substracting (7) from (8), we have

$$\mathbf{x\,[P_{n-1}(x) - xP_n(x)] = (1-x^2)P'_n(x)}$$

12.7.6 $\quad$ $\mathbf{(1-x^2)\,P'_n(x) = (n+1)\,[xP_n(x) - P_{n+1}(x)]}$

Proof. From recurrence formula 12.7.1, we have

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

or $\quad$ $(n+1)xP_n(x) + xnP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$

or $(n+1)[xP_n(x) - P_{n+1}(x)] = n[P_{n-1}(x) - xP_n(x)]$ .....(9)

From formula 12.7.5 we have

$$(1-x^2)P'_n(x) = n[P_{n-1}(x) - xP_n(x)] \quad .....(10)$$

From (9) and (10), we easily get

$$\mathbf{(1-x^2)P'_n(x) = (n+1)[xP_n(x) - P_{n+1}(x)]}$$

Self-Learning Exercise–I

(1) The solution of Legendre's differential equation is known as ......

(2) $(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n$ ......

(3) $\int_{-1}^{+1} P_n(x)P_m(x)\,dx =$ ..... (if $m \neq n$)

(4) $P_n(1) =$ ......

(5) $P_n(x)$ is a polynomial of degree ......

(6) $P_n(x) = \frac{1}{2^n \lfloor n} \frac{d^n(x^2-1)^n}{dx^n}$ is known as ......

(7) $x =$ ...... is an ordinary point for Legendre differential equations.

(8) The value of $P_2(x)$ is ....

(9) $(n+1)P_n(x) - P'_{n+1}(x) + xP'_n(x) =$ .....

(10) if n is even/odd, then $P_n(x)$ is .... function of x.

12.8 Cristoffel's Expansion

Result : Prove that

$$\mathbf{P'_n = (2n-1)P_{n-1} + (2n-5)P_{n-3} + (2n-9)P_{n-5} + ...}$$

where $\mathbf{P_n \equiv P_n(x)}$ **and** $\mathbf{P'_n \equiv P'_n(x)}$ .....(1)

The last term of the series will be $3P_1$ or P_0 according as n is even or odd.

Proof : Replacing n by $n-1$ in recurrence formula 12.7.3, we have

$$P'_n = (2n-1)P_{n-1} + P'_{n-2} \quad .....(2)$$

Writing $n-2$, $n-4$, and so on in place of n in (2), we find that

$$\left.\begin{aligned} P'_{n-2} &= (2n-5)P_{n-3} + P'_{n-4} \\ P'_{n-4} &= (2n-9)P_{n-5} + P'_{n-6} \\ &....... \quad .... \quad .... \\ &....... \quad .... \quad .... \\ P'_3 &= 5P_2 + P'_1 \\ P'_2 &= 3P_1 + P'_0 \end{aligned}\right\} \quad .....(A)$$

When n is even, then adding the relations in (A) and (2), we get

$$P'_n = (2n-1)\,P_{n-1} + (2n-5)\,P_{n-3} + (2n-9)\,P_{n-5} + \ldots.. + 3P_1 \quad (\because \quad P'_0(x) = 0)$$

and **when n is odd,** then

$$P'_n = (2n-1)\,P_{n-1} + (2n-5)\,P_{n-3} + (2n-9)\,P_{n-5} + \ldots.. + 5P_2 + P_0$$

$$(\because \quad P'_1 = 1 = P_0)$$

12.8.1 Cristoffel's Summation Formula

Result : Prove that

$$\sum_{r=0}^{n}(2r+1)P_r(x)P_r(y) = (n+1)\left[\frac{P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x)}{x-y}\right] \qquad .....(3)$$

Proof. Prom recurrence formula 12.7.1, we have

$$(2r+1)\,xP_r(x) = (r+1)\,P_{r+1}(x) + rP_{r-1}(x) \qquad .....(4)$$

$$(2r+1)\,yP_r(y) = (r+1)\,P_{r+1}(y) + rP_{r-1}(y) \qquad .....(5)$$

Now multiplying (4) by $P_r(y)$ and (5) by $P_r(x)$ and subtracting, we find that

$$(2r+1)(x-y)\,P_r(x)\,P_r(y) = (r+1)\,[P_{r+1}(x)\,P_r(y) - P_{r+1}(y)\,P_r(x)]$$
$$+ r[P_{r-1}(x)\,P_r(y) - P_r(x)\,P_{r-1}(y)] \qquad .....(6)$$

Taking $r = 0, 1, 2, \ldots. n$ in (6) and adding the relations column-wise, we get the required result (3).

12.9 Expression For P_n (cos θ) in Terms of Cosine Series

we know that

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n \qquad .....(1)$$

Taking
$$x = \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

in (1) we easily get

$$\sum_{n=0}^{\infty} P_n(\cos\theta)t^n = (1-te^{i\theta})^{-1/2}(1-te^{-i\theta})^{-1/2} \qquad .....(2)$$

$$= \left[1+\frac{1}{2}te^{i\theta}+\frac{1\cdot 3}{2\cdot 4}t^2e^{2i\theta}+\ldots+\frac{1\cdot 3.....(2n-1)}{2\cdot 4\cdot 6.....2n}t^ne^{in\theta}+\ldots\right]$$

$$\times\left[1+\frac{1}{2}te^{-i\theta}+\frac{1\cdot 3}{2\cdot 4}t^2e^{-2i\theta}+\ldots+\frac{1\cdot 3.....(2n-1)}{2\cdot 4\cdot 6.....2n}t^ne^{-in\theta}+\ldots\right]$$

Now equating coefficients of t^n both sides, we get

$$P_n(\cos\theta) = \frac{1\cdot 3\cdot 5....(2n-1)}{2\cdot 4\cdot 6....(2n)}\left(e^{in\theta}+e^{-in\theta}\right)+\frac{1\cdot 3\cdot 5....(2n-3)}{2\cdot 4\cdot 6...(2n-2)}\cdot\frac{1}{2}\left(e^{i(n-2)\theta}+e^{-i(n-2)\theta}\right)$$

$$+\frac{1\cdot3\cdot5.....(2n-5)}{2\cdot4\cdot6....(2n-4)}\cdot\frac{1\cdot3}{2\cdot4}\left(e^{i(n-4)\theta}+e^{-i(n-4)\theta}\right)+.....$$

or $$P_n(\cos\theta)=\frac{1\cdot3\cdot5....(2n-1)}{1\cdot4\cdot6....2n}$$

$$\times\left[2\cos n\,\theta+\frac{n}{2n-1}\cdot2\cos(n-2)\theta+\frac{n(n-1)}{(2n-1)(2n-3)}\cdot\frac{1\cdot3}{1\cdot2}\cdot2\cos(n-4)\theta+...\right]$$

the above formula is useful in obtaining the integrals involving the products of $P_n(\cos\theta)$ and sine and cosine multiple of θ.

Ex.*1. *Prove that $\dfrac{1+z}{z\sqrt{1-xz+z^2}}-\dfrac{1}{z}=\sum_{n=0}^{\infty}(P_n+P_{n+1})z^n$

Sol. We have $$\text{RHS}=\sum_{n=0}^{\infty}(P_n+P_{n+1})z^n$$

$$=\sum_{n=0}^{\infty}z^nP_n+\frac{1}{z}\sum_{n=0}^{\infty}z^{n+1}P_{n+1} \qquad .....(3)$$

Also $$\sum_{n=0}^{\infty}z^{n+1}P_{n+1}=zP_1+z^2P_2+z^3P_3+.. \qquad .....(4)$$

and $$\sum_{n=0}^{\infty}z^nP_n=P_0+zP_1+z^2P_2+z^3P_3+.. \qquad .....(5)$$

Substracting (5) from (4), we get

$$\sum_{n=0}^{\infty}z^{n+1}P_{n+1}=\sum_{n=0}^{\infty}z^nP_n-P_0 \qquad .....(6)$$

Using (6) in (3), we get

$$\text{RHS}=\sum_{n=0}^{\infty}z^nP_n+\frac{1}{z}\left[\sum_{n=0}^{\infty}z^nP_n-P_0\right]$$

$$=\left(1+\frac{1}{z}\right)\sum_{n=0}^{\infty}z^nP_n(x)-\frac{P_0}{z}$$

$$=\left(1+\frac{1}{z}\right)\left(1-2xz+z^2\right)^{-1/2}-\frac{1}{z} \qquad [\because\ P_0=1]$$

= L.H.S.

***Ex.*2. *Prove that* $P_n(1)=1$ and $P_n(-1)=(-1)^n$**

Sol. We have $$\sum_{n=0}^{\infty}h^nP_n(x)=(1-2xh+h^2)^{-1/2} \qquad .....(7)$$

For $x = 1$, we have

$$\sum_{n=0}^{\infty} h^n P_n(1) = (1-h)^{-1}$$

$$= 1 + h + h^2 + ... h^n = \sum_{n=0}^{\infty} h^n$$

Equating coefficients of h^n on both sides, we find that $P_1(1) = 1$

Also for $x = -1$, equation (7) gives

$$\sum_{n=0}^{\infty} h^n P_n(-1) = (1+h)^{-1} = 1 - h + h^2 - .... + (-1)^n h^n .... + ...$$

$$= \sum_{n=0}^{\infty} (-1)^n h^n$$

Equating coefficients of h^n on both sides, we get $P_n(-1) = (-1)^n$

Ex.3. Prove that

$$\mathbf{(2n+1)(x^2-1)\, P_n' = n(n+1)(P_{n+1} - P_{n-1})}$$

and hence deduce that

$$\int_{-1}^{1} \left(x^2-1\right) P_{n+1}(x) P_n'(x)\, dx = \frac{2n(n+1)}{(2n+1)(2n+3)}$$

Sol. From recurrence relation 12.7.5 and 12.7.6, we have

$$(1-x^2) P_n' = n(P_{n-1} - xP_n) \qquad .....(8)$$

$$(1-x^2) P_n' = (n+1)(xP_n - P_{n+1}) \qquad .....(9)$$

Eliminating xP_n from (8) and (9), we get

$$\frac{\left(1-x^2\right)P_n'}{n} + \frac{\left(1-x^2\right)P_n'}{(n+1)} = P_{n-1} - P_{n+1}$$

or
$$\frac{(n+1)(1-x)^2 P_n' + n\left(1-x^2\right)P_n'}{n(n+1)} = P_{n-1} - P_{n+1}$$

or
$$(2n+1)(1-x^2) P_n' = n(n+1)[P_{n-1} - P_{n+1}]$$

or
$$(2n+1)(x^2-1) P_n' = n(n+1)[P_{n+1} - P_{n-1}] \qquad .....(10)$$

This result is known as **Beltrami's** relation.

Deduction

Multiplying both sides of (10) by $P_{n+1}(x)$ and integrating w.r.t. 'x' from -1 to 1, we find that

$$\int_{-1}^{1} \left(x^2-1\right) P_{n+1}(x) P_n'(x)\, dx = \frac{n(n+1)}{(2n+1)} \int_{-1}^{1} P_{n+1}(x)\left[P_{n+1}(x) - P_{n-1}(x)\right] dx$$

Using orthogonal property for Legendre's polynomials, we get the required integral

Ex.4. Show that $\quad \boldsymbol{P_n(x)=\frac{1}{\pi}\int_0^{\pi}\left[x\pm\sqrt{(x^2-1)}\cos\theta\right]^n d\theta}$

where n is a positive integer.

This result is also known that **Laplace's first integral** for $P_n(x)$.

Proof. We know that

$$\int_0^{\pi}\frac{d\theta}{a\pm b\cos\theta}=\frac{\pi}{\sqrt{a^2+b^2}},\text{ we have } a^2>b^2 \qquad .....(11)$$

Taking $a = 1 - hx$ and $b=h\sqrt{x^2-1}$, then

$$a^2 - b^2 = (1 - hx)^2 - h^2(x^2 - 1) = 1 - 2xh + h^2$$

Thus (11) becomes
$$\int_0^{\pi}\frac{d\theta}{(1-hx)\pm h\sqrt{(x^2-1)}\cos\theta}=\frac{\pi}{\sqrt{1-2xh+h^2}}$$

or
$$\pi(1-2xh+h^2)^{-1/2}=\int_0^{\pi}\frac{d\theta}{(1-hx)\pm h\sqrt{(x^2-1)}\cos\theta}$$

or
$$\pi\sum_{n=0}^{\infty}h^nP_n(x)=\int_0^{\pi}\left[1-h\left\{x\pm\sqrt{(x^2-1)}\cos\theta\right\}\right]^{-1}d\theta$$

$$=\int_0^{\pi}(1-ht)^{-1}\,d\theta,\quad\text{where } t=x\pm\sqrt{x^2-1}\cos\theta$$

$$=\int_0^{\pi}\left(1+ht+h^2t^2+\ldots\;+h^nt^n+\ldots\right)d\theta$$

$$=\sum_{n=0}^{\infty}\int_0^{\pi}h^nt^n\,d\theta$$

Equating coefficients of h^n from both sides, we get

$$\pi P_n(x)=\int_0^{\pi}t^n d\theta$$

or
$$P_n(x)=\frac{1}{\pi}\int_0^{\pi}\left[x\pm\sqrt{x^2-1}\cos\theta\right]^n d\theta$$

Ex.5. Prove that

$$\int_0^{\pi}x^2P_{n+1}P_{n-1}dx=\frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$$

Proof. From Recurrence formulae 12.7.1 we have

$$(2n+1)\, xP_n = (n+1)\, P_{n+1} + np_{n-1}$$

Put $(n-1)$ and $(n+1)$ in place of n respectively, we get

$$(2n-1)\, xP_{n-1} = nP_n + (n-1)\, P_{n-2} \qquad .....(12)$$

$$(2n+3)\, xP_{n+1} = (n+2)\, P_{n+2} + (n+1)\, P_n \qquad .....(13)$$

Multiplying (12) and (13), we get

$$(2n-1)\,(2n+3)\, x^2 P_{n-1} P_{n+1} = n(n+2)\, P_n P_{n+2} + n(n+1)\, P^2_n$$
$$+ (n+2)\,(n-1)\, P_{n+2} P_{n-2} + (n^2-1)\, P_n P_{n-2}$$

Integrating w.r.t. x between limit -1 to $+1$, we have

$$(2n-1)(2n+3)\int_{-1}^{+1} x^2 P_{n-1} P_{n+1}\, dx = n(n+1)\int_{-1}^{+1} P_n^2\, dx$$

(other integrals on the RHS vanish due to integral $\int_{-1}^{+1} P_m P_n\, dx = 0$ if $m \neq n$)

or $$(2n-1)(2n+3)\int_{-1}^{+1} x^2 P_{n+1} P_{n+1}\, dx = \frac{2n(n+1)}{(2n+1)}$$

or $$\boldsymbol{\int_{-1}^{+1} x^2 P_{n-1}\, P_{n+1}\, dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}}$$

Ex.6. Show that $\boldsymbol{\int_{-1}^{+1} x P_n P_{n-1}\, dx = \frac{2n}{4n^2-1}}$

Proof. From Recurrence relation 12.7.1 we have

$$(2n+1)\, xP_n = (n+1)\, P_{n+1} + nP_{n-1} \qquad .....(14)$$

Multiplying (14) by P_{n-1} and then integrating w.r.t. x from -1 to $+1$, we get.

$$(2n+1)\int_{-1}^{+1} xP_n\, P_{n-1}\, dx = (n+1)\int_{-1}^{+1} P_{n-1}\, P_{n+1}\, dx + n\int_{-1}^{+1} [P_{n-1}]^2\, dx$$

Using orthogonal property for Legendre polynomial, we get

$$(2n+1)\int_{-1}^{1} xP_n P_{n-1}\, dx = \frac{2n}{(2n-1)}$$

$$\therefore \qquad \int_{-1}^{+1} xP_n\, P_{n-1} dx \cdot = \frac{2n}{4n^2-1}$$

Ex.7. Prove that $\boldsymbol{P_n(x) = \frac{1}{\pi}\int_0^{\pi} \frac{d\phi}{\left[x \pm \sqrt{(x^2-1)}\cos\phi\right]^{n+1}}}$ .....(15)

Sol. Taking $a = xt - 1$ and $b = t\sqrt{x^2-1}$, then $a^2 - b^2 = 1 - 2xt + t^2$

$$\therefore \qquad \frac{\pi}{\sqrt{a^2-b^2}} = \frac{\pi}{t}\left(1-\frac{2x}{t}+\frac{1}{t^2}\right)^{-1/2} = \pi\sum_{n=0}^{\infty} P_n(x)\,t^{-n-1} \text{ for large } t \qquad .....(16)$$

Also
$$\frac{1}{a \pm b\cos\phi} = \left[t\left\{x \pm \sqrt{x^2-1}\cos\phi\right\}-1\right]^{-1}$$

$$= \left[t\left\{x \pm \sqrt{x^2-1}\cos\phi\right\}\right]^{-1}\left[1-\frac{1}{t\left\{x \pm \sqrt{x^2-1}\cos\phi\right\}}\right]^{-1}$$

$$= \sum_{n=0}^{\infty}\frac{t^{-n-1}}{\left[x \pm \sqrt{x^2-1}\cos\phi\right]^{n+1}} \qquad .....(17)$$

Now integrating (17) both sides w.r.t. ϕ in $(0, \pi)$, we get

$$\int_0^{\pi}\left[\sum_{n=0}^{\infty}\frac{t^{-n-1}}{\left[x \pm \sqrt{x^2-1}\cos\phi\right]^{n+1}}\right]d\,\phi = \int_0^{\pi}\frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{\sqrt{a^2-b^2}}$$

Using (16) in the above expression, we find that

$$\sum_{n=0}^{\infty}\left[\int_0^{\pi}\frac{d\phi}{\left[x \pm \sqrt{x^2-1}\cos\phi\right]^{n+1}}\right]t^{-n-1} = \pi\sum_{n=0}^{\infty}P_n(x)\,t^{-n-1} \qquad .....(18)$$

Equating coefficicents of t^{-n-1} in (18), we get the required integral (15).

Remark. The integral given by (15) is known as **Laplace's second integral**.

Ex.8. Evaluate $\displaystyle\int_0^{\pi} P_n(\cos\theta)\cos n\theta\, d\theta$

Sol. By §12.9, we have

$$P_n(\cos\theta) = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 5\ldots 2n}\left[2\cos n\theta + \frac{n}{2n-1}2\cos(n-2)\theta\right.$$

$$\left. + \frac{n(n-1)}{(2n-1)(2n-3)}\cdot\frac{1\cdot 3}{1\cdot 2}2\cos(n-4)\theta + \ldots\right] \qquad .....(19)$$

Multiplying (19) both sides by $\cos n\,\theta$ and integrating w.r.t θ in $(0, \pi)$ we get

$$I=\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 5\ldots 2n}\int_0^{\pi}\left[2\cos^2 n\,\theta+\frac{2n}{2n-1}\cdot\cos n\theta\cos(n-2)\theta+\right.$$

$$\left.+\frac{2n(n-1)}{(2n-1)(2n-3)}\cdot\frac{1\cdot 3}{1\cdot 2}\cos n\,\theta\cos(n-4)\theta+\ldots\right]d\theta$$

Using the following orthogonal property for cosine function

$$\int_0^{\pi}\cos m\theta\cos n\theta\,d\theta=\begin{cases}0, & m\neq n\\ \pi/2, & m=n\end{cases},$$

we find that
$$I=\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}\cdot 2\frac{\pi}{2}=\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\left(\frac{2n-1}{2}\right)}{1\cdot 2\cdot 3\ldots\quad n}\cdot\pi$$

$$=\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(n+1)}=B\left(n+\frac{1}{2},\ \frac{1}{2}\right).$$

12.10 Recurrence Formulae for Q_n (x)

We have already defined that

$$\therefore\qquad Q_n(x)=\frac{2^n(\underline{n})^2}{\underline{2n+1}}\left[x^{-(n+1)}+\frac{(n+1)(n+2)}{2\cdot(2n+3)}x^{-(n+3)}+\ldots\ldots\right]\qquad\ldots\ldots(1)$$

Again above relation can be written as

$$Q'_n(x)=\frac{2^n\underline{n}}{\underline{2n+1}}\sum_{r=0}^{\infty}\frac{\underline{n+2r}\,x^{-(n+2r+1)}}{2^r\underline{r}(2n+3)\ldots\ldots(2n+2r+1)}\qquad\ldots\ldots(2)$$

Differentiating (2) with respect x, we get

$$Q'_n(x)=-\frac{2^n\underline{n}}{\underline{2n+1}}\sum_{r=0}^{\infty}\frac{\underline{(n+2r+1)}\,x^{-(n+2r+2)}}{2^r\underline{r}(2n+3)\ldots\ldots(2n+2r+1)}\qquad\ldots\ldots(3)$$

Putting $n-1$ for n, then we get

$$Q'_{n-1}(x)=-\frac{2^n(\underline{n})}{\underline{2n+1}}\sum_{r=0}^{\infty}\frac{\underline{(n+2r)}\,x^{-(n+2r+1)}}{2^r\underline{r}(2n+3)\ldots\ldots(2n+2r-1)}\qquad\ldots\ldots(4)$$

Again putting $n+1$ for n in (3), we get

$$Q'_{n+1}(x)=-\frac{2^n\underline{n}}{\underline{2n}}\sum_{r=0}^{\infty}\frac{\underline{(n+2r+2)}\,x^{-(n+2r+3)}}{2^r\underline{r}(2n+1)(2n+3)\ldots\ldots(2n+2r+3)}\qquad\ldots\ldots(5)$$

12.10.1 $\qquad \boldsymbol{Q'_{n+1}-Q'_{n-1}=(2n+1)Q_n}$

Proof. Using (1) and (4) above, we get

$$Q'_{n-1}+(2n+1)\,Q_n=-\frac{2^n\underline{|n}}{\underline{|2n+1}}\sum_{r=0}^{\infty}\frac{\underline{|n+2r}\;x^{-(n+2r+1)}}{2^r\underline{|r}\,(2n+3)\ldots\ldots(2n+2r-1)}$$

$$+(2n+1)\cdot\frac{2^n\underline{|n}}{\underline{|2n+1}}\sum_{r=0}^{\infty}\frac{\underline{|n+2r}\;x^{-(n+2r+1)}}{2^r\underline{|r}\,(2n+3)\ldots\ldots(2n+2r+1)}$$

$$=-\frac{2^n\underline{|n}}{\underline{|2n+1}}\sum_{r=0}^{\infty}\frac{\underline{|n+2r}\;x^{-(n+2r+1)}}{2^r\underline{|r}\,(2n+3)\ldots\ldots(2n+2r+1)}\left[2n+2r+1-(2n+1)\right]$$

$$=-\frac{2^n\underline{|n}}{\underline{|2n+1}}\sum_{r=0}^{\infty}\frac{\underline{|n+2r}\;x^{-(n+2r+1)}}{2^r\underline{|r}\,(2n+3)\ldots\ldots(2n+2r+1)}\times(2r)$$

$$=-\frac{2^n\underline{|n}}{\underline{|2n+1}}\sum_{r=1}^{\infty}\frac{\underline{|n+2r}\;x^{-(n+2r+1)}}{2^{r-1}\underline{|r-1}\,(2n+3)\ldots\ldots(2n+2r+1)}$$

Putting $r-1=s\Rightarrow r=s+1$, therefore

$$Q'_{n-1}+(2n+1)\,Q_n=-\frac{2^n\underline{|n}}{\underline{|2n+1}}\sum_{r=0}^{\infty}\frac{\underline{|n+2s+2}\;x^{-(n+2s+3)}}{2^s\underline{|s}\,(2n+3)\ldots\ldots(2n+2s+3)}$$

$$=Q'_{n+1}(x)=LHS$$

12.10.2 $\boldsymbol{nQ'_{n+1}+(n+1)Q'_{n-1}=(2n+1)\,x\,Q'_n}$

Proof. Using (1) and (4) above, we get

$$(2n+1)x\,Q'_n-(n+1)\,Q'_{n-1}=(2n+1)x\cdot\frac{-2^n\underline{|n}}{\underline{|2n+1}}\sum_{r=0}^{\infty}\frac{\underline{|n+2r+1}\;x^{-(n+2r+2)}}{2^r\underline{|r}\,(2n+3)\ldots\ldots(2n+2r+1)}$$

$$-(n+1)\,\frac{-2^n\underline{|n}}{\underline{|2n+1}}\sum_{r=0}^{\infty}\frac{\underline{|n+2r}\;x^{-(n+2r+1)}}{2^r\underline{|r}\,(2n+3)\ldots\ldots(2n+2r-1)}$$

$$=\frac{(-1)2^n\underline{|n}}{\underline{|2n+1}}\sum_{r=0}^{\infty}\frac{\underline{|n+2r}\;x^{-(n+2r+1)}}{2^r\underline{|r}\,(2n+3)\ldots\ldots(2n+2r+1)}$$

$$\left[(2n+1)x(n+2r+1)x^{-1}-(n+1)\cdot(2n+2r+1)\right]$$

$$=\frac{(-1)2^n\underline{|n}}{\underline{|2n+1}}\sum_{r=0}^{\infty}\frac{\underline{|n+2r+1}\;x^{-(n+2r+1)}}{2^r\underline{|r}\,(2n+3)\ldots\ldots(2n+2r+1)}\times 2nr$$

$$= \frac{(-n)2^n \lfloor n}{\lfloor 2n+1} \sum_{r=1}^{\infty} \frac{\lfloor n+2r \; x^{-(n+2r+1)}}{2^{r-1} \lfloor r-1 (2n+3) \ldots\ldots (2n+2r+1)}$$

$$= \frac{(-n)2^n \lfloor n}{\lfloor 2n+1} \sum_{s=0}^{\infty} \frac{\lfloor n+2s+2 \; x^{-(n+2s+3)}}{2^{s} \lfloor s (2n+3) \ldots\ldots (2n+2s+3)}$$

$$= nQ'_{n+1}(x) = L.H.S.$$

12.10.3 $\qquad (2n+1)xQ_n = (n+1)Q_{n+1} + nQ_{n-1}$

Proof. Integrateing the recurrence relation 12.10.2 w.r.t. x from x to ∞, we get

$$\int_x^{\infty} \left[nQ'_{n+1} + (n+1)Q'_{n-1} \right] dx = (2n+1) \int_x^{\infty} xQ'_n dx$$

or $\quad \left[nQ_{n+1} + (n+1)Q_{n-1} \right]_x^{\infty} = (2n+1) \left[(xQ_n)_x^{\infty} - \int_x^{\infty} Q_n(x)\, dx \right]$

$$= (2n+1)[xQ_n]_x^{\infty} - (2n+1) \int_x^{\infty} \frac{\left[Q'_{n+1} - Q'_{n-1} \right] dx}{(2n+1)} \qquad \text{(by relation 12.10.1)}$$

$$= (2n+1)[xQ_n]_x^{\infty} - [Q_{n+1}]_x^{\infty} + [Q_{n-1}]_x^{\infty}$$

The value of Q_{n-1}, Q_n or Q_{n+1} is zero when x is infinity since they contain only negative integral power of x, therefore

$$-nQ_{n+1} - (n+1)\, Q_{n-1} = -(2n+1)\, x\, Q_n + Q_{n+1} - Q_{n-1}$$

Solving it we easily get the required ralation 12.10.3

12.10.4 $\qquad (2n+1)(1-x^2)Q'_n = n(n+1)(Q_{n-1} - Q_{n+1})$

Sol. Since Q_n is a solution of Legendre's equation, namely

$$\frac{d}{dx}\left[(1-x^2) \frac{dy}{dx} \right] + n(n+1)y = 0$$

Therefore $\qquad \dfrac{d}{dx}\left[(1-x^2) Q'_n \right] = -n(n+1)Q_n \qquad \ldots..(5)$

Integrating w.r.t. x both sides of (5) between the limits, ∞ to x, we have

$$\left[(1-x^2) Q'_n \right]_{\infty}^{x} = -n(n+1) \int_{\infty}^{x} Q_n dx$$

or $\qquad (1-x^2) Q'_n(x) = -n(n+1) \int_{\infty}^{x} Q_n dx \qquad \ldots\ldots(6)$

Integrating both sides of recursence ralation 12.10.1 between the limit ∞ to x, we get

$$Q_{n+1} - Q_{n-1} = \int_{\infty}^{x} (2n+1) Q_n dx \qquad \ldots..(7)$$

Now, from (6) and (7), we get

$$\left(1-x^2\right)Q_n'(x) = -n(n+1)\left[\frac{Q_{n+1}(x)-Q_{n-1}(x)}{(2n+1)}\right]$$

$$(2n+1)\left(1-x^2\right)Q_n'(x) = n(n+1)\left[Q_{n-1}(x)-Q_{n+1}(x)\right]$$

12.11 Cristoffel's Second Summation Formula

Result. $$(y-x)\sum_{r=1}^{n}(2r+1)P_r(x)Q_r(y)$$

$$= 1-(n+1)\left[P_{n+1}(x)Q_n(y)-P_n(x)Q_{n+1}(y)\right] \qquad \text{......(1)}$$

Proof : From recurrence formulas for $P_n(x)$ and $Q_n(x)$, we have

$$(2n+1)\,x\,P_n(x) = (n+1)P_{n+1}(x)+nP_{n-1}(x) \qquad \text{.....(2)}$$

$$(2n+1)\,y\,Q_n(y) = (n+1)Q_{n+1}(y)+nQ_{n-1}(y) \qquad \text{.....(3)}$$

Multiplying (2) by $Q_n(y)$ and (3) by $P_n(x)$ and subtracting, we have

$(2n+1)(x-y)P_n(x)Q_n(y)+n\{P_{n-1}(x)Q_{n-1}(y)-Q_{n-1}(y)P_n(x)\}$

$$= (n+1)\{P_{n+1}(x)Q_n(y)-P_n(x)Q_{n+1}(y)\} \qquad \text{.....(4)}$$

Taking $n = 1, 2, 3$ ...... , n in (4) and adding, we get

$$(y-x)\sum_{r=1}^{n}(2r+1)P_r(x)Q_r(y)+\{Q_1(x)P_0(y)-Q_0(y)P_1(x)\}$$

$$= -(n+1)\{P_{n+1}(x)Q_n(y)-P_n(x)Q_{n+1}(y)\} \qquad \text{.....(5)}$$

Now since $Q_1(y) = y$, $Q_0(y) = 1$, $P_1(x) = x$, $P_0(x) = 1$, therefore (5) gives the required result (1).

12.12 Relations Between $P_n(x)$ and $Q_n(x)$

Result. Prove that $$\frac{1}{y-x} = \sum_{m=0}^{\infty}(2m+1)P_m(x)Q_m(y)$$

and hence deduce that

$$Q_m(y) = \int_{-1}^{1}\frac{P_m(x)}{y-x}dx, \quad (y>1)$$

Proof : Let $$f(x) = \frac{1}{y-x} = \frac{1}{y}\left(1-\frac{x}{y}\right)^{-1} = y^{-1}\left(1+\frac{x}{y}+\frac{x^2}{y}+.....+\frac{x^m}{y^m}+......\right)$$

$$= y^{-1}+x\cdot y^{-2}+x^2y^{-3}+......+x^m\cdot y^{-m-1}+.......$$

$$= A_0+A_1x+A_2x^2+....... \quad \text{(Suppose that)} \qquad \text{......(1)}$$

where A's are constants.

Further suppose that $f(x) = \sum_{m=0}^{\infty} B_m P_m(x),$

then we know that $B_m = \frac{1.2.3.......m}{1.3.5......(2m-1)}\left[A_m + \frac{(m+1)(m+2)}{2(2m+3)}A_{m+2} + ...\right]$ ......(2)

Comparing (1) and (3) we get

$$A_0 = y^{-1},\ A_1 = y^{-2},\ ....,\ A_m = y^{-(m+1)},.....$$

$$\therefore \qquad B_m = \frac{m!}{1\cdot 3\cdot 5\cdot (2m-1)}\left[y^{-(m+1)} + \frac{(m+1)(m+2)}{2(2m+3)}\cdot y^{-(m+3)} + ...\right]$$

$$= (2m+1)\, Q_m(y)$$

Hence $$\frac{1}{y-x} = \sum_{m=0}^{\infty}(2m+1)Q_m(y)P_m(x) \qquad .....(3)$$

Now multiplying (3) by $P_m(x)$ and integrating w.r.t x in the interval (–1, 1), we find that

$$\int_{-1}^{1} P_m(x)\cdot\frac{1}{y-x}dx = \int_{-1}^{1} P_m(x)\left[\sum_{m=0}^{\infty}(2m+1)P_m(x)Q_m(y)\right]dx$$

$$= Q_m(y)\int_{-1}^{1}\left[P_m(x)\right]^2(2m+1)\,dx \qquad \left[\because \int_{-1}^{1} Pm\,(n)\,P_n(x)\,dx = 0, m \neq n\right]$$

$$= Q_n(y)\cdot(2m+1)\cdot\frac{2}{2m+1} \qquad \left[\because \int_{-1}^{1} P_m^2(x)\,d_n = \frac{2}{2m+1}\right]$$

$$\therefore \quad \frac{1}{2}\int_{-1}^{1} P_m(x)\frac{1}{y-x}dx = Q_n(y)$$

This integral is called the **Neumann's integral for $Q_n(y)$.**

Ex.1. Prove that $\left(x^2-1\right)\left(Q_nP_n' - P_nQ_n'\right) = c$ ***and deduce that***

(i) $$\frac{Q_n}{P_n} = \int_x^{\infty}\frac{dx}{\left(x^2-1\right)P_n^2}$$

(ii) $$Q_0(x) = \frac{1}{2}\log\frac{x+1}{x-1}$$

(iii) $$Q_0(x) = \frac{x}{2}\log\frac{x+1}{x-1} - 1$$

Sol. The Legendre's equation is

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

Since $P_n(x)$ and $Q_n(x)$ are both the solution of this equation, therefore

$$\left(1-x^2\right)\frac{d^2P_n(x)}{dx^2} - 2x\frac{dP_n(x)}{dx} + n(n+1)P_n(x) = 0 \qquad .....(4)$$

and $$\left(1-x^2\right)\frac{d^2}{dx^2}Q_n(x)-2x\frac{d}{dx}Q_n(x)+(n+1)Q_n(x)=0 \qquad .....(5)$$

Multiplying (2) by $Q_n(x)$, and (3) by $P_n(x)$ and then substracting, we get

$$\left(x^2-1\right)\left[\frac{d^2}{dx^2}P_n(x)Q_n(x)-P_n(x)\frac{d^2}{dx^2}Q_n(x)\right]$$

$$+2x\left[Q_n(x)\frac{d}{dx}P_n(x)-P_n(x)\frac{d}{dx}Q_n(x)\right]=0$$

that is $$\frac{d}{dx}\left[\left(-1+x^2\right)\left\{\frac{d}{dx}P_n(x)\cdot Q_n(x)-P_n(x)\frac{d}{dx}Q_n(x)\right\}\right]=0$$

Integrating the above w.r.t x, we get

$$\left(x^2-1\right)\left\{P_n'(x)Q_n(x)-P_n(x)Q_n'(x)\right\}=c \qquad .....(6)$$

Deduction. *(i)* $$P_n'(x)Q_n(x)-Q_n'(x)P_n(x)=\frac{c}{x^2-1}=\frac{c}{x^2}\left(1-\frac{1}{x^2}\right)^{-1}$$

$$=c\left(\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^6}+......\right) \qquad .....(7)$$

Now $$P_n(x) = \frac{1.3....(2n-1)}{\lfloor n}\left[x^{-n}-\frac{n(n-1)x^{n-2}}{2(2n-1)}+...\right]$$

and $$Q_n(x) = \frac{\lfloor n}{1.3.....(2n+1)}\left[x^{-n-1}+\frac{(n+1)(n+2)\,x^{-n-3}}{2(2n+3)}+......\right]$$

Putting these values in (7), we get

$$\left[\frac{1.3.....(2n-1)}{\lfloor n}\left\{nx^{n-1}-\frac{n(n-1)(n-2)x^{n-3}}{2(2n-1)}+....\right\}\right]\times\frac{\lfloor n}{1.3....(2n+1)}$$

$$\times\left[x^{-n-1}+\frac{(n+1)(n+2)x^{-n-3}}{2(2n+3)}+......\right]-\frac{\lfloor n}{1.3....(2n+1)}$$

$$\times\left[\left\{-(n+1)x^{-n-2}-\frac{(n+1)(n+2)(n+3)x^{-n-4}}{2(2n+3)}+.......\right\}\right]$$

$$\times\left[\frac{1.3......(2n-1)}{\lfloor n}\left\{x^n-\frac{n(n-1)x^{n-2}}{2(2n-1)}+......\right\}\right]=c\left[\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^6}+......\right]$$

Equating the coefficients of $1/x^2$ from both sides, we get

$$\frac{1\cdot3\cdot(\lfloor 2n-1)}{\lfloor n}\cdot n\times\frac{\lfloor n}{1\cdot3\cdots(2n+1)}+\frac{\lfloor n(n+1)}{1.3....(2n+1)}\times\frac{1.3....(2n-1)}{\lfloor n}=c$$

$$\Rightarrow c = \frac{n}{(2n+1)} + \frac{(n+1)}{2n+1} = 1$$

Substituting c = 1 in (6), we get

$$P_n'(x)Q_n(x) - Q_n'(x)P_n(x) = \frac{1}{x^2-1}$$

$$\Rightarrow \frac{P_n'(x)Q_n(x) - Q_n'(x)P_n(x)}{P_n^2(x)} = \frac{1}{(x^2-1)P_n^2(x)}$$

$$\Rightarrow -\frac{d}{dx}\left[\frac{Q_n(x)}{P_n(x)}\right] = \frac{1}{(x^2-1)P_n^2(x)}$$

Integrating both sides w.r.t. x between the limit x to ∞, we get

$$\left[-\frac{Q_n}{P_n}\right]_x^\infty = \int_x^\infty \frac{dx}{(x^2-1)P_n^2(x)}$$

or $$\frac{Q_n(x)}{P_n(x)} - \lim_{x\to\infty}\frac{Q_n(x)}{P_n(x)} = \int_x^\infty \frac{dx}{(x^2-1)P_n^2(x)} \qquad .....(8)$$

Now $$\lim_{x\to\infty}\frac{Q_n(x)}{P_n(x)} = \lim_{x\to\infty}\frac{\frac{d^n}{dx^n}Q_n(x)}{\frac{d^n}{dx^n}P_n(x)}$$

$$= \lim_{x\to\infty}\frac{\frac{n!}{1.3.5....(2n-1)}\left\{(-1)^n(n+1)(n+2).....2nx^{-(2n+1)} + ...\right\}}{\frac{1.3.5.....(2n-1)}{n!}\cdot n!}$$

$$= 0$$

Thus (8) reduces to

$$\frac{Q_n(x)}{P_n(x)} = \int_x^\infty \frac{dx}{(x^2-1)P_n^2(x)} \qquad .....(9)$$

(ii) Putting $n = 0$ in (9) and using $P_0(x) = 1$, we get

$$Q_0(x) = \int_x^\infty \frac{dx}{x^2-1} = \frac{1}{2}\left[\log\frac{x-1}{x+1}\right]_x^\infty$$

$$= \frac{1}{2}\left(\log\frac{x+1}{x-1}\right)_x^\infty = \frac{1}{2}\log\left(\frac{x+1}{x-1}\right)$$

$$\because \qquad \lim_{x\to\infty} \log\left(\frac{x+1}{x-1}\right) = \lim_{x\to\infty} \log\frac{\left(1+(1/x)\right)}{1-(1/x)} = 0$$

(iii) Taking $n = 1$ and using $P_1(x) = x$ in (9), we get

$$Q_1(x) = x\int_x^\infty \frac{dx}{x^2\left(x^2-1\right)} = x\int_x^\infty\left(\frac{1}{x^2-1}-\frac{1}{x^2}\right)dx$$

$$= x\left[\frac{1}{2}\log\frac{x-1}{x+1}+\frac{1}{x}\right]_x^\infty$$

$$= -\frac{x}{2}\log\frac{x-1}{x+1}-1 = \frac{x}{2}\log\frac{x+1}{x-1}-1$$

Ex.12. Show that $n\left[P_nQ_{n-1}-Q_nP_{n-1}\right]=1$

Sol. We know that

$$(2n+1)\,x\,P_n = (n+1)\,P_{n+1} + n\,P_{n-1} \qquad .....(10)$$

and

$$(2n+1)\,x\,Q_n = (n+1)\,Q_{n+1} + n\,Q_{n-1} \qquad .....(11)$$

Multiplying (1) by Q_n and (2) by P_n and then substracting, we get

or $\quad 0 = (n+1)\left[P_{n+1}Q_n - Q_{n+1}P_n\right] + n\left[P_{n-1}Q_n - Q_{n-1}P_n\right]$

or $\quad n[P_n\,Q_{n-1} - Q_n P_{n-1}] = (n+1)[P_{n+1}\,Q_n - Q_{n+1}\,P_n]$

$\Rightarrow \quad f(n+1) = f(n) \qquad .....(12)$

where $\quad f(n) = n\left[P_nQ_{n-1}Q_nP_{n-1}\right]$

Replacing n by $n-1$ in (12), we get

$$f(n) = f(n-1)$$

Similarly $\quad f(n-1) = f(n-2) = \ldots = f(1)$

Hence $\quad f(n+1) = f(n) = f(n-1) = \ldots = f(1)$

But $\quad f(1) = \left[P_1Q_0 - Q_1P_0\right] \qquad \left[\because \ P_0(x)=1,\ P_1(x)=x\right]$

$$= xQ_0 - Q_1$$

$$= xQ_0 - (xQ_0 - 1) \qquad \left[\because \ Q_1 = xQ_0 - 1\right]$$

$$= 1$$

Thus $\quad f(n) = 1$

or $\quad n\left[P_nQ_{n-1} - Q_nP_{n-1}\right] = 1$

Self – Learning Exercise–II

1. $\frac{d}{dx}\left[\left(1-x^2\right)\frac{dQ_n}{dx}\right]=$........

2. $Q'_{n+1}-Q'_{n-1}=$...........

3. Legendre's function of second kind is ....

12.13 Summary

In this unit we studied the Legendre's differential equation and its solution as Legendre function of first and second kinds. We also studied the recurrence relation, generating function, orthogonal property, Rodrigues formulae and other important formulas for these functions.

12.14 Answers to Self Learning Exercises

Exercise - I

1. Legendre function of first kind
2. $P_n(x)$
3. 0
4. 1
5. n
6. Rodrigues formulae
7. 0
8. $\frac{1}{2}\left(5x^2-3x\right)$
9. 0
10. Even / odd

Exercise - II

1. $-n(n+1)\,Q_n(x)$
2. $(2n+1)\,Q_n(x)$
3. $Q_n(x)$

12.15 Exercise

1. Prove that $P_n(-x)=(-1)^n P_n(x)$ and $P_n(-1)=(-1)^n$.

2. Express $P(x)=x^4+2x^3+2x^2-x-3$ in terms of Legendre's polynomial

[Ans : $P(x)=\frac{8}{35}P_4(x)+\frac{4}{5}P_3(x)+\frac{40}{21}P_2(x)+\frac{1}{5}p_1(x)-\frac{224}{105}P_o(x)$**]**

3. Show that $\int_{-1}^{+1}P_n(x)\,dx=0$ except when $n=0$ in which case the value of integral is 2.

4. Prove that $\int_{-1}^{+1}\left(1-x^2\right)\left(P'_n(x)\right)^2dx=\frac{2n(n+1)}{(2n+1)}$

5. Show that $P_n(x)\,Q_{n-2}(x) - Q_n(x)P_{n-2}(x) = x\dfrac{(2n-1)}{n(n-1)}$

6. Prove that $xQ_n' - Q_{n-1}' = nQ_n$

7. Prove that

$$\sum_{n=0}^{\infty} P_n(x)t^n = (1-xt)^{-1}\ {}_1F_0\left[\frac{1}{2}\ ;\ -\ ;\ \frac{t^2(x^2-1)}{(1-xt)^2}\right]$$

8. Prove that

$$P_n(x) = x^n\ {}_2F_1\left(-\frac{n}{2},\ -\frac{n}{2}+\frac{1}{2}\ ;\ 1\ ;\ 1-\frac{1}{x^2}\right)$$

9. Show that

$$\sum_{n=0}^{\infty} \frac{P_n(x)t^n}{n!} = e^{xt}\ {}_0F_1\left(-\ ;\ 1\ ;\ \frac{t^2(x^2-1)}{4}\right)$$

10. Find the values of $P_{2n+1}(0)$, $P_{2n}(0)$, $P'_{2n}(0)$ and $P'_{2n+1}(0)$

[Ans. $0, \dfrac{(-1)^n(1/2)_n}{n!}, 0, \dfrac{(-1)^n(3/2)_n}{n!}$ **]**

11. Establish the Murphy's formula

$$P_n(x) = {}_2F_1\left(-n,\ n+1\ ;\ 1\ ;\ \frac{1-x}{2}\right)$$ and deduce that

(a) $P_n(x) = (-1)^n\ {}_2F_1\left(-n,\ n+1\ ;\ 1\ ;\ \dfrac{1+x}{2}\right)$

(b) $P_n(x) = \left(\dfrac{1+x}{2}\right)^n\ {}_2F_1\left(-n,\ -n\ ;\ 1\ ;\ \dfrac{x-1}{x+1}\right)$

(c) $P_n(x) = \left(\dfrac{x-1}{2}\right)^n\ {}_2F_1\left(-n,\ -n;1;\ \dfrac{x+1}{x-1}\right)$

(d) $P_n(\cos\theta) = {}_2F_1\left(-n,\ n+1\ ;\ 1\ ;\ \sin^2(\theta/2)\right)$

12. Prove that

$$P_n(x) = \frac{2^n(1/2)_n x^n}{n!}\ {}_2F_1\left(-\frac{n}{2},\ \frac{1}{2}-\frac{n}{2}\ ;\ \frac{1}{2}-n\ ;\ \frac{1}{x^2}\right)$$

13. Prove that

$$\sum_{n=0}^{\infty} \frac{P_n(x)t^n}{n!} = e^{xt} J_0\left(t\sqrt{1-x^2}\right)$$

14. Prove that

$$xP_n' = nP_n + (2n-3)P_{n-2} + (2n-7)P_{n-4} + ...$$

and hence or other wise show that

(a) $\int_{-1}^{1} xP_n P_n' \, dx = \frac{2n}{2n+1}$

(b) $\int_{-1}^{1} xP_n P_m' \, dx = 0$ or $\frac{2n}{2n+1}$

15. Show that $\int_{-1}^{1} \left[P_n'(x)\right]^2 dx = \frac{n}{n+1}$

16. Show that $\int_{-1}^{1} P_n(x)dx = \frac{(-1)^{(n-1)/2} \lfloor (n-1)}{2^n \lfloor (n+1)/2 \; \lfloor (n-1)/2}$

17. Prove that

$\int_{-1}^{1} P_n(x)dx = 0$, $n \neq 0$ and $\int_{-1}^{1} P_0(x)dx = 2$

18. Prove that

$$P_0^2 + 3P_1^2 + ...+(2n-1)^2 P_n^2 = (n+1)^2 P_n^2 + \left(1-x^2\right)\left(P_n'\right)^2$$

□ □ □

Unit 13 : Bessel's Functions

Structure of the Unit

13.0 Objective

In this unit you will learn about Bessel function which besides the solution of the well-known Bessel's equation may also be introduced through a generating function. You will also study important properties for this function.

13.1 Introduction

No other special function have received such detailed treatment in readily available treatises as have the Bessel functions. These functions were first introduced by F.W. Bessel, who is regarded as the founder of the modern practical Astronomy. In fact several problems of mathematical physics lead to Laplace's equation and in turn converts into Bessel's equation when there is a cylindrical symmetry. Therefore Bessel's function and Bessel's equation have received great attention.

In this unit, we introduce the Bessel function through the Bessel's differential equation and generating function. We then discuss the important properties (such as Recurrence formulae, orthogonal property, Addition theorem, integral representations etc.) for this function.

13.2 Definition

13.2.1 Bessel Differential Equation

The differential equation

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right) y = 0 \qquad .....(1)$$

is called Bessel's differential equation of order n where n is non-negative real number.

13.2.2 Bessel's function of the first kind of order *n*

It denoted by $J_n(x)$ and is defined as

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} \qquad .....(2)$$

(where n is any non-negative constant)

$$= \frac{(x/2)^n}{\Gamma(n+1)} \, {}_0F_1\left(-; n+1; -\frac{x^2}{4}\right) \qquad .....(3)$$

It n is a negative integer, then we put

$$J_n(x) = (-1)^n J_{-n}(x) \qquad .....(4)$$

$$J_n(-x) = (-1)^n J_n(x) \qquad .....(5)$$

Equations (3) and (4) together define $J_n(x)$ for all finite x and n.

Replacing n by 0 and 1 in (2), we find that

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^4} - \frac{x^6}{2^4 4^4 6^2} + \cdots \qquad .....(6)$$

and $$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 3} + \frac{x^5}{2^2 4^2 6} - \cdots \qquad ......(7)$$

13.3 Bessel's Equation and its Solution

Bessel differential equation is

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right) = 0 \qquad .....(1)$$

The equation (1) has a regular singular point at $x = 0$, and an irregular singular point at $x = \infty$, while all other points are ordinary points. The solution of equation (1) called Bessel's function will depend upon n. This index n may be non-integer, *a* positive integer or zero. We discuss three possibilities :

Case I. Solution of (1) for non-integral values of *n*

Here the equation (1) is solved in series by using the well-known method of Frobenius.

Let the series solution of (1) be

$$y = \sum_{r=0}^{\infty} a_r x^{c+r}, \quad a_0 \neq 0 \qquad .....(2)$$

From (2), we get $\quad \dfrac{dy}{dx} = \sum_{r=0}^{\infty} a_r (c+r) x^{c+r-1}$

and $\quad \dfrac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (c+r)(c+r-1) x^{c+r-2}$

Substitution for $y, \dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$ in (1) gives

$$x^2 \sum_{r=0}^{\infty} a_r (c+r)(c+r-1) x^{c+r-2} + x \sum_{r=0}^{\infty} a_r (c+r) x^{c+r-1} + \left(x^2 - n^2\right) \sum_{r=0}^{\infty} a_r \, x^{c+r} = 0$$

or $$\sum_{r=0}^{\infty} a_r (c+r)(c+r-1) x^{c+r} + \sum_{r=0}^{\infty} a_r (c+r) x^{c+r} + \sum_{r=0}^{\infty} a_r \, x^{c+r+2} - n^2 \sum_{r=0}^{\infty} a_r \, x^{c-r} = 0$$

or $$\sum_{r=0}^{\infty} \left[(c+r)(c+r-1)(c+r) - n^2\right] a_r \, x^{c+r} + \sum_{r=0}^{\infty} a_r \, x^{c+r+2} = 0$$

or $$\sum_{r=0}^{\infty} \left[(c+r+n)(c+r+n)\right] a_r \, x^{c+r} + \sum_{r=0}^{\infty} a_r \, x^{c+r+2} = 0 \qquad .....(3)$$

Equating to zero the lowest power x *i. e*, x^r, we get the indical equation as

$$(c+n)\,(c-n)\,a_0 = 0$$

$\Rightarrow \qquad c = n, -n \quad$ as $\quad a_0 \neq 0$

So roots of the indical equation are $c = n, -n$.

Now equating to zero, the coefficient of x^{c+1}, we find that

$$(c+1+n)\,(c+1-n)\,a_1 = 0$$

so that $\qquad a_1 = 0 \quad$ for $\quad c = n \quad$ and $\quad -n$.

Finally equating to zero the coefficient of x^{c+r}, we get

$$(c+r+n)\,(c+r-n)\,a_r + a_{r-2} = 0$$

or $$a_r = -\frac{1}{(c+r+n)(c+r-n)} a_{r-2} \qquad .....(4)$$

Putting $r = 3, 5, 7, ...$ in (4) and using $a_1 = 0$ we find that

$$a_1 = a_3 = a_5 = a_7 = ... = 0 \qquad .....(5)$$

Also putting $r = 2, 4, 6, ...$ in (4) gives

$$a_2 = -\frac{1}{(c+2+n)(c+2-n)} a_0$$

$$a_4 = \frac{1}{(c+2+n)(c+2-n)(c+4+n)(c+4-n)} a_0 \text{ and so on .}$$

Putting these values in (2), we get

$$y = \sum_{r=0}^{\infty} a_r x^{c+r} = a_0 x^c + a_2 x^{c+2} + a_4 x^{c+4} + \ldots\ldots [\text{as } a_1 = a_3 = a_5 = 0]$$

or $$y = a_0 x^c \left[1 - \frac{x^2}{(c+2+n)(c+2-n)} + \frac{x^4}{(c+2+n)(c+2-n)(c+4+n)(c+4-n)} - \ldots\ldots \right]$$

Replacing c by n and $-n$, we get

$$y = a_0 x^n \left[1 - \frac{x^2}{2\cdot(2n+2)} + \frac{x^4}{2\cdot 4(2n+2)(2n+4)} - \ldots\ldots \right] \qquad \ldots\ldots(5)$$

and $$y = a_0 x^{-n} \left[1 - \frac{x^2}{(-2n+2)\cdot 2} + \frac{x^4}{(2n+2)(-2n+4)\cdot 2\cdot 4} - \ldots\ldots \right] \qquad \ldots\ldots(6)$$

The particular solution of the equation (1) obtained from (5) above by taking the arbitrary constant $a_0 = \frac{1}{2^n \Gamma(n+1)}$ is called the Bessel function of the first kind of order n. It will be denoted by $J_n(x)$. Thus we have

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4\cdot 8(n+1)(n+2)} - \ldots\ldots \right] \qquad \ldots\ldots(7)$$

or $$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r\, \Gamma(n+r+1)} \left(\frac{x}{2}\right) x^{n+2r} \qquad \ldots\ldots(8)$$

Similarly taking $a_0 = \frac{1}{2^n \Gamma(n+1)}$ in (6), we get

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r\, \Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n} \qquad \ldots\ldots(9)$$

Let n be non-integral. Since n is not an integer and r is always integral, the factor $\Gamma(-n+r+1)$ in (9) is always finite and non-zero ($\Gamma(m)$ is always finite for $m \neq 0$ or a negative integer.) Again for $2r < n$, (9) shows that $J_{-n}(x)$ contains negative powers of x. On the other hand, (8) shows that $J_n(x)$ does not contain negative power of x at all. Therefore for $x = 0$, $J_n(x)$ is finite. While $J_{-n}(x)$ is infinite, and so one can not be expressed as constant multiple of the other. Thus we conclude that $J_n(x)$ and $J_{-n}(x)$ are independent solutions of (1) when n is not an integer. Thus general solution of Bessel's equation (1) when n is not an integer is

$$y = AJ_n(x) + BJ_{-n}(x)$$

where A and B one arbitrary constants.

Case-II. Solution for positive integral values of n and for $n = 0$.

It n is a positive integer, then for $c = -n$, the recurrence relation (4) gives

$$a_r = \frac{1}{r(2n-r)} a_{r-2}$$

which breaks when $r = 2n$.

Also if n = 0, the two roots of the indical equation becomes equal and in that case the afore-mentioned method is not applicable.

In both the cases, the second solution of (1) can be found by using methods mentioned in unit 9.

13.4 Relation between $J_n(x)$ and $J_{-n}(x)$, *n being* an integer

Result. $$J_{-n}(x) = (-1)^n J_n(x) \qquad .....(1)$$

Proof. We consider two cases :

Case I. Let *n* be a positive integer

We have $$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r\, \Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n} \qquad .....(2)$$

Since $n > 0$, so $\Gamma(-n + r + 1)$ is infinite. for r = 0, 1, ..., n – 1, therefore (2) becomes

$$J_{-n}(x) = \sum_{r=n}^{\infty} \frac{(-1)^r}{\lfloor r\, \Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^{m+n}}{\lfloor m+n\, \Gamma(m+1)} \left(\frac{x}{2}\right)^{2m+n} \qquad \text{(taking } r = m + n\text{)}$$

$$= (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r\, \Gamma(r+n+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= (-1)^n J_n(x) \quad \text{(by definition)}$$

Case II. Let *n* < 0.

Putting $n = -p$, where p is a tive integer

Since P > 0, therefore form Case I, we have

$$J_{-p}(x) = (-1)^p J_p(x)$$

or $$J_p(x) = (-1)^{-p} J_{-p}(x)$$

Putting $p = -n$, we get the required result.

Hence the relation (1) is true for any integer.

13.5 Generating Function

Theorem. Prove that when n is a positive integer J_n (x) is the coefficient of z^n in the expansion of $\exp\left\{\frac{x}{z}\left(z - \frac{1}{z}\right)\right\}$ ***in ascending and decending power of z.***

Proof. We have $\exp\left\{\frac{x}{z}\left(z-\frac{1}{z}\right)\right\}=\exp\left(\frac{xz}{2}\right)\cdot\exp\left(-\frac{x}{2z}\right)$

$$=\left[1+\left(\frac{x}{2}\right)z+\left(\frac{x}{2}\right)^2\frac{z^2}{\underline{|2}}+....+\left(\frac{x}{2}\right)^n\cdot\frac{z^n}{\underline{|n}}+\left(\frac{x}{2}\right)^{n+1}\frac{z^{n+1}}{\underline{|n+1}}+....\right]$$

$$\times\left[1-\left(\frac{x}{2}\right)z^{-1}+\left(\frac{x}{2}\right)^2\frac{z^{-2}}{\underline{|2}}+....+\left(\frac{x}{2}\right)^n\frac{(-1)^n z^{-n}}{\underline{|n}}+\left(\frac{x}{2}\right)^{n+1}\frac{(-1)^{n+1}}{\underline{|n+1}}z^{-(n+1)}+....\right] \quad ....(1)$$

Multiplying the R.H.S. of (1) term by term, we find that coefficient of z^n is

$$=\frac{1}{\underline{|n}}\left(\frac{x}{2}\right)^x-\frac{1}{\underline{|n+1}}\left(\frac{x}{2}\right)^{n+2}+\frac{1}{\underline{|2}\,\underline{|n+2}}\left(\frac{x}{2}\right)^{n+4}-....$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r}{\underline{|r}\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}=J_n(x) \quad .....(2)$$

Similarly the coefficient of z^{-n} in the expansion (1) is

$$=\frac{(-1)^n}{\underline{|n}}\left(\frac{x}{2}\right)^n+\frac{(-1)^{n+1}}{\underline{|n+1}}\left(\frac{x}{2}\right)^{n+2}+\frac{(-1)^{n+2}}{\underline{|2}\,\underline{|n+2}}\left(\frac{x}{2}\right)^{n+4}+....=(-1)^n J_n(x) \quad .....(3)$$

Further, the term independent of z is

$$=1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-....=J_0(x) \quad .....(4)$$

Hence relation (1) with help of (2), (3) and (4) may be written as

$$\exp\left\{\frac{x}{2}\left(z-\frac{1}{z}\right)\right\}=J_0(x)+\left(z-\frac{1}{z}\right)J_1(x)+\left(z^2-\frac{1}{z^2}\right)J_2(x)+....$$

Since $J_{-n}(x)=(-1)^n J_n(x)$, therefore

$$\exp\left\{\frac{x}{2}\left(z-\frac{1}{z}\right)\right\}=\sum_{n=-\infty}^{\infty}J_n(x)z^n \quad .....(5)$$

13.6 Recurrence Formulae for $J_n(x)$

13.6.1 $xJ'_n(x)=nJ_n(x)-xJ_{n+1}(x)$

Proof. We have

$$J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{\underline{|r}\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

Differentiating above *w.r.t.* x, we get

$$J'_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r(n+2r)}{\underline{|r}\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}\cdot\frac{1}{2}$$

$$= n\sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}\times\frac{x}{x} + 2r\sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}$$

$$= \frac{n}{x}\sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} + \sum_{r=1}^{\infty} \frac{(-1)^r}{\lfloor (r-1)\,\Gamma(n+r-1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \frac{n}{x}J_n(x) - \sum_{s=0}^{\infty} \frac{(-1)^s}{\lfloor s\,(n+s+2)}\left(\frac{x}{2}\right)^{n+2s+1}$$

$$= \frac{n}{x}J_n(x) - J_{n+1}(x)$$

Hence $\qquad xJ_n'(x) = nJ_n(x) - x\,J_{n+1}(x)$

13.6.2 $xJ'_n(x) = xJ_{n-1}(x) - nJ_n(x)$

Proof. We have as in formulae 13.6.1

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r(n+2r)}{\lfloor r\,\Gamma(n+r+1)}\times\left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r(2n+2r-n)}{\lfloor r\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r(n+r)}{\lfloor r\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1} - n\sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}\times\left(\frac{x}{x}\right)\times\frac{1}{2}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r(n+r)}{\lfloor r\,(n+r)\Gamma(n+r)}\left(\frac{x}{2}\right)^{n+2r-1} - \frac{n}{x}\sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r(n+r)}{\lfloor r\,\Gamma(n+r)}\left(\frac{x}{2}\right)^{n+2r-1} - \frac{n}{x}\,J_n(x)$$

$$= J_{n-1}(x) - \frac{n}{x}J_n(x)$$

Hence $\qquad xJ_n'(x) = x\,J_{n-1}(x) - nJ_n(x)$

13.6.3 $2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$

Proof. Adding recurrence formulae 13.6.1 and 13.6.2, we get the formula 13.6.3.

13.6.4 $\qquad \mathbf{2nJ_n(x) = x\left[J_{n-1}(x) + J_{n+1}(x)\right]}$

Proof. Substracting recurrence formula 13.6.2 from 13.6.1, we easily get recurrence formula 13.6.4.

13.6.5 $\qquad \mathbf{\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)}$

Proof. By formulas 13.6.1, we have

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

Multiplying both sides of above by x^{-n-1}, we have

$$x^{-n}J_n'(x) = nx^{-n-1}J_n(x) - x^{-n}J_{n+1}(x)$$

or $$x^{-n}J_n'(x) - nx^{-n-1}J_n(x) = -x^{-n}J_{n+1}(x)$$

or $$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = -x^{-n}J_{n+1}(x)$$

13.6.6 $$\boldsymbol{\frac{d}{dx}\left[x^{n}J_n(x)\right] = x^{n}J_{n-1}(x)}$$

Proof. By formula 13.6.2, we have

$$xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$$

Multiplying both sides of above by x^{n-1}, we have

$$x^{n}J_n'(x) = x^{n}J_{n-1}(x) - nx^{n-1}J_n(x)$$

or $$x^{n}J_n'(x) + nx^{n-1}J_n(x) = x^{n}J_{n-1}(x)$$

or $$\frac{d}{dx}\left[x^{n}J_n(x)\right] = x^{n}J_{n-1}(x)$$

Ex.1. Prove that $\boldsymbol{J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \sin x}$

Sol. We know that

$$J_n(x) = \frac{x^n}{2^n\Gamma(n+1)}\left[1 - \frac{x^2}{2\cdot(2n+2)} + \frac{x^4}{2\cdot 4\cdot(2n+2)(2n+4)} - \dots\dots\dots\dots\right]$$

Puting $n = \frac{1}{2}$ and using $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$, we get

$$J_{1/2}(x) = \sqrt{\frac{2x}{\pi}}\left[1 - \frac{x^2}{2\cdot 3} + \frac{x^4}{3\cdot 5\cdot 2\cdot 4} - \dots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[x - \frac{x^3}{31} + \frac{x^5}{51} - \dots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\sin x$$

Ex.2. Show that J_n (x) is even and odd function for even n and for odd n respectively.

Sol. Replacing x by $-x$ in the definition for Bessel function, we get

$$J_n(-x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{\underline{|r}\,\Gamma(n+r+1)}\left(-\frac{x}{2}\right)^{n+2r}$$

$$=(-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} = (-1)^n J_n(x)$$

(i) If n is even then $J_n(-x) = J_n(x)$, therefore $J_n(x)$ is even.

(ii) If n is odd then $J_n(-x) = -J_n(x)$, therefore $J_n(x)$ is odd.

Ex.3. By using generating function, for Bessel function, show that

(i) $\cos(x\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + ...$

(ii) $\sin(x\sin\theta) = 2J_1\sin\theta + 2J_3\sin 3\theta + ...$

(iii) $\cos x = J_0 - 2J_2 + 2J_4 - ....$

(iv) $\sin x = 2J_1 - 2J_3 + 2J_5 - ....$

Sol. We have $\exp\left\{\frac{x}{2}\left(z - \frac{1}{z}\right)\right\} = \sum_{n-\infty}^{\infty} z^n J_n(x)$

$$= J_0(x) + \left(z - \frac{1}{z}\right)J_1(x) + \left(z^2 - \frac{1}{z^2}\right)J_2(x) + ... \quad .....(1)$$

Let us put $z = e^{i\theta}$. Then

$$z^n - \left(\frac{1}{z^n}\right) = 2i\sin\theta$$

and
$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

From (1), we have

$$\exp(x\,i\sin\theta) = J_0 + (2i\sin\theta)J_1(x) + (2\cos 2\theta)J_2(x) + ...$$

$$\Rightarrow \quad \cos(x\sin\theta) + i\sin(x\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + i(2J_1\sin\theta + 2J_3\sin 3\theta + ...)$$

Separating real and imaginary parts, we easily arive at relations *(i)* and *(ii)*.

Also on putting $\theta = \frac{\pi}{2}$ in *(i)* and *(ii)*, we get easily the selations *(iii)* and *(iv)*.

Ex.4. Prove that $\frac{d}{dx}\left[x J_n(x) J_{n+1}(x)\right] = x\left[J_n^2(x) - J_{n+1}^2(x)\right]$ ..... (2)

and deduce that

$$x = 2J_0J_1 + 6J_1J_2 + ...+ 2(n+1)J_nJ_{n+1} + ...$$

Sol. we have L.H.S of (2) $= x J_n(x)J'_{n+1}(x) + x J'_n(x)J_{n+1}(x) + J_n(x)J_{n+1}(x)$ .....(3)

From recurrence relations 13.6.1 and 13.6.2 we have

$$x J'_n(x) = n J_n(x) - x J_{n+1}(x) \quad .....(4)$$

$$x J'_n(x) = -n J_n(x) + x J_{n-1}(x) \quad .....(5)$$

Putting n as $(n+1)$ in (5), we get

$$xJ'_{n+1}(x) = -(n+1)J_{n+1}(x) + xJ_n(x) \qquad .....(6)$$

Substituting the value of $xJ'_n(x)$ and $xJ'_{n+1}(x)$ from (4) and (6) in (3), we get.

$$\text{L.H.S of (2)} = J_n(x)\left[-(n+1)J_{n+1}(x) + xJ_n(x)\right]$$

$$+J_{n+1}(x)\left[nJ_n(x) - xJ_{n+1}(x)\right] + J_n(x)J_{n+1}(x)$$

$$= x\left[J_n^2(x) - J_{n+1}^2(x)\right] = \text{ R.H.S of (2)}$$

This completes the solution of the problem.

Deduction. Putting $n = 0, 1, 2$ ..... respectively in (2) and adding after multiplying by 1, 3, 5 res, we get

$$\frac{d}{dx}\left[x\left\{J_0(x)J_1(x) + 3J_1(x)J_2(x) + 5J_2(x)J_3(x) + ...\right\}\right] = x \qquad .....(7)$$

Integrating (7) in the interval (0, x), we get the required result. [after using Ex. 6 *(i)*]

Ex.5. Prove that $\dfrac{d}{dx}\left[J_n^2(x) + J_{n+1}^2(x)\right] = 2\left[\dfrac{n}{x}J_n^2(x) - \dfrac{(n+1)}{x}J_{n+1}^2(x)\right]$

Sol. We have $\dfrac{d}{dx}\left[J_n^2(x) + J_{n+1}^2(x)\right]$

$$= 2J_n(x)J'_n(x) + 2J_{n+1}(x)J'_{n+1}(x) \qquad .....(8)$$

From recurrence relation 13.6.1, we have $J'_n(x) = \dfrac{n}{x}J_n(x) - J_{n+1}(x)$ .....(9)

Replacing n by $n+1$ in recurrence relation 13.6.2, we find that

$$J'_{n+1}(x) = -\frac{(n+1)}{x}J_{n+1}(x) + J_n(x) \qquad .....(10)$$

Using (9) and (10) in (8), we get

$$\frac{d}{dx}\left[J_n^2(x) + J_{n+1}^2(x)\right] = 2J_n(x)\left[\frac{n}{x}J_n(x) - J_{n+1}(x)\right] + 2J_{n+1}(x)\left[-\frac{n+1}{x}J_{n+1}(x) + J_n(x)\right]$$

$$= 2\left[\frac{n}{x}J_n^2(x) - \frac{n+1}{x}J_{n+1}^2(x)\right]$$

which completes the solution of the problem.

Ex.6. Prove : (i) $\boldsymbol{J_0^2 + 2\left(J_1^2 + J_2^2 + J_3^2 + ...\right) = 1}$

(ii) $\boldsymbol{|J_0(x)| \le 1}$

(iii) $\boldsymbol{|J_n(x)| \le 2^{-1/2}, (n \ge 1)}$

Sol. From Ex.5 we have

$$\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = 2\left(\frac{n}{x}J_n^2 - \frac{(n+1)}{x}J_{n+1}^2\right) \qquad .....(11)$$

Replacing n by 0,1,2,3, ... in (1), we get

$$\frac{d}{dx}\left[J_0^2+J_1^2\right]=2\left[0-\frac{1}{x}J_1^2\right]$$

$$\frac{d}{dx}\left[J_1^2+J_2^2\right]=2\left[\frac{1}{x}J_1^2-\frac{2}{x}J_2^2\right]$$

$$\frac{d}{dx}\left[J_2^2+J_3^2\right]=2\left[\frac{2}{x}J_2^2-\frac{3}{x}J_3^2\right]$$

... ...

... ...

and so on.

Adding column-wise and using $\lim_{n\to\infty} J_n(x)=0,$ we get

$$\frac{d}{dx}\left[J_0^2+2J_1^2+2J_2^2+....\right]=0 \qquad .....(12)$$

Integrating the result (12), we get

$$J_0^2(x)+2\left[J_1^2(x)+J_2^2(x)+...\right]=c \qquad ......(13)$$

Putting $n=0$ in (13) and using

$$J_0(0)=1 \text{ and } J_n(0)=0 \text{ for } n\geq 1,$$

we obtain $1 + 2(0 + 0 + ...) = c$, Thus c = 1

Hence (13) gives $J_0^2+2(J_1^2+J_2^2+.....)=1$ ....(14)

(ii) From (14) we have $J_0^2=1-2\left(J_1^2+J_2^2+J_3^2+...\right)$ ....(15)

Since $J_1^2, J_2^2, J_3^2,......$ are all positive or zero, (15) gives

$$J_0^2\leq 1 \text{ so that } |J_0(x)|\leq 1$$

(iii) Also from (14) we have

$$J_0^2=1-2\left(J_1^2+J_2^2+J_3^2+.....+J_{n-1}^2+J_n^2+J_{n+1}^2+...\right)$$

Solving for J_n^2 we have

$$J_n^2=\frac{1}{2}\left(1-J_0^2\right)-\left(J_1^2+J_2^2+...\right) \qquad ....(16)$$

Since $J_0^2, J_1^2, J_2^2,....$ are all positive or zero, therefore

(16) gives that $J_n^2\leq\frac{1}{2}$ or $|J_n(x)|\leq 2^{-1/2}$, where $n\geq 1$

Ex.7. Prove that $\frac{d}{dx}\left\{\frac{J_{-n}(x)}{J_n(x)}\right\}=-\frac{2\sin n\pi}{\pi x J_n^2}$

or $J_n J'_{-n}=\frac{-2\sin n\pi}{\pi x}$

Sol. Since $J_n(x)$ and $J_{-n}(x)$ are solutions of

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0,$$

therefore $$J_n''+\frac{1}{x}J_n'+\left(1-\frac{n^2}{x^2}\right)J_n=0, \qquad \text{...(17)}$$

and $$J_{-n}''+\frac{1}{x}J_{-n}'+\left(1-\frac{n^2}{x^2}\right)J_{-n}=0 \qquad \text{...(18)}$$

Multiplying (17) by J_{-n} and (18) by J_n and substracting, we get

$$\left(J_{-n}\,J_n''-J_nJ_{-n}''\right)+\frac{1}{x}\left(J_{-n}\,J_n'-J_nJ_{-n}'\right)=0 \qquad \text{...(19)}$$

Let $u=J_{-n}\,J_n'-J_nJ_{-n}'.$

Then (19) reduces to $u'+\frac{1}{x}u=0 \;\Rightarrow\; \frac{u'}{u}=-\frac{1}{x}$

Integrating we get $\log u=\log\frac{a}{x}$ or $u=\frac{a}{x}$

where a is arbitrary constant or

$$J_{-n}\,J_n'-J_nJ_{-n}'=\frac{a}{x}$$

$$\frac{1}{2^{-n}\Gamma(-n+1)}\left[x^{-n}-\frac{x^{-n+2}}{2\cdot(-2n+2)}+\frac{x^{-n+4}}{2.4(-2n+2)(-2n+4)}-....\right]$$

$$\times\frac{1}{2^{n}\Gamma(n+1)}\left[nx^{n-1}-\frac{(n+2)x^{n+1}}{2\cdot(2n+2)}+\frac{(n+4)x^{n+3}}{2.4(2n+2)(2n+4)}-....\right]$$

$$-\frac{1}{2^{n}\Gamma(n+1)}\left[x^{n}-\frac{x^{n+2}}{2\cdot(2n+2)}+\frac{x^{n+4}}{2.4(2n+2)(2n+4)}-....\right]$$

$$\times\frac{1}{2^{-n}\Gamma(-n+1)}\left[-nx^{-n-1}-\frac{(2-n)x^{1-n}}{2\cdot(2-2n)}+\frac{(4-n)x^{2-n}}{2.4(2-2n)(4-2n)}-...\right]=\frac{a}{x} \qquad \text{.....(20)}$$

Comparing the coefficients of $\frac{1}{x}$ on both sides of (20), we get

$$a=\frac{1}{\Gamma(n+1)\Gamma(-n+1)}\left[n-(-n)\right]=\frac{2n}{n\Gamma(n)\Gamma(1-n)}=\frac{2\sin n\pi}{\pi}$$

(using $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$)

Thus $$\frac{J_{-n}J_n'-J_nJ_{-n}'}{J_n'}=\frac{2\sin n\pi}{\pi\,x\,J_n^2}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{J_{-n}}{J_n}\right) = -\frac{2 \sin n\pi}{\pi x\, J_n^2}$$

Self-Learning Exercise-I

1. $J_0(x)$ is a Bessel's function of order ....
2. $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n$...... .
3. Write generating function for Bessel function $J_n(x)$.
4. $J_{-n}(x) = (-1)^n$ ....
5. Write differential equation for the Bessel function $J_n(x)$.
6. $x\left[J_{n-1}(x) + J_{n+1}(x)\right] =$........... .
7. $J_n(x)$ is even function if n is ....
8. $\lim_{x\to 0} x^{-n} J_n(x) =$...........

13.7 Addition Theorem

Statement : It n is a positive integer, then

$$\boldsymbol{J_n(x+y) = \sum_{r=0}^{n} J_n(x) J_{n-r}(y) + \sum_{r=1}^{\infty} (-1)^r \{J_r(x) J_{n+r}(y) + J_r(y) J_{n+r}(x)\}} \quad ...(1)$$

Proof : we have $\quad \sum_{n=-\infty}^{\infty} J_n(x) z^n = \exp\left\{\frac{x}{2}\left(z - \frac{1}{z}\right)\right\}$

$$\therefore \sum_{n=-\infty}^{\infty} J_n(x+y) z^n = \exp\left\{\frac{x}{2}\left(z-\frac{1}{z}\right)\right\} exp\left\{\frac{y}{2}\left(z-\frac{1}{z}\right)\right\}$$

$$= \sum_{r=-\infty}^{\infty} z^r J_r(x) \sum_{s=-\infty}^{\infty} z^s J_s(y)$$

Now equating the coefficient of z^n on both sides, keeping in mind that the terms containing z^n on R.H.S. are obtained by taking $s = n - r$ and by making r vary from $-\infty$ to ∞ thus

$$J_n(x+y) = \sum_{r=-\infty}^{\infty} J_r(x) J_{n-r}(y) \quad ...(2)$$

where n is any integer.

or $$J_n(x+y) = \sum_{r=-\infty}^{-1} J_r(x) J_{n-r}(y) + \sum_{r=0}^{n} J_r(x) J_{n-r}(y) + \sum_{r=n+1}^{\infty} J_r(x) J_{n-r}(y) \quad ...(3)$$

Now $$\sum_{r=-\infty}^{-1} J_r(x) J_{n-r}(y) = \sum_{p=\infty}^{1} J_{-p}(x) J_{n+p}(y) \quad \text{(writing } -r = p)$$

$$= \sum_{p=1}^{\infty} (-1)^p J_p(x) J_{n+p}(y)$$

$$= \sum_{r=1}^{\infty} (-1)^r J_r(x) J_{n+r}(y) \text{ (replacing dummy index } p \text{ by } r) \quad ...(4)$$

Also $\quad \displaystyle\sum_{r=n+1}^{\infty} J_r(x) J_{n-r}(y) = \sum_{q=1}^{\infty} J_{n+q}(x) J_{-q}(y)$ (taking $r = n+q$)

$$= \sum_{q=1}^{\infty} (-1)^q J_{n+q}(x) J_q(y)$$

$$= \sum_{r=1}^{\infty} (-1)^r J_{n+r}(x) J_r(y) \quad ....(5)$$

Using (4) and (5) in (3), we easily arrive at the addition thorem given by (1).

13.8 Orthogonal Property

Result : If λ_i and λ_j are the roots of the equation $J_n(\lambda a) = 0$

then $$\int_0^{\alpha} x\, J_n(\lambda_i x) J_n(\lambda_i x)\, dx = \begin{cases} 0, & \text{if } i \neq j \\ \dfrac{a^2}{2} J_{n+1}^2(\lambda_i a), & \text{if } i = j \end{cases}$$

Proof : Case I : Let $i \neq j$ *i.e.* let λ_i and λ_j are different roots of $J_n(\lambda a) = 0$

$$\therefore J_n(\lambda_i a) = 0 \text{ and } J_n(\lambda_j a) = 0 \quad ....(1)$$

Let $\quad u(x) = J_n(\lambda_i x)$ and $v(x) = J_n(\lambda_j x) \quad ....(2)$

then u and v are Bessel functions satisfying the modified Bessel equation

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + \left(\lambda^2 x^2 - n^2\right) y = 0$$

or $$x^2 y'' + xy' + \left(\lambda^2 x^2 - n^2\right) y = 0 \quad ...(3)$$

or $$x^2 u'' + xu' + \left(\lambda_i^2 x^2 - n^2\right) u = 0 \quad ...(4)$$

or $$x^2 v'' + xv' + \left(\lambda_j^2 x^2 - n^2\right) v = 0 \quad ...(5)$$

Multiplying (4) by v and (5) by u and then substracting we get

$$x^2 \left(vu'' - uv''\right) + x\left(vu' - uv'\right) + x^2\left(\lambda_i^2 - \lambda_j^2\right) uv = 0$$

or $$x\left(vu'' - uv''\right) + \left(vu' - uv\right) = x\left(\lambda_j^2 - \lambda_i^2\right) uv$$

or $$x\frac{d}{dx}\left(vu' - uv'\right) + \left(vu' - uv'\right) = x\left(\lambda_j^2 - \lambda_i^2\right) uv$$

or $$x\frac{d}{dx}\left[x\left(vu' - uv'\right)\right] = x\left(\lambda_j^2 - \lambda_i^2\right) uv \quad ...(6)$$

Integrating (6) w.r.t. x from 0 to a, we get

$$\left(\lambda_j^{\,2}-\lambda_i^{\,2}\right)\int_0^a x\,u\,v\,dx=\left[x\left(vu'-uv'\right)\right]_0^a \qquad ...(7)$$

Using (2), (7) gives $\left(\lambda_j^{\,2}-\lambda_i^{\,2}\right)\int_0^a xJ_n\left(\lambda_i x\right)J_n\left(\lambda_j x\right)dx$

$$=\left[x\left\{J_n\left(\lambda_j x\right)J_n'\left(\lambda_i x\right)-J_n\left(\lambda_i x\right)J_n'\left(\lambda_j x\right)\right\}\right]_0^a$$

$$=a\left[J_n\left(\lambda_j a\right)J_n'\left(\lambda_i a\right)-J_n\left(\lambda_i a\right)J_n'\left(\lambda_j a\right)\right]$$

$$=0 \quad \text{[using (1)]}$$

Since $\lambda_i \neq \lambda_j$ the above equation gives

$$\int_0^a xJ_n\left(\lambda_i x\right)J_n\left(\lambda_j x\right)dx=0 \quad \text{if} \quad i\neq j \qquad ....(8)$$

Case II : Let $i=j$ (equal roots). Multiplying (4) by $2u'$, we have

$$2x^2u''u'+2x\left(u'\right)^2+2\left(\lambda_i^{\,2}x^2-n^2\right)uu'=0$$

or
$$\frac{d}{dx}\left[x^2\left(u'\right)^2-n^2u^2+\lambda_i^{\,2}x^2u^2\right]-2\lambda_i^{\,2}xu^2=0$$

or
$$2\lambda_i^{\,2}xu^2=\frac{d}{dx}\left[x^2\left(u'\right)^2-n^2u^2+\lambda_i^{\,2}x^2u^2\right] \qquad ....(9)$$

Integrating (9) *w.r.t.* x from 0 to a, we get

$$2\lambda_i^{\,2}\int_0^a xu^2dx=\left[x^2\left(u'\right)^2-n^2u^2+\lambda_i^{\,2}x^2u^2\right]_0^a \qquad ....(10)$$

Using the relation $J_n(0)=0$ and (1) and (2), we have

$$2\lambda_i^{\,2}\int_0^a xJ_n^2\left(\lambda_i x\right)dx=\left[x^2\left\{J_n'\left(\lambda_i x\right)\right\}^2-n^2\left\{J_n\left(\lambda_i x\right)\right\}^2+\lambda_i^2x^2\left\{J_n\left(\lambda_i x\right)\right\}^2\right]_{x=0}^a$$

or
$$2\lambda_i^{\,2}\int_0^a xJ_n^2\left(\lambda_i x\right)dx=a^2\left[\left\{J_n'\left(\lambda_i x\right)\right\}^2\right]_{\text{at } x=a} \qquad ....(11)$$

From recurrence relation 13.6.1, we have

$$\frac{d}{dx}\left[J_n\left(x\right)\right]=\frac{n}{x}J_n\left(x\right)-J_{n+1}\left(x\right) \qquad ...(12)$$

Replace x by $\lambda_i x$ in (12), we have

or
$$\frac{d\left[J_n\left(\lambda_i x\right)\right]}{d\left(\lambda_i x\right)}=\frac{n}{\left(\lambda_i x\right)}J_n\left(\lambda_i x\right)-J_{n+1}\left(\lambda_i x\right)$$

or
$$\frac{1}{\lambda_i}\cdot\frac{d\left[J_n\left(\lambda_i x\right)\right]}{dx}=\frac{n}{\lambda_i x}J_n\left(\lambda_i x\right)-J_{n+1}\left(\lambda_i x\right)$$

$$\Rightarrow J_n'\left(\lambda_i x\right)=\frac{n}{x}J_n\left(\lambda_i x\right)-\lambda_iJ_{n+1}\left(\lambda_i x\right)$$

Now $\left[\left\{J_n'(\lambda_i x)\right\}^2\right]_{\text{at } x=a} = \left[\left\{\frac{n}{x}J_n(\lambda_i x) - \lambda_i J_{n+1}(\lambda_i x)\right\}^2\right]_{\text{at } x=a}$

$= \left[0 - \lambda_i J_{n+1}(\lambda_i a)\right]^2$ (by (1))

$$= \lambda_i^2 J_{n+1}^2(\lambda_i a) \qquad \text{...(13)}$$

Using it in (11), we get

$$\int_0^a x J_n^2(\lambda_i x)\,dx = \frac{a^2}{2} J_{n+1}^2(\lambda_i a)$$

Combining these two results we can write

$$\int_0^a x J_n(\lambda_i x) J_n(\lambda_j x)\,dx = \frac{a^2}{2} J_{n+1}^2(\lambda_i a)\,\delta_{ij}$$

where δ_{ij} = (kronecker delta) = $\begin{cases} 0, i \neq j \\ 1, i \neq j \end{cases}$.

Ex.1. Prove that $J_n(x) = \frac{1}{\pi}\int \cos(n\phi - x\sin\phi)\,d\phi$ ***where n is a positive integer***

Sol. We shall use the following results :

$$\int_0^\pi \cos m\phi \cos n\phi d\phi = \int_0^\pi \sin m\phi \sin n\phi d\phi = \begin{cases} \pi/2, & \text{if } m = n \\ 0, & \text{if } m = n \end{cases} \qquad \text{....(14)}$$

We also proved in Ex. 3(§13.6) that

$$\cos(x\sin\phi) = J_0 + 2J_2\cos 2\phi + 2J_4\cos 4\phi + ... \qquad \text{....(15)}$$

and $$\sin(x\sin\phi) = 2J_1\sin\phi + 2J_3\sin 3\phi + 2J_5\sin 5\phi + ... \qquad \text{....(16)}$$

Multiplying (15) by $\cos n\phi$ and integrating between the limit 0 to π, and using (14) we get

$$\int_0^\pi \cos(x\sin\phi)\cos n\phi d\phi = 0 \quad \text{(if } n \text{ is odd)} \qquad \text{....(17)}$$

and $$\int_0^\pi \cos(x\sin\phi)\cos n\phi d\phi = 2J_n \int_0^\pi \cos^2 n\phi d\phi = 2J_n \frac{\pi}{2} = \pi J_n \quad \text{(if } n \text{ is even)} \qquad \text{....(18)}$$

Again multiplying (16) by $\sin n\phi$ and integrating between the limit 0 to π and using (14), we get

$$\int_0^\pi \sin(x\sin\phi)\sin n\phi d\phi = 0 \quad \text{(if } n \text{ is even)} \qquad \text{....(19)}$$

and $$\int_0^\pi \sin(x\sin\phi)\sin n\phi\,d\phi = 2J_n \int_0^\pi \sin 2n\phi\,d\phi = 2J_n(\pi/2) = \pi J_n \quad \text{(if } n \text{ is odd)} \qquad \text{....(20)}$$

Let n be odd. Adding (17) and (20), we get

$$\int_0^\pi \left[\cos(x\sin\phi)\cos n\phi + \sin(x\sin\phi)\sin n\phi\right] d\phi = \pi J_n$$

or $$\int_0^\pi \cos(n\phi - x\sin\phi)\,d\phi = \pi J_n$$

or $$J_n = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x\sin\phi)\,d\phi \qquad \text{....(21)}$$

If n is even, then add (18) and (19) to get the required result.

Thus (21) holds for each positive integer n (even as well as odd)

Remark : If n is negative integer so that $n = -p$, where p is a positive integer. Putting $n = -p$ in (21) we get

$$J_{-p}(x) = \frac{1}{\pi}\int_0^{\pi} \cos(-p\phi - n\sin\phi)\,d\phi \qquad \text{....(22)}$$

Let $\phi = \pi - \theta$ so that $d\phi = -d\theta$

$\because$ R.H.S. of (22) $$= \frac{1}{\pi}\int_{\pi}^{0} \cos\{(-p(\pi-\theta) - x\sin(\pi-\theta))\}(-d\theta)$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos\{(p\theta - x\sin\theta) - p\pi\}\,d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \left[\cos(p\theta - x\sin\theta)\cos p\pi + \sin(p\theta - x\sin\theta)\sin p\pi\right]d\theta$$

$$= \frac{(-1)^p}{\pi}\int \cos(p\theta - x\sin\theta)\,d\theta$$

Thus (22) becomes

$$(-1)^p J_p(x) = \frac{(-1)^p}{\pi}\int_0^{\pi} \cos(p\theta - x\sin\theta)\,d\theta$$

or $$J_{-n}(x) = \frac{1}{\pi}\int_0^{\pi} \cos(-n\theta - x\sin\theta)\,d\theta$$

Hence the result (22) holds for each integer.

Ex.2. Prove that $\boldsymbol{J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\phi)\,d\phi = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\phi)\,d\phi}$

Sol. From Ex.3(§13.6), we have

$$\cos(x\sin\phi) = J_0 + 2J_2\cos 2\phi + 2J_4\cos 4\phi + \ldots\ldots.. \qquad \text{.....(23)}$$

Integrating (23)w.r.t 'ϕ' between the limit 0 to π, we get

or $$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\phi)\,d\phi$$

Again replacing ϕ by $\left(\frac{\pi}{2} - \phi\right)$ in (23) and simplifying,we get

$$\cos(x\cos\phi) = J_0 - 2J_2\cos 2\phi + 2J_4\cos 4\phi\ldots\ldots.. \qquad \text{....(24)}$$

Thus $$\int_0^{\pi} \cos(x\cos\phi)\,d\phi = \pi J_0(x)$$

$$\therefore J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\phi)\,d\phi$$

Ex.3. Prove that $\boldsymbol{J_n(x) = (-2)^n x^n \frac{d^n}{d(x^2)^n} J_0(x)}$

Sol. Substituting the value of $J_0(x)$ in series in R.H.S, we have

$$\text{R.H.S} \quad = (-2x)^n \left[\frac{d^n}{d(x^2)^n} \left\{ \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r\, \Gamma(r+1)} \left(\frac{x}{2} \right)^{2r} \right\} \right]$$

$$= (-2x)^n \left[\frac{d^n}{dt^n} \left\{ \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r\, \Gamma(r+1)} \left(\frac{t^r}{2^{2r}} \right) \right\} \right]$$

$$= (-2x)^n \sum_{r=0}^{\infty} \frac{(-1)^r t^{r-n}}{\lfloor r \lfloor r-n 2^{2r}}$$

$$= (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r\, \Gamma(-n+r+1)} \left(\frac{x}{2} \right)^{-n+2r}$$

$$= (-1)^n J_{-n}(x) = J_n(x)$$

Ex.3. If, Prove that $\boldsymbol{\int_0^{\infty} e^{-ax} J_0(bx)\, dx = \frac{1}{\sqrt{a^2+b^2}}}$

Sol. Using series representation for the Bessel function and changing the order of integration and summation, we find that

$$I = \int_0^{\infty} e^{-ax} J_0(bx)\, dx = \sum_{r=0}^{\infty} \frac{(-1)^r (b/2)^{2r}}{(\lfloor r)^2} \int_0^{\infty} x^{2r} e^{-ax}\, dx$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (b/2)^{2r}\, \Gamma(2r+1)}{(\lfloor r)^2 a^{2r+1}} \qquad \text{(using the def. of gamma function)}$$

Applying gamma duplication formula for $\Gamma(2r+1)$ and simplifying, we find that

$$I = \frac{1}{a} \sum_{r=0}^{\infty} \frac{(1/2)_r}{\lfloor r} \left(-\frac{b^2}{a^2} \right)^r$$

$$= \frac{1}{a} \left(1 + \frac{b^2}{a^2} \right)^{-1/2} = \frac{1}{\sqrt{a^2+b^2}}$$

13.9 Integral Representation of Bessel Functions

Theorem : Prove that

$$\sqrt{\pi}\left(\frac{x}{2}\right)^{-n}\Gamma\left(n+\frac{1}{2}\right)J_n(x)=\int_{-1}^{1}\exp(ixt)\left(1-t^2\right)^{n-(1/2)}dt,\ \left(n>-\frac{1}{2}\right) \qquad .....(1)$$

Proof : We have

$$\text{R.H.S. of (1)} \qquad =\sum_{r=0}^{\infty}\frac{(i\,x)^r}{\lfloor r}\int_0^1 t^r\left(1-t^2\right)^{n-(1/2)}dt \qquad .....(2)$$

Since the integrand in (2) is even or odd according as r is even or odd respectively, therefore

$$\text{R.H.S} =\sum_{k=0}^{\infty}\frac{(ix)^{2k}}{\lfloor(2k)}\times 2\int_0^1 t^{2k}\left(1-t^2\right)^{n-(1/2)}dt$$

Putting $t^2 = u$ and using the formula

$$\lfloor 2k=\Gamma(2k+1)=2^{2k}\pi^{-1/2}\Gamma(k+1)\Gamma\left(k+\frac{1}{2}\right),$$

We get

$$\text{R.H.S. of (1)} \qquad =\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}\sqrt{\pi}}{2^{2k}\Gamma(k+1)\Gamma(k+1/2)}\int_0^1 u^{k-1/2}(1-u)^{n-(1/2)}du \qquad .....(3)$$

Now evaluating the integral by using the well known definition of Beta function, we get

$$\text{R.H.S. of (1)} \qquad =\sqrt{\pi}\,\Gamma\left(n+\frac{1}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\ \Gamma(k+n+1)}\cdot\left(\frac{x}{2}\right)^{2k}$$

$$=\sqrt{\pi}\left(\frac{x}{2}\right)^{-n}\Gamma\left(n+\frac{1}{2}\right)J_n(x)$$

Similarly we have

$$\sqrt{\pi}\ \Gamma\left(n+\frac{1}{2}\right)\left(\frac{x}{2}\right)^{-n}J_n(x)=\int_{-1}^{1}e^{-ixt}\left(1-t^2\right)^{n-(1/2)}dt \qquad .....(4)$$

Adding (1) and (4) we get

$$J_n(x)=\frac{2}{\sqrt{\pi}\ \Gamma[n+(1/2)]}\left(\frac{x}{2}\right)^n\int_0^1\cos xt\left(1-t^2\right)^{n-(1/2)}dt,\ (n>-1/2) \qquad .....(5)$$

For $t=\sin\phi$, eq. (5) gives

$$J_n(x)=\frac{2}{\sqrt{\pi}\ \Gamma[n+(1/2)]}\left(\frac{x}{2}\right)^n\int_0^{\pi/2}\cos(x\sin\phi)\cos^{2n}\phi\,d\phi$$

Replacing ϕ by $\left(\frac{\pi}{2}-\phi\right)$ in the above relation, we get

$$J_n(x)=\frac{2}{\sqrt{\pi}\,\Gamma\left[n+(1/2)\right]}\left(\frac{x}{2}\right)^n\int_0^{\pi/2}\cos(x\cos\phi)\sin^{2n}\phi\;d\phi$$

13.10 An Important Integral

Theorem : Prove that

$$\int_0^a x\left(a^2-x^2\right)J_0(kx)\,dx=\frac{4a}{k^3}J_1(ak)-\frac{2a^2}{k^2}J_0(ak) \qquad .....(1)$$

Proof : We know that $\frac{d}{dx}\left\{x^nJ_n(x)\right\}=x^nJ_{n-1}(x)$

Replacing x by kx, we get

$$\frac{d}{dx}\left\{x^nJ_n(kx)\right\}=kx^nJ_{n-1}(kx) \qquad .....(2)$$

Integrating (2) *w.r.t.* x in the interval (0, a), we get

$$\int_0^a x^nJ_{n-1}(kx)\,dx=a^nJ_n(ka) \qquad .....(3)$$

Now,

$$\int_0^a x\left(a^2-x^2\right)J_0(ax)\,dx=a^2\int_0^a x\,J_0(kx)\,dx-\int_0^a x^3J_0(kx)\,dx$$

$$=\frac{a^3}{k}J_1(ak)-\int_0^a\frac{x^2}{k}\frac{d}{dx}\left[x\,J_1(kx)\right]dx$$

[Using (3) with $n = 1$ for first integral and (2) with $n = 1$ for second integral]

$$=\frac{a^3}{k}J_1(ak)-\frac{1}{k}\left[\left\{x^2\cdot xJ_1(kx)\right\}_0^a-2\int_0^a x^2J_1(kx)\,dx\right]$$

$$=\frac{a^3}{k}J_1(ak)-\frac{a^3}{k}J_1(ak)+\frac{2}{k^2}\int_0^a\frac{d}{dx}\left\{x^2J_2(kx)\right\}dx$$

$$=\frac{2a^2}{k^2}J_2(ax) \qquad ....(4)$$

Also we have the recurrence relation

$$2nJ_n(x)=x\left[J_{n+1}(x)+J_{n+1}(x)\right] \qquad ....(5)$$

Taking n = 1 and replacing *x by kx* in (5), we find that

$$J_2(kx)=\frac{2}{kx}J_1(kx)-J_0(kx)$$

Substituting the value of J_2(kx) in (4), we easily get the integral (1).

Self-Learning Exercise-II

1. $\left[J_{1/2}(x)\right]^2+\left[J_{-1/2}(x)\right]^2=$ ....

2. The relation $J_0^1(x)=-J_1(x)$ is true /false

3. $\left[J_{1/2}(x)\right]=$ ....

4. $\int_0^{\pi}\cos(n\theta-x\sin\theta)\,d\theta=$ ....

5. $|J_0(x)|\le$ ...., $n\ge 1$

6. $|J_n(x)|\le$ ...., $n\ge 1$

13.11 Summary

In this unit we studied the Bessel's differential equation and its solution. Also we proved the important properties such as recurrence relations, generating function, orthogonal property, integrals representation for the Bessel function.

13.12 Answers to Self- Learning Exercises

Exercise-1

1. 0
2. $J_{n-1}(x)$
3. $exp\left\{\frac{x}{2}\left(z-\frac{1}{z}\right)\right\}=\sum_{n=-\infty}^{\infty}J_n(x)z^n$
4. $Jn\,(x)$
5. $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-n^2\right)y=0$
6. $2nJ_n(x)$
7. even
8. $\frac{1}{2^n n!}$

Exercise-II

1. $\frac{2}{\pi x}$
2. true
3. $\sqrt{\frac{2}{\pi x}}\sin x$
4. $\pi J_n(x)$
5. 1
6. $2^{-1/2}$

13.13 Exercise

1. Prove that $J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$

2. Prove that $\int_0^{\pi/2} \sqrt{\pi x}\, J_{1/2}(2x)\, dx = 1$

3. Prove that $\int_0^t J_0 \sqrt{x(t-x)}\; dx = 2\sin\frac{t}{2}$

4. Prove that

(i) $\int_0^x x^{n+1} J_n(x)\, dx = x^{n+1}\, J_{n+1}(x),\; n > -1$

(ii) $\int_0^x x^{-n} J_{n+1}(x)\, dx = \frac{1}{2^n \lfloor n} - x^{-n}\, J_n(x)$

5. Use recurrence relations for Bessel's functions to show that

(i) $J_2(x) = -\frac{J_0'(x)}{x} + J_0''(x)$

(ii) $4J_0'''(x) + 3J_0'(x) + J_3(x) = 0$

(iii) $2J_0''(x) = J_2(x) - J_0(x)$

6. Using generating function, prove that

$$J_n(x+y) = \sum_{r=-\infty}^{\infty} J_r(x) J_{n-r}(y)$$

7. Prove that

(i) $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left\{ \frac{\sin x}{x} - \cos x \right\}$

(ii) $J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}} \left\{ \frac{\cos x}{x} + \sin x \right\}$

(iii) $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \left\{ \frac{3}{x}\left(\frac{\sin x}{x} - \cos x \right) - \sin x \right\}$

(iii) $J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}} \left\{ \frac{3}{x}\left(\frac{\cos x}{x} + \sin x \right) - \cos x \right\}$

8. Prove that $$J_{n-1}(x) = \frac{2}{x}\left[nJ_n(x) - (n+2)J_{n+2}(x) + (n+4)J_{n+4}(x) - \ldots\ldots \right]$$

9. prove that $\int_0^a x \sin(ky)\left(y^2 - x^2\right)^{-1/2} dx = \frac{\pi y}{2} J_1(ky)$

10. show that $$J_n'(x) = \frac{2}{x}\left[\frac{n}{2} J_n(x) - (n+2)J_{n+2}(x) + (n+4)J_{n+4}(x) \ldots\ldots \right]$$

□□□

Unit 14 : Hermite Polynomials

Structure of the Unit

14.0 Objective

Here you will study Hermite polynomials its definition and important properties such as recurrence relations, generating function, orthogonal property, Rodrigue's formula etc.

14.1 Introduction

Hermite polynomials occur in the study of wave mechanics and other physical problems. We start with the Hermite differential equation and its solution. Then we develop and study properties of Hermite polynomials. We also illustrate the properties with the help of solved problems.

14.2 Hermite Differenential Equation and Its Solution

Hermite's equation is

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2 \quad ny = 0 \qquad .....(1)$$

where n is any integer For solving equation (1), we use Frobenius method.

Let $$y = \sum_{r=0}^{\infty} a_r x^{k+r}, \quad a_0 \neq 0 \qquad .....(2)$$

Now obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from (2) and substitute in (1), we get

$$\sum_{r=0}^{\infty} a_r\left[(k+r)(k+r-1)x^{k+r-2} - 2(k+r-n)x^{k+r}\right] = 0 \qquad .....(3)$$

Equation (3) is an identity. We equate to zero the coefficient of smallest power of x, viz. x^{k-2} in (3) and obtain the indical equation as

$$a_0\, k(k-1) = 0$$

$$k(k-1) = 0 \quad \because\ a_0 \neq 0 \qquad .....(4)$$

So roots of indical equation are $k = 0$, 1. They are distinct and differ by an integer.

Again equating to zero the next snallest power of x *i.e* x^{k-1}. So we get

$$a_1(k+1)\,k = 0 \qquad .....(5)$$

When $k = 0$, (5) shows that a_1 is indeterminate. Hence a_0 and a_1 can be taken as arbitrary constants.

Equating to zero the coeffcient of x^{k+r-2}, (3) gives

$$a_r = \frac{2(k+r-n-2)}{(k+r)(k+r-1)} a_{r-2} \qquad .....(6)$$

Putting $k = 0$, we get

$$a_r = \frac{2(r-n-2)}{r(r-1)} a_{r-2} \qquad .....(7)$$

For $r = 2, 4, 6, ..., 2\,r$ in (7), we get

$$a_2 = -\frac{2n}{2\cdot 1}\ a_0 = -\frac{(-1)^1\cdot 2^1\cdot n}{\underline{|2}} a_0,$$

$$a_4 = -\frac{2(2-n)}{4\cdot 3}\ a_2 = -\frac{(-1)^2\, 2^2 n(n-2)}{\underline{|4}} a_0$$

....

and $$a_{2r} = -\frac{(-1)^r\, 2^r \cdot n(n-2).....(n-2r+2)}{\underline{|2r}}\ a_0$$

Next, putting $r = 3, 5, 7, ..., 2r + 1$, in (7) we get

$$a_3 = \frac{(-1)^1\, 2^1 (n-1)}{\underline{|3}}\ a_1$$

$$a_5 = \frac{(-1)^2\, 2^2 (n-1)(n-3)}{\underline{|5}}\ a_1$$

....

and $$a_{2r+1} = \frac{(-1)^r 2^r (n-1)(n-3)......(n-2r+1)}{\underline{|2r+1}} \; a_1$$

Putting the above values in (2) with $k = 0$, we get

$$y = a_0\left[1 - \frac{2n}{\underline{|2}}x^2 + \frac{2^2 n(n-2)}{\underline{|4}} \; x^4 - ..... + \frac{(-2)^r n(n-2)...(n-2r+2)}{\underline{|2r}} \; x^{2r} + ...\right]$$

$$+a_1\left[x - \frac{2(n-1)}{\underline{|3}}x^3 + \frac{2^2(n-1)(n-3)}{\underline{|5}}x^5 + ... + \frac{(-2)^r(n-1)(n-3)...(n-2r+1)}{\underline{|2r+1}} \; x^{2r+1} + ...\right]$$

....(8)

or $y = a_0 v + a_1 w$, say .....(9)

Since v or w is not merely a constant, v and w form a fundamental set (*i.e.* linearly independent) of solutions of (1). Hence (8) or (9) is the most general solution of (1) with a_0 and a_1 as two arbitrary constants.

Remark : In practice we require solution of (1) such that

(i) it is finite for all finite values of x and

(ii) $\exp(1/2x^2)\, y(x) \to 0$ as $x \to \infty$

The solution (8) does not satisfy the condition *(ii)*. However, if the series terminate then this condition will be satistied. Replacing r by $r + 2$ in (7), we get

$$a_{r+2} = \frac{2(r-n)}{(r+1)(r+2)} \; a_r \qquad .....(10)$$

If r is a positive integer, then for $r = n$, $a_{r+2} = 0$ *ie* the series terminates. We now find the solution of (1) in descending powers of x for $n \in I^+$ (set of positive integers)

For $k = 0$, the equation (2) becomes

$$y = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} + ... \qquad .....(11)$$

From (10) we get $a_r = -\dfrac{(r+1)(r+2)}{2(n-r)} \; a_{r+2}$

Let r = $n - 2$, $n - 4$, ... Then

$$a_{n-2} = -\frac{n(n-1)}{2 \cdot 2} \; a_n .$$

$$a_{n-4} = \frac{n(n-1)(n-2)(n-3)}{2^2 \cdot 2 \cdot 4} \; a_n \text{ and so on}$$

Putting these values in (11) we find that

$$y = a_n\left[x^n - \frac{n(n-1)}{2 \cdot 2}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2^2 \cdot 2 \cdot 4}x^{n-4} + ...\right.$$

$$+\frac{(-1)^r n(n-1)...(n-2r+1)}{2^r \cdot 2\cdot 4...2r} x^{n-2r} + ...\Bigg]$$

$$= a_n \sum_{r=0}^{[n/2]} \frac{(-1)^r n(n-1)....(n-2r+1)}{2^r \cdot 2 \cdot 4.....2r} x^{n-2r}$$

Where $$\left[\frac{n}{2}\right] = \begin{cases} \frac{n}{2}, \text{ if } n \text{ is even} \\ \frac{1}{2}(n-1), \text{ if } n \text{ is odd} \end{cases}$$

Thus $$y = a_n \sum_{r=0}^{[n/2]} (-1)^r \frac{\lfloor n}{2^{2r} \lfloor r \lfloor n-2r} x^{n-2r}$$

Taking $a_n = 2^n$, then we get

$$y = H_n(x) \sum_{r=0}^{[n/2]} (-1)^r \frac{n!}{\lfloor r \lfloor n-2r} (2x)^{n-2r} \quad .....(12)$$

where $H_n(x)$ is called the Hermite polynamial of order n.

14.3 Generating function

Result. $e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} H_n(x)$ ***valid for all finite x and t.***

Proof. We have

$$e^{2xt-t^2} = e^{2xt} \cdot e^{-t^2}$$

$$= \sum_{r=0}^{\infty} \frac{(2xt)^r}{\lfloor r} \sum_{s=0}^{\infty} \frac{(-t^2)^s}{\lfloor s}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(2x)^r (-1)^s}{\lfloor r \quad \lfloor s} t^{r+2s}$$

Let $r + 2s = n$ so that $r = n - 2s$.

Hence the coefficient of t^n (for fixed value of s) is given by

$$= \frac{(-1)^s (2x)^{n-2s}}{\lfloor n-2s \quad \lfloor s}$$

The total value of t^n is obtained by summing over all admisible value of s, and since $r = n - 2s$, $r \geq 0$.

Now as $n - 2s \geq 0$ or $s \leq n/2$, therefore s goes from 0 to $n/2$ or from 0 to $(n-1)/2$ according as n is even or odd.

So total coefficient of t^n in the expansion of exp $(2xt - t^2)$ is given by

$$\sum_{s=0}^{[n/2]} \frac{(-1)^s (2x)^{n-2s}}{\lfloor n-2s \ \lfloor s} = \frac{H_n(x)}{\lfloor n} \qquad \text{(From equation (12) of §14.2)}$$

$$\therefore \qquad e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{\lfloor n} \ t^n$$

14.4 Hypergeometric Form

We have
$$H_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s \lfloor n}{\lfloor s \ \lfloor n-2s} (2x)^{n-2s} \qquad .....(1)$$

Now
$$\frac{n}{\lfloor n-2s} = \frac{\Gamma(n-1)}{\Gamma(n-2s+1)} = (-1)^{2s} \frac{\Gamma(-n+2s)}{\Gamma(-n)}$$

$$= \frac{2^{-n+2s-1} \ \pi^{-1/2} \ \Gamma\left(-\frac{n}{2}+s\right)\Gamma\left(-\frac{n}{2}+\frac{1}{2}+s\right)}{2^{-n-1} \ \pi^{-1/2} \ \Gamma\left(-\frac{n}{2}\right)\Gamma\left(-\frac{n}{2}+\frac{1}{2}\right)}$$

$$= 2^{2s}\left(-\frac{n}{2}\right)_s \left(-\frac{n}{2}+\frac{1}{2}\right)_s$$

Thus
$$H_n(x) = (2x)^n \sum_{s=0}^{[n/2]} \frac{(-1)^s \ x^{-2s} \ (-n/2)_s \ (-n+1/2)_s}{\lfloor s}$$

$$= (2x)^n \ {}_2F_0\left(-\frac{n}{2}, \ \frac{1-n}{2}; -; -\frac{1}{x^2}\right) \qquad .....(2)$$

14.5 Recurrence Formulae

14.5.1. $2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$

Proof. We know that

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} H_n(x)$$

Differentiating both sides w.r.t. 't', we have

or
$$e^{2xt-t^2}(2x-2t) = \sum_{n=0}^{\infty} n \frac{t^{n-1}}{\lfloor n} H_n(x)$$

or
$$2(x-t)\sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} H_n(x) = \sum_{n=1}^{\infty} \frac{t^{n-1}}{\lfloor n-1} H_n(x)$$

Equating the coefficients of t^n on both sides, we get

or
$$\frac{2x}{\lfloor n} H_n(x) - \frac{2}{\lfloor n-1} H_{n-1}(x) = \frac{1}{\lfloor n} H_{n+1}(x)$$

or $\quad 2x\, H_n(x) - 2n\, H_{n-1}(x) = H_{n+1}(x)$

or $\quad 2x\, H_n(x) = 2n\, H_{n-1}(x) + H_{n+1}(x).$

14.5.2. $\boldsymbol{H'_n(x) = 2nH_{n-1}(x)\ (n \geq 1)}$

Proof. We know that
$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} H_n(x)$$

Differentiating both side w.r.t. 'x' we have

$$2t \cdot e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} H'_n(x)$$

or
$$2t\sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} H_n(x) = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} H'_n(x)$$

Equating the coetticients of t^n on both sides, we get

or
$$\frac{2}{\lfloor n-1} H_{n-1}(x) = \frac{1}{\lfloor n} H'_n(x) = \frac{1}{n\lfloor n-1} H'_n(x)$$

or $\quad H'_n(x) = 2n\, H_{n-1}(x)$

14.5.3. $\boldsymbol{H'_n(x) = 2xH_{n-1}(x) - H_{n+1}(x)}$

Proof. Form Recurrence relations 14.5.1 and 14.5.2, we have

$$2x\, H_n(x) = 2n\, H_{n-1}(x) + H_{n+1}(x) \quad .....(1)$$

$$\mathrm{H}'_n(x) = 2n\, H_{n-1}(x) \quad .....(2)$$

Shbstracting (2) from (1), we have

or $\quad \mathrm{H}'_n(x) - 2x\, H_n(x) = -H_{n+1}(x)$

or $\quad \mathrm{H}'_n(x) = 2x\, H_n(x) - \mathrm{H}_{n+1}(x)$

14.5.4 $\quad \boldsymbol{H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0}$

Proof. Hermite's differential equation is

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0$$

$\because$ $H_n(x)$ is the solution of above differential equation, therefore.

$$H''_n(x) - 2x\, \mathrm{H}'_n(x) + 2n\, \mathrm{H}_n(x) = 0.$$

Self-Learning Exercise-I

1. $H_0(x) = ...$ $\qquad$ 2. $H_1(x) = ...$

3. $\sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} H_n(x) = ...$

4. Write down Hermite differential equation.

5. $\mathrm{H}'_n(x) = ...$ $\qquad$ 6. $H_n(-x) = ...$

Ex.1. Prove that $\boldsymbol{H_n''(x) = 4n(n-1)\, H_{n-2}(x)}$

Sol. From rcurrence relation 14.5.2, we have

$$H'_n(x) = 2n\, H_{n-1}(x)$$

Differentiating with respect to x, we get

$$H''_n(x) = 2n\, H'_{n-1}(x)$$

Again using recurrence relation 14.5.2, we find that

$$H''_n(x) = 2n \times 2(n-1)\, H_{n-2}(x)$$

$$= 4n\,(n-1)\, H_{n-2}(x)$$

Ex.2. Prove that if m < n

$$\frac{d^m\left[H_n(x)\right]}{dx^m} = \frac{2^m \lfloor n}{\lfloor n-m} H_{n-m}(x)$$

Sol. We know that

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} H_n(x)$$

or $$\frac{d^m}{dx^m}\left[e^{2xt-t^2}\right] = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} \frac{d^m}{dx^m}\left[H_n(x)\right] \cdot \frac{d^m}{dx^m}$$

or $$(2t)^m \cdot e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} \cdot \frac{d^m\left[H_n(x)\right]}{dx^m} \cdot \frac{d^m}{dx^m}$$

or $$(2t)^m \sum_{r=0}^{\infty} \frac{t^r}{\lfloor r} H_r(x) = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} \cdot \frac{d^m\left[H_n(x)\right]}{dx^m} \cdot \frac{d^m}{dx^m}$$

or $$2^m \sum_{r=0}^{\infty} \frac{t^{r+m}}{\lfloor r} H_r(x) = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} \cdot \frac{d^m\left[H_n(x)\right]}{dx^m}$$

If $$r + m = n$$

[Note that $r \geq 0 \Rightarrow n - m \geq 0$ or $m \leq n$]

or $$2^m \cdot \sum_{n=m}^{\infty} \frac{t^n}{\lfloor n-m} H_{n-m}(x) = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} \cdot \frac{d^m\left[H_n(x)\right]}{dx^m}$$

Equating the coefficient of t^n on both sides, we get

$$\frac{2^m}{\lfloor n-m} H_{n-m}(x) = \frac{1}{\lfloor n} \frac{d^m\left[H_n(x)\right]}{dx^m}$$

or $$\frac{2^m \lfloor n}{\lfloor n-m} H_{n-m}(x) = \frac{d^m\left[H_n(x)\right]}{dx^m}$$

Ex.3. Prove that

***(i)* $H_{2n}(0) = (-1)^n \cdot \dfrac{\lfloor 2n}{\lfloor n}$**

***(ii)* $H_{2n+1}(0) = 0$**

Sol. We have

$$\sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} H_n(x) = e^{2xt-t^2}$$

Putting $x = 0$ in this relation, we get

$$\sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} H_n(0) = e^{-t^2}$$

$$= \sum_{r=0}^{\infty} \frac{(-t^n)^r}{\lfloor r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r t^{2r}}{\lfloor r}$$

Note that R.H.S. contain only the terms of even powers of t. Equating the coefficient of t^{2n} on both the sides, we get

$$\frac{1}{\lfloor 2n} H_{2n}(0) = \frac{(-1)^n}{\lfloor n}$$

or $$H_{2n}(0) = \frac{(-1^n)\lfloor 2n}{\lfloor n} = (-1)^n \cdot 2^{2n}\left(\frac{1}{2}\right)_n$$

Further equating the coefficient of t^{2n+1} on both the sides, we obtain

$$H_{2n+1}(0) = 0$$

***Ex.4. Prove that* $H'_{2n}(0) = 0$ *and* $H'_{2n+1}(0) = (-1)^n 2^{2n+1}\left(\frac{3}{2}\right)_n$**

Sol. We have

$$H_n(x) = \sum_{s=0}^{[n/2]} \frac{(-1)^s \lfloor n}{\lfloor s \lfloor n-2s} (2x)^{n-2s}$$

Differentiating w.r.t. x, we get

$$H'_n(x) = \sum_{s=0}^{[(n-1)/2]} \frac{2(-1)^s \lfloor n (2x)^{n-2s-1}}{\lfloor s \lfloor n-2s} (n-2s)$$

Thus $$H'_{2n}(x) = 2\sum_{s=0}^{n-1} \frac{(-1)^s \lfloor 2n (2x)^{2n-2s-1}}{\lfloor s \lfloor 2n-2s-1}$$

and $$H'_{2n+1}(x) = 2\sum_{s=0}^{n} \frac{(-1)^s \lfloor (2n+1) (2x)^{2n-2s}}{\lfloor s \lfloor 2n-2s}$$

Hence $H'_{2n}(0) = 0$

and $H'_{2n+1}(0) = \dfrac{(-1)^n \lfloor(2n+1)}{\lfloor n}$

$$= (-1)^n \, 2^{2n+1} \left(\frac{3}{2}\right)_n \qquad \text{(by using gamma duplication formula)}$$

14.6 Rodrigues Formula for $H_n(x)$

To Prove that $\boldsymbol{H_n(x) = (-1)^n e^{x^2} \dfrac{d^n\left(e^{-x^2}\right)}{dx^n}}$

Proof. We have

$$f(x,t) = \frac{H_0(x)}{\lfloor 0} + \frac{H_1(x)t}{\lfloor 1} + \ldots.. + \frac{H_n(x)}{\lfloor n} t^n + \ldots..$$

where $f(x,t) = e^{2xt-t^2} = e^{x^2} e^{-(x-t)^2}$

$$\therefore \qquad \left[\frac{\partial^n f(x,t)}{\partial t^n}\right]_{t=0} = \frac{H_n(x)}{\lfloor n} \lfloor n = H_n(x)$$

$$\Rightarrow \qquad H_n(x) = \left[\frac{\partial^n \left\{e^{-(x-t)^2} \cdot e^{x^2}\right\}}{\partial t^n}\right]_{t=0}$$

$$= \left[\frac{\partial^n e^{-(x-t)^2}}{\partial t^n}\right]_{t=0} \cdot e^{x^2} \qquad \ldots..(1)$$

Let $x - t = u$ that is $t = x - u \Rightarrow x = u$ at $t = 0$

Also $x - t = u \Rightarrow \dfrac{\partial}{\partial t} \equiv -\dfrac{\partial}{\partial u}$

$$\therefore \qquad \left[\frac{\partial^n e^{-(x-t)^2}}{\partial t^n}\right] = (-1)^n \cdot \frac{\partial^n \left(e^{-u^2}\right)}{\partial u^n}$$

$$\Rightarrow \qquad \left[\frac{\partial^n e^{-(x-t)^2}}{\partial t^n}\right]_{t=0} = (-1)^n \cdot \frac{\partial^n \left(e^{-x^2}\right)}{\partial x^n}$$

$$= (-1)^n \cdot \frac{d^n\left(e^{-x^2}\right)}{dx^n}$$

From (1), we get $\quad H_n(x) = (-1)^n e^{x^2} \frac{d^n\left[e^{-x^2}\right]}{dx^n}$

14.7 Orthogonal Property of Hermite Polynomials

Theorem. Prove that $\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 2^n \lfloor n \sqrt{\pi}\,\delta_{mn}$ ***where*** δ_{mn} ***is Kronicar delta***

or $$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \begin{cases} 0 & \text{if} \quad m \neq n \\ \sqrt{\pi}\, 2^n \lfloor n & \text{if} \quad m = n \end{cases}$$

Proof. We know that

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} H_n(x)$$

$$e^{2xs-s^2} = \sum_{m=0}^{\infty} \frac{s^m}{\lfloor s} H_m(x)$$

$\Rightarrow$ $$e^{2xt-t^2} \cdot e^{2xs-s^2} = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} H_n(x) \cdot \sum_{m=0}^{\infty} \frac{s^m}{\lfloor s} H_m(x)$$

$\Rightarrow$ $\frac{1}{\lfloor n} \frac{1}{\lfloor m} H_n(x) H_m(x)$ = Coefficent of $t^n s^m$ in the expansion of

$$e^{2xt-t^2} \cdot e^{2xs-s^2}$$

So $\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \lfloor n \lfloor m$ times the coefficent of $t^n s^m$ in the expansion of

$$\int_{-\infty}^{\infty} e^{-x^2} e^{2xt-t^2} e^{2xs-s^2}\, dx \qquad .....(1)$$

Now $$\int_{-\infty}^{\infty} e^{-x^2} e^{2xt-t^2} e^{2xs-s^2}\, dx = e^{-t^2-s^2} \int_{-\infty}^{\infty} e^{-x^2+2xt+2rs}$$

$$= e^{-t^2-s^2} \int_{-\infty}^{\infty} e^{-x^2+2x(t+s)+(t+s)^2-(t+s)^2}\, dx$$

$$= e^{-t^2-s^2} \int_{-\infty}^{\infty} e^{-\left[x^2-2x(t+s)+(t+s)^2\right]} \cdot e^{(t+s)^2}\, dx$$

$$= e^{2st} \int_{-\infty}^{\infty} e^{-\left[x-(t+s)^2\right]} dx$$

$$= e^{2st} \int_{-\infty}^{\infty} e^{-u^2} du \qquad \text{[where } x-(t+s)=u \text{ and hence } dx = du]$$

$$= e^{2st} \cdot \sqrt{\pi} = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(2st)^n}{\lfloor n}$$

$$= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{2^n}{\lfloor n} s^n t^n$$

Here the series on right-hend side contains the terms having the equal powers of t and s. Therefore the coefficent of $t^n s^m$, $(m \neq n)$ will be zero. Equating the coefficent of $t^n s^m$ on both sides of above result,we get

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 0 \text{ where } m \neq n$$

and from (1), we have

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = \lfloor n \lfloor m \; \frac{2^n \sqrt{\pi}}{\lfloor n}$$

$$= \lfloor n\, 2^n \sqrt{\pi} \text{, where } m = n$$

Hence $$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = 2^n \lfloor n \sqrt{\pi}\, \delta_{mn}$$

Ex.1. Prove that $\boldsymbol{H_n(x) = 2^n \left\{ \exp.\left(-\frac{1}{4} \frac{d^2}{dx^2} \right) \right\} x^n}$

Sol. We have

$$\frac{d\left(e^{2tx}\right)}{dx} = 2te^{2tx}$$

$$\Rightarrow \qquad \frac{1}{2} \frac{d\left(e^{2tx}\right)}{dx} = te^{2tx}$$

Differentiating w.r.t. x

$$\frac{d}{dx}\left[\frac{1}{2}\frac{d}{dx}\left(e^{2tx}\right)\right] = 2t^2 e^{2tx}$$

$$\Rightarrow \qquad \frac{1}{2}\frac{d}{dx}\left[\frac{1}{2}\frac{d}{dx}\left(e^{2tx}\right)\right]=t^2e^{2tx}$$

$$\Rightarrow \qquad \left(\frac{1}{2}\frac{d}{dx}\right)^2 e^{2tx}=t^2e^{2tx}$$

Hence by symmetry for n terms, we get

$$\left(\frac{1}{2}\frac{d}{dx}\right)^n e^{2tx}=t^ne^{2tx} \qquad .....(2)$$

Now,

$$\left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\}e^{2tx}=\left[\sum_{n=0}^{\infty}\frac{1}{\lfloor n}\left(-\frac{1}{4}\frac{d^2}{dx}\right)^n\right]e^{2tx}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{\lfloor n}\left(\frac{1}{2}\frac{d}{dx}\right)^{2n}e^{2tx}$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{\lfloor n}t^{2n}e^{2tx} \qquad \text{[from (2)]}$$

$$=e^{2tx}\sum_{n=0}^{\infty}\frac{(-1)^n}{\lfloor n}t^{2n}$$

$$=e^{2tx}\sum_{n=0}^{\infty}\frac{1}{\lfloor n}\left(-t^2\right)^n$$

$$=e^{2tx}e^{-t^2}=e^{2xt-t^2}$$

or $\quad \left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\}\sum_{n=0}^{\infty}\frac{1}{\lfloor n}(2tx)^n=\sum_{n=0}^{\infty}\frac{t^n}{\lfloor n}H_n(x)$

Equating the coefficent of t^n on both sides we get

$$\left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx}\right)\right\}\frac{1}{\lfloor n}2^nx^n=\frac{1}{\lfloor n}H_n(x)$$

$$\Rightarrow \qquad H_n(x)=2^n\left\{\exp.\left(-\frac{1}{4}\frac{d^2}{dx}\right)\right\}x^n$$

which completes the solution of the problem.

Ex.2. Expand x^n in a series of Hermite polynomials

Sol. We have

$$e^{2xt-t^2}=\sum_{n=0}^{\infty}\frac{t^n}{\lfloor n}H_n(x)$$

$$e^{2xt} = e^{t^2} \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} H_n(x)$$

$$\Rightarrow \qquad \sum_{n=0}^{\infty} \frac{(2xt)^n}{\lfloor n} = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} H_n(x) \left[\sum_{s=0}^{\infty} \frac{t^{2s}}{\lfloor s} \right]$$

$$\Rightarrow \qquad \sum_{\eta=0}^{\infty} \frac{2^n x^n}{\lfloor n} t^n = \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{H_n(x)}{\lfloor n \lfloor s} t^{n+2s}$$

Put $n + 2s = m \Rightarrow n = m - 2s$ since $m - 2s \geq 0$.

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{[m/2]} \frac{H_{m-2s}(x) \cdot t^m}{\lfloor s \lfloor m-2s}$$

Equating coefficent of t^n on both sides

$$\frac{2^n x^n}{\lfloor n} = \sum_{s=0}^{[n/2]} \frac{H_{n-2s}(x)}{\lfloor s \lfloor n-2s}$$

$$\Rightarrow \qquad x^n = \sum_{s=0}^{[n/2]} \frac{\lfloor n \, H_{n-2s}(x)}{2^n \lfloor s \lfloor n-2s}$$

Ex.3. Prove that $\boldsymbol{P_n(x) = \frac{2}{\lfloor n \sqrt{\pi}} \int_0^{\infty} e^{-t^2} t^n H_n(xt)\, dt}$

This result is also known as Curzen's integral.

Sol. We know that $H_n(x) = \sum_{s=0}^{[n/2]} \frac{(-1)^s \lfloor n (2x)^{n-2s}}{\lfloor s \lfloor n-2s}$

$$\Rightarrow \qquad H_n(xt) = \sum_{s=0}^{[n/2]} \frac{(-1)^s \lfloor n (2xt)^{n-2s}}{\lfloor s \lfloor n-2s}$$

Now,

$$\text{RHS} = \frac{2}{\lfloor n \sqrt{\pi}} \int_0^{\infty} e^{-t^2} t^n \left\{ \sum_{s=0}^{[n/2]} \frac{(-1)^s \lfloor n (2xt)^{n-2s}}{\lfloor s \lfloor n-2s} \right\} dt$$

$$= \frac{2}{\lfloor n \sqrt{\pi}} \sum_{s=0}^{[n/2]} \frac{\lfloor n (-1)^s (2x)^{n-2s}}{\lfloor s \lfloor n-2s} \int_0^{\infty} e^{-t^2} t^{2n-2s}\, dt$$

Put $t^2 = \phi \Rightarrow dt = \frac{1}{2} \phi^{-1/2} d\phi$

$$\therefore \qquad \text{R.H.S.} = \frac{1}{\sqrt{\pi}} \sum_{s=0}^{[n/2]} \frac{(-1)^s (2x)^{n-2s}}{\lfloor s \lfloor n-2s} \int_0^{\infty} e^{-\phi} \phi^{n-s+\frac{1}{2}-1}\, d\phi$$

$$= \frac{1}{\sqrt{\pi}} \sum_{s=0}^{[n/2]} \frac{(-1)^s (2x)^{n-2s}}{\lfloor s \lfloor n-2s} \Gamma\left(n - s + \frac{1}{2}\right)$$

$$= \sum_{s=0}^{[n/2]} \frac{(-1)^s (2x)^{n-2s}}{\lfloor s \, \lfloor n-2s} \times \frac{\Gamma\left(n-s+\frac{1}{2}\right)}{\Gamma(1/2)} \qquad \left[\because \Gamma(1/2) = \sqrt{\pi}\ \right]$$

$$= \sum_{s=0}^{[n/2]} \frac{(-1)^s (2x)^{n-2s} (1/2)_{n-2s}}{\lfloor s \, \lfloor n-2s}$$

$= P_n(x)$ (by definition of Legendre polynomials)

Ex.4. Show that $\displaystyle \sum_{n=0}^{\infty} \frac{H_{n+s}(x)\, t^n}{\lfloor n} = \exp\left(2xt - t^2\right) H_s(x-t)$

Sol. Consider

$$\sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{H_{n+s}(x)\, t^n v^s}{\lfloor n \, \lfloor s} = \sum_{n=0}^{\infty} \sum_{s=0}^{n} \frac{H_n(x)\, t^{n-s} v^s}{\lfloor n-s \, \lfloor s}$$

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{n} \frac{(-n)_s (-1)^s H_n(x)\, t^n u^s}{\lfloor n \, \lfloor s} \left(\frac{v}{t}\right)^s$$

$$= \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{\lfloor n} \sum_{s=0}^{n} \frac{(-n)_s}{\lfloor s} \left(-\frac{v}{t}\right)^s$$

$$= \sum_{n=0}^{\infty} \frac{H_n(x)\, t^n}{\lfloor n} \left(1 + \frac{v}{t}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{H_n(x)(v+t)^n}{\lfloor n}$$

$$= e^{2x(t+v) - (t+v)^2}$$

$$= e^{2xt - t^2} \cdot e^{2v(x-t) - v^2}$$

$$= e^{2xt - t^2} \sum_{s=0}^{\infty} \frac{H_s(x-t)\, v^s}{\lfloor s}$$

Comparing the coeffiecient of $\dfrac{v^s}{s!}$, we get the required result.

Ex.5. Establish

$$\sum_{n=0}^{\infty} \frac{(c)_n H_n(x)\, t^n}{\lfloor n} = (1-2xt)^{-c} \, {}_2F_0\left(\frac{c}{2}, \frac{c}{2} + \frac{1}{2}\ ;\ -;\ -\frac{4t^2}{(1-2xt)^2}\right) \qquad .....(3)$$

Sol. We have

$$\text{L.H.S. of (3)} = \sum_{n=0}^{\infty} \sum_{s=0}^{[n/2]} \frac{(-1)^s (c)_n (2x)^{n-2s} t^n}{\lfloor s \lfloor n-2s}$$

Now using a well-known result

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n+2k), \text{ we get}$$

$$\text{L.H.S. of (3)} = \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s (c)_{n+2s} (2x)^n t^{n+2s}}{\lfloor s \lfloor n}$$

$$\because \quad (c)_{n+2s} = (c+2s)_n (c)_{2s}$$

$$\therefore \quad \text{L.H.S. of (3)} = \sum_{s=0}^{\infty} \frac{(-1)^s (c)_{2s} t^{2s}}{\lfloor s} \sum_{n=0}^{\infty} \frac{(c+2s)_n (2xt)^n}{\lfloor n}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s (c)_{2s} t^{2s}}{\lfloor s} (1-2xt)^{-c-2s}$$

$$\text{But} \quad (c)_{2s} = 2^{2s} \left(\frac{c}{2}\right)_s \left(\frac{c}{2} + \frac{1}{2}\right)_s$$

$$\text{Hence L.H.S. of (3)} = (1-2xt)^{-c} \sum_{s=0}^{\infty} \frac{(c/2)_s (c+1/2)_s}{\lfloor s} \left(-\frac{4t^2}{(1-2xt)^2}\right)^s$$

$$= (1-2xt)^{-c} \, {}_2F_0\left(\frac{c}{2}, \frac{c}{2} + \frac{1}{2}; -; -\frac{4t^2}{(1-2xt)^2}\right)$$

The relation (3) is called the **Braf man's generating function.**

Ex.6. Prove that $$\int_0^x e^{-y^2} H_n(y)\, dy = H_{n-1}(0) - e^{-x^2} H_{n-1}(x) \quad \text{.....(4)}$$

Sol. Using Rodrigue's formula in the left–hand side of (4), we get

$$\int_0^x e^{-y^2} H_n(y)\, dy = \int_0^x (-1)^n \frac{d^n}{dy^n}\left(e^{-y^2}\right) dy = (-1)^n \left[\frac{d^{n-1}}{dy^{n-1}}\left(e^{-y^2}\right)\right]_0^x$$

$$= -\left\{e^{-y^2} H_{n-1}(y)\right\}_0^x$$

(Using again the Rodrigue's formula)

$$= H_{n-1}(0) - e^{-x^2} H_{n-1}(x)$$

Self-Learning Exercise-II

1. $\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \ldots\ldots\ldots\ldots (if\ m \neq n)$

2. Write down Rodriques formulas for $H_n(x)$.

3. $H_{2n+1}(0) = \ldots\ldots\ldots\ldots$

14.8 Summary

In this unit, we studied the Hermite differential equation and Hermile polynomials. We also studied recurrence relation, generating function, Rodrigue, formula and orthogonal property for Hermite polynomials.

14.9 Answers to Self-Learning Exercises

Exercise I

1. e^{2xt-t^2}

2. $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2\lambda y = 0$

3. $2nH_{n-1}(x)$

Exercise II

1. 0

2. $H_{n(x)} = (-1)^n e^{x^2} \frac{d^n\left(e^{-x^2}\right)}{dx^n}$

3. 0

14.10 Exercise

1. Evaluate $\int_{-\infty}^{\infty} xe^{-x^2} H_m(x) H_n(x)\, dx$ (m ≠ n) [**Ans :** 0]

2. Prove that $H_5(x) = 32x^5 - 160x^3 + 120x$

3. Prove that $H_2(x) = 4x^2 - 2$

4. Express $H(x) = x^4 + 2x^3 + 2x^2 - x - 3$ in terms of Hermite polynomials.

5. Prove that $x H'_n(n) = n H'_{n-1}(x) + n H_n(x)$

6. Prove that $\int_{-\infty}^{\infty} x^2 e^{-x^2} \{H_n(x)\}^2 \, dx = \sqrt{\pi}\, 2^n \lfloor n \left(n + \frac{1}{2}\right)$

7. Show that $\sum_{k=0}^{n} \frac{H_k(x) H_k(y)}{2^k\, k!} = \frac{H_n(x) H_{n+1}(y) - H_{n+1}(x) H_n(y)}{2^{n+1}(y-x)\lfloor n}$

8. Evaluate $2^{n+1} e^{x^2} \int_x^{\infty} e^{-t^2} t^{n+1} P_n(x/t)\, dt$ **[Ans :** $H_n(x)$**]**

9. Evaluate $\int_{-\infty}^{\infty} x e^{-x^2} H_m(x) H_n(x)\, dt, m \neq n$ **[Ans :** 0**]**

10. If $\psi_n(x) = e^{-x2/2} Hn(x)$, then prove that

(i) $\int_{-\infty}^{\infty} \psi_m(x) \psi_n(x)\, dx = 2^n \lfloor n \sqrt{\pi}\, \delta_{m,n}$ if $m \neq n \pm 1$

(ii) $\int_{-\infty}^{\infty} \psi_m(x) \psi_n'(x)\, dx \begin{cases} 0, & \text{if } m \neq n \pm 1 \\ 2^{n-1} \lfloor n \sqrt{\pi}, & \text{if } m = n-1 \\ -2^n \lfloor n+1 \sqrt{\pi}, & \text{if } m = n+1 \end{cases}$

11. Using the expansion of x^n in a series of Hermite polynomials, show that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} x^n H_{n-2k}(x)\, dx = 2^{-2k} \frac{\lfloor n \sqrt{\pi}}{\lfloor k}$$

□ □ □

Unit 15 : Laguerre Polynomials

15.0 Objective

In this unit you will study Laguerre and associated Laguerre polynomials and their important properties such as generating function, orthogonal property, Rodrigue's formula, recurrence relations etc.

15.1 Introduction

The purpose of this unit is to introduce and study the Laguerre and associated Laguerre polynomials. We shall state and prove certain important properties associated with these classes of polynomials.

15.2 Laguerre's Differential Equation and Its Solution

THe Laguerre differential equation of order n is

$$x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+ny=0, \qquad .....(1)$$

where n is a positive integer

Now we apply the method of Frobenius for its solution which is finite for all values of x and which tends to ∞ no faster than $e^{x/2}$ as $x \to \infty$.

Proceeding on lines similar to explained in the case of Legendre, and Hermite polynomials, we find that if we assume the solution of (1) in the form

$$y = \sum_{r=0}^{\infty} a_r x^r \qquad .....(2)$$

then

$$y = a_0 \sum_{r=0}^{n} (-1)^r \frac{\lfloor n}{\lfloor n-r \, (\lfloor r)^2} x^r \qquad .....(3)$$

will be solution of equation (1). Taking $a_0 = 1$, the corresponding solution of equation (1) is known as Laguerre polynomial of order n, and which is denoted by $L_n(x)$. Thus

$$L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{\lfloor n}{\lfloor n-r \, (\lfloor r)^2} x^r$$

$$= {}_1F_1(-n;\, 1;\, x) \qquad .....(4)$$

Some times we take a_0 as $\lfloor n$, then alternative definition of Laguerre polynomials is

$$L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{(\lfloor n)^2}{\lfloor n-r \, (\lfloor r)^2} x^r \qquad .....(5)$$

15.3 Generating Function for $L_n(x)$

Theorem : ***Show that***

$$\frac{e^{-\frac{xt}{1-t}}}{1-t} = \sum_{n=0}^{\infty} L_n(x) \cdot t^n$$

Proof : Using the exponential series we have

$$\frac{e^{-\frac{xt}{1-t}}}{1-t} = \frac{1}{1-t} \sum_{r=0}^{\infty} \frac{1}{\lfloor r} \left(\frac{-xt}{1-t} \right)^r$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r} x^r t^r (1-t)^{-(r+1)}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r x^r t^r}{\lfloor r} \sum_{s=0}^{\infty} \frac{(r+1)_s\, t^s}{\lfloor s}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^r \lfloor (r+s)\, x^r\, t^{r+s}}{(\lfloor r)^2 \lfloor s} \qquad ....(1)$$

For a fixed r, the coefficient of t^n is

$$= (-1)^r \frac{\lfloor n \, x^r}{\left(\lfloor r\right)^2 \lfloor (n-r)}$$

Taking $n = r + s$.

Now $s = n - r$ and $s \geq 0$, so $r \leq n$.

Hence the total coefficient of t^n in (1) is

$$= \sum_{s=0}^{\infty} \frac{(-1)^r \lfloor n \, x^r}{\left(\lfloor r\right)^2 \lfloor (n-r)} = L_n(x) \qquad \text{(By definition)}$$

Hence $$\frac{e^{-\frac{xt}{1-t}}}{1-t} = \sum_{n=0}^{\infty} L_n(x) t^n$$

15.4 Recurrence Relations for $L_n(x)$

15.4.1 $(n+1) L_{n+1}(x) = (2n+1-x) L_n(x) - n L_{n-1}(x)$

Proof : From generating function, we have

$$\frac{1}{(1-t)} e^{-\frac{xt}{1-t}} = \sum_{n=0}^{\infty} t^n L_n(x) \qquad \text{.....(1)}$$

Differentiating (1) w.r.t. 't' we get

$$\sum_{n=0}^{\infty} n t^{n-1} L_n(x) = \frac{1}{(1-t)^2} e^{-\frac{xt}{1-t}} + \frac{1}{(1-t)} e^{-\frac{xt}{1-t}} \left\{ -\frac{x}{(1-t)^2} \right\}$$

$$= \frac{1}{(1-t)} \sum_{n=0}^{\infty} t^n L_n(x) - \frac{x}{(1-t)^2} \sum_{n=0}^{\infty} t^n L_n(x)$$

Multiplying both the side by $(1-t)^2$ we get

$$\left(1 - 2t + t^2\right) \sum_{n=0}^{\infty} n t^{n-1} L_n(x) = (1-t) \sum_{n=0}^{\infty} t^n L_n(x) - x \sum_{n=0}^{\infty} t^n L_n(x)$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} n t^{n-1} L_n(x) - 2 \sum_{n=0}^{\infty} n t^n L_n(x) + \sum_{n=0}^{\infty} n L_n(x) t^{n+1}$$

$$= \sum_{n=0}^{\infty} t^n L_n(x) - \sum_{n=0}^{\infty} t^{n+1} L_n(x) - x \sum_{n=0}^{\infty} t^n L_n(x)$$

Now equating the coefficent of t^n on both sides, we get

$$(n+1) L_{n+1}(x) - 2n L_n(x) + (n-1) L_{n-1}(x) = L_n(x) - L_{n-1}(x) - x L_n(x)$$

$$\Rightarrow \quad (n+1) L_{n+1}(x) = (2n+1-x) L_n(x) - n L_{n-1}(x)$$

15.4.2 $x\,L'_n(x) = n\,L_n(x) - n\,L_{n-1}(x)$

Proof : From generating function

$$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{(1-t)} e^{-\frac{xt}{1-t}} \qquad .....(2)$$

Differentiating w.r.t. '*x*' we get

$$\sum_{n=0}^{\infty} t^n L'_n(x) = \frac{1}{(1-t)} e^{-\frac{xt}{1-t}} \left(\frac{-t}{1-t}\right)$$

or $$\sum_{n=0}^{\infty} t^n L'_n(x) = -\frac{t}{1-t} \sum_{n=0}^{\infty} t^n L_n(x)$$

or $$(1-t)\sum_{n=0}^{\infty} t^n L'_n(x) = -t \sum_{n=0}^{\infty} t^n L_n(x)$$

or $$\sum_{n=0}^{\infty} t^n L'_n(x) - \sum_{n=0}^{\infty} t^{n+1} L'_n(x) = -\sum_{n=0}^{\infty} t^{n+1} L_n(x)$$

Equating the coefficients of t^n on both sides, we get

$$L'_n(x) - L'_{n-1}(x) = -L_{n-1}(x)$$

or $$L'_{n-1}(x) = L'_{n-1}(x) + L'_{n-1}(x) \qquad .....(3)$$

Differentiating Recurrence relation 15.4.1, we find that

$$(n+1)L'_{n+1}(x) = (2n+1-x)L'_n(x) - L_n(x) - n\,L'_{n-1}(x) \qquad .....(4)$$

Replacing *n* by (*n* + 1) in (3), we obtain

$$L'_n(x) = L'_{n+1}(x) + L_n(x) \qquad .....(5)$$

Putting the value of $L'_{n-1}(x)$ and $L'_{n+1}(x)$ from (3) and (5) in (4) we get

$$(n+1)\left[L'_n(x) - L_n(x)\right] = (2n+1-x)L'_n(x) - L_n(x) - n\left[L'_n(x) + L_{n-1}(x)\right]$$

$$n\,L'_n(x) - n\,L_n(x) + L'_n(x) - L_n(x)$$

$$= 2nL^{1}_n(x) + L'_n(x) - x\,L'_n(x) - L_n(x) - n\,L'_n(x) - n\,L_{n-1}(x)$$

On simplification, we get $x\,L'_n(x) = n\,L_n(x) - n\,L_{n-1}(x)$

15.4.3 $L'_n(x) = -\sum_{r=0}^{n-1} L_r(x)$

Proof : From generating function

$$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{(1-t)} e^{-\frac{xt}{1-t}} \qquad .....(6)$$

Differentiating (6) w.r.t. '*x*', we get

or $$\sum_{n=0}^{\infty} t^n L'_n(x) = \frac{1}{(1-t)} e^{-\frac{xt}{1-t}} \left[\frac{-t}{1-t}\right]$$

$$= -t(1-t)^{-1}\sum_{r=0}^{\infty} L_n(x)t^r \qquad \text{(using Bionomial theorem)}$$

$$= -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} L_n(x)\, t^{r+s+1} \qquad .....(7)$$

Taking $r + s + 1 = n$, we have s $= n - s - 1$. But $s \geq 0$ therefore $r \leq n - 1$

So the total coefficient of t^n R.H.S. is $-\sum_{r=0}^{n-1} L_r(x)$

Now equating coefficient of t^n on both sides in (7), we arrive at the required recurrence relation 15.4.3.

15.5 Rodrigue's Formula for L_n (*x*)

Prove that

$$\boldsymbol{L_n(x) = \frac{e^x}{\lfloor n} \frac{d^n}{dx^n}\left(x^n e^{-x}\right)}$$

Proof : Using Leibnitz's theorem for *n* times differentiation, we have

$$\text{R.H.S.} = \frac{e^x}{\lfloor n} D^n\left(x^n e^{-x}\right)$$

$$= \frac{e^x}{\lfloor n} \sum_{r=0}^{n} {}^nc_r\, D^{n-r}\left\{x^n\right\} D^r e^{-x}$$

$$= \frac{e^x}{\lfloor n} \sum_{r=0}^{n} {}^nc_r \cdot \frac{\lfloor n}{\lfloor\{n-(n-r)\}} x^{n-(n-r)}(-1)^r e^{-x}$$

$$= \sum_{r=0}^{n} \frac{e^x}{\lfloor n} \frac{(\lfloor n)^2}{(\lfloor r)^2 \lfloor n-r} x^r (-1)^r e^{-x}$$

$$= \sum_{r=0}^{n} \frac{(-1)^r \lfloor n\, x^r}{(\lfloor r)^2 \lfloor n-r} = L_n(x)$$

15.6 Orthogonal Property

Prove that

$$\boldsymbol{\int_0^{\infty} e^{-x} L_m(x) L_n(x)\, dx = \delta_{mn} = \begin{cases} 0, & if \quad m \neq n \\ 1, & if \quad m = n \end{cases}}$$

Proof : From generating function, we have

$$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{1-t} e^{-\frac{xt}{1-t}} \qquad .....(1)$$

$$\sum_{m=0}^{\infty} s^m L_m(x) = \frac{1}{1-s} e^{-\frac{xs}{1-s}} \qquad .....(2)$$

Multiplying (1) and (2), we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} L_n(x) t^n L_m(x) s^m = \frac{1}{(1-t)} \times \frac{1}{(1-s)} e^{-x\left[\frac{t}{1-t}+\frac{s}{1-s}\right]}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \left[\int_0^{\infty} e^{-x} L_m(x) L_n(x)\,dx\right] t^n s^m$$

$$= \frac{1}{(1-t)(1-s)} \int_0^{\infty} e^{-x} \cdot e^{-x\left[\frac{t}{1-t}+\frac{s}{1-s}\right]} dx$$

$$= \frac{1}{(1-t)(1-s)} \int_0^{\infty} e^{-x\left[1+\frac{t}{1-t}+\frac{s}{1-s}\right]} dx$$

$$= \frac{1}{(1-t)(1-s)} \left[-\frac{e^{-x\left[1+\frac{t}{1-t}+\frac{s}{1-s}\right]}}{\left[1+\frac{t}{1-t}+\frac{s}{1-s}\right]} \right]_0^{\infty}$$

$$= \frac{1}{(1-t)(1-s)} \times \frac{(1-t)(1-s)}{\left[(1+t)(1-s)+t-ts+s-st\right]} \times \left[e^{-x\left[1+\frac{t}{1-t}+\frac{s}{1-s}\right]} \right]_0^{\infty}$$

$$= \frac{1}{\left[1-s-t+ts+t-ts+s-st\right]} \times [0-1]$$

$$= \frac{1}{(1-st)} = (1-st)^{-1}$$

$$= 1 + st + (st)^2 + ... + (st)^n + ...$$

Equating the coefficients of $t^n s^n$ on both sides, we get

$$\int_0^{\infty} e^{-x} L_m(x) L_n\,dx = 0 \text{ if } m \neq n \qquad .....(3)$$

and equating the coefficient of $t^n s^m$, we get

$$\int_0^{\infty} e^{-x} \left[L_n(x)^2\right] dx = 1$$

That is $\int_0^{\infty} e^{-x} L_n(x) L_m(x)\,dx = 1$ (when $m = n$) .....(4)

Combining (3) and (4), we get $\int_0^{\infty} e^{-x} L_m(x) L_n(x)\,dx = \delta_{mn}$

Ex.*1. *Prove that

$$\int_0^\infty e^{-st} L_n(t)\,dt = \frac{1}{s}\left(1-\frac{1}{s}\right)^n$$

Sol. L.H.S. $= \int_0^\infty e^{-st} \sum_{r=0}^{n} \frac{(-1)^r \lfloor n\, t^r}{(\lfloor r)^2 \lfloor n-r} dt$

$$= \sum_{r=0}^{n} \frac{(-1)^r \lfloor n}{(\lfloor n-r)(\lfloor r)^2} \int_0^\infty e^{-st} t^{r+1-1} dt$$

$$= \sum_{r=0}^{n} \frac{(-1)^r \lfloor n}{(\lfloor n-r)(\lfloor r)^2} \frac{\Gamma(r+1)}{s^{r+1}}$$

$$= \frac{1}{s} \sum_{r=0}^{n} \frac{(-1)^r \lfloor n}{\lfloor n-r\ (\lfloor r)} \times \frac{1}{s^r}$$

$$= \frac{1}{s} \sum_{r=0}^{n} {}^n c_r \left(-\frac{1}{s}\right)^r = \frac{1}{s}\left(1-\frac{1}{s}\right)^n$$

= R.H.S.

***Ex.*2. *Prove that (i)* L_n (0) = 1, *(ii)* $L_n'(0) = -n$ *and (iii)* $L_n''(0) = \frac{n(n-1)}{2}$**

Sol. We know that

$$\frac{1}{1-t} e^{-\frac{xt}{1-t}} = \sum_{n=0}^{\infty} t^n L_n(x) \qquad \text{....(5)}$$

Taking $x = 0$ in (5), we get

or $$(1-t)^{-1} = \sum_{n=0}^{\infty} t^n L_n(0)$$

or $$\sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} t^n L_n(0)$$

Equating coefficients of t^n on both sides, we get

$$1 = L_n(0)$$

(ii) From Laguerre differential equation, we have

$$xy'' + (1 - x)\, y' + ny = 0$$

If $L_n(x)$ is the solution of this equation then

$$x L_n''(x) + (1-x) L_n'(x) + n L_n(x) = 0$$

Putting $x = 0$, we get

$$L_n'(0) = -n L_n(0)$$

$$= -n \cdot 1 \qquad \text{[from } (i)\text{]}$$

Thus $$L_n'(0) = -n$$

(iii) Differentiating twice w.r.t '*x*', (1) gives

$$\frac{e^{-xt/(1-t)}}{1-t}\cdot\left(-\frac{t}{1-t}\right)^2 = \sum_{n=0}^{\infty} L_n''(x)t^n$$

Putting $x = 0$, we get

$$\sum_{n=0}^{\infty} L_n''(0)t^n = t^2(1-t)^{-3} \qquad .....(6)$$

Equating the coefficients of t^n on both the sides of (6), we find that

$$\begin{aligned} L_n''(0) &= \text{Coeff. of } t^n \text{ in } t^2(1-t)^{-3} \\ &= \text{Coeff. of } t^{n-2} \text{ in } (1-t)^{-3} \\ &= \frac{(-3)(-3-1).....\{-3-(n-2)+1\}}{\lfloor(n-2)}(-1)^{n-2} \\ &= \frac{3.4.\ .....n}{\lfloor(n-2)} = \frac{\lfloor n}{\lfloor 2\ \lfloor n-2} = \frac{n(n-1)}{2} \end{aligned}$$

Self-Learning Exercise–1

1. Laguerre's differential equation is ....

2. $\int_0^{\infty} e^{-x} L_m(x) L_n\, dx$ = ... if $m \neq n$

3. $\int_0^{\infty} e^{-x}\left[L_n(x)^2\right]dx$ = ...

4. $L_n(0)$ = ...

5. .... = $nL_n(x) - nL_{n-1}(x)$

6. $L_0(x)$ = ...

7. $L_1(x)$ = ...

8. $L_2(x)$ = ...

15.7 Associated Laguerre Polynomial : Definition

Associated Laguerre polynomials of degree *n* and order *k* is denoted and defined as

$$L_n^k(x) = (-1)^k \frac{d^k}{dx^k} L_{n+k}(x) \qquad .....(1)$$

Now using the series representation for Laguerre polynomials we find that

$$L_n^k(x) = (-1)^k \frac{d^k}{dx^k} \sum_{r=0}^{n+k} (-1)^r \frac{\lfloor(n+k)}{\lfloor(n+k-r)\ (\lfloor r)^2} x^r$$

$$= (-1)^k \sum_{r=0}^{n+k} (-1)^r \frac{\underline{|(n+k)}}{\underline{|(n+k-r)}\ (\underline{|r})^2} \frac{d^k}{dx^k} x^r \qquad .....(2)$$

Now
$$\frac{d^k}{dx^k} x^r = \begin{cases} 0, & if \quad r < k \\ \dfrac{\underline{|r}}{\underline{|r-k}} x^{r-k}, & if \quad r \geq k \end{cases}$$

Hence breaking $\sum_{r=0}^{n+k}$ into two sums as $\sum_{r=0}^{k-1}$ and $\sum_{r=k}^{n+k}$, we find that

$$L_n^k(x) = (-1)^k \sum_{r=k}^{n+k} (-1)^{r+k} \frac{\underline{|(n+k)}}{\underline{|(n+k-r)}\ \underline{|r}\ \underline{|(r-k)}} x^{r-k}$$

Let $r - k = s$, so that $r = s + k$ and when $r = k$, $s = 0$ and $r = n + k$, $s = n$. Then

$$L_n^k(x) = \sum_{s=0}^{n} (-1)^{s+2k} \frac{\underline{|(n+k)}}{\underline{|(n-s)}\ \underline{|(s+k)}\ \underline{|s}} x^s$$

or
$$L_n^k(x) = \sum_{r=0}^{n} (-1)^r \frac{\underline{|(n+k)}}{\underline{|(n-r)}\ \underline{|(k+r)}\ \underline{|r}} x^r \qquad .....(3)$$

15.8 Generating Function for Associated Laguerre Polynomials

Prove that

$$\frac{1}{(1-t)^{k+1}} \exp\left\{-\frac{xt}{(1-t)}\right\} = \sum_{n=0}^{\infty} L_n^k(x) t^n$$

Proof : By generating function for Laguerre polynomial, we have

$$\frac{1}{(1-t)} \exp\left\{\frac{-xt}{1-t}\right\} = \sum_{n=0}^{\infty} L_n(x) t^n \qquad .....(1)$$

Differentiation both sides of (1) 'k' times w.r.t. 'x', gives

$$\frac{1}{(1-t)} \frac{d^k}{dx^k}\left[\exp\left\{-\frac{xt}{1-t}\right\}\right] = \sum_{n=0}^{\infty} t^n \frac{d^k}{dx^k}\{L_n(x)\}$$

or
$$\frac{1}{(1-t)}\left(-\frac{t}{1-t}\right)^k \exp\left\{-\frac{xt}{1-t}\right\} = \sum_{n=0}^{k-1} t^n \frac{d^k}{dx^k}\{L_n(x)\} + \sum_{n=k}^{\infty} t^n \frac{d^k}{dx^k}\{L_n(x)\}$$

or
$$(-1)^k \frac{t^k}{(1-t)^{k+1}} \exp\left\{-\frac{xt}{1-t}\right\} = 0 + \sum_{n=k}^{\infty} t^n \frac{d^k}{dx^k}\{L_n(x)\} \qquad .....(2)$$

Here we use that $L_n(x)$ is a polynomial of degres n so that

$$\frac{d^k}{dx^k}\{L_n(x)\} = \begin{cases} 0 & \text{if} \quad n < k \\ \text{non-zero} & \text{if} \quad n \geq k \end{cases}$$

Multiplying by (2) by $(-1)^k$ then we get

$$\frac{t^k}{(1-t)^{k+1}}\exp\left\{-\frac{xt}{1-t}\right\} = (-1)^k \sum_{n=k}^{\infty} t^n \frac{d^k}{dx^k}\{L_n(x)\}$$

$$\Rightarrow \qquad \frac{t^k}{(1-t)^{k+1}}\exp\left\{-\frac{xt}{1-t}\right\} = (-1)^k \sum_{s=0}^{\infty} t^{s+k} \frac{d^k}{dx^k}\{L_{s+k}(x)\}$$

(Taking s as new variable such that $n = s + k$ i.e. $s = n - k$ so when $n = k$, $s = 0$ and when n tends to∞, s also tends to ∞)

$$\therefore \qquad \frac{t^k}{(1-t)^{k+1}}\exp\left\{-\frac{xt}{1-t}\right\} = t^k \sum_{s=0}^{\infty} (-1)^k \frac{d^k}{dx^k}\{L_{s+k}(x)\}t^s$$

$$\text{or} \qquad \frac{1}{(1-t)^{k+1}}\exp\left\{-\frac{xt}{1-t}\right\} = \sum_{n=0}^{\infty} (-1)^k \frac{d^k}{dx^k}\{L_{n+k}(x)\}t^n$$

($\because$ The limit remain same so we can change the variable from s to n)

$$\frac{1}{(1-t)^{k+1}}\exp\left\{-\frac{xt}{1-t}\right\} = \sum_{n=0}^{\infty} L_n^k(x)t^n$$

15.9 Recurrence Relations for $L_n^k(x)$

15.9.1 $L_{n-1}^k(x) + L_n^{k-1}(x) = L_n^k(x)$

Proof : We know that

$$L_n^k(x) = \sum_{r=0}^{n} (-1)^r \frac{\lfloor\underline{(n+k)}}{\lfloor\underline{n-r}\ \lfloor\underline{k+r}\ \lfloor\underline{r}} \qquad .....(1)$$

Replacing n by $(n-1)$ in (1), we find that

$$L_{n-1}^k(x) = \sum_{r=0}^{n-1} (-1)^r \frac{(-1)^r \lfloor\underline{(n+k-1)}}{\lfloor\underline{(n-r-1)}\ \lfloor\underline{r}\ \lfloor\underline{k+r}} x^r \qquad .....(2)$$

Replacing k by $(k-1)$ in (1), we get

$$L_n^{k-1}(x) = \sum_{r=0}^{n} \frac{(-1)^r \lfloor\underline{(n+k-1)}}{\lfloor\underline{n-r}\ \lfloor\underline{k+r-1}\ \lfloor\underline{r}} x^r \qquad .....(3)$$

Using (2) and (3), we have

$$L_{n-1}^k(x) + L_n^{k-1}(x) = \sum_{r=0}^{n-1} \frac{(-1)^r \lfloor\underline{(n+k-1)}}{\lfloor\underline{(n-r-1)}\ \lfloor\underline{k+r}\ \lfloor\underline{r}} x^r + \sum_{r=0}^{n} \frac{(-1)^r \lfloor\underline{(n+k-1)}}{\lfloor\underline{n-r}\ \lfloor\underline{k+r-1}\ \lfloor\underline{r}} x^r$$

$$= \sum_{r=0}^{n-1} \frac{(-1)^r \lfloor\underline{(n+k-1)}}{\lfloor\underline{(n-r-1)}\ \lfloor\underline{k+r}\ \lfloor\underline{r}} x^r + \sum_{r=0}^{n-1} \frac{(-1)^r \lfloor\underline{(n+k-1)}}{\lfloor\underline{n-r}\ \lfloor\underline{k+r-1}\ \lfloor\underline{r}} x^r$$

$$+ \frac{(-1)^n \lfloor\underline{(n+k-1)}\, x^n}{\lfloor\underline{(n-n)}\ \lfloor\underline{k+n-1}\,\lfloor\underline{n}}$$

$$= \sum_{r=0}^{n-1} \frac{(-1)^r \lfloor\underline{(n+k-1)}}{\lfloor\underline{(n-r-1)}\ \lfloor\underline{(k+r-1)}\ \lfloor\underline{r}} x^r \times \frac{n+k}{(k+r)(n-r)} + \frac{(-1)^n x^n}{\lfloor\underline{n}}$$

$$= \sum_{r=0}^{n-1} \frac{(-1)^r \lfloor\underline{n+k}\ x^r}{\lfloor\underline{n-r}\ \ \lfloor\underline{k+r}\ \lfloor\underline{r}} + \frac{(-1)^n x^n}{\lfloor\underline{n}}$$

$$= \sum_{r=0}^{n} \frac{(-1)^r \lfloor\underline{n+k}\ \ x^r}{\lfloor\underline{n-r}\ \ \lfloor\underline{k+r}\ \lfloor\underline{r}}$$

$$= L_n^k(x) \qquad \text{[by (1)]}$$

15.9.2 $\boldsymbol{(n+1)L_{n+1}^k(x) = (2n+k+1-x)L_n^k(x) - (n+k)L_{n-1}^k(x)}$

Proof : From recurrence relation 15.4.1 for Laguerre polynomial we have

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x) \qquad ...(4)$$

Replacing n by $(n + k)$ in (4), we get

$$(n+k+1)L_{n+k+1}(x) = (2n+2k+1-x)L_{n+x}(x) - (n+k)L_{n+k-1}(x)$$

Differentiating k times, the above equation becomes

$$(n+k+1)\frac{d^k}{dx^k}\{L_{n+k+1}(x)\} = (2n+2k+1)\frac{d^k}{dx^k}\{L_{n+k}(x)\}$$

$$-\frac{d^k}{dx^k}\{xL_{n+k}(x)\} - (n+k)\frac{d^k}{dx^k}\{L_{n+k-1}(x)\} \qquad ...(5)$$

Using Leibnitz's theorem, we get

$$\frac{d^k}{dx^k}\{xL_{n+k}(x)\} = \frac{d^k}{dx^k}\{L_{n+k}(x)\}x + {}^kc_1 \frac{d^{k-1}}{dx^{k-1}}\{L_{n+k}(x)\}$$

$$= x\frac{d^k}{dx^k}\{L_{n+k}(x)\} + k\frac{d^{k-1}}{dx^{k-1}}\{L_{n+k}(x)\} \qquad ...(6)$$

Using (6) in (5) and then multiplying both sides by $(-1)^k$, we get

$$(-1)^k (n+k+1)\frac{d^k}{dx^k}L_{n+k+1}(x)$$

$$= (-1)^k (2n+2k+1)\frac{d^k}{dx^k}\{L_{n+k}(x)\} - (-1)^k x\frac{d^k}{dx^k}\{L_{n+k}(x)\}$$

$$+(-1)^{k-1} k\frac{d^{k-1}}{dx^{k-1}}\{L_{n+k-1+1}(x)\} - (-1)^k (n+k)\frac{d^k}{dx^k}\{L_{n+k-1}(x)\} \qquad ....(7)$$

But from definition $\quad L_n^k(x) = (-1)^k \dfrac{d^k}{dx^k}\{L_{n+k}(x)\} \qquad ....(8)$

Using (8) in (7), we get

$$(n+k+1)L_{n+1}^{k}(x)=(2n+2k+1)L_{n}^{k}(x)$$

$$-xL_{n}^{k}(x)+kL_{n+1}^{k-1}(x)-(n+k)L_{n-1}^{k}(x) \quad .....(9)$$

Replaceing n by $n + 1$ in 15.9.1, we get

$$L_{n}^{k}(x)+L_{n+1}^{k-1}(x)=L_{n+1}^{k}(x)$$

or $$L_{n+1}^{k-1}(x)=L_{n+1}^{k}(x)-L_{n}^{k}(x) \quad .....(10)$$

Eliminating L_{n+1}^{k-1} from (10) and (9), we get

$$(n+k+1)L_{n+1}^{k}(x)=(2n+2k+1)L_{n}^{k}(x)-x\,L_{n}^{k}(x)+k\left\{L_{n+1}^{k}(x)-L_{n}^{k}(x)\right\}-(n+k)L_{n-1}^{k}(x)$$

That is $(n+1)L_{n+1}^{k}(x)=(2n+k+1-x)L_{n}^{k}(x)-(n+k)L_{n-1}^{k}(x)$

15.9.3 $\frac{d}{dx}L_{n}^{k}(x)=-L_{n-1}^{k+1}(x)$

Proof : We know that $$L_{n}^{k}(x) = \sum_{r=0}^{n}\frac{(-1)^{r}\,\lfloor n+k\;\; x^{r}}{\lfloor n-r\;\lfloor k+r\;\lfloor r} \quad .....(11)$$

Differentiating both side of (11) w.r.t. 'x' we get

$$\text{L.H.S.}=\frac{d}{dx}L_{n}^{k}(x) = \sum_{r=0}^{n}\frac{(-1)^{r}\,\lfloor(n+k)\;\; rx^{r-1}}{\lfloor(n-r)\;\lfloor(k+r)\;\lfloor r}$$

$$=\sum_{r=1}^{n}\frac{(-1)^{r}\,\lfloor n+k\;\; x^{r-1}}{\lfloor n-r\;\lfloor k+r\;\lfloor r-1}$$

$$=\sum_{s=0}^{n-1}\frac{(-1)^{s+1}\,\lfloor n+k\;\; x^{s}}{\lfloor n-s-1\;\lfloor k+s+1\;\lfloor s} \qquad \text{(Taking } r-1=s\text{)}$$

$$=(-1)\sum_{s=0}^{n-1}\frac{(-1)^{s}\,\lfloor(n-1+k+1)\;\; x^{s}}{\lfloor n-s-1\;\lfloor k+s+1\;\lfloor s}$$

$$\frac{d}{dx}L_{n}^{k}(x) = -L_{n-1}^{k+1}(x)=\text{R.H.S}$$

15.10 Rodrigue's Formula for $L_{n}^{k}(x)$

Theorem : ***Prove that***

$$L_{n}^{k}(x)=\frac{e^{x}x^{-k}}{\lfloor n}\frac{d^{n}}{dx^{n}}\left(x^{n+k}\;x^{-x}\right)$$

Sol :

$$\text{R.H.S.} =\frac{e^{x}\;x^{-k}}{\lfloor n}D^{n}\left(e^{-x}\;x^{n+k}\right)$$

$$= \frac{e^x \, x^{-k}}{\lfloor n} \sum_{r=0}^{n} {}^n c_r D^{n-r} x^{n+k} \cdot D^r e^{-x} \qquad \text{(by Leibnitz theorem)}$$

$$= \frac{e^x \, x^{-k}}{\lfloor n} \sum_{r=0}^{n} {}^n c_r \frac{\lfloor n+k \; x^{n+k-(n-r)}}{\lfloor n+k-(n-r)} (-1)^r e^{-x}$$

$$= \sum_{r=0}^{n} \frac{e^x x^{-k}}{\lfloor n} \cdot \frac{\lfloor n}{\lfloor r \, \lfloor n-r} \times \frac{\lfloor n+k \; x^{k+r}}{\lfloor k+r} \times (-1)^r e^{-x}$$

$$= \sum_{r=0}^{n} \frac{(-1)^r \lfloor n+k \; x^r}{\lfloor n-r \; \lfloor k+r \; \lfloor r}$$

$$= L_n^k(x) = \text{L.H.S}$$

15.11 Orthogonal Property for Associated Laguerre Polynomial

Theorem : ***Prove that***

$$\int_0^\infty e^{-x} x^k L_n^k(x) \, L_m^k(x) \, dx = \frac{\lfloor n+k}{\lfloor n} \delta_{mn}$$

Proof : Associated Laguerre differential equations is

$$x \frac{d^2 y}{dx^2} + (1 - x + k) \frac{dy}{dx} + ny = 0 \qquad .....(1)$$

Multiplying by $x^k e^{-x}$ we have

$$x \, x^k e^{-x} \frac{d^2 y}{dx^2} + (1 - x + k) x^k e^{-x} \frac{dy}{dx} + ny \; x^k e^{-x} = 0$$

or

$$\frac{d}{dx}\left[x^{k+1} e^{-x} \frac{dy}{dx} \right] + n \, x^k e^{-x} y = 0 \qquad .....(2)$$

Since associated Laguerre polynomial $L_m^k(x)$ and $L_m^k(x)$ satisfy the equation, therefore

So

$$\frac{d}{dx}\left[x^{k+1} e^{-x} D L_n^k(x) \right] + n x^k e^{-x} L_n^k(x) = 0$$

and

$$\frac{d}{dx}\left[x^{k+1} e^{-x} D \, L_m^k(x) \right] + m \, x^k \, e^{-x} \, L_m^k(x) = 0 \qquad .....(4)$$

Multiplying (3) by $L_m^k(x)$ and (4) by $L_n^k(x)$ and then substracting, we have

$$L_m^k(x) \frac{d}{dx}\left[e^{-x} x^{k+1} \, D \, L_n^k(x) \right] - L_n^k(x) \frac{d}{dx}\left[e^{-x} x^{k+1} \, D \, L_m^k(x) \right]$$

$$= (m-n) x^k e^{-x} L_m^k(x) L_n^k(x)(x) \qquad .....(5)$$

Integrating both sides of (5) *w.r.t.* '*x*' from 0 to ∞, we have

$$(m-n)\int_0^\infty x^k e^{-x} L_m^k(x) L_n^k(x)\,dx = \int_0^\infty L_m^k(x)\frac{d}{dx}\left[e^{-x}x^{k+1}D\,L_n^k(x)\right]dx -$$

$$\int_0^\infty L_n^k(x)\frac{d}{dx}\left[e^{-x}x^{k+1}D\,L_m^k(x)\right]dx$$

$$=\left[L_m^k(x)e^{-x}x^{k+1}D\,L_n^k(x)\right]_0^\infty - \int_0^\infty L_m^k(x)e^{-x}x^{k+1}D\,L_n^k(x)\,dx$$

$$-\left[L_n^k(x)e^{-x}x^{k+1}D\,L_m^k(x)\right]_0^\infty + \int_0^\infty L_n^k(x)e^{-x}x^{k+1}D\,L_n^k(x)\,dx \quad .....(6)$$

$= 0$ if $m \neq n$

Hence $$\int_0^\infty x^k e^{-x} L_m^k(x) L_n^k(x)\,dx = 0, \text{ if } m \neq n.$$

If $m = n$ then we find value of

$$\int_0^\infty x^k e^{-x} L_n^k(x) L_n^k(x)\,dx = \int_0^\infty x^k e^{-x} L_n^k(x)\frac{e^x x^{-k}}{\lfloor n}\frac{d^n}{dx^n}\left(e^{-x}x^{n+k}\right)dx$$

$$=\frac{1}{\lfloor n}\int_0^\infty L_n^k(x)D^n\left(e^{-x}x^{n+k}\right)dx$$

$$=\frac{1}{\lfloor n}\left\{\left[L_n^k(x)D^{n-1}\left(x^{n+k}e^{-x}\right)\right]_0^\infty\right.$$

$$\left.-\int_0^\infty D\,L_n^k(x)D^{n-1}\left(x^{n+k}e^{-x}\right)dx\right\}$$

$$=0-\frac{1}{\lfloor n}\int_0^\infty D\,L_n^k(x)D^{n-1}\left(x^{n+k}e^{-x}\right)dx$$

$$=\frac{(-1)^n}{\lfloor n}\int_0^\infty D^n L_n^k(x)\left(x^{n+k}e^{-x}\right)dx \text{ (by symmetry for } n \text{ terms)}$$

$$=\frac{(-1)^n}{\lfloor n}\int_0^\infty (-1)^n x^{n+k+1-1}e^{-x}dx$$

$$=\frac{1}{\lfloor n}\int_0^\infty x^{n+k}\,e^{-x}dx$$

$$=\frac{\lfloor n+k}{\lfloor n} \quad .....(7)$$

Combining (6) and (7), we have

$$\int_0^\infty e^{-x}x^k L_n^k(x) L_n^k(x)\,dx = \frac{\lfloor n+k}{\lfloor n}\delta_{mn}$$

Ex.1. Prove that $\int_x^\infty e^{-t} L_n^k(t)\,dt = e^{-x}\left[L_n^k(x) - L_{n-1}^k(x)\right]$

Sol. Integrating by parts taking e^{-t} as second function, we get

$$\int_x^\infty e^{-t} L_n^k(t)\,dt = \left[-e^{-t} L_n^k(t)\right]_x^\infty + \int_x^\infty e^{-t} DL_n^k(t)\,dt$$

$$= e^{-x} L_n^k(x) + \int_x^\infty e^{-t} DL_n^k(t)\,dt$$

$$= e^{-x} L_n^k(x) + \int_x^\infty e^{-t}\left\{-\sum_{r=0}^{n-1} L_r^k(t)\right\} dt \qquad \left[\because DL_n^k(t) = -\sum_{r=0}^{n-1} L_r^k(t)\right]$$

$\therefore$
$$\int_x^\infty e^{-t} L_n^k(t)\,dt + \sum_{r=0}^{n-1}\int_x^\infty e^{-t} L_r^k(t)\,dt = e^{-x} L_n^k(x) \qquad .....(8)$$

or
$$\sum_{r=0}^{n}\int_x^\infty e^{-t} L_n^k(t)\,dt = e^{-x} L_n^k(x) \qquad .....(9)$$

Subtracting (9) from (8), we get

or
$$\sum_{r=0}^{n}\int_x^\infty e^{-t} L_r^k(t)\,dt - \int_x^\infty e^{-t} L_n^k(t)\,dt - \sum_{r=0}^{n-1}\int_x^\infty e^{-t} L_r^k(t)\,dt = 0$$

or
$$\int_x^\infty e^{-t} L_n^k(t)\,dt = \sum_{r=0}^{n}\int_x^\infty e^{-t} L_r^k(t)\,dt - \sum_{r=0}^{n-1}\int_x^\infty e^{-t} L_r^k(t)\,dt$$

or
$$\int_x^\infty e^{-t} L_n^k(t)\,dt = e^{-x} L_n^k(x) - e^{-x} L_{n-1}^k \qquad \text{[using (9)]}$$

or
$$\int_x^\infty e^{-t} L_n^k(t)\,dt = e^{-x}\left[L_n^k(x) - L_{n-1}^k(x)\right]$$

Ex.4. Establish the generating functions :

(i) $$\Gamma(1+\alpha)(xt)^{-\alpha/2} e^t J_n\left(2\sqrt{xt}\right) = \sum_{n=0}^{\infty}\frac{1}{(1+\alpha)_n} L_n^\alpha(x)\,t^n$$

(ii) $$\frac{1}{(1-t)^c}\,{}_1F_1\left(c\,;1+\alpha\,;-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty}\frac{(c)_n}{(1+\alpha)_n} L_n^\alpha(x)\,t^n$$

Sol. ***(i)*** We have

$$\sum_{n=0}^{\infty}\frac{1}{(1+\alpha)_n} L_n^\alpha(x)\,t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k x^k t^n}{\lfloor k \; \lfloor n-k \; (1+\alpha)_k}$$

Using $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+k)$, we get

$$\therefore \quad \sum_{n=0}^{\infty}\frac{1}{(1+\alpha)_n} L_n^{\alpha}(x)\, t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k\, t^{n+k}\, x^n}{\lfloor k \;\lfloor n\,(1+\alpha)_k}$$

$$= \sum_{n=0}^{\infty}\frac{t^n}{\lfloor n}\sum_{k=0}^{\infty}\frac{(-xt)^k}{\lfloor k\,(1+\alpha)_k}$$

$$= e^t\; {}_0F_1(-\,;1+\alpha\;;-xt) \qquad .....(10)$$

We know that

$$J_n(z) = \frac{(z/2)^n}{\Gamma(n+1)}\; {}_0F_1\left(-\,;1+n\;;-\frac{z^2}{4}\right) \qquad .....(11)$$

Using (11) in (9) we get the required generating function *(i)*

(ii) We have

$$\sum_{n=0}^{\infty}\frac{(c)_n}{(1+\alpha)_n} L_n^{(\alpha)}(x)\,t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(c)_n(-1)^k x^k t^n}{\lfloor k\;\lfloor n-k\,(1+\alpha)_k}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(c)_{n+k}(-1)^k x^k t^{n+k}}{\lfloor k\;\lfloor n\,(1+\alpha)_k}$$

$$= \sum_{k=0}^{\infty}\frac{(c)_k(-xt)^k}{\lfloor k\,(1+\alpha)_k}\sum_{n=0}^{\infty}\frac{(c+k)_n\,t^n}{\lfloor n}$$

$$= \sum_{k=0}^{\infty}\frac{(c)_k(-xt)^k}{\lfloor k(1+\alpha)_k}(1-t)^{-c-k}$$

$$= \frac{1}{(1-t)^c}\;{}_1F_1\left(c\;;1+\alpha\;;-\frac{xt}{1-t}\right)$$

Ex.5. Prove that $\boldsymbol{L_n^{(\alpha+\beta+1)}(x+y) = \sum_{r=0}^{n} L_r^{\alpha}(x)\,L_{n-r}^{\beta}(y)}$

Sol. We have

$$(1-t)^{-1-\alpha}\exp\left(-\frac{xt}{1-t}\right)(1-t)^{-1-\beta}\exp\left(-\frac{yt}{1-t}\right) = (1-t)^{-1-(\alpha+\beta+1)}\exp\left(-\frac{(x+y)t}{1-t}\right)$$

Therefore

$$\sum_{n=0}^{\infty} L_n^{\alpha+\beta+1}(x+y)\,t^n = \sum_{n=0}^{\infty} L_n^{\alpha}(x)\,t^n\sum_{r=0}^{\infty} L_r^{\beta}(y)\,t^n$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n} L_{n-r}^{\alpha}(x)\,L_r^{\beta}(y)\,t^n$$

Comparing the coefficients of t^n, we get the required result.

Ex.6. Prove that $L_n^\alpha(xy) = \sum_{r=0}^{n} \frac{(1+\alpha)_n (1-y)^{n-r} y^r L_r^\alpha(x)}{\lfloor n-r \, (1+\alpha)_r}$

Sol. We know that

$$e^t \, {}_0F_1(- \,;\, 1+\alpha \,;\, -xyt) = \sum_{n=0}^{\infty} \frac{L_n^\alpha(xy)\, t^n}{(1+\alpha)_n}$$

Now,

$$e^t \, {}_0F_1(- \,;\, 1+\alpha \,;\, -xyt) = e^{(1-y)t} e^{yt} \, {}_0F_1(- \,;\, 1+\alpha \,;\, -xyt)$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(1-y)^n L_r^\alpha(x)\, y^r t^{n+r}}{\lfloor n \, (1+\alpha)_r}$$

$$\Rightarrow \qquad \sum_{n=0}^{\infty} \frac{L_n^\alpha(xy)\, t^n}{(1+\alpha)_n} = \sum_{n=0}^{\infty} \frac{\{(1-y)t\}^n}{\lfloor n} \sum_{r=0}^{\infty} \frac{L_r^\alpha(x)(yt)^r}{(1+\alpha)_r}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(1-y)^{n-r}\, t^n L_r^\alpha(x)\, y^r}{\lfloor n-r \, (1+\alpha)_r}$$

Comparing the coefficients of t^n we get the required result required.

Self-Learning Exercise-II

1. Associated Laguerre differential equation is ....

2. $\int_0^\infty e^{-x} x^{-k} L_m^k(x) L_n^k(x)\, dx =$........ if $m \neq n$.

3. L_{n+k} is a Laguerre polynomial of degree ....

4. $L_{n-1}^k(x) + L_n^{k-1}(x) =$........

15.12 Summary

In this unit we studied the Laguerre and asociated Laguerre polynomials. we also studied the recurrence relation, generating funciton and orthogonal property for these polynomials.

15.13 Answer to Self-Learning Exercises

Exercise-I

1. $x y'' + (1-x) y' + n y = 0$

2. 0

3. 1

4. 1

5. $x L_n'(x)$

6. 1

7. $1 - x$

8. $\frac{1}{21}\left(2 - 4x + x^2\right)$

Exercise-II

1. $x\,y'' + (1 - x + k)\,y' + n\,y = 0$

2. 0

3. $n + k$

4. $L_n^k(x)$

15.14 Exercise

1. Find the value of

(i) $\int_0^\infty e^{-x} L_3(x) L_5(x)\,dx$ [**Ans.** 0]

(ii) $\int_0^\infty e^{-x} \left[L_4(x)\right]^2 dx$ [**Ans.** 1]

2. Express $10 - 23x + 10x^2 - x^3$ in terms of Laguerre polynomials.

[**Ans.** $L_0(x) + L_1(x) + 2L_2(x) + 6L_3(x)$]

3. Prove that $\int_x^\infty e^{-y} L_n(y)\,dy = e^x \left[L_n(x) - L_{n-1}(x)\right]$

4. Show that $\int_0^t L_n\{n(t - x)\}\,dx = \dfrac{(-1)^n H_{2n+1}(t/2)}{2^{2n}(3/2)_n}$

5. Show that $L_n^k(x) = \sum_{r=0}^{n} \dfrac{(-1)^{n-r}\,\Gamma(k + n + 1)\,x^{n-r}}{\lfloor r \;\lfloor n - r \;\Gamma(k + n - r + 1)} (n = 1, 2, 3, ...)$

6. Prove that

(i) $H_{2n}(x) = (-1)^n\, 2^{2n} \lfloor n\, L_n^{(-1/2)}(x^2)$

(ii) $H_{2n+1}(x) = (-1)^n\, 2^{2n+1} \lfloor n\; L_n^{1/2}(x^2)$

7. Show that $\int_0^t \{x(t - x)\}^{-1/2} H_{2n}\{x(t - x)\}^{1/2} dx = (-1)^n\, \pi\, 2^{2n} \left(\dfrac{1}{2}\right)_n L_n\left(\dfrac{t^2}{4}\right)$

8. Show that $L_n^\alpha(x) = \sum_{s=0}^{n} \dfrac{(\alpha - \beta)_s\, L_{n-s}^\beta(x)}{s!}$

9. Show that $\int_0^\infty e^{-x} x^{k+1} \left\{L_n^k(x)\right\}^2 dx = \dfrac{\lfloor (n + k)}{\lfloor n} (2n + k + 1)$

10. Prove that $\int_0^x (x - t)^m L_n(t)\,dt = \dfrac{\lfloor m \;\lfloor n}{\lfloor m + n + 1} x^{m+1}\, L_n^{m+1}(x)$

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