MA/MSc MT-02

Vardhaman Mahaveer Open University, Kota

Real Analysis and Topology

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Real Analysis and Topology

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COURSE INTRODUCTION

*The Present book entitled **"Real Analysis and Topology"** has been designed so as to cover the unit-wise syllabus of Mathematics-Second paper for M.A./M.Sc. (Previous) students of Vardhaman Mahaveer Open University, Kota. It can also be used for competitive examinations. The basic principles and theory have been explained in a simple, concise and lucid manner. Adequate number of illustrative examples and exercises have also been included to enable the students to grasp the subject easily. The units have been written by various experts in the field. The unit writers have consulted various standard books on the subject and they are thankful to the authors of these reference books.*

UNIT 1 : Algebra and Algebras of Sets

Structure of the Unit

1.0 Objectives

1.1 Introduction

1.2 Algebra and algebras of sets

1.2.1 Semirings

1.2.2 Algebra (Algebra of sets)

1.2.3 Ring of sets

1.2.4 σ-algebra

1.2.5 σ-ring

1.2.6 F_σ and Y_δ-sets

1.2.7 Borel sets

1.3 Axiom of choice

1.4 Summary

1.5 Answers to self-learning exercises

1.6 Exercises

1.0 Objectives

The study of real analysis affords the students an opportunity not only to impart necessary mathematical contents but also expose themselves to both rigor and abstraction. The purpose of writing this unit has been for students possible to reach this level with enough knowledge of sets and their algebras. This unit presents the basic material about sets in most non-axiomatic way and will help students pursuing the forth coming units.

1.1 Introduction

The unit begins with the definition of set and various elements of set theory, topological preliminaries and extended real number system. Next part of the unit consists of algebras of sets and related theorems are also the part of the unit. A brief introduction of axiom of choice concludes the unit.

1.2 Algebra and algebras of sets

At first we define set and some important results (without proof) related to the sets which will help students pursuing the forth coming units. It is not possible to define every term used in mathematics, yet all of the mathematics can be defined in terms of few undefined concepts. One of the basic undefined notion with which we would be dealing with is that of a **set**. The words like family, collection and aggregate are used as synonyms for the word **set**.

A collection of distinct objects such that there exists a clear rule by which we can predict the presence or absence of a given object in that collection is termed as a **set**. Thus a **set** is a well defined collection of distinct objects. The objects that belong to the set are called its **elements** or **points** or **members**. For any set A, the symbol $a \in A$ denote that a is an element of the set A and the symbol $a \notin A$ denote that a is not an element of A.

A convenient and more compact method to describe a set is the defining-property method. A defining-property of a set is one that is satisfied by every element of the set. The standard notation for a set using this property is $\{x : ....\}$, where the dotted line is filled by the defining-property. For example $\{x : x$ is a positive integer less than $10\}$. For any two sets A and B, A is said to be a subset of B if $x \in A \Rightarrow x \in B$. We also say that B is a super set of A and write it as $A \subset B$. The sets A and B are said to be equal, written as $A = B$ if $A \subset B$ and $B \subset A$. The set consisting of no elements is called the **empty set** or **null set** and is denoted by ϕ. Note that ϕ is the subset of every set. A set U is said to be the **universal set** if all the sets under consideration are the subset of U.

The collection of all the subsets of a set A is called the **power set** of A and is denoted by $P(A)$. If A and B are any two sets, then their **union** denoted by $A \cup B$ is a set which consists of all elements that are in A or in B (or in both). The **intersection** of two sets A and B denoted by $A \cap B$ is a set consisting of all elements that are in A as well as in B. The **difference** of the set A and B is denoted by $A - B$ and consists of all elements that are in A but not in B. The **Cartesian product** of any two non empty sets A and B, denoted by $A \times B$, is a set that consists of all ordered pairs (a, b) where $a \in A$ and $b \in B$. The **complement** of a set A with respect to the universal set U is denoted by A^c and is a set consisting of all those elements that are in U but not in A.

A set $A \subset R$ is said to be **bounded above** if there is a real number M such that $a \leq M$ for all $a \in A$. The number M is called an **upper bound** of A. Note that every number $M' \geq M$ is also an upper bound of A. The upper bound M of the set A is said to be the **least upper bound** or **supremum** of A if no number smaller than M is an upper bound of A. In other words M is the least upper bound of the non-empty set A, if

(i) $a \leq M$, for all $a \in A$ and

(ii) given any positive number $\in$, there is an element $a_o \in A$ such that $a_o > M - \in$.

The set $A \subset R$ is said to be **bounded below** if there is a real number m such that $a \geq m$ for all $a \in A$. The number m is called a **lower bound** of A. Every number $m' \leq m$ is also a lower bound of A. The lower bound m of the non-empty set A is said to be the **greatest lower bound** or infimum of a if no number larger that m is a lower bound of A. We can also say that m is the greatest lower bound of the non-empty set A, if

(i) $a \geq m$, for all $a \in A$ and

(ii) given $\in > 0$, there exists an element $a_o \in A$ such that $a_o < m + \in$.

We denote the least upper bound of the set A by sup (A) or $\sup_{a \in A} a$ or $\sup\{a : a \in A\}$. Similarly the greatest lower bound of A is denoted by inf (A) or $\inf_{a \in A} a$ or $\inf \{a : a \in A\}$.

Note that $\quad \inf (A) = -\sup (-A)$

It must by remembered that a set A is said to be bounded if it has a lower bound and an upper bound. In such a case it has a unique infimum (greatest lower bound) and a unique supremum (least upper bound).

Two most important relations for the complement of the union and intersection of a collection $\{A_\lambda : \lambda \in \Lambda\}$ of sets are due to De-Morgan and are known as De-Morgan's laws. These laws are stated below

$$\left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)^c = \bigcap_{\lambda \in \Lambda} A_\lambda^c$$

and

$$\left(\bigcap_{\lambda \in \Lambda} A_\lambda\right)^c = \bigcup_{\lambda \in \Lambda} A_\lambda^c.$$

The extended real number system denoted by E^*, consists of E^1 (set R of all real numbers) together with symbols ∞ and $-\infty$ satisfying the following

(i) $\infty + \infty = \infty \cdot \infty = (-\infty)(-\infty) = \infty,$

(ii) $-\infty + (-\infty) = \infty(-\infty) = -\infty < \infty$

(iii) For each $X \in E^1$

$$-\infty < x < \infty$$

and

$$x + \infty = \infty, \; x - (-\infty) = \infty, \; x + (-\infty) = -\infty$$

$$x - \infty = -\infty$$

Also

$$\frac{x}{\infty} = \frac{x}{-\infty} = 0$$

(iv) $0 \cdot \infty = 0 \cdot (-\infty) = 0$

(v) If $0 < x < \infty$, then $x \cdot \infty = \infty$ and $x \cdot (-\infty) = -\infty$

If $\quad -\infty < x < 0$, then $x(\infty) = -\infty$

and $\qquad x(-\infty) = \infty$

(vi) If $X \in E^1$ and $y \in E^*$, then

$$x + y = y + x \quad \text{and} \quad x \cdot y = y \cdot x$$

(vii) None of the following are defined

$$\infty + (-\infty), -\infty + \infty, \frac{x}{0}, \quad \text{for } X \in E^*.$$

It can be observed that if $a, b \in E^*$ and if $a < c$ for each $c > b$, then $a \leq b$. This is just the extension of the fact that if $a, b \in E^1$ and $a < b + \in$ for each $\in > 0$, then $a \leq b$.

A set N is said to be a neighborhood (*nhd*) of *a* point p, if there exists an open interval I containing p and contained in N, *i.e.*, $p \in I \subset N$. In this case the set $N - \{p\}$ is called the deleted neighborhood of p.

It may be observed that :

(i) Every open interval I is a *nhd* of each of its points.

(ii) The set N of natural numbers is not a *nhd* of any of its points.

(iii) The set R of real numbers is a hnd of each of its points.

A point p is said to be the limit point (or cluster point or accumulation point) of a set S if every *nhd* of the point p contains atleast one point of S other than p.

From the definition of limit point of a point it can be deduced that a point p is a limit point of a set S if and only if every *nhd* of p contains infinitely many points of S. Symbolically, a point p is a limit point of the set S, if for every nhd N of p, we have

$$(N \cap S) - \{p\} \neq \phi$$

However in order to show that p is not a limit point of the set S, we need to show that there exists a *nhd* N of p such that

$$N \cap S = \phi$$

The set of all the limit points of a set S is called the derived set of S and is denoted by $D(S)$. Note that

(i) Every point of the set R of real numbers is a limit point of R and hence $D(R) = R$.

(ii) The set Z of integers has no limit point.

(iii) The set N of natural numbers has no limit point.

(iv) A finite set has no limit point.

A set $G \subset R$ is said to be an open set if for each point $x \in G$, there exists a *nhd* N of x such that $N \subset G$. Equivalently, the set $G \subset R$ is open if for each $x \in G$, there exists a number $\in > 0$ such that $]x - \in, x + \in[\subset G$. It can be shown that the set G is open if it is a nhd of each of its points.

Note that *(i)* the empty set ϕ is always an open set. Also every open interval is an open set.

(ii) The union of arbitrary family of open sets is an open set.

(iii) Finite intersection of open sets is an open set.

(iv) The intersection of arbitrary family of open sets need not be an open set.

A subset $F \subset R$ is said to be a closed set if its complement F^c is an open set in R. A closed set may also be defined as :

A subset $F \subset R$ is said to be a closed set if every limit point of F is in F, *i.e.*, if $D(F) \subset F$.

The empty set ϕ is always a closed set (Note that ϕ is also an open set). Note that

(i) The union of finite number of closed sets is a closed set.

(ii) The union of arbitrary family of closed sets need not be a closed set.

(iii) The intersection of arbitrary family of closed sets is a closed set.

Let S be a subset of R and $F = \{G_\lambda : \lambda \in \Lambda\}$ be a family of subsets of R. Then F is said to be a cover (covering) of S if

$$S \subset \bigcup_{\lambda \in \Lambda} G_\lambda .$$

If the family F consists of all the open sets (*i.e.* each $G_\lambda : \lambda \in \Lambda$ is an open sets), then F is said to be an open cover (open covering) of S if

$$S \subset \bigcup_{\lambda \in \Lambda} G_\lambda$$

A finite sub collection $\{G_{n_i} \; ; i = 1, 2, ..., k\}$ of the family $F = \{G_\lambda : \lambda \in \Lambda\}$ of open subsets of R is said to be a finite sub cover of S if

$$S \subset \bigcup_{i=1}^{k} G_{n_i} .$$

Two sets A and B are said to be equipotent if there exists a one-to-one correspondence between A and B. We generally write it $A \sim B$. This relation between the sets is an equivalence relation. Equipotence is, in some sense, a measure of number of elements in a set. It is natural to try to order sets according to the number of elements they contain.

A set S is said to be **finite** if it is empty or equipotent to a set of the form $\{1, 2, ..., n\}$ for some natural number n. A set that is not **finite** is said to be an **infinite** set.

A set S is said to be a **denumerable** set (or an **enumerable** set) if it is equipotent to the set N of natural numbers. In other words the set S is denumerable if there exists a one-to-one correspondence from the set N of all natural numbers onto the set S.

A set S which is finite or denumerable is said to be a **countable** set. If S is not a **countable** set, then it is said to be an **uncountable** set.. Denumerable sets are sometime, referred to as **countably infinite** sets. If S is a **denumerable set**, then S can be written as the indexed set $\{x_i \; ; i \in N\}$, where $x_i \neq x_j$ if $i \neq j$ for all $i, j \in N$.

Expressing a denumerable set in this form is called **enumeration**. This leads us to notice that every countable set is equivalent to the set N of natural number.

Note that *(i)* A set is infinite if and only if it contains a denumerable subset.

(ii) The union of two countable sets is a countable set.

(iii) Every infinite set is equipotent (equivalent) to one of its proper sub sets.

(iv) The family of all finite subsets of a countable set is countable.

(v) The union of countable collection of countable sets is countable (*i.e.* if $A_1, A_2, ..., A_n, ...$ are countable sets, then $\bigcup_{n=1}^{\infty} A_n$ is countable).

(vi) Every subset of a countable set is countable.

(vii) The set Q of all rational numbers is countable.

(viii) The set [0, 1] is uncountable.

(ix) The set of all real numbers is uncountable (*i.e.* the set R is not countable).

(x) The set of all irrational numbers is uncountable.

Self-Learning Exercise–1

1. The set of positive integers is ....
2. The set of prime numbers is ....
3. The set $\{2, 2^2, 2^3, ..., 2^n, ...\}$ is ....
4. The set of positive irrational numbers is ....
5. If S is an uncountable set and T a countable subset of S, then $S - T$ is ..... .
6. Every uncountable set contains an infinite .... subset.
7. The set of positive transcendental numbers is ....
8. Every superset of an uncountable set is ....
9. The set of all rational numbers in [0, 1] is ....

Cantor set : The cantor set, denoted by C in a subset of the closed interval [0, 1] and is constructed as follows :

Let C_0 denoted the interval [0, 1] then trisect [0, 1] and remove the middle third open interval $\left]\frac{1}{3}, \frac{2}{3}\right[$ from C_0. The set so left behind is $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Notice that C_1 is the union of $2^1 = 2$ disjoint closed intervals. Trisect each closed interval in C_1 and remove from each of them, the middle third open interval, *i.e.*, remove the open intervals $\left]\frac{1}{9}, \frac{2}{9}\right[$ and $\left]\frac{7}{9}, \frac{8}{9}\right[$ from the closed intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ respectively. Let the remaining set after the removal of these open intervals be

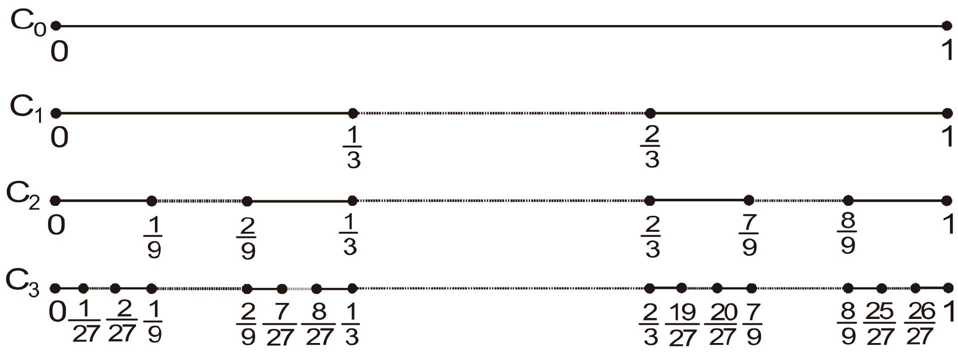
$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

Notice that C_2 is the union of $2^2 = 4$ disjoint closed intervals. Trisect each closed interval in C_2 and remove from each of them the middle third open intervals

$$\left]\frac{1}{27}, \frac{2}{27}\right[, \left]\frac{7}{27}, \frac{8}{27}\right[, \left]\frac{19}{27}, \frac{20}{27}\right[\text{ and } \left]\frac{25}{27}, \frac{26}{27}\right[\text{ respectively. Let}$$

$$C_3 = \left[0, \frac{1}{27}\right] \cup \left[\frac{2}{27}, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{19}{27}\right] \cup \left[\frac{20}{27}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, \frac{25}{27}\right] \cup \left[\frac{26}{27}, 1\right]$$

be the set left behind. We see that C_3 is the union of $2^3 = 8$ disjoint closed intervals. The inductive process of constructing C_{n+1} form C_n is clear now. Trisect each of the 2^n disjoint closed intervals of C_n, and remove from each one of them the middle third open interval. This gets us C_{n+1} from C_n, which is the union of 2^{n+1} disjoint closed intervals. Following figure shows the constructions of the sets C_0, C_1, C_2 and C_3 :



Clearly $C_{n+1} \subset C_n$ for all $n \in N$. The Cantor set of [0, 1] is thus defined by $C = \bigcap_{n=1}^{\infty} C_n$. We see that :

1. the points which are never removed from the interval [0, 1] constitute the Cantor set C.
2. the set C is a closed subset of R.
3. the total length of the removed intervals from [0, 1] to obtain C is equal to 1.

[Since at the n^{th} step, we remove 2^{n-1} open intervals each of length 3^{-n}, therefore a total length $2^{n-1} \cdot 3^{-n}$. Thus we remove altogether total length equal to $\sum_{n=1}^{\infty} 2^{n-1} \cdot 3^{-n} = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = 1$].

As such the set remaining in [0, 1] which infact is the cantor set may seem to be insignificant. Intuitively, it appears that the only points left in the Cantor set are the end points, *i.e.*,

$0, 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}, \ldots$, which are countable in numbers. But it happens to be a wrong impression. Actually the Cantor set is not countable, *i.e.*, the Cantor set is uncountable.

Now we start this section by introducing the concept of a semiring of sets and its properties. A semiring of sets is perhaps the simplest family of sets for which theory of measure can be built.

1.2.1 Semirings :

Let X be a nonempty set. A collection $\mathcal{S}$ of subsets of X is said to be a semiring if it satisfies the following properties :

(i) $\phi \in \mathcal{S}$

(ii) If $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$, *i.e.*, $\mathcal{S}$ is closed under finite intersections.

(iii) For every subsets $A, B \in \mathcal{S}$, there exist subsets $C_1, C_2, \ldots, C_n$ in $\mathcal{S}$ (depending on A and B) such that $A - B = \bigcup_{i=1}^{n} C_i$ and $C_i \cap C_j = \phi$ if $i \neq j$.

Now let $\mathcal{S}$ be a semiring of subsets of X. Then a subset A of X is said to be a σ-set with respect to $\mathcal{S}$ (or simply a σ-set) if there exists a sequence $\{A_n\}$ of pairwise disjoint members of $\mathcal{S}$ such that $A = \bigcup_{i=1}^{\infty} A_n$. If $A = \bigcup_{i=1}^{n} A_i$, with $A_1, A_2, \ldots, A_n \in \mathcal{S}$ and $A_i \cap A_j = \phi$ for $i \neq j$, then A is a σ–set (To see it we can put $A_i = \phi$ for all $i > n$). Thus it follows from the above definition of σ–set, that if $A, B \in \mathcal{S}$, then $A - B$ is a σ-set.

Theorem 1 : *If $\mathcal{S}$ is a semiring, then the following statements hold :*

(i) *If $A \in \mathcal{S}$ and $A_1, A_2, \ldots, A_n \in \mathcal{S}$, then $A - \bigcup_{i=1}^{n} A_i$ is a finite union of pairwise disjoint sets of $\mathcal{S}$ (and hence is a σ-set).*

(ii) *For every $\{A_n\}$ of pairwise disjoint members of $\mathcal{S}$, the set $A = \bigcup_{n=1}^{\infty} A_n$ is a σ-set.*

(iii) *Countable union and finite intersection of σ-set are σ-sets.*

Proof : ***(i)*** We prove the statement by mathematical induction. For $n = 1$, the statement holds true from the definition of semiring. So let the statement be true for some k. Let $A \in \mathcal{S}$ and $A_1, A_2, \ldots, A_k, A_{k+1} \in \mathcal{S}$. Then there exist sets $B_1, B_2, \ldots, B_p \in \mathcal{S}$ such that $B = A - \bigcup_{i=1}^{k} A_i = \bigcup_{i=1}^{p} B_i$ and $B_i \cap B_j = \phi$ if $i \neq j$. Consequently $A - \bigcup_{i=1}^{k+1} A_i = B - A_{k+1} = \bigcup_{i=1}^{p} (B_i - A_{k+1})$. Thus each $B - A_{k+1}$ can be written as a finite union of disjoint sets of $\mathcal{S}$. Now since $B_i \cap B_j = \phi$ for $i \neq j$, it follows that $A - \bigcup_{i=1}^{k+1} A_i$ can be written as a finite union of pair wise disjoint sets of $\mathcal{S}$. Thus the statement holds true for $k + 1$ sets also. Hence the statement is true for all n.

(ii) Let $\{A_n\}$ be a sequence of pairwise disjoint sets of $\mathcal{S}$ and $A = \bigcup_{n=1}^{\infty} A_n$.

Let $B_1 = A_1$ and $B_{n+1} = A_{n+1} - \bigcup_{i=1}^{n} A_i$; $n \geq 1$.

Then $A = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$.

It can be observed that $B_i \cap B_j = \phi$ if $i \neq j$. Thus by statement each B_i is a σ-set and so A is a σ-set.

(iii) The statement follows using the property *(ii)* of the definition of semiring and the above statement *(ii)*.

There are some natural collection of sets that satisfy properties which are more stronger than those of a semiring. One such a collection is algebra of sets.

1.2.2 Algebra (Algebra of sets) :

A nonempty collection $\mathcal{S}$ of subsets of a set X is said to be an algebra of sets (or simply an algebra) if $\mathcal{S}$ is closed under finite intersections and complementation. Thus the nonempty collection $\mathcal{S}$ is an algebra of sets if

(i) $A \in \mathcal{S}, B \in \mathcal{S} \Rightarrow A \cap B \in \mathcal{S}$

(ii) $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$.

We shall now show three basic properties in the following theorem.

Theorem 2 : *For an algebra of sets $\mathcal{S}$, the following statement hold true.*

(i) $\phi \in \mathcal{S}$ *and* $X \in \mathcal{S}$

(ii) $\mathcal{S}$ *is closed under finite unions and intersections.*

(iii) $\mathcal{S}$ *is a semiring.*

Proof : ***(i)*** Since $\mathcal{S}$ is nonempty, therefore, there exists some set $A \in \mathcal{S}$. Thus by the definition of algebra of sets, $A^c \in \mathcal{S}$ and so $A \cap A^c \in \mathcal{S}$, *i.e.*, $\phi \in \mathcal{S}$. Again $\phi \in \mathcal{S} \Rightarrow \phi^c = X \in \mathcal{S}$.

(ii) Let A, $B \in \mathcal{S}$. Then by definition A^c, $B^c \in \mathcal{S}$ and so $A^c \cap B^c \in \mathcal{S}$. Consequently $(A^c \cap B^c)^c \in \mathcal{S}$, *i.e.*, $A \cup B \in \mathcal{S}$. Thus if $A, B \in \mathcal{S}$, then $A \cup B$ and $A \cap B \in \mathcal{S}$.

By taking unions of two sets at a time, we see that if $A_1, A_2, ..., A_n \in \mathcal{S}$, then $A_1 \cup A_2 \cup .... \cup A_n$ *i.e.* $\bigcup_{i=1}^{n} A_i \in \mathcal{S}$. Similarly $A_1 \cap A_2 \cap .... \cap A_n$ *i.e.* $\bigcap_{i=1}^{n} A_i \in \mathcal{S}$.

(iii) In order to show that $\mathcal{S}$ is a semiring we are only to show that for every $A, B \in \mathcal{S}$, there exist sets $C_1, C_2, ..., C_n \in \mathcal{S}$ (depending on A and B) such that $A - B = \bigcup_{i=1}^{n} C_i$ and $C_i \cap C_j = \phi$ for all $i \neq j$. This is obvious in view of the identity $A - B = A \cap B^c$.

Ex.1. *For every nonempty set X the collection* $\mathcal{S} = \{\phi, X\}$ *of subsets of X is an algebra of sets. This is the smallest algebra (with respect to the inclusion).*

Ex.2. *For every nonempty set X, the power set P(X) of X forms an algebra of sets. This is perhaps the largest possible algebra.*

Ex.3. *Let* $a, b \in R$ *and let*

$$[a,b[= \begin{cases} \phi , & if \;\; a \geq b \\ \{x \in R \; ; \; a \leq x < b\} , & if \;\; a < b \end{cases}$$

Then the collection $\mathcal{S} = \{[a, b[\, : a, b \in R\}$ is a semiring of subsets of R, but it is not an algebra of sets, since for instance $[0, 1[\, \in \mathcal{S}$ and $[2, 3[\, \in \mathcal{S}$ but $[0, 1[\cup [2, 3[\, \in \mathcal{S}$.

We now give below the definition of ring of sets, that is something an intermediate notion between the semirings and algebras of sets.

1.2.3 Ring of sets :

A nonempty collection $\mathcal{R}$ of subsets of a set X is said to be a **ring of sets** (or simply a **ring**) if it satisfies the following properties :

(i) $A \in \mathcal{R}, B \in \mathcal{R} \Rightarrow A \cup B \in \mathcal{R}$

(ii) $A \in \mathcal{R}, B \in \mathcal{R} \Rightarrow A - B \in \mathcal{R}$.

The property *(i)* above, can immediately be extended by mathematical induction to any finite number of sets in $\mathcal{R}$, *i.e.* if $A_1, A_2, ..., A_n \in \mathcal{R}$ then $A = \bigcup_{i=1}^{n} A_i \in \mathcal{R}$. As an example, if X is a nonempty set, then the power set $P(X)$ of X forms a ring of sets.

An immediate consequence from the definition of ring of sets is :

(i) $\phi \in \mathcal{R}$, for if $A \in R$, then $A - A$ *i.e.* $\phi \in \mathcal{R}$.

(ii) The ring of sets $\mathcal{R}$ is closed under the finite intersections of sets, for if $A, B \in \mathcal{R}$, then $A - (A - B)$ *i.e.* $A \cap B \in \mathcal{R}$, which can be extended by induction method to any finite number of sets in $\mathcal{R}$.

Theorem 3 : *A nonempty collection* $\mathcal{R}$ *of subsets of a set X is an algebra of sets if and only if* $\mathcal{R}$ *is a ring of sets and* $X \in \mathcal{R}$.

Proof : First we suppose that $\mathcal{R}$ is an algebra of sets. Then for any $A \in \mathcal{R}$, we have A^c *i.e.* $X - A \in \mathcal{R}$ and so by definition $A \cup A^c \in \mathcal{R}$, *i.e.,* $X \in \mathcal{R}$.

Thus in order to show that $\mathcal{R}$ is a ring of sets, we are only to show that if $A, B \in \mathcal{R}$, then $A - B \in \mathcal{R}$.

Now let $A, B \in \mathcal{R}$, then since

$$A - B = A \cap (X - B) = X - ((X - A) \cup B)$$

and $\quad A, B \in \mathcal{R} \;\Rightarrow\; X - A \in \mathcal{R}, B \in \mathcal{R}$

$\Rightarrow (X - A) \cup B \in \mathcal{R}$

$\Rightarrow X - ((X - A) \cup B) \in \mathcal{R}$

Thus $A, B \in \mathcal{R} \Rightarrow A - B \in \mathcal{R}$.

Hence $\mathcal{R}$ is a ring of sets.

Conversely, let $\mathcal{R}$ be the ring of sets and $X \in \mathcal{R}$. Then by definition of ring of sets,

$$X \in \mathcal{R},\ A \in \mathcal{R} \Rightarrow X - A \quad i.e.\ A^c \in \mathcal{R}$$

Thus $X \in \mathcal{R} \Rightarrow A^c \in \mathcal{R}$

Hence $\mathcal{R}$ is an algebra of sets.

Theorem 4 : *For any collection $\mathcal{C}$ of subsets of a set X there exists a smallest algebra $\mathcal{S}$ of sets which contains $\mathcal{C}$ (i.e. there exists an algebra $\mathcal{S}$ containing $\mathcal{C}$ which is such that if F is any algebra containing $\mathcal{C}$, then F contains $\mathcal{S}$.*

Proof : Let **F** be the family of all algebras of subsets of the set X that contain $\mathcal{C}$ and let

$$\mathcal{S} = \cap \{F, F \in \mathbf{F}\}.$$

Since each F in **F** contains $\mathcal{C}$, therefore $\mathcal{C}$ is a sub collection of $\mathcal{S}$.

Also $\mathcal{S}$ is an algebra, for if $A, B \in \mathcal{S}$, then $A, B \in F$ for each $F \in \mathbf{F}$ and so $A \cup B \in F$ for each $F \in \mathbf{F}$

$\Rightarrow \quad A \cup B \in \cap \{F ; F \in \mathbf{F}\}$

$\Rightarrow \quad A \cup B \in \mathcal{S}.$

Similarly it can be shown that if $A \in \mathcal{S}$, then $A^c \in \mathcal{S}$.

Hence $\mathcal{S}$ is an algebra of sets.

Now from the definition of $\mathcal{S}$, it is clear that F is an algebra containing $\mathcal{C}$, then $\mathcal{S} \subset F$.

Thus $\mathcal{S}$ is the smallest algebra of sets containing $\mathcal{C}$.

Theorem 5 : *Let \$ be an algebra of sets of a set X and $\{A_n\}$ be a sequence of sets in \$. Then there exists a sequence $\{B_n\}$ of sets in \$ such that $B_i \cap B_j = \phi$ if $i \neq j$ and $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$.*

Proof : If the sequence $\{A_n\}$ is a sequence containing finite number of sets of $\mathcal{S}$, then the theorem is trivially true. So let $\{A_n\}$ be an infinite sequence of sets of $\mathcal{S}$.

Let $B_1 = A_1$ and $B_n = A_n - (A_1 \cup A_2 \cup ... \cup A_{n-1})$ for each $n > 1$.

Then $B_n = A_n \cap A_1^c \cap A_2^c \cap .... \cap A_{n-1}^c$.

Since intersections and the complements of the sets in the algebra $\mathcal{S}$ are in $\mathcal{S}$, therefore, each $B_n \in \mathcal{S}$; $n \geq 1$. Also $B_n \subset A_n$ for all $n > 1$.

Now let B_m and $B_n \in \mathcal{S}$ where $m < n$. Then, $B_m \cap B_n \subset A_m \cap B_n$ (since $B_m \subset A_m$)

$$= A_m \cap \left(A_n \cap A_1^c \cap A_2^c \cap .... \cap A_m^c \cap .... \cap A_{n-1}^c\right)$$

$$= \left(A_m \cap A_m^c\right) \cap \left(A_n \cap A_1^c \cap \ldots. \cap A_{n-1}^c\right)$$

$$= \phi \cap \left(A_n \cap A_1^c \cap \ldots \cap A_{n-1}^c\right) = \phi$$

Thus $B_m \cap B_n = \phi$ for all $m < n$

Now since $B_i \subset A_i$ for all $i > 1$,

therefore, $$\bigcup_{i=1}^{\infty} B_i \subset \bigcup_{i=1}^{\infty} A_i \quad \ldots..(1)$$

Now let $x \in \bigcup_{i=1}^{\infty} A_i$.

$\Rightarrow$ $x \in A_i$ for some $i \in N$.

Let n be the smallest value of i for which $x \in A_i$.

Then $x \in B_n$ (from the definition of B_n)

$\Rightarrow$ $$x \in \bigcup_{i=1}^{\infty} B_i$$

Thus $$\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{i=1}^{\infty} B_i \quad \ldots..(2)$$

from (1) and (2) we have

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i.$$

1.2.4 σ-Algebra :

An algebra $\mathcal{S}$ of subsets of some set X is said to be a **σ-algebra** if every union of a countable collection of members of $\mathcal{S}$ is in $\mathcal{S}$. That is in addition to $\mathcal{S}$ being an algebra, if for every sequence $\{A_n\}$ of members of $\mathcal{S}$, the countable union $\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$.

Now since $\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c$, it easily follows that a σ-algebra of sets is also closed under countable intersections. Thus a σ-algebra can be defined as :

An algebra of sets $\mathcal{S}$ is called a σ-algebra (or a σ-Boolean algebra or a Borel field) if for every sequence $\{A_n\}$ of members of $\mathcal{S}$, the union $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n \in \mathcal{S}$.

Every collection $\mathbf{F}$ of subsets of a nonempty set X is contained in a smallest σ-algebra. This smallest σ-algebra which is the intersection of all σ-algebras that contain $\mathbf{F}$, is called the **σ-algebra generated** by $\mathbf{F}$ and is denoted by σ ($\mathbf{F}$). Thus

$\sigma(\mathcal{F}) = \cap\{\mathcal{S} : \mathcal{S}$ is a σ-algebra and $\mathcal{F} \in \mathcal{S}\}$.

If $\mathcal{S}$ is a σ-algebra and a family of sets $\mathcal{F}$ of $\mathcal{S}$, satisfies $\sigma(\mathcal{F}) = \mathcal{S}$, then $\mathcal{F}$ is called a **family of generators**.

Theorem 6 : *If $f : X \to Y$ is a function and $\mathcal{F}$ is a nonempty family of subsets of y, then*

$$\sigma(f^{-1}(\mathcal{F})) = f^{-1}(\sigma(\mathcal{F})).$$

Proof : Since $f^{-1}(A^c) = (f^{-1}(A))^c$,

$$f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(A_i) \text{ and}$$

$$f^{-1}\left(\bigcap_{i=1}^{\infty} A_i\right) = \bigcap_{i=1}^{\infty} f^{-1}(A_i),$$

The above shows that

$$f^{-1}(\mathcal{S}) = \left\{f^{-1}(A) : A \in \mathcal{S}\right\}$$

is a σ-algebra whenever $\mathcal{S}$ is a σ-algebra thus $f^{-1}[\sigma(\mathcal{F})]$ is a σ-algebra, where

$$f^{-1}(\mathcal{F}) f^{-1}[\sigma(\mathcal{F})].$$

This implies that

$$\sigma[f^{-1}(\mathcal{F})] \subseteq f^{-1}[\sigma(\mathcal{F})] \qquad .....(1)$$

Now let $\mathcal{C} = \{A \in \sigma(\mathcal{F}) : f^{-1}(A) \in \sigma(f^{-1}(\mathcal{F}))\}$.

Then clearly $\mathcal{C}$ is a σ-algebra, where $\mathcal{F} \subseteq \mathcal{C} \subseteq \sigma(\mathcal{F})$.

Hence $\mathcal{C} = \sigma(\mathcal{F})$. This implies that

$$f^{-1}(\sigma(\mathcal{F})) \subseteq \sigma(f^{-1}(\mathcal{F})) \qquad .....(2)$$

from (1) and (2) we get

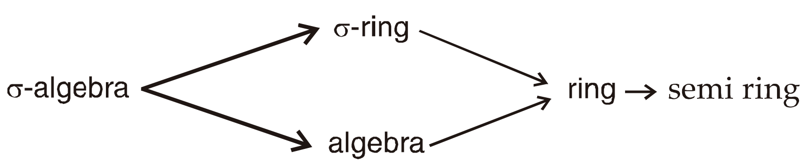
$$\sigma(f^{-1}(\mathcal{F})) \subseteq f^{-1}(\sigma(\mathcal{F})).$$

1.2.5 σ-ring :

A ring $\mathcal{R}$ is said to be a **σ-ring** if $\mathcal{R}$ is closed under countable unions and intersections of the collections of subsets in $\mathcal{R}$. *i.e.* if $\{A_n\} \subset \mathcal{R}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$ and $\bigcap_{n=1}^{\infty} A_n \in \mathcal{R}$.

It can be observed that a ring (*a* σ-ring) $\mathcal{R}$ is an algebra (*a* σ-algebra) if and only if $X \in \mathcal{R}$.

Following is a simple chart that gives the relationships between various families of sets :



1.2.6 $\mathbf{F}_\sigma$ and G_δ -sets :

We know that the intersection of any collection (countable or uncountable) of sets in R is closed and the union of a finite collection of closed sets is closed, but the union of a countable collection of closed sets need not be closed. Similarly the intersection of a countable collection of open sets need not be open. This intends us define two new classes of sets $\mathbf{F}_\sigma$-sets and G_σ-sets.

A set, which is a countable union of closed sets is said to be a **$\mathbf{F}_\delta$-set**.

Thus a closed set, an open interval] a, b [(since] a, b [= $\bigcup_{n=1}^{\infty}\left[a-\frac{1}{n}, b+\frac{1}{n}\right]$) and a countable union of $\mathbf{F}_\sigma$-sets, are all the $\mathbf{F}_\sigma$-sets.

A set which is a countable intersection of open sets is called a **G_δ-set**.

Thus an open set, a closed interval [a, b] $\left(\text{Since } [a,b]=\bigcap_{n=1}^{\infty}\left]a-\frac{1}{n}, b+\frac{1}{n}\right[\right)$ and a countable intersection of G_δ-sets, are all the G_δ-sets.

We can notice that the complement of an $\mathbf{F}_\sigma$-set is a G_δ-set and conversely also.

1.2.7 Borel sets :

The σ-algebra generated by the family of all open sets in R, denoted by B, is called the class of **Borel sets** in R. The sets in B are called the Borel sets or R. Thus it can be said that

The collection B of **Borel sets** is the smallest σ-algebra which contains all of the open sets. Such a smallest σ-algebra does exist. It is also the smallest σ-algebra that contains all closed sets and the smallest σ-algebra that contains all open intervals.

The open sets and closed sets are the simple examples of Borel sets. Similarly $\mathbf{F}_\sigma$-sets and G_δ-sets are Borel sets.

1.3 Axiom of choice

Although we shall not present here an axiomatic treatment of set theory, one obvious axiom "axiom of choice" deserves mention. It is clear that when we have a nonempty set, we may always choose a point from it. This holds for any finite number of non-empty sets. We may choose a point from each set. Strangely enough when there are infinite number of non-empty sets, the assumption that one can always choose a point from each set leads to some unintuitive conclusions. The principle can be stated more precisely as follows :

Let $\{A_\lambda ; \lambda \in \Lambda\}$ be a collection of non-empty sets. Then for each $\lambda \in \Lambda$, we may choose a point $a_\lambda \in A_\lambda$. Equivalently we can say that there exists a function $f : \Lambda \to \bigcup_{\lambda\in\Lambda} A_\lambda$ such that

$$f(\lambda) = X_\lambda \in A_\lambda.$$

The function f is called the choice function. This can be stated in set-theoretic terms as follows :

The cartesian product of a non-empty family of non-empty sets is non-empty.

There are few equivalent ways of stating the principle of **axiom of choice** of which the following is historically important. "If $\{A_\lambda : \lambda \in \Lambda\}$ is a non-empty family of pair wise disjoint nonempty sets, then there exists a set $E \subset \bigcup_{\lambda \in \Lambda} A_\lambda$ such that $E \cap A_\lambda$ consists of precisely one point for each $\lambda \in \Lambda$.

Self-learning exercise-2

1. A ring $\mathcal{R}$ is an algebra of subsets of X if and only if ....
2. A σ-ring $\mathcal{R}$ is a σ-algebra if and only if ....
3. Every algebra of sets is a ....
4. Every ring $\mathcal{R}$ is a ....
5. .... is closed under symmetric differences and finite intersections.
6. If $\mathcal{R}$ is nonempty collection of subsets of a set X, then $\mathcal{R}$ is a .... if $A, B \in \mathcal{R} \Rightarrow A \cup B \in \mathcal{R}$ and $A - B \in \mathcal{R}$.
7. A ring $\mathcal{R}$ is a σ-ring if it is closed under ....
8. Every open set and closed set is a ....
9. If $\mathcal{R}$ is a nonempty collection of subsets of set X, then $\mathcal{R}$ is .... if $A, B \in \mathcal{R}$ $\Rightarrow A \cap B \in \mathcal{R}$ and $A \in \mathcal{R} \Rightarrow A^c \in \mathcal{R}$.

1.4 Summary

The entire unit has been written assuming that the student already has acquaintance with the concepts of sets, their nature and various set-theoretic operations. It were also presupposed that the student has enough knowledge of theory of real numbers. In the beginning with a brief introduction of extended real number system we delt with the problems concerning the size of a set which includes the definitions and theorems on countable and uncountable sets. We then had introduction to semirings, rings, algebra of sets, σ-algebra and related theorems. Finally the unit ended with the axiom called the axiom of choice that has a very important place in the development of set theory.

1.5 Answers to self-learning exercises

Self-learning exercise-1

1. countable	**2.** countable	**3.** countable
4. uncountable	**5.** uncountable	**6.** countable
7. uncountable	**8.** uncountable	**9.** countable.

Self-learning exercise-2

1. $X \in \mathcal{R}$	**2.** $X \in \mathcal{R}$	**3.** ring
4. semiring	**5.** every ring	**6.** ring
7. countable unions	**8.** Borel set	**9.** an algebra.

1.6 Exercises

1. Let $\mathbb{S}$ be a semiring of subsets of the set X, and $Y \subset X$. Show that $\mathbb{S}_Y = \{Y \cap A \; ; \; A \in \mathbb{S}\}$ is a semiring of Y.

2. Prove that every open interval is a Borel set.

3. Prove that a ring $\mathcal{R}$ of subsets of a set X is an algebra if and only if $X \in \mathcal{R}$.

4. Prove that a σ-ring $\mathcal{R}$ of subsets of a set X is a σ-algebra if and only if $X \in \mathcal{R}$.

5. If $\mathcal{R}$ is a ring then show that the collection $\mathrm{F} = \left\{A \in \mathcal{R} \; : \; A \text{ or } A^c \in \mathcal{R}\right\}$ is an algebra.

□ □ □

Unit 2 : Lebesgue Measures and Measurable Sets

Structure of the Unit

2.0 Objectives

2.1 Introduction

2.2 Length of an interval

2.3 Length of sets

2.4 Measure of a set

  2.4.1 Outer measure of set

2.5 Measurable sets

  2.5.1 Definition of measurable set

  2.5.2 Lebesgue measure of a set

2.6 Non measurable sets

2.7 Summary

2.8 Answers to self-learning exercises

2.9 Exercises

2.0 Objectives

The objective of this unit is to generalize the idea of length by defining the **measure** on a class of sets on real line. In real analysis the **length** is an example of a **set function** *i.e.* a function that associates an extended real number to each set in some class of sets. The concept of the measure of set and measurable sets plays an important role in measure theory. We shall discuss various properties of measurable sets that will help us, define measurable function and theory of Leabesgue integration.

2.1 Introduction

In the present unit the definitions of length of an interval and set are given. The outer measure of an arbitrary set, having certain properties is also defined, which is closely related to the concept of length of sets and intervals. Next part of the unit consists of the definition of Lebesgue measure of a set. Certain properties of measurable sets are discussed in detail. Definition of σ-algebra and further properties of measurable sets are also described. Finally the existence of non-measurable sets is discussed through a theorem.

2.2 Length of an interval

Irrespective of whether an interval I is open or closed, its length, denoted by $l(I)$ is defined to be the difference of the end points of I. Thus if a and b are the end points of I, then its length $l(I)$ is $b - a$. In case when $a = b$, the interval I converts into a point and then its length is zero, where as the infinite interval has its length ∞. Therefore the "length" can be considered as a "set function" *i.e.* a function that associates an extended real number to each set in some collection of sets. The domain of this set function is the collection of all intervals. The set function satisfies the following properties :

(i) $l(I) \geq 0$

(ii) If $\{I_n\}$ is the countable collection of mutually disjoint intervals such that $\bigcup_n I_n$ is an interval, then $l\left(\bigcup_n I_n\right) = \sum_n l(I_n)$

(iii) For any fixed number x, $l(I + x) = l(I)$.

The above notion of length can be extended to arbitrary sets. The length of a set can now be defined as follows :

2.3 Length of sets

Let O be an open set in R. Then O can always be expressed as a countable union of mutually disjoint open intervals $\{I_n\}$, *i.e.*, $O = \bigcup_n I_n$. We define the length $l(O)$ of the set O as

$$l(O) = \sum_n l(I_n).$$

It obviously follows that if O_1 and O_2 are any two open sets in R, such that $O_1 \subset O_2$, then $l(O_1) \leq l(O_2)$ and so for any open set $O \subset [a, b]$, we have

$$0 \leq l(O) \leq b - a.$$

Also for any closed set $F \subset]\, a, b\, [$,

we have $\qquad l(F) = b - a - l(F^c)$,

where $\qquad F^c =]\, a, b\, [- F$

(*i.e.* the complement of F with respect to the open interval $]\, a, b\, [$). Here we can see that $l(O_1) \leq l(F) \geq 0$.

The generalization of concept of length to a wider class of sets in R, (the class of all sets in R) were introduced by "Henri Lebesgue" in 1904, which is both intuitive and has many applications, extensions and abstractions. The following definition is the first step in a generalization, equivalent to Lebesgue's, of length.

2.4 Measure of a set *E*

Let m be a function which associates to each set E in R, a nonnegative extended real number $m(E)$, called the measure of E and satisfies the following properties :

(i) $m(E)$ is defined for all sets E in $P(R)$,

(ii) If I is an interval in $P(R)$, then $m(I) = l(I)$,

(iii) For any sequence $< E_i >$ of disjoint sets in $P(R)$, $m\left(\bigcup_i E_i\right) = \sum_i m(E_i)$ (countable additivity property),

(iv) For any fixed number x, $m(E + x) = m(E)$ (Translation invariance property)

Construction of a set function which could satisfy all the above four properties was virtually not feasible. As a result one of the above four properties, were necessarily to be weakened. Henri Lebesgue weakened the first property and retained the last three of the four properties, saying that measure of the set need not be defined for all sets E in R.

The measure of the set could also be defined by replacing the property *(iii)* of countable additivity by a weaker property, the finite additivity for each finite sequence $< E_i >$ of disjoint sets. One more possible alternative of property *(iii)* is countable sub addititivity that is satisfied by the "outer measure". Thus it is required to introduce first the "outer measure" which is defined for all sets in R.

2.4.1 Outer measure of a set : Let A be a subset of real-numbers and $\{I_n\}$ be a countable collection of open intervals which covers A $\left(\text{i.e. } A \subset \bigcup_n I_n\right)$. For each such countable collection, let us consider the sum of the lengths of the intervals in the collection. We define the outer measure m* (A) of the set A to be the infimum of all such sums and write it as

$$m^*(A) = \begin{cases} 0 & \text{if } A = \phi \\ \text{Inf}\left\{ \sum_n l(I_n), \quad \text{where } A \subset \bigcup_n I_n \right\}, & \text{if } A \neq \phi. \end{cases}$$

Hence for each set A in R, $m^*(A) \geq 0$ is a unique number and for $A \neq \phi$.

(i) $m^*(A) \leq \sum_n l(I_n)$, where $\{I_n\}$ is any countable family of open intervals such that

$$A \subset \bigcup_n I_n, \text{ and}$$

(ii) for each number $\in > 0$, there is a countable family $\{I_n\}$ of open intervals such that

$$A \subset \bigcup_n I_n, \quad \text{and} \quad \sum_n l(I_n) < m^*(A) + \in.$$

Theorem 1. ***(i)*** *If A and B are two sets such that $A \subset B$, then $m^*(A) \le m^*(B)$.*

(ii) *for every singleton set A, $m^*(A) = 0$.*

(iii) *The outer measure is translation invariant i.e. for every set A and for each $x \in R$,*

$$m^*(A + x) = m^*(A).$$

Proof : ***(i)*** Given that $A \subset B$. Let $\{I_n\}$ be a countable family of open intervals such that $B \subset \bigcup_n I_n$.

Then $$A \subset B \subset \bigcup_n I_n$$

or $$A \subset \bigcup_n I_n$$

Hence $$m^*(A) \le \sum_n l(I_n)$$

Now since the above inequality is true for any countable family of open intervals that covers B, therefore $m^*(A) \le m^*(B)$.

(ii) Let $A = \{a\}$ be an arbitrary singleton set.

Let $$I_n = \left] a - \frac{1}{n}, a + \frac{1}{n} \right[$$

then clearly $a \in I_n \quad , \quad n \in N$

therefore $A = \{a\} \subset I_n \quad , \quad n \in N$

Also $l(I_n) = \dfrac{2}{n}$ for each $n \in N$

therefore,
$$\begin{aligned} m^*(A) = m^*(\{a\}) &= \inf \{l(I_n) ; n \in N\} \\ &= \inf \left\{ \frac{2}{n} ; n \in N \right\} \\ &= 0. \end{aligned}$$

(iii) Since for each $\in > 0$, there exists a countable collection $\{I_n\}$ of open intervals such that $A \subset \bigcup_n I_n$ and

$$\sum_n l(I_n) < m^*(A) + \in \qquad .....(1)$$

Also for any $x \in R$

$$A \subset \bigcup_n I_n \Rightarrow A + x \subset \bigcup_n (I_n + x)$$

therefore,
$$\begin{aligned} m^*(A + x) &\le \sum_n l(I_n + x) \\ &= \sum_n l(I_n) \\ &< m^*(A) + \in \qquad \text{(from (1))} \end{aligned}$$

Now since $\in > 0$ is an arbitrary number, so from above we have

$$m^*(A + x) < m^*(A) \qquad .....(2)$$

But $A = (A + x) - x$, therefore using the above result, we have

$$m^*((A + x) - x) \le m^*(A + x)$$

or $$m^*(A) \le m^*(A + x) \qquad .....(3)$$

From (2) and (3), we get

$$m^*(A + x) = m^*(A), \qquad \text{for any real number } x.$$

The following theorem answers affirmatively the question, whether the outer measure is a generalizatiohn of the length function defined for the intervals ?

Theorem 2. *The outer measure of an interval is its length.*

Proof : We first prove the result, when I is a closed finite interval say $[a, b]$. We know that for each $\in > 0$

$$I = [a, b] \subset \left]a - \frac{\in}{2}, a + \frac{\in}{2}\right[$$

therefore, $m^*(I) \le l\left(\left]a - \frac{\in}{2}, a + \frac{\in}{2}\right[\right) = b - a + \in.$

The above is true for each $\in > 0$, therefore, we have

$$m^*(I) \le b - a = l(I) \qquad .....(1)$$

Thus it remains to prove that

$m^*(I) \ge b - a$.

Let $\in > 0$ be any given positive numbers. Then there exists a countable collection $\{I_n\}$ of open intervals that covers $[a, b]$ (*i.e.* $[a, b] \subset \bigcup_n I_n$) such that

$$m^*(I) = m^*([a, b]) > \sum_n l(I_n) - \in \qquad .....(2)$$

Now by Heine-Borel theorem, any countable collection of open intervals that covers $[a, b]$ contains a finite sub-collection of open intervals, which also covers $[a, b]$, and since the sum of the lengths of the finite sub-collection cannot be greater than the sum of the lengths of the original countable collection, therefore, it is sufficient to show that the inequality (2) holds good for the finite sub-collection $\{I_n\}$ which covers $[a, b]$.

Now $a \in [a, b]$ and $[a, b] \subset \bigcup_n I_n$, therefore, $a \in \bigcup_n I_n$ and so there must be one of the intervals I_n's, which contains a.

Let this interval be $]a_1, b_1[$.

Then $$a_1 < a < b_1.$$

If $b_1 \le b$, then $b_1 \in [a, b]$ and since $b_1 \notin \,]a_1, b_1[$, there must exist an interval $]a_2, b_2[$ in the finite sub-collection $\{I_n\}$ such that

$$b_1 \in \,]a_2, b_2[\,, \text{ i.e. } a_2 < b_1 < b_2.$$

Continuing this way, we get a sequence $]a_1, b_1[$, $]a_2, b_2[$,..., $]a_k, b_k[$, of open intervals from the collection $\{I_n\}$ such that

$$a_i < b_{i-1} < b_i \qquad ; i = 1, 2, ...,$$

where $b_0 = a$.

Since $\{I_n\}$ is a finite collection, therefore the above process will terminate with some interval $]a_k, b_k[$, which is possible only when $b \in \,]a_k, b_k[$, *i.e.* when $a_k < b < b_k$.

Thus, $$\sum_n l(I_n) \ge \sum_{i=1}^{k} l(]a_i, b_i[)$$

$$= (b_1 - a_1) + (b_2 - a_2) + ... + (b_k - a_k)$$

$$= b_k - (a_k - b_{k-1}) - (a_{k-1} - b_{k-2}) - ... - (a_2 - b_1) - a_1$$

$$> b_k - a_1 \qquad (\because a_i < b_{i-1} < b_i)$$

$$> b - a \qquad (\because b_k > b \text{ and } a_1 < a)$$

Thus $$\sum_n l(I_n) > b - a$$

From (2), this shows that

$$m^*(I) > b - a - \in$$

Hence $$m^*(I) > b - a \qquad .....(3)$$

From (1) and (3) we have

$$m^*(I) = b - a.$$

Now we consider teh case when I is any finite interval. Then for a given $\in > 0$, we can always find a closed finite interval $J \subset I$ such that $l(J) > l(I) - \in$

Then $\quad l(I) - \in < l(J) = m^*(J) < m^*(I) \quad$ (since J is a closed finite interval)

But $\quad m^*(I) \le l(I)$

therefore, $\quad l(I) - \in < m^*(I) \le l(I)$

The above is true for all $\in > 0$, Hence $m^*(I) = l(I)$.

Finally, we consider the case, when I is an infinite interval. In this case for any positive number $k > 0$, these can be found a closed finite interval $J \subset I$ such that $l(J) = k$.

Then $$m^*(I) \ge m^*(J) = l(J) = k$$

or $$m^*(I) \ge k.$$

Since $k > 0$ is any arbitrary real number, therefore,

$$m^*(I) = \infty = l(I)$$

or $$m^*(I) = l(I).$$

The following theorem is related to the **countable sub additivity property** of outer measure.

Theorem 3. *Let $\{E_n\}$ be a countable collection of sets of real numbers. Then*

$$m^*\left(\bigcup_n E_n\right) \le \sum_n m^*(E_n)$$

Proof : If any one of the sets E_n in the countable collection is of infinite outer measure, then the inequality holds true. So let us assume that $m^*(E_n) < \infty$ for each $n \in N$.

Then for each $n \in N$, and for a given $\in > 0$ there exists a countable collection $\{I_{n,i}\}_{n,i}$ of open intervals such that

$$E_n \subset \bigcup_i I_{n,i} \quad \text{and} \quad \sum_i l(I_{n,i}) < m^*(E_n) + \frac{\in}{2^n}$$

Now
$$\bigcup_n E_n \subset \bigcup_n \left(\bigcup_i I_{n,i}\right)$$

Also since the countable union of countable sets is countable, so the collection $\{I_{n,i}\}_{n,i}$ is a countable collection of open intervals.

Therefore,
$$m^*\left(\bigcup_n E_n\right) \le m^*\left(\bigcup_n \left(\bigcup_i I_{n,i}\right)\right)$$

$$= \sum_n \left(\sum_i l(I_{n,i})\right)$$

$$\le \sum_n \left(m^*(E_n) + \frac{\in}{2^n}\right)$$

$$= \sum_n m^*(E_n) + \in$$

Since $\in > 0$ is arbitrary, therefore, we have

$$m^*\left(\bigcup_n E_n\right) \le \sum_n m^*(E_n).$$

Theorem 4. *If E is a countable set, then $m^*(E) = 0$.*

Proof : Let E be a countable set. Then E can be expressed as

$$E = \{x_1, x_2, \ldots, x_n, \ldots\}$$

Let
$$E_n = \{x_n\}. \text{ Then}$$

$$E = \bigcup_n E_n = \bigcup_n \{x_n\} \quad \text{(Countable union of singleton sets } \{x_n\}).$$

Thus
$$m^*(E) = m^*\left(\bigcup_n E_n\right) \le \sum_n m^*(E_n).$$

But the outermeasure of the singleton set is always zero, therefore $m^*(E_n) = 0$. Hence $m^*(E) \leq 0$.

Now since for any set E, $m^*(E) \geq 0$, therefore we have $m^*(E) = 0$.

Note : The sets N, Z and Q of natural numbers, integers and rational numbers are all countable sets, therefore each of these sets has outer measure zero.

Theorem 5. *If A and B are any two disjoint subsets of R, then*

$$m^*(A \cup B) = m^*(A) + m^*(B) \qquad \text{.....(1)}$$

Proof : From theorem 3, we know that

$$m^*(A \cup B) \leq m^*(A) + m^*(B)$$

Thus we are only to show that

$$m^*(A \cup B) \geq m^*(A) + m^*(B).$$

Let $\{I_n\}$ be the countable collection of open intervals such that

$$m^*(A \cup B) = \sum_n l(I_n). \qquad \text{.....(2)}$$

Since $A \cap B = \phi$, therefore, we can always split this countable collection into two disjoint sub-collections $\{I'_n\}$ and $\{I''_n\}$ which cover A and B respectively

i.e. $$A \subset \bigcup_n I'_n \quad \text{and} \quad B \subset \bigcup_n I''_n$$

clearly $$\{I_n\} = \{I'_n\} \cup \{I''_n\} \quad \text{and} \quad \{I'_n\} \cap \{I''_n\} = \phi$$

Therefore, $\sum_n l(I'_n) + \sum_n l(I''_n) = \sum_n l(I_n)$

and $$m^*(A) \leq \sum_n l(I'_n)$$

$$m^*(B) \leq \sum_n l(I''_n)$$

or $$m^*(A) + m^*(B) \leq \sum_n l(I'_n) + \sum_n l(I''_n)$$

$$= \sum_n l(I_n)$$

$$= m^*(A \cup B) \qquad \text{(from (2))}$$

or $$m^*(A \cup B) \geq m^*(A) + m^*(B) \qquad \text{.....(3)}$$

from (1) and (3) we have

$$m^*(A \cup B) = m^*(A) + m^*(B).$$

It can be verified that the converse of the theorem 4 is not always true. We shall show it by an example given below :

Problem 1. *The cantor's set C is uncountable with outer measure zero.*

Proof : Let C_n be the union of intervals left at the n^{th} stage during the construction of the cantor's set C. Then clearly C_n consists of 2^n closed intervals, each having length 3^{-n}. Thus

$$m^*(C_n) \leq 2^n\, 3^{-n} = (2/3)^n$$

Also each point in C lies in one of the intervals of the union C_n, for each $n \in N$, therefore

$$C \subset C_n \text{ for all } n \in N.$$

Thus
$$m^*(C) \leq m^*(C_n) \leq (2/3)^n$$

The above is true for each $n \in N$, therefore taking the limits when $n \to \infty$, we have

$$m^*(C) = 0.$$

Problem 2. *The closed interval* [0, 1] *is uncountable.*

Proof : If possible let the closed interval [0, 1] be countable. Then

$m^*([0, 1]) = 0.$ (since the outer measure of a countable set is zero.)

But $m^*([0, 1]) = l([0, 1]) = 1,$

which is a contradiction. Hence the closed interval [0, 1] is uncountable.

Problem 3. *For any two sets A and B, if* $m^*(A) = 0$, *then* $m^*(A \cup B) = m^*(B)$. *In particular, if* $B \subset A$, *then* $m^*(B) = 0$.

Proof : Let $m^*(A) = 0$. We know that

$$m^*(A \cup B) \leq m^*(A) + m^*(B)$$
$$= 0 + m^*(B)$$

$\therefore$ $m^*(A \cup B) \leq m^*(B)$ .....(1)

But $B \subset A \cup B$, therefore

$m^*(B) \leq m^*(A \cup B)$ .....(2)

From (1) and (2), we have

$$m^*(A \cup B) = m^*(B).$$

Now if $B \subset A$, then

$$m^*(B) \leq m^*(A) = 0$$

or $m^*(B) = 0$ (since $m^*(B) \geq 0$)

Problem 4. *For any set A and any* $\in\, \geq 0$, *there exists an open set O such that* $A \subset O$ *and* $m^*(O) \leq m^*(A) + \in$.

Proof : Let A be any given set, and $\in\, \geq 0$ be any positive numbers. Then there exists a countable collection $\{I_n\}$ of open intervals such that $A \subset \bigcup_n I_n$ and $\sum_n l(I_n) \leq m^*(A) + \in$ .....(1)

We know that every open interval is an open set and the arbitrary union of open sets is an open set, therefore if we take $O = \bigcup_n I_n$, then O is an open set. Thus $A \subset \bigcup_n I_n = O$ and

$$m^*(O) = m^*\left(\bigcup_n I_n\right) \le \sum_n l(I_n) \le m^*(A) + \in \quad \text{(from (1))}$$

Hence $m^*(O) \le m^*(A) + \in$.

Problem 5. *Given any set A in R, there exists G_δ – set $G \supset A$ such that $m^*(A) = m^*(G)$.*

Proof : Let A be any set of real numbers. Let $\in = \frac{1}{n} > 0, n \in N$. Then for each $n \in N$ there exists an open set O_n such that $A \subset O_n$ and

$$m^*(O_n) \le m^*(A) + \frac{1}{n} \quad .....(1)$$

(using the earlier example)

Now we know that a G_δ – set is the intersection of countable collection of open sets. Thus if we assume that

$G = \bigcap_n O_n$, then clearly G is a G_δ – set and also $A \subset G$.

Thus $\quad m^*(A) \le m^*(G) = m^*\left(\bigcap_n O_n\right)$

$$\le m^*(O_n) \qquad \left(\text{since } \bigcap_n O_n \subset O_n\right)$$

$$\le m^*(A) + \frac{1}{n};\ n \in N \quad \text{[from (1)]}$$

The above in true for each n, therefore letting $n \to \infty$, we have $m^*(A) = m^*(G)$.

Self-learning exercise-1

1. If $\{I_n\}$ is a countable collection of mutually disjoint intervals such that $\bigcup_n I_n$ is an interval, then

$$l\left(\bigcup_n I_n\right) = ...$$

2. If F is a closed subset of the open interval $]\,a, b\,[$, then the length $l(F)$ of F is defined by ..... .

3. For the sequence $< E_i >$ of disjoint sets in the power set $P(R)$, the property

$$m\left(\bigcup_i E_i\right) = \sum_i m(E_i) \text{ is called ..... .}$$

4. Let A be a non-empty set of real numbers and $\{I_n\}$ be the countable collection of open intervals. Then the outer measure $m^*(A)$ of A is defined by

$$m^*(A) = \inf\left(\sum_i l(I_n)\right) \text{ where..... .}$$

5. The outer measure $m^*(A)$ of the set A is translation invariant, *i.e.*, for each $x \in R$, we have...
6. The outer measure of an interval is..... .
7. If E is a countable set, then its outer measure is ..... .
8. The outer measure of the set of natural numbers is ..... .
9. The cantor's set C is uncountable. Its outer measure is ..... .
10. For any two sets A and B, if $m^*(A) = 0$, then $m^*(A \cup B) =$..... .
11. For any set A and any given $\in > O$, there exists an open set 0 such that

$$A \subset O \quad \text{and} \quad m^*(A) + \in \text{ ..... .}$$

2.5 Measurable sets

To begin with, we maintain the assumption that all sets are contained in some bounded interval X. This restriction will be removed presently. As we have seen, the outer measure $m^*(E)$ of a set $E \subset X$ is defined as :

$$m^*(E) = \underset{E \subset \bigcup_n I_n}{Inf} \left\{\sum_i l(I_n)\right\},$$

where, $\{I_n\}$ [the countable collection of open intervals] is a member of the family of countable collection of open intervals.

The inner measure $m_*(E)$ of the set E is defined by

$$m_*(E) = m(X) - m^*(X - E)$$

We say that the set E is measurable if

$$m^*(E) = m_*(E)$$

In other words, the set E is measurable if

$$m^*(E) = m(X) - m^*(X - E)$$

or $$m(X) = m^*(E) + m^*(X - E) \quad \text{.....(1)}$$

The above definition of measurability of the set E is given by Henri Lebesgue.

Since Lebesgue started with the set E contained in the bounded interval X, appropriate modifications were needed for the unbounded sets. Following definition due to caratheodory does not require such modifications :

2.5.1 Definition of measurable sets : A set E is said to be **Lebesgue measurable** or simply **measurable** if for each set A,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \quad \text{.....(2)}$$

Now since $$A = (A \cap E) \cup (A \cap E^c)$$

and the function m^* is sub additive, we have

$$m^* (A) \le m^* (A \cap E) + m^* (A \cap E^c) \qquad .....(3)$$

Thus in order that the set E is measurable, we need only to show that

$$m^* (A) \ge m^* (A \cap E^c) \qquad .....(4)$$

If in Henri Lebesgue's definition given by (1), we take $A = X$, then the equality (1) converts into the form

$$m^* (A) = m^* (A \cap E) + m^* (A \cap E^c)$$

which is same as that of given by (2). Thus the definition due to Lebesgue is a special case of the definition given by caratheodory.

The outer measure is defined for all sets in R, but it does not satisfy, in general, the countable additivity property. In order that the countable additivity is satisfied, we restrict the domain of definition for the function m^* to some suitable subset M of the power set $P(R)$. The members of this subset M are called **measurable sets** that are defined earlier and shown by (2).

The set function $m : M \to R^*$ (R^* being the extended real number system) obtained by restricting the set function m^* to the subset M of the domain of definition $P(R)$ of m^* is called the **Lebesgue measure function** for the sets in M. For each $E \in M$, $m(E)$ is called the **Lebesgue measure** or simply **measure** of the set E.

Theorem 6. ***(i)*** *If E is a measurable set then E^c is also measurable*

(ii) *The sets ϕ and R, are measurable sets.*

Proof :*(i)* Let E be a measurable set. Then for each set $A \subset R$,

$$\begin{aligned} m^* (A) &= m^* (A \cap E) + m^* (A \cap E^c) \\ &= m^* (A \cap (E^c)^c) + m^* (A \cap E^c) \\ &= m^* (A \cap E^c) + m^* (A \cap (E^c)^c). \end{aligned}$$

This shows that E^c is a measurable set.

(ii) For each set $A \subset R$, we have

$$\begin{aligned} m^* A \cap \phi) + m^* (A \cap \phi^c) &= m^* (\phi) + m^* (A) \ (\because\ \phi^c = R \text{ and } A \subset R) \\ &= 0 + m^* (A) \end{aligned}$$

or $$m^* (A) = m^* (A \cap \phi) + m^* (A \cap \phi^c)$$

showing that ϕ is measurable.

Again $$\begin{aligned} m^* (A \cap R) + m^* (A \cap R^c) &= m^* (A) + m^* (\phi) \ (\because\ R^c = \phi) \\ &= m^* (A) + 0 \end{aligned}$$

Therefore $$m^* (A) = m^* (A \cap R) + m^* (A \cap R^c).$$

Thus R is measurable.

Theorem 7. *If E is a set such that $m^*(E) = 0$, then E is measurable.*

Proof : For any set A, we have $A \cap E \subset E$

Therefore $$m^*(A \cap E) \leq m^*(E) = 0 \quad .....(1)$$

Also $$A \cap E^c \subset A$$

therefore $$m^*(A \cap E^c) \leq m^*(A) \quad .....(2)$$

From (1) and (2),

$$m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A)$$

this shows that E is measurable.

Theorem 8. *Every countable set is measurable.*

Proof : Let E be a countable set. Then $m^*(E) = 0$

This implies that E is measurable [by theorem 8]

Theorem 9. *The union of two measurable sets is a measurable set.*

Proof : Let E_1 and E_2 be any two measurable sets. Then for any set A, applying the measurablity on E_1,

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1{}^c) \quad .....(1)$$

Now we select $A \cap E_1{}^c$ for A and apply the measurablity on E_2. We get

$$m^*(A \cap E_1{}^c) = m^*((A \cap E_1{}^c) \cap E_2) + m^*((A \cap E_1{}^c) \cap E_2{}^c)$$

or $$m^*(A \cap E_1{}^c) = m^*((A \cap E_1{}^c) \cap E_2) + m^*(A \cap (E_1{}^c \cap E_2{}^c))$$

or $$m^*(A \cap E_1{}^c) = m^*(A \cap (E_1{}^c \cap E_2)) + m^*(A \cap (E_1 \cup E_2)^c) \quad .....(2)$$

[since $E_1{}^c \cap E_2{}^c = (E_1 \cup E_2)^c$]

Again selecting $A \cap (E_1 \cup E_2)$ for A and applying the measurablity on E_1, we get

$$m^*(A \cap (E_1 \cup E_2)) = m^*(A \cap (E_1 \cup E_2) \cap E_1) + m^*(A \cap (E_1 \cup E_2) \cap E_1{}^c)$$

But $$(E_1 \cup E_2) \cap E_1 = E_1 \quad .....(3)$$

and $$(E_1 \cup E_2) \cap E_1{}^c = (E_1 \cap E_1{}^c) \cup (E_2 \cap E_1{}^c)$$
$$= \phi \cup (E_1{}^c \cap E_2)$$
$$= E_1{}^c \cap E_2$$

therefore from (3)

$$m^*(A \cap (E_1 \cup E_2)) = m^*(A \cap E_1) + m^*(A \cap (E_1{}^c \cap E_2))$$
$$= m^*(A \cap E_1) + m^*(A \cap E_1{}^c) - m^*(A \cap (E_1 \cup E_2)^c)$$

[using (2)]

$$= m^*(A) - m^*(A \cap (E_1 \cup E_2)^c) \text{ [using (1)]}$$

or $$m^*(A) = m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$$

this implies that $E_1 \cup E_2$ is measurable.

Theorem 10. *The intersection of two measurable sets is also a measurable set.*

Proof : Let E_1 and E_2 be any two measurable sets. Then E_1^c and E_2^c are also measurable. Consequently $E_1^c \cup E_2^c$ *i.e.* $(E_1 \cap E_2)^c$ is measurable. This implies that $((E_1 \cap E_2)^c)^c$ *i.e.* $E_1 \cap E_2$ is measurable.

Theorem 11. *If E is a measurable set, then for any set A*

$$m^* (E \cup A) + m^* (E \cap A) = m^* (E) + m^* (A).$$

Proof : Since E is a measurable set, therefore for any set B,

$$m^* (B) = m^* (B \cap E) + m^* (B \cap E^c) \qquad .....(1)$$

First we take $B = A$ and then $B = E \cup A$. Then (1) becomes

$$m^* (A) = m^* (A \cap E) + m^* (A \cap E^c) \qquad .....(2)$$

and $$m^* (E \cup A) = m^* ((E \cap A) \cap E) + m^* ((E \cup A) \cap E^c)$$

Now $$(E \cup A) \cap E = E \qquad .....(3)$$

and $$(E \cup A) \cap E^c = (E \cap E^c) \cup (A \cap E^c)$$
$$= \phi \cup (A \cap E^c) = A \cap E^c$$

Therefore from (3)

$$m^* (E \cup A) = m^* (E) + m^* (A \cap E^c) \qquad .....(4)$$

From (2) and (4)

$$m^* (E \cup A) = m^* (E) + m^* (A) - m^* (A \cap E)$$

or $$m^* (E \cup A) + m^* (E \cap A) = m^* (E) + m^* (A).$$

Theorem 12. *The class M of measurable sets is an algebra, i.e., the complement of a measurable set is measurable and the union (and the intersection) of finite collection of measurable sets is measurable.*

Proof : By symmetry between the sets E and E^c in the definition of measurablity of the set E, we know that if $E \in M$ (*i.e.* if E is measurable), then $E^c \in M$. Thus in order to prove the theorem, it is sufficient to prove that if $\{E_1, E_2,..., E_n\}$ is a finite collection of measurable sets, then $\bigcup_{k=1}^{n} E_k$ and $\bigcap_{k=1}^{n} E_k \in M$.

First we prove that if E_1 and $E_2 \in M$, then $E_1 \cup E_2 \in M$ (*i.e.* E_1 and E_2 are measurable then $E_1 \cup E_2$ is also measurable). This we have already proved in theorem 9.

Thus $E_1 \cup E_2 \in M$.

Now if $E = \bigcup_{k=1}^{n} E_k = E_1 \cup E_2 \cup ... \cup E_n$ and if $E_i \in M$ for $i = 1, 2,..., n$, then by mathematical induction, it follows that E is measurable, *i.e.* $E \in M$.

Now we prove that if $E_k \in M$; $k = 1, 2, \ldots, n$, then $\bigcap_{k=1}^{n} E_k \in M$. We have

$$E_k \in M \,;\, k = 1, 2, \ldots, n \Rightarrow \bigcup_{k=1}^{n} E_k \in M$$

$$\Rightarrow \bigcup_{k=1}^{n} E_k^c \in M \qquad \text{(since } E \in M \Rightarrow E^c \in M\text{)}$$

$$\Rightarrow \left(\bigcup_{k=1}^{n} E_k^c \right)^c \in M$$

$$\Rightarrow \left(\left(\bigcap_{k=1}^{n} E_k \right)^c \right)^c \in M$$

$$\Rightarrow \bigcap_{k=1}^{n} E_k \in M$$

Hence M is an algebra.

Theorem 13. *Let $E_1, E_2, \ldots, E_n$ be a finite sequence of disjoint measurable sets. Then for any set A,*

$$m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i \right) \right) = \sum_{i=1}^{n} m^*(A \cap E_i).$$

Proof : We shall prove the theorem by in induction.

For $n = 1$, the result holds true, since

$$\text{L.H.S} = m^*(A \cap E_1) = \text{R.H.S.}$$

Assume that the result holds true for $(n-1)$ sets, *i.e.*, let

$$m^*\left(A \cap \left(\bigcup_{i=1}^{n-1} E_i \right) \right) = \sum_{i=1}^{n-1} m^*(A \cap E_i) \qquad \text{.....(1)}$$

Adding $m^*(A \cap E_n)$ on both sides of (1), we have,

$$m^*\left(A \cap \left(\bigcup_{i=1}^{n-1} E_i \right) \right) + m^*(A \cap E_n) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

or $$m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i \right) \cap E_n^c \right) + m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i \right) \cap E_n \right) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

(since E_i ; $i = 1, 2, \ldots, n$ are all disjoint)

Also E_n is measurable, therefore by taking the set $A \cap \left(\bigcup_{i=1}^{n} E_i \right)$, we have

$$m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i \right) \right) = m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i \right) \cap E_n \right) + m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i \right) \cap E_n^c \right)$$

Thus from (2)

$$m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i \right) \right) = \sum_{i=1}^{n} m^*\left(A \cap E_i \right).$$

Result : If $E_1, E_2, ..., E_n$ is a finite sequence of disjoint measurable sets, then

$$m^*\left(\bigcup_{i=1}^{n} E_i \right) = \sum_{i=1}^{n} m^*\left(E_i \right).$$

The above result can be proved by taking $A = R$ in theorem 14.

Theorem 14. *If $\{En : n \in N\}$ is a sequence of disjoint measureble sets, then*

$$m^*\left(\bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} m^*\left(E_i \right)$$

Proof : We know that for any $n \in N$ and for any set A.

$$m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i \right) \right) = \sum_{i=1}^{n} m^*\left(A \cap E_i \right)$$

Replacing A by R, the above becomes

$$m^*\left(\bigcup_{i=1}^{n} E_i \right) = \sum_{i=1}^{n} m^*\left(E_i \right) \qquad .....(1)$$

Where $E_i'^s$; $i = 1, 2, ..., n$ are all disjoint.

Also for any $n \in N$

$$\bigcup_{i=1}^{n} E_i \subset \bigcup_{i=1}^{\infty} E_i$$

Therefore $$m^*\left(\bigcup_{i=1}^{n} E_i \right) \leq m^*\left(\bigcup_{i=1}^{\infty} E_i \right)$$

or $$\sum_{i=1}^{n} m^*\left(E_i \right) \leq m^*\left(\bigcup_{i=1}^{\infty} E_i \right) \qquad \text{[from (1)]}$$

The right hand side of the above inequality is independent of n, therefore letting $n \to \infty$, we get

$$\sum_{i=1}^{\infty} m^*\left(E_i \right) \leq m^*\left(\bigcup_{i=1}^{\infty} E_i \right) \qquad .....(2)$$

But we know that

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} m^*(E_i) \qquad .....(3)$$

Therefore, in view of (2) and (3), we have

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i).$$

Problem 6. *If E_1 and E_2 are measurable sets then $E_1 - E_2$ is measurable. Also*

$$m^*(E_1 - E_2) = m^*(E_1) - m^*(E_2), \quad \text{if} \quad E_2 \subset E_1$$

Proof : Since E_1 and E_2 are measurable, therefore,

$$E_1 - E_2 = E_1 \cup E_2^c$$

is also measurable. Further

$$E_1 = (E_1 - E_2) \cup E_2 \qquad (\text{if } E_2 \subset E_1)$$

which is a union of two disjoint measurable sets $E_1 - E_2$ and E_2. Therefore,

$$m^*(E_1) = m^*((E_1 - E_2) \cup E_2)$$
$$= m^*(E_1 - E_2) + m^*(E_2)$$

or $$m^*(E_1 - E_2) = m^*(E_1) - m^*(E_2).$$

Problem 7. *If E_1 and E_2 are two measurable sets, then*

$$m^*(E_1 \cup E_2) + m^*(E_1 \cap E_2) = m^*(E_1) + m^*(E_2).$$

Proof : Since E_1 is a measurable set, therefore for any set A,

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c) \qquad .....(1)$$

Taking $A = E_1 \cup E_2$ and adding $m^*(E_1 \cap E_2)$ on both sides, (1) becomes

$$m^*(E_1 \cup E_2) + m^*(E_1 \cap E_2) = m^*((E_1 \cup E_2) \cap E_1)$$
$$+ m^*((E_1 \cup E_2) \cap E_1^c) + m^*(E_1 \cap E_2) \qquad .....(2)$$

But $E_2 = ((E_1 \cup E_2) \cap E_1^c) \cup (E_1 \cap E_2)$ is the union of two disjoint measurable sets $(E_1 \cup E_2) \cap E_1^c$ and $E_1 \cap E_2$, therefore,

$$m^*(E_2) = m^*(((E_1 \cup E_2) \cap E_1) \cup (E_1 \cap E_2))$$
$$= m^*((E_1 \cup E_2) \cap E_1^c) + m^*(E_1 \cap E_2)$$

Therefore, (2) is

$$m^*(E_1 \cup E_2) + m^*(E_1 \cap E_2) = m^*(E_1) + m^*(E_2) \qquad (\text{since } (E_1 \cap E_2) \cap E_1 = E_1).$$

Problem 8. *If $E \subset \,]\, a, b\, [$ is Lebesgue measurable set, then prove that*

$$m(E) + m(E^c) = b - a.$$

Proof : We know that, if $E \subset \,]\, a, b\, [$, then

$$m_*(E) = b - a - m^*(E^c)$$

But since E is meaurable, therefore,

$$m_* (E) = m^* (E) = m (E)$$

and $$m_* (E^c) = m^* (E^c) = m (E^c)$$

Thus we have,

$$m (E) = b - a - m (E^c)$$

or $$m (E) + m (E^c) = b - a.$$

Problem 9. *If M is a measurable set, then for any set E, prove that*

$$m^* (E) = m^* (EM) + m^* (E - EM), \quad \text{where} \quad EM = E \cap M.$$

Proof : We know that

$$E = (E \cap M) \cup [E - (E \cap M] = EM \cup (E - EM)$$

Also $$(E \cap M) \cap [E - (E \cap M)] = EM \cap (E - EM) = \phi$$

that is EM and $E - EM$ are disjoint, therefore, using theorem 5,

$$m^* (E) = m^* (EM \cup (E - EM)) = m^* (EM) + m^* (E - EM)$$

or $$m^* (E) = m^* (EM) + m^* (E - EM).$$

Theorem 15. *The collection M of measurable sets is a σ-algebra.*

Proof : We have proved in theorem 12 that the collection M of measurable sets is an algebra. In order to show that M is a σ-algebra, we need to show that if E is the union of a countable collection of measurable sets, then E is measurable.

Let $\{E_n ; n \in N\}$ be a countable collection of measurable sets in M and let $E = \bigcup_{i=1}^{\infty} E_i$.

Then there always exists a sequence $\{F_n\}$ of mutually disjoint measurable sets in M such that

$$E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$$

Now let $B_n = \bigcup_{i=1}^{n} F_i$. Then the sets B_n are measurable (Since the finite union of measurable sets in M, is measurable, as M is an algebra) and so B_n^c are also measurable. Also $B_n \subset E$, therefore $E^c \subset B_n^c$.

Now choosing A as the test set and applying the masurability on the set B_n $(n \in N)$, we have

$$\begin{aligned} m^* (A) &= m^* (A \cap B_n) + m^* (A \cap B_n^c) \\ &\geq m^* (A \cap B_n) + m^* (A \cap E^c) \qquad (\text{since } A \cap E^c \subset A \cap B_n) \\ &= m^* \left(A \cap \left(\bigcup_{i=1}^{n} F_i \right) \right) + m^* \left(A \cap E^c \right) \end{aligned}$$

$$= \sum_{i=1}^{n} m^*\left(A \cap F_i\right) + m^*\left(A \cap E^c\right) \qquad \text{(by theorem 13)}$$

The left hand side of the above inequality is independent of n, therefore, letting $n \to \infty$

$$m^*(A) \geq \sum_{i=1}^{\infty} m^*\left(A \cap F_i\right) + m^*\left(A \cap E^c\right)$$

$$\geq m^*\left(\bigcup_{i=1}^{\infty} \left(A \cap F_i\right)\right) + m^*\left(A \cap E^c\right)$$

$$= m^*\left(A \cap \left(\bigcup_{i=1}^{\infty} F_i\right)\right) + m^*\left(A \cap E^c\right)$$

$$= m^*(A \cap E) + m^*(A \cap E^c),$$

showing that the set E is measurable, *i.e.*, $E \in M$. Hence M is a σ-algebra.

Theorem 16. *Every interval is measurable.*

Proof : First of all we shall prove that the interval $]\,a, \infty\,[$ is measurable. Let A be any subset of R and let

$$A_1 = A \cap\,]\,a, \infty\,[\text{ and } A_2 = A \cap\,]-\infty, a\,].$$

Then we shall show that

$$m^*(A) \geq m^*(A_1) + m^*(A_2).$$

Now if $m^*(A) = \infty$, then the result is trivially true, so let $m^*(A) < \infty$. Then for given $\in\, > 0$, there exists a countable collection $\{I_n\}$ of open intervals that covers A (*i.e.* $A \subset \bigcup_n I_n$) such that

$$\sum_n l\left(I_n\right) < m^*\left(A\right) + \in \qquad .....(1)$$

Let $\qquad I'_n = I_n \cap\,]\,a, \infty\,[\quad$ and $\quad I''_n = I_n \cap\,]-\infty, a\,]$.

Then $\qquad I'_n \cup I''_n = (I_n \cap\,]\,a, \infty\,[\,) \cup (I_n \cap\,]-\infty, a]\,)$

$$= I_n \cap\,]-\infty, \infty\,[$$

$$= I_n$$

and $I'_n \cap I''_n = \phi$. Therefore,

$$l(I_n) = l(I'_n) + l(I''_n). \qquad .....(2)$$

Now $\qquad A_1 = A \cap\,]\,a, \infty\,[\, \subset \left(\bigcup_n I_n\right) \cap\,]\,a, \infty\,[$

$$= \bigcup_n \left(I_n \cap\,]\,a, \infty[\,\right)$$

$$= \bigcup_n I'_n\,.$$

Similarly $A_2 \subset \bigcup_n I''_n$. Therefore,

$$m^*(A_1) + m^*(A_2) \leq m^*\left(\bigcup_n I'_n\right) + m^*\left(\bigcup_n I''_n\right)$$

$$= \sum_n l(I'_n) + \sum_N l(I''_n)$$

$$= \sum_n \left(l(I'_n) + l(I''_n)\right)$$

$$= \sum_n l(I_n) \qquad \text{[from (2)]}$$

$$< m^*(A) + \in \qquad \text{[from (1)]}$$

Since $\in > 0$ is an arbitrary positive number, therefore

$$m^*(A_1) + m^*(A_2) \leq m^*(A)$$

or $$m^*(A) \geq m^*(A \cap\ ]\, a, \infty\, [\,) + m^*(A \cap\ ]-\infty, a\,]\,)$$

Thus the interval $]\, a, \infty\, [$ is measurable.

Now since $]-\infty, a] = (\,]\, a, \infty\, [\,)^c$ and the complement of a measurable set is measurable, therefore the interval $]-\infty, a]$ is measurable.

Also $]-\infty, a\, [\ = \bigcup_{n=1}^{\infty}\left(\,]-\infty, a - \frac{1}{n}]\,\right)$, which is a countable union of measurable sets and so a measurable set. Therefore, $]-\infty, a\, [$ is a measurable set.

Finally every open interval $]\, a, b\, [$ can be expressed as $]\, a, b\, [\ = (\,]-\infty, b\, [\,) \cap (\,]\, a, \infty[\,)$ and the intersection of two measurable sets is a measurable set, therefore the open interval $]\, a, b\, [$ is measurable.

In a similar way we can prove that the intervals $[a, b]$, $[a, b\, [$and $]\, a, b]$ are measurable.

Hence every interval is measurable.

2.5.2 Lebesgue measure of a set :

If E is a measurable set, then the outer measure $m^*(E)$ of E is called the Lebesgue measure of the set E and is denoted by $m(E)$. Thus if E is a measurable set, then $m(E) = m^*(E)$. Therefore the set function m : $M \to R^*$ is an additive set function an the σ -algebra class M of measurable sets which is the restriction of the set function m^* to the family M of measurable sets. For each $E \in M$, $m(E) = m^*(E)$. The extended real number m (E) is said to be the Lebesgue measure or simply measure of the set $E \in M$.

Problem 10. *Every open set is measurable.*

Proof : Since every open set is the union of finite or a countable collection of open intervals, and each open interval is measureable, also the finite union or a countable union of measurable sets is a measurable set, therefore, each open set is measurable.

Problem 11. *Every closed set is measurable.*

Proof : Every closed set is the complement of an open set and each open set is measurable, also the complement of every measurable set is a measurable set, therefore every closed set is measurable.

Problem 12. *For any set A there exists a measurable set E containing A such that* $m^*(A) = m(E)$.

Proof : We know that for a given set A and a given positive number $\in = \frac{1}{n}$, there exists an open set G_n such that $A \subset G_n$ and

$$m^*(G_n) \le m^*(A) + \frac{1}{n}; \quad n \in N \qquad .....(1)$$

Now, since each G_n is an open set ($n \in N$) therefore each G_n is measurable ($n \in N$).

Thus $m^*(G_n) = m(G_n)$

Let $$E = \bigcap_{n=1}^{\infty} G_n$$

Since the countable intersection of measurable sets is a measurable set, therefore it is clear that E is a measurable set

Now $$E = \bigcap_{n=1}^{\infty} G_n \subset G_n$$

therefore $m^*(E) \le m^*(G_n)$

or $m(E) \le m^*(A) + \frac{1}{n}; n \in N$ [using (1)]

[Since E is measurable so $m^*(E) = m(E)$].

The left hand side of the above inequality is independent of n, therefore letting $n \to \infty$

$$m(E) \le m^*(A) \qquad .....(2)$$

But $A \subset E$, therefore

$$m^*(A) \le m^*(E) = m(E) \qquad .....(3)$$

In view of (2) and (3), we have

$$m^*(A) = m(E).$$

Theorem 17. *Every Borel set is measurable.*

Proof : Let B be a Borel set. Then B is obtained by using countable union and intersection of open sets or closed sets. Since every open interval is measurable and a countable union of measurable sets is measurable, therefore every open set is measurable. Also a set is closed if and only if its complement is open and the complement of a measuable set is measurable, therefore every closed set is measurable. Thus every open set and closed set is measurable. Now B is obtained by using the countable union and intersection of open and closed sets, therefore B is measurable.

Self-learning exercise-2

1. If X is a bounded interval and $E \subset X$, then $m^* (E) + m_* (X - E) = ...$.
2. E is a measurable, if for any set A, $m^* (A \cap E) + m^* (A \cap E^c) \leq ....$.
3. The set R of reals is .... .
4. If E is a countable set, then E is .... .
5. If E is a measurable set, then for any set A,

$$m^* (E \cup A) + m^* (E \cap A) = ...$$

6. If $E_1, E_2, ..., E_n$ is a finite sequence of disjoint measurable sets, then

$$m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) = ...$$

7. For the finite sequence $E_1, E_2,..., E_n$ of disjoint measurable sets,

$$m^*\left(\bigcup_{i=1}^{n} E_i\right) = ...$$

8. If E_1 and E_2 are measurable sets, then $m^* (E_1) - m^* (E_2) = ...$.
9. If I is an interval, then I is .... .
10. Open set is a .... .
11. Closed set is a .... .

Theorem 18. *Let $< E_i >$ be an infinite decreasing sequence of measurable sets, that is, $E_1 \supset E_2 \supset E_3$ .... . Let $m (E_i) < \infty$ for at least one $i \in N$. Then*

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n).$$

Proof : Let k be the least positive integer such that $m (E_k) < \infty$. Then clearly $m (E_i) < \infty$ for all $i \geq k$.

Let $$E = \bigcap_{i=1}^{\infty} E_i$$

and $F_i = E_i - E_{i+1}$.

Then the sets F_i are pairwise disjoint and measurable. Also

$$E_k - E = \bigcup_{i=k}^{\infty} F_i$$

Therefore, $$m(E_k - E) = m\left(\bigcup_{i=k}^{\infty} F_i\right) = \sum_{i=k}^{\infty} m(F_i)$$

$$= \sum_{i=k}^{\infty} m(E_i - E_{i+1}) \qquad .....(1)$$

But $E_k = E \cup (E_k - E)$ and $E_i = E_{i+1} \cup (E_i - E_{i+1})$

therefore $m(E_k) = m(E) + m(E_k - E)$ and $m(E_i) = m(E_{i+1}) + m(E_i - E_{i+1})$

Thus (1) becomes

$$m(E_k) - m(E) = \sum_{i=k}^{\infty}\left(m(E_i) - m(E_{i+1})\right)$$

$$= \lim_{n\to\infty} \sum_{i=k}^{n}\left(m(E_i) - m(E_{i+1})\right)$$

$$= \lim_{n\to\infty} \left(m(E_k) - m(E_{n+1})\right)$$

$$= m(E_k) - \lim_{n\to\infty} m(E_n)$$

Now, since $m(E_k) < \infty$, therefore

$$m(E) = \lim_{n\to\infty} m(E_n)$$

or $$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n).$$

Theorem 19. *Let $< E_i >$ be an infinite increasing sequence of measurable sets, that is, $E_1 \subset E_2 \subset E_3 \subset .....$ Then*

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n).$$

Proof : If $m(E_k) = \infty$ for some $k \in N$, then the result holds good, since

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \ge m(E_k) = \infty$$

and $m(E_n) = \infty$ for each $n \ge k$. So let $m(E_i) < \infty$ for each $i \in N$.

Let $$E = \bigcup_{i=1}^{\infty} E_i \quad \text{and} \quad F_i = E_{i+1} - E_i$$

Then the sets F_i are pairwise disjoint and measurable sets. Also

$$E - E_1 = \bigcup_{i=1}^{\infty} F_i$$

therefore, $$m\,(E - E_1) = m\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} m(F_i)$$

$$= \sum_{i=1}^{\infty} m(E_{i+1} - E_i)$$

or $$m\,(E) - m\,(E_1) = \lim_{n\to\infty} \sum_{i=1}^{n} \left[m(E_{i+1}) - m(E_1)\right]$$

$$= \lim_{n\to\infty} \left[m(E_{n+1}) - m(E_1)\right]$$

$$= \lim_{n\to\infty} m(E_{n+1}) - m(E_1)$$

Thus $$m\,(E) = \lim_{n\to\infty} m(E_n)$$

or $$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n).$$

Theorem 20. *Let E be a measurable set. Then for any real number x, the translation E + x is also measurable. Further more m(E + x) = m (E).*

Proof : Since E is measurable, therefore for any set A,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

or $$m^*(A + x) = m^*[(A \cap E) + x] + m^*[(A \cap E^c) + x]$$

(since m^* is translation invariant)

Also we can varify that

$$(A \cap E) + x = (A + x) \cap (E + x)$$

and $$(A \cap E^c) + x = (A + x) \cap (E^c + x)$$

therefore, $$m^*(A + x) = m^*[(A + x) \cap (E + x)] + m^*[(A + x) \cap (E^c + x)]$$

Now replacing A with $A - x$, we obtain

$$m^*(A) = m^*[A \cap (E + x)] + m^*[A \cap (E^c + x)]$$

or $$m^*(A) = m^*[A \cap (E + x)] + m^*[A \cap (E + x)^c].$$

This implies that $E + x$ is measurable.

Now since $m^*(E + x) = m^*(E)$

therefore, $m(E + x) = m(E)$.

2.6 Non-measurable sets

In the study of real analysis, most of the sets, we come across are measurable. But there are non-measurable sets also given by many mathematicians. The non-measurable sets can not be constructed without assuming the validity of axiom of choice. We shall discuss an example of non-measurable set through a theorem. But before proving it we shall define certain preliminaries.

Definition (sum modulo 1) : Let x and y be any two real numbers in [0, 1 [, then the **sum modulo 1** of x and y, denoted by $\overset{\circ}{+}$ is defined by

$$x \overset{\circ}{+} y = \begin{cases} x+y & \text{if } x+y<1 \\ x+y-1 & \text{if } x+y \geq 1 \end{cases}$$

Definition (Translate modulo 1) : Let E be a subset of [0, 1 [. Then the **translate modulo 1** of E by y is the set defined by

$$E \overset{\circ}{+} y = \left\{ z : z = x \overset{\circ}{+} y,\ x \in E \right\}.$$

It can be verified that :

(i) $x, y \in [0,1[\Rightarrow x \overset{\circ}{+} y \in [0,1[$.

(ii) $\overset{\circ}{+}$ is commutative and associative also.

Theorem 21. *There exists a non-measurable set in the interval* [0, 1[.

Proof : Let [0, 1 [be the interval and let $x, y \in$ [0, 1 [. We define an equivalence relation on the set [0, 1 [by saying that x and y are equivalent and write $x \sim y$ if and if $x - y$ is a rational number. This relation partitions the interval [0, 1 [into mutually disjoint classes E_α with the property that any two elements of the some class differ by a rational number, whereas those belonging to different classes differ by an irrational number. Thus

$$[0, 1 [\cup E_\alpha .$$

Since the set of rational numbers in [0, 1 [is countable, therefore each E_α is a countable set. Also since [0, 1 [is not countable, therefore the class of sets E_α is uncountable.

By axiom of choice, we now construct a set P in [0, 1 [consisting just one element x_α from each E_α. Then $P \subset$ [0, 1 [. We shall show that this set P is non-measurable.

Let $<r_i>$ be a sequence of rational numbers in [0, 1 [, with $r_0 = 0$. We define

$$P_i = P \overset{\circ}{+} r_i, \quad i \in N \quad \text{where} \quad P_0 = P$$

We shall first show that $\{P_i\}$ is a sequence of pairwise disjoint sets.

Let $x \in P_i \cap P_j$, then there exist elements $p_i, p_j \in P$ such that

$$x = p_i \overset{\circ}{+} r_i = p_j \overset{\circ}{+} r_j$$

inplies that $p_i - p_j$ is a rational number. This shows that p_i and p_j belong to the some set E_α. But since P consists of only one element from each E_α, therefore we must have $i = j$. Thus if $i \neq j$, then $P_i \cap P_j = \phi$.

Now since $P_i \subset [0, 1[$, therefore $\bigcup_{i=1}^{\infty} P_i \subset [0,1[,$ .....(1)

Let $x \in [0, 1[$. Then $x \in E_\alpha$ for some α. Since P consists of just one element from each E_α, therefore either $x \in P = P_0$ or there exists an element $p_i \in P$ such that $x - p_i$ is a rational number say r_i

or $\quad x = p_i \overset{\circ}{+} r_i \Rightarrow x \in P_i$ for some i

$\Rightarrow \quad x \in \bigcup_{i=1}^{\infty} P_i$

Thus $\quad [0,1[\subset \bigcup_{i=1}^{\infty} P_i$ .....(2)

In view of (1) and (2), we have

$$\bigcup_{i=1}^{\infty} P_i = [0,1[.$$

Now we show that P is non-measurable. If P is measurable, then each P_i is measurable and

$m(P_i) = m(P + r_i) = m(P)$; $i = 0, 1, 2,...$ (since m is translation invariant.)

Thus $\quad m\left(\bigcup_{i=1}^{\infty} P_i\right) = m\left([0,1[\right)$

or $\quad \sum_{i=1}^{\infty} m(P_i) = 1$

or $\quad \sum_{i=1}^{\infty} m(P) = 1$

But $\quad \sum_{i=1}^{\infty} m(P) = \begin{cases} 0 & \text{if } m(P) = 0 \\ \infty & \text{if } m(P) > 0 \end{cases}$

This leads to contradictory statements.

Hence P is a non-measurable set.

2.7 Summary

In this unit we discussed the definitions of length of an interval, outer measure of the set, measurable sets and their measure. We also studied various properties of measurable sets through theorems and examples. In the last we proved that there exist non-measurable sets also.

2.8 Answers to self-learning exercises

Self-learning exercise-1

1. $\sum_n l(I_n)$ **2.** $b - a - l(F^c)$ **3.** countable additivity property

4. $A \subset \bigcup_n I_n$ **5.** $m^*(A + x) = m^*(A)$ **6.** its length

7. zero **8.** zero **9.** zero

10. $m^*(B)$ **11.** $m^*(O)$

Self-learning exercise-2

1. $m(X)$ **2.** $m^*(A)$ **3.** measurable

4. measurable **5.** $m^*(E) + m^*(A)$ **6.** $\sum_{i=1}^{n} m^*(A \cap E_i)$

7. $\sum_{i=1}^{n} m^*(E_i)$ **8.** $m^*(E_1 - E_2)$ **9.** measurable

10. measurable set **11.** measurable set.

2.9 Exercises

1. Show that is countably additive on disjoint measurable sets.

2. Show that family of measurable sets is a σ-algebra of sets in $P(R)$.

3. Show that the inner measure of a set E cannot exceed its outer measure.

4. Prove the existence of a subset of R that is not measurable.

5. If $< E_i >$ is a sequence of measurable sets with $m^*(E_i) = 0$; $i \in N$, then prove that $\bigcup_{i=1}^{\infty} E_i$ is a measurable set and has its measure zero.

□ □ □

UNIT 3 : Measurable Functions and Convergence of Sequences of Measurable Functions

Structure of the Unit

3.0 Objectives

3.1 Introduction

3.2 Measurable functions

  3.2.1 Definition of measurable function

3.3 Algebra of measurable functions

3.4 Borel measurable functions

3.5 Almost every where property

  3.5.1 Almost every where convergence

3.6 Supremum and infimum of a sequence

3.7 Convergence of sequences of functions

  3.7.1 Convergence in measure

  3.7.2 Almost uniform convergence

3.8 Summary

3.9 Answers to self-learning exercises

3.10 Exercise

3.0 Objectives

The purpose of writing this unit is to introduce the functions defined on a given measurable set. Actually for the existence of Lebesgue integral of a function, the function must be less restrictive than that of continuity. This condition gives rise to a new class of functions, known as measurable functions. The class of measurable functions thus plays an important role in the Lebesgue theory of integration.

3.1 Introduction

In the beginning of the unit the definition of measurable function is given in two different ways, one by a theorem and other in commonly used form. Operations on measurable functions, properties of functions that are continuous on measurable sets, convergence almost every where and convergence in measure of sequences of measurable functions are also discussed in sequal.

3.2 Measurable functions

We have seen in the earlier unit that all sets are not measurable. Therefore it is important to know that sets, which are constructed under certain condition are measurable or not. If we start with a function *f*, the most important sets that arise from it are listed in the following theorem :

Theorem 1. *Let f be an extended real valued function* (*a function whose values are in the set of extended real numbers, is called the extended real valued function.*), *whose domain is a measurable set E. Then following four statements are equivalent* :

(i) *For each real number* α, *the set* $\{x \in E : f(x) > \alpha\}$ *is measurable.*

(ii) *For each real number* α, *the set* $\{x \in E : f(x) \geq \alpha\}$ *is measurable.*

(iii) *For each real number* α, *the set* $\{x \in E : f(x) < \alpha\}$ *is measurable.*

(iv) *For each real number* α, *the set* $\{x \in E : f(x) \leq \alpha\}$ *is measurable.*

(v) *For each extended real number* α, *the set* $\{x \in E : f(x) = \alpha\}$ *is measurable.*

Proof : ***(i)*** $\Rightarrow$ ***(ii)*** Since

$$\{x \in E : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty}\left\{x \in E : f(x) > \alpha - \frac{1}{n}\right\}$$

and the intersection of countable collection of measurable sets is a measurable set, therefore $\{x \in E : f(x) \geq \alpha\}$ is a measurable set.

(ii) $\Rightarrow$ ***(i)*** Since

$$\{x \in E : f(x) > \alpha\} = \bigcup_{n=1}^{\infty}\left\{x \in E : f(x) \geq \alpha + \frac{1}{n}\right\}$$

and the countable union of measurable sets is a measurable set, therefore $\{x \in E : f(x) > \alpha\}$ is a measurable set.

(ii) $\Rightarrow$ ***(iii)*** Since

$$\{x \in E : f(x) < \alpha\} = E - \{x \in E : f(x) \geq \alpha\}$$

and $$\{x \in E : f(x) \geq \alpha\} = E - \{x \in E : f(x) < \alpha\}$$

and since the deference of two measurable sets is a measurable set, therefore, if the set on one of the sides is measurable, then the set on the other side is measurable.

Similarly we can show that *(i)* $\Rightarrow$ *(iv)*

(iii) $\Rightarrow$ ***(iv)*** Since

$$\{x \in E : f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty}\left\{x \in E : f(x) < \alpha + \frac{1}{n}\right\}$$

which is a countable intersection of measurable set, therefore $\{x \in E : f(x) \leq \alpha\}$ is a measurable.

(iv)* ⇒ *(iii) Since

$$\{x \in E : f(x) < \alpha\} = \bigcup_{n=1}^{\infty}\left\{x \in E : f(x) \le \alpha - \frac{1}{n}\right\}$$

is a countable union of measurable set, therefore $\{x \in E : f(x) < \alpha\}$ is a measurable set.

Thus all the first four statements are equivalent.

Now if α is any real number then

$$\{x \in E : f(x) = \alpha\} = \{x \in E : f(x) \ge \alpha\} \cap \{x \in E : f(x) \le \alpha\}$$

Thus if *(ii)* and *(iv)* hold true then *(v)* holds *i.e. (ii)* and *(iv)* ⇒ *(v)* for any real number α. Since

$$\{x \in E : f(x) = \infty\} = \bigcap_{n=1}^{\infty}\{x \in E : f(x) \ge n\}$$

which is a countable intersection of measurable sets if *(ii)* holds true. Thus *(ii)* ⇒ *(iv)* for $\alpha = \infty$. Similarly *(iv)* ⇒ *(v)* for $\alpha = -\infty$, and we have *(ii)* and *(iv)* ⇒ *(v)* for each extended real number α.

An extended real valued function f is said to be Lebesgue measurable or simply measurable on a measurable set E (of finite or infinite measure) if it satisfies any one of the first four statements of theorem 1.

Thus if one restricts himself to measurable functions, then the most important sets, connected with them are measurable.

We now give a formal and most commonly used definition of measurable function.

3.2.1 Definition of measurable function :

An extended real valued function f on a measurable set E is said to be Lebesgue measurable or more precisely measurable on E, if the set $\{x \in E : f(x) > \alpha\}$ is a measurable set for every real number α. The measure of the set $\{x \in E : f(x) > \alpha\}$ may be finite or infinite.

3.3 Algebra of measurable functions

In this section we shall show that the class of measurable functions is closed under the algebraic operations namely addition, subtraction, multiplication and division.

Theorem 2. *Let f and g be measurable functions defined on a measurable set E, and c be a constant. Then the functions $f \pm c$, cf, $-f$, $f \pm g$, $|f|$, f^2, $f \cdot g$ are measurable. Further if* $g(x) \neq 0$ *for each $x \in E$, then* $\frac{1}{g}$ and $\frac{f}{g}$ *are also measurable.*

Proof : Let α be any arbitrary real number. Since f is a measurable function and

$$\{x \in E : f(x) \pm c > \alpha\} = x\ \{x : f(x) > \alpha \pm c\},$$

therefore the function $f \pm c$ is measurable.

Let $c \neq 0$ (since if $c = 0$, then $cf = 0$, *a* constant function that is always measurable). Then

$$\{x \in E : (cf)(x) > \alpha\} = \{x \in E : cf(x) > \alpha\}$$

$$= \left\{x \in E : f(x) > \frac{\alpha}{c};\ c > 0\right\}$$

or

$$= \left\{x \in E : f(x) < \frac{\alpha}{c};\ c < 0\right\}$$

in both cases the set $\{x \in E : (cf)(x) > \alpha\}$ is measurable. Hence cf is a measurable function.

In case when $c = -1$, then cf is measurable implies that $-f$ is measurable.

In order to show that $f \pm g$ are measurable, we shall first show that if f and g are measurable functions, then the set $\{x \in E : f(x) > g(x)\}$ is a measurable set.

Now $f > g$ implies that there exists a rational number $r \in Q$ such that $f(x) > r > g(x)$ for each $x \in E$.

Thus $\{x \in E : f(x) > g(x)\} = \bigcup_{r \in Q} \left[\{x \in E : f(x) > r\} \cap \{x \in E : g(x) < r\}\right]$

= countable union of measurable set

= a measurable set

Therefore, the set $\{x \in E : f(x) > g(x)\}$ is a measurable set. Now for any real number α, $\{x \in E : (f + g)(x) > \alpha\} = \{x \in E : f(x) + g(x) > \alpha\} = \{x \in E : f(x) > \alpha - g(x)\}$.

Now g is measurable $\Rightarrow -g$ is measurable

$\Rightarrow \alpha + (-g)$ is measurable for all $\alpha \in R$

$\Rightarrow \alpha - g$ is measurable

Also f is measurable, therefore the set $\{x \in E : f(x) > \alpha - g(x)\}$ must be a measurable set, which implies that the set $\{x \in E : (f + g)(x) > \alpha\}$ is measurable, that is, $f + g$ is a measurable function.

Now f and g are measurable, so f and $-g$ are measurable. Thus $f + (-g)$, that is, $f - g$ is measurable. Hence $f \pm g$ are measurable functions.

For any real number α, we have

$$\{x \in E : |f|(x) > \alpha\} = \begin{cases} E & \text{if } \alpha < 0 \\ \{x \in E : f(x) > \alpha\} \cup \{x \in E : f(x) < -\alpha\} & \text{if } \alpha \geq 0 \end{cases}$$

Now since E is a measurable set and union of two measurable sets is also measurable, therefore, the set $\{x \in E : |f|(x) > \alpha\}$ is a measurable set. Thus $|f|$ is a measurable function on E.

To show that f^2 is a measurable function we proceed as follows :

for every real number α

$$\{x \in E : f^2(x) > \alpha\} = \begin{cases} E & \text{if } \alpha < 0 \\ \{x \in E : |f|(x) > \sqrt{\alpha}; & \text{if } \alpha \geq 0 \end{cases}$$

But $\left\{x \in E : |f|(x) > \sqrt{\alpha}\right\} = \left\{x \in E : f(x) > \sqrt{\alpha}\right\} \cup \left\{x \in E : f(x) < -\sqrt{\alpha}\right\}$, therefore, $\{x \in E : |f|(x) > \sqrt{\alpha}\}$ is a measurable set (being the union of two measurable sets). Also E is measurable, therefore the set $\{x \in E : f^2(x) > \alpha\}$ is measurable. Hence f^2 is measurable.

Now, f and g are measurable on E

$\Rightarrow$ $f + g$ and $f - g$ are measurable on E

$\Rightarrow$ $(f + g)^2$ and $(f - g)^2$ are measurable on E

$\Rightarrow$ $(f + g)^2 - (f - g)^2$ is measurable on E

$\Rightarrow$ $\frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$ are measurable on E

$\Rightarrow$ $f \cdot g$ is measurable on E.

Now let $g(x) \neq 0$, for all $x \in E$. Then $\frac{1}{g}$ exists for all $x \in E$. Thus for every real α

$$\left\{x \in E : \left(\frac{1}{g}\right)(x) > \alpha\right\} = \begin{cases} \{x \in E : g(x) > 0\} & \text{if } \alpha = 0 \\ \{x \in E : g(x) > 0\} \cap \left\{x \in E : g(x) < \frac{1}{\alpha}\right\} & \text{if } \alpha > 0 \\ \left[\{x \in E : g(x) < 0\} \cap \left\{x \in E : g(x) < \frac{1}{\alpha}\right\}\right] \cup \{x \in E : g(x) > 0\} & \text{if } \alpha < 0 \end{cases}$$

In all the three cases we observe that the set is $\left\{x \in E : \left(\frac{1}{g}\right)(x) > \alpha\right\}$ is measurable. Hence $\frac{1}{g}$ is a measurable function, if g does not vanish for all $x \in E$.

Finally if $g(x) \neq 0$ for all $x \in E$, then g is measurable implies $\frac{1}{g}$ is measurable.

Thus f and $\frac{1}{g}$ are measurable on E

$\Rightarrow$ $f \cdot \frac{1}{g}$ is measurable on E

$\Rightarrow$ $\frac{f}{g}$ is measurable on E.

Theorem 3. *If f and g are measurable function on a measurable set E, then the set $\{x \in E : f(x) > g(x)\}$ is a measurable set.*

Proof : For each rational number $r_i \in Q$, we define

$$A_i = \{x \in E : f(x) > r_i > g(x)\}; \quad i \in N$$

or $$A_i = \{x \in E : f(x) > r_i\} \cap \{x \in E : g(x) < r_i\}$$

Then A_i is the intersection of two measurable sets and so is a measurable set for each $i \in N$

Now $$\{x \in E : f(x) > g(x)\} = \bigcup_{i=1}^{\infty} A_i$$

= Countable union of measurable sets

= a measurable set

Hence $\{x \in E : f(x) > g(x)\}$ is a measurable set.

***Ex.*1.** *Show that a function f on a set E is measurable if and only if for any rational number $r \in Q$, the set $\{x \in E : f(x) < r\}$ is measurable.*

Sol. Firstly, let the function f be a measurable function on E. Then for any arbitrary real number α, the set $\{x \in E : f(x) < \alpha\}$ is a measurable set. But since $Q \subset R$, therefore $r \in Q \Rightarrow r \in R$, therefore $\{x \in E : f(x) < r\}$ is a measurable set for all $r \in Q$.

Conversely, let $\{x \in E : f(x) < r\}$ be a measurable set for all rational numbers $r \in Q$. Set α be any real number, then

$$\{x \in E : f(x) < \alpha\} = \bigcup_{r \in Q} \{x \in E : f(x) < r < \alpha\}$$

$$= \bigcup_{r < \alpha} \{x \in E : f(x) < r; r \in Q\}$$

= countable union of measurable set

= a measurable set

Thus the set $\{x \in E : f(x) < \alpha\}$ is a measurable set. Consequently f is measurable on E.

***Ex.*2.** *Let f be a measurable function on the measurable set E_n, for all $n \in N$. Then f is measurable on the set E, where $E = \bigcup_n E_n$.*

Sol. Let α be any arbitrary real number. Since f is measurable on E_n for all $n \in N$, therefore, $\{x \in E_n : f(x) > \alpha\}$ is a measurable set, for all $n \in N$.

Now let $E = \bigcup_n E_n$. Then

$$\{x \in E : f(x) > \alpha\} = \{x \in \bigcup_n E_n : f(x) > \alpha\}$$

$$= \bigcup_n \{x \in E_n : f(x) > \alpha\}$$

= Countable union of measurable sets

= a measurable set.

Thus $\{x \in E : f(x) > \alpha\}$ is measurable.

Hence f is measurable on $E = \bigcup_n E_n$.

***Ex.*3.** *Show that every constant function on a measurable set E is a measurable function on E.*

Sol. Let $f(x) = c$, for all $x \in E$ be a constant function on the measurable set E and α be any arbitrary real number. Then it is evident that

$$\{x \in E : f(x) > \alpha\} = \begin{cases} E, & \text{if } c > \alpha \\ \phi, & \text{if } c \leq \alpha \end{cases}$$

In both the cases, the $\{x \in E : f(x) > \alpha\}$ is a measurable set as E and ϕ both are measurable. Hence f is a measurable function on E.

***Ex.*4.** *Show that every function defined on a set E with measure zero is a measurable function.*

Sol. Let f be a function defined on a measurable set E, where $m(E) = 0$.

Let α be any real number. Then

$$\{x \in E : f(x) > \alpha\} \subset E$$

$\Rightarrow$ $m\{x \in E : f(x) > \alpha\} \leq m(E) = 0$

$\Rightarrow$ $m\{x \in E : f(x) > \alpha\} = 0$

$\Rightarrow$ $\{x \in E : f(x) > \alpha\}$ is measurable for all real number α.

$\Rightarrow$ f is measurable on E.

***Ex.*5.** *Prove that the characteristic function ϕ_A of a set A is measurable if and only if A is a measurable set.*

Sol. We know that the characteristic function ϕ_A of a set $A \subset E$ is defined by

$$\phi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in E - A \end{cases}$$

Let E be any set such that $A \subset E$. Then by definition of ϕ_A we have $A = \{x \in E : \phi_A(x) > 0\}$. Let ϕ_A be measurable. Then $\{x \in E : \phi_A(x) > 0\}$ is measurable *i.e.*, A is a measurable set.

Conversely, let A be a measurable set and α be any real number. Then

$$\{x \in E : f(x) > \alpha\} = \begin{cases} \phi, & \text{if } \alpha \geq 1 \\ A, & \text{if } 0 \leq \alpha < 1 \\ A \cup A^c = E, & \text{if } \alpha < 0 \end{cases}$$

Since every set on the right hand side is a measurable set, it is evident that the set $\{x \in E : \phi_A(x) > \alpha\}$ is measurable. Consequently ϕ_A is a measurable function defined on A.

***Ex.*6.** *Give an example to show that the function $|f|$ is measurable but f is not measurable.*

Sol. Let P be a nonmeasurable subset of $E = [0, 1[$. Let us define a function $f : E \to R$, where

$$f(x) = \begin{cases} 1, & \text{if } x \in P \\ -1, & \text{if } x \notin P \end{cases}$$

Then the set $\{x \in E : f(x) > 0\} = P$ is not measurable. Therefore f is not a measurable function on E. However $|f|$ is measurable on E, since the set

$$\{x \in E : |f(x)| > \alpha\} = \begin{cases} E, & \text{if } \alpha < 1 \\ \phi, & \text{if } \alpha \geq 1 \end{cases}$$

is a measurable set (as both the sets on the right hand side are measurable).

Thus $|f|$ is a measurable function on the set $E = [0, 1[$ where as f is not measurable on E.

***Ex.*7.** *Show that a function f is measurable on a measurable set E if and only if its positive part* f^+ *and negative part* f^- *are measurable.*

Sol. For every extended real valued function f, we define the positive part f^+ and the negative part f^- as follows :

$$f^+ = \frac{1}{2}[|f| + f]$$

$$f^- = \frac{1}{2}[|f| - f]$$

Since f is measurable on the measurable set E, therefore $|f|$ is also measurable on E. Consequently, from the definitions of f^+ and f^-, we observe that both are measurable on E.

Conversely, let f^+ and f^- be measurable on E. Then $f = f^+ - f^-$ is also measurable on E.

***Ex.*8.** *If f is a measurable function on the measurable set E, then prove that for every extended real number* α*, the set* $\{x \in E : f(x) = \alpha\}$ *is a measurable set but not conversely.*

Sol. Let f be a measurable function on the measurable set E. We first consider the case $\alpha < \infty$. Then

$$\{x \in E : f(x) = \alpha\} = \{x \in E : f(x) \geq \alpha\} \cap \{x \in E : f(x) \leq \alpha\}$$

= intersection of two measurable sets (as f is measurable)

= a measurable set.

Now we consider the case when $\alpha = \infty$. Then,

$$\{x \in E : f(x) = \alpha\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) > n\}$$

= a countable intersection of measurable sets

= a measurable set.

Similarly we can prove the result when $\alpha = -\infty$ (just replace n by $-n$)

Conversely, let us take a nonmeasurable subset A of R and define function f as :

$$f(x) = \begin{cases} x^2, & \text{if } x \in A \\ -x^2, & \text{if } x \in R - A \end{cases}$$

We see that the set $\{x \in E : f(x) = \alpha\}$ consists of exactly two elements for each real number α and so is a measurable set. But the set

$\{x \in R : f(x) > 0\} = A - \{0\}$, which is a nonmeasurable set. Hence f is not a measurable function on R.

Ex.9. *Let f be a measurable function on a measurable set D and the function g be defined by*

$$g(x) = \begin{cases} f(x), & \text{if } \ x \in D \\ 0, & \text{if } \ x \notin D \end{cases}$$

Then show that f is measurable if and only if g is measurable.

Sol. First of all, let f be measurable on D. Then for each $\alpha \in R$

$$\begin{aligned} \{x \in E : g(x) > \alpha\} &= \{x \in D : f(x) > \alpha\}, \qquad \text{if } \ \alpha \geq 0 \\ &= \{x \in D : f(x) > \alpha\} \cup D^c, \ \text{if } \ \alpha < 0 \end{aligned}$$

Since D is a measurable set, therefore D^c is also a measurable set. Also f is measurable on D, so is $\{x \in D : f(x) > \alpha\}$. Thus $\{x \in D : f(x) > \alpha\} \cup D^c$ is a measurable set. Consequently g is measurable on E.

Conversely, let the function g be measurable on E. Then

$$\{x \in D : f(x) > \alpha\} = \{x \in D : g(x) > \alpha\},$$

which is a measurable set as g is a measurable function on E. Hence f is also measurable on D.

Theorem 4. *A continuous function defined on a measurable set is always measurable. However its converse is not always true.*

Proof : Let f be a continuous function defined on a measurable set E. We consider the following set :

$A = \{x \in E : f(x) \geq \alpha\}$, where α is any arbitrary real number. We shall show that the above set A is a closed set, *i.e.,* $D(A) \subset A$ [$D(A)$ being the derived set of A]. Let $x_0 \in D(A)$ (*i.e.* x_0 be a limit point of A). Then for every neighbourhood G of x_0, we must have

$$[G - \{x_0\}] \cap A \neq \phi$$

Thus, if $x \in [G - \{x_0\}] \cap A$, then

$$\begin{aligned} x \in [G - \{x_0\}] \cap A &\Rightarrow x \in G, x \neq x_0 \ \text{ and } \ x \in A \\ &\Rightarrow f(x) \geq \alpha \end{aligned}$$

Thus, $$x \in G, x \neq x_0 \Rightarrow f(x) \geq \alpha$$

But f is given to be a continuous function, therefore,

$$\begin{aligned} f(x) \geq \alpha &\Rightarrow f(x_0) \geq \alpha \\ &\Rightarrow x_0 \in A \end{aligned}$$

Therefore, $$\begin{aligned} x_0 \in D(A) &\Rightarrow x_0 \in A \\ &\Rightarrow D(A) \subset A \end{aligned}$$

that is, every limit point of A, is in A, that is, A is a closed set.

But every closed set is measurable, therefore A is a measurable set.

i.e. $A = \{x \in E : f(x) \geq \alpha\}$ is a measurable set

$\Rightarrow f$ is measurable on E.

To prove that the converse of the theorem is not always true, we consider the function f on the interval [0, 2] defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1] = B \text{ (say)} \\ 2, & \text{if } x \in]1,2] = B^c \end{cases}$$

Then $\{x \in [0, 2] : f(x) > \alpha\} = \begin{cases} \phi, & \text{if } \alpha \geq 2 \\ B^c, & \text{if } 1 \leq \alpha < 2 \\ [0,2], & \text{if } \alpha < 1 \end{cases}$

Since the sets ϕ, B^c and [0, 2] all are measurable, it is clear that f is measurable on [0, 2], but it is not continuous in [0, 2] as $x = 1$ is a point of discontinuity.

Theorem 5. *A function f defined on a measurable set E is measurable if and only if for any open set $G \subset R$, $f^{-1}(G)$ is a measurable set.*

Proof : Let f be a function defined on a measurable set E and G be an open subset of the set R of real numbers.

First of all let f be measurable on E. We know that every open subset G of R can be expressed as a countable union of disjoint open intervals, so let

$$G = \bigcup_{n=1}^{\infty} I_n, \quad \text{where } I_n =]\, a_n, b_n [$$

then $$f^{-1}(G) = \bigcup_{n=1}^{\infty} \{x \in E : f(x) \in I_n\}$$

But $$f(x) \in I_n \Rightarrow f(x) \in]\, a_n, b_n [$$
$$\Rightarrow a_n < f(x) < b_n$$

Therefore, $$f^{-1}(G) = \bigcup_{n=1}^{\infty} \{x \in E : f(x) > a_n\} \cap \{x \in E : f(x) < b_n\}$$

Now f is measurable on E, so both sets $\{x \in E : f(x) > a_n\}$ and $\{x \in E : f(x) < b_n\}$ are measurable. Since the intersection of two measurable sets is measurable, also the countable union of measurable sets is a measurable, therefore, $f^{-1}(G)$ is a measurable set.

Conversely, let $f^{-1}(G)$ be a measurable set. We take the open set G as $G =]\, a, \infty [$, $a > 0$

Then $$f^{-1}(G) = \{x \in E : f(x) \in]\, a, \infty [\,\}$$
$$= \{x \in E : a < f(x) < \infty\}$$
$$= \{x \in E : f(x) > a\}$$

Now since $f^{-1}(G)$ is a measurable set, therefore, $\{x \in E : f(x) > a\}$ is measurable, which shows that f is a measurable function on E.

Theorem 6. *If f and g are real valued function defined and measurable on the set R, then the real valued function*

$$h(x) = F(f(x), g(x)), x \in R,$$

which is defined and continuous on the Euclidean space R^2, is measurable.

Proof : Let α be any arbitrary real number. Consider the set

$$G_\alpha = \{(p, q) : F(p, q) > \alpha\}$$

Then G_α is an open set in R^2 and so can be expressed as a countable union of open intervals as follows :

$$G_\alpha = \bigcup_n I_n$$

where $I_n = \{(p, q); p \in]a_n, b_n[, q \in]c_n, d_n[\}$ $a_n, b_n, c_n\, d_n \in R$, and $n \in N$

Now since f is a measurable function on R therefore, the sets $\{x \in R : f(x) > a_n\}$ and $\{x \in R : f(x) < b_n\}$ are measurable. Consequently the set

$$\{x \in R : f(x) \in]a_n, b_n[\} = \{x \in R : f(x) > a_n\} \cap \{x \in R : f(x) < b_n\}$$

is a measurable set (being the intersection of two measurable sets).

Similarly the set $\{x \in R : g(x) \in]c_n, d_n[\}$ is a measurable set. This implies that the set

$$\{x \in R : F(f(x), g(x)) \in I_n\} = \{x \in R : f(x) \in]a_n, b_n[\} \cap \{x \in R : g(x) \in]c_n, d_n[\}$$

is a measurable set.

Further we note that

$$\{x \in R : h(x) > \alpha\} = \{x \in R : (f(x), g(x)) \in G_\alpha$$
$$= \bigcup_n \{x \in R : (f(x), g(x)) \in I_n\}$$

which is a countable union of measurable sets and so is measurable.

Thus $\{x \in R : h(x) > \alpha\}$ is a measurable set. Hence h is a measurable function on R^2.

Theorem 7. *If g is a measurable function an a set E and f is a continuous function defined on the range of g, then the composite function fog is a measurable function on E.*

Proof : Let G be an open set and g be a measurable function on the set E. Let

$$A = \{x \in R : g(x) \in G\}.$$

Then we shall first prove that A is a measurable set,

Since G is an open set, therefore

$$G = \bigcup_n I_n \quad \text{where} \quad I_n =]a_n, b_n[$$

Thus $$A = \{x \in E : g(x) \in G\} = \bigcup_n \{x \in E : g(x) \in I_n\}$$

or $$A = \bigcup_n \{x \in E : g(x) > a_n\} \cap \{x \in E : g(x) < b_n\}$$

= Countable union of measurable sets

= a measurable set .....(1)

Now for any arbitrary number α

$$\{x \in E : (fog)(x) > \alpha\} = \{x \in E : g(x) \in H\}$$

where $H = \{y : f(y) > \alpha\}$

Now since f is a continuous function, therefore H is an open set. Consequently the set $\{x \in E : g(x) \in H\}$ is measurable (by (1)).

Hence $\{x \in E : (fog)(x) > \alpha\}$ is measurable. This proves that fog is measurable on E.

3.4 Borel measurable functions

A function f defined on a Borel set E is said to be a **Borel measurable function** or simply a **Borel function** on E if for every real number α, the $\{x \in E : f(x) > \alpha\}$ is a Borel set.

Since we know that every Borel set is measurable, therefore if $\{x \in E : f(x) > \alpha\}$ is a Borel set, then it is a measurable function on E. Thus we note that every Borel function is Lebesgue measurable. But every measurable function need not be a Borel function. For example the characteristic function of a set, which is Lebesgue measurable but a non-Borel set, is Lebesgue measurable but not a Borel function

It can be proved that

(i) A continuous function defined on a Borel set is Borel measurable function, but the converse is not always true.

(ii) If f is a Borel measurable function and B is a Borel set, then $f^{-1}(B)$ is a Borel set.

(iii) If f and g are Borel measurable functions, then their composite function fog is also Borel measurable, since if we take any arbitrary real number α, then

$$\{x \in E : (fog)(x) > \alpha\} = \{x : f(g(x)) > \alpha\}$$

$$= \{x : g(x) \in A\}$$

$$= g^{-1}(A)$$

where $A = \{u : f(u) > \alpha\}$.

The set A is a Borel set since f is a Borel function. Therefore $g^{-1}(A)$ is a Borel set (using *(ii)* above). Hence $\{x : (fog)(x) > \alpha\}$ is a Borel set. Consequently fog is a Borel function.

3.5 Almost everywhere property

A property P is said to hold **almost everywhere** (*a.e.*) on the set S, if the set of points, where it fails to hold good is a set of measure zero. Thus in particular we say that $f = g$ *a.e.* if the function f and g have the same domain and $m(\{x : f(x) \neq g(x)\}) = 0$. Similarly we say that the sequence $<f_n>$ of functions converges to the function f *a.e.* if there is a set E of measure zero, such that $<f_n>$ converges to g for each x not in E.

Theorem 8. *Suppose that f and g are two functions defined on the common domain E and f is measurable on E. If f = g a.e. on E, then g is also a measurable function on E.*

Proof : Let $E_1 = \{x \in E : f(x) = g(x)\}$

and $E_2 = \{x \in E : f(x) \neq g(x)\}$

then $E = E_1 \cup E_2$ and $m(E_2) = 0$

Now $m(E_2) = 0 \Rightarrow E_2$ is a measurable set.

Also since f is measurable on E, therefore E is measurable. Consequently $E_1 = E - E_2$ is measurable.

But f is measurable on E and $E_1 \subset E$, therefore f is measurable on E_1

$\Rightarrow$ $\{x \in E_1 : f(x) > \alpha\}$ is a measurable set for $\alpha \in R$

$\Rightarrow$ $\{x \in E_1 : g(x) > \alpha\}$ is a measurable set, since $f = g$ on E_1.

Now, $\{x \in E_2 : g(x) > \alpha\} \subset E_2$ and $m(E_2) = 0$, therefore, the set

$$\{x \in E_2 : g(x) > \alpha\}$$

is a measurable set.

Now, $\{x \in E : g(x) > \alpha\} = \{x \in E_1 : g(x) > \alpha\} \cup \{x \in E_2 : g(x) > \alpha\}$

$(\because E = E_1 \cup E_2)$

Both the set on the right hand side are measurable and the union of two measurable sets is a measurable set, therefore $\{x \in E : g(x) > \alpha\}$ is a measurable set. Hence g is a measurable function on E.

The sets of measure zero are just unimportant in the theory of Lebesgue measure. Since the behaviour of measurable functions on the sets of measure zero is of very less meaningful, it becomes necessary to introduce the following generalization of the ordinary concept of convergence of sequences of functions.

3.5.1 Almost everywhere convergence :

A sequence $<f_n>$ of functions defined on a set E is said to converge *a.e.* to a function defined on E if $\lim_{n\to\infty} f_n(x) = f(x)$, for all points $x \in E - E_1$, where $E_1 \subset E$ and $m(E_1) = 0$.

For example if we consider the sequence $<f_n>$ of functions on [0, 1], where

$$f_n(x) = (-1)^n x^n, \quad \text{for all } x \in [0, 1], \text{ then}$$

$<f_n>$ converges *a.e.* to the function $f \equiv 0$ (zero function), everywhere except at $x = 1$.

Self-learning exercise-1

1. If $|f|$ is a measurable function on a measurable set E, then f ....
2. The function $1/g$ is measurable on the set E only if ....
3. If f is measurable on a measurable set E, then whether f is continuous or not on E ?
4. For each extended real number α if the set $\{x \in E : f(x) = \alpha\}$ is a measurable set then f is measurable on E; true or false ?
5. If f is a measurable function on the measurable set E, then for any positive integer k, f^k is measurable or not ?
6. If f and g are two functions on a common domain E such that $f = g$ *a.e.* then f is measurable if ....
7. If the function f defined on the measurable set E is continuous *a.e.* on E, then ....

Theorem 9. (*E. Borel*) *Let f be a measurable function, finite almost everywhere defined on the closed interval E =* $[a, b]$. *Then for all* $\sigma > 0$ *and* $\in > 0$, *there exists a continuous function* ϕ *defined on E such that*

$$(\{x \in E : |f(x) - \phi(x)| \geq \sigma\}) < \in.$$

Proof : Let f be a bounded function. Then there exists a positive number K, such that

$$|f(x)| \leq K \quad \text{for all} \quad x \in E = [a, b].$$

Now let σ and $\in$ be any two arbitrary positive number. We choose a positive integer m, which is so large that $\dfrac{K}{m} < \sigma$

Let $$E_r = \left\{x \in E : \frac{r-1}{m} \cdot K \leq f(x) \leq \frac{r}{K} \cdot K\right\}$$

where $$r = 1-m, 2-m, \ldots, m-1$$

and $$E_m = \left\{x \in E : \frac{m-1}{m} K \leq f(x) \leq K\right\}.$$

Then all these sets are pairwise disjoint and

$$E = [a, b] = \bigcup_{r=1-m}^{m} E_r.$$

Now for each r, we choose a closed set $B_r \subset E_r$, such that

$$m(E_r) < m(B_r) + \frac{\in}{2m} \qquad .....(1)$$

and, let $$B = \bigcup_{r=1-m}^{m} B_r.$$

Then $$E - B = \bigcup_{r=1-m}^{m} (E_r - B_r)$$

therefore $$m(E-B) = m\left(\bigcup_{r=1-m}^{m} (E_r - B_r)\right)$$

or $$m(E) - m(E) = \sum_{r=1-m}^{m} m(E_r - B_r)$$

$$< \sum_{r=1-m}^{m} \frac{\in}{2m} = \frac{\in}{2m} \cdot m = \frac{\in}{2} < \in$$

Thus $$m(E) - m(B) < \in \qquad .....(2)$$

We now define a function ψ on the set B, where

$$\psi(x) = \frac{r}{m} K \quad \text{for all} \quad x \in B_r;\ r = 1-m, 2-m, \ldots, m.$$

Then ψ is a constant function on each B_r and since $B_i \cap B_j = \phi$ for $i \neq j$, therefore ψ is continuous on B.

Again $$|\psi(x)| = \left|\frac{r}{m} K\right| = \frac{r}{m} K \le K$$

therefore, $$|f(x) - \psi(x)| = \left|K - \frac{r}{m} K\right| = \frac{K}{m} |m - r| \le \frac{K}{m} < \sigma$$

or $$|f(x) - \psi(x)| < \sigma; \quad \text{for all} \quad x \in B$$

We now use the following lemma :

If B is a closed set contained in $[a, b]$ and if ψ is defined on B which is continuous on B, then there exists a function ϕ on $[a, b]$ such that :

(i) ϕ is continuous

(ii) $\phi(x) = \psi(x)$ for all $x \in B$ and

(iii) $\max |\phi(x)| = \max |\psi(x)|$

Using the above lemma, we can define a function ϕ on $E = [a, b]$ having all these properties.

Further, $\{x \in E : |f(x) - \phi(x)| \ge \sigma\} \subset E - B$

therefore, $m(\{x \in E : |f(x) - \phi(x)| \ge \sigma\}) \le m(E) - m(B)$

or $m(\{x \in E : |f(x) - \phi(x)| \ge \sigma\}) < \in$. [From (2)]

3.6 Supremum and infimum of a sequence

Let f_1 and f_2 be any two real valued functions defined on the common domain E. Then $f^* = \max(f_1, f_2)$ and $f_* = \min(f_1, f_2)$ are the real valued functions on E, whose values at any point $x \in E$ are :

$$f^*(x) = \max(f_1(x), f_2(x))$$

and $$f_*(x) = \min(f_1(x), f_2(x)) \text{ respectively.}$$

It can be observed that if f_1 and f_2 are measurable functions on the set E, then f^* and f_* are also measurable, for if, α is only arbitrary real number, then

$$\{x \in E : f^*(x) > \alpha\} = \{x \in E : f_1(x) > \alpha\} \cup \{x \in E : f_2(x) > \alpha\}$$

and $$\{x \in E : f_*(x) > \alpha\} = \{x \in E : f_1(x) > \alpha\} \cap \{x \in E : f_2(x) > \alpha\}$$

Since f_1 and f_2 are measurable functions, therefore, the sets $\{x \in E : f_1(x) > \alpha\}$ and $\{x \in R : f_2(x) > \alpha\}$ are measurable sets. Also since union and intersection of two measurable sets are measurable. This shows that f^* and f_* both are measurable functions on E.

If $<f_i>$ is a finite sequence of n functions defined on a common domain E, then the function

$$f^* = \max(f_1, f_2, \ldots, f_n)$$

and $$f_* = \min(f_1, f_2, \ldots, f_n)$$

are the functions on the same domain E. In case when the sequence $<f_n>$ is an infinite sequence, $\sup_n <f_n>$ and $\inf_n <f_n>$ can be defined likewise. For any $x \in E$, the supremum of $\{f_1(x), f_2(x), \ldots\}$ is denoted by $\sup_n <f_n>$. We also denote by $\lim \sup_n <f_n>$, the function whose value at $x \in E$ is $\lim \sup_n <f_n(x)>$. In a similar way we can define $\inf_n <f_n>$ and $\lim \inf_n <f_n>$.

It can be observed that

$$\lim \sup_n <f_n> = \inf_n \left(\sup_{k \geq n} <f_k> \right)$$

$$\inf_n <f_n> = -\sup_n (<-f_n>)$$

and $$\lim \inf_n <f_n> = -\lim \sup_n (<-f_n>)$$

$$= \sup_n \left(\inf_{k \geq n} <f_k> \right).$$

Theorem 10. *If $<f_n>$ is a sequence of measurable functions defined on a measurable set E, then $\sup_n <f_n>$ and $\inf_n <f_n>$ are also measurable on E.*

Proof : Since f_n is a measurable function on the measurable set E for each $n \in N$, therefore for any arbitrary real number α, the set $\{x \in E : f_n(x) > \alpha\}$ is measurable subset of E for each $n \in N$.

Now let $$g(x) = \sup_n <f_n(x)>; \qquad x \in E$$

and $$h(x) = \inf_n <f_n(x)>; \qquad x \in E$$

Now, $\{x \in E : g(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x \in E : f_n(x) > \alpha\}$

= Countable union of measurable sets

= a measurable set.

therefore g, *i.e.* $\sup_n < f_n >$ is a measurable function on E.

Now since $h(x) = -\sup_n < -f_n(x) >; \quad x \in E$

therefore, it is clear that h *i.e.* $\inf_n < f_n >$ is also a measurable function on E.

Theorem 11. *Let* $< f_n >$ *be a sequence of measurable functions defined on the measurable set E. Then* $\limsup_n < f_n >$ *and* $\liminf_n < f_n >$ *are also measurable on E.*

Proof : Let us define

$$g_n = \sup_{k \geq n} < f_k >$$

and

$$h_n = \inf_{k \geq n} < f_k >$$

then,

$$\limsup_n < f_n > = \inf_n < g_n >$$

and

$$\liminf_n < f_n > = \sup_n < h_n >$$

We have already proved in the last theorem that g_n and h_n are measurable on the set E. Once again with the help of same theorem $\inf_n < g_n >$ and $\sup_n < h_n >$ *i.e.* $\limsup_n < f_n >$ and $\liminf_n < f_n >$ are measurable on E.

Note : If $< f_n >$ is a sequence of measurable functions defined on the measurable set E and if $\lim f_n$ does exist, then

$$\limsup_n < f_n > = \liminf_n < f_n > = \lim f_n$$

Now since $\limsup_n < f_n >$ and $\liminf_n < f_n >$ are measurable if $< f_n >$ is a sequence of measurable function. Hence $\lim f_n$ is also a measurable function on E.

Theorem 12. *If* $< f_n >$ *is a convergent sequence of measurable functions defined on a measurable set E, then the limit function of* $< f_n >$ *is measurable.*

Proof : Suppose that $< f_n >$ is a convergent sequence of measurable functions defined on the measurable set E, that converges to the limit function f on E. We wish to prove that f is a measurable function on E.

Since $< f_n >$ is convergent, therefore, either $< f_n >$ is a monotonic increasing or a monotonic decreasing sequence.

If $< f_n >$ is a monotonic increasing sequence then we know that

$$\lim f_n = \sup_n < f_n >$$

Similarly if $< f_n >$ is a monotonic decreasing sequence, then

$$\lim f_n = \inf_n < f_n > .$$

Thus in order to prove the theorem we need to prove that $\sup_n < f_n >$ and $\inf_n < f_n >$ are measurable on E.

Now let $\quad g = \sup_n < f_n > \quad$ and $\quad h = \inf_n < f_n >$.

Then for any real number α

$$\{x \in E : g(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x \in E : f_n(x) > \alpha\}$$

= a countable union of measurable sets

= a measurable set.

Hence $\{x \in E : g(x) > \alpha\}$ is a measurable set and so g is measurable on E.

Now $\quad h = \inf_n < f_n > = -\sup_n < -f_n >$

Thus h is also measurable on E as g is measurable.

Therefore both $\sup_n < f_n >$ and $\inf_n < f_n >$ are measurable on E, that is, the limit function f is measurable on E.

Self-learning exercise-2

1. If f_1 and f_2 are real valued measurable on the common domain E, then $f^* = \max(f_1, f_2)$ is also measurable on E. Whether $f_* = \min(f_1, f_2)$ is measurable on E ?

2. Let $< f_n >$ be a sequence of measurable functions defined on a measurable set E. Then $\lim [\sup_n < f_n >]$ is defined by ....

3. Let $< f_n >$ be a sequence of measurable functions defined on the measurable set E. Then $\lim f_n$ is ..... .

4. If $< f_n >$ is a sequence of measurable functions on a measurable set E, then $\lim \sup_n < f_n >$ and $\lim \inf_n < f_n >$ are ..... .

3.7 Convergence of Sequences of Functions

When we speak of the theory of real functions, J.E. Littlewood's third principle about the measurable functions reminds us the following :

Every convergent sequence of (measurable) functions is nearly uniformly convergent. The following theorem gives one version of the third principle due to Littlewood.

Theorem 13. *Let E be a measurable set, with* $m(E) < \infty$, *and* $< f_n >$ *a sequence of measurable functions defined on E. Let f be a (real valued) measurable function such that for each* $x \in E$, $f_n(x) \to f(x)$. *Then for a given* $\in > 0$ *and* $\delta > 0$, *there is measurable set* $A \subset E$, *with* $m(A) < \delta$ *and an integer* n_0 *such that for all* $x \in E - A$ *and all* $n \geq n_0$

$$|f_n(x) - f(x)| < \in.$$

Proof : Let $\quad G_n = \{x \in E : |f_n(x) - f(x)| \geq \in\}$

Since the functions f_n and f are measurable, therefore it is clear that G_n are measurable.

Let
$$E_n = \bigcup_{n=k}^{\infty} G_n$$
$$= \{x : x \in G_n \text{ for some } n \geq k\}$$
$$= \{x \in E : |f_n(x) - f(x)| \geq \in \text{ for some } n \geq k\}.$$

It is clear that $E_{k+1} \subset E_k$ and for each $x \in E$, there must be some set E_k such that $x \notin E_k$, since if we assume that $x \in E_k$ for all k, then for any fixed k, we have

$$|f_n(x) - f(x)| \geq \in \quad \text{for some } n \geq k$$

which contradicts the fact that $f_n(x) \to f(x)$. Thus $< E_k >$ is a decreasing sequence of measurable sets such that $\bigcap_{k=1}^{\infty} E_k = \phi$ and so $\lim_{k \to \infty} m(E_k) = 0$.

Thus for a given positive number δ, there exists a positive integer n_0 such that

$$m(E_k) < \delta \quad \text{for all} \quad k \leq n_0$$

and in particular $m(E_{n_0}) < \delta$.

that is $\quad m\{x \in E : |f_n(x) - f(x)| \geq \in \text{ for some } n \geq n_0\} < \delta$

Now if we take $A = E_{n_0}$, then $m(A) < \delta$

and $\quad E - A = \{x \in E : |f_n(x) - f(x)| < \in, \text{ for all } n \geq n_0\}$

or $\quad |f_n(x) - f(x)| < \in, \quad$ for all $n \geq n_0$

and for all $\quad x \in E - A$.

The following theorem is a little modification of the above theorem :

Theorem 14. *Let E be a measurable set of finite measure* (*i.e. m* (*E*) < ∞) *and* $<f_n>$ *a sequence of measurable functions defined on E that converge to a real valued function f a.e. on E. Then for given* $\in > 0$ *and* $\delta > 0$*, there exists a set* $A \subset E$*, with* $m(A) < \delta$ *and a positive integer* n_0 *such that*

$$|f_n(x) - f(x)| < \in, \quad \text{for all} \quad x \in E - A \quad \text{and for all} \quad n \geq n_0.$$

Proof : If G is the set of all those points x for which $f_n(x) \not\to f(x)$, then clearly $m(G) = 0$ and $f_n(x) \to f(x)$ for all points $x \in E - G$. Let $G_1 = E - G$. Then applying the above theorem, there exists a set $A_1 \subset G_1$, with $m(A_1) < \delta$ and a positive integer n_0 such that

$$|f_n(x) - f(x)| < \in, \quad \text{for all} \quad x \in G_1 - A_1 \quad \text{and for all} \quad n \geq n_0$$

Now since $m(G) = 0$, therefore taking

$A = A_1 \cup G$ and $E - A = G_1 - A_1$, the above becomes

$$|f_n(x) - f(x)| < \in, \quad \text{for all} \quad x \in E - A \quad \text{and for all} \quad n \geq n_0.$$

The condition that $m(E) < \infty$ in the above two theorems is mandatory as can be verified from the following example.

***Ex.*10.** *Show that the condition in the theorem* 14 *that* $m(E) < \infty$, *can not be relaxed.*

Sol. Let $E = \{x : x \geq 0\}$. Then we have $m(E) = \infty$. Let us define a sequence $<f_n>$ of functions $f_n : E \to R$,

where $$f_n(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x \leq n \\ 1 & \text{if} \quad x > n \end{cases}$$

Construction of such functions shows that each f_n is a measurable function for $n \in N$. Thus $<f_n>$ is a sequence of measurable functions defined on E that converges to $f \equiv 0$. If we take $\in = 1$ and $0 < \delta < 1$, then there can not exist any set $A \subset E$ with $m(A) < \delta$ and a positive integer n_0 such that

$$|f_n(x) - f(x)| < \in, \quad \text{for all} \quad x \in E - A_1 \quad \text{and for all} \quad n \geq n_0$$

Hence $m(E)$ has to be finite.

Theorem 15. *Let* $<f_n>$ *be a sequence of measurable functions defined on a measurable set E, that converge pointwise to a function f defined on E. Then f is a measurable function.*

Proof : Let α be any real number and $m \in N$. Let us define the sets

$$W_m^{(k)} = \left\{ x \in E : f_k(x) > \alpha + \frac{1}{m} \right\}$$

and $$V_m^{(n)} = \bigcap_{k=n}^{\infty} W_m^{(k)}; \quad k, n \in N$$

Since f_k is a measurable function for all $k \in N$ and $\alpha + (1/m)$ is a real number, therefore each $W_m^{(k)}$ is a measurable set and so is $V_m^{(n)}$.

To prove the theorem, we shall show that $\{x \in E : f(x) > \alpha\} = \bigcup_{m,n} V_m^{(n)}$.

Let $y \in \{x \in E : f(x) > \alpha\}$. Then $f(y) > \alpha$ and as such we can always find a natural number m such that $f(y) > \alpha + \frac{1}{m}$.

Now since $\lim_{k\to\infty} f_k(y) = f(y)$,

therefore, there exists a positive integer n_0 such that

$$f_k(y) > \alpha + \frac{1}{m} \quad \text{for all} \quad k \geq n_0.$$

This implies that $\quad y \in W_m^{(k)} \quad$ for all $\quad k \geq n_0$

or $\quad y \in V_m^{(n)} \qquad \left(\text{since } V_m^{(n)} = \bigcap_{k=n}^{\infty} W_m^{(k)}\right)$

or $\quad y \in \bigcup_{n,m} V_m^{(n)}$

Thus $\quad y \in \{x \in E : f(x) > \alpha\} \Rightarrow y \in \bigcup_{n,m} V_m^{(n)}$

or $\quad \{x \in E : f(x) > \alpha\} \subset \bigcup_{n,m} V_m^{(n)} \qquad$ .....(1)

Now let $\quad y \in \bigcup_{n,m} V_m^{(n)}$.

Then $\quad y \in V_m^{(n)} \quad$ for some $\quad m, n \in N$

$\Rightarrow y \in W_m^{(k)} \quad$ for all $\quad k \geq n \qquad \left(\text{since } v_m^{(n)} \bigcap_{k=n}^{\infty} W_m^{(k)}\right)$

Thus $\quad f_k(y) > \alpha + \frac{1}{m} \quad$ for all $\quad k \geq n$

or $\quad f(y) = \lim_{k\to\infty} f_k(y) \geq \alpha + \frac{1}{m}$

and so $\quad f(y) \geq \alpha + \frac{1}{m}$

i.e. $\quad f(y) > \alpha$

or $\quad y \in \{x\ E : f(x) > \alpha\}$

Thus $\quad \bigcup_{n,m} V_m^{(n)} \subset \{x \in E : f(x) > \alpha\} \qquad$ .....(2)

from (1) and (2), we have

$$\{x \in E : f(x) > \alpha\} = \bigcup_{n,m} V_m^{(n)}$$

which shows that f is measurable.

Theorem 16. (*Lebesgue*) *$<f_n>$ be a sequence of measurable functions finite a.e. on a set E. Let $f_n(x) \to f(x)$ a.e. on E, and f be finite a.e. on E. Then for each $\in > 0$*

$$\lim_{n\to\infty}\left[m\left(\{x \in E : |f_n(x) - f(x)| \geq \in\}\right)\right] = 0.$$

Proof : Since $<f_n>$ converges to f *a.e.* on E, therefore f is a measurable function on E.

Now let, $A = \{x \in E : |f(x)| = \infty\}$

$A_n = \{x \in E : |f_n(x)| = \infty\}$

and $B = \left\{x \in E : \lim_{n\to\infty} f_n(x) = f(x)\right\}.$

Then $m(A) = m(A_n) = m(B) = 0$

therefore, if we assume $F = A \cup \left\{\bigcap_{n=1}^{\infty} A_n\right\} \cup B,$

then $m(F) = 0$

Now for $\in > 0$, let $E_k(\in) = \{x \in E : |f_k(x) - f(x)| \geq \in\}$

$$W_n(\in) = \bigcap_{k=n}^{\infty} E_k(\in)$$

and

$$V = \bigcap_{n=1}^{\infty} W_n(\in)$$

Then clearly all these sets are measurable, as each E_k is a measurable set. We also see that $W_1(\in) \supset W_2(\in) \supset \ldots$, that is, $< W_n(\in) >$ is a monotonically decreasing sequence of measurable sets, therefore

$$m\left(\bigcap_{n=1}^{\infty} W_n(\in)\right) = \lim_{n\to\infty} m(W_n(\in))$$

or $$m(V) = \lim_{n\to\infty} m(W_n(\in)) \qquad .....(1)$$

We shall prove that $V \subset F$

Let $x \notin F$, then $x \notin A$, $x \notin A_n$ and $x \notin B$.

This implies that $f(x) < \infty$, $f_k(x) < \infty$ for each $k \in N$ and also $\lim_{n\to\infty} f_n(x) = f(x)$.

Thus for given $\in > 0$, there exists a positive integer n_0 such that

$$|f_k(x) - f(x)| < \in \qquad \text{for all} \quad k \geq n_0$$

$$\Rightarrow x \notin E_k(\in) \quad \text{for} \quad k \geq n_0$$

$$\Rightarrow x \notin W_k(\in) \quad \text{for} \quad k \geq n_0$$

$$\Rightarrow x \notin V$$

Thus $\qquad x \notin F \Rightarrow x \notin V$

$$\Rightarrow V \subset F$$

$$\Rightarrow m(V) \leq m(F)$$

$$\Rightarrow m(V) = 0 \qquad \text{(since } m(F) = 0\text{)}$$

Therefore, from (1)

$$\lim_{n\to\infty}\left[m\left(W_n(\in)\right)\right] = 0$$

But since $\qquad E_k(\in) \subset W_k(\in)$, therefore

$$\lim_{n\to\infty}\left[m\left(x \in E : |f_n(x) - f(x)| \geq \in\right)\right] = 0.$$

3.7.1 Convergence in measure :

Let $<f_n>$ be a sequence of measurable functions finite *a.e.* on a measurable set E. Let f be a measurable function finite *a.e.* on E. If for each $\in > 0$

$$\lim_{n\to\infty}\left[m\left(x \in E : |f_n(x) - f(x)| \geq \in\right)\right] = 0,$$

then the sequence $<f_n>$ is said to **converge in measure** to the function f.

We can also say that, a sequence $<f_n>$ of measurable functions on a measurable set E is said to **converge in measure** to a measurable function f on the set E, if for each $\delta > 0$ and $\in > 0$, there exists a positive integer n_0 such that

$$m(\{x \in E : |f_n(x) - f(x)| \geq \in\}) < \delta, \quad \text{for all } n > n_0.$$

If the sequence $<f_n>$ of measurable functions, converges in measure to the measurable function f, then the limit f is always unique. This is evident from the following theorem.

Theorem 17. *If $<f_n>$ is a sequence of measurable functions defined on a measurable set E that converges in measure to the function f defined on E, then $<f_n>$ converges in measure to every function g which is equivalent to f.*

Proof : Let $\in > 0$ be any positive number, then we can observe that

$$\{x \in E : |f_n(x) - g(x)| \geq \in\} \subset \{x \in E : f(x) \neq g(x)\} \cup \{x \in E : |f_n(x) - f(x)| \geq \in\}$$

Therefore,

$$m(\{x \in E : |f_n(x) - g(x)| \geq \in\}) \leq m(\{x \in E : f(x) \neq g(x)\}) + m(\{x \in E : |f_n(x) - f(x)| \geq \in\})$$

But since g is equivalent to f, therefore $m(\{x \in E : f(x) \neq g(x)\}) = 0$,

Thus we have

$$m(\{x \in E : |f_n(x) - g(x)| \geq \in\}) \leq m(\{x \in E : |f_n(x) - f(x)| \geq \in\})$$

or, $\lim_{n\to\infty}\left[m\left(\left\{x\in E:|f_n(x)-g(x)|\geq\in\right\}\right)\right]\leq\lim_{n\to\infty}\left[m\left(\left\{x\in E:|f_n(x)-f(x)|\geq\in\right\}\right)\right]$

$= 0$ (since $<f_n>$ converges in measure to f)

or, $\lim_{n\to\infty}\left[m\left(\left\{x\in E:|f_n(x)-g(x)|\geq\in\right\}\right)\right]=0$

This shows that $<f_n>$ converges in measure to the function g.

Theorem 18. *If the sequence $<f_n>$ of measurable functions defined on a measurable set E converges in measure to two functions f and g, then these limit functions are equivalent, i.e.,*

$$m(\{x \in E : |f(x) \neq g(x)|\}) = 0.$$

Proof : Since $|f - g| \leq |f - f_n| + |f_n - g|$, therefore for each positive number $\in$, we observe that

$$\{x \in E : |f(x) - g(x)| \geq \in\} \subset \{x \in E : |f(x) - f_n(x)| \geq \frac{\in}{2}\}$$

$$\cup \{x \in E : |f_n(x) - g(x)| \geq \frac{\in}{2}\}$$

or $$m(\{x \in E : |f(x) - g(x)| \geq \in\}) \leq m(\{x \in E : |f(x) - f_n(x)| \geq \frac{\in}{2}\})$$

$$+ m(\{x \in E : |f_n(x) - g(x)| \geq \frac{\in}{2}\})$$

or $$\lim_{n\to\infty}\left[m\left(\left\{x\in E:|f(x)-g(x)|\geq\in\right\}\right)\right]\leq\lim_{n\to\infty}\left[m\left(\left\{x\in E:|f(x)-f_n(x)|\geq\frac{\in}{2}\right\}\right)\right]$$

$$+\lim_{n\to\infty}\left[m\left(\left\{x\in E:|f_n(x)-g(x)|\geq\frac{\in}{2}\right\}\right)\right]$$

But since $<f_n>$ converges is measure to f and g, therefore,

$$\lim_{n\to\infty}\left[m\left(\left\{x\in E:|f(x)-f_n(x)|\geq\frac{\in}{2}\right\}\right)\right]=0$$

and $$\lim_{n\to\infty}\left[m\left(\left\{x\in E:|f_n(x)-g(x)|\geq\frac{\in}{2}\right\}\right)\right]=0$$

Thus we have

$$m(\{x \in E : |f(x) - g(x)| \geq \in\}) = 0 \quad .....(1)$$

But $$\left\{x\in E:f(x)\neq g(x)\right\}\subset\bigcup_{n=1}^{\infty}\left\{x\in E:|f_n(x)-g(x)\geq\frac{1}{n}\right\}$$

$$=\sum_{n=1}^{\infty}m\left(\left\{x\in E:|f_n(x)-g(x)\geq\frac{1}{n}\right\}\right)$$

$$= 0$$

Therefore, $m(\{x \in E : f(x) \neq g(x)\}) = 0$

or $$g \sim f.$$

There are sequences of measurable functions that converge in measure but fail to converge at some point. This fact can be understood as "Convergence in measure is more general than convergence almost everywhere". First we take an example and then prove a theorem called **F. Riesz theorem.**

***Ex.*12.** *For each $n \in N$, consider n subintervals*

$$\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \ldots, \left[\frac{n-1}{n}, 1\right]$$

of the closed interval [0, 1] *and designate them as*

$$E_{nk} = \left[\frac{k-1}{n}, \frac{k}{n}\right]; \quad k = 1, 2, \ldots, n, \text{ for each } n \in N.$$

We arrange these subintervals as follows :

$$[0, 1]$$

$$\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right]$$

$$\left[0, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{2}{3}\right], \left[\frac{2}{3}, 1\right]$$

----

i.e. $E_{11}, E_{21}, E_{22}, E_{31}, E_{32}, E_{33}, E_{41}, \ldots$

Let $< E_n >$ denote the sequence of the above subintervals.

Now define the characteristic function of E_n as

$$f_n = \psi_{E_n}$$

Since $m(E_n) = 0$ as $n \to \infty$, we can observe that the sequence $< f_n >$ converges in measure to the zero function, but we also observe that for a given $\in > 0$

$$E_n = \{x \in E[0, 1] : |f_n(x)| \geq \in\} \qquad \text{for all } n \in N$$

Then for each $x \in [0, 1]$, $f_n(x) = 1$ for infinite many values of n (by definition of f_n). Thus $f_n(x) \not\to 0$ for any $x \in [0, 1]$, *i.e.,* the sequence $< f_n >$ does not converge point wise to zero function.

Theorem 19. **(*F. Riesz*)** *Let $< f_n >$ be a sequence of measurable functions on the set E, that converges in measure to the function f on E. Then there exists a subsequence $< f_{n_k} >$ of $< f_n >$ which converges to f a.e. on E, i.e.* $\lim f_{n_k} = f$ *a.e. on E.*

Proof : Let $< \in_n >$ be a monotonically decreasing sequence of positive numbers such that $\lim_{n\to\infty} \in_n = 0.$ Also let $\sum_{n=1}^{\infty} \delta_n$ be an infinite series of positive numbers such that $\sum_{n=1}^{\infty} \delta_n = 0.$

Now assume a strictly increasing sequence $< n_k >$ of positive integers.

Now since $< f_n >$ converges in measure to the function f, therefore, for given $\in_1 > 0$ and $\delta_1 > 0$, there is a positive integer n_1 in the sequence $< n_k >$ such that

$$m\left(\left\{x \in E : | f_{n_1}(x) - f(x) | \geq \in_1\right\}\right) < \delta_1$$

Similarly for $\in_2 > 0$ and $\delta_2 > 0$, there exists a positive integer $n_2 > n_1$ in the sequence $< n_k >$ such that

$$m\left(\left\{x \in E : | f_{n_2}(x) - f(x) | \geq \in_2\right\}\right) < \delta_2$$

and so on. In general for $\in_k > 0$ and $\delta_k > 0$ there exists $n_k > 0$ such that

$$m\left(\left\{x \in E : | f_{n_k}(x) - f(x) | \geq \in_k\right\}\right) < \delta_k$$

Let $$E_k = \left\{x \in E : | f_{n_k}(x) - f(x) | \geq \in_k\right\},$$

$$W_n = \bigcup_{k=n}^{\infty} E_k; \qquad n \in N$$

and $$V = \bigcap_{n=1}^{\infty} W_n$$

Then we observe that $< W_n >$ is a dcereasing sequence of measurable sets, *i.e.*,

$W_1 \supset W_2 \supset ... \supset W_n \supset ...$ and $m\,(W_1) < \infty$.

Therefore, $$m\,(V) = m\left(\bigcap_{n=1}^{\infty} W_n\right) = \lim_{n\to\infty} m(W_n)$$

But $$m\,(W_n) = m\left(\bigcap_{k=n}^{\infty} E_k\right) \leq \sum_{k=n}^{\infty} m(E_k) < \sum_{k=n}^{\infty} \delta_k$$

Thus $$m\,(W_n) < \infty$$

Hence $$m\,(V) = \lim_{n\to\infty} m(W_n) < \lim_{n\to\infty} \sum_{k=n}^{\infty} \delta_k = \lim_{k\to\infty} \delta_k = 0 \text{ (by our assumption)}$$

Now we shall prove that the sequence $< f_{n_k} >$ converges to f for all points in $E - V$.

Let $x_0 \in E - V$. Then $x_0 \notin V$, *i.e.*, $x_0 \notin W_{n_0}$ for some positive integer n_0. This implies that

$$x_0 \notin \left\{x \in E : | f_{n_k}(x) - f(x) | \geq \in_k\right\}, \ k \geq n_0$$

i.e. $$| f_{n_k}(x) - f(x_0) | < \in_k \qquad \text{for all} \quad k \geq n_0$$

But $\in_k \to 0$ as $k \to \infty$,

therefore, $\lim_{k\to\infty} f_{n_k}(x_0) = f(x_0)$; $x_0 \in E - V$

or $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ for all $x \in E - V$

But $m(V) = 0$, therefore the sequence $< f_{n_k} >$ converges to f *a.e.* on E.

Theorem 20. **(*D.F. Egorov*)** *Let E be a measurable set with m (E) < ∞ and* $< f_n >$ *be a sequence of measurable functions on E that converges to the measurable function f a.e. on E. Then for every* $\in > 0$, *there is a measurable set* $W \subset E$, *with* $m(W) < \in$ *such that* $< f_n >$ *converges to f uniformly on E – W.*

Proof : Let $< \sigma_i >$ be a monotonically decreasing sequence of positive numbers (*i.e.* $\sigma_1 > \sigma_2 > \sigma_3 > \ldots$.) which converges to 0.

Further let $\sum_{i=1}^{\infty} s_i$ be an infinite series of positive numbers converging to 0.

Now since for each real number $r > 0$

$$\lim_{n\to\infty}\left[m\left(\bigcup_{p=n}^{\infty}\{x \in E : |f_p(x) - f(x)| \geq r\}\right)\right] = 0$$

therefore for each $n \in N$ and for every pair (σ_n, s_n) there exists a positive integer $k \in N$ such that

$$m\left(\bigcup_{p=k}^{\infty}\{x \in E : |f_p(x) - f(x)| \geq \sigma_n\}\right) < s_n.$$

Let $$W = \bigcup_{n=j}^{\infty}\left[\bigcup_{p=k}^{\infty}\{x \in E : |f_p(x) - f(x)| \geq \sigma_n\}\right]$$

Then form above

$$m(W) = m\left(\bigcup_{n=j}^{\infty}\left[\bigcup_{p=k}^{\infty}\{x \in E : |f_p(x) - f(x)| \geq \sigma_n\}\right]\right)$$

$$= \sum_{n=j}^{\infty} m\left(\bigcup_{p=k}^{\infty}\{x \in E : |f_p(x) - f(x)| \geq \sigma_n\}\right)$$

$$< \sum_{n=j}^{\infty} s_n, \text{ which converges to zero.}$$

Therefore, for a given $\in > 0$, there exists a positive integer $j \in N$ such that

$$\sum_{n=j}^{\infty} s_n < \in$$

or $$m(W) < \in.$$

Further if $x_0 \in E - W$, then $x_0 \notin W$

$$\Rightarrow \quad x_0 \notin \bigcup_{n=j}^{\infty} \left[\bigcup_{p=k}^{\infty} \{x \in E : |f_p(x) - f(x)| \geq \sigma_n\} \right]$$

$$\Rightarrow \quad x_0 \notin \bigcup_{p=k}^{\infty} \{x \in E : |f_p(x) - f(x)| \geq \sigma_n\} \qquad \text{for all } n \geq j$$

$$\Rightarrow \quad x_0 \notin \{x \in E : |f_p(x) - f(x)| \geq \sigma_n\} \qquad \text{for all } n \geq j \text{ and } p \geq k$$

$$\Rightarrow \quad |f_p(x_0) - f(x_0)| < \sigma_n \qquad \text{for all } n \geq j \text{ and } p \geq k$$

But since $\lim_{n\to\infty} \sigma_n = 0$, therefore for a given positive number η, there exists a positive integer n such that

$$\sigma_n < \eta \qquad \text{for all } n \geq j.$$

This implies that

$$|f_p(x) - f(x)| < \eta \qquad \text{for all } p \geq k,$$

or $$|f_p(x) - f(x)| < \eta \qquad \text{for all } p \geq k \text{ and for all } x \in E - W$$

or $$\lim_{p\to\infty} f_p(x) = f(x)$$

Since η does not depend upon x, therefore $< f_n >$ converges to f uniformly on $E - W$.

The above theorem gives rise the concept of almost uniform convergence (*a.u.*) of sequences of measurable functions, which is defined below :

3.7.2 Almost uniform convergence :

Let $< f_n >$ be a sequence of measurable functions on the set E and f be a measurable functions on E. We say that $< f_n >$ converges almost uniformly to the function f and write $f_n \to f$ *a.u.*, if for a given $\in > 0$, there exists a set W, with $m(W) < \in$ such that $< f_n >$ converges to f uniformly on the set $E - W$.

Clearly the uniform convergence almost everywhere implies almost uniform convergence on the set $E - W$.

Structure of measurable functions : In real analysis, the study of complicated functions becomes little simple if they can be represented, exactly or approximately in the form of comparatively simpler functions, with simple nature. In this section, we shall study theorems on approximating measurable functions by means of continuous functions.

Theorem 21. *Let f be measurable function finite a.e. on a measurable set E. Then for any positive number* $\in$, *there exists a bounded measurable function g such that*

$$m\,(\{x \in E : f(x) \neq g(x)\}) < \in.$$

Proof : We define $\quad A_k = \{x \in E : f(x) > k\} \quad$ for $\quad k \in N$

and $\quad B = \{x \in E : f(x) = \infty\}$

then $\quad m(B) = 0 \quad$ (since f is finite *a.e.* on E)

Also since $A_1 \supset A_2 \supset A_3 \supset \ldots$

and $$B = \bigcap_{k=1}^{\infty} A_k$$

therefore, $$m(B) = m\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k \to \infty} m(A_k)$$

or $$\lim_{k \to \infty} m(A_k) = 0$$

Thus for a given $\in > 0$, there exists a positive integer n_0 such that $m\left(A_{n_0}\right) < \in$

Now let g be a function on E defined by

$$g(x) = \begin{cases} f(x), & \text{if} \quad x \in E - A_{n_0} \\ 0, & \text{if} \quad x \in A_{n_0} \end{cases}$$

Then g is a bounded measurable function since $|g(x)| \leq n_0$

Now since $\quad A_{n_0} = \{x \in E \,;\, f(x) \neq g(x)\}$ and $m\left(A_{n_0}\right) < \in$

therefore, $\quad m\,(x \in E : f(x) \neq g(x)) < \in.$

Theorem 22 (M. frechet). *Let f be a measurable function defined an a set E (or [a, b]). Then there exists a sequence* $< g_n >$ *of continous functions on R such that* $< g_n >$ *converges to f a.e. on E.*

Proof : Let us consider sequences $< \lambda_n >$ and $< \sigma_n >$ of positive numbers such that

$$\lim_{n \to \infty} \lambda_n = 0 \quad \text{and} \quad \lim_{n \to \infty} \sigma_n = 0.$$

Then by Borel's theorem, for each $n \in N$ and pair (λ_n, σ_n), there exists a sequence $< \phi_n >$ of continuous functions defined on E such that

$$m\left(\{x \in E : |f(x) - \phi_n(x)| \geq \sigma_n\}\right) < \lambda_n$$

Now since $\lim_{n \to \infty} \sigma_n = 0$, therefore for a given positive number σ there exists a positive integer integer $n_0 \in N$ such that

$$\sigma_n < \sigma \qquad \text{for all} \quad n \geq n_0$$

Thus $\quad \{x \in E : |f(x) - \phi_n(x)| \geq \sigma\} \subset \{x \in E : f(x) - \phi_n(x) | \geq \sigma_n\}$

or $$m\left(\left\{x \in E : | f(x)-\phi_n(x) \geq \sigma\right\}\right) \leq m\left(\left\{x \in E : f(x)-\sigma_n(x) | \geq \sigma_n\right\}\right)$$

or $$m\left(\left\{x \in E : | f(x)-\phi_n(x) \geq \sigma\right\}\right) < \lambda_n$$

$\Rightarrow$ $$\lim_{n\to\infty}\left[m\left(\left\{x \in E : | f(x)-\phi_n(x) \geq \sigma\right\}\right)\right] < \lim_{n\to\infty} \lambda_n = 0$$

or $$\lim_{n\to\infty}\left[m\left(\left\{x \in E : | f(x)-\phi_n(x) \geq \sigma\right\}\right)\right] = 0$$

i.e. the sequence $< \phi_n >$ converges in measure to the function *f*.

Thus by *F*. Riesz theorem, there exists a subsequence $\langle \phi_{n_k} \rangle$ of $< \phi_n >$ which also converges to *f a.e.* on *E*.

Now considering $g_k = \phi_{n_k}$, we get the required result.

Theorem 23 (Lusin). *Let f be a measurable function finite a.e. on E = [a, b]. Then given* $\in > 0$, *there exists a function* ϕ, *continous on [a, b] such that*

$$m\ (\{x \in E\ :\ f(x) \neq \phi\ (x)\}) < \in.$$

Proof : By frechet's theorem, there exists a sequence $< \phi_n >$ of continuous functions which converges to the function *f a.e.* on *E*. Let $\in > 0$ be any positive number. Then by Egorov's theorem there exists a measurable set $W \subset E = [a, b]$, with $m\ (W) < \frac{\in}{2}$, such that the sequence $< \phi_n >$ converges uniformly to *f* on the set $E - W$ ($E = [a, b]$). Then the function *f* is continuous on $E - W$.

Also for given $\in > 0$, there exists a closed subset $F \subset E - W$, such that

$$m\ (F) > m\ (E - W) - \frac{\in}{2}$$

$$= m\ (E) - m\ (W) - \frac{\in}{2}$$

$$> m(E) - \frac{\in}{2} - \frac{\in}{2} \qquad \left(\because m(W) < \frac{\in}{2}\right)$$

$$= m\ (E) - \in \qquad .....(1)$$

Then the function f is continuous on *F*.

We now use the following **lemma** :

"If *F* is a closed set contained in *E* and if *f* is a continuous function defined on *F*, then it is always possible to define a function ϕ on *E*, with the properties :

(i) ϕ is continuous on *E*.

(ii) for $x \in F$, $f(x) = \phi\ (x)$ and

(iii) If $|\ f(x)\ | \leq M$, then $|\ \phi\ (x)\ | \leq M$.

Making use of the above lemma, then

$$\{x \in E : f(x) \neq \phi(x)\} \subset E - F$$

$$\Rightarrow \quad m(\{x \in E : f(x) \neq \phi(x)\}) \leq m(E - F)$$

$$= m(E) - m(F)$$

$$< \in$$

Thus $\quad m(\{x \in E : f(x) \neq \phi(x)\}) < \in$ [using (1)]

And if $|f(x)| \leq M$, then $|\phi(x)| \leq M$.

Self-learning exercise-3

1. If a sequence $< f_n >$ converges in measure to the function f, then it converges in measure to every function g which is .... .
2. If a sequence $< f_n >$ of measurable functions converges in measure to the function f, then f is ... .
3. If the sequence $< f_n >$ of measurable functions converges in measure to the function f *a.e.* on E, then $< f_n >$ ....
4. If the sequence $< f_n >$ of measurable functions converges in measure to the function f on E, then $< f_n >$ .... converges pointwise.
5. If a sequence $< f_n >$ of measurable functions converges in measure to the function f, then $< f_n >$ is ..... .
6. If $< f_n >$ converges to f *a.e.*, $< g_n >$ converges to g *a.e.* and $f_n = g_n$ *a.e.* for all $n \in N$, then .... .

3.8 Summary

In this unit we learnt about the measurable functions and the relation between measurable function and measurable set. We also discussed the convergence of the sequences of measurable functions on measurable sets, with different nature, for example convergence *a.e.*, convergence in measure etc.

3.9 Answers to self-learning exercises

Self-learning exercise-1

1. is not necessarily measurable on E.
2. g does not vanish for all points in E.
3. not necessarily.
4. false
5. measurable
6. g is measurable
7. f is measurable on E.

Self-learning exercise-2

1. yes
2. the function whose value at $x \in E$ is $\limsup_{n} \langle f_n(x) \rangle$
3. measurable
4. also measurable and equal.

Self-learning exercise-3

1. equivalent to f.
2. unique.
3. converges in measure to f on E.
4. may not
5. fundamental in measure.
6. $f = g$ $a.e.$

3.10 Exercises

1. Prove that almost uniform convergence of sequence of measurable functions implies convergence in measure.
2. State and prove F. Riesz theorem for measurable functions.
3. Prove that a real valued function, continuous in an open interval is measurable.
4. Show that the step function defined on R is measurable.
5. If a function f^2 is measurable, then does f also measurable ?
6. If f is measurable on each set E_i in the countable collection $\{E_i\}$ of pairwise disjoint measurable sets. Then show that f is measurable on $\bigcup_i E_i$ also.
7. If $< f_n >$ is a sequence of measurable functions, then show that $\lim\sup_n \langle f_n \rangle$ and $\lim\inf_n \langle f_n \rangle$ are also measurable.

□ □ □

UNIT 4 : Weierstrass Approximation Theorem and Lebesgue Integral

Structure of the Unit

4.0 Objectives

4.1 Introduction

4.2 Weierstrass approximation theorem

4.3 Lebesgue integral of bounded function

  4.3.1 Partitions of a measurable set

  4.3.2 Upper and lower Lebesgue Darboux sums

  4.3.3 Upper and lower Lebesgue integrals

  4.3.4 Lebesgue integral

  4.3.5 Basic properties of Lebesgue integral

4.4 Lebesgue integral of bounded functions over subsets of measurable sets

4.5 Algebra of Lebesgue integrable functions

4.6 Limits of the sequences under the sign of integral

4.7 Summary

4.8 Answers to self-learning exercises

4.9 Exercises

4.0 Objectives

This unit has purposely been written to get students, acquainted with the theory of **Lebesgue integral** of bounded functions, in order to introduce a concept of integration applicable to a wider class of function than that of in Riemann integration.

4.1 Introduction

The unit starts with the Weierstrass's theorem on approximation of continuous function by a polynomial though it has nothing to do with the theory of Lebesgue integral. Introduction of Lebesgue integrable function along with various properties has been given as the next part of the unit. Theorem related to countable additivity of Lebesgue integral and finally the Lebesgue bounded convergence theorem are also the parts of the present unit.

4.2 Weierstrass approximation theorem

The Weierstrass approximation theorem establishes the fact that a continuous function can be approximated by a polynomial. Before we take up the Weierstrass theorem, we would prove the following lemma :

Lemma : For each positive integer n and real number x

$$\sum_{k=0}^{n} C_k^n \left(k-nx\right)^2 x^k \left(1-x\right)^{n-k} \le \frac{n}{4}.$$

Proof of the lemma : By the binomial theorem, we have the identity

$$(1+t)^n = \sum_{k=0}^{n} C_k^n\, t^k, \quad \text{where } C_k^n = \frac{\lfloor n}{\lfloor n \lfloor n-k}. \qquad .....(1)$$

Differentiating the above identity and multiplying by t

$$nt\,(1+t)^{n-1} = \sum_{k=0}^{n} k \cdot C_k^n\, t^k \qquad .....(2)$$

Once again differentiating the above and multiplying by t, we get

$$nt\,(1+t)^{n-1} + n\,(n-1)t^2\,(1-t)^{n-2} = \sum_{k=0}^{n} k^2 \cdot C_k^n\, t^k \qquad .....(2)$$

Now replacing t by $\frac{x}{1-x}$ in (1), (2) and (3) and multiplying each by $(1-x)^n$, we get

$$\sum_{k=0}^{n} C_k^n\, x^k \left(1-x\right)^{n-k} = 1 \qquad .....(4)$$

$$\sum_{k=0}^{n} k\, C_k^n\, x^k \left(1-x\right)^{n-k} = nx \qquad .....(5)$$

and

$$\sum_{k=0}^{n} k^2\, C_k^n\, x^k \left(1-x\right)^{n-k} = nx\left(1-x+nx\right) \qquad .....(6)$$

Multiplying (4) by $n^2 x^2$, (5) by $-2nx$ and adding the resulting equations to (6), we get

$$\sum_{k=0}^{n} \left(k-nx\right)^2 C_k^n\, x^k \left(1-x\right)^{n-k} = nx\left(1-x\right) \qquad .....(7)$$

But since for all real x

$$(2x-1)^2 \ge 0$$

or $\quad 4x\,(x-1) + 1 \ge 0$

or $\quad 4x\,(1-x) - 1 \le 0$

or $\quad x\,(1-x) \le \frac{1}{4}$

Therefore, the result (7) becomes

$$\sum_{k=0}^{n} C_k^n (k-nx)^2 x^k (1-x)^{n-k} \le \frac{n}{4}.$$

Theorem 1. (*Weierstrass approximation theorem*) *Let f be a real valued continuous function defined on* [0, 1]. *Then f can be approximated uniformly to a polynomial p on* [0, 1] *if for a given* $\in > 0$, *there exists a polynomial p such that* $|f(x) - p(x)| < \in$ *for all* $x \in [0, 1]$.

Or

If f is a real valued continuous function on [0, 1], *then* $B_n(x) \to f(x)$ *uniformly w.r.t. x as* $n \to \infty$, *where*

$$B_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_k^n x^k (1-x)^{n-k}$$

*is the **Bernstein polynomial** of degree n for the function f on* [0, 1].

Proof : Since the function f is continuous on [0, 1], it is uniformly continuous on [0, 1]. Thus for a given positive number $\in$, there exists a positive number δ such that $|f(x) - f(y)| < \frac{\in}{2}$ for all $x, y \in [0, 1]$ whenever $|x - y| < \delta$.

Now let x be any real number in [0, 1].

Then since $\sum_{k=0}^{n} C_k^n x^k (1-x)^{n-k} = 1$, for any positive integer n,

therefore, $f(x) = \sum_{k=0}^{n} f(x) C_k^n x^k (1-x)^{n-k}$

Thus

$$|f(x) - B_n(x)| = \left| \sum_{k=0}^{n} f(x) C_k^n x^k (1-x)^{n-k} - \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_k^n x^k (1-x)^{n-k} \right|$$

$$= \left| \sum_{k=0}^{n} \left(f(x) - f\left(\frac{k}{n}\right) \right) C_k^n x^k (1-x)^{n-k} \right|$$

$$\le \sum_{k=0}^{n} \left| f(x) - f\left(\frac{k}{n}\right) \right| C_k^n x^k (1-x)^{n-k}$$

$$= \left[\Sigma^1 + \Sigma^2 \right] \left| f(x) - f\left(\frac{k}{n}\right) \right| C_k^n x^k (1-x)^{n-k} \qquad .....(1)$$

where Σ^1 denotes the sum over those k for which $\left| x - \frac{k}{n} \right| < \delta$ and Σ^2 denotes the sum over the remaining k for which $\left| x - \frac{k}{n} \right| \geq \delta$.

Now $$\sum{}^1 \left| f(x) - f\left(\frac{k}{n}\right) \right| C_k^n\, x^k (1-x)^{n-k} < \frac{\in}{2} \sum_{k=0}^{n} C_k^n\, x^k (1-x)^{n-k} = \frac{\in}{2}$$

therefore, $$\sum{}^1 \left| f(x) - f\left(\frac{k}{n}\right) \right| C_k^n\, x^k (1-x)^{n-k} < \frac{\in}{2} \qquad .....(2)$$

Also f is continuous on [0, 1], therefore f is bounded in [0, 1]. Let $|f(x) \leq M$ for all $x \in [0, 1]$. Then

$$\left| f(x) - f\left(\frac{k}{n}\right) \right| \leq | f(x) | + \left| f\left(\frac{k}{n}\right) \right| \leq M + M = 2M$$

Then $$\sum{}^2 \left| f(x) - f\left(\frac{k}{n}\right) \right| C_k^n\, x^k (1-x)^{n-k} \leq 2M \sum_{k=0}^{n} C_k^n\, x^k (1-x)^{n-k}$$

$$= 2M \sum_{k=0}^{n} C_k^n \frac{\left(x - \frac{k}{n}\right)^2}{\left(x - \frac{k}{n}\right)^2} x^k (1-x)^{n-k}$$

$$< \frac{2M}{n^2 \delta^2} \sum_{k=0}^{n} C_k^n (x - nk)^2 x^k (1-x)^{n-k}$$

$$\leq \frac{2M}{n^2 \delta^2} \cdot \frac{n}{4} \qquad \text{[using the lemma]}$$

$$= \frac{M}{2n\delta^2}$$

Now let $n > \frac{M}{\in \delta^2}$, then

$$\sum{}^2 \left| f(x) - f\left(\frac{k}{n}\right) \right| C_k^n\, x^k (1-x)^{n-k} < \frac{\in}{2} \qquad .....(3)$$

from (1), (2) and (3), we get

$$| f(x) - B_n(x) | < \in, \qquad \text{for all} \quad x \in [0, 1]$$

If we take $\quad p(x) = B_n(x)$ for all $x \in [0, 1]$,

then we get the result.

Theorem 2. (*Weierstrass*) *Let f be a continuous function defined on the closed interval* $[a, b]$. *Then for each* $\in > 0$, *there is a polynomial p such that* $|f(x) - p(x)| < \in$ *for all* $x \in [a, b]$.

Proof : If $[a, b] = [0, 1]$, then theorem holds good (We have just proved in the earlier theorem). So let $[a, b] \neq [0, 1]$.

Let g be a polynomial function an $[0, 1]$, defined by

$$g(y) = f(a + y(b - a)); \qquad y \in [0, 1]$$

clearly $$g(0) = f(a) \text{ and } g(1) = f(b).$$

Also since f is continuous an $[a, b]$, therefore g is also continuous an $[0, 1]$. Thus by the previous theorem there exists a polynomial function q such that for given $\in > 0$,

$$|g(y) - q(y)| < \in \qquad \text{for all } y \in [0, 1] \qquad .....(1)$$

Now if $x \in [a, b]$, then $\dfrac{x-a}{b-a} \in [0, 1]$.

So by taking $$y = \frac{x-a}{b-a}, \text{ we have}$$

$$g(y) = g\left(\frac{x-a}{b-a}\right) = f\left(a + \frac{x-a}{b-a}(b-a)\right)$$

$$= f(x)$$

Then from (1)

$$\left| f(x) - q\left(\frac{x-a}{b-a}\right) \right| < \in \quad \text{for all } x \in [a, b]$$

Now defining $$p(x) = q\left(\frac{x-a}{b-a}\right), \text{ we see}$$

that p is a polynomial function for all $x \in [a, b]$. Then above becomes

$$|f(x) - p(x)| < \in \qquad \text{for all } x \in [a, b].$$

4.3 Lebesgue integral of a bounded function

The definition of Lebesgue integral of bounded function almost same as that of the definition of Riemann integral of functions, except that in Riemann integral we subdivide the closed interval $[a, b]$ into finite subintervals, where as for Lebesgue integral the subdivision of the interval $[a, b]$ are in much more general kind of measurable sets. Henry Lebesgue introduced a new concept of integral, called the Lebesgue integral, based upon the theory of measurable sets and functions, which generalizes the theory of Riemann integral. Unlike the Riemann integral, the Lebesgue integral proves several useful convergence theorems.

4.3.1 Partitions of a measurable set :

Let E be a measurable set. Then a finite collection $P = \{E_1, E_2, ..., E_i, ..., E_n\}$ of measurable subsets of E, where

$$E = \bigcup_{i=1}^{n} E_i \qquad E_i \cap E_j = \phi \quad \text{for all} \quad i, j = 1, 2, ..., n\ (i \neq j)$$

is said to be a measurable partition of the set E. The subsets $E_1, E_2, ..., E_n$ are called the components of the measurable partition P.

If P_1 and P_2 are any two different measurable partitions of the set E, then P_2 is said to be a refinement of the measurable partition P_1 if every component of P_2 is contained in some component of P_1, that is, if the components of P_2 are obtained by breaking up the components of P_1. We write as $P_1 \subset P_2$.

Now if $\{E_1, E_2, ..., E_m\}$ and $Q = \{E'_1, E'_2, ..., E'_n\}$ are any two measurable partitions of the same measurable set E, then the partition PQ, whose components are the sets $E_i \cap E'_j$, where $i = 1, 2, ..., m; j = 1, 2, ... n$, is called the common refinement of the partitions P and Q. Thus

$$PQ = \{E_i \cap E'_j;\ i = 1, 2, ... m; j = 1, 2, ..., n\}.$$

4.3.2 Upper and lower Lebesgue Darboux sums :

Let f be a bounded measurable function defined on a measurable set E.

Let $$P = \{E_1, E_2, ..., E_n\}$$

be any measurable partition of E. We define,

$$U(f, P) = \sum_{i=1}^{n} M_i \cdot m(E_i)$$

and $$L(f, P) = \sum_{i=1}^{n} m_i \cdot m(E_i), \text{ where}$$

$$M_i = \sup \{f(x); x \in E_i\},$$

$$m_i = \inf \{f(x); x \in E_i\}$$

and $$m(E_i) = \text{Lebesgue measure of } E_i.$$

The sums $U(f, P)$ and $L(f, P)$ are called the upper and the lower Lebesgue Darboux sums respectively, corresponding to the measurable partition P of E.

It is clear that $L(f, P) \leq U(f, P)$, for every measurable partition P of E. Also

$$L(-f, P) = -U(f, P) \text{ and } U(-f, P) = -L(f, P).$$

Theorem 3. *Let f be a bounded function defined on a measurable set E. If P and P' are two measurable partitions of E such that P' is a refinement of P, then*

(i) $L(f, P) \leq L(f, P')$; *(ii)* $U(f, P) \geq U(f, P')$

Proof : Since P' is the refinement of P, therefore $P \subset P'$. We shall prove the theorem for the case when P' is obtained from P by just partitioning one of the component E_j of P into two disjoint components E'_j and E''_j. So if

$$P = \{E_1, E_2, ..., E_j, ..., E_n\}$$

is any measurable partition of P, then

$$P' = \{E_1, E_2, ..., E'_j, E''_j, ..., E_n\}$$

is the refinement of P.

Now, let $\quad M_j = \sup \{f(x) ; x \in E_j\},$

$$M'_j = \sup \{f(x) : x \in E'_j\},$$

and $\quad M''_j = \sup \{f(x) : x \in E''_j\}$

Then clearly $\quad M'_j \leq M_j \quad \text{and} \quad M''_j \leq M_j$

Also $\quad m(E_j) = m(E'_j) + m(E''_j) \quad (\text{since } E'_j \cup E''_j = E_j \text{ and } E'_j \cap E''_j = \phi)$

Now
$$U(f, P') = \sum_{\substack{i=1 \\ i \neq j}}^{n} M_i \cdot m(E_i) + M'_j \, m(E'_j) + M''_j \, m(E''_j)$$

$$\leq \sum_{\substack{i=1 \\ i \neq j}}^{n} M_i \, m(E_i) + M_j \, m(E'_j) + M_j \cdot m(E''_j)$$

$$= \sum_{\substack{i=1 \\ i \neq j}}^{n} M_i \, m(E_i) + M_j \left[m(E'_j) + m(E''_j) \right]$$

$$= \sum_{\substack{i=1 \\ i \neq j}}^{n} M_i \, m(E_i) + M_j \, m(E_j)$$

$$= \sum_{i=1}^{n} M_i \, m(E_i)$$

$$= U(f, P)$$

Therefore, we have

$$U(f, P) \geq U(f, P') \qquad .....(1)$$

In a similar way, we can show that

$$L(f, P) \leq L(f, P') \qquad .....(2)$$

With the help of (1) and (2)

$$L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P).$$

Theorem 4. *The lower Lebesgue Darboux sums of any bounded measurable function f on a measurable set E can not exceed its upper Lebegue sum, i.e., if P_1 and P_2 are any two measurable partitions of the measurable set E, then*

(i) $L(f, P_1) \leq U(f, P_2)$

(ii) $L(f, P_2) \leq U(f, P_1)$

Proof : Let $P_1 = \{E_1, E_2, ..., E_m\}$

and $P_2 = \{E'_1, E'_2, ..., E'_n\}$

be any two measurable partitions of the measurable set E.

Let $P = \{E_i \cap E'_j;\ i = 1, 2, ..., m;\ j = 1, 2, ..., n\}$.

Then P is a common refinement of P_1 and P_2 and is a measurable partition of E. Therefore using theorem 3,

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_1)$$

and $$L(f, P_2) \leq L(f, P) \leq U(f, P) \leq U(f, P_2)$$

which implies that $$L(f, P_1) \leq U(f, P_2)$$

and $$L(f, P_2) \leq U(f, P_1).$$

From the above theorem, we observe that the family of all lower Lebesgue Darboux sums $L(f, P)$ corresponding to all the possible measurable partitions P of the measurable set E is bounded above by any upper Lebesgue Darboux sum $U(f, P)$.

i.e. $$U = \sup.\ \{L(f, P);\ P \text{ being a measurable partition of } E\} \leq U(f, P)$$

Similarly, we observe that the family of all upper Lebesgue Darboux sums $U(f, P)$ corresponding to all possible measurable partitions P of E is bounded below by any lower Lebesgue Darboux sum $L(f, P)$. *i.e.* $V = \inf.\ \{U(f, P);\ P \text{ being a measurable partition of } E\} \geq L(f, P)$.

Thus, U is a lower bound for family of upper Lebesgue Darboux sums and V is an upper bound for the family of lower Lebesgue Darboux sums. Consequently

$$L(f, P) \leq U \leq V \leq U(f, P).$$

4.3.3 Upper and lower Lebesgue integrals :

Let f be a bounded function defined on a measurable set $E = [a, b]$. Then the infimum of all the upper Lebesgue Darboux sums $U(f, P)$ of the function f corresponding to all the possible measurable partitions P of E is said to be the upper Lebegue integral of f over $E = [a, b]$ and is denoted by $L\int_a^{\bar{b}} f(x)\,dx$. Thus

$$L\int_a^{\bar{b}} f(x)\,dx = \inf\{U(f, P);\ P \text{ being a measurable partition of } E = [a, b]\}$$

Similarly, the supremum of all the lower Lebesgue Darboux sums $L(f, P)$ of the function f corresponding to all the possible measurable partitions P is said to be lower Lebesgue integral of f over $E = [a, b]$ and is denoted by $L\int_{\underline{a}}^{b} f(x)\,dx$. Thus

$$L\int_{\underline{a}}^{b} f(x)\,dx = \sup\{L(f, P); P \text{ being a measurable partition of } E = [a, b]\}.$$

Thus

(i) $L\int_{a}^{\overline{b}} f(x)\,dx \le U(f, P)$; for all measurable partitions P of $[a, b]$

(ii) $L\int_{\underline{a}}^{b} f(x)\,dx \ge L(f, P)$; for all measurable partitions P of $[a, b]$

(iii) $L\int_{\underline{a}}^{b} (-f)(x)\,dx = -L\int_{a}^{\overline{b}} f(x)\,dx$ (since $L(-f, P) = -U(f, P)$)

(iv) $L\int_{a}^{\overline{b}} (-f)(x)\,dx = -L\int_{\underline{a}}^{b} f(x)\,dx$ (since $U(-f, P) = -L(f, P)$)

(v) For every given positive number $\in$, there always exists a measurable partition P of the set $E = [a, b]$ such that

$$U(f, P) < L\int_{a}^{\overline{b}} f(x)\,dx + \in$$

(vi) For every given positive number $\in$, there always exists a measurable partition Q of $[a, b]$ such that

$$L(f, Q) < L\int_{\underline{a}}^{b} f(x)\,dx - \in$$

4.3.4 Lebesgue integral :

Let f be a bounded function defined on $[a, b]$. We say that f is Lebesgue integrable on $[a, b]$ if and only if

$$L\int_{\underline{a}}^{b} f(x)\,dx = L\int_{a}^{\overline{b}} f(x)\,dx.$$

The common value is said to be the Lebesgue integral or L-integral of the function f over $[a, b]$ and is denoted by $\int_a^b f(x)\,dx$. Thus the bounded function f on $[a, b]$ is L-integrable over $[a, b]$ if and only if

$$L\int_{\underline{a}}^{b} f(x)\,dx = \int_a^b f(x)\,dx = L\int_a^{\overline{b}} f(x)\,dx$$

The class of all the bounded functions that are L-integrable over $[a, b]$ is denoted by $L\,[a, b]$.

In order that a function f defined on an interval is integrable (L-integrable) over that interval, we must ensure that

(i) f is bounded and

(ii) the interval of integration is finite *i.e.* neither of the end points of the interval is infinite. If we say that integral $\int_a^b f(x)\,dx$ does exist, then this always means that the function f is bounded and is integrable over $[a, b]$.

It may also be observed that every bounded function may not be integrable over the interval $[a, b]$, *i.e.*, there may be a bounded function f on $[a, b]$ for which

$$L\int_{\underline{a}}^{b} f(x)\,dx \neq L\int_a^{\overline{b}} f(x)\,dx.$$

In the following theorem we shall derive necessary and sufficient condition for a bounded function f on $[a, b]$ to be L-integrable over $[a, b]$.

Theorem 5. *The necessary and sufficient condition for a bounded function f defined on the interval* $[a, b]$, *to be L-integrable over* $[a, b]$ *is that given* $\in > 0$, *there exists a measurable partition P of* $[a, b]$ *such that*

$$U(f, P) - L(f, P) < \in.$$

Proof : Necessary condition : Let the bounded function f on $[a, b]$ be L-integrable over $[a, b]$. Then

$$L\int_{\underline{a}}^{b} f(x)\,dx = L\int_a^{\overline{b}} f(x)\,dx \qquad .....(1)$$

Now from the definitions of upper and lower Lebesgue integrals, for any arbitrary positive number $\in$, there exist measurable partitions P_1 and P_2 of $[a, b]$, such that

$$U(f, P_1) < L\int_a^{\overline{b}} f(x)\,dx + \frac{\in}{2} \qquad .....(2)$$

and $$L(f, P_2) > L\underline{\int_a^b} f(x)\,dx - \frac{\in}{2}$$

or $$-L(f, P_2) < -L\underline{\int_a^b} f(x)\,dx + \frac{\in}{2} \qquad .....(3)$$

Let P be the common refinement of the measurable partitions P_1 and P_2. Thus P is a measurable partition of $[a, b]$.

Then we know that $U(f, P) \le U(f, P_1)$

and $L(f, P) \ge L(f, P_2)$

i.e. $-L(f, P) \le -L(f, P_2)$

Then (2) and (3) reduce to

$$U(f, P) < L\overline{\int_a^b} f(x)\,dx + \frac{\in}{2}$$

and $$-L(f, P) < -L\underline{\int_a^b} f(x)\,dx + \frac{\in}{2}$$

Adding the above inequalities, we get

$$U(f, P) - L(f, P) < L\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx + \in$$

or $U(f, P) - L(f, P) < \in$. [Using (1)]

Sufficient condition : Now let for any given $\in > 0$, there exists a measurable partition P of $[a, b]$, such that

$$U(f, P) - L(f, P) < \in \qquad .....(4)$$

From the definitions of upper and lower Lebesgue integrals, we know that

$$L\overline{\int_a^b} f(x)\,dx \le U(f, P)$$

and $$L\underline{\int_a^b} f(x)\,dx \ge L(f, P)$$

i.e. $$-L\underline{\int_a^b} f(x)\,dx \le -L(f, P)$$

Adding these two inequalities

$$L\overline{\int_a^b} f(x)\,dx - L\underline{\int_a^b} f(x)\,dx \le U(f, P) - L(f, P) < \in \qquad \text{[using (4)]}$$

or $$L\int_a^{\overline{b}} f(x)\,dx < L\int_{\underline{a}}^{b} f(x)\,dx + \in$$

or $$L\int_a^{\overline{b}} f(x)\,dx \le L\int_{\underline{a}}^{b} f(x)\,dx \qquad \text{(since } \in \text{ is arbitrary)} \qquad .....(5)$$

But $$L\int_a^{\overline{b}} f(x)\,dx \ge L\int_{\underline{a}}^{b} f(x)\,dx \qquad .....(6)$$

From (5) and (6)

$$L\int_a^{\overline{b}} f(x)\,dx = L\int_{\underline{a}}^{b} f(x)\,dx$$

i.e., f is Lebesgue integrable over $[a, b]$.

4.3.5 Basic properties of Lebesgue integrals :

In this subsection, we shall show the role played by the measurable functions defined on measurable sets in Lebesgue theory of integration.

Theorem 6. *Every bounded measurable function f defined on a measurable set E is L-integrable over E.*

Proof : Since f is bounded on the measurable set E, therefore, there exist numbers k and K such that

$$k \le f(x) \le K \quad \text{for all } x \in E.$$

We divide the interval $[k, K]$ by means of finite number of points $\alpha_0, \alpha_1, ..., \alpha_n$, such that

$$k \le \alpha_0 < \alpha_1 < ... < \alpha_n = K$$

and let $$\delta = \max\{\alpha_i - \alpha_{i-1}\}; \quad i = 1, 2, ..., n.$$

Also define $$E_i = \{x \in E : \alpha_{i-1} \le f(x) < \alpha_i; \quad i = 1, 2, ..., n\}$$

then each E_i is measurable for $i = 1, 2, ..., n$. Also E'_i are pairwise disjoint.

Thus $$P = \{E_1, E_2, ..., E_n\} \quad \text{is a measurable partition of } E.$$

Let $$M_i = \sup\{f(x) : x \in E_i\}; \quad i = 1, 2, ..., n$$

and $$m_i = \inf\{f(x) : x \in E_i\}; \quad i = 1, 2, ..., n$$

Then, $$M_i \le \alpha_i \text{ and } m_i \ge \alpha_{i-1} \quad \text{for all } i = 1, 2, ..., n.$$

Now $$U(f, P) = \sum_{i=1}^{n} M_i\, m(E_i) \le \sum_{i=1}^{n} \alpha_i\, m(E_i)$$

and $$L(f, P) = \sum_{i=1}^{n} m_i\, m(E_i) \ge \sum_{i=1}^{n} \alpha_{i-1}\, m(E_i)$$

Then, $\quad U(f, P) - L(f, P) \le \sum_{i=1}^{n} (\alpha_i - \alpha_{i-1})\, m(E_i)$

$$\le \delta \sum_{i=1}^{n} m(E_i)$$

$$= \delta \cdot m(E)$$

[Since E'_i s are pairwise disjoint subsets of E, therefore, $m(E) = \sum_{i=1}^{n} m(E_i)$]

Hence $\quad U(f, P) - L(f, P) \le \delta \cdot m(E)$.

For arbitrary positive number $\in$, taking $\delta < \dfrac{\in}{m(E)}$, the above reduces to

$$U(f, P) - L(f, P) < \in.$$

This proves that f is L-integrable over E.

Theorem 7. (*First mean value theorem*) *Let f be a bounded measurable function such that $a \le f(x) \le b$ on a measurable set E. Then*

$$a \cdot m(E) \le \int_E F(x)\,dx \le b \cdot m(E).$$

Proof : Since $a \le f(x) \le b$ for all $x \in E$ therefore, for some $m \in N$,

$$a \le f(x) \le b \Rightarrow \left(a - \frac{1}{m}\right) < f(x) < \left(b + \frac{1}{m}\right)$$

Let $\alpha = a - \dfrac{1}{m}$ and $\beta = b + \dfrac{1}{m}$, then

$$\alpha < f(x) < \beta \quad \text{for all} \quad x \in E.$$

We divide the closed interval $[\alpha, \beta]$ by means of points $\lambda_0, \lambda_1, ..., \lambda_n$, such that

$$\alpha = \lambda_0 < \lambda_1 < \lambda_n < ... < \lambda_n = \beta \text{ and define}$$

$$E_k = \{x \in E : \lambda_k \le f(x) < \lambda_{k+1};\ k = 0, 1, 2, ..., n-1\}$$

Evidently $\quad E = \bigcup_{k=0}^{n-1} E_k$ and $E_i \cap E_j = \phi$ for all $i \ne j$; $i, j = 0, 1, 2, ..., n-1$

Also since f is measurable on E, therefore each E_k; $k = 0, 1, 2..., n-1$ is measurable and so

$$m(E) = \sum_{k=0}^{n-1} m(E_k) \qquad .....(1)$$

Thus $P\{E_0, E_1, ..., E_{n-1}\}$ is a measurable partition of E.

Now $\quad \alpha \le \lambda_k \le \beta;\ \ k = 0, 1, 2, ..., n$

$\Rightarrow \quad \alpha \cdot m(E_k) \le \lambda_k\, m(E_k) \le \beta \cdot m(E_k)$

$$\Rightarrow \qquad \sum_{k=0}^{n-1} \alpha \cdot m(E_k) \leq \sum_{k=0}^{n-1} \lambda_k m(E_k) \leq \sum_{k=0}^{n-1} \beta \cdot m(E_k)$$

$$\Rightarrow \qquad \alpha \sum_{k=0}^{n-1} m(E_k) \leq \sum_{k=0}^{n-1} \lambda_k m(E_k) \leq \beta \sum_{k=0}^{n-1} m(E_k)$$

$$\Rightarrow \qquad \alpha \cdot m(E) \leq \sum_{k=0}^{n-1} \lambda_k m(E_k) \leq \beta \cdot m(E) \qquad \text{[from (1)]} \qquad .....(2)$$

Now making max $(\lambda_{k+1} - \lambda_k) \to 0$, we get $\lambda_{k+1} \to \lambda_k$. Hence for this partition P,

$$U(f, P) = \sum_{k=0}^{n-1} \lambda_{k+1} \cdot m(E_k) \to \sum_{k=0}^{n-1} \lambda_k \cdot m(E_k) = L(f, P)$$

i.e. $\qquad U(f, P) = L(f, P)$

Therefore, $\qquad \int_E f(x)dx \sum_{k=0}^{n-1} \lambda_k m(E_k)$

Thus (2) is

$$\alpha \cdot m(E) \leq \int_E f(x)dx \leq \beta \cdot m(E)$$

or $$\left(a - \frac{1}{m}\right) m(E) \leq \int_E f(x)dx \leq \left(b + \frac{1}{m}\right) m(E)$$

When $m \to \infty$, the above reduces to

$$a \cdot m(E) \leq \int_E f(x)dx \leq b \cdot m(E).$$

Self-learning exercise-1

1. If $P = \{E_1, E_2, ..., E_n\}$ is a measurable partition of the measurable set E, then it is necessary that $\sum_{i=1}^{n} E_i = E$ and ....

2. If P_1 and P_2 are any two measurable partitions of the measurable set E, then P_2 is said to be a refinement P_1 if ....

3. If f is a measurable function on the measurable set E. Then $U(f, P)$ and $L(f, P)$ can be defined only when ....

4. If P and Q are the measurable partitions of the measurable set E such that Q is the refinement of P, then $L(f, P)$ is always .... $L(f, Q)$ and $U(f, P)$ is always .... $U(f, Q)$.

5. If f is a bounded function on the measurable set E. Then for all measurable partitions P of E, $\sup_P \{L(f, P)\} ... \inf_P \{U(f, P)\}$.

6. If upper and lower Lebesgue integrals are defined for the bounded function f on $[a, b]$, then

$$L\int_a^{\bar{b}} (-f)(x)\,dx = \text{.......}.$$

7. If f is a bounded and measurable function defined on a measurable set E, then ....

4.4 Lebesgue integral of bounded functions over a subset of measurable set

In this section, we shall prove some theorems related to the Lebesgue integrals of bounded functions, over a subset of real numbers.

Theorem 8. *Let A and B be any two disjoint measurable subsets of the measurable set E and let f be a bounded measurable function (L-integrable) on E. Then*

$$\int_E f(x)\,dx = \int_A f(x)\,dx + \int_B f(x)\,dx.$$

Proof : Since A and B are disjoint measurable subsets of E, therefore,

$$E = A \cup B \quad \text{and} \quad A \cap B = \phi.$$

Let $P = \{E_1, E_2, \ldots, E_n\}$ be any measurable partition of E and suppose that

$$E'_i = E_i \cap A; \qquad i = 1, 2, \ldots, n$$

$$E''_i = \{E'_i \cap B; \qquad i = 1, 2, \ldots, n.$$

Then $\qquad P'_i = \{E'_i; \qquad i = 1, 2, \ldots, n\}$

and $\qquad P'' = \{E''; \qquad i = 1, 2, \ldots, n\}$

are measurable partitions of A and B respectively. Therefore

$$L(f, P') \le \int_A f(x)\,dx \le U(f, P')$$

and $$L(f, P'') \le \int_B f(x)\,dx \le U(f, P'')$$

Let $P^* = P' \cup P''$. Then P^* is a measurable partition of E and is a refinement of P. Thus

$$L(f, P^*) = L(f, P') + L(f, P'')$$

and $$U(f, P^*) = U(f, P') + U(f, P'')$$

Thus, $$L(f, P^*) = L(f, P') + L(f, P'') \le \int_A f(x)\,dx + \int_B f(x)\,dx$$

$$\le U(f, P') + U(f, P'') = U(f, P^*)$$

or $$L(f, P^*) \le \int_A f(x)\,dx + \int_B f(x)\,dx \le U(f, P^*)$$

But since P^* is a refinement of P, therefore,

$$L(f, P) \le L(f, P^*) \quad \text{and} \quad U(f, P) \ge U(f, P^*)$$

Thus above becomes

$$L(f, P) \le \int_A f(x)\,dx + \int_B f(x)\,dx \le U(f, P) \qquad \text{.....(1)}$$

Also $$L(f, P) \le \int_E f(x)\,dx \le U(f,P) \qquad .....(2)$$

From (1) and (2), we have

$$\int_E f(x)\,dx = \int_A f(x)\,dx + \int_B f(x)\,dx.$$

Theorem 9. *If E is the finite union of pairwise disjoint measurable sets $E_1, E_2, ..., E_n$, (i.e., $E = E_1 \cup E_2 \cup .... \cup E_n$ and $E_i \cup E_j = \phi$ for all $i \ne j$ and $i, j = 1, 2, ..., n$) and if f is a bounded measurable function defined on the measurable set E, then*

$$\int_E f(x)\,dx = \sum_{i=1}^{n} f(x)\,dx.$$

Proof : We shall prove the theorem using the principle of mathematical induction. We know that when $E = E_1 \cup E_2$, then

$$\int_E f(x)\,dx = \int_{E_1} f(x)\,dx + \int_{E_2} f(x)\,dx \qquad .....(1)$$

Thus the theorem holds for $n = 2$.

Let the theorem holds true when $E = E_1 \cup E_2 \cup ... \cup E_{n-1}$. Then we have

$$\int_E f(x)\,dx = \sum_{i=1}^{n-1} \int_{E_i} f(x)\,dx \qquad .....(2)$$

Then for $E = E_1 \cup E_2 \cup ... \cup E_n$, have,

$$E = \left(\bigcup_{i=1}^{n-1} E_i \right) \bigcup E_n$$

This implies that

$$\int_E f(x)\,dx = \int_{\bigcup_{i=1}^{n-1} E_i} f(x)\,dx + \int_{E_n} f(x)\,dx \qquad \text{[using (1)]}$$

$$= \sum_{i=1}^{n-1} \int_{E_i} f(x)\,dx + \int_{E_n} f(x)\,dx \qquad \text{[using (2)]}$$

$$= \sum_{i=1}^{n} \int_{E_i} f(x)\,dx.$$

Hence the theorem holds true when $E = E_1 \cup E_2 \cup ... \cup E_n$, *i.e.*

$$\int_E f(x)\,dx = \sum_{i=1}^{n} \int_{E_i} f(x)\,dx.$$

Theorem 10. ***(Countable additivity of integrals)*** *let f be a bounded measurable function defined on a measurable set E and E be the union of countable family $\{E_i; i \in N\}$ of pairwise disjoint measurable sets. Then*

$$\int_E f(x)\,dx = \sum_{i=1}^{\infty} \int_{E_i} f(x)\,dx.$$

Proof : We are given that, the measurable set E is the union of countable family $\{E_i;\, i \in N\}$ of pairwise disjoint measurable sets, *i.e.*,

$$E = \bigcup_{i=1}^{\infty} E_i; \qquad E_i \cap E_j = \phi \quad \text{for all} \quad i \neq j \quad \text{and} \quad i, j \in N.$$

E can be expressed as

$$E = \left(\bigcup_{i=1}^{n} E_i\right) \cup \left(\bigcup_{i=n+1}^{\infty} E_i\right)$$

or $$E = \bigcup_{i=1}^{n} E_i + R_{n+1}$$

where $$R_{n+1} = \bigcup_{i=n+1}^{\infty} E_i$$

Then $$\int_E f(x)\,dx = \int_{\bigcup_{i=1}^{n-1} E_i} f(x)\,dx + \int_{R_{n+1}} f(x)\,dx \qquad \text{(using theorem 8)}$$

$$= \sum_{i=1}^{n} \int_{E_i} f(x)\,dx + \int_{R_{n+1}} f(x)\,dx \qquad \text{(using theorem 9)} \quad .....(1)$$

Now f is bounded on E, therefore let there exist numbers α and β such that

$$\alpha \le f(x) \le \beta \qquad \text{for all} \quad x \in R_{n+1}.$$

Applying the first mean value theorem,

$$\alpha \cdot m(R_{n+1}) \le \int_{R_{n+1}} f(x)\,dx \le \beta \cdot m(R_{n+1}) \qquad .....(2)$$

But $$R_{n+1} = \bigcup_{i=n+1}^{\infty} E_i, \text{ therefore,}$$

$$m(R_{n+1}) = m\left(\bigcup_{i=n+1}^{\infty} E_i,\right) = \sum_{i=n+1}^{\infty} m(E_i) \quad (\because E_i \cap E_j = \phi \text{ for all } i \neq j)$$

Taking the limits on both sides when $n \to \infty$

$$\lim_{n\to\infty} m(R_{n+1}) = \lim_{n\to\infty}\left(\sum_{i=n+1}^{\infty} m(E_i)\right) = 0$$

[Since $\sum_{i=1}^{\infty} m(E_i) = m(E)$, therefore the series $\sum_{i=1}^{\infty} m(E_i)$ is a convergent series and since $m(R_{n+1}) = \sum_{i=n+1}^{\infty} m(E_i)$ is the remainder of the convergent series after n terms, therefore $\lim_{n\to\infty}\left(\sum_{i=n+1}^{\infty} m(E_i)\right)$ is 0]

Thus $$\lim_{n\to\infty} m(R_{n+1}) = 0 \qquad .....(3)$$

Now taking $n \to \infty$ in (2) and using (3)

$$\alpha \cdot 0 \le \lim_{n\to\infty} \int_{R_{n+1}} f(x)\,dx \le \beta \cdot 0$$

or $$\lim_{n\to\infty} \int_{R_{n+1}} f(x)\,dx = 0 \qquad .....(4)$$

Now taking $n \to \infty$ in (1)

$$\int_E f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \int_{E_i} f(x)\,dx + \lim_{n\to\infty} \int_{R_{n+1}} f(x)\,dx$$

$$= \sum_{i=1}^{\infty} \int_{E_i} f(x)\,dx + 0 \qquad \text{(using (4))}$$

Hence $$\int_E f(x)\,dx = \sum_{i=1}^{\infty} \int_{E_i} f(x)\,dx.$$

4.5 Algebra of Lebesgue integrable functions

Theorem 11. *Let f and g be two bounded measurable functions defined on a measurable set E. Then*

(i) for any arbitrary number $k \in R$, the function kf is L-integrable over E and

$$\int_E (kf)(x)\,dx = k\int_E f(x)\,dx$$

(ii) $f \pm g$ are L-integrable over E and

$$\int_E (f \pm g)(x)\,dx = \int_E f(x)\,dx \pm \int_E g(x)\,dx$$

Proof : *(i)* Let $k \in R$ be any arbitrary number.

Case 1 When $k = 0$. In this case the theorem obviously holds true.

Case 2 When $k > 0$. Since f is bounded and measurable on E, therefore f is L-integrable over E. Thus there exists a measurable partition P of E such that for given $\in > 0$

$$U(f,P) - L(f,P) < \frac{\in}{k} \qquad .....(1)$$

Now $k > 0$, therefore $(kf)(x) = kf(x)$

Thus $\quad U(kf, P) = k\, U(f, P)$

and $\quad L(kf, P) = k\, L(f, P)$

Then $\quad U(kf, P) - L(kf, P) = k\,[U(f, P) - L(f, P)] < k\dfrac{\in}{k} = \in$ [using (1)]

or $\quad U(kf, P) - L(kf, P) < \in$

$\Rightarrow$ kf is L-integrable over E, if $k > 0$.

Case 3 When $k < 0$. Let $k = -p$, where $p > 0$. Then since f is bounded and measurable on E, therefore there exists a measurable partition P of E such that

$$U(f, P) - L(f, P) < \frac{\in}{p} \qquad .....(2)$$

Now $\quad U(kf, P) = U(-pf, P) = -p\, L(f, p)$

and $\quad L(kf, P) = L(-pf, P) = -p\, U(f, p)$

therefore, $\quad U(kf, P) - L(kf, P) = -p\,[L(f, p) - U(f, P)]$

$$= p\,[U(f, p) - L(f, P)] < p\cdot\frac{\in}{p} \qquad \text{[using (2)]}$$

$$= \in$$

or $\quad U(kf, P) - L(kf, P) < \in$

$\Rightarrow$ kf is L-integrable over E, if $k < 0$.

Hence for every $k \in R$,

kf is L-integrable over E.

It remains to be shown that

$$\int_E (kf)(x)\,dx = k\int_E f(x)\,dx.$$

If $k = 0$, then obviously

$$\int_E (kf)(x)\,dx = k\int_E f(x)\,dx$$

If $k > 0$, then

$$\int_E (kf)(x)\,dx = \sup_P\{L(kf, P)\}$$

$$= k\sup_P\{L(f, P)\}$$

$$= k\int_E f(x)\,dx$$

and if $k < 0$, then

$$\int_E (k f)(x)\,dx = \sup_P \{L(k f, P)\}$$

$$= k \inf_P \{U(f, P)\}$$

$$= k\int_E f(x)\,dx$$

Thus, for every $k \in R$,

$$\int_E (k f)(x)\,dx = k\int_E f(x)\,dx.$$

(ii) Since the function f and g are bounded and measurable on the measurable set E, therefore the function $f \pm g$ are also bounded and measurable on E. Consequently $f \pm g$ are L-integrable over E.

Now let $P = \{E_1, E_2, ..., E_n\}$ be any measurable partition of E and

$$M_r = \sup \{(f + g)(x) ; x \in E_r ; r = 1, 2, ..., n\}$$

$$M'_r = \sup \{f(x) ; x \in E_r ; r = 1, 2, ..., n\}$$

$$M''_r = \sup \{g(x) ; x \in E_r ; r = 1, 2, ..., n\}$$

$$m_r = \inf \{(f + g)(x) ; x \in E_r ; r = 1, 2, ..., n\}$$

$$m'_r = \inf \{f(x) ; x \in E_r ; r = 1, 2, ..., n\}$$

$$m''_r = \inf \{g(x) ; x \in E_r ; r = 1, 2, ..., n\}$$

Then $\quad f(x) + g(x) \le M'_r + M''_r \quad$ for all $x \in E_r$

$\Rightarrow \quad (f + x)(x) \le M'_r + M''_r \quad$ for all $x \in E_r$

$\Rightarrow \quad M_r \le M'_r + M''_r \qquad$ .....(3)

Similarly $\quad m_r \ge m'_r + m''_r \qquad$ .....(4)

Now f and g are bounded and measurable on E, therefore, f, g and $f + g$ are L-integrable over E and so

$$L(f, P) \le \int_E f(x)\,dx \le U(f, P) \qquad .....(5)$$

$$L(g, P) \le \int_E g(x)\,dx \le U(g, P) \qquad .....(6)$$

and

$$L(f + g, P) \le \int_E (f + g)(x)\,dx \le U(f + g, P) \qquad .....(7)$$

From (5) and (6)

$$L(f, P) + L(g, P) \le \int_E f(x)\,dx + \int_E g(x)\,dx \le U(f, P) + U(g, P) \qquad .....(8)$$

Also

$$U(f + g, P) = \sum_{r=1}^{n} M_r m(E_r) \le \sum_{r=1}^{n} (M'_r + M''_r)\, m(E_r) \qquad \text{[from (3)]}$$

$$= \sum_{r=1}^{n} M'_r m(E_r) + \sum_{r=1}^{n} M''_r m(E_r)$$

$$= U(f, P) + U(g, P)$$

Thus $U(f+g, P) \leq U(f, P) + U(g, P)$ .....(9)

Similarly, we can show that

$$L(f+g, P) \geq L(f, P) + L(g, P) \quad .....(10)$$

In view of (9) and (10), inequality (7) becomes

$$L(f, P) + L(g, P) \leq \int_E (f+g)(x)\,dx \leq U(f,P) + U(g,P) \quad .....(11)$$

from (8) and (11)

$$\int_E (f+g)(x)\,dx = \int_E f(x)\,dx + \int_E g(x)\,dx \quad .....(12)$$

Now
$$\int_E (f-g)(x)\,dx = \int_E (f+(-g)(x))\,dx$$
$$= \int_E (f+(-1)g)(x)\,dx$$
$$= \int_E f(x)\,dx + \int_E ((-1)g)(x)\,dx \quad \text{[from (12)]}$$
$$= \int_E f(x)\,dx + (-1)\int_E g(x)\,dx \quad \text{[from result (i)]}$$
$$= \int_E f(x)\,dx - \int_E g(x)\,dx \quad .....(13)$$

From (12) and (13) we get

$$\int_E (f \pm g)(x)\,dx = \int_E f(x)\,dx \pm \int_E g(x)\,dx.$$

Theorem 12. *If f is a bounded measurable function defined on a measurable set E, then |f| is L-integrable over E and*

$$\left|\int_E f(x)\,dx\right| \leq \int_E |f(x)|\,dx$$

Proof : Since f is bounded and measurable on the measurable set E, therefore, $|f|$ is also bounded and measurable on E, and so $|f|$ is L-integrable over E.

Now, let $E_1 = \{x \in E\,; f(x) \geq 0\}$

and $E_2 = \{x \in E\,; f(x) < 0\}$

Then $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \phi$.

Thus by countable additivity of integrals

$$\int_E |f(x)|\,dx = \int_{E_1} |f(x)|\,dx + \int_{E_2} |f(x)|\,dx \quad .....(1)$$

and
$$\int_E f(x)\,dx = \int_{E_1} f(x)\,dx + \int_{E_2} f(x)\,dx$$
$$= \int_{E_1} |f(x)|\,dx - \int_{E_2} |f(x)|\,dx$$

$$\Rightarrow \quad \left|\int_E f(x)\,dx\right| = \left|\int_{E_1} |f(x)|\,dx - \int_{E_2} |f(x)|\,dx\right|$$

$$\leq \left| \int_{E_1} |f(x)|\,dx \right| + \left| \int_{E_2} |f(x)|\,dx \right|$$

$$= \int_{E_1} |f(x)|\,dx + \int_{E_2} |f(x)|\,dx$$

$$= \int_{E} |f(x)|\,dx \qquad \text{(from (1))}$$

Hence $$\left| \int_E f(x)\,dx \right| \leq \int_E |f(x)|\,dx.$$

Problem 1. *Let f be a constant function on a measurable set E, where f (x) = c for all x ∈ E. Then prove that*

$$\int_E f(x)\,dx = c \cdot m(E)$$

Sol. Here $f(x) = c$ for all $x \in E$,

therefore $c \leq f(x) \leq c$ for all $x \in E$

$$\Rightarrow \quad c \cdot m(E) \leq \int_E f(x)\,dx \leq c \cdot m(E) \qquad \text{(by first mean value theorem)}$$

$$\Rightarrow \quad \int_E f(x)\,dx = c \cdot m(E).$$

Problem 2. *If f is a bounded function defined on a measurable set E, and m (E) = 0. Then show that*

$$\int_E f(x)\,dx = 0.$$

Sol. Since f is bounded on E, therefore, there exist real numbers a and b such that

$$a \leq f(x) \leq b \text{ for all } x \in E$$

$$\Rightarrow \quad a \cdot m(E) \leq \int_E f(x)\,dx \leq b \cdot m(E) \qquad \text{(by first mean value theorem)}$$

$$\Rightarrow \quad a \cdot 0 \leq \int_E f(x)\,dx \leq b \cdot 0$$

$$\Rightarrow \quad \int_E f(x)\,dx = 0.$$

Problem 3. *Let f and g be bounded measurable functions on a measurable set E and f = g a.e. on E. Then show that*

$$\int_E f(x)\,dx = \int_E g(x)\,dx$$

Sol. Let $E_1 = \{x \in E : f(x) = g(x)\}$

and $E_2 = \{x \in E : f(x) \neq g(x)\}$

then $m(E_2) = 0$, $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \phi$.

Therefore, $\int_{E_2} f(x)dx = 0$ and $\int_{E_2} g(x)dx = 0.$ (problem 2)

Also since $f(x) = g(x)$ for all $x \in E_1$,

therefore, $\int_{E_1} f(x)dx = \int_{E_1} g(x)dx.$

Thus $\int_{E_1} f(x)dx + \int_{E_2} f(x)dx = \int_{E_1} g(x)dx + \int_{E_2} g(x)dx$

or, $\int_{E} f(x)dx = \int_{E} g(x)dx.$ (by countable additivity)

Problem 4. *Show by an example that the lebesgue integral of a nowhere zero function can be zero.*

Sol. Let $f : Q \to R$ be the function where $f(x) = 1$ for all $x \in Q$. Then f is nowhere zero. Also since Q is the set of rational numbers, so a countable set. Therefore $m(Q) = 0$.

Now $1 \leq f(x) \leq 1$, for all $x \in Q$

$\Rightarrow$ $1 \cdot m(Q) \leq \int_{Q} f(x)dx \leq 1 \cdot m(Q)$

$\Rightarrow$ $0 \leq \int_{Q} f(x)dx \leq 0$

$\Rightarrow$ $\int_{Q} f(x)dx \leq 0.$

Problem 5. *Let f be a bounded measurable function on a measurable set E and $f(x) \geq 0$ a.e. on E. If* $\int_{E} f(x)dx = 0$, then show $f(x) = 0$, *a.e.* on E.

Sol. Let $E_0 = \{x \in E : f(x) = 0\}$

and $E_k = \left\{x \in E; \frac{M}{k+1} < f(x) \leq \frac{M}{k}\right\}$

for all $k \in N$, where in is a positive number such that $f(x) \leq M$ for all $x \in E$.

Then $E = \bigcup_{k=0}^{\infty} E_k \,; E_i \cap E_j = \phi$ for all $i \neq j : i, j \in N$.

Obviously, for every $k \in N$, the set E_k is a measurable set, since f is measurable

on E and $E_k = \left\{x \in E; f(x) \leq \frac{M}{k}\right\} \cap \left\{x \in E : f(x) > \frac{M}{k+1}\right\}$

is the intersection of two measurable sets.

Now $$E - E_0 = \bigcup_{k=1}^{\infty} E_k$$

$\therefore$ $$m\,(E - E_0) = m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k) \qquad .....(1)$$

Since, $\frac{M}{k+1} < f(x)$ for all $x \in E_k : k \in N$

therefore, by first mean value theorem

$$\frac{M}{k+1} \cdot m(E_k) < \int_{E_k} f(x)\,dx \le \sum_{k=0}^{\infty} \int_{E_k} f(x)\,dx$$

$$= \int_E f(x)\,dx$$

$$= 0 \text{ (given)}$$

Thus $$\frac{M}{k+1} \cdot m(E_k) \le 0 \quad \text{for all } k \in N \qquad .....(2)$$

But $\frac{M}{k+1} \ge 0$ and $m\,(E_k) \ge 0$ for all $k \in N$

therefore, $$\frac{M}{k+1} \cdot m(E_k) \ge 0 \qquad .....(3)$$

From (2) and (3) we get $\frac{M}{k+1} \cdot m(E_k) = 0$

or $$m\,(E_k) = 0, \text{ for all } k \in N$$

or, $$\sum_{k=1}^{\infty} m(E_k) = 0$$

Thus form (1) $m\,(E - E_0) = 0$

Now $f(x) = 0$ for all $x \in E_0$

therefore $f(x) = 0$ *a.e.* on E.

Problem 6. *If the function f is L-integrable over the measurable set E and if f (x) ≥ 0 a.e. on E, then show that* $\int_E f(x)\,dx \ge 0.$

Sol. We can always assume that $f(x) \ge 0$ for all $x \in E$. Then clearly for every measurable partition P of E, we have $U(f, P) \ge 0$

Thus $$L\int_E f(x)\,dx = \inf_P U\{f, P\} \ge 0$$

or $$\int_E f(x)\,dx \ge 0.$$

Problem 7. *If the function f and g are Lebesgue integrable over the measurable set E and if f (x) < g (x) a.e. on E, then*

$$\int_E f(x)\,dx \le \int_E g(x)\,dx$$

Sol. Since f is L-integrable over E,

therefore, $-f$ is L-integrable over E.

Also g is L-integrable over E. Consequently

$$g+(-f) \quad i.e. \quad g-f \quad \text{is L-integrable over } E.$$

Now $f(x) \le g(x)$ *a.e.* on E

$\Rightarrow$ $g(x) - f(x) \ge 0$ *a.e.* on E

Thus using the result of problem (6), we have

$$\int_E (g-f)(x)\,dx \ge 0$$

or
$$\int_E \left[g(x) - f(x)\right] dx \ge 0$$

or
$$\int_E g(x)\,dx - \int_E f(x)\,dx \ge 0$$

or
$$\int_E f(x)\,dx \le \int_E g(x)\,dx.$$

Self-learning exercise-2

1. If f and g are bounded measurable functions on E of finite measure, then $f = g$ *a.e.* on $E \Rightarrow$ .... .
2. If $f(x) = k$ *a.e.* on E, then $\int_E f(x)\,dx = \ldots$.
3. If $f(x) = 1$ *a.e.* on E, then $\int_E f(x)\,dx = \ldots$.
4. If $f \le g$ *a.e.* on E, then ... .
5. Let f be a nonnegative measurable function defined on the measurable set E. Then $f = 0$ *a.e.* on E, if and only if ... .
6. Let f be L-integrable over the measurable set E, then a $\int_E |f(x)|\,dx = 0 \Rightarrow \ldots$.

4.6 Limits of the sequences under the sign of integral

In this section we shall study the nature of sequences of bounded and measurable functions defined on a measurable set of finite measure.

Theorem 13. (Lebesgue bounded convergence theorem) *Let* $< f_n >$ *be a sequence of bounded measurable functions defined on a set E of finite measure. If there exists a positive number M such that* $|f_n(x)| \le M$ *for all* $n \in N$ *and for all* $x \in E$ *and if* $< f_n >$ *converges in measure to a bounded measurable function f on E, then*

$$\lim_{n\to\infty} \int_E f_n(x)\,dx = \int_E f(x)\,dx.$$

Proof. Since $\forall\ n \in N, f_n$ is bounded and measurable on E and therefore integrable on E. By hypothesis, $\forall\ \delta > 0$

$$\lim_{n\to\infty}\Big[m\big(\{x \in E : |f_n(x) - f(x)| \ge \delta\}\big)\Big] = 0 \qquad .....(1)$$

Also $< f_n >$ is the sequence of bounded measurable functions on E such that $|f_n(x)| \le M$ for all $n \in N$ and for all $x \in E$,

therefore, we have $\quad |f(x)| \le M \quad$ for all $x \in E$

i.e. $\quad f$ is a bounded measurable function on E and so is integrable over E. Now for $\sigma > 0$

let, $\quad E_n = \{x \in E : |f_n(x) - f(x)| > \sigma\}$

and $\quad E'_n = \{x \in E : |f_n(x) - f(x)| < \sigma\}$

Then $\quad E = E_n \cup E'_n$ and $E_n \cap E'_n = \phi$

Also since $< f_n >$ is not convergent in E_n,

therefore, by definition of convergence in measure, we have

$$\lim_{n\to\infty} m(E_n) = 0 \qquad .....(2)$$

Applying the countable additivity of integrals, we have

$$\int_E |f_n(x) - f(x)|\,dx = \int_{E_n} |f_n(x) - f(x)|\,dx + \int_{E'_n} |f_n(x) - f(x)|\,dx$$

Also for all $x \in E'_n$ .....(3)

$|f_n(x) - f(x)| < \sigma$, therefore by first mean value theorem

$$\int_{E'_n} |f_n(x) - f(x)|\,dx < \sigma \cdot m(E'_n) \le \sigma \cdot m(E) \qquad (\text{since } E'_n \subset E)$$

Thus

$$\int_{E'_n} |f_n(x) - f(x)|\,dx < \sigma \cdot m(E) \qquad .....(4)$$

Now for $\in > 0$, we choose σ in such a way that $\sigma \cdot m(E) < \frac{\in}{2}\cdot$ Then (4) reduces to

$$\int_{E'_n} |f_n(x) - f(x)|\,dx < \frac{\in}{2} \qquad .....(5)$$

Again $\quad |f_n(x) - f(x)| \le |f_n(x)| + |f(x)| \le M + M$, for all $x \in E$

therefore, $|f_n(x) - f(x)| \le 2M$, for all $x \in E$

Thus applying first mean value theorem

$$\int_{E_n} \left| f_n(x) - f_n(x) \right| dx \le 2M \cdot m(E_n) \qquad .....(6)$$

But from (2), for given $\in \ge 0$, there exists a positive integer n_0 such that

$$\left| m(E_n) - 0 \right| < \frac{\in}{4m} \text{ for all } n \ge n_0$$

or $$m(E_n) < \frac{\in}{4m}$$

Thus (6) reduces to

$$\int_{E_n} \left| f_n(x) - f(x) \right| dx < 2M \cdot \frac{\in}{4M}$$

or $$\int_{E_n} \left| f_n(x) - f(x) \right| dx < \frac{\in}{2} \qquad .....(7)$$

Using (5) and (7) in (3), we get

$$\int_{E} \left| f_n(x) - f(x) \right| dx < \frac{\in}{2} + \frac{\in}{2} = \in, \quad \text{for all } n \ge n_0$$

Thus $$\left| \int_{E} \left[f_n(x) - f(x) \right] dx \right| \le \int_{E} \left| f_n(x) - f(x) \right| dx < \in, \quad \text{for all } n \ge n_0$$

$\Rightarrow$ $$\left| \int_{E} \left[f_n(x) - f(x) \right] dx \right| < \in, \quad \text{for all } n \ge n_0$$

$\Rightarrow$ $$\lim_{n\to\infty} \left[\int_{E} \left[f_n(x) - f(x) \right] dx \right] = 0$$

$\Rightarrow$ $$\lim_{n\to\infty} \int_{E} f_n(x)\,dx - \int_{E} f(x)\,dx = 0$$

or $$\lim_{n\to\infty} \int_{E} f_n(x)\,dx = \int_{E} f(x)\,dx.$$

Theorem 14. (Lebesgue dominated convergence theorem) *Let $< f_n >$ be a sequence of measurable functions defined on a measurable set E, such that $|f_n(x)| < \psi(x)$ for all $x \in E$ and for all $n \in N$, where $\psi(x)$ is an integrable function over E. If $< f_n >$ converges in measure to the measurable function f on E, then*

$$\lim_{n\to\infty} \int_{E} f_n(x)\,dx = \int_{E} f(x)\,dx.$$

Proof. Since $|f_n(x) < \psi(x)$ for all $x \in E$ and for all $n \in N$, therefore it is clear that f_n is bounded for all $n \in N$, *i.e.*, $<f_n>$ is a sequence of bounded measurable functions on E and so each f_n is integrable over E.

Again $<f_n>$ converges in measure to the measurable function f on E, therefore for every $\delta > 0$,

$$\lim_{n\to\infty}\left[m\left(\left\{x \in E : \left| f_n(x) - f(x) \right| \geq \delta\right\}\right)\right] = 0 \quad ....(1)$$

Now $<f_n>$ is a sequence of integrable functions over E that converges in measure to the function f on E, therefore f is also integrable over E. Hence $f_n - f$ is integrable over E, for all $n \in N$.

Now for any arbitrary number $\sigma > 0$, let

$$E_n = \{x \in E : |f_n(x) - f(x)| \geq \sigma\} \text{ and}$$
$$E'_n = \{x \in E : |f_n(x) - f(x)| < \sigma\}$$

Then it is clear that, $\lim_{n\to\infty} m(E_n) = 0$ .....(2)

and $$E = E_n \cup E'_n \;;\; E_n \cap E'_n = \phi$$

Thus using the countable additive property of integrals we get

$$\int_E \left| f_n(x) - f(x) \right| dx = \int_{E_n} \left| f_n(x) - f(x) \right| dx + \int_{E'_n} \left| f_n(x) - f(x) \right| dx \quad .....(3)$$

Now for all $x \in E'_n$

$$|f_n(x) - f(x)| < \sigma$$

therefore, $$\int_{E'_n} \left| f_n(x) - f(x) \right| dx < \sigma \cdot m(E'_n) \leq \sigma \cdot m(E)$$

Thus $$\int_{E'_n} \left| f_n(x) - f(x) \right| dx < \sigma \cdot m(E).$$

If we choose $\in > 0$ such that $\sigma \cdot m(E) < \frac{\in}{2}$, then above reduces to

$$\int_{E'_n} \left| f_n(x) - f(x) \right| dx < \frac{\in}{2} \quad .....(4)$$

Now $|f_n(x)| < \psi(x)$ for all $x \in E$ and $n \in N$ and by hypothesis

$$\lim_{n\to\infty}\left[m\left(\left\{x \in E : \left| f_n(x) - f(x) \right| \geq \sigma\right\}\right)\right] = 0$$

therefore $|f(x)| < \psi(x)$, for all $x \in E$

Thus $|f_n(x) - f(x)| < |f_n(x)| + |f(x)| < \psi(x) + \psi(x) = 2\,\psi(x)$, for all $x \in E$

or, $|f_n(x) - f(x)| < 2\,\psi(x)$, for all $x \in E$.

or $$\int_{E_n} \left| f_n(x) - f(x) \right| < 2 \int_{E_n} \psi(x)\, dx \quad .....(5)$$

But $|f_n(x)| < \psi(x)$, for all $x \in E$ and $n \in N$

therefore, $\psi(x) \geq 0$ for all $x \in E$

or
$$\int_{E_n} \psi(x)\,dx \geq 0$$

So
$$\left|\int_{E_n} \psi(x)\,dx\right| = \int_{E_n} \psi(x)\,dx$$

Thus from (5)

$$\int_{E_n} |f_n(x) - f(x)|\,dx < 2\left|\int_{E_n} \psi(x)\,dx\right| \quad .....(6)$$

Now from (2)

$\lim_{n\to\infty}\left[m(E_n)\right] = 0$ implies that for $\lambda > 0$, there exists a positive integer n_0 such that

$|m(E_n) - 0| < \lambda$ for all $n \geq n_0$

or $m(E_n) < \lambda$ for all $n \geq n_0$

Now using the theorem of absolute continuity, we can write

$$\left|\int_{E_n} \psi(x)\,dx\right| < \frac{\in}{4}$$

Then from (6)

$$\int_{E_n} |f_n(x) - f(x)|\,dx < \frac{\in}{2} \quad \text{for all } n \geq n_0 \quad .....(7)$$

Now using (4) and (7) in (3)

$$\int_{E} |f_n(x) - f(x)|\,dx < \frac{\in}{2} + \frac{\in}{2} = \in \quad ; \text{ for all } n \geq n_0$$

or
$$\int_{E} |f_n(x) - f(x)|\,dx < \in \quad ; \text{ for all } n \geq n_0$$

Thus
$$\left|\int_{E} \left[f_n(x) - f(x)\right]dx\right| \leq \int_{E} |f_n(x) - f(x)|\,dx < \in$$

or
$$\left|\int_{E} \left[f_n(x) - f(x)\right]dx\right| < \in \quad \text{for all } n \geq n_0$$

$\Rightarrow$
$$\lim_{n\to\infty}\left[\int_{E} \left[f_n(x) - f(x)\right]dx\right] = 0$$

$$\Rightarrow \qquad \lim_{n\to\infty}\left[\int_E f_n(x)\,dx - \int_E f(x)\,dx\right] = 0$$

$$\Rightarrow \qquad \lim_{n\to\infty}\int_E f_n(x)\,dx = \int_E f(x)\,dx.$$

Theorem 15. *Let $<f_n>$ be a sequence of measurable functions defined on a measurable set E, and $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. on E.*

Then f is measurable on E.

Proof. Let $B = \{x \in E \,; f_n(x) \neq f(x)\}$. Then by hypothesis, $m(B) = 0$ (since $f_n \to f$ *a.e.*)

We define $$g_n(x) = \begin{cases} f_n(x), & \text{if } x \notin B \\ 0, & \text{if } x \in B \end{cases}$$

and $$g(x) = \begin{cases} f(x), & \text{if } x \notin B \\ 0, & \text{if } x \in B \end{cases}$$

Then g_n is a measurable function for all $n \in N$.

Also for all $x \in B$, $\lim_{n\to\infty} g_n(x) = 0 = g(x)$ and for all $x \notin B$,

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f_n(x) = f(x) = g(x)$$

Therefore, $<g_n>$ converges point-wise to g on E.

Now since each g_n is measurable ($n \in N$), therefore, the limit function g is also measurable. Consequently f is measurable on E.

Problem 8. *Let $<f_n>$ be a sequence of functions, integrable over the measurable set E, which converges in mean to a function f on E. Then $<f_n>$ converges in measure to the function f.*

Sol. Let $E_n = \{x \in E \,; | f_n(x) - f(x) | \geq \delta\}$

where δ is any arbitrary positive real number. Then

$$\int_{E_n} | f_n(x) - f(x) | \, dx \geq \delta\, m(E_n)$$

or, $$\delta \cdot m(E_n) \leq \int_E | f_n(x) - f(x) | \, dx \qquad (\because E_n \subset E) \qquad .....(1)$$

Also $<f_n>$ converges in mean to f, so

$$\lim_{n\to\infty}\int_E | f_n(x) - f(x) | \, dx = 0 \qquad .....(2)$$

Thus from (1)

$$\lim_{n\to\infty}\left[\delta \cdot m(E_n)\right] \leq 0$$

or $$\lim_{n\to\infty}\left[\delta\cdot m(E_n)\right]=0 \qquad (\because\ \delta>0)$$

or $$\lim_{n\to\infty}\left[m(E_n)\right]=0$$

Hence $<f_n>$ converges to f in measure.

Problem 9. *Use Lebesgue dominated convergence theorem for the sequence* $< f_n >$ *of measurable functions on* [0, 1], *where* $f_n(x)=\dfrac{n^{3/2}x}{1+n^2x^2}$ for all $x\in[0, 1]$ and $n\in N$.

Sol. Here, $$f_n(x)=\frac{n^{3/2}x}{1+n^2x^2}=\frac{1}{x}\cdot\frac{n^{3/2}x}{1+n^2x^2}\le\frac{1}{x}$$

Therefore, $f_n(x)\le\psi(x)$, where $\psi(x)=\dfrac{1}{x}$ is Lebesgue integrable over [0, 1].

Hence by Lebesgue dominated convergence theorem

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx=\int_0^1\left[\lim_{n\to\infty}f_n(x)\right]dx$$

$$=\int_0^1\left(\lim_{n\to\infty}\frac{n^{3/2}x}{1+n^2x^2}\right)dx$$

$$=\int_0^1 0\,dx=0$$

Thus $$\lim_{n\to\infty}\int_0^1\frac{n^{3/2}x}{1+n^2x^2}\,dx=0.$$

4.7 Summary

In the beginning of the unit we studied about Weirstrass approximation theorem. With the introduction of upper and lower Lebesgue Darboux sums and then defining upper and lower Lebesgue integrals, we could define the Lebesgue integral of a bounded function over a measurable set. We also studied the basic properties of Lebesgue integrals and algebra of Lebesgue integrable functions over a measurable set E. Finally we were introduced with the Lebesgue bounded convergence and Lebesgue dominated convergence theorems for sequences of measurable functions on the measurable set.

4.8 Answers to self-learning exercises

Exercise 1

1. $E_i\cap E_j=\phi$ for all $i\ne j$.

2. Every component of P_2 is contained in some component of P_1.

3. f is bounded on E.

4. $\leq$; $\geq$.

5. $\leq$

6. $-L\int_a^b f(x)\,dx.$

7. f is Lebesgue integrable over E.

Exercise 2

1. $\int_E f(x)\,dx = \int_E g(x)\,dx.$

2. $k \cdot m(E)$

3. $m(E)$

4. $\int_E f(x)\,dx \leq \int_E g(x)\,dx.$

5. $\int_E f(x)\,dx = 0$

6. $f = 0$ *a.e.* on E.

4.9 Exercises

1. If f is a bounded measurable function on $[a, b]$ such that $f(x) \geq 0$ *a.e.* on $[a, b]$ and if E and F are measurable subsets of $[a, b]$ such that $E \subset F$, then prove that $\int_E f(x)\,dx \leq \int_F f(x)\,dx.$

2. If f is a nonnegative measurable function on a measurable set E such that $\int_E f(x)\,dx = 0,$ then prove that $f = 0$ *a.e.* on E.

3. If f is bounded measurable function on $[a, b]$ and if E_1 and E_2 are disjoint measurable subsets of $[a, b]$, then prove that $\int_a^b f(x)\,dx = \int_{E_1} f(x)\,dx + \int_{E_2} f(x)\,dx.$

4. If E is a measurable subset of $[a, b]$ and f, g are bounded Lebesgue integrable functions on $[a, b]$ such that $f = g$ *a.e.* on E, then prove that $\int_E f(x)\,dx = \int_F f(x)\,dx.$

5. If E is the union of a countable family $\{E_i\}$ of pairwise disjoint measurable sets and if f is Lebesgue integrable over E, then prove that

$$\int_E f(x)\,dx = \sum_{i=1}^{\infty} \int_{E_i} f(x)\,dx.$$

6. Let $< f_n >$ be a Cauchy sequence of functions integrable (in Lebesgue sense) over a measurable set E of finite measurable, and let $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in E$. Then prove that f is Lebesgue integrable over E and

$$\lim_{n\to\infty} \int_E \left| f_n(x) - f(x) \right| dx = 0.$$

□ □ □

Unit 5 : Summable Functions, Space of Square Summable Functions

Structure of the Unit

5.0 Objectives

5.1 Introduction

5.2 Summable functions

5.2.1 The Lebesgue integral of non-negative unbounded functions

5.2.2 The lebesgue integral of arbitrary functions

5.2.3 Absolutely equi-continuous integrals

5.3 The space L_2 of square summable functions

5.4 Summary

5.5 Answers to self-learning exercises

5.6 Exercises

5.0 Objectives

In this unit, we shall generalize the definition of Lebesgue integral to include unfounded measurable function **and domain can have infinite measure .**Those functions, whose Lebesgue integral is a finite real number are known as, summable functions. To generalize the definition of Lebesgue integral, we first study integral of a nonnegative unbounded function then we consider functions of arbitrary sign. Next , we introduce the concept of square summable functions , and establish that space of square summable functions is a normal linear space and complete space.

5.1 Introduction

In this unit, we will first define the integral of a nonnegative unbounded measurable functions. Then, in next part, we will define integral of arbitrary measurable functions *i.e.* those functions, which can be written as the difference of two nonnegative valued measurable functions. Next we will study the passage of limit under sign of integration. In the end of unit , we will define square summable function and prove that space of square summable functions is Banach space.

5.2 Summable functions

5.2.1 The Lebesgue integral of non-negative unbounded functions :

Given any non-negative unbounded function f on a set E, we will convert it into abounded function according to the following definition. Let f be a measurable and non-negative function on a measurable set E and let $n \in N$. Define the function $[f(x)]_n$ on E by

$$[f(x)]_n = \begin{cases} f(x) & \text{if } 0 \le f(x) \le n \\ n & \text{if } f(x) > n \end{cases}$$

i.e. $$[f(x)]_n = \min\,[f(x), n] \ \forall\, x \in E,$$

From the definition it is clean that $[f]_1 \le [f]_2 \le .....$ and for each $n \in N$, $[f(x)]_n$ is a bounded function. Also we have $\lim_{n\to\infty} [f(x)]_n = f(x)$ and $[f]_n$ is measurable therefore it is Lebesgue integrable on E.

Let f be a non-negative and measurable function defined on a measurable set E. The Limiting value (finite or infinite) $\lim_{n\to\infty} \int_E [f(x)]_n \, dx$ is called the **Lebesgue integral** of the function f on the set E and is denoted by the symbol $\int_E f(x)dx$ .Thus $\int_E f(x)dx = \lim_{n\to\infty} \int_E [f(x)]_n \, dx.$ Further if $\lim_{n\to\infty} \int_E [f(x)]_n \, dx$ exists, then we say that f is **Lebesgue integrable** or **summable** on the set E. If $E = [a, b]$, then we use notation $\int_a^b f(x)\, dx.$

Most of the results of the Lebesgue integral for arbitrary measurable function can be easily prove by the use of corresponding results for bounded measurable functions. The new definition for integral of a non-negative function f coincides with the definition given for bounded measurable function earlier because for sufficiently large n we have $[f(x)]_n \equiv f(x)$.

Thus every bounded measurable non-negative functions is summable. It is clear that if a function is summable of the set E, it is also summable on every subset of E. Further if $m\,(E) = 0$, every measurable function defined on E is summable and $\int_E f(x)\,dx = 0.$ Now we discuss other properties

Theorem 1. *If a function is summable on E, then it is finite almost every where on E .*

Proof. Let f be a summable function on the set E, we have to prove that $f(x) < \infty$ *a.e.* on E. Let $E_1 = \{x \in E \mid f(x) = \infty\}$, then $[f(x)]_n = n, \forall\ x \in E_1$. Now we know that is $A \subset B$ then $\int_A f(x)\,dx \le \int_B f(x)\,dx,$ so using this property, we get

$$\int_E [f(x)]_n \, dx \ge \int_{E_1} [f(x)]_n \, dx. \qquad .....(1)$$

On set E_1, $f(x) = \infty$, so $[f(x)]_n = n$, we get from (1)

$$\int_E [f(x)]_n \, dx \ge \int_{E_1} n\,dx = n.m\,(E_1)$$

Here we have to show that $m(E_1) = 0$. Let of possible $m(E_1) > 0$, then

$$\lim_{x\to\infty} \int_E [f(x)]_n \, dx \ge \lim_{x\to\infty} n.m(E_1) = \infty$$

$$\Rightarrow \qquad \int_E f(x)\, dx = \infty,$$

which contradicts the fact that f is summable on E. Hence $m(E_1)$ is not greater than zero

$$\Rightarrow \qquad m(E_1) = 0.$$

Thus f is finite *a.e.* on E.

Theorem 2. *Let f be a non-negative measurable function defined on a measurable set E. If $\int_E f(x)\,dx = 0$, then $f(x) = 0$ a.e. on E.*

Proof. By hypothesis for any $n \in N$

$$\int_E f(x)\,dx \ge \int_E [f(x)]_n \, dx \ge 0$$

But $$\int_E f(x)\,dx = 0, \text{ so} \int_E [f(x)]_n \, dx = 0,$$

therefor by theorem for bounded measurable function $[f(x)]_n = 0$ *a.e.* on E.

Let $$E_1 = \bigcup_{n=1}^{\infty} \{x \in E \mid [f(x)]_n \ne 0\}, \quad \text{then } m(E_1) = 0.$$

Since $\lim_{n\to\infty} [f(x)]_n = f(x) \,\forall\, x \in E$ and $[f(x)]_n = 0$ *a.e.* on E, therefore $f(x) = 0$ for $x \in E \sim E_1$. Since $m(E_1) = 0$, so we have $f(x) = 0$ *a.e.* on E.

Theorem 3. *Let f and g be two non-negative measurable functions on a measurable set E.*

(*i*) *If $f = g$ a.e. on E, then* $\int_E f(x)dx = \int_E g(x)\,dx$

(*ii*) *If $f \le g$ a.e. on E, then* $\int_E f(x)dx \le \int_E g(x)\,dx$.

Proof. (***i***) Since $f(x) = g(x)$ *a.e.* on E, therefore for each $n \in N$, $[f(x)]_n = [g(x)]_n$ *a.e.* on E and $[f]_n$, $[g]_n$ are bounded functions, so by theorem for bounded measurable functions, we have

$$\int_E [f(x)]_n \, dx = \int_E [g(x)]_n \, dx$$

Taking Limit $n \to \infty$, to both side, we get

$$\lim_{n\to\infty} \int_E [f(x)]_n \, dx = \lim_{n\to\infty} \int_E [g(x)]_n \, dx$$

$$\Rightarrow \qquad \int_E f(x)\,dx = \int_E g(x)\,dx$$

(ii) Since $f(x) \le g(x)$ *a.e.* on E, therefore for each $n \in N$, $[f(x)]_n \le [g(x)]_n$ *a.e.* on E. Then by theorem for bounded measurable function, we have

$$\int_E [\,f(x)]_n\,dx \le \int_E [\,g(x)]_n\,dx$$

Taking Limit $n \to \infty$, we obtain $\int_E f(x)\,dx \le \int_E g(x)\,dx.$

Theorem 4. *If $f(x)$ and $g(x)$ be two non-negative measurable functions on the set E. If $h(x) = f(x) + g(x)$, then*

$$\int_E h(x)\,dx = \int_E f(x)\,dx + \int_E g(x)\,dx.$$

Proof. Here $h(x) = f(x) + g(x)\ \forall\ x \in E$, so for any $n \in N$ we have

$$[\,h(x)]_n \le [\,f(x)]_n + [g(x)]_n \le [h(x)]_{2n}$$

$$\Rightarrow \qquad \int_E [h(x)]_n\,dx \le \int_E \left([f(x)]_n + [g(x)]_n\right) dx$$

$$= \int_E [f(x)]_n\,dx + \int_E [g(x)]_n\,dx$$

(using theorem for bounded measurable functions)

$$\le \int_E [h(x)]_{2n}\,dx$$

Taking Limit $n \to \infty$, we have

$$\lim_{n\to\infty} \int_E [h(x)]_n\,dx \le \lim_{n\to\infty} \int_E [f(x)_n]\,dx + \lim_{n\to\infty} \int_E [g(x)]_n\,dx$$

$$\le \lim_{n\to\infty} \int_E [h(x)]_{2n}\,dx$$

$$\Rightarrow \qquad \int_E h(x)dx \le \int_E f(x)\,dx + \int_E g(x)\,dx \le \int_E h(x)\,dx$$

$$\Rightarrow \qquad \int_E h(x)dx = \int_E f(x)\,dx + \int_E g(x)\,dx$$

$$\Rightarrow \qquad \int_E (f+h)x\,dx = \int_E f(x)\,dx + \int_E g(x)\,dx.$$

Corollary . *If $f_1, f_2, \dots, f_n$ are non-negative measurable functions defined on E, then*

$$\int_E \sum_{i=1}^{n} f_i(x)\,dx = \sum_{i=1}^{n} f_i(x)\,dx$$

Proof. On generalisation of above Theorem 4 for n function n functions we get the result.

Theorem 5. *If f is a non-negative measurable function on a measurable set E and λ is a real number them*

$$\int_E \lambda f(x)\,dx = \lambda \int_E f(x)\,dx.$$

Further if f is summable on E, then λf is also summable on E .

Proof. The theorem is self proved for $\lambda = 0$. If $\lambda = p \in N$ then from corollary of theorem 4 for p non-negative measurable functions $f_1, f_2, \ldots, f_p$, we have

$$\int_E \sum_{i=1}^{p} f_i(x)\,dx = \sum_{i=1}^{p} \int_E f_i(x)\,dx$$

Taping $f_1 = f_2 = \ldots. = f_p = f$, we get

$$\int_E p\, f(x)\,dx = p\int_E f\,(x)\,dx \qquad \text{.....(1)}$$

If $\lambda = \dfrac{1}{q}$ for some $q \in N$ then by equation (1)

$$q\int_E \frac{1}{q} f(x)\,dx = \int_E f\,(x)\,dx$$

$$\Rightarrow \qquad \int_E \frac{1}{q} f(x)\,dx = \frac{1}{q}\int_E f\,(x)dx \qquad \text{.....(2)}$$

If λ is a rational number $\dfrac{p}{q}$, then using (1) and (2)

$$\int_E \frac{p}{q} f(x)\,dx = \frac{p}{q}\int_E f\,(x)\,dx$$

Lastly let λ be any irrational number, then there exist rational numbers $r, s \in Q$ such that $r < \lambda < s$, then using theorem and using (3), we get

$$r\int_E f(x)\,dx \le \int_E \lambda\, f\,(x)\,dx \le s\int_E f(x)\,dx.$$

Taking limits $r \to \lambda$ and $s \to \lambda$, we have

$$\int_E \lambda\, f(x)\,dx = \lambda \int_E f\,(x)\,dx.$$

Finally, the summability of λf on E follows from the summability of f on E.

Now we shall discuss certain results related with integration of sequence . First we prove the following Lemma.

Lemma. *Let* $<f_n>$ *be a sequence of non-negative measurable functions. If* $\lim_{n\to\infty} f_n(x_0) = f(x_0)$ *at a point* x_0, *then for each* $m \in N$

$$\lim_{n\to\infty} [f_n(x_0)]_m = [f(x_0)]_m.$$

Proof. If $f(x_0) > m$, then there exists a number $n_0 \in N$ such that $f_n(x_0) > m$ for all $n > n_0$ and then

$$[f_n(x_0)]_m = m = [f(x_0)]_m, \quad \forall\ n > n_0.$$

If $f(x_0) < m$, then there exists a number $n_0' \in N$ such that $f_n(x_0) < m\ \forall\ n > n_0'$ and then

$$[f_n(x_0)]_m = f_n(x_0) \to f(x_0) = [f(x_0)]_m.$$

If $f(x_0) = m$, then for given $\in > 0$ there exists a number $n_0'' \in N$, such that

$$f_n(x_0) > m - \in \quad \forall\ n > n_0''.$$

[For $\lim_{n\to\infty} f_n(x_0) = f(x_0) = m \Rightarrow | f_n(x_0) - m | < \in \Rightarrow m - \in < f_n(x_0) < m + \in$]

Then $\quad m - \in < [f_n(x_0)]_m \leq m \quad \forall\ n > n_0''$

$\Rightarrow \quad m - \in < [f_n(x_0)]_m \leq m < m + \in$

$\Rightarrow \quad | [f_n(x_0)]_m - m | < \in \quad \forall\ n > n_0''$

$\Rightarrow \quad | [f_n(x_0)]_m - f(x_0) | < \in \quad \forall\ n > n_0''$

$\Rightarrow \quad | [f_n(x_0)]_m - [f(x_0)]_m | < \in \quad \forall\ n > n_0''$

[For $f(x_0) = m \Rightarrow [f(x_0)]_m = f(x_0)$]

$\Rightarrow \quad \lim_{n\to\infty} [f_n(x_0)]_m = [f(x_0)]_m.$

Thus the Lemma holds in all cases.

Theorem 6. (*Fatou's Lemma). Let* $<f_n>$ *be a sequence of non-negative measurable functions converger to* f *a.e. on the set E, then*

$$\int_E f(x)\, dx \leq \sup \left\{ \int_E f_n(x)\, dx \right\}$$

Proof. For any $m \in N$, we have from above Lemma

$$\lim_{n\to\infty} [f_n(x)]_m = [f(x)]_m \text{ on } E.$$

Since each of the function $[f_n(x)]_m$ is bounded by the number m, so by Lebergue bounded convergence theorem we have

$$\lim_{n\to\infty} \int_E [f_n(x)]_m\, dx = \int_E [f(x)]_m\, dx \quad .....(1)$$

Also $[f_n(x)]_m \leq f_n(x) \;\forall\; x \in E$

$$\Rightarrow \qquad \int_E [f_n(x)]_m\, dx \leq \int_E f_n(x)\, dx \leq \sup\left\{\int_E f_n(x)\, dx\right\} \qquad .....(2)$$

Taking Lim $n \to \infty$ and using (1) in (2), we get

$$\int_E [f(x)]_m\, dx \leq \sup\left\{\int_E f_n(x)\, dx\right\}$$

Again taking $n \to \infty$, we get

$$\int_E f(x)\, dx \leq \sup\left\{\int_E f_n(x)\, dx\right\}.$$

With the help of above theorem, it is easy to obtain another theorem concerning passage to the limit under the integral sign.

Theorem 7. *(Lebegue monotone convergence theorem). Let* $<f_n>$ *be an increasing sequence of non-negative measurable functions defined on the set E . If* $\lim_{n\to\infty} f_n(x) = f(x)$ *on E, then*

$$\lim_{n\to\infty} \int_E f_n(x)\, dx = \int_E f(x)\, dx.$$

Proof. Since $<f_n>$ is monotonically increasing sequence, so $f_1(x) \leq f_2(x) \leq f_3(x) \leq \ldots$.

It is given that $\lim_{n\to\infty} f_n(x) = f(x)$ on E and we know that a monotonically increasing sequence $<f_n>$ is convergent iff it is bounded and in that case $\lim_{n\to\infty} f_n(x) = \text{Sub} < f_n(x) >$

$$\Rightarrow \qquad f_n(x) \leq f(x), \qquad \forall\; x \in E$$

$$\Rightarrow \qquad \lim_{n\to\infty} \int_E f_n(x)\, dx \leq \int_E f(x)\, dx \qquad ......(1)$$

Now as $\lim_{n\to\infty} \int_E f_n(x)\, dx$ exists, so by Fatou's lemma we have

$$\int_E f(x)\, dx \leq \lim_{n\to\infty} \int_E f_n(x)\, dx \qquad .....(2)$$

From (1) and (2) we get

$$\lim_{n\to\infty} \int_E f_n(x)\, dx = \int_E f(x)\, dx$$

Corollary. *Let* $\{u_n\}$ *be a sequence of non-negative measurable functions defined on a measurable set E.*

If $$\sum_{n=1}^{\infty} u_n(x) = f(x),$$

then $$\int_E f(x)\, dx = \sum_{n=1}^{\infty} \int_E u_n(x)\, dx.$$

Proof. Let $f_n(x) = \sum_{n=1}^{n} u_i(x)$, $\forall\ x \in E$ and for each $n \in N$.

Then $\{f_n\}$ is on increasing sequence of non-negative measurable functions on E and the result follows from the main theorem.

Theorem 8. (*Countable additivity of the integral*) *Let E be union of a finite or countable family of pairwise disjoint measurable sets i.e.* $E = \cup E_i$, $E_i \cap E_j = \phi$, $i \neq j$. *Then for any non-negative measurable function f defined on the set E*

$$\int_E f(x)\,dx = \sum_i \int_{E_i} f(x)\,dx.$$

Proof. Let U_i be a function defined on E as

$$U_i(x) = \begin{cases} f(x) & \text{for } \ x \in E_i \\ 0 & \text{for } \ x \in E \sim E_i \end{cases}$$

Then $f(x) = \sum_i U_i(x)$ and by above corollary, we have

$$\int_E f(x)\,dx = \sum_i \int_E U_i(x)\,dx \qquad \text{......(1)}$$

Now
$$[U_i(x)]_n = \begin{cases} [f(x)]_n & \text{for } \ x \in E \\ 0 & \text{for } \ x \in E \sim E_i \end{cases}$$

and therefore $\int_E [U_i(x)]_n\,dx = \int_{E_i} [f(x)]_n\,dx$

Taking $n \to \infty$, we get $\int_E U_i(x)\,dx = \int_{E_i} f(x)\,dx$ .....(2)

From (1) and (2), we get

$$\int_E f(x)\,dx = \sum_i \int_{E_i} f(x)\,dx.$$

5.2.2 Integral of Arbitrary function :

In order to define the Lebergue integral for measurable function that take both positive and negative values, we shall show that such function can be written as the difference of two non-negative valued measurable functions.

Let f be any real valued measurable function on E . We define the functions f^+ and f^- called respectively the **positive** and **negative** parts of f, as $f^+(x) = \max(f(x), 0)$, $f^-(x) = \max(-f(x), 0)$, $x \in E$. For a fixed $x \in E$, we observe that

(*i*) $f(x) > 0 \Rightarrow f^+(x) = f(x)$ and $f^-(x) = 0$

(*ii*) $f(x) < 0 \Rightarrow f^+(x) = 0$ and $f^-(x) = -f(x)$

(*iii*) $f(x) = 0 \Rightarrow f^+(x) = 0$ and $f^-(x) = 0$

(*iv*) $f(x) = f^+(x) - f^-(x)$ and $|f(x)| = f^+(x) + f^-(x)$

and since f is a measurable function on E so both f^+ and f^- are non-negative measurable functions on E. All the results given in previous section are true for the functions f^+ and f^-. Now we define the Lebesgue integral for an arbitrary measurable function.

If f be an arbitrary measurable function defined on measurable set E and f^+, f^- are the positive and negative parts of the function f, then we define Lebesgue integral of f on E as

$$\int_E f(x)\,dx = \int_E f^+(x)\,dx - \int_E f^-(x)\,dx$$

The function f is said to be L-integrable or summable on E is both the functions f^+ and f^- are L-integrable or summable on E.

Theorem 9. *A measurable function f is summable on E if and only if* $|f|$ *is summable and in this case*

$$\left|\int_E f(x)\,dx\right| \le \int_E |f(x)|\,dx.$$

Proof. Let f is summable on E, so $\int_E f(x)\,dx < \infty$.

Now $\int_E f(x)\,dx = \int_E f^+(x)\,dx - \int_E f^-(x)\,dx \quad [\because f = f^+ - f^-]$

$$\int_E f(x)\,dx < \infty \Rightarrow \int_E f^+(x)\,dx < \infty \text{ and } \int_E f^-(x)\,dx < \infty$$

$$\Rightarrow \quad \int_E f^+(x)\,dx + \int_E f^-(x)\,dx < \infty$$

$$\Rightarrow \quad \int_E [f^+(x) + f^-(x)]\,dx < \infty$$

$$\Rightarrow \quad \int_E |f(x)|\,dx < \infty$$

$\therefore$ $|f|$ is summable on E.

Conversely let $|f|$ be summable on E.

$\because$ $|f| = f^+ + f^-$, so f^+ and f^- are summable on E. Therefore $f^+ - f^-$ is summable on E. Hence f is summable on E.

Now,
$$\left|\int_E f(x)\,dx\right| = \left|\int_E \left[f^+(x) - f^-(x)\right] dx\right|$$

$$= \left|\int_E f^+(x)\,dx - \int_E f^-(x)\,dx\right|$$

$$\le \left|\int_E f^+(x)\,dx\right| + \left|-\int_E f^-(x)\,dx\right|$$

$$= \int_E f^+(x)\,dx + \int_E f^-(x)\,dx$$

$$= \int_E \left[f^+(x) + f^-(x) \right] dx$$

$$= \int_E |f(x)|\,dx$$

Here $\left| \int_E f(x)\,dx \right| \leq | f(x) | dx.$

Theorem 10 : *If f and g are summable functions on a set E and c be a constant, then the functions f ± g and c.f are also summable and*

$$\int_E (f \pm g)(x)\,dx = \int_E f(x)\,dx \pm \int_E g(x)\,dx$$

and $$\int_E (cf)(x)\,dx = c\int_E f(x)\,dx.$$

Proof : Since f and g are summable functions on E.

$\Rightarrow$ $\int_E f(x)\,dx < \infty$ and $\int_E g(x)\,dx < \infty$

$\Rightarrow$ $\int_E f^+(x)\,dx - \int_E f^-(x)\,dx < \infty$ and $\int_E g^+(x)\,dx - \int_E g^-(x)\,dx < \infty$

$\Rightarrow$ $\int_E f^+(x)\,dx,\ \int_E f^-(x)\,dx,\ \int_E g^+(x)\,dx,\ \int_E g^-(x)\,dx$ are finite.

$\Rightarrow$ $\int_E f^+(x)\,dx + \int_E g^+(x)\,dx - \int_E f^-(x)\,dx - \int_E g^-(x)\,dx < \infty$

$\Rightarrow$ $\int_E (f^+ + g^+(x)\,dx - \int_E (f^- + g^-(x)\,dx < \infty$

$\Rightarrow$ $\int_E (f + g)(x)\,dx < \infty$

$\Rightarrow$ $f + g$ is summable on E.

Also $\int_E (f + g)(x)\,dx = \int_E (f + g)^-(x)\,dx$

$$= \int_E f^+(x)\,dx + \int_E g^+(x)\,dx - \int_E f^-(x)\,dx - \int_E g^-(x)\,dx$$

$$= \int_E f^+(x)\,dx - \int_E f^-(x)\,dx + \int_E g^+(x)\,dx - \int_E g^-(x)\,dx$$

$$= \int_E f(x) + \int_E g(x)\,dx$$

Let $\phi(x) = c\,f(x)\ \forall\ x \in E$

Case 1 : Let $c \geq 0$, $\phi + (x) = \max\{c\,f(x), 0\}$

$= c \max\{f(x), 0\} = c\,f^+(x)$

$\phi^-(x) = \max\{- \text{cf}(\text{x}), 0\}$

$= c \max\{-f(x), 0\}$

$= cf^-(x).$

$$\therefore \quad \int_E c\,f(x)\,dx = \int_E \phi(x)\,dx = \int_E \phi^+(x)\,dx - \int_E \phi^-(x)\,dx$$

$$= \int_E c\,f^+(x)\,dx - \int_E c\,f^-(x)\,dx$$

$$= c\int_E f^+(x)\,dx - c\int_E f^-(x)\,dx$$

$$= c\left[\int_E f^+(x)\,dx - \int_E f^-(x)\,dx\right]$$

$$= c\int_E f(x)\,dx.$$

Case II : If $c < 0,\ \phi^+(x) = \max\{c\,f(x), 0\} = |c| \max\{-f(x), 0\}$

$$= |c|\ f^-(x)$$

and $\phi^-(x) = \max\{-c\,f(x), 0\}$

$$= |c| \max\{f(x), 0\}$$

$$= |c|\,f^+(x).$$

Therefore $\int_E c\,f(x)\,dx = \int_E \phi(x)\,dx = \int_E \phi^+(x)\,dx - \int_E \phi^-(x)\,dx$

$$= \int_E |c|\,f^-(x)\,dx - \int_E |c|\,f^+(x)\,dx$$

$$= |c|\int_E f^-(x)\,dx - |c|\int_E f^+(x)\,dx$$

$$= -|c|\int_E f(x)\,dx$$

$$= c\int_E f(x)\,dx.$$

Thus if $c \in R$ then

$$\int_E cf(x)\,dx = c\int_E f(x)\,dx$$

$\therefore$ Summability of f implies the summability of cf.

Now $\int_E (f-g)(x)\,dx = \int_E [f+(-1)g](x)\,dx$

$$= \int_E f(x)\,dx + \int_E (-1)\,g(x)\,dx$$

$$= \int_E f(x)\,dx + (-1)\int_E g(x)\,dx$$

$$= \int_E f(x)\,dx - \int_E g(x)\,dx.$$

$\therefore$ Summability of f and g implies summability of $f-g$.

Theorem 11 : (*Finite additivity of the integral*) *Let the set E can be expressed as a finite union of pairwise disjoint measurable sets i.e.*

$$E = \bigcup_{i=1}^{n} E_i,\ E_i \cap E_j = \phi,\ i \neq j.$$

If function f is summable on each of the sets E_i, then it is also summable on E and

$$\int_E f(x)\,dx = \sum_{i=1}^{n} \int_{E_i} f(x)\,dx.$$

Proof : Since f is summable on each of the sets E_i so f^+ and f^- are summable on each of the sets E_i. f^+ and f^- are non-negative measurable functions on each E_i, therefore we

have $$\int_E f^+(x)\,dx = \sum_{i=1}^{n} \int_{E_i} f^+(x)\,dx \qquad .....(1)$$

and $$\int_E f^-(x)\,dx = \sum_{i=1}^{n} \int_{E_i} f^-(x)\,dx \qquad .....(2)$$

Since R.H.S. terms of (1) & (2) are finite so L.H.S. terms of (1) & (2) are finite. Subtracting (2) from (1), we get

$$\int_E [f^+(x) - f^-(x)]\,dx = \sum_{i=1}^{n} \int_{E_i} [f^+(x) - f^-(x)]\,dx$$

Hence $$\int_E f(x)\,dx = \sum_{i=1}^{n} \int_{E_i} f(x)\,dx.$$

Note : If $E = \bigcup_{i=1}^{\infty} E_i,\ E_i \cap E_j = \phi,\ i \neq j$ then summability of f on each E_i does not imply the summability of f on E. However, we have the following theorem.

Theorem 12. *Let a set E can be expressed as a countable union of pairwise disjoint measurable sets i.e.* $E = \bigcup_{i=1}^{\infty} E_i,\ E_i \cap E_j = \phi,\ i \neq j$, *then*

(i) If f is summable on E, then

$$\int_E f(x)dx = \sum_{i=1}^{\infty} \int_{E_i} f(x)dx,$$

(ii) If f is summable on each E_i, then f is summable on E if and only if the condition

$$\sum_{i=1}^{\infty} \int_{E_i} |f(x)|\,dx < \infty \text{ is satisfied.}$$

Proof. Since f is summable on E therefore f is summable on each set E_i and so f^+ and f^- are summable on each set E_i and are non negative, so by theorem 9 we have

$$\int_E f^+(x)dx = \sum_{i=1}^{\infty} \int_{E_i} f^+(x)dx \qquad .....(1)$$

$$\int_E f^-(x)dx = \sum_{i=1}^{\infty} \int_{E_i} f^-(x)dx \qquad .....(2)$$

Subtracting (2) from (1)

$$\int_E [f^+(x) - f^-(x)]dx = \sum_{i=1}^{\infty} \int_{E_i} f^+(x)dx - \int_{E_i} f^-(x)dx$$

$$= \sum_{i=1}^{\infty} \int_{E_i} [f^+(x) - f^-(x)]dx$$

$$\Rightarrow \qquad \int_E f(x)dx = \sum_{i=1}^{\infty} \int_{E_i} f(x)dx.$$

(ii) Since f is summable on each of the set $Ei \Rightarrow |f|$ is summable on each E_i.

$\because$ $|f|$ is non-negative measurable function on each E_i.

$\therefore$ By theorem 9, we have

$$\int_E |f(x)|\,dx = \sum_{i=1}^{\infty} \int_{E_i} |f(x)|\,dx \qquad .....(3)$$

$$\because \qquad \int_E |f(x)|\,dx < \infty \Rightarrow \sum_{i=1}^{\infty} \int_{E_i} |f(x)|\,dx < \infty$$

Conversely if $\sum_{i=1}^{\infty} \int_{E_i} |f(x)|\,dx < \infty$

$\Rightarrow$ $\int_E |f(x)|\,dx < \infty,$ [using (3)]

$\Rightarrow$ $|f|$ is summable on E

$\Rightarrow$ f is summable on E.

Theorem 13. *Let f be a summable function on set E. Then for given $\in > 0$, there exist a $\delta > 0$ such that*

$$\left|\int_e |f(x)|\,dx\right| < \in$$

where e is a measurable subset of E with $m(e) < \delta$.

Proof. Since f is summable on E, therefore $|f|$ is also summable on E. As $|f|$ is a non-negative valued function on E, given $\in > 0$ there exists a number $n_0 \in N$ such that

$$\int_E |f(x)|\,dx - \int_E [|f(x)|]_{n_0}\,dx < \frac{\in}{2}. \text{ [For } [f(x)]_n \leq f(x)]$$

$\Rightarrow$
$$\int_E (|f(x)| - [|f(x)|]_{n_0})dx < \frac{\in}{2} \qquad .....(1)$$

Choosing a real number δ such that $\delta < \frac{\in}{2n_0}$ and if e is measurable subset of E with $m(e) < \delta$, then we have

$$\int_E [|f(x)|]_{n_0}\,dx \leq n_0 m(e) < n_0\delta < \frac{\in}{2} \qquad .....(2)$$

From (1) and (2) we have

$$\left|\int_e f(x)dx\right| = \int_e (|f(x)| - [|f(x)|]_{n_0})dx + \int_e [|f(x)|]_{n_0}\,dx$$

$$\leq \int_E (|f(x) - [|f(x)|]_{n_0})dx + \int_e [|f(x)|]_{n_0}\,dx$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in$$

But $\left|\int_e f(x)\,dx\right| \leq \int_e |f(x)|\,dx$

Therefore $\int_e |f(x)dx| < \in.$

We shall now prove generalization of the Lebesgue theorem on the passage to limit under the integral sign for bounded function. The theorem is also true for summable function and know as dominated convergence theorem.

Theorem 14. (*Lebesgue dominated convergence theorem*) *Let* $\{f_n\}$ *be a sequence of measurable functions converging in measure to f. If there exists a non-negative summable function* ϕ *such that* $|f_n(x)| \leq \phi(x)$ *a.e. on E for each* $n \in N$, *then*

$$\lim_{n\to\infty} \int_E f_n(x)dx = \int_E f(x)dx.$$

Proof. Since $|f_n(x)| \leq \phi(x)$ *a.e.* on E, $\forall\, n \in N$

$\Rightarrow$ $\int_E |f_n(x)|\,dx \leq \int_E \phi(x)dx \quad \forall n \in N$

$\because$ ϕ is summable on E, so $\int_E \phi(x)dx < \infty$

$\Rightarrow$ $\int_E |f_n(x)|\,dx < \infty, \quad \forall\, n \in N$

$\Rightarrow$ Each function $f_n(x)$ is summable on E.

Now, we have $\{f_n\}$ is converging in measure to the function f on E, therefore by Reisz's theorem, $\exists$ a subsequence $\{f_{n_k}\}$ converging to f *a.e.* on E.

$\Rightarrow \quad |f(x)| \leq \phi\,(x)\ a.e.$ on E

$\Rightarrow \quad \int_E |f(x)|\,dx \leq \int_E \phi(x)dx$

$\Rightarrow \quad |f|$ is summable on E

$\Rightarrow \quad f$ is summable on E

Also we have $\quad |f_k(x) - f(x)| \leq |f_k(x)| + |f(x)|$

$\leq \phi\,(x) + \phi\,(x)$

$= 2\phi\,(x)\ a.e.$ on $E \quad \forall\, k \in N$

Since ϕ is non-negative summable function, therefore for every $\in > 0$, $\exists\, \delta > 0$ such that.

$$\int_E \phi(x)dx < \frac{\in}{4} \text{ for all subsets } e \subset E \text{ with } m\,(e) < \delta.$$

Take $\eta > 0$ be real number such that $\eta m(E) < \frac{\in}{4}$. Since $\{f_n\}$ converges in measure to f, so for given δ there exists a number n_0 such that for $k \geq n_0$

$$m\,(\{x \in E \,||f_k(x) - f(x)| \geq \eta\}) < \delta.$$

For each $k \in N$, let us break E into two subsets

$$A_k = \{x \in E \,||f_k(x) - f(x)| \geq \eta\}$$

$$B_k = \{x \in E \,||f_k(x) - f(x)| < \eta\}$$

Then $A_k \cup B_k = E$ and $A_k \cap B_k = \phi$ for each $k \in N$. If $k \geq n_0$ then $m\,(A_k) < \delta$, and according to the choice of δ

$$\int_{A_k} |f_k(x) - f(x)|\,dx \leq \int_{A_k} 2\phi(x)dx < 2 \cdot \frac{\in}{4} = \frac{\in}{2} \qquad .....(1)$$

Also for every $k \in N$, we have

$$\int_{B_k} |f_k(x) - f(x)|\,dx < \eta \cdot m\,(B_k) \leq \eta \cdot m(E) < \frac{\in}{2} \qquad .....(2)$$

Adding (3) and (4), we have $\forall\ k \geq n0$

$$\int_{A_k} |f_k(x) - f(x)|\,dx + \int_{B_k} |f_k(x) - f(x)|\,dx < \frac{\in}{2} + \frac{\in}{2} = \in$$

$$\Rightarrow \qquad \int_E |f_k(x) - f(x)|\,dx < \in \qquad .....(3)$$

Now $\quad \left| \int_E f_k(x)\,dx - \int_E f(x)\,dx \right| = \left| \int_E [f_k(x) - f(x)]dx \right|$

$$\leq \int_E |f_k(x) - f(x)|\,dx < \in \qquad \text{[using (3)]}$$

$$\Rightarrow \qquad \lim_{k \to \infty} \int_E f_k(x)\,dx = \int_E f(x)\,dx.$$

5.2.3 Absolutely equi-continuous integrals

If $< f_n >$ be a sequence of summable functions on the set E, then the sequence of integrals $\int_E f_n(x)dx$ of these functions is said to be **absolutely equi-continuous** if for each $\in > 0$, $\exists\ \delta > 0$ such that $\forall\ n \in N$

$$\int_A |f_n(x)|\, dx < \in$$

for any measurable set $A \subset E$ with $m(A) < \delta$.

Theorem 15. (*Vitali's Theorem*) : *Let* $< f_n >$ *be a sequence of summable functions on a set* E *with finite measure. If* $< f_n >$ *converges in measure to* f *and if the family of integrals of* f_n *is absolutely equi-continuous, then* f *is summable on* E *and*

$$\lim_{n\to\infty} \int_E f_n(x)\, dx = \int_E f(x)\, dx.$$

Proof. Let $\in > 0$ be a real number and let $\eta = \dfrac{\in}{m(E)}$. Since the family on integrals $\left\{\int_E f_n(x)\, dx\right\}$ is absolutely equi-continuous, for given $\in > 0$ there exist a $\delta > 0$ such that for all $n \in N$

$$\int_e |f_n(x)|\, dx < \in$$

for all measurable subsets e of E with $m(e) < \delta$. .....(1)

Since $\{f_n\}$ converging in measure to the function f, therefore by Reisz's theorem $\exists$ a subsequence $\{f_{n_k}\}$ which converges to f *a.e.* on E.

Now $\quad ||f_{n_k}(x)| - |f(x)|| \le |f_{n_k}(x) - f(x)|$.

It follows that $|f_{n_k}(x)|$ converges to $|f|$ *a.e.* on E.

$\therefore$ for $e \subset E$ with $m(e) < \delta$ we have by Fatou's Lemma

$$\int_e |f(x)|\, dx \le \sup_{n_k} \int_e |f_{n_k}(x)|\, dx < \in \qquad \text{[using (1)]}$$

$$\Rightarrow \qquad \int_e |f(x)|\, dx < \in \qquad .....(2)$$

Now for each $k \in N$, we define two sets

$$A_k = \{x \in E \,||f_k(x) - f(x)| \ge \eta\}$$

$$B_k = \{x \in E \,||f_k(x) - f(x)| < \eta\}$$

$$\Rightarrow \qquad A_k \cap B_k = \phi \text{ and } A_k \cup B_k = E \quad \forall\ n \in N.$$

For a given $\delta > 0$, $\exists\ n_0 \in N$ such that $\forall\ k \ge n_0$ with $m(A_k) < \delta$ we have

$$\int_{A_k} |f_k(x) - f(x)|\, dx \le \int_{A_k} |f_k(x)|\, dx + \int_{A_k} |f(x)|\, dx$$

$$< \in + \in = 2\in \qquad \text{[using (1) and (2)]}$$

$$\Rightarrow \qquad \int_{A_k} |f_k(x) - f(x)|\, dx < 2\in \qquad .....(3)$$

and $$\int_{B_k} |f_k(x) - f(x)|\, dx < \eta\, m\,(E) = \in \qquad .....(4)$$

$$\therefore \qquad \left| \int_E f_k(x)\, dx - \int_E f(x)\, dx \right| \leq \int_E |f_k(x) - f(x)|\, dx$$

$$= \int_{A_k} |f_k(x) - f(x)|\, dx + \int_{B_k} |f_k(x) - f(x)|\, dx$$

$$< 2\in + \in$$

$$= 3\in \qquad .....(5)$$

$\Rightarrow$ $|f_k - f|$ is summable on E

$\Rightarrow$ $f_k - f$ is summable on E,

$$f = f_k - (f_k - f)$$

$\Rightarrow$ f is summable on E

From (5), $\lim\limits_{k\to\infty} \int_E f_k(x)\, dx = \int_E f(x)\, dx.$

5.2.4 The space of summable functions

We denote by L_E or simply by L the space of summable functions on a measurable set $E \subset R$. Now we discuss some useful properties of L-space.

L-space is a linear space *i.e. (i)* $f \in L$, $g \in L \Rightarrow f + g \in L$ *(ii)* $c \in R, f \in L \Rightarrow c\,f \in L$ *(i)* & *(ii)* follows from the theorem 11.

We define a norm in the space L as

$$\|f\| = \int_E |f(x)|\, dx$$

for any function $f \in L$. It is obvious that $\|f\| \geq 0$ and $\|f\| = 0$ if and only if $f \sim 0$ *i.e.* $f = \theta$ as element of the set L (θ is zero element of space L).

If c is constant then $\|c\,f\| = |c|\,\|f\|$ obtain from theorem 11.

$$\|f + g\| = \int_E |f(x) + g(x)|\, dx$$

$$= \int_E |f(x)|\, dx + \int_E |g(x)|\, dx$$

$$= \|f\| + \|g\|$$

Hence L is a normed space.

Let $\{f_n\}$ be a sequence of functions in L. The sequence $\{f_n\}$ is said to be convergent to f in the **means of order on** or simply **convergent in mean** if $\lim\limits_{n\to\infty} \int_E |f_n(x) - f(x)|\, dx = 0$ and the sequence is said to be convergent in norm.

Theorem 16. *Let $\{f_n\}$ be a sequence in L. If $\{f_n\}$ converges in norm to a function f, then $\{f_n\}$ converges in measure.*

Proof. Let us assume that $\{f_n\}$ converges in norm to function f but $\{f_n\}$ does not converge to f in measure.

Then for some $\in > 0$ there is a $\delta > 0$ such that $m\,(\{x \in E : |f_k(x) - f(x)| \geq \in\}) \geq \delta$ for an infinite number of values of the index k; $k = k_1, k_2, \ldots, k_i, \ldots$.

If we write $A_k = \{x \in E : |f_k(x) - f(x)| \geq \in\}$, then

$$\int_E |f_{k_i}(x) - f(x)|\,dx \geq \int_{A_{k_i}} |f_{k_i}(x) - f(x)|\,dx \geq \delta$$

This means that the subsequence $\{f_{k_i}\}$ does not converge in mean to f, which is contradiction. Hence the theorem.

5.3 The space L_2 of square summable functions

In this section we study about the square summable functions and its space L_2. We establish that L_2 is a Banach space.

If f is a measurable function defined on a measurable set $E \subset R$ and $\int_E f^2(x)\,dx$ exist and finite then f is said to be a square summable function.

We denote by $L_2(E)$ or simply L_2 the space of all square summable functions on the set E. Generally E is taken as the closed interval $[a, b]$ and in such a case the integral $\int_E f^2(x)\,dx$ is written as $\int_a^b f^2(x)\,dx$.

Theorem 17. *The space L_2 of square summable functions is a linear space.*

Proof : To prove L_2 is a linear space we have to show that

(i) $f, g \in L_2 \Rightarrow f + g \in L_2$

(ii) $f \in L_2 \Rightarrow cf \in L_2, c \in R.$

(i) Let $f,\ g \in L_2$, therefore

$$\int_a^b f^2(x)\,dx < \infty \quad \text{and} \quad \int_a^b g^2(x)\,dx < \infty \qquad \text{......(1)}$$

$$\because \qquad [f(x) - g(x)]^2 \geq 0$$

$$\Rightarrow \qquad |f(x) \cdot g(x)| \leq \frac{f^2(x) + g^2(x)}{2} \quad \forall x \in E$$

$$\therefore \qquad \int_a^b |f(x)|\,dx \leq \frac{1}{2}\int_a^b [f^2(x) + g^2(x)]\,dx$$

$$= \frac{1}{2}\left[\int_a^b f^2(x)\,dx + \int_a^b g^2(x)\,dx\right] < \infty \text{ (2)}, \qquad \text{[using (1)]}$$

$\Rightarrow$ $|f.g|$ is summable, which means that $f.g.$ is summable. Now

$$\int_a^b [f(x)+g(x)]^2 dx = \int_a^b [f^2(x)+2f(x)g(x)+g^2(x)]\,dx$$

$$= \int_a^b [f^2(x)\,dx + 2\int_a^b f(x)\,g(x)\,dx + \int_a^b g^2(x)\,dx$$

$$< \infty + 2.\,\infty + \infty, \qquad \text{[using (1) \& (2)]}$$

$\Rightarrow$ $f + g \in L^2$.

(ii) For $c \in R$ and $f \in L_2 \Rightarrow \int_a^b [f^2(x)\,dx < \infty$

$\therefore$ $\int_a^b (cf)^2(x)\,dx = c^2 \int_a^b f^2(x)\,dx < \infty$

$\Rightarrow$ $cf \in L_2$.

Which establish that L_2 is a linear space.

We shall now establish two inequalities which play an important role in the study of L_2 space.

Theorem 18. *(Cauchy-Bunyakowski-Schwarty inequality or CBS inequality)*

If $f, g\, L_2$ then

$$\left[\int_a^b f(x)\,g(x)\,dx\right]^2 \le \left[\int_a^b f^2(x)\,dx\right]\left[\int_a^b g^2(x)\,dx\right]$$

Proof : For $\lambda \in R$ and $\lambda \neq 0$

$$\int_a^b [\lambda f(x)+g(x)]^2 dx = \lambda^2 \int_a^b f^2(x)\,dx + 2\lambda \int_a^b f(x)\,g(x)\,dx + \int_a^b g^2(x)\,dx \qquad .....(1)$$

We know that the quadratic expression $A\lambda^2 + B\lambda + C$ has non negative values for all $\lambda \in R$ if $B^2 - 4AC \le 0$ and $A > 0$. Since the expression on R.H.S. of (1) is non-negative as L.H.S. of (1) is non-negative for all $\lambda \in R$,

Hence $B^2 \le AC$

$$\Rightarrow \left[2\int_a^b f(x)\,g(x)\,dx\right]^2 \le 4\left[\int_a^b f^2(x)\,dx\right]\left[\int_a^b g^2(x)\,dx\right]$$

$$\Rightarrow \left[\int_a^b f(x)\,g(x)\,dx\right]^2 \le \left[\int_a^b f^2(x)\,dx\right]\left[\int_a^b g^2(x)\,dx\right].$$

Corollary : If $(b - a) < \infty$, then every square summable function is summable *i.e.* $L_2 \subset L$.

Proof : If we take $g(x) = 1, f(x) = |f(x)|$ in CBS inequality, we get

$$\left[\int_a^b |f(x)|\,dx\right]^2 \le \left[\int_a^b |f^2(x)\,dx|\right]\left[\int_a^b 1\cdot dx\right]$$

$$= \left[\int_a^b |f^2(x)\,dx|\right](b-a)$$

$$< \infty \qquad [\because f \in L_2 \text{ and } (b-a) < \infty]$$

$\Rightarrow \quad \left[\int_a^b | f(x)\, dx |\right] < \infty$

$\Rightarrow \quad | f(x) |$ is summable *i.e.* $|f| \in L \Rightarrow f \in L.$

Theorem 19. (*Minkowski's inequality*). *If f and g* $\in L_2$, *then*

$$\left[\int_a^b [f(x)+g(x)]^2\, dx\right]^{1/2} \le \left[\int_a^b f^2(x)\, dx\right]^{1/2} \left[\int_a^b g^2(x)\, dx\right]^{1/2}.$$

Proof : By CBS inequality, we have

$$\int_a^b f(x)\, g(x)\, dx \le \left[\int_a^b f^2(x)\, dx\right]^{1/2} \left[\int_a^b g^2(x)\, dx\right]^{1/2}$$

$$\Rightarrow \quad 2\int_a^b f(x) g(x)\, dx + \int_a^b f^2(x)\, dx + \int_a^b g^2(x)\, dx$$

$$\le 2\left[\int_a^b f^2(x)\, dx\right]^{1/2} \left[\int_a^b g^2(x)\, dx\right]^{1/2} + \int_a^b f^2(x)\, dx + \int_a^b g^2(x)\, dx$$

$$\Rightarrow \quad \int_a^b [f(x)+g(x)]^2\, dx \le \left[\left[\int_a^b f^2(x)\, dx\right]^{1/2} + \left[\int_a^b g^2(x)\, dx\right]^{1/2}\right]^2$$

$$\Rightarrow \quad \left[\int_a^b [f(x)+g(x)]^2\, dx\right]^{1/2} \le \left[\int_a^b f^2(x)\, dx\right]^{1/2} + \left[\int_a^b g^2(x)\, dx\right]^{1/2}$$

Theorem 20. *The space* L_2 *is a normed linear space.*

Proof : We define a function

$$\| \cdot \| : L_2 \to R \quad \text{as} \quad \| f \| = \left[\int_a^b f^2(x)\, dx\right]^{1/2}$$

We observe that

(i) $\|f\| \ge 0 \;\forall\; f \in L_2$ and

$$\|f\| = 0 \Leftrightarrow \left[\int_a^b f^2(x)\, dx\right]^{1/2} = 0$$

$$\Leftrightarrow f^2(x) = 0 \;\; \forall\, x \in E$$

$$\Leftrightarrow f(x) = 0 \;\; \forall\, x \in E$$

$$\Leftrightarrow f = \theta \text{ (zero function)}$$

(ii) For $c \in R$ and $$\| cf \| = \left[\int_a^b (cf)^2(x)dx\right]^{1/2}$$

$$= \left[\int_a^b c^2 f^2(x)\,dx\right]^{1/2}$$

$$= |c| \left[\int_s^b f^2(x)\,dx\right]^{1/2}$$

$$= |c| \, \|f\|$$

(iii) By Minkowsky inequality

$$f, g \in L_2 \Rightarrow \|f + g\| \leq \|f\| + \|g\|$$

Hence the function $\| \cdot \|$ is norm in the space L_2, consequently L_2 is normed linear space.

Now we introduce the notion of convergence in norm.

Let $\{f_n\}$ be a sequence in L_2. The sequence is said to converge in norm to a function $f \in L_2$ if for any arbitrary real number $\in > 0$ there exists a number $n_0 \in N$ such that $\|f_n - f\| < \in$ whenever $n \geq n_0$ and in such a case we write $\lim_{n\to\infty} f_n = f$ or $f_n \to f$.

The convergence in the mean is also termed as convergence in the mean of order two or the convergence in the mean square and the expression $\lim_{n\to\infty} f_n = f$ means that

$$\lim_{n\to\infty} \int_a^b [f_n(x) - f(x)]^2 \, dx = 0.$$

Theorem 21. *Let $\{f_n\}$ be a sequence in L_2. If $\{f_n\}$ converges in the mean square to a function $f \in L_2$, then $\{f_n\}$ converges in measure to f.*

Proof : Let us assume that $\{f_n\}$ converges in the mean square to a function f but $\{f_n\}$ does not converges in measure to f. This means that for some $\in > 0$ there is a $\delta > 0$ such that

$$m(\{x \in [a, b] : |f_k(x) - f(x)| \geq \in\}) \geq \delta$$

for an infinite number of values $k_1, k_2, ..., k_i, ...$. For the index k. If we write

$$e_k = \{x \in [a, b] \mid |f_k(x) - f(x)| \geq \in\}$$

then $$\int_a^b |f_{k_i}(x) - f(x)|^2 \, dx \geq \int_{e_{k_1}} |f_{k_i}(x) - f(x)|^2 \, dx$$

$$\geq \delta \in^2.$$

This means that the subsequence $\{f_{k_1}\}$ does not converge in the mean, which is contradiction. Hence the theorem.

A sequence $\{f_n\}$ in L_2 is a **Cauchy sequence** in L_2 if for every $\in > 0$ there exist a number $n_0(\in) \in N$, such that $m, n \geq n_0(\in)$, then $\|f_m - f_n\| < \in$.

Theorem 22. *If a sequence* $\{f_n\}$ *in* L_2 *converges in norm to a function f in* L_2 *then it is a Cauchy sequence.*

Proof : Let $\in > 0$ be a real number. Since $\lim_{n\to\infty} f_n = f$ there exists a number $n_0 (\in) \in N$ such that $\|f_n - f\| < \in$ whenever $n \geq n_0$. Now if $m \geq n_0$ and $n \geq n_0$, then

$$\|f_m - f_n\| = \|f_m - f + f - f_n\|$$

$$\leq \|f_m - f\| + \|f_n - f\| \qquad \text{[Minkowski's inequality]}$$

$$< \in + \in$$

$$= 2\in$$

Hence the theorem.

Theorem 23 : *A Cauchy sequence* $\{f_n\}$ *in* L_2 *converges to an element f in* L_2

or

Prove that L_2 *is a complete space.*

Proof : Since $\{f_n\}$ is a Cauchy sequence in L_2, so for each $\in > 0$ there exist a number $n_0 (\in) \in N$ such that if $m, n \geq n_0 (\in)$ then

$$\|f_m - f_n\| < \in \qquad .....(1)$$

Then there exist a subsequence $\{f_{n_k}\}$ such that

$$\|f_{n_{k+1}} - f_{n_k}\| < \frac{1}{2^k} \quad \text{for } k \in N \qquad ....(2)$$

In CBS inequality on putting $f(x) = |f_{n_{k+1}} - f_{n_k}|$ and $g(x) = 1$, we get

$$\int_a^b |f_{n_{k+1}}(x) - f_{n_k}(x)|\, dx \leq \sqrt{b-a}\, \|f_{n_{k+1}} - f_{n_k}\|$$

and therefore using (2)

$$\sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|\, dx \leq \sqrt{b-a} \sum_{k=1}^{\infty} \|f_{n_{k+1}} - f_{n_k}\|$$

$$< \sqrt{b-a} \sum_{k=1}^{\infty} \frac{1}{2^k}$$

$$\Rightarrow \quad \sum_{k=1}^{\infty} \int_a^b |f_{n_{k+1}}(x) - f_{n_k}(x)|\, dx < \infty$$

and consequently the series $\sum_{k=1}^{\infty} \int_a^b |f_{n_{k+1}} - f_{n_k}|\, dx$ is convergent.

$\Rightarrow$ The series $|f_{n_1}(x)| + \sum_{k=1}^{\infty} \left| f_{n_{k+1}}(x) - f_{n_k}(x) \right|$ converges almost everywhere on $[a, b]$. We know that every absolutely convergent series is convergent therefore the series

$$f_{n_1}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$$

converges *a.e.* on $[a, b]$. This is equivalent to

$$\lim_{k\to\infty} f_{n_k}(x) \text{ exists } a.e. \text{ on } [a, b]$$

Let $E_1 = \left\{ x \in [a,b] \middle| \lim_{k\to\infty} f_{n_k}(x) < \infty \right\}$

then $m([a, b] \sim E_1) = 0$.

Now we define a function f on $E = [a, b]$ as follows :

$$f(x) = \begin{cases} \lim_{k\to\infty} f_{n_k}(x) & \text{for } x \in E_1 \\ 0 & \text{for } x \in [a,b] \sim E \end{cases}$$

Then f is a measurable function and

$$\lim_{k\to\infty} f_{n_k}(x) = f(x) \ a.e. \text{ on } [a, b].$$

We shall now show that $f \in L_2$ and $\lim_{n\to\infty} f_n = f$. If k_0 is a number such that $n_{k_0} > n_0(\in)$, then from (1), we have

$$\int_a^b [f_n(x) - f_{n_k}(x)]^2 \, dx < \in^2 \quad \forall\ n > n_0 \text{ and } k > k_0.$$

Since sequence of functions $\{(f_n - f_{n_k})^2\}$ converges a.e. on E to $(f_n - f)^2$ so by Fatou's Lemma we have

$$\int_a^b [f_n(x) - f(x)]^2 \, dx \le \limsup_{n\to\infty} \int_a^b [f_n(x) - f_{n_k}(x)]^2 \, dx$$

$$\Rightarrow \quad \int_a^b [f_n(x) - f(x)]^2 \, dx < \in^2 \quad \forall\ n > n_0$$

i.e. $\quad \|f_n - f\| < \in \ \forall\ n > n_0$.

Consequently $\quad \lim_{n\to\infty} f_n = f$

Since $(f_n - f) \in L_2$ it follows that

$$f = f_n - (f_n - f) \in L_2.$$

Thus every Cauchy sequence in L_2 converges to point in L_2. Hence L_2 is complete space.

Note : Through this theorem we have proved that L_2 is a complete space. As L_2 is a normed linear space, therefore L_2 is a Banach space.

Self-Learning Exercise-1

1. Let $f(x) = \frac{1}{\sqrt{x}}$ for $x \in (0, 1]$ and $f(0) = 0$. Then define $[f(x)]_n$.
2. If f is a non-negative unbounded measurable function defined on a measurable set E, then $[f]_n$ is measurable on E for each $n \in N$. [True/False]
3. Let f be a measurable function defined on a measurable set E. If $|f|$ is summable on E, then it is not necessary that f is summable on E. True/False
4. The space L_2 of square summable functions is a linear space. True/False
5. Every square summable function is a summable function. True/False
6. L_2-space is not a Banach space. True/False

5.4 Summary

In this unit, we have discussed about summable functions, Lebsgue integral of non-negative unbounded functions, Lebsgue integral of arbitrary functions, the space L_2 of square summable functions and some important results on these topics.

5.5 Answers to self-learning exercises

1.

$$[f(x)]_n = \frac{1}{\sqrt{x}} \quad \text{for } \frac{1}{n^2} \le x \le 1$$

$$= n \quad \text{for } 0 < x < \frac{1}{n^2}$$

$$= 0 \quad \text{for } x = 0.$$

2. True. **3.** False. **4.** True.

5. True. **6.** False.

5.6 Exercises

1. Prove that a summable function is finite a.e.
2. If $m(E) = 0$, then every function f defined on E is summable on E and $\int_E f(x)\,dx = 0$.
3. If f is summable on E, then it is summable on every subset of E.
4. If f is non-negative valued measurable function on $[a, b]$ and if $f(x) \le g(x)\ \forall\ x \in [a, b]$ where g is a summable on $[a, b]$. Then prove that f is a summable function.
5. Let the functions f and g be equivalent. If one the integrals exists, then so does the other, and the two integrals are equal.

6. If f is summable on $[a, b]$ then show that

$$\int_a^b f(x)\,dx = \int_{-b}^{-a} f(-x)\,dx.$$

7. Show that the space L_2 of all square summable functions is a metric space.

8. Show that the product of two square summable functions is summable.

9. Let $\{f_n\}$ be a sequence of functions in L_2 converges in norm to f. Then for any $g \in L_2$ show that

$$\lim_{n\to\infty} \int_a^b f(x)\,g(x)\,dx = \int_b^a f(x)\,g(x)\,dx.$$

10. Show that the space L_2 of square summable functions is a Banach space.

□□□

Unit 6 : Fourier Series and Coefficients, Parseval's Identity, Riesz-Fisher Theorem

Structure of the Unit

6.0 Objectives

6.1 Introduction

6.2 Scalar product

6.3 Hilbert space

6.4 Orthogonal elements

6.5 Orthogonal system

6.6 Fourier series and coefficients

6.7 Closed orthogonal system

6.8 Complete orthogonal system

6.9 Summary

6.10 Answers to self-learning exercises

6.11 Exercises

6.0 Objectives

In this unit we will first define scalar product of two functions in L_2 space. With the help of scalar product we will define Hilbert space. Hilbert space play very important role in functional analysis. Next we will study Fourier series and its properties in L_2 space. The results related to Fourier series is very useful in mathematical physics. Parseval's identity, Bessel's inequality play an important role in wave mechanics.

6.1 Introduction

In this unit, we will first study the definition of scalar product of two functions and its properties in L_2 space. Next we will give definition of Hilbert space, orthogonal function and orthonormal system. Next we will study Fourier series, Bessel's inequality and Parseval identity. With the help of Parseval identity we will define closed system. In the last we study Riesz Fischer theorem, complete orthonormal system and its propertics.

6.2 Scalar product

The **scalar product** of two functions $f \in L_2$ and $g \in L_2$ denoted as $<f, g>$, is defined as the integral of the product of the functions :

$$<f, g> = \int_a^b f(x)\, g(x)\, dx.$$

By CBS inequality for f, g $\in L_2$ the scalar product $<f, g>$ has finite value for any two functions $f, g \in L_2$. It is obvious that if one or both functions in a scalar product are replaced by equivalent functions the scalar product does not change.

The scalar product has following properties :

(*a*) $<f, g> = <g, f>$

Proof: $<f, g> = \int_a^b f(x)\, g(x)\, dx = \int_a^b g(x)\, f(x)\, dx = <g, f>$

(*b*) $<f + g, h> = <f, h> + <g, h>$

Proof :

$$\begin{aligned} <f + g, h> &= \int_a^b (f(x) + g(x)) h(x)\, dx \\ &= \int_a^b f(x)\, h(x)\, dx + \int_a^b g(x)\, h(x)\, dx \\ &= <f, h> + <g, h> \end{aligned}$$

(*c*) $<cf, g> = c<f, g> = <f, cg>$ where $c \in R$

(*d*) $<f, f> \geq 0 \;\forall\, f \in L_2$ and

$<f, f> = 0$ iff $f = \theta$.

Theorem 1. *If $f, g \in L_2$ then*

$$|<f, g>| \leq \|f\|_{L_2} \|g\|_{L_2}$$

Proof. From CBS inequality we know that if $f, g \in L_2$ then

$$\left[\int_a^b f(x) g(x)\, dx\right]^2 \leq \left[\int_a^b f^2(x)\, dx\right]\left[\int_a^b g^2(x)\, dx\right]$$

$\Rightarrow$ $$<f, g>^2 \leq \|f\|_{L_2}^2 \|g\|_{L_2}^2$$

or equivalently $|<f, g>| \leq \|f\|_{L_2} \|g\|_{L_2}$.

6.3 Hilbert space

A Branch space is called a **Hilbert space** if for any two elements f and g of it there is associated a real number called their scalar product $<f, g>$ satisfying properties (*a*) to (*d*) and the **norm** of an element f of it is expressed in terms of scalar product $\|f\| = \sqrt{<f, f>}$.

Thus L_2 is a real Hilbert space.

Theorem 2. *The scalar product in L_2 is a continuous function of its argument that is if $\{f_n\}$ and $\{g_n\}$ are two convergent sequences in L_2 with $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$ then*

$$\lim_{h\to\infty} < f_n, g_n > = < f, g >$$

Proof. We have $\{f_n\}$ and $\{g_n\}$ are two convergent sequences in L_2 and

$$\lim_{n\to\infty} f_n = f,\ \lim_{n\to\infty} g_n = g.$$

Now $\lim_{n\to\infty} f_n = f \Rightarrow$ for every $\in > 0$, $\exists\, n_0 \in N$ such that

$$\| f_n - f \| < \in\ \forall\ n \geq n_0.$$

$\because$ $|\, \|f_n\| - \|f\|\, | \leq \|f_n - f\|$, so $\forall\ n > n_0$, $|\, \|f_n\| - \|f\|\, | < \in$

$\Rightarrow$ $\{\|f_n\|\}$ is a convergent sequence of real numbers and it is bounded, since every convergent sequence is bounded.

Similarly $\{\|g_n\|\}$ is a convergent and bounded sequence of real numbers.

$\Rightarrow$ $\exists\, K \in R$ *s.t.* $\|g_n\| < K,\ \forall\ n \in N$

For any $n \in N$, we have

$$<f_n, g_n> - <f, g> = <f_n, g_n> - <f, g_n> + <f, g_n> - <f, g>$$
$$= <f_n - f, g_n> + <f, g_n - g>$$

$\therefore$ $$|<f_n, g_n> - <f, g>| \leq |<f_n - f, g_n>| + |<f, g_n - g>|$$
$$\leq \|f_n - f\|\,\|g_n\| + \|f\|\,\|g_n - g\| \qquad \text{[by theorem 1]}$$
$$< \in K + \|f\| \cdot \in = \in [K + \|f\|] \to 0 \text{ as } n \to \infty$$

$\therefore$ $$\lim_{n\to\infty} <f_n, g_n> = <f, g>.$$

6.4 Orthogonal elements

Two functions f and g in $L_2\,[a, b]$ are said to be **orthogonal** on the closed interval $[a, b]$, written as $f \perp g$ if $<f, g> = 0$.

Equivalently, two measurable functions f and g defined on the closed interval $[a, b]$ are said to be orthogonal if

$$\int_a^b f(x)g(x)dx = 0.$$

Form the properties of the scaler product it follows immediately that

(i) The zero element θ is orthogonal to any element $f \in L_2$.

(ii) $$f \perp f \Leftrightarrow <f, f> = 0$$
$$\Leftrightarrow \|f\|^2 = 0$$
$$\Leftrightarrow \|f\| = 0$$
$$\Leftrightarrow f = \theta.$$

(*iii*) If f is the sum of functions f_i (the sum involves a finite or countable number of summands) such that

$$f = \sum_i cf_i \text{ and } g \perp f\,;\ \forall\ i, \text{ then } f \perp g.$$

Lemma : If $f = \sum_i f_i$ (the sum contains finite or countable number of summands) and if all the elements f_i are pairwise orthogonal, then

$$\|f\|^2 = \sum_i \| f_i \|^2$$

Proof. Using the distributivity property of the scaler product, we have

$$\|f\|^2 = <f, f>$$

$$= \left\langle \sum_i f_i, \sum_j f_j \right\rangle = \sum_i \sum_j \langle f_i, f_j \rangle \qquad .....(1)$$

Since f_i are pairwise orthogonal, so

$<f_i, f_j> = 0,\ i \neq j$ and therefore form (1) we have

$$\|f\|^2 = \sum_i \langle f_i, f_i \rangle$$

$$= \sum_i \| f_i \|^2.$$

Theorem 3. *A series* $\sum_{i=1}^{\infty} f_i$ *of pairwise orthogonal elements in* L_2 *is convergent iff the series of real numbers* $\sum_{i=1}^{\infty} \| f_i \|^2$ *is convergent.*

Proof. First suppose that the series $\sum_{i=1}^{\infty} f_i$ is convergent and converges to f (say).

Since f_i are pairwise orthogonal

$$\Rightarrow \qquad <f_i, f_j> = 0\ \forall\ i \neq j \qquad .....(1)$$

$$\therefore \qquad \|f\|^2 = <f, f> = \left\langle \sum_{i=1}^{\infty} f_i, \sum_{j=1}^{\infty} f_i \right\rangle = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle f_i, f_j \rangle$$

$$= \sum_{i=1}^{\infty} \langle f_i, f_i \rangle \qquad \text{using (1)}$$

$$= \sum_{i=1}^{\infty} \| f_i \|^2$$

Thus $\sum_{i=1}^{\infty} \| f_i \|^2 = \| f \|^2$.

$\Rightarrow$ The series of real numbers $\sum_{i=1}^{\infty} \| f_i \|^2$ converges to $\|f\|^2$.

Conversely, suppose that the series of real numbers $\sum_{i=1}^{\infty} \| f_i \|^2$ is convergent, we have to show that the series $\sum_{i=1}^{\infty} f_i$ is convergent.

We define the partial sum of the series as :

$$s_n = \sum_{i=1}^{n} f_i,$$

then $\{s_n\}$ is the sequence of partial sums of series $\sum_{i=1}^{\infty} f_i$

Let $p > n$,

$$\| s_p - s_n \|^2 = \left\| \sum_{i=1}^{p} f_i - \sum_{i=1}^{n} f_i \right\|^2$$

$$= \left\| \sum_{i=n+1}^{p} f_i \right\|^2$$

$$= \left\langle \sum_{i=n+1}^{p} f_i, \sum_{j=n+1}^{p} f_j \right\rangle$$

$$= \sum_{i=n+1}^{p} \sum_{j=n+1}^{p} < f_i, f_j >$$

$$= \sum_{i=n+1}^{p} < f_i, f_i > \qquad \text{[Since } f_i \text{ are pairwise orthogonal]}$$

$$= \sum_{i=n+1}^{p} \| f_i \|^2 .$$

$$\Rightarrow \qquad \| s_p - s_n \|^2 = \sum_{i=n+1}^{p} \| f_i \|^2 . \text{ for } p > n \qquad .....(1)$$

Since the series $\sum_{i=1}^{\infty} \| f_i \|^2$ as convergent, so for each $\in > 0$ there exists $n_0 \in N$ such that

$$s'_p - s'_n < \in \quad \forall p > n \geq n_0, \; where \;\; s'_p = \sum_{i=1}^{p} \| f_i \|^2 .$$

$$\Rightarrow \qquad \sum_{i=n+1}^{p} \| f_i \|^2 < \in \quad \forall \; p > n \qquad .....(2)$$

Using (2) in (1) we get

$$\| s_p - s_n \|^2 < \in \quad \forall \; p > n$$

$\Rightarrow$ $\| s_p - s_n \|^2 \to 0$ as $p, n \to \infty$ and hence $\| s_p - s_n \| \to 0$ as $p, n \to \infty$. This shows that $\{s_n\}$ is a Cauchy sequence in L_2. Since L_2 is a complete space, so every Cauchy sequence in L_2 converges to an element in L_2 and consequently the sequence $\{s_n\}$ in convergent which means that the series $\sum_{i=1}^{\infty} f_i$ is convergent.

6.5 Orthonormal system

A system of functions $\phi_1, \phi_2, \phi_i, ...$ (finite or countable) in $L_2 [a, b]$ is called a **orthonormal system** on the closed interval $[a, b]$ if

(i) $< \phi_i, \phi_j > = 0$ for $i \neq j$

(ii) $< \phi_i, \phi_i > = 1$ *i.e.* $\| \phi_i \| = 1 \; \forall \; i$

An important example of an orthonormal system on the interval $[-\pi, \pi]$ is the well known trigonometric system of functions,

$$\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \sin x, ..., \frac{1}{\sqrt{\pi}} \cos kx, \frac{1}{\sqrt{\pi}} \sin kx, ...$$

6.6 Fourier series and coefficients

Let $\{\phi_i\}$ be an orthonormal system of functions in L_2. For any function $f \in L_2$ the scalar product $a_i = < f, \phi_i >$, $i = 1, 2, 3...$ are called **Fourier coefficients** of the function f with respect to the orthonormal system $\{\phi_i\}$ and the series $\sum_{i=1}^{\infty} a_i \, \phi_i$ is called **Fourier series** of f with respect to that orthonormal system.

Theorem 5. *The Fourier series of any function $f \in L_2$ converses in norm. Further it converges to f if and only if*

$$\| f \|^2 = \sum_{i=1}^{\infty} a_i^2$$

where a_i are the Fourier coefficients of the function f. If $\Sigma a_i f_i$ is the Fourier series of the square summable function with respect to an orthonormal sequence $< f_n >$ of square summable functions then show that

$$\sum_{i=1}^{\infty} a_i^2 \le || f ||^2$$

Proof. Let $f = \sum_{i=1}^{\infty} a_i \phi_i$ be the Fourier series for the function f, where $a_i = < f, \phi_i >$, $i \in N$ and $\{\phi_i\}$ be an orthonormal system of functions in L_2.

For any $n \in N$, let us put

$$g(x) = f(x) - \sum_{i=1}^{n} a_i \phi_i(x).$$

Then for any $i = 1, 2, ..., n$, we have

$$< g, \phi_j > = \left\langle f(x) - \sum_{i=1}^{n} a_i \phi_i, \phi_j \right\rangle = < f, \phi_j > - \sum_{i=1}^{n} a_i < \phi_i, \phi_j >$$

$$= a_j - a_j || \phi_j ||^2$$

$$= a_j - a_j \cdot 1 \qquad \{\text{Since } < \phi_i, \phi_j > = 0 \text{ if } i \ne j, || \phi_i || = 1\}$$

$$= 0$$

$$< g, \phi_j > = 0 \quad \forall j = 1, 2, ..., n$$

$\Rightarrow g \perp \phi_j$ *i.e.* g is orthogonal to all ϕ_j, $j = 1, 2, ..., n$.

Thus we have $$f = g + \sum_{i=1}^{n} a_i \phi_i \qquad .....(1)$$

We know that if $f_1, f_2, ..., f_n, ...$. are pairwise orthogonal function such that $\sum f_i = f$ then $\sum_i || f_i ||^2 = || f ||^2$.

Now from (1) we have

$$||f||^2 = || g ||^2 + \sum_{i=1}^{n} || \alpha_i \phi_i ||^2$$

$$||f||^2 = || g ||^2 + \sum_{i=1}^{n} a_i^2 || \phi_i ||^2 = || g ||^2 + \sum_{i=1}^{n} a_i^2, \text{ since } || \phi_i || = 1$$

Consequently, $$\sum_{i=1}^{n} a_i^2 \le || f_i ||^2 \quad \text{for any } n \in N.$$

Since R.H.S. of above inequality is independent of n, so we have

$$\sum_{i=1}^{\infty} a_i^2 \leq \|f\|^2 \qquad \text{.....(2)}$$

$\Rightarrow$ The series $\sum_{i=1}^{\infty} a_i^2$ is convergent [$\because$ $\|f\|2 = <f, f> < \infty$]

$\Rightarrow$ The series $\sum_{i=1}^{\infty} a_i \phi_i$ converges in norm.

[By Theorem (3) $\sum_{i=1}^{\infty} a_i \phi_i$ is convergent iff the series $\sum_{i=1}^{\infty} \|a_i \phi_i\|^2$ is convergent *i.e.* $\sum_{i=1}^{\infty} a^2$ is convergent]

$\Rightarrow$ The Fourier series of $f \in L_2$ converges in norm.

Part II. Let as assume that the Fourier series $\sum_{i=1}^{\infty} a_i \phi_i$ is convergent and converges to f, therefore

$$f = \sum_{i=1}^{\infty} a_i \phi_i$$

$$\Rightarrow \qquad \|f\|^2 = \sum_{i=1}^{\infty} \|a_i \phi_i\|^2 \qquad \text{[By Lemma]}$$

$$\Rightarrow \qquad \|f\|^2 = \sum_{i=1}^{\infty} a_i^2 \|\phi_i\|^2 \qquad [\because \ \|\phi_i\| = 1 \ \forall \ i]$$

$$= \sum_{i=1}^{\infty} a_i^2 .$$

Conversely suppose that the condition

$$\|f\|^2 = \sum_{i=1}^{\infty} a_i^2 \text{ be fulfilled, writing}$$

$$h = f - \sum_{i=1}^{\infty} a_i \phi_i ,$$

we can easily prove that h is orthogonal to ϕ_i, $i = 1, 2, ...$.

$$\therefore \qquad f = h + \sum_{i=1}^{\infty} a_i \phi_i$$

Again from Lemma

$$\|f\|^2 = \|h\|^2 + \sum_{i=1}^{\infty} \|a_i\, \phi_i\|^2 = \|h\|^2 + \sum_{i=1}^{\infty} a_i^2.$$

Since $\quad \|f\|^2 = \sum_{i=1}^{\infty} a_i^2$, by hypothesis,

we have $h = \theta$ that is $\quad f = \sum_{i=1}^{\infty} a_i\, \phi_i$

$\Rightarrow \quad \sum_{i=1}^{\infty} a_i\, \phi_i$ converses to f.

Note : The inequality $\sum_{i=1}^{\infty} a_i^2 \le \|f\|^2$ is known as **Bessel's inequality** and its particular case $\sum_{i=1}^{\infty} a_i^2 = \|f\|^2$ is called **Parseval's identity.**

6.7 Closed orthonormal system

An orthonormal system $\{\phi_i\}$ is said to be **closed** if it satisfies Parseval's identity $\sum_{i=1}^{\infty} a_i^2 = \|f\|^2$ for the function f, where a_i are Fourier coefficients for f with respect to ϕ_i.

Theorem 6. *If the orthonormal system $\{\phi_i\}$ is closed and if f and g belong to L_2,*

then $$\int_a^b f(x)\, g(x)\, dx = \sum_{i=1}^{\infty} \alpha_i\, \beta_i$$

where $\quad \alpha_i = \langle f, \phi_i \rangle$ and $\beta_i = \langle g, \phi_i \rangle$.

Proof. Since α_i, ϕ_i are the Fourier coefficients for the function f and g respectively then $\alpha_i + \beta_i$ will be the Fourier coefficients of $f + g$.

Therefore $$\|f + g\|^2 = \sum_{i=1}^{\infty} (\alpha_i + \beta_i)^2$$

$$\Rightarrow \quad \int_a^b (f+g)^2(x)\, dx = \sum_{i=1}^{\infty} (\alpha_i + \beta_i)^2$$

$$\Rightarrow \quad \int_a^b [f(x)]^2\, dx + 2\int_a^b f(x)\, g(x)\, dx + \int_a^b [g(x)]^2\, dx$$

$$= \sum_{i=1}^{\infty} \alpha_i^2 + 2\sum_{i=1}^{\infty} \alpha_i\, \beta_i + \sum_{i=1}^{\infty} \beta_i^2$$

Using Parseval's identity we have

$$\int_a^b f(x)\, g(x)\, dx = \sum_{i=1}^{\infty} \alpha_i \, \beta_i .$$

Let $D \subset L_2$. The set D is said to be **everywhere dense** in L_2 is every element (function) in L_2 is the limit (in norm) of a sequence in D.

Thus a set $D \subset L_2$ is every where dense in L_2 iff for any $f \in L_2$ and for each $\in > 0$ there is an element $g \in D$ such that $\| f - g \| < \in$.

Theorem 7. *Let a set $D \subset L_2$ be everywhere dense in L_2. If Parseval's identity holds for all functions in D, then the system $\{\phi_i\}$ is closed.*

Proof. Let $f \in L_2$ and $\sum_{i=1}^{\infty} a_i \, \phi_i$ be Fourier series of f, where $a_i = \langle f, \phi_i \rangle$.

Let $S_n(f) = \sum_{i=1}^{n} a_i \, \phi_i$ be the partial sum of first n terms of the Fourier series. Then $S_n(f)$ satisfies following properties.

(i) $S_n(cf) = c\, S_n(f)$ for any $c \in R$,

(ii) $S_n(f_1 + f_2) = S_n(f_1) + S_n(f_2)$

(iii) $\| S_n f \| \le \| f \|$

and the last one follows from Bessel's inequality,

$$\| S_n \|^2 = \sum_{i=1}^{n} a_i^2 \le \| f \|^2 .$$

Since D is dense in L_2, so for $\in > 0$ $\exists$ a function $g \in D$ such that

$$\| f - g \| < \frac{\in}{3}.$$

Then $\qquad \| f - S_n(f) \| \le \| f - g \| + \| g - S_n(g) \| + \| S_n(g) - S_n(f) \|$

But $\qquad \| S_n(g) - S_n(f) = \| S_n(g - f) \| \le \| g - f \| < \frac{\in}{3}$ and therefore

$$\| f - S_n(f) \| < \frac{2\in}{3} + \| g - S_n(g) \|$$

and since Paseval's identity holds for $g(x)$, so for $\in > 0$, $\exists\, n_0 \in N$ such that

$$\| f - S_n(f) \| < \frac{\in}{3} \qquad \forall\, n > n_0.$$

Hence $\qquad \| f - S_n(f) \| < \frac{2\in}{3} + \frac{\in}{3} = \in \quad \forall\, n > n_0.$

This proves the result.

Theorem 8. (*Riesz-Fisher theorem*). *Let* $\{\phi_i\}$ *be an orthonormal system in* L_2 *and* $\{a_i\}$ *be sequence of real numbers such that the series* $\sum a_i^2$ *is convergent.*

Then there exists a function $f \in L_2$ *such that* $\| f \|^2 = \sum a_i^2$ *where* $a_i = \langle f_i\, \phi_i \rangle \ \ \forall\, i \in N$.

Proof. Since the series $\sum a_i^2$ is convergent, so for given $\in > 0$, there exists a number $n_0(\in) \in N$ such that $p > n > n_0$ imply

$$S'_p - S'_n < \in^2, \text{ where } S'_p = \sum_{i=1}^{p} a_i^2 \Rightarrow \sum_{i=n+1}^{p} a_i^2 < \in^2.$$

Consider a sequence $\{S_n\}$ in L_2 as

$$S_n(x) = \sum_{i=1}^{n} a_i\, \phi_i(x),$$

then
$$\| S_p - S_n \|^2 = \left\| \sum_{i=1}^{p} a_i\, \phi_i(x) - \sum_{i=1}^{n} a_i\, \phi_i(x) \right\|^2$$

$$= \left\| \sum_{i=n+1}^{p} a_i\, \phi_i(x) \right\|^2$$

$$= \int_a^b \left\{ \sum_{i=n+1}^{p} a_i\, \phi_i(x) \right\}^2 dx$$

$$= \sum_{j=n+1}^{p} \sum_{i=n+1}^{p} a_i\, a_j \int_a^b \phi_i(x)\, \phi_j(x)\, dx$$

$$= \sum_{j=n+1}^{p} a_j^2 \int_a^b \phi_j^2(x)\, dx \qquad (\because \{i\} \text{ is an orthonormal system})$$

$$= \sum_{j=n+1}^{p} a_j^2 \, \| \phi_j \|^2$$

$$= \sum_{j=n+1}^{p} a_j^2 < \in^2$$

$$\Rightarrow \qquad \| S_p - S_n \| < \in \qquad \forall\, p > n > n_0.$$

$\Rightarrow$ $\{S_n\}$ is a Cauchy sequence in L_2 and L_2 is a complete therefore $\exists$ a function $f \in L_2$ such that

$$\lim_{n\to\infty} S_n = f.$$

Now $$< f, \phi_i > = \int_a^b f(x)\,\phi_i(x)\,dx = \int_a^b \lim_{n\to\infty} S_n(x)\,\phi_i(x)\,dx$$

$$= \int_a^b \lim_{n\to\infty} \left(\sum_{j=1}^{n} a_j\,\phi_j(x) \right) \phi_i(x)\,dx$$

$$= \int_a^b \left(\sum_{j=1}^{\infty} a_j\,\phi_j(x) \right) \phi_i(x)\,dx$$

$$= \int_a^b a_i [\phi_i(x)]^2\,dx = a_i \,\|\,\phi_i\,\|^2 = a_i$$

$\Rightarrow$ $< f, \phi_i > = a_i, \quad \forall\, i \in N.$

$\because$ $$f = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i\,\phi_i(x)$$

$\Rightarrow$ $$\|f\|^2 = \lim_{n\to\infty} \int_a^b \left[\sum_{i=1}^{n} a_i\,\phi_i(x) \right]^2 dx$$

$$= \lim_{n\to\infty} \sum_{i,j=1}^{n} a_i\,a_j \int_a^b \phi_i(x)\,\phi_j(x)\,dx$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{\infty} a_i^2$$

Thus f is the required function in L_2.

6.8 Complete orthonormal system

An orthonormal system $\{\phi_i\}$ is said to be **complete** if there is no function in L_2 different from θ (zero function) which is orthogonal to all function ϕ_i.

Theorem 9. *An orthonormal system $\{\phi_i\}$ is complete iff it is closed.*

Proof. At first we assume that $\{\phi_i\}$ is closed *i.e.* Parseval's identity is satisfied. We have to prove that $\{\phi_i\}$ is complete.

Let f be orthogonal to each ϕ_i, then

$$a_i = < f, \phi_i > = 0, \quad \forall\, i \in N.$$

As ϕ_i satisfies Parseval's identity, so

$$\|f\|^2 = \sum a_i^2 = 0$$

$\Rightarrow$ $\|f\| = 0$

$\Rightarrow$ $f = \theta$ (zero function)

$\Rightarrow$ The system $\{\phi_i\}$ is complete.

Conversely, let $\{\phi_i\}$ is complete, we have to show that $\{\phi_i\}$ is closed.

On the contrary we suppose that $\{\phi_i\}$ is not closed *i.e.* Parseval's identity fails for some function $g \in L_2$

i.e. $$\sum_{i=1}^{\infty} a_i^2 \leq \| g \|^2 \qquad .....(1)$$

where a_i are Fourier coefficients of g with respect to $\{\phi_i\}$. Using Riesz-Fisher theorem $\exists f \in L_2$ such that

$$\| f \|^2 = \sum_{i=1}^{\infty} a_i^2 \quad \text{and} <f, \phi_i> = a_i \quad \forall\, i \in N \qquad .....(2)$$

Now $$<f - g, \phi_i> = <f, \phi_i> - <g, \phi_i>$$
$$= a_i - a_i = 0$$

$\Rightarrow$ $f - g$ is orthogonal to all ϕ_i. Since $\{\phi_i\}$ is complete, so $f - g = \theta \Rightarrow f = g$

$$\Rightarrow \qquad \|f\|^2 = \| g \|^2 = \sum_{i=1}^{\infty} a_i^2 \qquad \text{[using (2)]}$$

Using (1) we get

$$\|f\|^2 \leq \| g \|^2$$

which is contradiction.

Hence in (1) it should be

$$\| g \|^2 = \|f\|^2 = \sum_{i=1}^{\infty} a_i^2$$

$\Rightarrow$ $\{\phi_i\}$ is closed.

Corollary : The trigonometric system

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}} \ldots \frac{\cos kx}{\sqrt{\pi}}, \frac{\sin kx}{\sqrt{\pi}}, \ldots \right\}$$

in $L_2 [-\pi, \pi]$ is complete.

Proof. Let a function $f \in L_2 [-\pi, \pi]$ be orthogonal to all functions in the trigonometric system. Then it is also orthogonal to every trigonometric polynomial

$$P(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

But the class $P_T [-\pi, \pi]$ of trigonometric polynomials is everywhere dense in $L_2 [-\pi, \pi]$ and every trigonometric polynomial obviously satisfies Parseval's identity.

Then by theorem 8 it is closed. Since the system is orthonormal it follows from the above theorem that the system is complete.

Self- learning exercise-1

1. $<f, f> = 0 \quad \forall f \in L_2$ True/False
2. If f and g be any two elements of L_2, then $<f, g> \geq 0$ True/False
3. The zero element $\theta \in L_2$ is orthogonal to every element $f \in L_2$. True/False
4. If $f \in L_2$, then the scalar product $a_i = <f, \phi_i>$, $i = 1, 2, ...$ are called .... of function f with respect to orthonormal system $\{\phi_i\}$.
5. The inequality $\sum_{i=1}^{\infty} a_i^2 \leq \| f \|^2$ is known as ....

6.9 Summary

In this unit we have discussed about scalar product of elements in L_2-space, orthogonal elements, orthogonal system, Fourier series and coefficients, orthonormal system and some important results on these topics like Bessel's inequality, Parseval's identity and Riesz-Fisher Theorem.

6.10 Answers to self-learning exercises-1

1. False 2. False 3. True 4. Fourier coefficients
5. Bessel's inequality.

6.11 Exercises

1. Let $\{\phi_i\}$ be a complete orthonormal system of functions. If $\{\psi_i\}$ is a system of functions in L_2 such that

$$\sum_{i=1}^{\infty} \int_a^b [\phi_i(x) - \psi_i(x)]^2 \, dx \leq 1.$$

Then prove that the system $\{\psi_i\}$ is also complete.

2. Let $\{\phi_i\}$ be an orthonormal system and let $f \in L_2$. Show that of all linear combinations $\sum_{i=1}^{n} a_i \, \phi_i(x)$, the norm of the difference $\left\| f - \sum_{i=1}^{n} a_i \, \phi_i \right\|$ has the least value, where $a_i = <f, \phi_i>$, $i = 1, 2, 3, ..., n$.
3. State and prove Parseval's identity.
4. State and prove Bessel's inequality.
5. If Parseval's identity holds for all functions $1, x, x^2, x^3, ...$ then the system $\{\phi_i\}$ of orthonormal function is closed prove it.

□ □ □

UNIT 7 : L ^{p}-Spaces, Holder-Minkowski Inequalities, Completeness of L ^{p}-Spaces

Structure of the Unit

7.0 Objectives

7.1 Introduction

7.2 L^p-spaces

7.3 Holder-Minkowski inequalities

7.4 Convergence in norm and Cauchy sequence in L^p-space

7.5 Completeness of L^p-space

7.6 Summary

7.7 Answers to self-learning exercises

7.8 Exercises

7.0 Objectives

In this unit, we will study the spaces which are direct generalization of the space of square summable functions discussed in the previous unit. Many of classical spaces in analysis of measurable functions and most of the important norms on such spaces have been defined by integrals. The Lebesgue L^p-spaces is one of the such important class of spaces. A complete understanding of these spaces require a thorough knowledge of the Lebesgue theory of measure and integration, which we have developed in the proceeding units. These spaces have remarkable properties and are of enormous importance in analysis as well as its application.

7.1 Introduction

In this unit we will first define L^p-space and prove that the L^p-space is linear space. Next we will prove Holder-Minkowski inequalities. In the end, we will prove that L^p-space is complete normed linear space *i.e.* Banach space.

7.2 L^p-spaces

By $L^p [a, b]$ or $L^p [E]$, we mean a class of all function f such that

(i) f is measurable and finite almost everywhere over $[a, b]$,

(iii) $|f|^p$ is integrable over $[a, b]$ for $p > 0$

i.e. $$\int_a^b |f(x)|^p \, dx < \infty \quad \text{for} \quad p > 0.$$

We denote by L^p or $L^p[E]$ or L_E^p, the set of all such functions.

Theorem 1. *Every pth power summable function on set E is summable on E i.e.* $L^p[E] \subset L[E]$. *But the converse is not true.*

Proof : Let f be a p^{th} power summable function on E.

$$\Rightarrow \qquad \int_E |f(x)|^p \, dx < \infty \qquad \text{.....(1)}$$

Let $E_1 = \{x \in E \,|\, |f(x)| < 1$ and $E_2 = E \sim E_1$. Since $f(x)$ is bounded on the set E_1

$$\Rightarrow \qquad \int_{E_1} |f(x)| dx \le 1 \cdot m(E_1) \qquad \text{.....(2)}$$

$$\because \qquad |f(x)|^p \ge 0$$

$$\therefore \quad \text{From (1)} \qquad 0 \cdot m(E_1) \le \int_{E_1} |f(x)|^p \, dx < \infty$$

$\Rightarrow \quad 0 \cdot m(E_1) < \infty$

$\Rightarrow \quad m(E_1)$ is finite

From (2), $$\int_{E_1} |f(x)| dx < m(E_1) < \infty$$

$\Rightarrow \quad |f|$ is summable on E_1.

Now $$\forall x \in E_2, |f(x)| \le |f(x)|^p$$

$$\Rightarrow \qquad \int_{E_2} |f(x)| \, dx \le \int_{E_2} |f(x)|^p \, dx$$

$$\le \int_E |f(x)|^p \, dx < \infty \qquad [\because \; E_2 \subset E]$$

$\Rightarrow \quad |f|$ is summable on E_2 .....(4)

From (3) and (4) f is summable on $E_1 \cup E_2$ *i.e.* on E.

$\therefore \quad L^p \subset L.$

The converse of above result is not necessarily true. For example if we consider a function

$f(x) = x^{-1/4} \; \forall \; x \in E = [0, 16]$, then

$$\int_0^{16} |f(x)| \, dx = \int_0^{16} x^{-1/4} dx = \frac{32}{3} < \infty$$

$\Rightarrow \quad f$ is summable on $[0, 16]$.

But is we take $p = 4$ then

$$\int_0^{16} |f(x)|^4 \, dx = \int_0^{16} x^{-1} dx$$

$$= \text{Log}_e 16 - \log_e 0 = \infty$$

$\Rightarrow \quad f \notin L_E^4.$

Theorem 2. *The L^p-space is a linear space.*

Proof : In order to prove L^p-space is a linear space we will show that

(i) $f \in L^p, g \in L^p \Rightarrow f + g \in L^p$,

(ii) $f \in L^p, c \in R \Rightarrow cf \in L^p$.

Since $f, g \in L^p$ therefore f and g are measurable functions on E and

$$\int_E |f(x)|^p\, dx < \infty, \int_E |g(x)|^p\, dx < \infty \qquad .....(1)$$

We know that the sum of two measurable functions is a measurable function therefore $f + g$ is measurable function on E.

Let $E_1 = \{x \in E \mid |f(x)| \le |g(x)|\}$

$E_2 = \{x \in E \mid |f(x)| > |g(x)|\}$

then $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \phi$.

Now for all $x \in E_1$

$$|f(x) + g(x)|^p \le [|f(x)| + |g(x)|]^p$$
$$\le [|g(x)| + |g(x)|]^p$$
$$= 2^p |g(x)|^p$$

$$\therefore \quad \int_{E_1} |f(x)+g(x)|^p\, dx \le \int_{E_1} 2^p\, |g(x)|^p\, dx < \infty. \qquad [\because\ g \in L^p]$$

$$\Rightarrow \quad \int_{E_1} |f(x)+g(x)|^p\, dx < \infty \qquad .....(2)$$

Again, $\forall x \in E_2,\ |f(x)+g(x)|^p \le 2^p\, |f(x)|^p$

$$\therefore \quad \int_{E_2} |f(x)+g(x)|^p\, dx \le \int_{E_2} 2^p\, |f(x)|^p\, dx < \infty \qquad [\because\ f \in L^p]$$

$$\Rightarrow \quad \int_{E_2} |f(x)+g(x)|^p\, dx < \infty \qquad .....(3)$$

Using countable additive property we have

$$\int_E |f(x)+g(x)|^p\, dx = \int_{E_1} |f(x)+g(x)|^p\, dx + \int_{E_2} |f(x)+g(x)|^p\, dx$$
$$< \infty \qquad \text{[using (1) \& (2)]}$$

$\Rightarrow \quad f + g \in L^p$.

Also $$\int_E |c\, f(x)|^p\, dx = \int_E |c|^p\, |f(x)|^p\, dx$$
$$< \infty \qquad [\because\ f \in L^p]$$

$\Rightarrow \quad cf \in L^p$.

Let us now define a function $\|\cdot\|_p : L^p \to R,\ 0 < p < \infty$ as follows :

$$\|f\|_p = \left[\int_E |f(x)|^p\, dx\right]^{1/p}, \quad 0 < p < \infty,\ f \in L^p.$$

7.3 Holder-Minkowski inequalities

Before establishing that above defined mapping $\|\cdot\|_p$ defines a norm on L^p, we will prove Holder-Minkowski inequalities, which are useful in establishing that $\|\cdot\|_p$ is a norm on L^p. To prove Holder inequality we will require the following inequality which is generalization of the inequality between arithmetic and geometric means.

Lemma : Let $0 < \mu < 1$, then prove that $a^\mu b^{1-\mu} < \mu a + (1-\mu) b$ holds good for any pair of non-negative real numbers a, b with equality only if $a = b$.

Proof : If $a = 0 = b$ then inequality is trivial. Now let us take $a > 0$, $b > 0$. We define a function $\phi(t)$ such that

$$\phi(t) = (1-\mu) + \mu t - t^\mu, \quad \text{where } 0 \le t < \infty.$$

$$\Rightarrow \qquad \phi^1(t) = \mu(1 - t^{\mu-1}).$$

If $t < 1$ then $\phi'(t) < 0$ and if $t > 1$ then $\phi'(t) > 0$. This means that the function ϕ decreases in $[0, 1]$ and increases in $[1, \infty)$ and hence $\phi(t)$ is minimum at $t = 1$.

$$\therefore \qquad \phi(t) \ge \phi(1) \;\forall\; t \in [0, \infty)$$

$$\Rightarrow \qquad 1 - \mu + \mu t - t^\mu \ge 0$$

Put $\qquad t = a/b$

$$1 - \mu + \mu\frac{a}{b} - \left(\frac{a}{b}\right)^\mu \ge 0$$

$$\Rightarrow \qquad b^\mu(1-\mu) + \mu a b^{\mu-1} - a^\mu \ge 0$$

$$\Rightarrow \qquad b(1-\mu) + \mu a - a^\mu b^{1-\mu} \ge 0$$

$$\Rightarrow \qquad a^\mu b^{1-\mu} \le (1-\mu) b + \mu a.$$

The equality holds good for $t = 1$ *i.e.* $a = b$.

In the study of L^p-spaces an essential role is played by another space L^q where p and q are non-negative extended real numbers related as

$$\frac{1}{p} + \frac{1}{q} = 1,$$

such two numbers p and q are termed as conjugate numbers.

Theorem 4. (*Holder inequality*) *: Let $1 \le p \le \infty$ and q be a non-negative real number such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p$ and $g \in L^q$, then show that*

(i) $\quad f \cdot g \in L$ *i.e. $f \cdot g$ is summable*

(ii) $$\int_E |f(x)\, g(x)|\, dx \le \left[\int_E |f(x)|^p\, dx\right]^{1/q} \cdot \left[\int_E |g(x)|^q\, dx\right]^{1/q}$$

$$= \|f\|_p \cdot \|g\|_q$$

i.e. $\int_E |f(x)\,g(x)|\,dx \le \|f\|_p \cdot \|g\|_q$

Further the equality holds iff for some non zero constants α *and* β,

$$\alpha\,|f(x)|^p = \beta\,|g(x)|^q \quad a.e. \text{ on } E.$$

Proof : The proof is obvious when $p = 1, q = \infty$.

Now take $1 < p < \infty$ and consequently $1 < q < \infty$. If either $f = 0$ *a.e.* on E or $g = 0$ *a.e.* on E or both, then

$$\int_E |f(x)\,g(x)|\,dx = 0 = \left[\int_E |f(x)|^p\,dx\right]^{1/p}\left[\int_E |g(x)^q dx|\right]^{1/q}$$

Let $f(x) \ne 0$ and $g(x) \ne 0$ on E, then

$$\|f\|_p = \left[\int_E |f(x)|^p\,dx\right]^{1/p} > 0 \text{ and}$$

$$\|g\|_q = \left[\int_E |g(x)|^q\,dx\right]^{1/q} > 0.$$

We have from Lemma, for $0 < \mu < 1$ and non-negative real number a, b

$$a^{\mu}\,b^{1-\mu} \le \mu\,a + (1-\mu)\,b \qquad .....(1)$$

Let $\mu = \dfrac{1}{p},\; a = \dfrac{|f(x)|^p}{\|f\|_p^p},\; b = \dfrac{|g(x)|^q}{\|g\|_q^q}$

From (1), we get

$$\left[\frac{|f(x)|^p}{\|f\|_p^p}\right]^{1/p}\cdot\left[\frac{|g(x)|^q}{\|g\|_q^q}\right]^{1-\frac{1}{p}} \le \frac{1}{p}\frac{|f(x)|^p}{\|f\|_p^p} + \left(1-\frac{1}{p}\right)\frac{|g(x)|^q}{\|g\|_q^q}$$

$$\Rightarrow \quad \frac{|f(x)|\,|g(x)|}{\|f\|_p\,\|g\|_q} \le \frac{1}{p}\frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g(x)|^q}{\|g\|_q^q} \qquad \left[\because\; 1-\frac{1}{p} = \frac{1}{q}\right]$$

$$\Rightarrow \quad \frac{1}{\|f\|_p\,\|g\|_q}\int_E |f(x)\,g(x)|\,dx \le \frac{1}{p}\frac{1}{\|f\|_p^p}\int_E |f(x)|^p\,dx + \frac{1}{q}\frac{1}{\|g\|_q^q}\int_E |g(x)|^q\,dx$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \quad \int_E |f(x)\,g(x)|\,dx \le \|f\|_p\,\|g\|_q \qquad .....(2)$$

$\because \quad f \in L^p$ and $g \in L^q \Rightarrow fg \in L$

For equality in (1), we have $a = b$

$$\Rightarrow \quad |f(x)|^p\,\|g\|_g^g = |g(x)|^q\,\|f\|_p^p$$

If we take $\alpha = \|g\|_g^g,\; \beta = \|f\|_p^p$

then α, β will be non-negative constants so $\qquad \alpha\,|f(x)|^p = \beta\,|g(x)|^p.$

For equality in (1), we obtain equality in (2).

Theorem 5. (*Minkowski's inequality*) : *Let* $f(x) \in L^p$ *and* $g(x) \in L^p$ *where* $p \geq 1$, *then*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof : The theorem is obvious for $p = 1$. Consider the case when $1 < p < \infty$. Let q be conjugate to p then $\frac{1}{p}+\frac{1}{q}=1$. Since L^p is a linear space therefore

$$f, g \in L^p \Rightarrow f+g \in L^p \Rightarrow (f+g)^{p/q} \in L^p.$$

On applying the Holder inequality for the functions $f(x)$ and $(f(x)+g(x))^{p/q}$, we get

$$\int_a^b |f(x)|\;|f(x)+g(x)|^{p/q}\,dx \leq \left(\int_a^b |f(x)|^p\,dx\right)^{1/p}\left(\int_a^b |f(x)+g(x)|^{\frac{p}{q}\cdot q}\,dx\right)^{1/q}$$

$$\Rightarrow \int_a^b |f(x)|\;|f(x)+g(x)|^{p/q}\,dx \leq \left(\int_a^b |f(x)|^p\,dx\right)^{1/p}\left(\int_a^b |f(x)+g(x)|^p\,dx\right)^{1/q} \quad ....(1)$$

Similarly on applying Holder's inequality for the functions $g(x)$ and $(f(x)+g(x))^{p/q}$, we get

$$\int_a^b |g(x)|\;|f(x)+g(x)|^{p/q}\,dx \leq \left(\int_a^b |g(x)|^p\,dx\right)^{1/p}\left(\int_a^b |f(x)+g(x)|^p\,dx\right)^{1/q} \quad ....(2)$$

Adding (1) and (2), we get

$$\int_a^b |f(x)|\;|f(x)+g(x)|^{p/q}\,dx + \int_a^b |g(x)|\;|f(x)+g(x)|^{p/q}\,dx$$

$$\leq \left[\left(\int_a^b |f(x)|^p\,dx\right)^{1/p} + \left(\int_a^b |g(x)|^p\,dx\right)^{1/p}\right]\left(\int_a^b |f(x)+g(x)|^p\,dx\right)^{1/q} \quad .....(3)$$

Now $\frac{1}{p}+\frac{1}{q}=1 \Rightarrow p = 1+\frac{p}{q}$ so that

$$\begin{aligned}|f(x)+g(x)|^p &= |f(x)+g(x)|\;|f(x)+g(x)|^{p-1}\\ &= |f(x)+g(x)|\;|f(x)+g(x)|^{p/q}\\ &\leq (\,|f(x)|+|g(x)|\,)\,|f(x)+g(x)|^{p/q}\end{aligned}$$

$$\Rightarrow \int_a^b |f(x)+g(x)|^p\,dx \leq \int_a^b |f(x)|\;|f(x)+g(x)|^{p/q}\,dx$$

$$+\int_a^b |g(x)|\;|f(x)+g(x)|^{p/q}\,dx \quad .....(4)$$

From (4) in view of (3), we get

$$\int_a^b |f(x)+g(x)|^p\,dx \leq \left[\left(\int_a^b |f(x)|^p\,dx\right)^{1/p} + \left(\int_a^b |g(x)|^p\,dx\right)^{1/p}\right]$$

$$\left(\int_a^b |f(x)+g(x)|^b\,dx\right)^{1/q}$$

or $\left(\int_a^b |f(x)+g(x)|\,dx\right)^{1-\frac{1}{q}} \le \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} + \left(\int_a^b |g(x)|^p\,dx\right)^{1/p}$

or $\left(\int_a^b |f(x)+g(x)|\,dx\right)^{1/p} \le \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} + \left(\int_a^b |g(x)|^p\,dx\right)^{1/p}$

Hence $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

Theorem 6. *Show that the L^p-space is a metric space.*

Proof : Let $f,\ g \in L^p$. Define the distance function d on L^p by

$$d(f, g) = \|f-g\|_p = \left(\int_a^b |f(x)-g(x)|^p\right)^{1/p}, \text{ then}$$

$[M_1]$ $d(f, g) = \|f-g\|_p = \left(\int_a^b |f(x)-g(x)|^p\right)^{1/p} \ge 0$ $[\because\ |f(x)-g(x)\,1 \ge 0]$

$[M_2]$ $d(f, g) = 0 \Leftrightarrow \|f-g\|_p = 0$

$\Leftrightarrow \int_a^b |f(x)-g(x)|^p\,dx = 0$

$\Leftrightarrow |f(x)-g(x)|^p = 0$ *a.e.*

$\Leftrightarrow f(x) = g(x)$

$\Leftrightarrow f = g$ *a.e.*

***M* [3]** $d(f, g) = \|f-g\|_p = \left(\int_a^b |f(x)-g(x)|^p\,dx\right)^{1/p}$

$= \left(\int_a^b |g(x)-f(x)|^p dx\right)^{1/p}$ $[\because\ |f-g| = |g-f|]$

$= d(g, f)$

***M* [4]** $d(f, g) = \|f-g\|_p = \|f-h+h-g\|_p$

$\le \|f-g\|_p + \|h-g\|_p$

$= d(f, h) + d(h, g)$

or $d(f, g) \le d(f, h) + d(h, g)$

Hence L^p is a metric space.

Theorem 7 : *Show that the L^p-space is a normed metric space.*

Proof : By theorem 2, L^p is linear space. Now we define a function $\|\cdot\|_p$ in L^p as :

$\|f\|_p : L^p \to R$ such that

$$\|f\|_p = \left[\int_E |f(x)|^p\,dx\right]^{1/p} \quad \forall\ f \in L^p$$

(i) Since $|f(x)|^p \ge 0 \quad \forall\ x \in E$

$\Rightarrow \quad \left[\int_E |f(x)|^p \, dx\right]^{1/p} \geq 0 \Rightarrow \|f\|_p \geq 0$

$Lp \to R$ such that in L^p as :

$\|f\|_p : L^p \to R$ such that

Also $\quad \|f\|_p = 0 \Leftrightarrow \left[\int_E |f(x)|^p \, dx\right]^{1/p} = 0$

$$\Leftrightarrow |f(x)|^p = 0 \;\; \forall x \in E$$

$$\Leftrightarrow f(x) = 0 = \forall x \in E$$

$$\Leftrightarrow f = \theta \qquad \text{(Zero function in } L^p\text{)}$$

(ii) For $c \in R, \; f \in L^p$

$$\|cf\|_p = \left[\int_E |c\,f(x)|^p \, dx\right]^{1/p} = |c|\left[\int_E |f(x)|^p \, dx\right]^{1/p}$$

$$= |c| \, \|f\|_p$$

(iii) By Minkowsky inequality, we have

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p \qquad \forall \; f, g \in L^p$$

Hence the function $\|\cdot\|_p$ satisfies all the axioms of norm *i.e.* it is norm in L^p space.

Therefore L^p space is a normed linear space.

7.4 Convergence in norm and Cauchy sequence in L^p space

If $1 \leq p < \infty$ and $\{f_n\}$ be a sequence in L^p space, then sequence $\{f_n\}$ is said to converge in norm to a function $f \in L^p$, if for each $\in > 0$, $\exists \; n_0(\in) \in N$ such that

$$\|f_n - f\|_p < \in \qquad \forall \; n \geq n_0(\in).$$

This type of convergence is also known as convergence in the mean of order p when $1 \leq p < \infty$.

Let $\{f_n\}$ be a sequence of functions in L^p-space, then $\{f_n\}$ said to be a Cauchy sequence, if for any $\in > 0$, there exits $n_0 \in N$ such that

$$\|f_m - f_n\|_p < \in \qquad \forall \; m, n \geq n_0.$$

7.5 Completeness of L^p-spaces

Theorem 8 : (*Riexz-Fisher*). *The space L^p is complete for $p \geq 1$.*

Proof : In order to prove the theorem, we will show that every Cauchy sequence in L^p converges to some element f in L^p. Let $\{f_n\}$ be any Cauchy sequence in L^p-space, then for given $\in > 0$ there exists $n_0 \in N$ such that

$$\|f_m - f_n\|_p < \in \qquad \forall \; m, n \geq n_0.$$

Since $\in > 0$ is an arbitrary so take $\in = \frac{1}{2}$, we can find natural number n_1 such that

$$\| f_m - f_n \|_p < \frac{1}{2} \quad \forall\ m, n \geq n_1.$$

Similarly taking $\in = \frac{1}{2^k}$, $\forall\ k \in N$, we can find a natural number n_k such that

$$\| f_m - f_n \|_p < \frac{1}{2^k} \quad \forall\ m, n \geq n_k$$

In particular $\| f_m - f_{n_k} \|_p < \frac{1}{2^k} \quad \forall\ m \geq n_k$.

Let $g_k = f_{n_k}$, then we have

$$\| g_{k+1} - g_k \|_p = \| f_{n_{k+1}} - f_{n_k} \|_p < \frac{1}{2^k} \qquad .....(1)$$

$$\Rightarrow \quad \sum_{k=1}^{\infty} \| g_{k+1} - g_k \| < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 \qquad .....(1)$$

$$\Rightarrow \quad \sum_{k=1}^{\infty} \| g_{k+1} - g_k \| \text{ is convergent series :}$$

Define g such that

$$g(x) = | g_1(x) | + \sum_{k=1}^{\infty} | g_{k+1} - g_k |_p \qquad .....(2)$$

and $g(x) = \infty$, if R.H.S. is divergent.

Now,

$$\left(\int_a^b | g(x) |^p \, dx \right)^{1/p} = \lim_{n \to \infty} \left\{ \int_a^b \left| g_1(x) + \sum_{k=1}^{n} | g_{k+1} - g_k | \right|^p dx \right\}^{1/p}.$$

or

$$\| g \|_p \leq \lim_{n \to \infty} \left(\| g_1 \|_p + \sum_{k=1}^{\infty} \| g_{k+1} - g_k \|_p \right) \quad \text{[By Minkowski's inequality]}$$

$$= \| g_1 \|_p + \sum_{k=1}^{\infty} \| g_{k+1} - g_k \| \leq \| g_1 \|_p + 1 \qquad \text{[by (1)]}$$

$\Rightarrow$ $\| g \|_p$ is a finite quantity $\Rightarrow$ $g \in L^p [a, b]$

Let $E' = \{x \in [a, b] : g(x) = \infty\}$.

Now we define a function f by

$$f(x) = 0 \quad \forall\ x \in E'$$

and

$$f(x) = g_1(x) + \sum_{k=1}^{\infty} (g_{k+1} - g_k) \qquad \text{for } x \notin E'$$

or $$f(x) = \lim_{m\to\infty}\left[g_1 + \sum_{k=1}^{m-1}(g_{k+1} - g_k)\right] \quad \text{for} \quad x \notin E'$$

$$= \lim_{n\to\infty} g_m(x) \qquad \text{for} \quad x \notin E'.$$

Thus $$f(x) = \begin{cases} \lim\limits_{m\to\infty} g_m(x) & \text{for } x \notin E' \\ 0 & x \in E' \end{cases}$$

$\therefore$ $$f(x) = \lim_{m\to\infty} g_m(x) \quad a.e. \text{ in } [a, b]$$

or $$\lim_{m\to\infty} |g_m - f| = 0 \quad a.e. \text{ in } [a, b] \qquad .....(3)$$

Also $$g_m(x) = g_1 + \sum_{k=1}^{m-1} (g_{k+1} - g_k)$$

$$\Rightarrow \quad |g_m| \le |g_1| + \sum_{k=1}^{m-1} (g_{k+1} - g_k)$$

$$\le |g_1| + \sum_{k=1}^{\infty} (g_{k+1} - g_k) = g,$$

$$\Rightarrow \quad |g_m| \le g \quad \forall\ m \in N \qquad \text{[by (2)]}$$

$$\Rightarrow \quad \sum_{m=1}^{\infty} |g_m(x)| \le g$$

$$\Rightarrow \quad |f| \le g$$

Again $|g_m - f| \le |g_m| + |f| \le g + g = 2g$

$\Rightarrow \quad || g_m - f| \le 2g$

Thus there exists a function $g \in L^p$ such that

$$|g_m - f| \le 2g \quad \forall\ m,$$

and $$\lim_{m\to\infty} |g_m - f| = 0 \quad a.e. \text{ in } [a, b] \qquad .....(4)$$

Applying Lebesgue dominated convergence theorem, we get

$$\lim_{m\to\infty} \int_a^b |g_m - f|^p\, dx = \int_a^b \lim_{m\to\infty} |g_m - f|^p\, dx$$

$$= \int_a^b 0 \cdot dx = 0, \text{ by (4)}$$

$$\Rightarrow \quad \left(\lim_{m\to\infty} \int_a^b |g_m - f|^p\, dx\right)^{1/p} = 0$$

$$\Rightarrow \quad \lim_{m\to\infty} \| g_m - f \|_p = 0$$

$$\Rightarrow \quad \lim_{m\to\infty} \| f_{n_m} - f \|_p = 0 \qquad [\because \; g_m = f_{n_m}]$$

$$\Rightarrow \quad \| f_{n_m} - f \|_p < \frac{\in}{2}$$

Also $\| f_m - f_{n_m} \|_p < \dfrac{\in}{2}$

$$\therefore \quad \| f_m - f \|_p = \| f_m - f_{n_m} + f_{n_m} - f \|_p$$

$$\leq \| f_m - f_{n_m} \|_p + \| f_{n_m} - f \|_p$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in$$

$$\Rightarrow \quad \lim_{m\to\infty} \| f_m - f \|_p = 0$$

or $\lim_{m\to\infty} f_m = f \in L^p$

Hence L^p is complete space.

Note : By theorem we have proved that L^p is normed linear space and from above theorem we have proved that L^p is complete space so L^p is Banach space.

***Ex* 1.** *Show that a sequence of functions in* L^p*-space has a unique limit.*

Sol. Let $\{f_n\}$ be a sequence of functions in L^p-space. If possible let $f_n \to f$ and $f_n \to g$.

Then $\| f - f_n \|_p = 0$, $\| f_n - g \|_p = 0$ as $n \to \infty$.

New $\quad \| f - g \|_p = \| f - f_n + f_n - g \|_p$

$$= \| f - f_n \|_p + \| f_n - g \|_p$$

$$= 0 \text{ as } n \to \infty \qquad [\because f_n \to f, \; g_n \to g]$$

$$\Rightarrow \quad \| f - g \|_p < 0 \Rightarrow \| f - g \|_p = 0, \quad \text{since} \quad \| f - g \|_p \geq 0.$$

$$\Rightarrow f = g$$

$\Rightarrow \lim f_n$ is unique.

***Ex* 2.** *Let* $< f_n >$ *be a sequence of functions belonging to* L^p*-space. If this sequence is convergent, then it is a Cauchy sequence.*

Sol. Let $\lim_{n\to\infty} f_n = f$. Then for each $\in > 0$, there exists a number $n_0 \in N$ such that

$$\| f_n - f \|_p < \frac{\in}{2} \qquad \forall n \geq n_0 \qquad \text{.....(1)}$$

Now, if $n, m \geq 0$, then

$$\| f_n - f_m \|_p = \| f_n - f + f - f_m \|_p$$

$$\leq \| f_n - f \|_p + \| f - f_m \|_p$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in, \qquad \text{[using (1)]}$$

$\Rightarrow$ $\| f_n - f_m \|_p < \ \forall\, m, n \geq n_0$ and hence $< f_n >$ is a Cauchy sequence.

Self-learning exercise-1

1. If $f, g \in L^p [a, b]$, then $f - g \in L^p [a, b]$. True/False
2. 2 is a self conjugate number. True/False
3. If $f \in L^p [a, b]$ and $g \leq f$, then $g \notin L^p [a, b]$. True/False
4. $\| cf \|_p = c \| f \|_p \ \forall \ c \in R$. True/False
5. If $f, g \in L^p [a, b]$ for $p \geq 1$, then $\| f + g \|_p \geq \| f \|_p + \| g \|_p$. True/False

7.6 Summary

In this unit we have discussed about L^p-spaces, Holder-Minkowaski inequalities, completeness of L^p-spaces and some important results related to these topics.

7.7 Answers to self-learning exercises

1. True **2.** True **3.** False **4.** False **5.** False.

7.8 Exercises

1. Define L^p-space and prove that L^p is a Banach space.
2. A sequence $\{f_n\}$ of functions in L^p converges in mean to a function $f \in L^p$ iff $\| f_m - f_n \| \to 0$ as $n \to \infty$.
3. If $0 < p < 1$ and f, g are non-negative function in L^p, then prove that

   $\int_E | f(x) g(x) | dx \geq \| f \|_p \| g \|_q$, provided that

   $\int_E | g(x) |^q \, dx \neq 0$, prove it.
4. Prove that the sequence of functions in L^p-space has at most one limit.

□ □ □

UNIT 8 : Topological Spaces

Structure of the Unit

8.0 Objectives

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8.3 Closed sets

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8.5 Sub space

- 8.5.1 Subspace
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8.7 Summary

8.8 Answer to self-learning exercises

8.9 Exercises

8.0 Objectives

In this unit, we define what a topological space is, and we study a number of ways of constructing a topology on a set as to make it into a topological space. We also study some of the elementary concepts associated with topological spaces. Open and closed sets, limit points are introduced as natural generalization of the corresponding ideas for the real line and Euclidean space.

8.1 Introduction

The concept of topological space grew out of the study of the real line and Euclidean space and the study of the continuous function on these spaces. The definition of a topological space that is given in this unit, was a long time in being formulated. Various mathematicians – Frechet, Hausdorff and others proposed different definitions over a period of years. It took quite a while before mathematicians settled on the one that seemed most suitable. The definition finally settled on may seem a bit abstruct but as you learn the various ways of constructing topological spaces, you will get a better understanding for what the concept is.

8.2 Topological space

8.2.1. Definition :

A **topological space** is a pair $(X ; \tau)$, where X is non-empty set and τ is a family of subsets of X satisfying :

(T_1) $\phi \in \tau$ and $X \in \tau$

(T_2) If $\{G_\lambda : \lambda \in \wedge\}$ is a family of subsets of X in τ, where $\wedge$ is an arbitrary index set, then

$$G = \bigcup_{\lambda \in \Lambda} G_\lambda \quad \text{is also in } \tau.$$

(T_3) If $\{G_m : m = 1, 2, ..., n , n \in N\}$ is a finite collection of subsets of X in τ, then

$$H = \bigcap_{i=1}^{n} G_i \quad \text{is also in } \tau.$$

The family τ is said to be a topology on the set X. Members of τ are said to be τ-open or simply open subsets of X.

Note 1: The property (T_2) and (T_3) are also stated as

(T'_2) τ is closed under arbitrary union

(T'_3) τ is closed under finite intersection.

Note 2 : The same set X may have different topologies.

Let τ_1 and τ_2 be any two topologies on the same set X.

If $\tau_1 \subset \tau_2$, then τ_1 is called **weaker** or **coarser** then τ_2 or τ_2 is called **stronger** or **finer** than τ_1. If $\tau_1 \subset \tau_2$ and $\tau_1 \neq \tau_2$, then τ_1 is called **strictly coarser** then τ_2 or τ_2 is called **strictly finer** then τ_1. τ_1 and τ_2 are said to be **comparable** if either $\tau_1 \subset \tau_2$ or $\tau_2 \subset \tau_1$.

8.2.2. Examples of topologies :

***(i)* Discrete topology :**

Let X be any set and $P(X)$ be the power set of X, then $P(X)$ is a topology on X called **discrete topology** on X. This topology is **finest** topology on X.

***(ii)* Indiscrete topology (Trivial topology) :**

Let X be any set, then $\tau = \{\phi, X\}$ is a topology on X, it is called **indiscrete** or **trivial topology** on X. This topology is weakest or coarest topology on a set X. For a singleton set $X = \{a\}$, discrete topology and indiscrete topology coincide.

***(iii)* Sierpinski space :**

Let $X = \{0, 1\}$, then $\tau = \{\phi, X, \{0\}\}$ is topology on X. The topological space (X, τ) is called **Sierpinski space.**

(iv) Let $X = \{a, b\}$, then $P(X) = \{\phi, X, \{a\}, \{b\}\}$, if we take

(1) $\tau = \{\phi, X\}$ (2) $\tau = P(X)$

(3) $\tau = \{\phi, X, \{a\}\}$ (4) $\tau = \{\phi, X, \{b\}\}$

the result is always a topology. If we take

(1) $\tau = \{\phi, \{a\}, \{b\}\}$ (2) $\tau = \{X, \{a\}, \{b\}\}$

(3) $\tau = \{\phi\ \{a\}\}$ (4) $\tau = \{X, \{a\}\}$

the result is not a topology.

(v) Let $X = \{a, b, c\}$, then

$$\tau_1 = \{\phi, X, \{a\}\}$$
$$\tau_2 = \{\phi, X, \{a, b\}\}$$
$$\tau_3 = \{\phi, X, \{a\}, \{a, b\}\}$$
$$\tau_4 = \{\phi, X, \{a\}, \{b, c\}\}$$
$$\tau_5 = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$$
$$\tau_6 = \{\phi, X, \{b\}, \{b, c\}, \{a, b\}\}$$

are topologies on X and clearly τ_3 is strictly finer than τ_1 and τ_2, but τ_1 and τ_2 are not comparable.

Also τ_3 and τ_4 are not comparable. τ_5 is strictly finer than τ_3.

Now, $A_1 = \{\phi, X, \{a\}, \{b\}\}$ is not a topology on X as $\{a\}, \{b\} \in A_1$ but their union $\{a\} \cup \{b\} = \{a, b\} \notin A_1$.

Similarly $A_2 = \{\phi, X, \{a, b\}, \{b, c\}\}$ is not a topology on X as $\{a, b\}, \{b, c\} \in A_2$ but their intersection $\{a, b\} \cap \{b, c\} = \{b\} \notin A_2$.

***(vi)* Cofinite topology :**

Let X be an infinite set. Let τ be the family consisting of ϕ, X and all subsets G of X, *s.t.* $X \sim G$ is finite, then τ is a topology on X as

(T_1) $\phi, X \in \tau$ (by definition)

(T_2) Let $\{G_\lambda : \lambda \in \wedge\}$ be a family of τ-open subsets where $\wedge$ is an arbitrary index set, then we wish to show $G = \bigcup_{\lambda \in \Lambda} G_\lambda$ is also τ-open subset of X. Now $G \in \tau$ iff $X \sim G$ is finite.

So $$X \sim G = X \sim \bigcup_{\lambda \in \Lambda} G_\lambda = \bigcap_{\lambda \in \Lambda} (X \sim G_\lambda)$$ [Using De-Morgan's law]

Now each $(X \sim G_\lambda)$ is finite, since $G_\lambda \in \tau$ and arbitrary intersection of finite sets is also finite so $X \sim G$ is finite and hence $G \in \tau$.

(T_3) Let $G_1, G_2 \in \tau$, we will show $G_1 \cap G_2 \in \tau$.

To show $G_1 \cap G_2 \in \tau$ we have to show that $X \sim (G_1 \cap G_2)$ is finite.

Now $X \sim (G_1 \cap G_2) = (X \sim G_1) \cup (X \sim G_2)$ [by D'Morgan law]

$(X \sim G_1)$ and $(X \sim G_2)$ is finite, since $G_1, G_2 \in \tau$ and union of two finite sets is also finite so $X \sim (G_1 \cap G_2)$ is finite and hence $G_1 \cap G_2 \in \tau$.

Now by above argument it is easy to show that If $\{G_m : m = 1, 2,..., n\}$ is a finite collection of subsets of X in τ, then $\bigcap_{i=1}^{n} G_i \in \tau$.

Thus by above, we can say that τ is a topology on X.

Note 1 : In proof of a collection of subsets of X is a topology on X, to show the (T_3) condition in the definition it is sufficient to show that whenever $G_1, G_2 \in \tau$ then $G_1 \cap G_2 \in \tau$.

Note 2 : One can similarly define **co-countable topology** on an **uncountable** set X.

Note 3 : If X is a finite set, then cofinite topology is same as discrete topology on X. Similarly if X is a countable set, then co-countable topology on X is same as discrete topology on X.

Note 4 : Cofinite topology is also known as **finite complement topology.**

(vii) Let (X, d) be metric space. $G \subset X$ is called an open set if $\forall\ x \in G, \exists\ r \in \boldsymbol{R}^+$ such that open ball $B(x, r) \subset G\ \ \forall\ x \in G$. Let τ be the family of subsets of X, which are open in the above sense. Then τ is a topology on X and is called **usual topology** on X or **metric topology** on X.

***(viii)* Usual topology on *R* :**

A subset $G \subset \boldsymbol{R}$ is called an open set if $\exists\ \delta \in \boldsymbol{R}^+$ such that the open interval $(x - \delta, x + \delta) \subset G$ $\forall\ x \in G$. Let U be the family of subsets G of X, which are open in the above sense. Then U is a topology on X and is called usual topology on $\boldsymbol{R}$.

(ix) **Semi interval topology on *R* :**

Let τ be the collection of subsets G of $\boldsymbol{R}$, such that $\forall\ x \in G\ \exists\ r \in \boldsymbol{R}^+$ and $[x, x + r) \subset G$. Then τ is called semi interval topology on $\boldsymbol{R}$. One can of course consider semi-open intervals of the form $(x - r, x]$ instead of $[x, x + r)$ in the definition and get another topology on $\boldsymbol{R}$.

(x) **Right | Left hand topology on *R* :**

The topology generated by the family of intervals of the form $(a, \infty) = \{x \in \boldsymbol{R} : x > a\}$ is called right hand topology. Similarly the topology generated by the family of intervals of the form $(-\infty, a) = \{x \in \boldsymbol{R} : x < a\}$ is called left hand topology.

Self-learning exercise-1

1. If $X = \{a, b\}$, $P(X) = \{\phi, x, \{a\}, \{b\}\}$, then which one of the following is a topology ?

(a) $T = \{\phi, \{a\}, \{b\}\}$ *(b)* $T = \{X, \{a\}, \{b\}\}$

(c) $T = \{\phi, \{a\}\}$ *(d)* $T = \{\phi, X\}$

2. If $X = \{a, b\}$, then which one of the following is not a topology ?

(a) $T = \{\phi, X\}$ *(b)* $T = \{\phi, X, \{a\}\}$

(c) $T = \{\phi, X, \{b\}\}$ *(d)* $T = \{\phi, \{a\}, \{b\}\}$

3. Which of the following is a topology on $X = \{1, 2, 3, 4\}$

(a) $T = \{\phi, X, \{1\}, \{2\}\}$ *(b)* $T = \{\phi, X, \{1\}, \{2, 3\}\}$

(c) $T = \{\phi, X, \{2\}, \{1, 4\}\}$ *(d)* $T = \{\phi, X, \{1\}\}$

4. If τ_1 and τ_2 are topologies on the same set X, then prove that

(i) $\tau_1 \cap \tau_2$ is also a topology

(ii) $\tau_1 \cup \tau_2$ is not a topology on X.

8.3 Closed sets

8.3.1. Definition :

Any set $F \subset X$ is called closed subset of a topological space (X, τ) if $X \sim F$ is open subset of X.*i.e.* $X \sim F \in \tau$.

A topological space (X, τ) is said to be a **door space** if every subset of X is either τ-open or τ-closed.

8.3.2. Example of door space :

Let $X = \{a, b, c\}$ and

$$\tau = \{\phi, X, \{a, b\}, \{b, c\}, \{b\}\}$$

then $P(X) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$

τ-open sets : $\phi, X, \{a, b\}, \{b, c\}, \{b\}$

τ-closed sets : $\phi, X, \{c\}, \{a\}, \{a, c\}$

Thus every subset of X is either τ-open or τ-closed.

Note : In analogy with everyday usage, a beginner is likely to think that "closed" is the negation of "open", that is to say, a set is closed if and only if it is not open. But this is not so. The fact is that the possibilities of a set being open and its being closed are neither mutually exclusive not exhaustive. For example the null set ϕ and the whole set X are always open as well as closed in every topological space. On the other hand the set of rational numbers $\boldsymbol{Q}$ is neither open nor closed in the usual topology on $\boldsymbol{R}$. A set which is both open and closed is sometimes called a **Clopen set.**

Theorem 1. *Let* $\boldsymbol{C}$ *be the family of all τ-closed sets in a topological space* (X, τ). *Then* $\boldsymbol{C}$ *has the following properties*

(C_1) $\phi \in \boldsymbol{C}, X \in \boldsymbol{C}$

(C_2) *$\boldsymbol{C}$ is closed under arbitrary intersections.*

(C_3) *$\boldsymbol{C}$ is closed under finite unions.*

Proof. (C_1) since $X, \phi \in \tau \Rightarrow X \sim X = \phi$ is τ-closed

$$\Rightarrow \phi \in \boldsymbol{C}.$$

Also $X \sim \phi = X$ is τ-closed

$$\Rightarrow X \in \boldsymbol{C}$$

(C_2) Let $\{C_\lambda : \lambda \in \wedge\}$ be an arbitrary family of closed sets in $\boldsymbol{C}$.

Let $F = \bigcap_{\lambda \in \Lambda} C_\lambda$, to prove $F \in \boldsymbol{C}$.

Now let $G_\lambda = X \sim C_\lambda$

$\because$ C_λ is τ-closed subset of X $\forall$ $\lambda \in \wedge$

$\therefore$ G_λ is τ-open subset of X $\forall$ $\lambda \in \wedge$

$\Rightarrow$ $\bigcup_{\lambda \in \Lambda} G_\lambda$ is also τ-open subset of X [from (T_2) property of defi 8.2.1]

$\Rightarrow$ $\bigcup_{\lambda \in \Lambda} (X - C_\lambda)$ is τ-open subset of X

$\Rightarrow$ $X \sim \bigcap_{\lambda \in \Lambda} C_\lambda$ is τ-open subset of X [De-Morgan's Law]

$\Rightarrow$ $F = \bigcap_{\lambda \in \Lambda} G_\lambda$ is also τ-open subset of X

$\Rightarrow$ $F \in \boldsymbol{C} \Rightarrow \boldsymbol{C}$ is closed under arbitrary intersection.

(C_3) Let C_1 and $C_2 \in \boldsymbol{C}$, to prove $C_1 \cup C_2 \in \boldsymbol{C}$.

$\because$ $C_1, C_2 \in \boldsymbol{C} \Rightarrow X \sim C_1$ and $X \sim C_2$ are τ-open

$\Rightarrow$ $(X \sim C_1) \cap (X \sim C_2)$ is τ-open

$\Rightarrow \quad X \sim (C_1 \cup C_2)$ is τ-open [De'Morgan's Law]

$\Rightarrow \quad C_1 \cup C_2$ is τ-closed subset of X

$\Rightarrow \quad C_1 \cup C_2 \in \boldsymbol{C} \Rightarrow \boldsymbol{C}$ is closed under finite union.

Theorem 2. *Let X be any set and **C** is a family of subsets of X which satisfy the property* $(C_1) - (C_3)$ *of Theorem* 1. *Then there exists a unique topology τ on X such that **C** coincides with the family of closed subsets of* (X, τ).

Proof. Here we are given a set X (just a bare set with no topology on it) and some collection $\boldsymbol{C} \subset P(X)$ of it's subsets. We are given that property $(C_1) - (C_3)$ holds for $\boldsymbol{C}$. We do not know how $\boldsymbol{C}$ originated, nor do we know whether its members are closed subsets of X. Actually it is meaningless to talk about closed subsets of X, unless a topology on X is specified.

Now we define a topology τ on X consist of complements (in X) of members of $\boldsymbol{C}$ *i.e.*

$$\tau = \{B \subset X : X \sim B \in \boldsymbol{C}\}$$

First we show that τ is a topology on X

(T_1) $\phi, X \in \tau$ since $\phi \in \boldsymbol{C} \Rightarrow X \sim \phi = X \in \tau$

and $X \in \boldsymbol{C} \Rightarrow X \sim X = \phi \in \tau$.

(T_2) τ is closed under arbitrary union :

Let $\{G_\lambda : \lambda \in \wedge\}$ be any arbitrary collection of subsets of X in τ *i.e.* $G_\lambda \in \tau \ \forall \ \lambda \in \wedge$

$\Rightarrow \quad X \sim G_\lambda \in \boldsymbol{C} \ \forall \ \lambda \in \wedge$

$\because \quad \boldsymbol{C}$ is closed under arbitrary intersection

$$\Rightarrow \quad \bigcap_{\lambda \in \Lambda} (X \sim G_\lambda) \in \boldsymbol{C}$$

$$\Rightarrow \quad X \sim \left(\bigcup_{\lambda \in \Lambda} G_\lambda \right) \in \boldsymbol{C} \qquad \text{[De-Morgan's Law]}$$

$$\Rightarrow \quad \bigcup_{\lambda \in \Lambda} G_\lambda \in \tau$$

$\Rightarrow$ τ is closed under arbitrary union.

(T_3) τ is closed under finite intersection :

Let $G_1, G_2 \in \tau$. To prove $G_1 \cap G_2 \in \tau$.

$\because \quad G_1, G_2 \in \tau \Rightarrow X \sim G_1$ and $X \sim G_2 \in \boldsymbol{C}$

$\because \quad \boldsymbol{C}$ is closed under finite union

$\because \quad (X \sim G_1) \cup (X \sim G_2) \in \boldsymbol{C}$

$\Rightarrow \quad X \sim (G_1 \cap G_2) \in \boldsymbol{C}$ [De'Morgan's Law]

$\Rightarrow \quad G_1 \cap G_2 \in \tau$

$\Rightarrow$ τ is closed under finite intersection.

It is clear that τ-closed subsets of X are precisely the members of ***C***. Thus τ is the required topology.

8.3.3. Closure :

The **closure** of a subset of a topological space is defined as the intersection of all closed subsets containing it. Or in other words the smallest closed set containing it. If $A \subset X$ then closure of A is denoted as $\overline{A}$ and defined as

$$\overline{A} = \cap \{C \subset X : C \text{ is closed in } X \text{ and } A \subset C\}$$

Theorem 3. *Let A and B be subsets of a topological space* (X, τ).

(i) $\overline{A}$ *is a closed subset of X. More over it is the smallest closed subset of X containing A i.e. if F is closed in X and* $A \subset F$ *then* $\overline{A} \subset F$

(ii) $\overline{\phi} = \phi$

(iii) A is closed in X iff $\overline{A} = A$

(iv) $\overline{\overline{A}} = \overline{A}$

(v) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Proof. ***(i)*** and ***(ii)*** are immediate consequence of the definition and properties of closed set.

(iii) Let A is closed in X, then A itself is the smallest closed set containing A thus $\overline{A} = A$ conversely let $\overline{A} = A$, then clearly A is closed as it is equal to the smallest closed set containing A. Thus we have A is closed in X iff $\overline{A} = A$.

(iv) Since $\overline{A}$ is a closed set thus using *(iii)*

$$\overline{\left(\overline{A}\right)} = \overline{A}$$

(v) As $A \subset \overline{A}$ and $B \subset \overline{B} \Rightarrow A \cup B \subset \overline{A} \cup \overline{B}$ .....(1)

Now by definition $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$ and $\overline{A} \cup \overline{B}$ is a closed set being union of two closed sets, thus $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ .....(2)

Now since when $A_1 \subset A_2 \Rightarrow \overline{A}_1 \subset \overline{A}_2$

therefore $A \subset A \cup B \Rightarrow \overline{A} \subset \overline{A \cup B}$ .....(3)

$B \subset A \cup B \Rightarrow \overline{B} \subset \overline{A \cup B}$ .....(4)

(3) and (4) $\Rightarrow \overline{A} \cup \overline{B} \subset \overline{A \cup B}$ .....(5)

(2) and (5) $\Rightarrow \overline{A \cup B} = \overline{A} \cup \overline{B}$

8.3.4. Dense subset :

A subset $A \subset X$ of a topological space (X, τ) is called a **dense subset** of X if

$$\overline{A} = X$$

Trivially, the entire set X is always dense in itself.

Theorem 4. *A subset A of a space X is dense in X iff for every non empty open subset G of X, $A \cap G \neq \phi$.*

Proof. *Let A is dense in X and G is a non empty open set in X. If $A \cap G = \phi$, then $A \subset X \sim G$.*

$\Rightarrow \quad \overline{A} \subset \overline{X \sim G} = X \sim G$ [$\because$ G is open so $X \sim G$ is closed in X]

but since A is dense in X so $\overline{A} = X$

$\Rightarrow \quad X \subset X \sim G$ which is a contradiction, thus $A \cup G \neq \phi$

Conversely assume that A meets every non-empty open subset of X. This clearly means that the only closed set containing A is X and consequently $\overline{A} = X$.

8.4 Neighbourhood

8.4.1. Neighbourhood of a point x :

Let (X, τ) be a topological space. A subset $A \subset X$ is called a **neighbourhood of a point** $x \in X$ if $\exists$ $G \in \tau$ with $x \in G$ *s.t.* $G \subset A$. The word neighbourhood is, in short, written as 'nbd'. From the definition of nbd it is clear that any open set $G \subset X$ is nbd of each of its point $x \in G$.

8.4.2. Deleted neighbourhood :

If A is a nbd of a point $x \in X$, then $A \sim \{x\}$ is called deleted neithbourhood of x.

8.4.3. Open neighbourhood :

In any topological space nbd of a point need not be an open set. On the other hand every open set is nbd of each of its points, such a nbd of a point is called open neighbourhood of that point.

8.4.4. Neighbourhood of set :

A set $N \subset X$ is called a nbd of a set $A \subset X$ if $\exists$ $G \in \tau$ *s.t.* $A \subset G \subset N$.

8.4.5. Interior point :

Let (X, τ) be a topological space let $x_0 \in A \subset X$. Then x_0 is called τ-interior or interior point of A if $\exists$ $G \in \tau$ such that $x_0 \in G \subset A$ *i.e.* if A is nbd of x_0.

8.4.6. Interior of a set :

Let (X, τ) be a topological space and $A \subset X$. Then the interior of A is defined to be the set of all interior points of A. If is denoted as A° or int (A) or $\text{int}_\tau (A)$. Thus

$$A^\circ = \{x \in A : A \text{ is a nbd of } x\}$$

Theorem 5. *A subset of a topological space is open iff it is nbd of each of its points.*

Proof. Let (X, τ) be a topological space and $G \subset X$. First assume G is open. Then by definition of nbd, G is nbd of each of its points. Conversely assume G is a nbd of each of its point. Then for each $x \in G$, there is an open set V_x such that $x \in V_x \subset G$. Clearly then $G = \bigcup_{x \in G} V_x$. Since each V_x is open and G is arbitrary union of open subsets of X thus by property (T_2) of definition 8.2.1 G is open.

Corollary 5.1 : A subset A of a topological space is open iff

$$A^{\circ} = A$$

Theorem 6. *Let* (X, τ) *be a topological space and* $A \subset X$. *Then* A° *is the union of all open sets contained in A. It is also the largest open subset of X contained in A.*

Proof : Let U be the family of all open sets contained in A (U is non-empty since $\phi \in U$). Let $V = \bigcup_{G \in U} G$. We wish to show $V = \text{int}(A)$ or A°

Now if $x \in V$, then $x \in G$ for some $G \in U$. This means A is nbd of x and so $x \in A^{\circ}$. Conversely let $x \in A^{\circ}$, then there is an open set H such that $x \in H \subset A$. But then $H \in U$ and so $H \subset V$, so $x \in H \Rightarrow x \in V$. Thus we have $V = \bigcup_{G \in U} G = A^{\circ}$.

This proves first assertion of the theorem and also shows that A° is an open set contained in A.

Now suppose G is an open set contained in A. Then $G \in U$ and so $G \subset A^{\circ}$, thus A° is the largest open set contained in A.

Theorem 7. *Let* (X, τ) *be a topological space and* $x \in X$ *be arbitrary. Then*

(i) there is at least one nbd for x

(ii) for each nbd N of x, $x \in N$

(iii) if M is a super set of a nbd N of x, then M is also a nbd of x.

(iv) if N_1 and N_2 be neighbourhoods of x, then $N_1 \cap N_2$ is also a nbd of x.

Proof : ***(i)*** $x \in X \subset X$ and $X \in \tau$, thus by definition X is a nbd of x. Hence $\exists$ at least one nbd for x.

(ii) Let N be a nbd of $x \Rightarrow \exists\ G \in \tau$ s.t. $x \in G \subset N \Rightarrow x \in N$

(iii) Let N be a nbd of $x \Rightarrow \exists\ G \in \tau$ s.t. $x \in G \subset N$

Now $M \supset N$ thus $x \in G \subset N \subset M \Rightarrow M$ is also nbd of x.

(iv) Let N_1 and N_2 be nbds of the same point x, then

$\exists\ G_1\ G_2 \in \tau$ s.t. $x \in G_1 \subset N_1$, $x \in G_2 \subset N_2$

$\Rightarrow x \in G_1 \cap G_2 \subset N_1 \cap N_2$ [$\because$ for topology τ on X if $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$]

thus $\exists\ G = G_1 \cap G_2$ s.t. $x \in G \subset N_1 \cap N_2$, and $G \in \tau$.

$\Rightarrow N_1 \cap N_2$ is also a nbd of x. [by definition]

8.4.7 Neighbourhood system :

Let (X, τ) be a topological space. Let η_x be the set of all neighbourhoods of x in X (with respect to given topology τ). The family η_x is called the neighbourhood system at x.

Now if η_x is a nbd system at x, then using Theorem 7 we can show that η_x has following properties :

$[N_0]$: $\eta_x \neq \phi \quad \forall\ x \in X$

$[N_1]$: $N \in \eta_x \Rightarrow \ x \in N$

$[N_2]$: $N \in \eta_x, M \supset N \Rightarrow \ M \in \eta_x$

$[N_3]$: $N \in \eta_x, M \in \eta_x \Rightarrow \ N \cap M \in \eta_x$

$[N_4]$: $N \in \eta_x \Rightarrow \exists\ M \in \eta_x$ s.t. $M \subset N$ and $M \in \eta_y \ \forall\ y \in M$

Theorem 8. [*Characterization of a topological space in terms of neighbourhoods*] *Let X be a non-empty set and $x \in X$, let there be associated family $N(x)$ of subsets of X, satisfying the conditions $[N_0]$ to $[N_4]$ mentioned above. Then there exist a unique topology τ on X such that if η_x is the collection of nbds of x, defined by the topology τ on X, then $N(x) = \eta_x$*

Proof : Here given that X is a non-empty set, and $N(x)$ be a family of subsets of X satisfying the condition

$[N_0]$: $N(x) \neq \phi \quad \forall\ x \in X$

$[N_1]$: $N \in N(x) \Rightarrow \ x \in N$

$[N_2]$: $N \in N(x), M \supset N \Rightarrow \ M \in N(x)$

$[N_3]$: $N \in N(x), M \in N(x) \Rightarrow \ N \cap M \in N(x)$

$[N_4]$: $N \in N(x) \Rightarrow \exists\ M \in N(x)$ s.t. $M \subset N$ and $M \in N(y) \ \forall\ y \in M$.

We define τ as follows :

$$G \in \tau \Leftrightarrow G \in N(x) \qquad \forall\ x \in G$$

To prove τ is a topology on X

(T_1) Since ϕ contains no point, the statement $\phi \in N(x)$ for all $x \in \phi$ is trivially true.

Since by $[N_0]$ $N(x) \neq \phi \qquad \forall\ x \in X$

$\Rightarrow \quad \exists\ G \in N(x) \qquad \forall\ x \in X$

since $\quad X \supset G$ thus from $[N_2]$

$X \in N(x) \qquad \forall\ x \in X$

$\Rightarrow \quad X \in \tau$

(T_2) Let $\{G_\lambda : \lambda \in \wedge\}$ be an arbitrary family in τ

$\Rightarrow \quad G_\lambda \in N(x) \qquad \forall\ x \in G_\lambda \quad$ and $\quad \forall\ \lambda \in \wedge$

$\Rightarrow \quad G_\lambda \in N(x) \qquad \forall\ x \in \cup \{G_\lambda : \lambda \in \wedge\}$

$$\Rightarrow \quad \bigcup_{\lambda \in \wedge} G_\lambda \in N(x) \quad \forall x \in \bigcup_{\lambda \in \wedge} G_\lambda$$

$$\Rightarrow \quad \bigcup_{\lambda \in \wedge} G_\lambda \in \tau$$

(T_3) Let $G1\ G2$

$\Rightarrow \quad G_1 \in N(x) \qquad \forall\ x \in G_1 \quad$ and $\quad G_2 \in N(x) \quad \forall\ x \in G_2$

$\Rightarrow \quad G_1 \cap G_2 \in N(x) \qquad \forall\ x \in G_1 \cap G_2, \qquad$ by $[N_3]$

$\Rightarrow \quad G_1 \cap G_2 \in \tau$ by defi. of τ.

Hence τ is a topology on X.

Second part : It remains to prove that

$$N(x) = \eta_x$$

Let $N \in N(x)$ be arbitrary. Then by $[N_4]$ $\exists\, M \in N(x)$

such that $M \subset N$ and $M \in N(y)\ \forall\ y \in M$

$\Rightarrow \quad M \in \tau$ by defi. of τ

Also $\quad M \in N(x) \Rightarrow x \in M \quad$ by $[N_1]$

Now $\quad x \in M \subset N \quad$ where $\quad M \in \tau$

$\Rightarrow \quad N$ is a τ-nbd of x and $N \in \eta_x$

thus $\quad N(x) \subset \eta_x \qquad$ .....(1)

conversely let $P \in \eta_x \Rightarrow P$ is a τ-nbd of x

$\Rightarrow \quad \exists\, Q \in \tau \quad$ s.t. $\quad x \in Q \subset P$

Now $\quad Q \in \tau \Rightarrow Q \in N(x) \qquad \forall\ x \in Q$

Now $\quad Q \in N(x)$ and $P \supset Q \Rightarrow P \in N(x)$ using $[N_2]$

$\Rightarrow \quad \eta_x \subset N(x) \qquad$ .....(2)

from (1) and (2) $\Rightarrow \eta_x = N(x)$

8.4.8 Limit point :

Let A be a subset of a topological space X and $x_0 \in X$. Then x_0 is called limit point of A if every open set containing x_0 contains at least one point of A other then x_0. Limit point is also known as accumulation point or cluster point.

As examples, in a discrete space no point is a limit point of any set while at the other extreme, in an indiscrete space, a point x_0 is a limit point of any set A provided only that A contains at least one point besides x_0. In the usual topology on $\boldsymbol{R}$, every real number is a limit of the set of rational numbers, while the set of integers has no limit point.

8.4.9 Derived set :

Let A be a subset of a topological space (X, τ). Then the derived set of A, denoted by A', is the set of all limit points of A in X.

Obviously A' depends not only on A but also on the topology under consideration.

Theorem 9. *For a subset A of a topological space* (X, τ), $\bar{A} = A \cup A'$

Proof : First we claim that $A \cup A'$ is closed or that $X \sim (A \cup A')$ in open. We do so by showing that $X \sim (A \cup A')$ is *nbd* of each of its points. Let

$$y \in X \sim (A \cup A') \Rightarrow y \notin A \cup A' \Rightarrow y \notin A' \text{ and } y \notin A$$

$\Rightarrow y$ is not a limit point of A, there exist an open set $V \in \tau$ containing y such V contains no point of A except possibly y. But $y \notin A$, so we have $A \cap V = \phi$. We claim $A' \cap V$ is also empty. For, let $z \in A' \cap V$. Then V is an open set containing z, which is a limit point of A, so $V \cap A \neq \phi$, which is a contradiction, so $A' \cap V = \phi$ and hence $V \subset X \sim (A \cup A')$. This proves that $A \cup A'$ is closed and obviously contains. A *i.e.*

$$A \subset A \cup A' \Rightarrow \overline{A} \subset \overline{A \cup A'} = A \cup A' \qquad [\because \ A \cup A' \text{ is closed}]$$

$$\Rightarrow \overline{A} \subset A \cup A' \qquad .....(1)$$

For the other way inclusion $A \cup A' \subset \overline{A}$, it suffices to show that $A' \subset \overline{A}$, since we already have $A \subset \overline{A}$.

Let $y \in A'$, if $y \notin \overline{A}$, then $y \in X \sim \overline{A}$ which is an open set, since $\overline{A}$ is always a closed set. But y is limit point of A so $\left(X \sim \overline{A}\right) \cap A \neq \phi$, which is a contradiction since $A \subset \overline{A} \Rightarrow \left(X \sim \overline{A}\right) \subset \left(X \sim A\right)$, so

$$y \in \overline{A} \Rightarrow A' \subset \overline{A} \quad \text{thus} \quad A \cup A' \subset \overline{A} \qquad .....(2)$$

from (1) and (2) $\Rightarrow \ \overline{A} = A \cup A'$.

Theorem 10. *Let* (X, τ) *be a topological space and let A, B be non-empty subsets of X, then*

(i) $\phi' = \phi$

(ii) $x \in A' \Rightarrow x \in (A \sim \{x\})'$

(iii) $A \subset B \Rightarrow A' \subset B'$

(iv) $(A \cup B)' = A' \cup B'$

(v) $(A \cap B)' \subset A' \cap B'$

here A' *means derived set of A.*

Proof : ***(i)*** Let $x \in X$ be arbitrary and let G be an open set s.t. $x \in G$, then $(G \sim \{x\}) \cap \phi = \phi$

$$\Rightarrow x \text{ is not a limit point of } \phi$$

$$\Rightarrow x \notin \phi' \quad \forall \ x \in X \ \Rightarrow \ \phi' = \phi$$

(ii) Let $x \in A'$, then $(G \sim \{x\}) \cap A \neq \phi \ \forall \ G \in \tau$, such that $x \in G$. .....(1)

Now $\qquad (G \sim \{x\}) \cap (A \sim \{x\})$

$$= (G \cap \{x\}^c) \cap (A \cap \{x\}^c) \qquad [\text{Here } \{x\}^c = X \sim \{x\}]$$

$$= G \cap A \cap \{x\}^c \cap \{x\}^c$$

$$= G \cap A \cap \{x\}^c$$

$$= (G \cap \{x\}^c) \cap A \neq \phi \qquad \text{by (1)}$$

$$\Rightarrow x \in (A \sim \{x\})'$$

Thus $\qquad x \in A' \Rightarrow x \in (A \sim \{x\})'$

(iii) Let $x \in A'$, then

$$(G \sim \{x\}) \cap A \neq \phi \quad \forall \ G \in \tau \quad \text{such that } x \in G \qquad .....(2)$$

$$A \subset B \Rightarrow A \cap (G \sim \{x\}) \subset B \cap (G \sim \{x\})$$

$$\Rightarrow B \cap (G \sim \{x\}) \neq \phi \qquad \text{by (2)}$$

$$\Rightarrow x \in B'$$

thus $\quad x \in A' \Rightarrow x \in B' \Rightarrow A' \subset B'$

thus $\quad A \subset B \Rightarrow A' \subset B'.$

(iv) Since $A \subset A \cup B,\ B \subset A \cup B$

$\therefore \quad A' \subset (A \cup B)',\ \ B' \subset (A \cup B)'$ from *(iii)*

$\Rightarrow \quad A' \cup B' \subset (A \cup B)'$ .....(1)

For the converse part let

$x \in (A \cup B)'$, then we must show that $x \in A' \cup B'$

we will prove the contra positive of above *i.e.*

if $\quad x \notin A' \cup B'$ then $x \notin (A \cup B)'$

Now $\quad x \notin A' \cup B' \Rightarrow x \notin A'$ and $x \notin B'$

$\Rightarrow \quad \exists$ open sets G_1 and $G_2 \in \tau$ such that $x \in G_1$ and $x \in G_2$

but $\quad (G_1 \sim \{x\}) \cap A = \phi = (G_2 \sim \{x\}) \cap B$

Now $x \in G_1 \cup G_2 \in \tau$

$[(G_1 \sim \{x\}) \cap A] \cup [(G_2 \sim \{x\}) \cap B] = \phi$

$\Rightarrow \quad [(G_1 \cup G_2) \sim \{x\}] \cap (A \cup B) = \phi$

$\Rightarrow \ \exists$ an open set $G_1 \cup G_2 \in \tau$ s.t. $x \in G_1 \cup G_2$ and

$[(G_1 \cup G_2 \sim \{x\})] \cap (A \cup B) = \phi$

$\Rightarrow \quad x \notin (A \cup B)'$

thus $\quad (A \cup B)' \subset A' \cup B'$ .....(2)

from (1) and (2) $\Rightarrow (A \cup B)' = A' \cup B'.$

(v) Since $A \cap B \subset A$ and $A \cap B \subset B$

$\Rightarrow \quad (A \cap B)' \subset A'$ and $(A \cap B)' \subset B'$ by (iii)

$\Rightarrow \quad (A \cap B)' \subset A' \cap B'.$

8.4.10 Exterior of a set :

The **exterior** of a subset A of a topological space (X, τ) is defined as interior of $(X \sim A)$. Thus symbolically ext $(A) = (X \sim A)^\circ$, elements of ext (A) are called **exterior points** of A.

8.4.11 Boundary set :

The **boundary set** of a subset A of a topological space (X, τ) is the set of all points which belong neither to the interior of A nor to the exterior of A and is denoted by $b(A)$. Thus symbolically

$$b(A) = X \sim (A^\circ \cup \text{ext}(A))$$

elements of $b(A)$ are called **boundary points** of A.

8.5 Sub Space

8.5.1 Sub space :

Let (X, τ) be a topological space and $Y \subset X$ such that $Y \neq \phi$. It is natural to inquire whether τ induces a topology on Y and if so how the two topologies are related. Now we define a topology S on Y by following way, if G is an τ-open set in X then $H = G \cap Y$ is S-open set in Y *i.e.*

$$S = \{G \cap Y : G \in \tau\}$$

First we verify that S is a topology on Y.

(T_1) $\phi \in S$ and $Y \in S$

since $\quad \phi \in \tau \Rightarrow \phi \cap Y = \phi \in S$

$$X \in \tau \Rightarrow X \cap Y = Y \in S \qquad [\because Y \subset X \Rightarrow X \cap Y = Y]$$

(T_2) Let $\{H_\lambda : \lambda \in \wedge\}$ be a family of subsets of Y in S ***i.e.*** $H_\lambda \in S, \forall \lambda \in \wedge$

To show $\quad H = \bigcup_{\lambda \in \wedge} H_\lambda \in S$

Since $H_\lambda \in S \; \forall \lambda \in \wedge \Rightarrow \exists \, G_\lambda \in \tau, \forall \lambda \in \wedge$ such that $H_\lambda = G_\lambda \cap Y, \forall \lambda \in \wedge$

$\Rightarrow \quad \{G_\lambda : \lambda \in \wedge\}$ be arbitrary family of τ-open sets in X

$$\Rightarrow \quad G = \bigcup_{\lambda \in \wedge} G_\lambda \in \tau \qquad [\tau \text{ is a topology on } X]$$

$$\Rightarrow \quad G \cap Y \in S \Rightarrow \left(\bigcup_{\lambda \in \wedge} G_\lambda\right) \cap Y \in S$$

$$\Rightarrow \quad \bigcup_{\lambda \in \wedge} (G_\lambda \cap Y) \in S \Rightarrow \bigcup_{\lambda \in \wedge} H_\lambda \in S$$

$$\Rightarrow \quad H = \bigcup_{\lambda \in \wedge} H_\lambda \in S$$

(T_3) Let H_1 and $H_2 \in S \Rightarrow G_1, G_2 \in \tau$ such that

$$H_1 = G_1 \cap Y, \quad H_2 = G_2 \cap Y$$

$\because \quad \tau$ is a topology on $X \Rightarrow G_1 \cap G_2 \in \tau$

$\Rightarrow \quad (G_1 \cap G_2) \cap Y \in S$

$\Rightarrow \quad (G_1 \cap Y) \cap (G_2 \cap Y) \in S$

$\Rightarrow \quad H_1 \cap H_2 \in S$

Thus S is a topology on Y called **relative topology** on Y **induce** by topology τ on X. The space (Y, S) is called **subspace** of (X, τ).

8.5.2 Hereditary property :

A property of topological space is said to be hereditary if whenever a space has that property, then so does every subspace of it. A trivial example of a hereditary property is the property of being either an indiscrete or discrete space *i.e.* every subspace of indiscrete or discrete space are indiscrete or discrete space respectively.

8.6 Solved Examples

***Ex.*1.** *Suppose* τ *is a family consisting of* ϕ *and all subsets* A_n *of* $\boldsymbol{N}$ *of the form*

$A_n = \{n, n+1, n+2, ...\} \quad \forall\, n \in \boldsymbol{N}$

(i) Show that τ *is a topology on* $\boldsymbol{N}$

(ii) Find open sets containing 2 *and* 7 *respectively.*

Sol. *(i)* To prove that τ is a topology on $\boldsymbol{N}$

(T_1) $\phi \in \tau$ (Given)

$$\boldsymbol{N} = A_1 = \{1, 2, 3, ...\} \in \tau.$$

(T_2) Let $\{A_i : i \in \wedge\}$ be the family of τ-open subsets of $\boldsymbol{N}$. Let $A = \cup \{A_i : i \in \wedge\}$ here $\wedge$ being a subsets of $\boldsymbol{N}$ contains a smallest positive integer n_0

$$\therefore \quad A = \cup \{A_i : i \in \wedge\} = \{n_0, n_0 + 1, n_0 + 2, ...\}$$

$$= A_{n_0} \in \tau$$

(T_3) Let $A_n, A_m \in \tau$ for same $n, m \in \boldsymbol{N}$

Now $\quad A_n \cap A_m = \{n, n+1, n+2, ...\} \cap \{m, m+1, m+2, ...\}$

if $n < m \quad A_n \cap A_m = \{m, m+1, m+2, ...\} = A_m \in \tau$

if $m < n \quad A_n \cap A_m = \{n, n+1, n+2, ...\} = A_n \in \tau$

$\Rightarrow A_n \cap A_m \in \tau$ in every case.

$\Rightarrow \tau$ is a topology on $\boldsymbol{N}$.

(ii) The open sets containing 2 are

$$A_1 = \boldsymbol{N} = \{1, 2, 3, ...\}$$
$$A_2 = \{2, 3, 4, ...\}$$

The open sets containing 7 are

$$A_1 = \{1, 2, 3, 4, 5, 6, 7, ...\}$$
$$A_2 = \{2, 3, 4, 5, 6, 7, 8, ...\}$$
$$A_3 = \{3, 4, 5, 6, 7, 8, ...\}$$
$$A_4 = \{4, 5, 6, 7, 8, ...\}$$
$$A_5 = \{5, 6, 7, 8, 9, ...\}$$
$$A_6 = \{6, 7, 8, 9, ...\}$$
$$A_7 = \{7, 8, 9, ...\}$$

***Ex.*2.** *Let U be the collection of all subsets $G \subset \boldsymbol{R}$, having the property that to each $x \in G$ $\exists\, \delta > 0$ such that open interval $(x - \delta, x + \delta) \subset G$, $\forall\, x \in G$.*

Show that U is a topology on $\boldsymbol{R}$ (*Usual topology*).

Sol. (T_1) $\phi \in U$, since $\phi \subset \boldsymbol{R}$ and ϕ does not contain any element and therefore the condition $x \in (x - \delta, x + \delta) \subset \phi$ is vacuously true.

$\boldsymbol{R} \in U$, since $\forall\, x \in \boldsymbol{R}\ \exists$ open interval *i.e.* $\exists\, \delta > 0$ such that $x \in (x - \delta, x - \delta) \subset \boldsymbol{R}$.

(T_2) Let $\{G_\lambda : \lambda \in \wedge\}$ be the arbitrary family of subsets of $\boldsymbol{R}$ such that $G_\lambda \in U\ \forall\, \lambda \in \wedge$.

To prove that $\quad G = \bigcup_{\lambda \in \wedge} G_\lambda \in U. \quad$ Let $x \in G$

$\Rightarrow x \in G_\lambda$ to same $\lambda \in \wedge$. Since $G_\lambda \in U$ and $x \in G_\lambda$ thus $\exists\, \delta > 0$ *s.t.*

$$x \in (x - \delta, x + \delta) \subset G_\lambda \subset \bigcup_{\lambda \in \wedge} G_\lambda = G$$

$\Rightarrow \qquad G \in U$

(T_3) Let $G_1, G_2 \in U$, to prove $G_1 \cap G_2 \in U$

if $G_1 \cap G_2 = \phi$, then obviously $G_1 \cap G_2 \in U$, so let us assume $G_1 \cap G_2 \neq \phi$ and let $x \in G_1 \cap G_2$

$\Rightarrow \qquad x \in G_1$ and $x \in G_2$ where $G_1, G_2 \in U$

$\Rightarrow \qquad \exists\, \delta_1 > 0$ and $\delta_2 > 0$ such that

$x \in (x - \delta_1, x + \delta_1) \subset G_1$ and $x \in (x - \delta_2, x + \delta_2) \subset G_2$

Let $\delta = \min\{\delta_1, \delta_2\}$ then $\delta > 0$ and

$\qquad x \in (x - \delta, x + \delta) \subset (x - \delta_1, x + \delta_1) \subset G_1$

and $\quad x \in (x - \delta, x + \delta) \subset (x - \delta_2, x + \delta_2) \subset G_2$

$\Rightarrow \qquad x \in (x - \delta, x + \delta) \subset G_1 \cap G_2 \Rightarrow G_1 \cap G_2 \in U.$

***Ex.*3.** *Let $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, b, e\}, \{a, c, d\}, \{a, b, c, d\}\}$ be a topology on $X = \{a, b, c, d, e\}$, then :*

(i) List all τ-open subsets of X

(ii) List all τ-closed subsets of X

(iii) Find the τ-open nbds of a

(iv) Find the closure of the sets $\{a\}$, $\{b\}$ and $\{c\}$

(v) Find the interior points of the subset $A = \{a, b, c\}$

(vi) Which of the sets $\{a\}$, $\{b\}$, $\{c, e\}$ are dense in X.

Sol. ***(i)*** τ-open subsets of X are the elements of τ namely

$$\phi, X, \{a\}, \{a, b\}, \{a, b, e\}, \{a, c, d\}, \{a, b, c, d\}$$

(ii) τ-closed subsets of X are

$$X - \phi, X - X, X - \{a\}, X - \{a, b\}, X - \{a, b, e\}, X - \{a, c, d\}, X - \{a, b, c, d\}$$

ie. $X, \phi, \{b, c, d, e\}, \{c, d, e\}, \{c, d\}, \{b, e\}, \{e\}$

(iii) τ-open nbds of a are open sets containing a

$\{a\}, \{a, b\}$ $\{a, b, e\}, \{a, c, d\}$ $\{a, b, c, d\}, X$

(iv) $\overline{\{a\}} = \cap \{F : F$ is τ-close subset s.t. $\{a\} \subset F\} = X,$

since X is the only closed subset which contains a

$\overline{\{b\}} = X \cap \{b, c, d, e\} \cap \{b, e\} = \{b, e\}$

$\overline{\{c\}} = X \cap \{b, c, d, e\} \cap \{c, d, e\} \cap \{c, d\} = \{c, d\}$

(v) $A^\circ = \cup \{G : G \in \tau, G \subset A\} = \{a\} \cup \{a, b\} = \{a, b\}$

(vi) A is called dense in X if $\overline{A} = X$, from (iv) it is clear that $\overline{\{a\}} = X$ so $\{a\}$ is dense in X, since $\overline{X} = X$ $\{\because X$ is closed$\}$

$\Rightarrow$ X is dense in X

but $\overline{\{b\}} = \{b, e\} \neq X \Rightarrow \{b\}$ is not dense in X

also $\overline{\{c, e\}} = \{c, d, e\} \neq X \Rightarrow \{c, e\}$ is not dense in X.

Ex.4. *Give examples to show that arbitrary union of closed sets is not necessarily closed and arbitrary intersection of open sets is not necessarily open in a topological space.*

Sol. Let τ-denote the usual topology U on $\boldsymbol{R}$

Let $$F_n = \left[0, \frac{n}{n+1}\right] \quad \forall\, n \in \boldsymbol{N}$$

The F_n is τ-closed subset of $\boldsymbol{R}$ $\forall\, n \in \boldsymbol{N}$

[Since closed intervals are τ-closed sets in ($\boldsymbol{R}$, U)]

but $$\bigcup_{n=1}^{\infty} F_n = \left[0, \frac{1}{2}\right] \cup \left[0, \frac{2}{3}\right] \cup \ldots = [0,1)$$

= semi open set $\neq$ closed subset of $\boldsymbol{R}$

Now let $$G_n = \left(\frac{-1}{n}, \frac{1}{n}\right) \quad \forall\, n \in \boldsymbol{N}$$

The G_n is τ-open subsets of $\boldsymbol{R}$ $\forall\, n \in \boldsymbol{N}$

[Since open intervals are τ-open sets in ($\boldsymbol{R}$, U)]

but $$\bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \left(\frac{-1}{n}, \frac{1}{n}\right) = (-1, 1) \cap \left(\frac{-1}{2}, \frac{1}{2}\right) \cap \ldots$$

= $\{0\} \neq$ open subset of $\boldsymbol{R}$.

Ex.5. *Give two examples of topologies on $X = \{a, b, c\}$ in which every open set is also a closed set.*

Sol. *(a)* Here $X = \{a, b, c\}$. Now consider discrete topology (X, D) on X, in which every subset of X is open *i.e.*

D-open sets are $\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}$

D-closed sets are $X, \phi, \{b, c\}, \{a, c\}, \{a, b\}, \{c\}, \{b\}, \{a\}$

thus every D-open set is also D-closed sets.

(b) Consider the topology $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ on X

τ-open sets are $\phi, X, \{a\}, \{b, c\}$

τ-closed sets are $X, \phi, \{b, c\}, \{a\}$

$\Rightarrow$ every τ-open set is also τ-closed set.

***Ex.*6.** *Find three mutually non-comparable topologies for the set $X = \{a, b, c\}$.*

Sol. Let

$$\tau_1 = \{\phi, X, \{a\}\}$$
$$\tau_2 = \{\phi, X, \{a, c\}\}$$
$$\tau_3 = \{\phi, X, \{b, c\}\}$$

Then the topologies τ_1, τ_2 and τ_3 are mutually non-comparable.

***Ex.*7.** *Show that any finite subset of **R** is closed set for the usual topology U on **R**.*

Sol. Let $A = \{a_1, a_2, ..., a_n\}$ be a finite subset of $\boldsymbol{R}$. First we shall show that $\{a_1\}$ is closed. Since

$$R - \{a_1\} = \{x \in \boldsymbol{R} : x \neq a_1\}$$
$$= \{x \in \boldsymbol{R} : \text{ either } \; x < 1 \; \text{ or } \; x > a_1\}$$
$$= \{x \in \boldsymbol{R} : x < a_1\} \cup \{x \in \boldsymbol{R} : x > a_1\}$$
$$= (-\infty, a_1) \cup (a_1, \infty)$$

= union of two open rays, since open rays are U-open sets in $\boldsymbol{R}$

$\therefore$ $\boldsymbol{R} - \{a_1\}$ = union of two U-open sets

= U-open set

$\Rightarrow \{a_1\}$ is U-closed set

Thus every singleton subset of $\boldsymbol{R}$ is closed

Now $A = \{a_1\} \cup \{a_2\} \cup ... \cup \{a_n\}$

= finite union of U-closed sets

= U-closed set [by Theorem 1]

$\Rightarrow$ Every finite subset of $\boldsymbol{R}$ is a U-closed set.

***Ex.*8.** *Is $\overline{A \cap B} = \overline{A} \cap \overline{B}$? Give reason in support of your answer.*

Sol. We know from Theorem 3(v) that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ but $\overline{A} \cap \overline{B} \not\subset \overline{A \cap B}$.

Let $A = (0, 1)$, $B = (1, 2)$ be two open subset of $\boldsymbol{R}$ in usual topology, then $A \cap B = \phi$ and so $\overline{A \cap B} = \overline{\phi} = \phi$

but $\overline{A} = [0, 1], \overline{B} = [1, 2]$

$\therefore$ $\overline{A} \cap \overline{B} = [0, 1] \cap [1, 2] = \{1\} \neq \phi = \overline{A \cap B}$

$\Rightarrow$ $\overline{A} \cap \overline{B} \not\subset \overline{A \cap B}$

thus $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$

Ex.9. *Let* $\tau = \{\phi, X, \{1\}, \{1, 2\}, \{1, 2, 5\}, \{1, 2, 3, 4\}, \{1, 3, 4\}\}$ *be the topology on* $X = \{1, 2, 3, 4, 5\}$

Determine limit points, closure, interior, exterior and boundary of the following sets :

(i) $A = \{3, 4, 5\}$ ***(ii)*** $B = \{2\}$.

Sol. τ-open sets are : $\phi, X, \{1\}, \{1, 2\}, \{1, 2, 5\}, \{1, 2, 3, 4\}, \{1, 3, 4\}$

τ-closed sets are : $\phi, X, \{2, 3, 4, 5\}, \{3, 4, 5\}, \{3, 4\}, \{5\}, \{2, 5\}$

(i) $A = \{3, 4, 5\}$

$\overline{A} = \cap \{F \subset X : F \text{ is closed}, F \supset A\} = \{3, 4, 5\}$

The following sets are open nbds of 1 :

$X, \{1\}, \{1, 2\}, \{1, 2, 5\}, \{1, 2, 3, 4\}, \{1, 3, 4\}$

Now $(\{1\} - \{1\}) \cap A = \phi \cap A = \phi$

thus $\exists$ open set $G = \{1\}$ such that

$$(G \sim \{1\}) \cap A = \phi$$

$\Rightarrow$ 1 is not limit point of A.

Now $G = \{1, 2\}$ is the open set containing 2 and

$$(G \sim \{2\}) \cap A = \phi$$

$\Rightarrow$ 2 is not limit point of A

τ-open sets containing 3 are : $\{1, 2, 3, 4\}, \{1, 3, 4\}, X$

Let $G_1 = \{1, 2, 3, 4\}, G_2 = \{1, 3, 4\}$.

Now $(G_1 \sim \{3\}) \cap A = \{4\} \neq \phi$

$(G_2 \sim \{3\}) \cap A = \{4\} \neq \phi$

$\Rightarrow$ 3 is limit point of A

Similarly 4 is limit point of A [prove by your own]

τ–open set containing 5 are : $\{1, 2, 5\}, X$

$G = \{1, 2, 5\}$ then

$$(G \sim \{5\}) \cap A = \phi$$

$\Rightarrow$ 5 is not limit point of A

$\therefore$ derived set $A' = \{3, 4\}$

Now $A^\circ = \cup \{G : G \subset A, G \in \tau\} = \cup \{\phi\} = \phi$

$$\text{ext}(A) = (X \sim A)^\circ = \{1, 2\}^\circ = \cup \{\{1\}, \{1, 2\}, \phi\} = \{1, 2\}$$

$$b(A) = X \sim (A^\circ \cup \text{ext}(A)) = X - (\phi \cup \{1, 2\})$$

$$= \{3, 4, 5\} = A$$

(ii) $B = \{2\}$

τ-open sets containing 1 : $\{1\}, \{1, 2\}, \{1, 2, 5\}, \{1, 2, 3, 4\}, \{1, 3, 4\}$

Let $G = \{1\}$ so $(G \sim \{1\}) \cap B = \phi$

$\Rightarrow$ 1 is not limit point of B.

τ-open sets containing 2 : $\{1, 2\}, \{1, 2, 5\}, \{1, 2, 3, 4\}$

Let $G = \{1, 2\}$ so $(G \sim \{2\}) \cap B = \phi$

$\Rightarrow$ 2 is not limit point of B

Similarly you can show 3, 4 are also not limit points of B

Now τ-open sets containing 5 : $G_1 = \{1, 2, 5\}$ and X

$$(G_1 \sim \{5\}) \cap B = \{2\} \neq \phi$$

$$(X \sim \{5\}) \cap B = \{2\} \neq \phi$$

$\Rightarrow$ 5 is a limit point of B

$\therefore$ derived set of $B = B' = \{5\}$

$$\overline{B} = \cap \{F \subset X : F \text{ is } \tau\text{-closed and } F \supset B\}$$

$$= \{2, 3, 4, 5,\} \cap \{2, 5\} = \{2, 5\}$$

$$B^\circ = \cup \{G : G \in \tau \text{ and } G \subset B\} = \cup \{\phi\} = \phi$$

$$\text{ext}(B) = (X \sim B)^\circ = \{1, 3, 4, 5\}^\circ = \cup \{\phi, \{1\}, \{1, 3, 4\}\} = \{1, 3, 4\}$$

$$b(B) = X - (B^\circ \cup \text{ext}(B)) = X - (\phi \cup \{1, 3, 4\})$$

$$= \{2, 5\}$$

***Ex.*10.** *Let τ be a topology on a set X consisting of four sets i.e. $\tau = \{\phi, X, A, B\}$, where A and B are non-empty distinct proper subsets of X. What conditions must A and B satisfy* ?

Sol. Since $A \cap B$ must also belong to τ, there are two possibilities :

Case 1 : $A \cap B = \phi$

Thus $A \cup B$ can not be A or B; hence $A \cup B = X$.

Thus the class $\{A, B\}$ is a partition of X.

Case 2 : $A \cap B = A$ or $A \cap B = B$

In either case, one of the sets is a subset of the other, and the members of τ are totally ordered by inclusion : $\phi \subset A \subset B \subset X$ or $\phi \subset B \subset A \subset X$.

***Ex.*11.** *Consider the following topology on*

$X = \{a, b, c, d, e\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c\}, \{a, b, c, d\}\}$

List the member of the relative topology τ_A *on* $A = \{a, c, e\}$

Sol. $\tau_A = \{A \cap G : G \in \tau\}$ so the members of τ_A are

$A \cap X = A$ $A \cap \{a\} = \{a\}$ $A \cap \{a, c, d\} = \{a, c\}$ $A \cap \{a, b, e\} = \{a, e\}$

$A \cap \phi = \phi$ $A \cap \{a, b\} = \{a\}$ $A \cap \{a, b, c, d\} = \{a, c\}$

i.e. $\tau_A = \{\phi, A, \{a\}, \{a, c\}, \{a, e\}\}$

observe that $\{a, c\}$ and $\{a, e\}$ are not open in X, but are relatively open in A ***i.e.*** τ_A open.

Self-learning exercise-2

1. Let τ be the topology on $\boldsymbol{N}$ consisting of ϕ and all subsets A_n of the form

$$A_n = \{n, n+1, n+2, ...\}$$

when $n \in \boldsymbol{N}$:

(i) Determine the closed subsets of $(\boldsymbol{N}, \tau)$.

(ii) Determine the closure of the sets $\{7, 24, 47, 85\}$ and $\{3, 6, 9, 12, ...\}$.

(iii) Determine those subsets of $\boldsymbol{N}$, which are dense in $\boldsymbol{N}$.

2. Let $X = \{a, b, c, d, e\}$. Determine wether or not each of the following classes of subsets of X is a topology on X

(i) $\tau_1 = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$.

(ii) $\tau_2 = \{\phi, X, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$.

(iii) $\tau_3 = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}\}$.

3. *(i)* In discrete topology every point $p \in X$ is a limit point of every subset $A \subset X$ T/F

(ii) Discrete topology on a set $X \neq \phi$ is door topology. T/F

(iii) In indiscrete topology every non-empty subset $A \subset X$ is dense in X. T/F

8.7 Summary

In this chapter we have studied the concept of a topology on a non-empty set X. We have also discussed various examples of topologies and studied that it is possible to define different topologies on the same set. We also studied how these topologies are related to each other. We have also studied about closed sets, closures, derived set, interior, exterior and boundary of a set in a topological space and proved various theorems on their properties. We have studied a topology can also be defined in terms of closed sets and neighbourhood systems. Finally we have studied a way of defining a topology on a subset of a topological space by relative topology.

8.8 Answers to self-learning exercises

Self-learning exercise-1

1. *(d)** **2.** *(d)* **3.** *(d)*

4. *(ii)* Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}\}$, $\tau_2 = \{\phi, X, \{b\}\}$

Then τ_1 and τ_2 are topology on X but

$$\tau_1 \cup \tau_2 = \{\phi, X, \{a\}, \{b\}\} \text{ is not a topology on } X$$

Self- learning exercise-2

1. *(i)* A set is closed iff its complement is open. Hence the closed subsets of $\boldsymbol{N}$ are as follows :

$$\phi, \boldsymbol{N}, \{1\}, \{1, 2\}, \{1, 2, 3\},..., \{1, 2, 3,..., m\}, ...$$

(ii) The closure of a set is the smallest closed super set. So

$$\overline{\{7, 24, 47, 85\}} = \{1, 2, 3, ..., 85\}$$

$$\overline{\{3, 6, 9, 12, ...\}} = \{1, 2, 3, ...\} = \boldsymbol{N}$$

(iii) If a subset A of $\boldsymbol{N}$ is infinite, or equivalently unbounded, then $\overline{A} = \boldsymbol{N}$ *i.e.* A is dense in $\boldsymbol{N}$. If A is finite, then its closure is not $\boldsymbol{N}$, *i.e.* A is not dense in $\boldsymbol{N}$.

2. *(i)* τ_1 is not a topology since $\{a, b\} \cup \{a, c\} \notin \tau_1$

(ii) τ_2 is not a topology since $\{a, b, c\} \cap \{a, b, d\} \notin \tau_2$

(iii) τ_3 is a topology

3. *(i)* F *(ii)* T *(iii)* T

8.9 Exercises

1. Consider the collection τ consisting of ϕ, $\boldsymbol{N}$ and all subsets of $\boldsymbol{N}$ of from $G_n = \{1, 2, 3,..., n\}$, $\forall\, n \in \boldsymbol{N}$. Show that τ is a topology on $\boldsymbol{N}$.

2. Consider that topology τ on $\boldsymbol{N}$ given in *Q*.1,

(i) List all closed subsets of $\boldsymbol{N}$

(ii) Find the closure of $\{2, 3, 6, 12\}$, $\{2, 4, 6, ...\}$ and $\{1, 2, 3, 5, 7, 11, 13, ...\}$

(iii) Determine those subsets of $\boldsymbol{N}$, which are dense in $\boldsymbol{N}$.

(iv) Find the derived set of $\{1, 3, 5, 7, ...\}$, $\{1, 2, 3, 4,\}$ and $\{1, 4, 9, 16, ...\}$

(v) Determine interior of $\{1, 3, 5, 7, ...\}$ and $\{2, 4, 6, ...\}$

(vi) Find τ-open nbds of 5 and 11.

3. Let X be a topological space and let Y and Z be subspaces of X such that $Y \subset Z$. Show that the topology which Y has as a subspaces of X is the same as that which it has as a subspace of Z.

4. Show that *a* subset A of a topological space X is closed iff $\overline{A} = A$

5. Show that a subset A of a topological space X is open iff $A^\circ = A$.

6. Show that if A is τ-closed subset of X and $x \in X \sim A$, then $\exists$ τ-nbd M of x such that

$$M \cap A = \phi.$$

7. Show that closed intervals are closed set in usual topology on ***R***.

8. Let (X, τ) be a topological space, and $A, B \subset X$. Then show that

   (i) $\phi^\circ = \phi$ *(ii)* $X^\circ = X$

   (iii) $A \subset B \Rightarrow A^\circ \subset B^\circ$ *(iv)* $(A^\circ)^\circ = A^\circ$

   (v) $(A \cap B)^\circ = A^\circ \cap B^\circ$ *(vi)* $(A \cup B)^\circ \supset A^\circ \cup B^\circ$.

9. Let (Y, τ_y) be the subspace of a topological space (X, τ) and $A \subset Y$. Let $\mathrm{cl}_Y(A)$ and $\mathrm{cl}_X(A)$ denote closure of A in τ_y and τ topologies on Y and X respectively, then show that

$$cl_Y(A) = cl_X(A) \cap Y.$$

10. Let (Y, τ_y) be the subspace of a topological space (X, τ). Then show that every τ_Y-open set is also τ-open iff Y is τ-open.

11. In any topological space, prove that $b(A) = \phi$ iff A is both open as well as closed.

□□□

Unit 9 : Bases, Sub-bases and Continuity

Structure of the Unit

9.0 Objectives

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9.2 Base for a topology

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9.6 Continuity at a point

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9.9 Summary

9.10 Answers to self learning exercises

9.11 Exercises

9.0 Objectives

In this unit we have define very important concept of bases and subbases in topology. After reading this unit, you will learn how the concept of bases is very useful in defining and discussing the properties of a topological space. You will also learn that how the concept of continuity can be generalized in a topological space. In the end, you will also learn about homeomorphism between topological spaces.

9.1 Introduction

In all the examples of topological spaces in previous chapter, we were able to specify the entire collection of open-sets. A topology on a set can be a complicated collection of subsets of a set, and it can be difficult to describe the entire collection, so instead we specify a sub-collection of open sets that generates the topology. One such collection is called a basis and another is called a sub-basis.

Continuity is of fundamental importance in topology. Indeed it is a basic to much of mathematics. A topology on a set is a structure that establishes a notion of proximity on the set. Continuous functions between topological spaces preserve proximity, reflecting the idea that a continuous function sends points that are close in one space to point that are close in the other. A continuous bijective function that has a continuous inverse is called a homeomorphism. Such function provide us with the main notion of topological equivalence.

9.2 Base for a topology

Let (X, τ) be a topological space. *A* class $\boldsymbol{B}$ of open subsets of X. *i.e.* $\boldsymbol{B} \subset \tau$, is a base for the topology τ iff every open set $G \in \tau$ is the union of members of $\boldsymbol{B}$ or equivalently, $\boldsymbol{B} \subset \tau$ is a base for τ iff for any point p belonging to an open set G, there exist $B \in \boldsymbol{B}$ with $p \in B \subset G$. The elements of $\boldsymbol{B}$ are referred to as basic open sets.

***Ex.*1.** *The set of all open intervals in* $\boldsymbol{R}$ *form a base for the usual topology on* $\boldsymbol{R}$. *For if* $G \subset \boldsymbol{R}$ *is open and* $p \in G$, *then by definition, there exists an open interval* (a, b) *with*

$$p \in (a, b) \subset G.$$

Similarly the set of all open intervals (r, s) *with r and s as rationals also forms a base for the usual topology on* $\boldsymbol{R}$.

***Ex.*2.** *The collection of all open circular discs* (*i.e., not containing the points on circumference*) *in* $\boldsymbol{R}^2$ *forms a base for the usual topology on* $\boldsymbol{R}^2$.

***Ex.*3.** *Consider any discrete space* (X, D). *Then the class* $\boldsymbol{B} = \{\{p\} : p \in X\}$ *of all singleton subsets of X is a base for the discrete topology D on X. For each singleton set* $\{p\}$ *is D-open, since every* $A \subset X$ *is D-open, furthermore, every set is the union of singleton sets.*

***Ex.*4.** *Let* $X = \{a, b, c\}$, $\boldsymbol{B} = \{\{a,b\}, \{b,c\}\}$ *cannot be a base for any topology on X. Since* $\{a,b\}$ *and* $\{b,c\}$ *would them selves be open and therefore their intersection* $\{a, b\} \cap \{b, c\} = \{b\}$ *would also be open, but* $\{b\}$ *cannot written as union, of members of* $\boldsymbol{B}$.

Theorem 1. *Let* $\boldsymbol{B}$ *be a collection of subsets of a non empty set X. Then* $\boldsymbol{B}$ *is a base for some topology on X iff it satisfy the following two conditions* :

($\boldsymbol{B_1}$) $X = \cup \{B : B \in \boldsymbol{B}\}$

($\boldsymbol{B_2}$) *For any* $B_1, B_2 \in \boldsymbol{B}$, *if* $x \in B_1 \cap B_2$ *then* $\exists\, B_3 \in \boldsymbol{B}$ *such that* $x \in B_3 \subset B_1 \cap B_2$

Proof. First we assume that $\boldsymbol{B}$ is a base for a topology τ on X. Since X is open subset of X, X is the union of members of $\boldsymbol{B}$. Hence X is the union of all members of $\boldsymbol{B}$, ***i.e.*** $X = \cup \{B : B \in \boldsymbol{B}\}$

Furthermore if $B_1, B_2 \in \boldsymbol{B} \Rightarrow B_1, B_2$ are τ-open subsets of $X \Rightarrow B_1 \cap B_2$ is also τ-open subset of X, since $\boldsymbol{B}$ is a base for τ, therefore by definition if $x \in B_1 \cap B_2$ $\exists\, B_3 \in \boldsymbol{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. Thus both the condition **($\boldsymbol{B_1}$)** and **($\boldsymbol{B_2}$)** are satisfied.

Conversely we assume $\boldsymbol{B}$ is the collection of subsets of X satisfying **($\boldsymbol{B_1}$)** and **($\boldsymbol{B_2}$)**. Let τ be the collection of all subsets of X which are unions of members of $\boldsymbol{B}$. We claim that τ is a topology on X. Observe that $\boldsymbol{B} \subset \tau$ will be the base for this topology.

[T_1] By (B_1) $X = \cup \{B : B \in \boldsymbol{B}\}$ so $X \in \tau$. Note that ϕ is the union of the empty subclass of $\boldsymbol{B}$, *i.e.* $\phi = \cup \{B : B \in \phi \subset \boldsymbol{B}\}$; hence $\phi \in \tau$ and so τ setisfies **[T_1]**.

[T_2] Now let $\{G_\lambda : \lambda \in \wedge\}$ be a class of members of τ. By definition each G_λ is the union of members of $\boldsymbol{B}$ hence the union $\bigcup_{\lambda \in \Lambda} G_\lambda$ is also the union of members $\boldsymbol{B}$ and so $\bigcup_{\lambda \in \Lambda} G_\lambda \in \tau$. Thus τ satisfies **[T_2]**.

[T_3]. Let $G_1, G_2 \in \tau$. By definition of τ there exist two subclasses $\{B_i : i \in I\}$ and $\{B_j : j \in J\}$ of $\boldsymbol{B}$ such that $G_1 = \bigcup_{i \in I} B_i$ and $G_2 = \bigcup_{j \in J} B_j$ [Here I and J are some index sets]. Then

$$G_1 \cap G_2 = \left(\bigcup_{i \in I} B_i\right) \cap \left(\bigcup_{j \in J} B_j\right) = \cup \{B_i \cap B_j : i \in I, j \in J\}$$

let by **[B_2]**, $B_i \cap B_j$ is the union of members of $\boldsymbol{B}$ hence $G_1 \cap G_2 = \cup \{B_i \cap B_j : i \in I, j \in J\}$ is also the union of members of $\boldsymbol{B}$ and so belongs to τ, which therefore satisfies **[T_3]**.

Hence τ is a topology on X, with base $\boldsymbol{B}$.

Theorem 2. *If* $\boldsymbol{B}$ *is a sub class of* τ, *then the following statements are equivalent* (*i.e. the two definition given for* $\boldsymbol{B}$ *to be a base are equivalent*)

(*i*) *Each* $G \in \tau$ *is the union of members of* $\boldsymbol{B}$

(*ii*) *For* any *point* $p \in G$, *where* G *is an open set* $\exists\, B \in \boldsymbol{B}$ *such that* $p \in B \subset G$.

Proof. ***(i)* $\Rightarrow$ *(ii)***

Let $G \in \tau$ is the union of members of $\boldsymbol{B}$ *i.e.*

$$G = \bigcup_{i \in I} B_i \text{ when } B_i \in \boldsymbol{B}\ \forall\, i \in I \text{ (Index set)}$$

then each point $p \in G \Rightarrow p \in \bigcup_{i \in I} B_i$

$$\Rightarrow \quad \exists\, i_0 \in I \text{ such that } p \in B_{i_0}, \text{ so } \quad p \in B_{i_0} \subset \bigcup_{i \in I} B_i = G$$

(ii)* $\Rightarrow$ *(i) Let for each $p \in G$, $\exists\, B_p \in \boldsymbol{B}$ such that

$$p \in B_p \subset G$$

then $\quad G = \cup \{B_p : p \in G\}$ and G is the union of members of $\boldsymbol{B}$.

Theorem 3. *Let* $\boldsymbol{B}$ *be a base for a topology* τ *on* X *and let* $\boldsymbol{B}^*$ *be a class of open sets containing* $\boldsymbol{B}$ ***i.e.*** $\boldsymbol{B} \subset \boldsymbol{B}^* \subset \tau$. *Then* $\boldsymbol{B}^*$ *is also a base for* τ.

Proof. Let G be an open subset of X. Since $\boldsymbol{B}$ is a base for (X, τ), G is the union of member of $\boldsymbol{B}$ *i.e.* $G = \bigcup_{i \in I} B_i$ where $B_i \in \boldsymbol{B}$. But $\boldsymbol{B} \subset \boldsymbol{B}^*$ hence each $B_i \in \boldsymbol{B}$ also belongs to $\boldsymbol{B}^*$. So G is the union of members of $\boldsymbol{B}^*$ and therefore $\boldsymbol{B}^*$ is also a base for (X, τ).

Theorem 4. *Let **B** and $\boldsymbol{B}^*$ be bases, respectively for topologies* τ *and* τ^* *on a set X. Let each* $B \in \boldsymbol{B}$ *is the union of members of* $\boldsymbol{B}^*$ *then* τ^* *is finer than* τ, *i.e.* $\tau \subset \tau^*$.

Proof. Let $G \in \tau$ be any τ-open set, since **B** is a base for τ, G is the union of members of **B**

i.e. $$G = \bigcup_{i \in I} B_i \text{ where } B_i \in \boldsymbol{B}\ \forall i \in I \text{ (Index set)}$$

But, by hypothesis, each $B_i \in \boldsymbol{B}$ is the union of members of $\boldsymbol{B}^*$ and so $G = \bigcup_{i \in I} B_i$ is also the union of members of $\boldsymbol{B}^*$, which are τ^*-open sets. Hence G is also τ^*-open set *i.e.* $G \in \tau^*$ and thus

$$\tau \subset \tau^*.$$

9.3 Subbases

Let (X, τ) be a topological space. *A* class **S** of open subsets of X, *i.e.* $\boldsymbol{S} \subset \tau$ is a subbase for the topology τ on X iff finite intersections of members of **S** form a base for τ. The elements of **S** are referred to as sub-basic open sets.

Example. *Let* $a, b \in \boldsymbol{R}$ *be arbitrary such that* $a < b$.

clearly $$(-\infty, b) \cap (a, \infty) = (a, b)$$

The open intervals (a, b) *form a base for the usual topology on* **R.** *Hence by definition the family of infinite open intervals form a subbase for the usual topology on* **R.**

Theorem 5. *Any collection **A** of subsets of a non-empty set X is the subbase for a unique topology on X. That is, finite intersections of members of **A** form a base for a topology* τ *on X.*

Proof. Let **B** is the class of finite intersections of member of **A.** We show that **B** satisfies the two conditions [$\boldsymbol{B}_1$] and [$\boldsymbol{B}_2$] in Theorem 1.

[$\boldsymbol{B}_1$] Since X is the intersection of empty collection of members of **A** and so $X = \cup \{B : B \in \boldsymbol{B}\}$

[$\boldsymbol{B}_2$] Let $B_1, B_2 \in \boldsymbol{B}$ and $x \in B_1 \cap B_2$. Then B_1, B_2 are finite intersection of members of **A.**

Hence $B_1 \cap B_2$ is also a finite intersection of members of **A** and so $B_1 \cap B_2 \in B$. Hence **B** is a base for a unique topology on X for which **A** is subbase.

9.4 Local Base

Let p be any arbitrary point in a topological space X. *A* collection $\boldsymbol{B}_p$ of open sets containing p is called a local base at p iff for each open set G containing p, $\exists\, B \in \boldsymbol{B}_p$ such that $p \in B \subset G$.

Example 1. *Consider usual topology U on **R** and point* $x \in \boldsymbol{R}$. *Then the collection of all open intervals* $(x - \in, x + \in)\ \forall\ \in > 0$, *is a local base at x. Since any open set containing x also contains an open set* $(x - \in, x + \in)$ *for some* $\in > 0$.

Example 2. *Let **B** be a base for a topology* τ *on X and let* $x \in X$. *Then the members of the base **B** which contains x form a local base at x.*

Theorem 6. *A point x in a topological space* (X, τ) *is an limit point of* $A \subset X$ *iff each member of some local base* ***B*** *at x contains a point of A different from x.*

Proof. Since $x \in X$ is a limit point of A iff

$$(G \sim \{x\}) \cap A \neq \phi \ \forall\, G \in \tau \ s.t.\ x \in G.$$

But $\boldsymbol{B} \subset \tau$, so in particular

$$(B \sim \{x\}) \cap A \neq \phi \ \forall\ B \in \boldsymbol{B}$$

Conversely, we assume $(B \sim \{x\}) \cap A \neq \phi \ \forall\ B \in \boldsymbol{B}$ and let G be any open subset of X containing x. Then

$\exists\, B_0 \in \boldsymbol{B}$ such that $x \in B_0 \subset G$. But then

$$(G \sim \{x\}) \cap A \supset (B_0 - \{x\}) \cap A \neq \phi$$

$\Rightarrow \quad (G \sim \{x\}) \cap A \neq \phi \ \forall\, G \in \tau \ s.t.\ x \in G.$

$\Rightarrow \quad x$ is a limits point of A.

9.4.1 First countable space : Let (X, τ) be a topological space. The space X is said to satisfy the first axiom of countability if X has a countable local bare at each $x \in X$. The space X, in this case is called first countable or first axiom space.

9.4.2 Second countable space : Let (X, τ) be a topological space. The space X is said to satisfy the second axiom of countability if there exists a countable base for τ on X. In this case, the space X is called second countable or second axiom space.

Example. *The collection of all open intervals* (r, s) *with r and s as rational numbers from a base* ***B*** *for the usual topology U of* ***R****. Since* ***Q*** *is a countable set, so* ***B*** *is a countable base for U on* ***R****.*

$\Rightarrow \quad$ *(**R**, U) is second countable space.*

Theorem 7. *A second countable space is always first countable space, but converse is not true.*

Proof. Let (X, τ) be a second countable space with a countable base $\boldsymbol{B}$.

Let $\boldsymbol{B} = \{B_n : n \in \boldsymbol{N}\}$ when $\boldsymbol{N}$ is set of natural numbers, let $x \in X$ be arbitrary and

$$L_x = \{B_n \in \boldsymbol{B} : x \in B_n\}$$

Then

(i) L_x, being a subset of a countable set $\boldsymbol{B}$ is countable.

(ii) Since members of $\boldsymbol{B}$ are τ-open sets and so members of L_x, as $L_x \subset \boldsymbol{B}$.

(iii) Any $G \in L_x \Rightarrow x \in G$ [By definition of L_x]

(iv) Let $G \in \tau$ be arbitrary such that $x \in G$.

$\therefore \quad x \in G \in \tau \Rightarrow \exists\, B_1 \in \boldsymbol{B}$ s.t. $x \in B_1 \subset G$

$\Rightarrow \exists\, B_1 \in L_x$ s.t. $x \in B_1 \subset G \qquad [\because x \in G \in B_1 \Rightarrow B_1 \in L_x]$

$\therefore \quad x \in G \in \tau \Rightarrow \exists\, B_1 \in L_x$ *s.t.* $B_1 \subset G$

$\Rightarrow \quad L_x$ is a countable local base at $x \in X$

$\Rightarrow \quad X$ is first countable

Now to show that converse is not true we will give an example of a first countable space which is not second countable space.

Let τ be a discrete topology on an uncountable set X, so that every subset of X is open in X. Clearly $\boldsymbol{B} = \{\{x\} : x \in X\}$ is a base for topology τ on X and $\boldsymbol{B}$ is not countable. Hence (X, τ) is not a second countable space.

But (X, τ) is first countable, since if we take $L_x = \{\{x\}\}$, then evidently L_x is a local base at $x \in X$. Since for any $G \in \tau$ with $x \in G$

$\exists\, \{x\}$ such that $x \in \{x\} \subset G$.

Also L_x is a countable local base at $x \in X$ as L_x contains only one member $\{x\}$.

$\Rightarrow \quad (X, \tau)$ is first countable.

Illustrative Examples

***Ex.*1.** *Let* $X = \{1,2,3,4\}$ *and* $A = \{\{1, 2\}, \{2, 4\}, \{3\}\}$

Determine the topology on X generated by the elements of A and hence determine the base for this topology.

Sol. Finite intersections of the members of A form the class $\boldsymbol{B}$ given by

$$\boldsymbol{B} = \{\{1, 2\}, \{3\}, \{2, 4\}, \{2\}, \phi, X\}$$

Now $\boldsymbol{B}$ is a base for some topology on X. The union of the members of $\boldsymbol{B}$ form the topology τ on X given by

$$\tau = \{\{1, 2\}, \{3\}, \{2, 4\}, \phi, \{2\}, X, \{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, \{2, 3\}\}$$

***Ex.*2.** *Determine the topology* τ *on the real line* $\boldsymbol{R}$ *generated by the class A of all closed intervals* $[a, a + 1]$.

Sol. Let $p \in \boldsymbol{R}$, *clearly* $[p, p + 1]$ *and* $[p - 1, p] \in A$

Hence $\quad [p - 1, p] \cap [p, p + 1] = \{p\}$

belongs to the topology τ, *i.e.* all singleton sets $\{p\}$ are τ-open and so τ is the discrete topology on $\boldsymbol{R}$.

Self-learning exercise-1

1. Let $X = \{a, b, c, d, e\}$ and $A = \{\{a, b, c\}, \{c, d\}, \{d, e\}\}$.

   Find the topology generated by A. Also find the base of this topology.

2. Let $X = \{1, 2, 3, 4, 5\}$ and $\tau = \{\phi, \{1\}, \{1, 2,\}, \{1, 2, 3\}, X\}$

Then which of following is a local base at 1,2, and 3 respectively.

(i) $B_1 = \{\{1, 2\}, X\}$ *(ii)* $B_2 = \{\{2, 3\}, X\}$ *(iii)* $B_3 = \{\{1, 2, 3\}\}$.

9.5 Continuous mappings

Let (X, τ) and (Y, μ) are topological spaces. A mapping f from X in to Y is continuous relative to τ and μ, or $\tau - \mu$ continuous or simply continuous iff the inverse image $f^{-1}[H]$ of every μ-open subset $H \subset Y$ is a τ-open subset of X, *i.e.* iff $H \in \mu$ implies $f^{-1}[H] \in \tau$.

Ex.1. *Consider the following topologies on*

$X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d\}$ *respectively* :

$$\tau = \{X, \phi, \{1\}, \{1, 2\}, \{1, 2, 3\}\}, \mu = \{Y, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}\}$$

Also consider the functions $f : X \to Y$ and $g : X \to y$ defined by $f(1) = b$, $f(2) = c$, $f(3) = d$, $f(4) = c$ and $g(1) = a = g(2)$, $g(3) = c$, $g(4) = d$ then f is continuous since the inverse of each member of the topology μ on Y is a member of the topology τ on X, we can **see**

$$f^{-1}(Y) = X, \quad f^{-1}(\phi) = \phi,$$

$$f^{-1}(\{a\}) = \phi, \quad f^{-1}(\{b\}) = \{1\}$$

$$f^{-1}(\{a, b\}) = \{1\}, \quad f^{-1}(\{b, c, d\}) = X$$

but g is not continuous since $\{b, c, d\} \in \mu$, *i.e.* an open subset of Y, but its inverse image

$g^{-1}(\{b, c, d\}) = \{3, 4\}$ is not an open subset of X, *i.e.* $\{3, 4\} \notin \tau$.

Ex.2. *Every function $f : X \to Y$ (where (X, D) is a discrete topological space and (Y, τ) be any space) is $D - \tau$ continuous function* ***i.e.*** *every function from a discrete space is always a continuous function, since if H is any open subset of Y, its inverse $f^{-1}[H]$ is an open subset of X as every subset of discrete space is open.*

Similarly every function $g : X \to Y$ (where (X, τ) is any space and (Y, I) is an indiscrete space) is $\tau - I$ continuous, since in indiscrete topology there is only two open subsets of Y, namely Y and ϕ and for any function $g : X \to Y$ $g^{-1}[Y] = X$ and $g^{-1}(\phi) = \phi$ which are open subsets of X.

Ex.3. *The identity map from (X, τ) in to (X, τ) is a continuous map.*

Sol. Identity map given by $f : X \to X$ such that

$$f(x) = x \quad \forall \; x \in X$$

Let $G \subset X$ be an arbitrary open set.

Then $\quad f^{-1}(G) = \{x \in X : f(x) \in G\}$

$\quad = \{x \in X : x \in G\} \qquad [\because f(x) = x \;\forall x \in X]$

$\quad = G$ (an open set in X)

Thus inverse image of any open set G in X is open in X.

Hence f is a continuous map.

Theorem 8. *A function* $f : X \to Y$ *is continuous iff the inverse of each member of a base* $\boldsymbol{B}$ *for* Y *is an open subset of* X.

Proof. Let (X, τ) and (Y, μ) be topological spaces and $f : X \to Y$ be a map. First we assume that f is continuous map and let $B \subset Y$ such that $B \in \boldsymbol{B}$, since $\boldsymbol{B}$ is a base for Y every member of $\boldsymbol{B}$ is a member of μ, *i.e.* $B \in \mu$

$\Rightarrow$ B is an open subset of Y

Since $f : X \to Y$ is continuous, thus $f^{-1}[B]$ is open subset of X.

Conversely we assume that $f^{-1}[B]$ is open subset of X $\forall B \in \boldsymbol{B}$. To prove : f is continuous.

Let G *be* any open subset of Y and since $\boldsymbol{B}$ is base for topology on Y; then $G = \bigcup_{i \in \wedge} B_i$ a union *of* member of $\boldsymbol{B}$. But

$$f^{-1}[G] = f^{-1}\left[\bigcup_{i \in \wedge} B_i\right] = \bigcup_{i \in \wedge} f^{-1}[B_i]$$

and each $f^{-1}[B_i]$ is open by hypothesis; hence $f^{-1}[G]$ is the union of open sets and therefore open. Accordingly, f is continuous.

Note : Similarly we can prove that if $\boldsymbol{S}$ is a subbase for a topology on Y, then a function $f : X \to Y$ is continuous iff the inverse of each member of $\boldsymbol{S}$ is an open subset of X.

Note 2 : Continuous functions can be characterized by their behavior with respect to closed sets as followes :

Theorem 9. *A function* $f : X \to Y$ *is continuous iff the inverse image of every closed subset of* Y *is a closed subset of* X.

Proof. Let (X, τ) and (Y, μ) be topological spaces and $f : X \to Y$ be a map, first we assume that f is continuous map and let $F \subset Y$ be a closed subset of Y, then $Y \sim F$ is μ-open subset of Y and since f is continuous map $f^{-1}[Y \sim F]$ is open subset of X

$\Rightarrow$ $X \sim f^{-1}[F]$ is open subset of X

$\Rightarrow$ $f^{-1}[F]$ is closed subset of X

Conversely we assume that inverse image of a closed subset of Y is closed subset of X. Let G be an open set of Y

$\Rightarrow$ $Y \sim G$ is a closed subset of Y, by hypothesis

$f^{-1}[Y \sim G]$ is a closed subset of X

$\Rightarrow$ $X \sim f^{-1}[G]$ is a closed subset of X

$\Rightarrow$ $f^{-1}[G]$ is an open subset of X

Accordingly f is a continuous map.

Theorem 10. *Let $f : X \to Y$ be a constant function, say $f(x) = x_0 \ \forall \ x \in X$, then f is continuous relative to any topology τ on X and any topology μ on Y.*

Proof. We need to show that the inverse image of any μ–open subset of Y is a τ-open subset of X. Let $G \in \mu$ be any open subset of Y,

Now $$f^{-1}[G] = \begin{cases} X, & \text{if} \quad x_0 \in G \\ \phi, & \text{if} \quad x_0 \notin G \end{cases}.$$

In either case $f^{-1}[G]$ is an open subset of X, since X and ϕ belong to every topology τ on X.

Theorem 11. *Let the functions $f : X \to Y$ and $g : Y \to Z$ are continuous functions. Then the composition function $gof : X \to Z$ is also continuous.*

Proof. Let G be an open subset of Z. Then $g^{-1}[G]$ is open in Y since g is continuous. But f is also continuous so $f^{-1}[g^{-1}[G]] = (gof)^{-1}[G]$ is open in X.

Thus $(gof)^{-1}[G]$ is open in X for every open subset G of Z, accordingly gof is continuous.

Theorem 12. *Let $\{\tau_i : i \in \wedge\}$ be a collection of topologies on a set X. If a function $f : X \to Y$ is continuous with respect to each τ_i, then f is continuous with respect to the intersection topology $\tau = \bigcap_{i \in \wedge} \tau_i$*

Proof. Let G be an open subset of Y. Then by hypothesis, $f^{-1}[G]$ belongs to each τ_i. Hence $f^{-1}[G]$ belongs to the intersection, *i.e.* $f^{-1}[G] \in \bigcap_{i \in \wedge} \tau_i = \tau$, and so f is continuous with respect to τ.

Theorem 13. *A function $f : X \to Y$ is continuous iff, for every subset $A \subset X, f[\overline{A}] \subset \overline{f[A]}$*

Proof. First we assume that $f : X \to Y$ is continuous, Now

$$f[A] \subset \overline{f[A]} \qquad [\because B \subset \overline{B} \text{ always}]$$

$$\Rightarrow \quad A \subset f^{-1}[f[A]] \subset f^{-1}\left[\overline{f[A]}\right]$$

But $\overline{f[A]}$ is closed and since f is continuous so $f^{-1}\left[\overline{f[A]}\right]$ is also closed, also $\overline{A}$ is the smallest closed set containing A therefore $A \subset \overline{A} \subset f^{-1}\left[\overline{f[A]}\right]$

$$\Rightarrow \quad f[A] \subset f[\overline{A}] \subset f\left[f^{-1}\left[\overline{f[A]}\right]\right] \subset \overline{f(A)}$$

$$\Rightarrow \quad f[\overline{A}] \subset \overline{f[A]}$$

Conversely, assume $f[\overline{A}] \subset \overline{f[A]}$ for any $A \subset X$, and let F be a closed subset of Y, set

$$A = f^{-1}[F].$$

We wish to show that A is closed subset of X or equivalently $A = \overline{A}$

Now $A \subset \overline{A}$ is always true

$$f\left[\overline{A}\right] = f\left[\overline{f^{-1}[F]}\right] \subset \overline{f\left[f^{-1}[F]\right]} \subset \overline{F} = F \quad \text{.....(i)}$$

Hence $\overline{A} \subset f^{-1}\left[f\left[\overline{A}\right]\right] \subset f^{-1}[F] = A$

$\Rightarrow \quad \overline{A} \subset A$ *.....(ii)*

from *(i)* and *(ii)* $\Rightarrow A = \overline{A}$ and f is continuous.

9.6 Continuity at a point

A function $f : X \to Y$ is continuous at $a \in X$ iff the inverse image $f^{-1}[H]$ of every open subset $H \subset Y$ containing $f(a)$ is a superset of an pen set $G \subset X$ containing a or, equivalently, iff the inverse image of every neighbourhood of $f(a)$ is a neighbourhood of a *i.e.*,

$$N \in N_{f(a)} \Rightarrow f^{-1}[N] \in N_a$$

***Ex.*1.** *Consider the following topology τ on $X = \{1, 2, 3, 4\}$*

$$\tau = \{\phi, X, \{1\}, \{2\}, \{1, 2\}, \{2, 3, 4\}\}$$

Let the function $f : X \to X$ defined as $f(1) = 2 = f(3)$, $f(2) = 4$ and $f(4) = 3$. Show that f is continuous at 4 but not continuous at 3.

Sol. The only open sets containing $f(4) = 3$ are $\{2, 3, 4\}$ and X. Now $f^{-1}(\{2, 3, 4\}) = X = f^{-1}(X)$ which are open in X. Hence f is continuous at 4 since the inverse of each open set $f(4)$ is an open set containing 4.

Now observe that $\{1, 2\}$ is an open set containing $f(3) = 2$ and $f^{-1}[\{1, 2\}] = \{1, 3\}$. Hence f is not continuous at 3 since these exist no open set containing 3 which is contained in $\{1, 3\}$.

***Ex.*2.** *Let $\{p\}$ is an open subset of X. Show that for any topological space Y and any function $f : X \to Y$, f is continuous at $p \in X$*

Sol. Let $H \subset Y$ be an open set containing $f(p)$. But

$$f(p) \in H \Rightarrow p \in f^{-1}[H] \Rightarrow \{p\} \subset f^{-1}[H]$$

Hence f is continuous at p.

Theorem 14. *Let X and Y be topological spaces. Then a function $f : X \to Y$ is continuous iff it is continuous at every point $p \in X$.*

Proof. First we assume that f is continuous and let $p \in X$ be any point. Let H be an open subset of Y containing $f(p) \Rightarrow f(p) \in H \Rightarrow p \in f^{-1}(H)$ and $f^{-1}(H)$ is open as f is continuous. Thus f is continuous at p.

Now suppose f is continuous at every point $p \in X$ and let $H \subset Y$ be open. Now for every $p \in f^{-1}[H]$ these exist an open set $G_p \subset X$ such that $p \in G_p \subset f^{-1}[H]$. Hence $f^{-1}[H] = U\{G_p : p \in f^{-1}[H]\}$ a union of open sets and thus an open set. Accordingly f is continuous.

9.7 Open and closed functions

Open function : If X and Y are topological spaces, then a function $f : X \to Y$ is called an open (or interior) function if the image of every open set is open.

Closed function : If X and Y are topological spaces, then a function $g : X \to Y$ is called a closed function if the image of every closed set is closed.

In general, functions which are open need not be closed and vice versa.

Example. *Let $f : \mathbf{R} \to \mathbf{R}$ be a constant function, say $f(x) = 1$ $\forall x \in \mathbf{R}$, where the topology on both $\mathbf{R}$ is usual topology. Then if $A \subset R$ is a closed subset of $\mathbf{R}$ then $f(A) = \{1\}$ which is always a closed subset of $\mathbf{R}$, thus f is a closed mapping.*

But f is not open, since if we take $B = (0, 1)$ which is open in $\mathbf{R}$, then $f(B) = \{1\}$, but $\{1\}$ is closed in $\mathbf{R}$, so $\{1\}$ is not open in $\mathbf{R}$, thus f is not an open mapping.

9.8 Homeomorphism

Let X and Y be topological spaces. A function $f : X \to Y$ is called a **homeomorphism** between X and Y if

(i) f is bijective,

(ii) f is continuous on X to Y,

(iii) f^{-1} is continuous on Y to X.

In other words we can say that f is a homeomorphism if and only if f is bijective, continuous and open. Y is said to be a **homeomorphic image** or simply a **homeomorph** of X and we write $X \cong Y$.

A property which when satisfied by a topological space is also satisfied by every homeomorphic image of this space, is called a **topological property or a topological invariant property.**

Theorem 15. *Homeomorphism is an equivalence relation in the family of topological spaces.*

Proof. Homeomorphism is reflexive :

Let (X, τ) be a topological space. Then the identity map $f : X \to X$ given by $f(x) = x$ is bijective and continuous, for, if $G \in \tau$, then $f^{-1}(G) = G \in \tau$, f^{-1} is also an identity map, which is also continuous, thus f is a homeomorphism, *i.e.*, every topological space is homeomorphic to itself.

Thus homeomorphism is reflexive.

(ii) Homeomorphism is symmetric :

Let $f : (X, \tau) \to (Y, \mu)$ be a homeomorphism.

If we show that $f^{-1} : (Y, \mu) \to (X, \tau)$ is a homeomorphism we can conclude the symmetry.

Since f is a homeomorphism

$\Rightarrow$ (1) f is one-one and onto

(2) f and f^{-1} are continuous map

(1) $\Rightarrow f^{-1}$ is one-one and onto

(2) $\Rightarrow f^{-1}$ and $(f^{-1})^{-1} = f$ are continuous map

$\Rightarrow f^{-1}$ is homeomorphism

$\Rightarrow$ Homeomorphism is symmetric.

(iii) Homeomorphism is transitive :

Let $f : (X, \tau) \to (Y, \mu)$ and $g : (Y, \mu) \to (Z, \nu)$ be homeomorphism. If we show that $gof : (X, \tau) \to (Z, \nu)$ is a homeomorphism, we can conclude that homeomorphism is transitive. Now

f and g are homeomorphism

$\Rightarrow$ (a) f and g are one-one and onto

$\Rightarrow$ (b) f and g are continuous maps

$\Rightarrow$ (c) f^{-1} and g^{-1} are continuous maps

Now *(a)* $\Rightarrow gof$ is one-one and onto

(b) $\Rightarrow gof$ is continuous

(c) $\Rightarrow (gof)^{-1} = f^{-1} og^{-1}$ is continuous

$\Rightarrow gof$ is a homeomorphism.

$\Rightarrow$ homeomorphism is transitive.

Thus homeomorphism is an equivalence relation.

Theorem 16. *A one-one onto map* $f : (X, \tau) \to (Y, \mu)$ *is a homeomorphism iff* $\overline{f(A)} = f(\overline{A})$ *for any* $A \subset X$

Proof. Let $f : (X, \tau) \to (Y, \mu)$ be one-one onto map

Let $f(\overline{A}) = \overline{f(A)}$ for any $A \subset X$

To prove that f is a homeomorphism, For this we must show that

(a) f is one-one onto map (Given) *(b)* f is continuous *(c)* f^{-1} is continuous

Let $A \subset X$ be arbitrary.

By hypothesis $f(\overline{A}) = \overline{f(A)}$

$$\Rightarrow \quad f(\overline{A}) \subset \overline{f(A)} \quad .....(i)$$

$$\text{and} \quad \overline{f(A)} \subset f(\overline{A}) \quad .....(ii)$$

Theorem 13 *(i)* shows that f is continuous map.

Let $B = f(A) \Rightarrow f^{-1}(B) = A$ [$\because$ f is one-one]

Now from *(ii)* $\overline{f\left[f^{-1}(B)\right]} \subset f\left(\overline{f^{-1}(B)}\right)$

$$\Rightarrow \overline{B} \subset f\left(\overline{f^{-1}(B)}\right)$$

$$\Rightarrow f^{-1}\left(\overline{B}\right) \subset \overline{f^{-1}(B)} \quad \text{where} \quad B \subset Y$$

Again by Theorem 13 *(iii)* shows that f^{-1} is continuous map. Thus *(a)*, *(b)* and *(c)* have been proved and therefore f is a homeomorphism.

Conversely, suppose that $f : (X, \tau) \to (Y, \mu)$ is one-one onto and a homeomorphism.

To prove that $f\left(\overline{A}\right) = \overline{f(A)}$ for any $A \subset X$

Let $A \subset X$ be arbitrary and $B = f(A)$

$$B = f(A) \Rightarrow f^{-1}(B) = A \qquad [\because f \text{ is one-one}]$$

since f is continuous, thus by Theorem 13

$$f\left(\overline{A}\right) \subset \overline{f(A)} \qquad \text{(iv)}$$

since f^{-1} is continuous, thus by Theorem 13

$$f^{-1}\left(\overline{B}\right) \subset \overline{f^{-1}(B)}$$

$$\Rightarrow \qquad f^{-1}\left[\overline{f(A)}\right] \subset \overline{f^{-1}f(A)}$$

$$\Rightarrow \qquad f^{-1}\left[\overline{f(A)}\right] \subset \overline{A} \qquad [\because f(f^{-1}(A)) = A]$$

$$\Rightarrow \qquad \overline{f(A)} \subset f\left(\overline{A}\right) \qquad \text{(v)}$$

Thus from *(iv)* and *(v)*

$$f\left(\overline{A}\right) = \overline{f(A)}$$

Theorem 17. *A one-one onto continuous map* $f : (X, \tau) \to (Y, \mu)$ *is a homeomorphism iff* f *is either open or closed.*

Proof. Let $f : (X, \tau) \to (Y, \mu)$ is one-one, onto and continuous map. Also let f is either open or closed. To prove that f is a homeomorphism, it is enough to show that f^{-1} is continuous. For this we have to show that $f^{-1}(\overline{B}) \subset \overline{f^{-1}(B)}$ for any $B \subset Y$

$B \subset Y \Rightarrow f^{-1}(B) \subset X$ and $\overline{f^{-1}(B)}$ is a closed subset of X

$\because$ f is a closed mapping $\Rightarrow f\left[\overline{f^{-1}(B)}\right]$ is closed

$$\Rightarrow \qquad f\left[\overline{f^{-1}(B)}\right] = \overline{f\left[\overline{f^{-1}(B)}\right]} \qquad \text{(i)}$$

Since $f^{-1}(B) \subset \overline{f^{-1}(B)}$ $\qquad [\because A \subset \overline{A}$ *for any set* $A \subset X]$

$$\Rightarrow \qquad f\left[f^{-1}(B)\right] \subset f\overline{\left[f^{-1}(B)\right]}$$

$$\Rightarrow \qquad B \subset f\left[\overline{f^{-1}(B)}\right]$$

$$\Rightarrow \qquad \bar{B} \subset \overline{f\left[\overline{f^{-1}(B)}\right]}$$

$$\Rightarrow \qquad \bar{B} \subset f\left[\overline{f^{-1}(B)}\right] \qquad \text{[using (i)]}$$

$$\Rightarrow \qquad f^{-1}(\bar{B}) \subset \overline{f^{-1}(B)}$$

$\Rightarrow$ f^{-1} is continuous

Similarly we can show that if f is open, than f^{-1} is continuous

Illustrative Examples

***Ex.*1.** *Show that characteristic function of $A \subset X$ is continuous on X iff A is both open and closed in X.*

Sol. Let (X, τ) be a topological space and but $A \subset X$ be arbitrary. The characteristic function f of A is defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

Now we assume that A is both open and closed.

To prove f is continuous $f : X \to \boldsymbol{R}$ (where $\boldsymbol{R}$ is real line with usual topology)

Let G be an open subset of $\boldsymbol{R}$

$$f^{-1}(G) = \{x \in X : f(x) \in G\}$$

$$\Rightarrow \qquad f^{-1}(G) = \begin{cases} A & \text{if} \quad 1 \in G,\ 0 \notin G \\ X \sim A & \text{if} \quad 0 \in G, 1 \notin G \\ X & \text{if} \quad 0, 1 \in G \\ \phi & \text{if} \quad 0, 1 \notin G \end{cases}$$

In all case, $f^{-1}(G)$ is an open set.

$\Rightarrow$ f is continuous.

Conversely, suppose that f is continuous.

To prove that A is both open and closed.

Let G be an open subset of $\boldsymbol{R}$ *s.t.* $0 \in G$, $1 \notin G$.

Then $\qquad f^{-1}(G) = X \sim A$

$\because$ f is continuous $\quad \Rightarrow f^{-1}(G) = X \sim A$ is open

$\qquad \Rightarrow A$ is closed in X.

Let H be on open subset of $\boldsymbol{R}$ *s.t.* $1 \in H$, $0 \notin H$

Then $f^{-1}(H) = A$

f is continuous $\Rightarrow$ $f^{-1}(H) = A$ is open in X

$\Rightarrow$ A is both open and closed.

Ex.2. *Let* $X = \{0, 1, 2\}$

$\tau = \{\phi, X, \{0\}, \{0, 1\}$

Let f be a continuous map of X in to itself such that $f(1) = 0$ and $f(2) = 1$, what is $f(0)$ =?

Sol. Let $f(0) = a$, then $a = 0, 1$, or 2

If $a = 1$ or 2, then

$$f^{-1}(0) = \{x \in X : f(x) = 0\}$$
$$= \{x \in X : x = 1 \text{ or } 2\}$$
$$= \text{either } \{1\} \text{ or } \{2\}$$

but $\{1\} \notin \tau$ and $\{2\} \notin \tau$ but $\{0\} \in \tau$.

Contrary to the fact that f is continuous

If $a = 0$, then $f^{-1}(0) = \{x \in X : f(x) = 0\}$

$$= \{0, 1\} \in \tau$$

Hence f is continuous since inverse of open set $\{0\}$ is open set $\{0, 1\} \in \tau$.

Hence $f(0) = 0$

Self-learning exercise-2

1. Let $X = (-1, 1)$. Show that X with subspace topology of usual topology on $\boldsymbol{R}$ is homeomorphic to usual topology on $\boldsymbol{R}$.
2. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}\}$ let the function $f : X \to X$ be defined as

$$f(a) = b, f(b) = d, f(c) = b, f(d) = c$$

(i) Show that f is not continuous at c *(ii)* Show that f is continuous at d.

9.9 Summary

In this chapter you have learnt the important concepts of bases and subbases of a topological space. You have learnt that many times it is convenient to define a topological space with the help of bases and subbases. You have also learnt that we can always define a topology on a set with any collection of subsets of a set.

This chapter also belongs to the concept of continuity and homeomorphism. You have learnt that how can we generalize the concept of continuity to any arbitrary topological spaces. You also learnt that if two topological spaces are homeomorphic then many properties (known as topological properly) in those spaces are identical.

9.10 Answers to self-learning exercises

Self-learning exercise–1

1. $\tau_A = \{\phi, X, \{c\}, \{d\}, \{c, d\}, \{c, d, e\}, \{a, b, c, d\}\}$

$B_A = \{\{c\}, \{d\}, \{c, d, e\}, \{a, b, c, d\}\}$

2. Local base at 1 : None of B_1, B_2 and B_3

Local base at 2 : B_1

Local base at 3 : B_3

Self-learning exercise–2

1. Hint : Define $f : X \to \boldsymbol{R}$ s.t. $f(x) = \tan \frac{\pi}{2} x$

Show that f is one-one onto and continuous then also show that f^{-1} is continuous

2. *(i)* **Hint** : Let $G = \{a, b\} \in \tau$ then

$f^{-1}(G) = \{a, c\} \notin \tau$.

9.11 Exercises

1. Show that the map

$f : (\boldsymbol{R}, U) \to (\boldsymbol{R}, U)$ given by

$f(x) = x^2 \ \forall \ x \in \boldsymbol{R}$ is not open

2. Show that the map

$f : (\boldsymbol{R}, U) \to (\boldsymbol{R}, U)$ given by

$$f(x) = \begin{cases} x, & x < 1 \\ 1, & 1 \le x \le 2 \\ x^2/4, & x > 2 \end{cases}$$

is continuous but not open

3. Let $f : (X, \tau) \to (Y, \mu)$ be a map. Show that f is continuous if μ is an indiscrete topology.

4. Show that the identity function $I : (X, \tau) \to (X, \tau^*)$ is continuous iff τ is finer then τ^*, *i.e.* $\tau^* \subset \tau$.

5. Consider the discrete topology D on $X = \{1, 2, 3, 4, 5\}$, find a subbase $\boldsymbol{S}$ of D which does not contain any singleton set.

□ □ □

UNIT 10 : Separation Axioms (T_0, T_1, T_2, T_4 Spaces)

Structure of the Unit

10.0 Objectives

10.1 Introduction

10.2 T_0-axiom of separation (Kolomogorov space)

10.3 T_1-axiom or Frechet axiom of separation

10.4 T_2-axiom of separation (Hausdorff space)

10.5 Regular space

10.5.1 T_3-space

10.5.2 Example of regular space which is not T_3-space

10.6 Normal space

10.6.1 T_4-space

10.6.2 Example of a normal space which is not T_4-space

10.7 Summary

10.8 Answers to self-learning exercises

10.9 Exercise

10.0 Objectives

In this chapter you will read about the separation axioms of Alexandroff and Hopf. You will also learn about various topological spaces like T_0, T_1, T_2, T_3 and T_4.

10.1 Introduction

The T_i space nomenclature for $i = 1, 2, 3, 4$ was introduced by Alexandroff and Hopf. The word "T" referes to the German word "Ternugs axiom" which means "Separation axiom". Many properties of a topological space X depand upon the distribution of the open sets in the space. A space is more likely to be separable, or first or second countable, if there are "few" open sets; on the other hand, an arbitrary function on X to some topological space is more likely to be continuous, or a sequence to have a unique limit, if the space has "many" open sets.

10.2 T_0-Axiom of separation (Kolomogorov space)

A topological space (X, τ) is said to satisfy the T_0-axiom of separation if given a pair of distinct points $x, y \in X$, either

$$\exists\, G \in \tau \quad \text{s.t.} \quad x \in G, \quad y \notin G$$

or

$$\exists\, H \in \tau \quad \text{s.t} \quad x \notin H, \quad y \in H$$

In this case the space (X, τ) is called a T_0-space (Kolomogorov space)

Examples to T_0-space :

1. Every discrete space is T_0-space.
2. An indiscrete space containing only one point is a T_0-space
3. A cofinite topological space (X, τ) on an infinite set X is T_0-space.
4. Every metric space is T_0-space.

Theorem 1. *A topological space (X, τ) is a T_0-space if for any distinct arbitrary points $x, y \in X$, the closure of singleton set $\{x\}$ and $\{y\}$ are distinct.*

Proof : Let (X, τ) be a T_0-space and let $x, y \in X$ such that $x \neq y$.

To prove $\quad \overline{\{x\}} \neq \overline{\{y\}}$

Now $x, y \in X$ so by definition of T_0-space,

$$\exists\, G \in \tau \text{ such that } x \in G, y \notin G \Rightarrow y \in X \sim G, x \notin X \sim G,$$

by definition of closure

$$\overline{\{y\}} = \cap \{F : F \text{ is a closed set such that } y \in F\}$$

Also $X \sim G$ is a closed set containing y

$\therefore \quad \overline{\{y\}} \subset X \sim G$

but $x \notin X \sim G \Rightarrow x \notin \overline{\{y\}}$ .....(1)

Also $\{x\} \subset \overline{\{x\}}$ thus $x \in \overline{\{x\}}$ .....(2)

from (1) and (2) $\Rightarrow \overline{\{x\}} \neq \overline{\{y\}}$

Conversely : Let x, y be any two distinct points of a topological space (X, τ).

Also let $\quad \overline{\{x\}} \neq \overline{\{y\}}$ .....(3)

To prove (X, τ) is a T_0-space.

(3) $\Rightarrow \exists\, p \in X$ such that $p \in \overline{\{x\}}$ and $p \notin \overline{\{y\}}$,

we claim $x \notin \overline{\{y\}}$.

Let if possible $\quad x \in \overline{\{y\}} \Rightarrow \{x\} \subset \overline{\{y\}}$

$$\Rightarrow \overline{\{x\}} \subset \overline{\overline{\{y\}}} = \overline{\{y\}} \qquad [\because \overline{\overline{A}} = \overline{A}]$$

$$\Rightarrow \overline{\{x\}} \subset \overline{\{y\}},$$

Also $\quad p \in \overline{\{x\}} \Rightarrow p \in \overline{\{y\}}$.

A contradiction, since $p \notin \overline{\{y\}}$

Hence $x \notin \overline{\{y\}}$

$$\Rightarrow x \in X \sim \overline{\{y\}}, \text{ also } y \in \overline{\{y\}}, \Rightarrow y \notin X - \overline{\{y\}}$$

$\because \overline{\{y\}}$ is closed so $G = x \sim \overline{\{y\}}$ is open, thus we have found an open set G such that $x \in G$ but $y \notin G$.

$$\Rightarrow (X, \tau) \text{ is a } T_0\text{-space.}$$

10.3 T_1-Axiom or Frechet axiom of separation

A topological space (X, τ) is said to satisfy the T_1-axiom of separation if given a pair of distinct points $x, y \in X$, $\exists\ G, H \in \tau$ s.t. $x \in G, y \notin G$ and $y \in H, x \notin H$. In this case the space (X, τ) is called T_1-space or Frechet space.

Example of T_1-space :

1. Every metric space is T_1-space.
2. If (X, τ) is a cofinite topological space on an infinite space X, then it is T_1-space.

Example of T_0-space which is not a T_1-space :

We define a topology τ on $\boldsymbol{N}$ such that

(a) $\phi, \boldsymbol{N} \in \tau$

(b) $A_n \in \tau, \forall n \in \boldsymbol{N}$, where $A_n = \{1, 2, 3,..., n\}$.

Consider $m, n \in N$ such that $m < n$, then $m \in A_m, n \notin A_m$. Thus given any two distinct numbers $m, n \in \boldsymbol{N}$ such that $m \neq n$ and $m < n$, $\exists$ open set $A_m \in \tau$ such that $m \in A_m, n \notin A_m$

$\therefore (N, \tau)$ is a T_0-space

But if $m \neq n$ and $m < n$, there is no open set, which contains n but does not contain m., Thus (N, τ) is not a T_1-space.

Theorem 2. *A topological* (X, τ) *is a* T_1*-space iff* $\{x\}$ *is closed* $\forall x \in X$.

Proof : Let (X, τ) is a topological space such that $\{x\}$ is closed $\forall x \in X$. To prove (X, τ) is a T_1-space.

Consider $x, y \in X$ such that $x \neq y$, then by our assumption $\{x\}$ and $\{y\}$ are closed sets such that $\{x\} \cap \{y\} = \phi$.

$$\Rightarrow X \sim \{x\} \text{ and } X \sim \{y\} \text{ are open sets.}$$

Let $\quad G = X \sim \{y\}, H = X \sim \{x\}, \quad$ then $\quad G, H \in \tau$ such that

$$x \in G, y \notin G \quad \text{and} \quad y \in H, x \notin H.$$

$$\Rightarrow (X, \tau) \text{ is a } T_1\text{-space.}$$

Conversely, suppose that (X, τ) is a T_1-space.

Now we have to prove that $\{x\}$ is a closed set for each $x \in X$. For that it is sufficient to show that its complement $X \sim \{x\}$ is open. Let y be any element of $X \sim \{x\}$, then $x \neq y$. Since (X, τ) is a T_1-space and we know that every T_1-space is a T_0-space, so there exists $G_y \in \tau$ such that $y \in G_y$ but $x \notin G_y$ and consequently $y \in G_y \subset X \sim \{x\}$. This shows that $X \sim \{x\}$ is a nbd of each of its points and hence it is open, that is $\{x\}$ is closed.

Corollary 1 : *A topological space X is a T_1-space if and only if every finite subset of X is closed.*

Proof. Let (X, τ) be a T_1-space and let $A = \{a_1, a_2,..., a_n\}$ be any finite subset of X. Then $a_i \in X$ for each $i = 1, 2,..., n$. Since (X, τ) is a T_1-space, so every singleton subset of X is closed.

Now $$A = \{a_1\} \cup \{a_2\} \cup ... \cup \{a_n\}$$

$\Rightarrow$ A is finite union of closed subsets of X and hence A is closed.

Conversely, suppose that every finite subset of X is closed. Then in particular every singleton subset of X is closed and hence X is a T_1-space.

Corollary 2. *Finite T_1-space is a discrete space.*

Proof : Let (X, τ) be a finite T_1-space. Then by Corollary 1, every finite subset of X is a closed set.

$\Rightarrow$ All subsets of X are closed, since X is finite.

$\Rightarrow$ All subsets of X are open.

$\Rightarrow$ X is a discrete space.

Theorem 3. *A topological space (X, τ) is a T_1-space iff τ contains the co-finite topology on X. (i.e. τ is finer then co-finite topology on X)*

Proof : Let (X, τ) be a T_1-space.

To prove that τ contains co-finite topology on X, we have to show that $A \in \tau$ such that $X \sim A$ is finite, where $A \subset X$.

Now if $A \subset X$ such that $X \sim A$ is finite, then by Corollary 1 of Theorem 2, $X \sim A$ is a closed subset of $X \Rightarrow A \in \tau$. Thus τ contains co-finite topology.

Conversely : Suppose that τ contains co-finite topology on X. To prove (X, τ) is a T_1-space.

Now $\{x\}$ is a finite subset of X

$\Rightarrow$ $X \sim \{x\}$ is open in co-finite topology

$\Rightarrow$ $X \sim \{x\} \in \tau$

$\Rightarrow$ $\{x\}$ is τ-closed subset of X

Thus $\{x\}$ is closed subset of (X, τ) $\forall$ $x \in X$, by Theorem 2 (X, τ) is a T_1-space.

Theorem 4. *A finite subset of a T_1-space has no limit point.*

Proof : Let (X, τ) be a T_1-space let

$$A = \{a_1, a_2,..., a_n\} \text{ be a finite subset of } X$$

To prove that A has no limit point.

$$X \text{ is a } T_1\text{-space} \Rightarrow A \text{ is closed set}$$

$$\Rightarrow A \text{ contains all its limit point} \quad .....(1)$$

Let $a_i \in A$ be arbitrary and, we write

$$G_i = A \sim \{a_i\} = \{a_1, a_2,..., a_{i-1}, a_{i+1},..., a_n\}$$

G_i is finite subset of $X \Rightarrow G_i$ is closed $\Rightarrow X \sim G_i$ is open.

$$a_i \notin G_i \Rightarrow a_i \in X \sim G_i$$

Thus $X \sim G_i$ is an open set with $a_i \in X \sim G_i$

Also $\quad (X \sim G_i) \cap A = \{a_i\}$

By definition of limit point a_i is not a limit point, since there is an open subset of X containing a_i does not contain any point of A other than a_i.

But a_i is an arbitrary point of A.

$\Rightarrow$ Every point of A is not a limit point of A. Now (1) declares that A has no limit point.

Theorem 5. *T_1-axiom of separation is hereditary property or Every subspace of a T_1-space is also a T_1-space.*

Proof : Let (X, τ) be a T_1-space and (Y, U) is a subspace of (X, τ).

i.e. $\quad U = \{G \cap y : G \in \tau\}$ and $Y \subset X$. .....(1)

To prove (Y, U) is a T_1-space.

Let $x, y \in Y$ be arbitrary s.t. $x \neq y$

$\Rightarrow \quad x, y \in X$ s.t. $x \neq y$ $\quad [\because Y \subset X]$

$\because$ (X, τ) is a X_1-space.

$\Rightarrow \quad \exists\, G, H \in \tau$ s.t. $x \in G, y \notin G; y \in H, x \notin H$

Consequently $\quad x \in G \cap Y, y \notin G \cap y$

and $\quad x \notin H \cap y, y \in H \cap Y$

Let $\quad G_1 = G \cap Y$ and $H_1 = H \cap Y$

$\because \quad G, H \in \tau \Rightarrow G_1, H_1 \in U$

Thus given a pair of distinct points $x, y \in Y, \exists\, G_1, H_1 \in U$ such that $x \in G_1, y \notin G_1$, and $x \notin H_1$, $y \in H_1$.

$\Rightarrow$ (Y, U) is a T_1-space.

Theorem 6. *The property of a space being a T_1-space is a topological property i.e. the homeomorphic image of a T_1-space is a T_1-space.*

Proof : Let (X, τ) be a T_1-space and let

$$f : (X, \tau) \to (Y, U) \text{ is a homeomorphism.}$$

To prove that the T_1-space is a topological property, it is enough to prove that $(f(X) = Y, U)$ is a T_1-space.

Let $y_1, y_2 \in Y$ be two distinct points of Y such that $y_1 \neq y_2$. Since f is a homeomorphism (*i.e.* f is one-one onto and bicontinuous).

$$\exists\, x_1, x_2 \in X \quad \text{s.t.} \quad y_1 = f(x_1), \quad y_2 = f(x_2) \quad \text{and} \quad x_1 \neq x_2.$$

Since the space (X, τ) is a T_1-space $\exists$ open sets $G, H \in \tau$ such that $x_1 \in G$, $x_2 \notin G$ and $x_1 \notin H$, $x_2 \in H$.

Since f is a homeomorphism therefore f-image of an open set is also an open set.

$$\Rightarrow \qquad G_1 = f(G) \quad \text{and} \quad H_1 = f(H)$$

are U-open subset of Y *i.e.* $G_1, H_1 \in U$.

Now $\quad x_1 \in G, x_2 \notin G \Rightarrow y_1 = f(x_1) \in f(G) = G_1$

and $\quad y_2 = f(x_2) \notin f(G) = G_1$

and $\quad x_1 \notin H, x_2 \in H \Rightarrow y_1 = f(x_1) \notin f(H) = H_1$

and $\quad y_2 = f(x_2) \in f(H) = H_1$

Thus given a pair of distinct points $y_1, y_2 \in Y$, $\exists\, G_1, H_1 \in U$ such that $y_1 \in G_1$, $y_2 \notin G_1$ and $y_1 \notin H_1$, $y_2 \in H_1$

$\Rightarrow \quad (Y, U)$ is a T_1-space.

10.4 T_2-Axiom of separation (Hausdorff space)

A topological space (X, τ) is said to satisfy the T_2-axiom of separation if given a pair of distinct points $x, y \in X$, $\exists\, G, H \in \tau$ s.t. $x \in G$, $y \in H$, $G \cap H = \phi$. In this case the space (X, τ) is called a T_2-space or Hausdorff space.

Examples of T_2-space

***Ex.*1.** *Every metric space is a T_2-space.*

Sol. Let (X, d) be a metric space, then metric topology τ on X is defined as any subset $U \subset X$ is τ-open subset of X if $\forall\, x \in U\, \exists \in > 0$ such that open ball $B(x, \in) \subset U$.

To prove (X, τ) is a T_2-space.

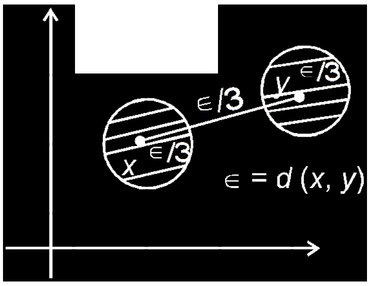
Let $x, y \in X$ be any pair of distinct points *i.e.* $x \neq y$,

$\because \qquad x \neq y \Rightarrow d(x, y) > 0.$ Let $\in = d(x, y)$.

Let $\qquad G = B(x, \in/3) \quad \text{and} \quad H = B(y, \in/3)$

Then G and H are open subsets of X [$\because$ open balls are open subsets] clearly $x \in G$, $y \in H$ and $G \cap H = \phi$.

Thus every metric space is a T_2-space.



Ex.2. *Every discrete space is a T_2-space.*

Sol. Let (X, τ) be a discrete space and $x, y \in X$ be arbitrary such that $x \neq y$.

By definition of discrete space $\{x\}$ and $\{y\}$ are τ-open sets, obviously

$$\{x\} \cap \{y\} = \phi \quad \text{and} \quad x \in \{x\}, y \in \{y\}.$$

Thus $\exists$ disjoint open sets $\{x\}$ and $\{y\}$ containing x and y respectively. Consequently (X, τ) is a T_2-space.

Ex.3. *Cofinite topology τ on any infinite set X is not a T_2-space.*

Sol. Let $G, H \in \tau$ be arbitrary. Then by definition of cofinite topology, $X \sim G$ and $X \sim H$ are finite subsets of X. Here we have to show that X is not a T_2-space. For this it is sufficient to show that $\exists$ no pair of disjoint open sets in cofinite topology on X.

Let if possible G and H are disjoint open sets so that

$$G \cap H = \phi \Rightarrow (G \cap H)^c = \phi^c$$

$$\Rightarrow G^c \cup H^c = X \qquad .....(1)$$

but $\left.\begin{matrix} G^c = X \sim G = \text{finite set} \\ H^c = X \sim G = \text{finite set} \end{matrix}\right\}$ [$\because$ G and H are open in (X, τ)]

$\Rightarrow$ L.H.S. of (1) is union of two finite sets, thus finite set but R.H.S. of (1) is an infinite set X (Given), which is a contradiction. Thus $\exists$ no pair of disjoint open sets in cofinite topology on X.

Theorem 7. *Every T_2-space is a T_1-space but the converse in not true.*

Proof : Let (X, τ) is a T_2-space. To prove (X, τ) is a T_1-space. Let $x, y \in X$ such that $x \neq y$ then by definition of T_2-space we can find disjoint open sets $G, H \in \tau$ such that $x \in G$, $y \in H$ and $G \cap H = \phi \Rightarrow x \in G, y \notin G$ and $x \notin H, y \in H$.

Hence given a pair of distinct points $x, y \in X$, such that $x \neq y$ $\exists$ $G, H \in \tau$ such that $x \in G$, $y \notin G$ and $x \notin H, y \in H$.

$\Rightarrow$ (X, τ) is a T_1-space.

To prove converse is not true, consider a cofinite topology τ on an infinite set X. Then by Theorem 3 (X, τ) is a T_1-space. But by Example 3 of §10.4 (X, τ) is not a T_2-space. Thus every T_1-space is not a T_2-space.

Corollary 1. *Every singleton set in a T_2-space is closed.*

Theorem 8. *In a T_2-space, a convergent sequence has a unique limit.*

Proof : Let (X, τ) be a T_2-space and $<a_n>$ is a convergent sequence in X. To prove that the sequence $<a_n>$ has a unique limit.

Let if possible $<a_n>$ does not have a unique limit. Let $<a_n>$ converge to two distinct points, say $a_0, b_0 \in X$. Then $a_0 \neq b_0$, By definition of T_2-space $\exists$ open sets G, H such that

$$a_0 \in G, b_0 \in H \text{ such that } G \cap H = \phi.$$

By definition of convergence

$$a_0 \in G \in \tau \Rightarrow \exists\, n_0 \in N \text{ such that } \forall\, n \geq n_0 \Rightarrow a_n \in G$$

$$b_0 \in H \in \tau \Rightarrow \exists\, k_0 \in N \text{ such that } \forall\, n \geq k_0 \Rightarrow a_n \in H$$

Let $\quad m_0 = \max \{n_0, k_0\}.$

Then $\quad \forall\, n \geq m_0 \Rightarrow a_n \in G, a_n \in H \Rightarrow a_n \in G \cap H$

$$\Rightarrow G \cap H \neq \phi$$

which is a contradictions since $G \cap H = \phi$.

Thus the sequence $<a_n>$ has a unique limit.

Theorem 9. *Let (X, τ) be any topological space and let (Y, U) be a Hausdorff space. Let f and g be continuous mappings of X into Y. Then the set $\{x \in X : f(x) = g(x)\}$ is a closed subset of X.*

Proof : Given (X, τ) be any topological space and (Y, U) a Hausdorff space. Let $f : X \rightarrow Y$ and $g : X \rightarrow Y$ are continuous maps. Let $A = \{x \in X : f(x) = g(x)\}$. Now we have to show that A is closed. For this is sufficient to show that

$$X \sim A = \{x \in X : f(x) \neq g(x)\} \text{ is open} \qquad .....(1)$$

Let $x \in X \sim A$ be arbitrary, then $x \notin A$.

Let $\quad f(x) = y_1$ and $g(x) = y_2$ then $y_1 \neq y_2$ $\quad$ [by (1)]

Further more $y_1, y_2 \in Y$ and (Y, U) is a T_2-space, hence $\exists\, G, H \in U$ such that $y_1 \in G$, $y_2 \in H$, $G \cap H = \phi$.

Since f and g are continuous maps. Hence by definition $f^{-1}(G)$, $g^{-1}(H)$ are open in X, write

$$W = f^{-1}(G) \cap g^{-1}(H)$$

W is also an open set in X [finite intersection of open sets].

$$y_1 \in G \Rightarrow f^{-1}(y_1) \in f^{-1}(G) \Rightarrow x \in f^{-1}(G)$$

$$y_2 \in H \Rightarrow g^{-1}(y_2) \in g^{-1}(H) \Rightarrow x \in g^{-1}(H)$$

$$\Rightarrow \quad x \in f^{-1}(G) \cap g^{-1}(H) \Rightarrow x \in W \qquad .....(2)$$

$$W = f^{-1}(G) \cap g^{-1}(H) \Rightarrow W \subset f^{-1}(G), W \subset g^{-1}(H)$$

$$\Rightarrow \quad f(W) \subset G, g(W) \subset H$$

$$\Rightarrow \quad f(W) \cap g(W) \subset G \cap H = \phi$$

$\Rightarrow \quad f(W) \cap g(W) \subset \phi$

but $\quad \phi \subset f(W) \cap g(W) \qquad$ [Always]

$\Rightarrow \quad f(W) \cap g(W) = \phi$

$\Rightarrow \quad f(y) \neq g(y) \ \forall \ y \in W$

$\Rightarrow \quad y \in X \sim A \quad \forall \ y \in W$

$\Rightarrow \quad W \subset X \sim A$

Thus for any $x \in X \sim A$, $\exists\ W \in \tau$ such that $x \in W \subset X \sim A$

$\Rightarrow$ every $x \in X \sim A$ is an interior point of $X \sim A$

$\Rightarrow$ $X \sim A$ is open subset of X

$\Rightarrow$ A is closed subset of X.

Theorem 10. *If f and g are continuous functions on a topological space X with values in T_2-space, Y. Then the set of all points $x \in X$ such that $f(x) = g(x)$ is closed. Deduce that if f and g agree on a dense subset on X, then $f = g$ on the whole X.*

Proof : ***(i)*** Proof of I[st] part is same as Theorem 9.

(ii) Suppose f and g agree on a dense subset $P \subset X$ so that

$$f(x) = g(x) \quad \forall x \in P, \qquad \overline{P} = X$$

$f(x) = g(x)\ \forall\ x \in P \Rightarrow P$ is closed by case (i)

$\Rightarrow \overline{P} = P$ Also $\overline{P} = X$

$\Rightarrow P = X$

$\Rightarrow \quad f(x) = g(x) \quad \forall\ x \in X \Rightarrow f = g$ on the whole X.

Theorem 11. *For any space (X, τ), following conditions are equivalent.*

(i) X is a T_2-space

(ii) For each pair $x, y \in X$, $\exists$ a nbd N_y of y such that x is not in $\overline{N}_y$.

(iii) For each $x \in X$, $\{x\} = \bigcap \overline{N}_x$, where the intersection is taken over all the nbds of x.

Proof : ***(i) ⇒ (ii)*** Let (X, τ) be a T_2-space and $x, y \in X$ be arbitrary such that $x \neq y$.

To prove that $\exists$ nbd N_y of y such that $x \notin \overline{N}_y$.

By definition of T_2-space $\exists\ G, H \in \tau$ such that

$$x \in G, y \in H, \qquad G \cap H = \phi.$$

$y \notin G, x \in G \in \tau \Rightarrow y \in G^c, x \notin G^c, G^c$ is closed.

$G \cap H = \phi \Rightarrow H \subset X \sim G = G^c \Rightarrow y \in H \subset G^c$

$\Rightarrow y \in N_y, x \notin N_y$, where $N_y = G^c$

$\Rightarrow N_y$ is a closed nbd of y such that $x \notin N_y$

But $\quad N_y = \overline{N}_y$, since N_y is closed

$\Rightarrow N_y$ is a closed nbd of y such that $x \notin \overline{N}_y$.

(ii)* ⇒ *(iii) Suppose any $x, y \in X \Rightarrow \exists$ nbd N_y of y such that $x \notin \bar{N}_y$.

To prove $\bigcap \bar{N}_x = \{x\}$, where the intersection is then over all the nbd of x.

By over assumption, there also exist nbd N_x of x such that $y \notin \bar{N}_x$.

Let $\boldsymbol{B} = \{ \bar{N}_x : N_x$ is a nbd of x such that $y \notin \bar{N}_x \}$

$$\bigcap \bar{N}_x = \bigcap \{B : B \in \boldsymbol{B}\} = \{x\}$$

(iii)* ⇒ *(i) Let $\forall x \in X$, $\{x\} = \bigcap \bar{N}_x$, where the intersection is taken over all the nbd of x.

To prove X is a T_2-space.

Let $x, y \in X$ be arbitrary such that $x \neq y$.

Now $\{x\} = \bigcap \bar{N}_x \Rightarrow y \notin \bigcap \bar{N}_x$. Also $\bar{N}_x$ is a closed nbd of x

$\Rightarrow y \notin \bar{N}_x = N$

$\Rightarrow \exists$ a closed nbd $\bar{N}_x$ of x such that $y \notin \bar{N}_x = N$

$\Rightarrow x \in N, y \notin N$ where N is closed.

By definition of nbd $\exists\, G \in \tau$ such that $x \in G \subset N, y \notin N$

$H = X \sim N$ then H is open and $x \notin H$

$y \notin N \Rightarrow y \in H$

$G \subset N \Rightarrow G \cap (X - N) = \phi \Rightarrow G \cap H = \phi$.

∴ Given $x, y \in X$, $\exists\, G, H \in \tau$ such that $x \in G, y \in H, G \cap H = \phi$

⇒ X is a T_2-space.

Theorem 12. *The property of a space being a Haudorff space is a hereditary property*

or

Every subspace of a T_2-space is a T_2-space.

Proof : Let (X, τ) be a Hausdoff space and (Y, U) be a sub space of (X, τ).

To prove (Y, U) is a Hausdorff space.

Let a pair of elements $y_1, y_2 \in Y$ such that $y_1 \neq y_2$.

Then $y_1, y_2 \in X$ such that $y_1 \neq y_2$. For $Y \subset X$

(X, τ) is a T_2-space, $\exists$ disjoint sets

$G_1, G_2 \in \tau$ such that $y_1 \in G_1, y_2 \in G_2, G_1 \cap G_2 = \phi$.

$G_1, G_2 \in \tau \Rightarrow \exists\, H_1, H_2 \in U$, such that $H_1 = G_1 \cap Y, H_2 = G_2 \cap Y$

$H_1 \cap H_2 = (G_1 \cap Y) \cap (G_2 \cap Y) = (G_1 \cap G_2) \cap Y = \phi$.

$y_1 \in Y; y_1 \in G_1 \Rightarrow y_1 \in G_1 \cap Y = H_1$

$y_2 \in Y; y_2 \in G_2 \Rightarrow y_2 \in G_2 \cap Y = H_2$

Thus given a pair of distinct points $y_1, y_2 \in Y$ such that $y_1 \neq y_2$ $\exists$ disjoint sets $H_1, H_2 \in U$ such that $y_1 \in H_1, y_2 \in H_2, H_1 \cap H_2 = \phi$.

⇒ (Y, U) is a T_2-space.

Theorem 13. *The property of a space being a T_2-space is a topological property.*

Proof : Let (X, τ) be a T_2-space and let

$$f : (X, \tau) \to (Y, U)$$

be a homeomorphism so that any $G \in \tau \Rightarrow f(G) \in U$.

To prove that T_2-space is a topological property, it suffices to prove that $(f(X) = Y, U)$ is a T_2-space.

Let y_1, y_2 is a distinct pair of points in Y such that $y_1 \neq y_2$.

Since f is a bijection $\Rightarrow \exists\, x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $y_1 = f(x_1)$ and $y_2 = f(x_2)$.

Since (X, τ) is a T_2-space $\exists\, G, H \in \tau$ such that

$$x_1 \in G, x_2 \in H, G \cap H = \phi.$$

Let $\quad G_1 = f(G), H_1 = f(H) \Rightarrow G_1, H_1 \in U$

and $\quad y_1 = f(x_1) \in f(G) = G_1$

$\quad y_2 = f(x_2) \in f(H) = H_1$ and $f(G \cap H) = G_1 \cap H_1 = \phi$

Thus given a pair of distinct points $y_1, y_2 \in Y\ \exists\, G_1, H_1 \in U$ such that $y_1 \in G_1, y_2 \in H_1$, $G_1 \cap H_1 = \phi$.

$\Rightarrow$ (Y, U) is a T_2-space.

10.5 Regular space

A topological space (X, τ) is said to be a regular space if given an element $x \in X$ and closed set $F \subset X$ such that $x \notin F$, $\exists$ disjoint open sets $G_1, G_2 \subset X$ such that $x \in G_1, F \subset G_2, G_1 \cap G_2 = \phi$.

10.5.1 T_3-space :

A regular T_1-space is called a T_3-space.

10.5.2 Example of regular space which is not T_3-space :

Let $X = \{a, b, c\}$ and topology τ on X is $\tau = \{\phi, X, \{a\}, \{b, c\}\}$. Then it is clear that (X, τ) is a topological space.

Clearly $\{a\}, \{a, c\}$ are open as well closed subset of (X, τ). Now consider a pair of distinct elements $b, c \in X$. Then only open sets containing either of the elements b, c are X, $\{b, c\}$ such that $b \in X,\ b \in \{b, c\}; c \in X, c \in \{b, c\}$.

Thus there is no open sets $G, H \in \tau$ such that

$$b \in G, c \notin G \text{ and } b \notin H, c \in H$$

$\Rightarrow$ (X, τ) is not T_1-space

$\Rightarrow$ (X, τ) is not a T_3-space [by definition]

Now : Let $\quad a \in X, \{b, c\} \subset X$ such that $a \notin \{b, c\}$

where $\{b, c\}$ is closed subset of $X \exists\, \{a\}, \{b, c\}$ τ-open sets such that

$$a \in \{a\}, \{b, c\} \subset \{b, c\} \text{ such that } \{a\} \cap \{b, c\} = \phi.$$

Also to be any such point $x \in X$ and $F \subset X$ such that $x \notin F$

$$\exists\, G, H \in \tau \text{ such that } x \in G, F \subset G \text{ and } G \cap H = \phi.$$

$\Rightarrow$ (X, τ) is a regular space.

Examples of regular space :

1. Every discrete space is regular.
2. Every indiscrete space is regular.
3. Every metric space is regular.

Theorem 14. *Every T_3-space is a T_2-space.*

Proof : Let (X, τ) be a T_3-space. Then be definition of T_3-space, it is a regular T_1-space. Now we have to show that (X, τ) is a T_2-space. Let x, y be any two elements of X such that $x \neq y$. Since (X, τ) is a T_1-space, so every singletion subset $\{x\}$ of X is closed. Again, since (X, τ) is regular, so corresponding to closed set $\{x\}$ and the point $y \notin \{x\}$ there exist open sets G and H such that

$$\{x\} \subset G, y \in H \text{ and } G \cap H = \phi.$$

$$\Rightarrow \qquad x \in G, y \in H \text{ and } G \cap H = \phi.$$

Thus for $x, y \in X$ with $x \neq y$ there exist $G, H \in \tau$ such that

$$x \in G, y \in H \text{ and } G \cap H = \phi.$$

Hence (X, τ) is a T_2-space.

Theorem 15. *A topological space (X, τ) is regular iff for every point x of X and every nbd N of x $\exists$ a nbd M of x such that $\overline{M} \subset N$.*

OR

A topological space is regular iff the collection of all τ-closed nbds from a local base at x.

Proof : Let (X, τ) be a regular space.

To prove that given a nbd N of x $\exists$ a nbd M of x s.t. $\overline{M} \subset N$.

$\because$ (X, τ) is a regular space, therefore given a closed set F and an element $x \in X$ such that $x \notin F$ $\exists$ disjoint open sets $G_1, G_2 \subset X$ such that $x \in G_1$, $F \subset G_2$, $G_1 \cap G_2 = \phi$.

$$x \in G_1 \Rightarrow G_1 \text{ is a nbd of } x$$

$$G_1 \cap G_2 = \phi \Rightarrow G_1 \subset X \sim G_2$$

$$\Rightarrow \overline{G_1} \subset \overline{X \sim G_2} = X \sim G_2 \qquad [\because X \sim G_2 \text{ is closed}]$$

$$\because \qquad F \subset G_2 \Rightarrow X \sim G_2 \subset X \sim F$$

$$\therefore \qquad \overline{G_1} \subset X \sim F = H \text{ (say)}.$$

$$\because \qquad F \text{ is closed} \Rightarrow H \text{ is open}.$$

$$\therefore \qquad \overline{G_1} \subset H$$

$$x \notin F \Rightarrow x \in X \sim F \Rightarrow x \in H.$$

Thus given a nbd H of x, $\exists$ a nbd G_1 of x such that

$$x \in G_1 \subset \overline{G_1} \subset H$$

Conversely, assume that (X, τ) is a topological space such that given a nbd G of an element $x \in X$, $\exists$ a nbd H of x such that

$$x \in H \subset \overline{H} \subset G.$$

To prove (X, τ) is a regular space.

Let $F \subset X$ be a closed set and $x \in X$ such that $x \notin F$.

Now $\quad x \notin F$, F is closed $\Rightarrow x \in X \sim F$ is open

$$\Rightarrow X \sim F \text{ is nbd of } x$$

$\therefore$ By our assumption, $\exists$ a nbd G of x such that

$$x \in G \subset \overline{G} \subset X \sim F \qquad \text{.....(1)}$$

Let $\quad G = G_1$ and $X \sim \overline{G} = G_2$

Then $\quad G_1 \cap G_2 = G \cap \left(X \sim \overline{G}\right) = G \cap X \sim G \cap \overline{G}$

$$= G \sim G = \phi$$

$$x \in G \Rightarrow x \in G_1$$

From (1) $\quad \overline{G} \subset X \sim F \Rightarrow X \sim (X \sim F) \subset X \sim \overline{G}$

$$\Rightarrow F \subset G_2$$

$\therefore$ $\overline{G}$ is closed $\Rightarrow X \sim \overline{G}$ is open $\Rightarrow G_2$ is open. Thus we have shown that given a closed set $F \subset X$ and a point $x \in X$ such that $x \notin F$ $\exists$ disjoint open sets G_1, G_2 such that

$$x \in G_1, \quad F \subset G_2, \quad G_1 \cap G_2 = \phi$$

$\Rightarrow$ (X, τ) is a regular space.

Theorem 16. *The property of a space being regular is hereditary property.*

Proof : Let (X, τ) be a regular space and (Y, U) be a sub space of (X, τ). To prove (Y, U) is a regular space.

Let F be a U-closed subset of Y and $p \in Y$ such that $p \notin F$.

$\because$ F is U-closed subset of $Y \Rightarrow \exists$ K, τ-closed subset of X such that

$$F = K \cap Y \text{ also } p \in Y \subset X \Rightarrow p \in X$$

such that $\quad p \notin K \cap Y \Rightarrow p \notin K. \qquad [\because p \in Y]$

Now K is τ-closed subset of X and $p \in X$ such that $p \notin K$. As (X, τ) is a regular space $\Rightarrow$ $\exists$ G_1, G_2 τ-open sets such that

$$p \in G_1, K \subset G_2 \text{ and } G_1 \cap G_2 = \phi$$

$\because \quad G_1$, G_2 are τ-open sets of X

$\Rightarrow \quad p \in G_1 \cap Y = H_1$ and $G_2 \cap Y = H_2$ (say)

$\Rightarrow$ H_1, H_2 are U-open subsets of Y such that

$$p \in H_1, K \cap Y \subset G_2 \cap Y = H_2$$

or $p \in H_1$ and $F \subset H_2$ and

$$H_1 \cap H_2 = (G_1 \cap Y) \cap (G_2 \cap Y) = (G_1 \cap G_2) \cap Y = \phi$$

$\Rightarrow$ $\exists\, H_1, H_2$ U-open subsets of Y such that $p \in H_1, F \subset H_2, H_1 \cap H_2 = \phi$

$\Rightarrow$ (Y, U) is regular.

Corollary : The property of being a T_3-space is hereditary.

Theorem 17. *Regularity is a topological property.*

Proof : Let (X, τ) be a regular space and (Y, U) be any topological space. Let $f : (X, \tau) \to (Y, U)$ be a homeomorphism. To prove (Y, U) is a regular space.

Let $y \in Y$ and F-U-closed subset of Y such that $y \notin F$.

$\because$ $f : X \to Y$ is one-one and onto

$\Rightarrow$ $\exists\, x \in X$ s.t. $f(x) = y \Rightarrow x = f^{-1}(y)$

f is homeomorphism

$\Rightarrow$ f and f^{-1} both are continuous.

$F \subset Y$ is U-closed, f is continuous

$\Rightarrow$ $f^{-1}(F) \subset X$ is τ-closed.

Now $y \notin F \Rightarrow f^{-1}(y) \notin f^{-1}(F) \Rightarrow x \notin f^{-1}(F)$.

Now $x \in X$ such that $x \notin f^{-1}(F), f^{-1}(F)$ is τ-closed. By definition of regularity

$$\exists\, G, H \in \tau \text{ such that } x \in G, f^{-1}(F) \subset H, \; G \cap H = \phi$$

$\Rightarrow$ $f(x) \in f(G), F \subset f(H), f(G \cap H) = \phi$

$\Rightarrow$ $y \in f(G), F \subset f(H), f(G) \cap f(H) = \phi$

Since f^{-1} is continuous and $G, H \in \tau$

$\Rightarrow$ $f(G), f(H) \in U$

$\Rightarrow$ for $y \in Y$ and F-U-closed subset of Y such that $y \notin F$

$$\exists\, G_1 = f(G), \; G_2 = f(H) \in U \text{ such that } y \in G_1, F \subset G_2 \text{ such that } G_1 \cap G_2 = \phi$$

$\Rightarrow$ (Y, U) is regular.

Corollary : The property of being a T_3-space is a topological property.

10.6 Normal space

A topological space (X, τ) is said to be normal space if given a pair of disjoint closed sets $C_1, C_2 \subset X \,\exists$ disjoint open sets $G_1, G_2 \subset X$ such that $C_1 \subset G_1, C_2 \subset G_2, G_1 \cap G_2 = \phi$.

10.6.1 T_4-space :

A normal T_1-space is called a T_4-space.

10.6.2 Example of a normal space which is not T_4-space :

Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b, c\}\}$ then we have already proved that (X, τ) is not a T_1-space. Thus (X, τ) is not a T_4-space.

Now it is easy to show that given a pair of disjoint closed sets $\{a\}, \{b, c\} \subset X$, we can find a pair of disjoint open sets $\{a\}, \{b, c\} \subset X$ such that closed set $\{a\} \subset$ open set $\{a\}$ and closed set $\{b, c\} \subset$ open set $\{b, c\}$.

$\Rightarrow$ (X, τ) is a normal space.

Examples of normal space.

1. Every discrete space is normal
2. Every indiscrete space is normal
3. Every metric space is normal.

Theorem 18. *A T_4-space is a T_3-space.*

Proof : Let (X, τ) be a T_4-space, thus

(i) (X, τ) is a T_1-space

(ii) (X, τ) is a normal space.

To prove (X, τ) is a T_3-space, it is sufficient to show that (X, τ) is regular, as a regular T_1-space is T_3-space.

Let $x \in X$, and F be a τ-closed subset of X such that

$$x \notin F \;\because\; x \in X \text{ and } (X, \tau) \text{ is } T_1\text{-space}$$

$$\Rightarrow \{x\} \text{ is a closed subset of } X$$

$$\because \quad x \notin F \Rightarrow \{x\} \cap F = \phi$$

$$\Rightarrow \{x\} \text{ and } F \text{ are disjoint closed subsets of } X.$$

Since (X, τ) is a normal space

$\Rightarrow$ $G_1, G_2 \in \tau$ such that $\{x\} \subset G_1, F \subset G_2$ such that $G_1 \cap G_2 = \phi$

i.e. given a point $x \in X$ and a closed subset $F \subset X$ such that $x \notin F$

$\exists$ open sets $G_1, G_2 \in \tau$ such that $x \in G_1, F \subset G_2, G_1 \cap G_2 = \phi$

$\Rightarrow$ (X, τ) is a regular space and hence (X, τ) is a T_3-space.

Note : T_4-space $\Rightarrow$ T_3-space

but normal space $\not\Rightarrow$ regular space.

Consider $X = \{a, b\}$ and $T = \{\phi, X, \{a\}\}$ then the space (X, τ) is a normal space, as there does not exist any pair of disjoint closed subsets of X.

(X, τ) is not regular as $a \in X$ and $F = \{b\}$ is a closed subset of X such that $a \notin F$. But there does not exist any pair of disjoint open subsets of X, G_1 and G_2 such that

$$a \in G_1, F \subset G_2 \quad \text{and} \quad G_1 \cap G_2 = \phi.$$

Thus (X, τ) is not regular.

Theorem 19. *A closed sub space of normal space is a normal space.*

Proof : Let (X, τ) be a topological space, which is normal and (Y, U) is a closed sub space of $(X, \tau) \Rightarrow Y$ is τ-closed subset of X. To prove (Y, U) is normal space.

Let $F_1, F_2 \subset Y$ be disjoint sets which are closed in Y. Since Y is closed in X, a subset $F \subset Y$ is closed in Y iff F is closed in X.

$\Rightarrow$ F_1 and F_2 are disjoint closed subsets of X.

By the property of normal space (X, τ)

$$\exists \text{ open sets } G_1, G_2 \in \tau \text{ such that } F_1 \subset G_1, F_2 \subset G_2, G_1 \cap G_2 = \phi$$

$$F_1 \subset Y, F_1 \subset G_1 \Rightarrow F_1 \subset G_1 \cap Y = H_1 \text{ (say)}$$

Similarly $$F_2 \subset G_2 \cap Y = H_2 \text{ (say)}$$

By definition of sub space topology $H_1 = G_1 \cap Y$ and $H_2 = G_2 \cap Y$ are U-open subsets of Y.

Also $$H_1 \cap H_2 = (G_1 \cap Y) \cap (G_2 \cap Y) = (G_1 \cap G_2) \cap Y = \phi \cap Y = \phi.$$

Thus given a pair of disjoint closed sets F_1, F_2 in Y, $\exists$ disjoint U-open subsets H_1, H_2 of Y such that $F_1 \subset H_1, F_2 \subset H_2$ and $H_1 \cap H_2 = \phi$.

This shows that (Y, U) is a normal space.

Note : Normality is not necessary a hereditary property, but above theorem 19 is a weaker statement for normal spaces. But property of being a T_4-space is hereditary (Proof left as an exercise)

Theorem 20. *Normality is a topological property.*

Proof : Let $f : (X, \tau) \rightarrow (Y, U)$ is a homeomorphism and let (X, τ) is a normal space. Then (Y, U) is homomorphic image of (X, τ). To prove (Y, U) is also a normal space.

Let F_1, F_2 be disjoint U-closed subsets of Y *i.e.* $F_1 \cap F_2 = \phi$. Since f is continuous

$\Rightarrow$ $$E_1 = f^{-1}(F_1), E_2 = f^{-1}(F_2) \text{ are } \tau\text{-closed subsets of } X.$$

$$E_1 \cap E_2 = f^{-1}(F_1) \cap f^{-1}(F_2) = f-1(F_1 \cap F_2) = f^{-1}(\phi) = \phi.$$

$\Rightarrow$ E_1 and E_2 are disjoint τ-closed subsets of X, also (X, τ) is a normal space $\Rightarrow \exists$ τ-open sets G_1, G_2 such that $E_1 \subset G_1$ and $E_2 \subset G_2, G_1 \cap G_2 = \phi$

$\Rightarrow$ $$f^{-1}(F_1) \subset G_1, \quad f^{-1}(F_2) \subset G_2$$

$\Rightarrow$ $$F_1 \subset f(G_1) = H_1, \quad F_2 \subset f(G_2) = H_2 \text{ (say).}$$

Since f is a homeomorphism and G_1, G_2 are τ-open subsets of X

$$\Rightarrow H_1 = f(G_1) \quad \text{and} \quad H_2 = f(G_2) \text{ are } U\text{-open subsets of } Y.$$

$$H_1 \cap H_2 = f(G_1) \cap f(G_2) = f(G_1 \cap G_2) = f(\phi) = \phi.$$

Thus given a pair F_1, F_2 of disjoint U-closed sets in $Y \exists$ U-open sets $H_1, H_2 \in U$ such that $F_1 \subset H_1, F_2 \subset H_2, H_1 \cap H_2 = \phi$.

This shows that (Y, U) is also a normal space.

Corollary : The property of being a T_4-space is a topological property.

Theorem 21. *A topological space* (X, τ) *is a normal space iff for any closed set F and an open set G containing F, there exist an open set V such that* $F \subset V \subset \overline{V} \subset G$.

Proof : First we assume that (X, τ) is a normal space and let $G \subset X$ be an open set containing a closed set F *i.e.* $F \subset G \subset X$. To prove $\exists$ an open set $V \in \tau$ such that $F \subset V \subset \overline{V} \subset G$.

$$G \subset X \text{ is } \tau\text{-open} \Rightarrow X \sim G \text{ is } \tau\text{-closed.}$$

$$F \cap (X \sim G) = F \cap X \sim F \cap G = F \sim F = \phi. \qquad [\because F \subset G]$$

$\therefore$ F and $(X \sim G)$ are disjoint closed sets in X.

Using normality of (X, τ), we can find a pair of disjoint open sets $H_1, H_2 \in \tau$ s.t. $F \subset H_1$, $X \sim G \subset H_2$, $H_1 \cap H_2 = \phi$.

Now $\qquad H_1 \cap H_2 = \phi \Rightarrow H_1 \subset X \sim H_2$

$$\Rightarrow \overline{H_1} \subset \overline{X \sim H_2} = X \sim H_2 \qquad [\because X \sim H_2 \text{ is closed}]$$

$$\Rightarrow \overline{H_1} \subset X \sim H_2$$

since $\qquad X \sim G \subset H_2 \Rightarrow X \sim H_2 \subset X \sim (X \sim G)$

$$\Rightarrow X \sim H_2 \subset G$$

Thus $\overline{H_1} \subset X \sim H_2 \subset G$.

Thus the set H_1 has the following properties :

(a) H_1 is τ-open

(b) $F \subset H_1$

(c) $\overline{H_1} \subset G$

Thus $\exists$ an τ-open set H_1 s.t. $F \subset H_1 \subset \overline{H_1} \subset G$.

Conversely, suppose that (X, τ) is a topological space such that given a closed set F and an open set G containing F, $\exists$ an open set V such that $F \subset V \subset \overline{V} \subset G$.

To prove (X, τ) is a normal space.

Let F_1 and F_2 be a pair of disjoint closed sets in X *i.e.*

$$F_1 \cap F_2 = \phi \Rightarrow F_1 \subset X \sim F_2.$$

$\Rightarrow$ $X \sim F_2$ is an open set containing a closed set F_1.

By hypothesis, $\exists$ another open set V containing F_1 such that

$$F_1 \subset V \subset \overline{V} \subset X \sim F_2$$

Let $\qquad U = X \sim \overline{V}$

$\because$ $\overline{V}$ is a closed set $\therefore$ $U = X \sim \overline{V}$ is an open set

$$\overline{V} \subset X \sim F_2 \Rightarrow X \sim (X \sim F_2) \subset X \sim \overline{V}$$

$$\Rightarrow F_2 \subset U$$

$$U \cap V = V \cap (X \sim \overline{V}) = V \cap X \sim V \cap \overline{V} = V \sim V = \phi.$$

Thus given a pair of disjoint closed sets F_1 and F_2 in X, $\exists$ open sets V, U such that $F_1 \subset V$, $F_2 \subset U$ and $U \cap V = \phi$.

This shows that (X, τ) is a normal space.

Self-learning exercise-1

1. Select true or false :

(a) Every discrete space is a T_2-space

(b) If τ is cofinite topology on an infinite set X, then (X, τ) is a T_2-space

(c) A cofinite topology on an infinite set X is not T_1-space

(d) A T_3-space is a T_4-space

(e) A singleton subset of T_1-space is closed

(f) Every metric space is normal

10.7 Summary

In this unit you have learnt about various separation axioms. If R = Regular, N = Normal, then we have seen that :

1. $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$

2. T_0, T_1, T_2, T_3, R, N, T_4, are all topological properties.

3. T_0, T_1, T_2, T_3, R, T_4, are all hereditary properties. Normality is not hereditary.

10.8 Answers to self-learning exercises

Self-learning exercise-1

(a) T *(b)* F *(c)* F *(d)* F

(e) T *(f)* T

10.9 Exercises

1. Prove that every metric space is normal space

2. Prove that every second countable regular space is normal space.

3. Let (X, τ) be a T_1-space. If τ^* is a topology on X such that $\tau \subset \tau^*$, show that (X, τ^*) is also a T_1-space.

4. Let (X, τ) be a Hausdroff space and let $f : X \to X$ be continuous. Show that the set $\{x \in X : f(x) = x\}$ is closed in X.

□ □ □

Unit 11 : Compact and Locally Compact Spaces

Structure of the Unit

11.0 Objectives

11.1 Introduction

11.2 Compact Topological spaces

11.2.1 Open cover

11.2.2 Sub-cover

11.2.3 Compact Topological space

11.3 Finite intersection property (FIP)

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11.5 Locally compact space

11.6 Summary

11.7 Answers to self-learning exercises

11.8 Exercises

11.0 Objectives

In this unit we shall study about the compactness of the topological spaces. For this we have to understand the concepts of open cover and Sub-cover. There are some types of compactness such as countable, sequential and local compactness. Only local compactness will be discussed in this unit.

11.1 Introduction

There are some closed surfaces contained in a finite part of three dimensional Euclidean space like sphere and ellipsoid. The concept of topological compactness is based on this type of surfaces, on the other-hand some surfaces are not contained in a finite part of the space like paraboloid. In this unit, compactness of a topological space is studied in terms of open cover and its Sub-cover.

11.2 Compact Topological spaces

11.2.1 Open cover : Let (X, τ) be a topological space and let A be a subsets of X.A collection $\boldsymbol{C} = \{G_\alpha \mid \alpha \in \Lambda\}$ of open subsets of X is said to be an open cover of A if

$$A \subset \bigcup_{\alpha \in \Lambda} G_\alpha .$$

If $\boldsymbol{C}$ is an open cover of A, then we say that $\boldsymbol{C}$ covers A. Sometimes $\boldsymbol{C}$ is called simply cover of A.

11.2.2 Sub cover : Let (X, τ) be a topological space and $\boldsymbol{C}$ be an open cover of subset A of X. A sub-collection (subset) $\boldsymbol{C}_1$ of $\boldsymbol{C}$ is said to be a sub-cover of $\boldsymbol{C}$ if $\boldsymbol{C}_1$ covers A.

A cover of A is said to be finite cover if it consists of finite number of open sets. If a cover $\boldsymbol{C}$ has a finite sub-cover then $\boldsymbol{C}$ is said to be **reducible to a finite sub-cover.**

11.2.3 Compact Topological space : Let (X, τ) be a topological space. A subset A of X is said to be compact iff every open cover of A has a finite sub-cover, that is, iff every open cover of A is reducible to a finite sub-cover.

The topological space X is said to be compact iff every open cover of X is reducible to a finite sub-cover, that is, iff for every collection $\boldsymbol{C} = \{G_\alpha \mid \alpha \in \Lambda\}$ of τ-open sets for which

$$X = \bigcup_{\alpha \in \Lambda} G_\alpha,$$

there exist finitely many open sets G_{α_i} $(1 \le i \le n)$ form $\boldsymbol{C}$ such that

$$X = G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n}$$

or

$$X = \bigcup_{i=1}^{n} G_{\alpha_i}.$$

***Ex.*1.** *Let $X = \{a, b, c\}$ and τ be a topology on X such that*

$$\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$$

then the collection $\boldsymbol{C} = \{X\}$ is an open cover of X, where as the collection $\{\{a\}, \{a,b\}\}$ is not a open cover as it does not cover X. Also the collection $\{\{a\}, \{b\}, \{c\}\}$ is not an open cover of X as $\{c\}$ is not t-open set although union of this collection is equal to X.

***Ex.*2.** $\boldsymbol{C}_1 = \{(-n, n) \mid n \in N\}$

and $\boldsymbol{C}_2 = \{(-3n, 3n) \mid n \in N\}$

are $\boldsymbol{U}$-open covers of R, where $\boldsymbol{U}$ is usual topology on R, the set of real numbers.

***Ex.*3.** *Every finite topological space is compact.*

Let (X, τ) be a topological space where X is finite. Since X is finite therefore τ is finite and hence every open cover of X is finite. We may say that every open cover of X is reducible to a finite sub-cover. Thus X is compact.

***Ex.*4.** *Every indiscrete space is compact.*

Let (X, I) be an indiscrete space. For indiscrete space, topology $I = \{\phi, X\}$, thus the only open cover of X is $\{X\}$, which is finite, so X is compact. Here X may be infinite.

Ex.5. *Let (X, D) be a discrete topological space, where X is infinite. Let A be an infinite subset of X, then we can easily verify that A is not compact. Consider a collection* ***C*** *such that*

$$\boldsymbol{C} = \{\{x\} \mid x \in A\}.$$

Obviously ***C*** *is an open cover of A as* $\{x\} \in D \ \forall \ x \in A$ *and* $A = \cup \{\{x\} \mid x \in A\}$.

This cover is infinite. Evidently it has no finite sub-cover as any sub-collection obtained by deletion of any member from ***C*** *will not cover A. Hence A is not compact.*

If we replace A by X, then we may say that infinite discrete space is not compact.

Theorem 1. *Compactness is not a relative property,*

OR

Let Y be a subspace of a topological space (X, τ) *and A is a subset of Y. Then A is compact relative to X iff A is compact relative to Y.*

Proof. Let (X, τ) be a topological space and let Y be the subspace of X for the relativized topology τ_Y given by

$$\tau_Y = \{G \cap Y \mid G \in \tau\} \qquad .....(1)$$

Let $A \subset Y$ and let A be compact relative to X. We shall show that A is compact relative to Y. For this, let

$\boldsymbol{C} = \{H_\alpha \mid \alpha \in \Lambda\}$ be a collection of τ_Y-open sets such that

$$A \subset \cup \{H_\alpha \mid \alpha \in \Lambda\} \qquad .....(2)$$

that is $\boldsymbol{C}$ is τ_Y-open cover of A. Since $H_\alpha \in \tau_Y$, $\exists\, G_\alpha \in \tau$ such that

$$H_\alpha = G_\alpha \cap Y, \ \forall \ \alpha \in \Lambda \qquad .....(3)$$

from (2) and (3) it follows that $A \subset \cup \{G_\alpha \cap Y \mid \alpha \in \Lambda\}$

or $A \subset \cup \{G_\alpha \mid \alpha \in \Lambda\}, \qquad (\because A \subset Y)$

so that the collection $\{G_\alpha \mid \alpha \in \Lambda\}$ of τ-open subsets of X is an open cover of A. Since A is compact relative to X, this cover is reducible to a finite subcover, that is, there exist finitely many open sets $G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}$ such that

$$A \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n}.$$

Since $A \subset Y$, therefore

$$A \subset Y \cap \left[G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n}\right] = \left(Y \cap G_{\alpha_1}\right) \cup \left(Y \cap G_{\alpha_2}\right) \cup \ldots \cup \left(Y \cap G_{\alpha_n}\right)$$

(Distributive-law)

$$= H_{\alpha_1} \cup H_{\alpha_2} \cup \ldots \cup H_{\alpha_n} \qquad \text{(from (3))}$$

or $$A \subset H_{\alpha_1} \cup H_{\alpha_2} \cup \ldots \cup H_{\alpha_n}.$$

This shows that cover $\boldsymbol{C}$ is reducible to finite subcover. Thus A is compact relative to Y.

Converse : Let A be compact relative to Y. Let

$$\boldsymbol{C'} = \{G_\alpha \mid \alpha \in \Lambda\}$$

be a collection of τ-open subsets of X, which covers A that is

$$A \subset \cup \{G_\alpha \mid \alpha \in \Lambda\}$$

or $$A \subset \bigcup_{\alpha \in \Lambda} G_\alpha. \qquad .....(4)$$

Since $A \subset Y$, then by (4), we have

$$A \subset Y \cap \left[\bigcup_{\alpha \in \Lambda} G_\alpha,\right] = \bigcup_{\alpha \in \Lambda} [Y \cap G_\alpha]. \text{ (Distributive law)} \qquad .....(5)$$

Since $G_\alpha \in \tau$, therefore $Y \cap G_\alpha \in \tau_Y$ (by the definition of τ_Y)

Let $Y \cap G_\alpha = H\alpha$, then by (5), we have

$$A \subset \bigcup_{\alpha \in \Lambda} H_\alpha,$$

where $$H_\alpha \in \tau_Y \ \forall \ \alpha \in \Lambda \qquad .....(6)$$

(6) shows that the collection $\{H_\alpha \mid \alpha \in \Lambda\}$ is an open cover of A relative to Y. Since A is compact relative to Y, this cover is reducible to a finite subcover, that is, there exist finitely many subsets $(1 \le i \le m)$ such that

$$A \subset H_{\alpha_1} \cup H_{\alpha_2} \cup .... \cup H_{\alpha_m} = \left(Y \cap G_{\alpha_1}\right) \cup \left(Y \cap G_{\alpha_2}\right) \cup .... \cup \left(Y \cap G_{\alpha_m}\right)$$

$$= Y \cap \left[G_{\alpha_1} \cup G_{\alpha_2} \cup .... \cup G_{\alpha_m}\right]$$

thus $$A \subset G_{\alpha_1} \cup G_{\alpha_2} \cup .... \cup G_{\alpha_m} \qquad (\because A \subset Y)$$

which shows that cover $\boldsymbol{C'}$ is reducible to a finite sub-cover and hence A is compact relative to X.

Theorem 2. *A closed subset of a compact space is compact.*

Proof. Let (X, τ) be a topological space and let A be a subset of X such that A is closed. We shall show that A is compact.

Let $\boldsymbol{C} = \{G_\alpha \mid \alpha \in \Lambda\}$ be an open covering of A, where $G_\alpha \in \tau$, $\forall \ \alpha \in \Lambda$, then

$$A \subset \bigcup_{\alpha \in \Lambda} G_\alpha \qquad .....(1)$$

Since A is closed therefore $(X - A)$ is open. Now $X = (X \sim A) \cup A$, so

$$(X - A) \cup \left(\bigcup_{\alpha \in \Lambda} G_\alpha\right) = X, \qquad \text{(by (1))} \qquad .....(2)$$

which shows that the collection $\boldsymbol{C}$ together with $X - A$ is an open cover of X. Since X is compact, therefore this cover is reducible to a finite subcover, that is,

$$X = (X - A)\left(\bigcup_{i=1}^{n} G_{\alpha_i}\right), \text{ for some } \alpha_i \text{ among } \alpha' s. \qquad .....(3)$$

From (3), we may conclude that

$$A \subset \bigcup_{i=1}^{n} G_{\alpha_i}.$$

Hence A is compact.

Theorem 3. *Every compact subset of a Hausdorff space is closed.*

Proof. Let A be a compact subset of a Hansdorff topological space (X, τ). We shall show that A is closed. For this it is sufficient to show that A^c is open. Let $x \in A^c$. Given that X is Hausdorff, so for every $y \in A, y \neq x$, there exist open neighbourhoods M_y and N_y of x and y respectively such that

$$M_y \cap N_y = \phi \qquad .....(1)$$

Consider the collection $\boldsymbol{C} = \{N_y \mid y \in A\}$. Obviously this is an open cover of A. Since A is compact, therefore this cover is reducible to a finite sub-cover such that

$$A \subset \bigcup_{i=1}^{n} N_{y_i}, \text{ for some } y_i \in A \qquad .....(2)$$

Associated with each of $N_{y_1}, N_{y_2}, ..., N_{y_n}$ we also have open sets $M_{y_1}, M_{y_2}, ..., M_{y_n}$ such that $x \in M_{y_1}, x \in M_{y_2}, ..., x \in M_{y_n}$ and $M_{y_i} \cap N_{y_i} = \phi$ for $i = 1, 2, ..., n$.

Let $\bigcap_{i=1}^{n} M_{y_i} = M$ and $\bigcup_{i=1}^{n} N_{y_i} = N.$

Since M is intersection of finite number of open neighbourhoods of x, therefore M is also an open neighbourhood of x.

Now Let $a \in N$ $\Rightarrow a \in \bigcup_{i=1}^{n} N_{y_i}$

$\Rightarrow a \in N_{y_i}$ for some $y_i \in A$

$\Rightarrow a \notin M_{y_i}$ $\left(\text{since } M_{y_i} \cap N_{y_i} = \phi\right)$

$\Rightarrow a \notin M.$

Since a is an arbitrary point of N, so $M \cap N = \phi$. By (2), we have $A \subset N$. Since $M \cap N = \phi$, therefore $A \cap M = \phi$, which shows that $M \subset A^c$. Thus, we have obtained that for $x \in A^c$, $\exists$ an open neighbourhood M of x such that $x \in M \subset A^c$, that is, x is an interior point of A^c. Since x is an arbitrary point of A^c so it is open and hence A is closed.

Corollary : Let A be a compact subset of a Hausclorff space X and let $x \in A^c$. Then, there exist open sets G and H such that $x \in G, A \subset H$ and $G \cap H = \phi$.

Theorem 4. *A continuous image of a compact space is compact.*

Proof. Let (X, τ) and (Y, V) be two topological spaces and let f be a continuous mapping of X into Y. We have to show that $f(X)$ is compact. Let $\{H_\alpha \mid \alpha \in \Lambda\}$ be an open over of $f(X)$, then

$$f(X) \subseteq \bigcup_{\alpha \in \Lambda}^{n} (H_\alpha), \qquad \text{.....(1)}$$

Since f is continuous, therefore $f^{-1}(H_\alpha)$ is τ-open subset of X for each α. From (1) we have

$$X \subseteq \bigcup_{\alpha \in \Lambda} f^{-1}(H_\alpha),$$

which shows that the collection $\{f^{-1}(H_\alpha) \mid \alpha \in \Lambda\}$ is an open cover of X. Since X is compact, therefore this cover is reducible to a finite sub cover, that is,

$$X = \bigcup_{i=1}^{n} f^{-1}\left(H_{\alpha_i}\right) \text{ for some } \alpha_i \text{ among } \alpha\text{' } s.$$

or
$$X = f^{-1}\left(\bigcup_{i=1}^{n}\left(H_{\alpha_i}\right)\right)$$

$\Rightarrow$
$$f(X) = \bigcup_{i=1}^{n} H_{\alpha_i},$$

which shows that the cover $\{H_\alpha \mid \alpha \in \Lambda\}$ of $f(X)$ is reducible to a finite subcover. Hence $f(X)$ is compact.

Theorem 5. *The space (R, U) is not compact, where U is usual topology on* ***R****, the set of real numbers.*

Proof. To show (R, U) is not compact, we have to show that there exists an open cover of R which is not reducible to a finite subcover. Consider the collection

$$\boldsymbol{C} = \{(-n, n) \mid n \in N\},$$

obviously this collection is an open cover of R. Now, we shall show that no finite sub collection of $\boldsymbol{C}$ can cover R. Let

$$\boldsymbol{C}' = \{(-n_1, n_1), (-n_2, n_2), ..., (-n_m, n_m)\}$$

be any finite subcollection of $\boldsymbol{C}$. Let $max\ \{n_1, n_2, ..., n_m\} = n_0$, then obviously $n_0 \notin (-n_i, n_i)$, $1 \le i \le m$, but $n_0 \in R$.

Thus $\boldsymbol{C}'$ does not cover R. So $\boldsymbol{C}$ is not reducible to a finite subcover. Hence (R, U) is not compact.

Theorem 6. (Heine-Borel Theorem) : *A subset of (R, U) is compact iff it bounded and closed.*

Proof. Let A be a subset of R. First suppose that A is closed and bounded. Since A is bounded therefore there exist two real numbers a and b such that $a < b$ and $A \subset [a, b]$. Now, we shall show that every closed and bounded interval $[a, b]$ on R is compact. Let $I_1 = [a, b]$ and suppose, if possible, I_1 is not compact.

Then there exists an open cover $\boldsymbol{C} = \{G_\alpha \mid \alpha \in \Lambda\}$ of I_1 which is not reducible to a finite subcover of I_1. Now, one of the closed intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ must have no finite subcover. Let us denote this interval as I_2. Again bisecting I_2, we obtain a closed interval I_3, in a similar manner, which has no finite subcover. Continuing this process of bisection of intervals, we obtained nested sequence of closed intervals $\{I_n\}$. As $n \to \infty$, we have length of I_n as $|I_n| \to 0$ and I_n has no finite subcover for every n. By **cantor's** intersection theorem, $\exists\, x \in R$ such that $x \in \cap \{I_n \mid n \in N\}$.

Also

$$x \in I_n \subset [a, b] \subset \bigcup_{\alpha \in \Lambda} G_\alpha$$

$\Rightarrow \quad x \in G_\alpha$ for some $\alpha \in \Lambda$.

Since G_α is $\cup$-open, $\exists\, \varepsilon > 0$ such that

$$x \in (x - \varepsilon, x + \varepsilon) \subset G_\alpha$$

Take n so large such that

$$I_n \subset (x - \varepsilon, x + \varepsilon) \subset G_\alpha$$

$$(\because\ |I_n| \to 0,\ n \to \infty \text{ and } x \in I_n,\ \forall\ n \in N)$$

which shows that I_n is covered by G_α, a single member of cover $\boldsymbol{C}$. This is a contradiction as I_n has no finite subcover for all $n \in N$. Hence $[a, b]$ is compact. Now since A is closed in R, therefore, by the theorem 2, A is compact.

Converse : Let us suppose that A is compact and consider a collection $\boldsymbol{C} = \{(x-1, x+1) \mid x \in A\}$. Obviously $\boldsymbol{C}$ is an open cover of A. Since A is compact therefore this cover is reducible to a finite subcover, that is,

$$A \subset (x_1 - 1, x_1 + 1) \cup (x_2 - 1, x_2 + 1) \cup \ldots \cup (x_n - 1, x_n + 1) \text{ for some } x_i \in A \quad .....(1)$$

Let $\qquad p = max\ \{x_1, x_2, \ldots, x_n\}$

and $\qquad q = min\ \{x_1, x_2, \ldots, x_n\}$.

then $\quad (x_1 - 1, x_1 + 1) \cup (x_2 - 1, x_2 + 1) \cup \ldots \cup (x_n - 1, x_n + 1) \subset [q - 1, p + 1] \quad .....(2)$

By (1) and (2), we have

$$A \subset [q - 1, p + 1]$$

Thus A is bounded. Since R is a Housdorff space and A is compact, therefore by the theorem 3, A is closed. This completes the proof.

Note : Since compactness is not a relative property (theorem 1) therefore we can consider $\cup$-open cover of set A instead of relativized $\cup$-open cover.

***Ex*.6.** *A compact subset of a non-Hausdorff space need not be closed. Give an example in favour of this statement.*

Sol. Let (X, I) be an indiscrete topology and X has more then one element in it. Since $I = \{\phi, X\}$, that is, only closed subsets of X are ϕ and X it self, so no proper subset of X can be closed. Let A be any proper subset of X, then it is not closed, but it is compact as only open cover of A is $\{X\}$ which is finite.

***Ex*.7.** *Give an example of a compact space which is not Haudorff.*

Sol. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Since X is finite therefore it is compact. It is not Hausdorff because a and b are two disjoint points such that they have no disjoint neighbourhoods.

***Ex*.8.** *If* (X, τ) *be a compact topological space then* (X, τ') *is compact if* τ' *is coarser that* τ.

Sol. Since τ' is coarser then τ so that $\tau' \subset \tau$. Let $\{G_\alpha \mid \alpha \in \Lambda\}$ be τ'-open cover for X. Since $\tau' \subset \tau$ therefore this collection is also τ-open cover for X. But X is τ-compact, therefore this τ-open cover is reducible to a finite subcover $\{G_{\alpha_i} \mid 1 \le i \le n\}$ which is also τ'-open. Thus X is τ'-compact, that is, (X, τ') is compact.

11.3 Finite intersection property (FIP)

Let $\boldsymbol{C}$ be a collection of sets. Then $\boldsymbol{C}$ is said to have the **finite intersection property (FIP)** iff the intersection of members of each finite subcollection of $\boldsymbol{C}$ is non-empty, that is, if $\boldsymbol{C}_1 \subset \boldsymbol{C}$ and $\boldsymbol{C}_1$ is finite then

$$\cap \{A \mid A \in \boldsymbol{C}_1\} \neq \phi.$$

This collection $\boldsymbol{C}$ of sets is called **fixed** if it has a non-empty intersection, that is if

$$\cap \{A \mid A \in \boldsymbol{C}\} \neq \phi$$

and called **free** if its intersection is empty, that is, if

$$\cap \{A \mid A \in \boldsymbol{C}\} = \phi.$$

11.4 Bolzano-weierstrass property (BWP)

A topological space X is said to have Bolzano-weiertrass property (BWP) if every in finite subset of X has a limit point. A space with BWP is also known as **Frechet compact space.**

Theorem 7. *A topological space X is compact iff every collection of closed subsets of X with the FIP is fixed, that is, has a non-empty intersection.*

Proof. Let (X, τ) be a compact topological space and let $F = \{F_\alpha \mid \alpha \in \Lambda\}$ be a collection of closed subsets of X having finite intersection property. We shall show that F is fixed, that is, it has a non-empty intersection. Let, if possible, F is not fixed, that is

$$\cap \{F_\alpha \mid \alpha \in \Lambda\} = \phi \qquad .....(1)$$

Taking complement to both sides and using De-Morgan's law, we have

$$\cup \{F_\alpha^c \mid \alpha \in \Lambda\} = X \qquad .....(2)$$

Since F_α is closed, therefore F_α^c is open, $\forall\ \alpha \in \Lambda$. So, by (2), the collection $\{F_\alpha^c \mid \alpha \in \Lambda\}$ is an open cover of X. Since X is compact, therefore, we have

$$\cup \{F_{\alpha_i}^c \mid 1 \le i \le n\} = X \qquad .....(3)$$

Again, using complement and De-Morgan law, form (3) we have

$$\cap \{F_{\alpha_i} \mid 1 \le i \le n\} = \phi \qquad .....(4)$$

which shows that finite subcollection of the collection ***F*** has empty intersection. This contradicts the FIP of ***F***. Thus

$\cap \{F_\alpha \mid \alpha \in \Lambda\} \neq \phi$ that is ***F*** is fixed.

Converse : Let every collection of closed subsets of topological space X, with FIP, is fixed. We shall show that X is compact. Let ***C*** $= \{G_\alpha \mid \alpha \in \Lambda\}$ be an open cover of X, then we have

$$\bigcup_{\alpha\in\Lambda} G_\alpha = X \qquad .....(5)$$

Taking complements and using De-Morgans law, we have

$$\bigcap_{\alpha\in\Lambda} G_\alpha^c = \phi \qquad .....(6)$$

Thus the collection $\{G_\alpha^c \mid \alpha \in \Lambda\}$ of closed sets is free, that is, not fixed. So this collection does not have FIP. For, if it has FIP, then it must be fixed (by our assumption). Hence there exist a finite subcollection of the collection $\{G_\alpha^c \mid \alpha \in \Lambda\}$ having empty intersection, that is,

$$\bigcap_{i=1}^{n} G_{\alpha_i}^c = \phi, \quad \text{for some } \alpha_i \text{ among } \alpha' s.$$

$$\Rightarrow \qquad \bigcup_{i=1}^{n} G_{\alpha_i} = X \quad \text{(De-Morgan's law)}$$

which shows that cover ***C*** of X is reducible to a finite sub-cover of X. Hence X is compact.

Theorem 8. *A topological space is compact if and only if every class of closed sets with empty intersection has a finite subclass with empty intersection.*

Proof. Let (X, τ) be a compact topological space and let $\{F_\alpha : \alpha \in I\}$ be a family of closed sets of X such that

$$\bigcap_{\alpha\in I} F_\alpha = \phi$$

Taping complements to both sides and using De-Morgan's law we get

$$\bigcup_{\alpha\in I} F_\alpha^c = X \qquad .....(1)$$

Since F_α is a closed set for each $\alpha \in I$, so F_α^c is an open set for each $\alpha \in I$. Therefore, form (1) we can say that $\{F_\alpha^c : \alpha \in I\}$ is an open covering for a compact space X. So, by compactness of X there exist finite number of inchices $\alpha_1, \alpha_2, \ldots, \alpha_n$ in I such that

$$X=\bigcup_{i=1}^{n} F_\alpha^c$$

$$\Rightarrow \qquad X^c = \left(\bigcup_{i=1}^{n} F_\alpha^c\right)^c$$

$$\Rightarrow \qquad \phi = \bigcap_{i=1}^{n} F_a.$$

Conversely, suppose that every family of closed sets with empty intersection has a finite sub-family with empty intersection. Now we have to show that X is compact. Let $\{G_\alpha : \alpha \in I\}$ be an open covering of X. Then

$$X = \bigcup_{\alpha\in I} G_\alpha$$

$$\Rightarrow \qquad X^c = \left(\bigcup_{\alpha\in I} G_\alpha\right)^c$$

$$\Rightarrow \qquad \phi = \bigcap_{\alpha\in I} G_\alpha^c.$$

This shows that $\{G_\alpha^c : \alpha \in I\}$ is a family of closed sets with empty intersection, since each G_α in open so each G_α^c is closed for $\alpha \in I$. So, by our assumption there exist a finite subfamily $\{G_{\alpha_i}^c : i = 1, 2, ..., n\}$ such that

$$\bigcap_{i=1}^{n} F_{\alpha_i}^c = \phi$$

$$\Rightarrow \qquad \left(\bigcap_{i=1}^{n} G_{\alpha_i}^c\right)^c = \phi$$

$$\Rightarrow \qquad \bigcup_{i=1}^{n} G_{\alpha_i} = X.$$

Thus every open covering of X has a finite subcover and hence (X, τ) is compact.

Theorem 9. *A compact space has Bolzario-weierstrass property.*

Proof. Let X be compact space and let A be an infinite subset of X. We have to show that A has a limit point. Let, if possible, A has no limit point in X. Then for every $x \in X$, there exist an open neithbourhood G_x of x such that it does not contain any point of A other than (possibly) x. The collection $\{N_x \mid x \in X\}$ forms an open cover of X. Since X is compact, therefore this cover is reducible to a finite subcover of X, that is,

$$X = \bigcup_{i=1}^{n} N_{x_i}, \text{ for some } x_i \in X.$$

Since $A \subset X$, *therefore*

$$A \subset \bigcup_{i=1}^{n} N_{x_i}$$

which shows that A is finite having at most n elements as each N_{x_i} has at most one element of A. Which is contradiction as A is infinite. Hence A has a limit point. So X has BWP.

Theorem 10. *In a Hausdorff topological space disjoint compact sets can be separated by disjoint open sets.*

Proof. Let (X, τ) be a Hausdorff space and let A, B be any two compact subsets of X such that $A \cap B = \phi$. Let $a \in A$, then $a \notin B$. We thus have a point a disjoint from the compact set B. By corollary of Theorem 3 there exist open sets G_a, H_a such that

$$a \in G_a, B \subset H_a \quad and \quad G_a \cap H_a = \phi$$

As a vanis in A form above we get

$$A \subset \bigcup_{a \in A} G_a$$

This shows that $\{G_a : a \in A\}$ is an open covering for the compact set A. By compactness of A there exist $a_1, a_2, ..., a_n$ all in A such that

$$A \subset G_{a_1} \cup G_{a_2} \cup ... \cup G_{a_n}.$$

Associated with each of $G_{a_1}, G_{a_2}, ..., G_{a_n}$, we have open sets $H_{a_1}, H_{a_2}, ..., H_{a_n}$, such that $B \subset H_{a_1}, B \subset H_{a_2}, ..., B \subset H_{a_n}$ and $G_{a_i} \cap H_{a_i} = \phi$ for $i = 1, 2, ..., n$.

Thus we have $\quad A \subset G_{a_1} \cup G_{a_2} \cup ... \cup G_{a_n} = G$ (say)

and $\quad B \subset H_{a_1} \cap H_{a_2} \cap ... \cap H_{a_n} = H$ (say)

Clearly G and H are open subsets of X.

Since $\quad G_{a_i} \cap H_{a_i} = \phi$ for $i = 1, 2, ..., n$

and $\quad H \subset H_{a_i}$ for $i = 1, 2, ..., n$

$\therefore \quad H \cap G_{a_i} = \phi$ for $i = 1, 2, ..., n$

Now
$$G \cap H = H \cap \left(G_{a_1} \cup G_{a_2} \cup ... \cup G_{a_n}\right)$$
$$= \left(H \cap G_{a_1}\right) \cup \left(H \cap G_{a_2}\right) \cup ... \cup \left(H \cap G_{a_n}\right)$$
$$= \phi \cup \phi \cup ... \cup \phi$$
$$= \phi$$
$$\Rightarrow \quad G \cap H = \phi$$

Thus if A and B are disjoint compact sets in a Hausdorff space X, then there exist disjoint open sets G, H in X such

that $\quad A \subset G, B \subset H$ and $G \cap H = \phi$.

Theorem 11. *A compact Hausderff space in normal.*

Proof. Let (X, τ) be a compact Hausdorff space. We have to show that (X, τ) is normal. Let F_1 and F_2 be two disjoint closed sets in X. Since F_1 and F_2 are closed subsets of a compact space X, so by Theorem 2, F_1 and F_2 are two disjoint compact subsets of X. Again, since F_1 and F_2 are disjoint compact subsets of a Hausdorff space X, so by Theorem 10 there exist two open subsets G and H of X such that

$$F_1 \subset G, F_2 \subset H \quad \text{and} \quad G \cap H = \phi.$$

Thus disjoint closed sets in X have been separated by disjoint open sets in X and hence (X, τ) is normal.

11.5 Locally compact space

A topological space (X, τ) is said to be locally compact if and only if every point of X has a compact neithbourhood.

Thus X is locally compact space if for every $p \in X$, there is an open set G and a compact set K such that

$$p \in G \subset K.$$

or

A topological space (X, τ) is said to be locally compact if and only if every point in X has atleast one neighbourhood whose closure is compact.

Ex.9. *The real line **R** with usual topology u on **R** is locally compact, since for each $x \in$ **R** we have*

$$x \in (x - 1, x + 1) \subset [x - 1, x + 1],$$

*where $[x - 1, x + 1]$ is compact being closed and bounded subset of **R**.*

Theorem 12. *Every compact topological space is locally compact, but converse is not necessarily true.*

Proof. Let (X, τ) be a compact topological space and let x be an arbitrary point in X. Since X is an open set, so it is neighbourhood for each $x \in X$. As X is given to be compact, so every point of X has a compact neighbourhood and hence X is locally compact.

The converse of above theorem is not necessarily true because if we consider the discrete topological space $(X, D = P(X))$, where X is an infinite set is not compact space but it is locally compact because D-open set $\{x\}$ is neighourhood of x. Also $\{x\}$ being a finite subset of a topological space is always compact. Hence for each $x \in X$ there is a D-open set $\{x\}$ which is compact such that $x \in \{x\} \subseteq \{x\}$. Hence (X, D) is locally compact but not compact.

Theorem 13. *Every closed subset of locally compact space is locally compact*

or

Every closed subspace of a locally compact space is locally compact.

Proof. Let (X, τ) be a locally compact topological space and let Y be any closed subset of X. Then (Y, τ_y) is a closed subspace of X. Now we have to prove that y is a locally compact space.

Let a be any arbitrary element of Y, then $a \in X$ as $Y \subset X$. Since X is a locally compact space and $a \in X$, therefore there exist $G_a \in \tau$ and a compact subset K of X such that

$$a \in G_a \subset K.$$

$$\Rightarrow \qquad a \in G_a \cap Y \subset Y \cap K, \text{ since } a \in Y \qquad .....(1)$$

Since Y is a closed subset of X and $Y \cap K \subset K$, so $Y \cap K$ is a closed subset of K. Again, since K is a compact set and we know that every closed subset of a compact space is compact, so $Y \cap K$ is a compact subset of K and hence of Y as $Y \cap K \subset Y$. Now

$$G_a \in \tau \Rightarrow G_a \cap Y \in \tau_Y, \text{ i.e., } G_a \cap Y$$

is an open subset of Y. Form (1) we have

$$a \in G_a \cap Y \subset Y \cap K$$

$\Rightarrow Y \cap K$ is a neighourhood of a in Y and we have shown that $Y \cap K$ is compact. Thus corresponding to each a in Y there is a compact neighourhood of a in Y and hence Y is locally compact.

Theorem 14. *Every open continuous image of a locally compact space is locally compact.*

Proof. Let (X, τ) be a locally compact space and let (Y, U) be any topological space. Also, let f be a an open and continuous function form X into Y. Now we have to show that $f(X)$ is locally compact subspace of Y. Let y be any element of $f(X)$, then there exists $x \in X$ such that $f(x) = y$.

Since (X, τ) is locally compact space and $x \in X$, so there exist an open set G and a compact set K in X such that

$$x \in G \subset K$$

$\Rightarrow$ $$y = f(x) \in f(G) \subset f(K) \qquad .....(1)$$

Since f is an open mapping and G is an open subset of X, so $f(G)$ is an open subset of $f(X)$. Again, since K is a compact subset of X and f is a continuous mapping form X into Y, so $f(K)$ is a compact subset of $f(X)$, as continuous image of a compact set is compact. Hence form (1) we can say that every point of $f(X)$ has a compact neighourhood in $f(X)$ and hence $f(X)$ is locally compact subspace of Y.

Self-learning exercise-1

1. Define following :

(i) Open-cover

(ii) Compact space

(iii) Finte intersection property

(iv) Bolzano-weierstrass property

2. Which of the following statements are true :

(a) Every indiscrete space is compact.

(b) Every discrete space is compact.

(c) Every finite space is compact.

(d) A closed subset of a compact space is compact.

(e) Every locelly compact space is compact.

(f) Every closed subsets of a locally compact space is locally compact.

3. Give an example to show that a compact subset of a non- Housdroff space need not to be closed.

11.6 Summary

In this unit, we hove studied have about compactness of the topological space. We observed that compactness is an absolute property . We have also studied about local compactness and related theorems.

11.7 Answers to self-learning exercises

Self-learning exercise–1

2. (*a*), (*c*), (*d*), (*f*) are true.

3. Indiscrete space (X, I), where X consists of more than one point. Let A be proper subset of X, then A is not closed as only closed sets are ϕ and X. A is compact as only open cover of A is $\{X\}$, which is finite. Also X is not housdroff.

11.8 Exercises

1. Show that a topological space (X, τ) is compact iff every basic open cover of X has a finite subcover.
2. Show that cantor's set is compact.
3. Show that compactness is a topological property.
4. Show that a coofinite topological space (X, t) is compact.
5. If f be a mapping of a locally compact space X onto a housdorff space Y such that f is continuous as well as open, then Y is locally compact.
6. Show that every closed interval $[a, b]$ is compact with respect to relativised U-topology for $[a, b]$.
7. Show that no infinite discrete space is compact.
8. Show that the intersection of the members of an arbitrary family of closed and compact subsets is also closed and compact.

□ □ □

Unit 12 : One Point Compactification

Structure of the Unit

12.0 Objectives

12.1 Introduction

12.2 Compactification

12.3 One-point compactification

12.4 Summary

12.5 Answers to self-learning exercises

12.6 Exercises

12.0 Objectives

In this unit, you will learn an important concept of compactness named compactification. The one-point compactification was introduced by Alexandroff and Urysohn in 1924. One-point compactification allows us to add a single point to a locally compact Hausdorff space X, in order to obtain a compact Hausdorff space Y containing X as a subspace.

12.1 Introduction

As we have already studied, compact space and sets have a number of useful properties. For example;

(i) Compact sets are closed and bounded in a metric space,

(ii) Sequences have convergent subsequences in a compact subset of a metric space,

(iii) Compact metric spaces are complete, and

(iv) Continuous functions on compact spaces attain minimum and maximum values.

Furthermore, we have also studied the useful properties possessed by a Hausdorff space. For example

(i) Single point sets are closed in a Hausdorff space, and

(ii) Convergent sequences converges to a unique limit in a Hausdorff space.

Unfortunately, we do not always have the advantages afforded by a compact and Hausdorff space in the topological spaces, we use.

It is possible to embed a non-compact topological space (X, τ) into a compact space (Y, u) and then use the properties of Y to gain information about X. Such a space Y is called a compactification of X.

12.2 Compactification

A topological space X, is said to be **embedded** in a topological space Y, if X is homeomorphic to a subspace of Y. If Y is a compact space, then Y is called a compactification of X. Frequently, the compactification of a space X is accomplished by adjoining one or more points to X and then defining an appropriate topology on the enlarged set, so that the enlarged space is compact and contains X as a subspace.

Ex. *Consider the real line* $\mathbf{R}$ *with the usual topology* U*. We know that the space* $(\mathbf{R}, U)$ *is not compact. We adjoin two new points, denoted by* ∞ *and* $-\infty$*, to* $\mathbf{R}$ *and call the enlarged set* $\mathbf{R}^* = \mathbf{R} \cup \{-\infty, \infty\}$ *the* ***extended real line****. The order relation in* $\mathbf{R}$ *can be extended to* $\mathbf{R}^*$ *by defining* $-\infty < a < \infty$, $\forall\ a \in \mathbf{R}$*. The class of subsets of* $\mathbf{R}^*$ *of the form* $(a, b) = \{x \in \mathbf{R}^* : a < x < b\}$, $(a, \infty] = \{x \in \mathbf{R}^* : a < x\}$ *and* $[-\infty, a) = \{x \in \mathbf{R}^* : x < a\}$ *is a base for a topology* u^* *on* $\mathbf{R}^*$*. Furthermore the space* $(\mathbf{R}^*, u^*)$ *is compact space and contain* $(\mathbf{R}, U)$ *as a subspace, and so it is a compactification of* $(\mathbf{R}, U)$*.*

12.3 One-point compactification

Let (X, τ) be any topological space. We shall define the **Alexandroff** or **one-point compactification** of (X, τ), which we denote by (X_∞, T_∞). Here :

1. $X_\infty = X \cup \{\infty\}$, where ∞, called the **point at infinity** is distinct from every other point in X, *i.e.* $\infty \notin X$.

2. T_∞ be the collection of all sets U in X_∞ such that *(i)* U is open in X or *(ii)* $X_\infty \sim U$ is a closed and compact subset of X,

i.e. $T_\infty = \{U \in P(X_\infty) : U \in \tau \text{ or } X_\infty \sim U \text{ is a closed and compact subset of } X\}$,

where $P(X_\infty)$ is power set of X_∞.

Theorem 1. *T_∞ is a topology on X_∞.*

Proof. : [*T.* 1] Since $X_\infty \sim X_\infty = \phi$ is closed and compact in X,

Thus $X_\infty \in T_\infty$. As ϕ is open in $X \Rightarrow \phi \in \tau$

$\Rightarrow \quad \phi \in T_\infty$

thus $X_\infty \in T_\infty$ and $\phi \in T_\infty$.

[*T.* 2] Let $\{G_\lambda : \lambda \in \wedge\}$ be any collection of open sets in X_∞. To show

$$G = \bigcup_{\lambda \in \Lambda} G_\lambda \in T_\infty$$

First assume that $\infty \notin G_\lambda\ \forall\ \lambda \in \wedge$, then $G_\lambda \in \tau$, $\forall\ \lambda \in \wedge$ and so

$$\bigcup_{\lambda \in \Lambda} G_\lambda = G \in \tau \qquad \{\because\ \tau \text{ is a topology an } X\}$$

$\Rightarrow$ $\quad G = \bigcup_{\lambda \in \Lambda} G_\lambda \in T_\infty$ [By definition of T_∞]

If $\quad \infty \in \bigcup_{\lambda \in \Lambda} G_\lambda \Rightarrow \infty \in G_\mu$ for some $\mu \in \wedge$. But then by definition. *(ii)* $X_\infty \sim G_\mu$ is closed and compact in X.

Now $\quad \bigcap_{\lambda \in \Lambda} (X_\infty \sim G_\lambda) \subset X_\infty \sim G_\mu$.

thus $\quad \bigcap_{\lambda \in \Lambda} (X_\infty \sim G_\lambda)$ is closed and compact in X.

So $\quad X_\infty \sim \bigcap_{\lambda \in \Lambda} (X_\infty \sim G_\lambda) = \bigcup_{\lambda \in \Lambda} G_\lambda = G \in T_\infty$

[*T*. 3] Let $A, B \in T_\infty$

Now if $\infty \notin A$ and $\infty \notin B$, then $A, B \in \tau$ and so

$$A \cap B \in \tau \Rightarrow A \cap B \in T_\infty$$

If $\infty \notin A$ and $\infty \in B$, then $A \in \tau, B \notin \tau$ so that $X_\infty \sim B$ is a closed and compact subset of X. But $X_\infty \sim B = X \sim B$ and so $X \sim B$ is a closed in X, thus $B \in X$ and hence $A \cap B \in \tau \Rightarrow A \cap B \in T_\infty$.

If $\infty \in A$ and $\infty \notin B$, then similarly $A \cap B \in T_\infty$.

If $\infty \in A$ and $\infty \in B$, then $X_\infty \sim A$ and $X_\infty \sim B$ are both closed and compact subsets of X. But then $(X_\infty \sim A) \cup (X_\infty \sim B)$ is closed and compact in X. Also

$$(X_\infty \sim A) \cup (X_\infty \sim B) = X_\infty \sim (A \cap B)$$

$\Rightarrow$ $\quad X_\infty \sim (A \cap B)$ is closed and compact in X

$\Rightarrow$ $\quad A \cap B \in T_\infty$.

Thus is all cases $A \cap B \in T_\infty$, and therefore by [*T*.1], [*T*. 2] and [*T*. 3], (X_∞, T_∞) is a topological space.

Theorem 2. *Let* (X_∞, T_∞) *be the one- point compactifiction of a topological space* (X, τ), *then* (X_∞, T_∞) *is a compact space.*

Proof. : Let $\mathbf{C} = \{G_\lambda : \lambda \in \wedge)$ be an open cover for X_∞. Then $\infty \in G_\mu$ for some $\mu \in \wedge$, so that

$$G_\mu = X_\infty \sim F, \quad \text{where } F \text{ is compact and closed in } X$$

[By definition of (X_∞, T_∞)]

Since $\mathbf{C}$ is an open cover of X_∞ and $F \subset X \subset X_\infty$, thus $\mathbf{C}$ is an open cover of F also and F is compact, thus there exist a finite sub-cover $\{G_{\lambda_1}, G_{\lambda_2}, ..., G_{\lambda_n}\}$ of $\mathbf{C}$ such that

$$F = X_\infty \sim G_\mu = G_{\lambda_1} \cup G_{\lambda_2} \cup ... \cup G_{\lambda_n}.$$

But then $\{G_\mu, G_{\lambda_1}, G_{\lambda_2}, ..., G_{\lambda_n}\}$ is a finite sub-cover of $\mathbf{C}$ covering X_∞. Hence X_∞ is compact.

Theorem 3. *Let* (X_∞, T_∞) *be the one- point compactifiction of a topological space* (X, τ)*, then* X *is a subspace of* X_∞.

Proof. : Let $U \in T_\infty$, if $\infty \notin U$, then $U \in \tau$ [By definition] and so we write $U = U \cap X$.

If $\infty \in U$, Then $U = X_\infty \sim F$, where F is closed and compact in X [By definition], then

$$U \cap X = (X_\infty \sim F) \cap X = X \sim F, \qquad \text{which is open in } X.$$

Thus if $U \in T_\infty$, then $U \cap X \in \tau$ .....(1)

Now let $V \in \tau$, then $\infty \notin V$ and so $V \in T_\infty$.

Hence $\tau \subset T_\infty$ .....(2)

Combining (1) and (2) we find that (X, τ) is a subspace of (X_∞, T_∞).

Theorem 4. *Let* (X_∞, T_∞) *be the one-point compactifiction of a topological space* (X, τ)*, then* $\bar{X} = X_\infty$

Proof. Let G be any T_∞-open nbd of ∞. Then $G = X_\infty - F$, where F is compact and closed in X. But X is non compact and so $F \neq X$. Hence $G \cap X \neq \phi$. Thus $\infty \in \bar{X}$. Accordingly $\bar{X} = X_\infty$.

Theorem 5. *Let* (X_∞, T_∞) *be the one-point compactification of a topological space* (X, τ)*, then* X_∞ *is a Hausdorff space if and only if is* X *Hausdroff and locally compact.*

Proof. First assume that X_∞ is a Hausdorff space. By Theorem 3, X is a subspace of X_∞ and the property of being Hausdroff is hereditary. Hence X is also Hausdroff. Let $x \in X$, then x and ∞ are distinct points in X_∞. Since X_∞ is Hausdroff, thus there are T_∞-open nbds U of x and V of ∞ such that $U \cap V = \phi$. Hence $U \subset X_\infty \sim V$ so that $x \in U \cap X \subset (X_\infty \sim V) \cap X = X \sim V$. Since $U \cap X$ is τ- open it follows that $X \sim V$ is a τ-open nbd of x. But $\infty \in V$ and $V \in T_\infty$ imply that $X \sim V = X_\infty \sim V$ is closed and compact in X. Thus each point x of X has a compact nbd $X \sim V$ in X and so X is locally compact. Hence we have shown that X is Hausdorff and locally compact.

Conversely, suppose that X is Hausdorff and locally compact. We show that X_∞ is Hausdroff. Let x, y be distinct point of X_∞.

Case 1 : Suppose $x, y \in X$. Since X is Hausdroff we can find τ-open nbds G of x and H of y such that $G \cap H = \phi$. Since ∞ does not belong to G and H we have G and H are also T_∞-open. Hence X_∞ is also Hausdorff.

Case 2 : Suppose $x \neq \infty$ and $y = \infty$. Then $x \in X$. Since X is locally compact, we can find a compact nbd C of x in X. But X is Hausdroff and so the compact subset C of X is closed in X. Hence by definition $V = X_\infty \sim C$ is T_∞-open nbd of ∞. Since C is a nbd of x we have $x \in U \subset C$, For some τ-open set U in X. But then $U \cap V = \phi$. Again, By definition $U \in T_\infty$. Thus x and y have T_∞- nbds U and V respectively with $U \cap V = \phi$ and so X_∞ is Hausdorff.

Theorem 6. *Let* (X_∞, T_∞) *be the one-point compactification of a topological space* (X_∞, τ)*. Then* (X, τ) *is uniquely embedded into* (X_∞, T_∞) *such that* $X_\infty \sim X$ *is a singleton.*

Proof. Suppose X_∞ and Y_∞ are two one point compactifications of X. We then have $X_\infty \sim X = \{\infty\}$ and $Y_\infty \sim X = \{\infty_y\}$.

Define a map $h : X_\infty \to Y_\infty$ by

$$h(x) = \begin{cases} x, \forall x \in X \\ \infty_y, x = \infty \in X_\infty. \end{cases}$$

Then h is a continuous bijection. We show that h is a homeomorphism. Let U be open in X_∞. If $U \subset X$, the $h(U) = U$, which is open in Y_∞. If U is an open nbd of ∞, then $C = X_\infty \sim U$ is closed and compact in X. Since C is compact, by continuity of h, it follows that $h(C)$ is a compact subset of X. But X is Hausdorff and so $h(C)$ is closed in X. Thus $Y_\infty \sim h(C)$ is open in Y_∞. But h is a bijection and so $h(X_\infty \sim C) = Y_\infty - h(C)$. Thus $h(U) = h(X_\infty \sim C)$ is also open in Y_∞, whence h is an open map. Therefore h is a homeomorphism of X_∞ on to Y_∞. This proves the desired result.

Ex.1. *Let **R** be the set of all real numbers. Show that the set* $S = \{(x, y); x^2 + y^2 = 1\}$ *in* $\boldsymbol{R}^2$ *is the one point compactification of **R** and that* $\infty = (0, 1)$ *is the point at infinity.*

Sol. For all $(x, y) \in \boldsymbol{R}^2$, let $f(x, y) = x^2 + y^2 - 1$. Then f is continuous function on $\boldsymbol{R}^2$ to $\boldsymbol{R}$. Also $S = f^{-1}(\{0\})$. But $\{0\}$ is closed in $\boldsymbol{R}$. Hence by continuity of f, S is closed in $\boldsymbol{R}^2$.

Moreover, S is a subset of the closed rectangle $[-1, 1] \times [-1, 1] \subset \boldsymbol{R}^2$. Hence S is bounded in $\boldsymbol{R}^2$. *Thus* S is closed and bounded subset of $\boldsymbol{R}^2$ consequently S is compact in $\boldsymbol{R}^2$ by Heine-Borel Theorem.

We have, $\boldsymbol{R}$ is locally compact, since $x \in \boldsymbol{R}$ lies in some open interval (a, b), *which* is contained in the compact set $[a, b]$.

Now we have to show that $\boldsymbol{R}$ is embedding in some subset Y of S such that $Y = S \sim \{\infty\}$ where $\infty = (0, 1)$. We define $h : \boldsymbol{R} \to S - \{\infty\}$ by

$$h(x) = \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1} \right) \forall\, x \in \boldsymbol{R}$$

and $\quad g : S - \{\infty\} \to \boldsymbol{R}$, by

$$g(x, y) = \frac{x}{1-y} \quad \forall\, (x, y) \in S - \{\infty\}, \quad \text{where } \infty = (0, 1)$$

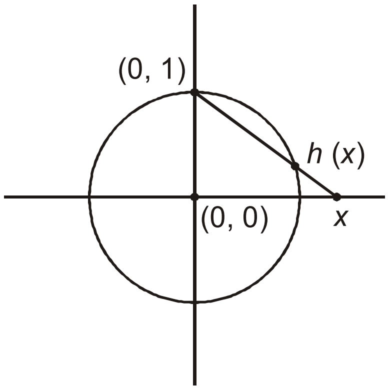
then h and g are continuous,

Also $$h(g(x, y)) = h\left(\frac{x}{1-y} \right) = (x, y) \quad [\text{since } x^2 + y^2 = 1] \text{ and}$$

$$g(h(x)) = g\left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1} \right) = x$$

These relation imply that $g = h^{-1}$ and so h^{-1} is continuous. Hence h is a homeomorphism of $\boldsymbol{R}$ on

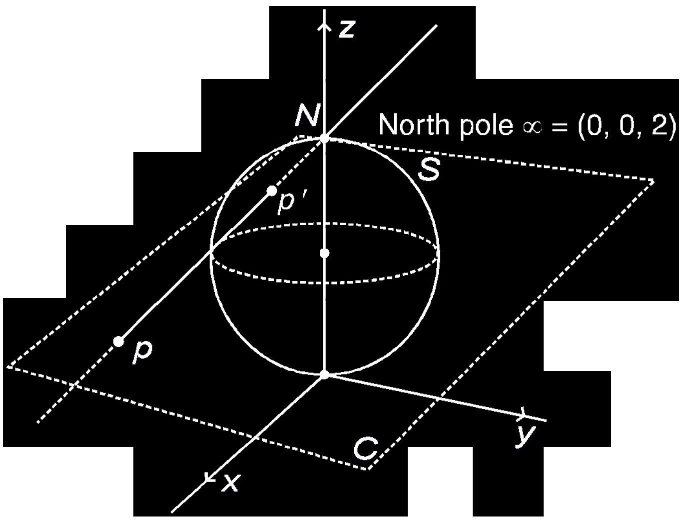
$$h(\boldsymbol{R}) = S - \{\infty\} \subset S.$$



Therefore, S is the one-point compactifiction of $\boldsymbol{R}$ and $\infty = (0, 1)$ is the point at-infinity of this compactification.

Ex.2. *The one- point compactification of the plane is homeomorphic to the sphere.*

Sol. Let $\boldsymbol{C}$ denote the $\langle x, y \rangle$ – plane in Euclidean 3-space $\boldsymbol{R}^3$, and let S denote the sphere with center (0, 0, 1) on the z-axis and radius 1. The line passing through the **"North-pole"** $\infty = (0, 0, 2) \in S$ and any point $p \in \boldsymbol{C}$ intersects the sphere S in exactly one point $\boldsymbol{p'}$ distinct from ∞, as show in the figure.



Let $f : \boldsymbol{C} \to S$ be defined by $f(p) = p'$. Then f is, in fact a homeomorphism from the plane $\boldsymbol{C}$ (which is not compact) on to the subset $S - \{\infty\}$ of the sphere S (which is compact). Hence S is a one-point compactification of $\boldsymbol{C}$.

Note : When the plane $\boldsymbol{C}$ is considered to be the complex plane $\mathbb{C}$, then the one-point compactification $\mathbb{C} \cup \{\infty\}$ is called the **Riemann sphere** or the **extended complex plane** and the mapping f is known as **stereographic projection.**

Self-learning exercise-1

1. The one-point compactification of the interval (0, 1] is ....
2. The one-point compactifiction of the interval (0, 1) is homeomorphic to ....
3. The one-point compactifiction of the set of complex numbers $\mathbb{C}$ is called ..... .
4. In one-point compactifiction of the set of complex numbers $\mathbb{C}$, the point at infinity is mapped to ....
5. Let $Y = X \cup \{\infty\}$ be the one point compactification of X, then $\overline{X}$ = ...

12.4 Summary

You have learnt a very useful concept of one- point compactification of a topological space in this unit. You have learnt that by adjoining a point at infinity, the spaces $\boldsymbol{R}$ and $\mathbb{C}$ can be made compact spaces, and with the help of which, important consequences can be drawn.

12.5 Answers to self-learning exercises

1. [0, 1]
2. Circle $C = \{(\cos 2\pi t, \sin 2\pi t) : t \in (0, 1)\} \cup \{(1, 0)\}$
3. Extended complex plane
4. North pole $N = (0, 0, 2)$ of the sphere S
5. Y.

12.6 Exercises

1. Show that the one-point compactification of unit open interval (0, 1) is homeomorphic to the circle.
2. Show that the one-point compactification of set of rational numbers Q is not Hausdorff.
3. Let (X, τ) be a topological space and let (X_∞, T_∞) be its one-point compactifiction, then X is a dense subset of X_∞ if and if X is not compact.

□□□

UNIT 13 : Connected and Locally Connected Spaces

Structure of the Unit

13.0 Objectives

13.1 Introduction

13.2 Separated sets

13.3 Connected space

13.3.1 Connected and disconnected set

13.3.2 Connected and disconnected space

13.4 Locally connected space

13.5 Summary

13.6 Answers to self-learning exercises

13.7 Exercises

13.0 Objectives

In this unit, we shall study about connectedness of a topological space. For this purpose we shall study separated subsets of a topological space. We shall also discuss locally connectedness of a topological space.

13.1 Introduction

Connectedness is an important property of the topological space which is significant in the study of continuity of curves. The connectedness is a topological invariant property. Connected space means a single piece and when it is stretched or bent without tearing, then it remains a single piece. In this unit, mathematical formulation of this concept is discussed.

13.2 Separated Sets

Let (X, τ) be a topological space. Let A and B be two non-empty subsets of X, then A and B are said to be **τ-separated** or simply **separated** sets if and only if

$$A \cap \overline{B} = \phi \quad \text{and} \quad \overline{A} \cap B = \phi.$$

where $\overline{A}$ and $\overline{B}$ are closures of set A and set B respectively.

Notes :

1. Two conditions $A \cap \overline{B} = \phi$ and $\overline{A} \cap B = \phi$ can be written as single condition

$$(A \cap \overline{B}) \cup (\overline{A} \cap B) = \phi.$$

2. From the conditions of the separated sets it is clear that *A* and *B* are disjoint and neither of them contains limit point of the other, for, if *A* and *B* are separated sets, then

$$A \cap \overline{B} = \phi \text{ and } \overline{A} \cap B = \phi.$$

Now $\quad A \subset \overline{A} \Rightarrow A \cap B \subset \overline{A} \cap B = \phi$

$\Rightarrow A \cap B = \phi.$

We know that $\quad \overline{A} = A \cup A'$

$\Rightarrow \quad B \cap \overline{A} = B \cap (A \cup A')$

$= (B \cap A) \cup (B \cap A') = \phi \cup (B \cap A') = B \cap A'$

$\phi = B \cap A'$

Similarly we can show that $A \cap B' = \phi.$

***Ex.*1.** *Consider the topological space R of real numbers with usual topology U.*

Let $\quad A = (1, 2), B = (2, 3], C = [2, 3),$

then $\quad \overline{A} = [1, 2], \overline{B} = [2, 3], \overline{C} = [2, 3],$

so $\overline{A} \cap B = \phi$ and $A \cap \overline{B} = \phi,$ hence *A* and *B* are separated.

Also, *A* and *C* are not separated as $\overline{A} \cap C = \{2\} \neq \phi,$ although they are disjoint. Hence disjoint sets need not be separated.

***Ex.*2.** *In (R U), the set* $A = (-\infty, 0)$ *are disjoint but not separated as*

$$\overline{A} \cap B = [0, \infty) \cap [0, \infty) = \{0\} \neq \phi.$$

Theorem 1. *Let (X, τ) be a topological space. Let A and B be separated subsets of X and C ⊂ A, D ⊂ B, then C and D are also separated, where C and D are non empty.*

Proof : Since *A* and *B* are separated so

$A \cap \overline{B} = \phi$ and $\overline{A} \cap B = \phi$ .....(1)

Given that, $\quad C \subset A \Rightarrow \overline{C} \subset \overline{A}$ .....(2)

and $\quad D \subset B \Rightarrow \overline{D} \subset \overline{B}$ .....(3)

From (1), (2) and (3), we have

$$C \cap \overline{D} = \phi \text{ and } \overline{C} \cap D = \phi$$

and thus *C* and *D* are separated sets.

Theorem 2. *Let (X, τ) be a topological space and (Y,* τ_Y*) be its subspace. Let A and B be two subsets of Y, then A and B are* τ_Y*-separated, if and only if A and B are τ-separated.*

Proof : Let $cl_X(A)$, $cl_Y(A)$ be τ-closure and τ_Y-closure of *A* respectively and $cl_X(B)$, $cl_Y(B)$ be τ-closure and τ_Y-closure of *B* respectively.

We know that

$$cl_Y(A) = cl_X(A) \cap Y \text{ and } cl_Y(B) = cl_X(B) \cap Y \quad .....(1)$$

Now,

$$A \cap cl_Y(B) = A \cap [cl_X(B) \cap Y] \quad \text{[by (1)]}$$

$$= A \cap cl_X(B) \quad (\because \ A \subset Y)$$

Similarly $cl_Y(A) \cap B = cl_X(A) \cap B$

Thus, we have

$$A \cap cl_Y(B) = \phi \Leftrightarrow A \cap cl_X(B) = \phi \quad .....(2)$$

$$cl_Y(A) \cap B = \phi \Leftrightarrow cl_X(A) \cap B = \phi \quad .....(3)$$

From (2) and (3), we may conclude that A and B are τ-separated iff A and B are τ_Y-separated.

Theorem 3. *Two open subsets of a topological space are separated iff they are disjoint.*

Proof : Let A and B be two open subsets of the topological space X. First suppose that A and B are separated then by the definition, A and B are necessarily disjoint.

Conversely, let A and B are disjoint. Since A and B are open then A^c and B^c are closed, then

$$\overline{A}^c = A^c \text{ and } \overline{B}^c = B^c \quad .....(1)$$

and $A \cap B = \phi \Rightarrow A \subset B^c$ and $B \subset A^c$

$$\Rightarrow \overline{A} \subset \overline{B}^c \text{ and } \overline{B} \subset \overline{B}^c \quad \left(G \subset H \ \Rightarrow \overline{G} \subset \overline{H}\right)$$

$$\Rightarrow \overline{A} \subset B^c \text{ and } \overline{B} \subset A^c \quad \text{[by (1)]}$$

$$\overline{A} \cap B = \phi \text{ and } \overline{B} \cap A = \phi.$$

Thus A and B are separated.

Theorem 4. *Two closed subsets of a topological space are separated iff they are disjoint.*

Proof : Let A and B be two closed subsets of the topological space X. Since separated sets are always disjoint therefore we shall prove that if A and B are disjoint then they are separated. Suppose A and B are disjoint, then

$$A \cap B = \phi \quad .....(1)$$

Since A and B are closed therefore

$$A = \overline{A} \text{ and } B = \overline{B} \quad .....(2)$$

From (1) and (2). we have

$$A \cap \overline{B} = \phi, \quad \overline{A} \cap B = \phi.$$

Thus A and B are separated.

Theorem 5. *Two disjoint sets A and B are separated in a topological space X iff they are both open and closed in the subspace $A \cup B$.*

Proof : Let A and B be two disjoint subsets of the topological space X. First suppose that A and B are separated.

Since A and B are separated in X, therefore

$$A \cap cl_X(B) = \phi \quad \text{and} \quad cl_X(A) \cap B = \phi \qquad .....(1)$$

Let $A \cup B = Y$; then closure of A in Y,

$$\begin{aligned} cl_Y(A) &= cl_X(A) \cap Y \\ &= cl_X(A) \cap (A \cup B) \\ &= (cl_X(A) \cap A) \cup (cl_X(A) \cap B) && \text{(by distribution law)} \\ &= A \cap \phi && [\because\ A \subset cl_X(A) \text{ and by (1)}] \\ &= A \end{aligned}$$

Thus, $cl_Y(A) = A \Rightarrow A$ is closed in Y *i.e.* in $A \cup B$

Similarly, we can show that B is closed in $A \cup B$.

Now, Since A and B are disjoint therefore both are complement of each other in $A \cup B$, thus both are open in $A \cup B$ also.

Conversely, suppose that A and B are both open and closed in $A \cup B = Y$.

Since A is closed in Y, therefore

$$\begin{aligned} A &= cl_Y(A) \\ &= cl_X(A) \cap Y \\ &= cl_X(A) \cap (A \cup B) \\ &= (cl_X(A) \cap A) \cup (cl_X(A) \cap B) && \text{(by distribution law)} \\ &= A \cup (cl_X(A) \cap B) && [\because\ A \subset cl_X(A)] \qquad .....(2) \end{aligned}$$

$\Rightarrow$ either $\quad cl_X(A) \cap B = \phi \quad$ or $\quad (cl_X(A) \cap B) \subset A$

but $\quad A \cap (cl_X(A) \cap B) = (A \cap B) \cap cl_X(A)$

$$\begin{aligned} &= \phi \cap cl_X(A) && (\because\ A \cap B = \phi) \\ &= \phi \end{aligned}$$

so $cl_X(A) \cap B$ and A are disjoint so $cl_X(A) \cap B$ can not be a subset of A so

$$cl_X(A) \cap B = \phi. \qquad .....(3)$$

Similarly, we can show that

$$A \cap cl_X(B) = \phi \qquad .....(4)$$

From (3) and (4), A and B are separated in X.

13.3 Connected space

13.3.1 Connected and disconnected set :

Let (X, τ) be a topological space. A subset A of X is said to be **τ-disconnected** or **simply disconnected** iff $\exists\ G, H \subset X$ such that G and H are τ-separated and $A = G \cup H$, that is, iff $\exists$ two non-empty subsets G and H of X such that *(i)* $G \cap \bar{H} = \phi, \ \bar{G} \cap H = \phi$

(ii) $A = G \cup H$

The set A is said to be **connected** iff it is not disconnected.

13.3.2 Connected and disconnected space :

A topological space X is said to be disconnected iff it is the union of two separated sets, that is, iff there exists two non-empty subsets A and B of X such that $A \cap \bar{B} = \phi, \ \bar{A} \cap B = \phi$ and $X = A \cup B$. Space X is said to be connected iff it is disconnected. Here $(A \cup B)$ is called disconnection of X.

Theorem 6. *Let* (X, τ) *be a topological space and* (Y, τ_Y) *be its subspace. A subset A of Y is* τ_Y*-disconnected iff it is* τ*-disconnected.*

Proof : First suppose that A is τ_Y-disconnected, then there exists two τ_Y-separated sets B and C such that $A = B \cup C$. By the theorem 2, if B and C are τ_Y-separated, then they are τ-separated. Thus A can be expressed as the union of two τ-separated sets. Hence A is τ-disconnected. Similarly, using theorem 2, we can prove the converse part of the theorem.

Theorem 7. *A topological X is disconnected iff there exists a proper subset of X which is both open and closed in X.*

Proof : First suppose that G is a proper subset of X such that it is both open and closed. Let $G^c = H$, then H is also proper subset of X $(G \neq \phi, G \neq X \Rightarrow G^c \neq X, G^c \neq \phi)$.

Also, $\quad G \cup G^c = X \Rightarrow G \cup H = X \quad$ .....(1)

and $\quad G \cap H = \phi. \quad$ .....(2)

Since G is both open and closed therefore H is also both open and closed.

Now, G is closed $\Rightarrow \bar{G} = G$

H is closed $\Rightarrow \bar{H} = H$,

thus,

$$G \cap H = \phi \Rightarrow G \cap \bar{H} = \phi \text{ and } \bar{G} \cap H = \phi \quad .....(3)$$

that is, G and H are separated.

From (1) and (3) we can conclude that X can be expressed as union of two separated sets. Hence X is disconnected.

Conversely, let X be disconnected, then $\exists$ two non-empty subsets G and H of X such that

$$G \cap \bar{H} = \phi, \ \bar{G} \cap H = \phi \quad .....(1)$$

and $\quad X = G \cup H. \quad$ .....(2)

We know that separated sets are disjoint so

$$G \cap H = \phi \quad .....(3)$$

From (2) and (3), $G^c = H$ and G, H are proper subsets of X.

Now, $$G \cup H = X \Rightarrow G \cup \bar{H} = X \qquad (\because\ H \subset \bar{H}) \qquad .....(4)$$

From (1) and (4), $G^c = \bar{H}$ and since $\bar{H}$ is closed, G^c is closed and hence G is open. Similarly H is open. Since both are complement of each other, therefore both are closed also, thus G is a non-empty proper subset of X which is both open and closed.

Theorem 8. *A topological space X is disconnected iff X is the union of two non-empty disjoint open (closed) sets.*

Proof : Let X be a topological space. First suppose that X is disconnected. Then, by theorem 7, there exists a non-empty proper subset A of X which is both open and closed. Then A^c is also open closed. Thus, A and A^c are two non-empty disjoint open (closed) sets such that

$$X = A \cup A^c.$$

Conversely, let X be union of two non-empty disjoint open (closed) sets. Let A and B be two non-empty open (closed) subsets of X such that $A \cap B = \phi$ and $X = A \cup B$. Then $A = B^c$, so A is closed (open) also as B is open (closed). Thus A is a non-empty proper subset of X which is both open and closed. Hence by theorem 7, X is disconnected.

***Ex.*3.** *Let* $X = \{1, 2, 3\}$ *and* $\tau = \{\phi, \{1\}, \{2, 3\}, X\}$, *then* $\{1\}$ *is a proper subset of X which is both open and closed. Hence by theorem* 7, *X is disconnected.*

***Ex.*4.** *Let* $X = \{1, 2, 3\}$ *and* $\tau = \{\phi, \{1\}, \{2\}, \{1, 2\}, X\}$, *then closed sets are* X, $\{2, 3\}$, $\{1, 3\}$, $\{3\}$, ϕ *thus there is no proper subset of X which is both open and closed. Hence X is not disconnected, that is, X is connected space.*

***Ex.*5.** *Let* $X = \{a, b, c, d, e\}$ *and* $\tau = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}\ \{b, c, d, e\}, X\}$, *then X is disconnected as* $\{a\}$ *is a proper subset of X which is both open and closed.*

Let $Y = \{b, d, e\}$ be a subset of X then $\tau_Y = \{\phi, \{d\}, Y\}$ is relativized topology for Y. Here ϕ and Y are only subsets of Y which are both τ_Y-open and τ_Y-closed. Hence is no proper subset of Y which is both open and closed. Thus Y is connected subset of X.

***Ex.*6.** *Every discrete space containing more than one point is disconnected.*

Proof : Let (X, D) be a discrete space containing more than one point. We know that D contains all the possible subsets of X, hence every singleton subset of X is proper subset of X which is both open and closed. Hence X is disconnected.

***Ex.*7.** *Every indiscrete topological space is connected.*

Proof : Let (X, I) be a indiscrete space then $I = \{\phi, X\}$, that is, no proper subset of X is both open and closed. Hence X is not disconnected, so X is connected.

Theorem 9. *Closure of a connected set is connected.*

Proof : Let (X, τ) be a topological space and Y is subset of X such that Y is connected.

Let, if possible, closure of Y, *i.e.* $\overline{Y}$ is disconnected. Then there exists a proper subset A of $\overline{Y}$ such that A is both open and closed in $\overline{Y}$.

A is open in $\overline{Y} \Rightarrow \exists\, G \in \tau$, such that

$$A = G \cap \overline{Y} \qquad .....(1)$$

A is closed in $\overline{Y} \Rightarrow \exists\, H \subset X$, H is τ-closed such that

$$A = H \cap \overline{Y} \qquad .....(2)$$

Now,

$$\begin{aligned} G \cap Y &= G \cap \left(Y \cap \overline{Y}\right) \qquad \left(\because Y \subset \overline{Y}\right) \\ &= \left(G \cap \overline{Y}\right) \cap Y \\ &= A \cap Y \qquad ....(3) \end{aligned}$$

Similarly,

$$H \cap Y = A \cap Y \qquad .....(4)$$

from (3) and (4)

$$G \cap Y = H \cap Y \qquad .....(5)$$

Also, since G is τ-open, therefore $G \cap Y$ is open in Y and since H is τ-closed, therefore $H \cap Y$ is closed in Y. Thus $G \cap Y$ and $H \cap Y$ both are open as well as closed in Y. Since Y is connected, therefore $G \cap Y$ and $H \cap Y$ can not be proper subset of Y, that is, either

$$G \cap Y = \phi \quad \text{or} \quad G \cap Y = Y$$

Case 1. If $G \cap Y = \phi$ and by (3) $G \cap Y = A \cap y$

$\Rightarrow \quad A \cap Y = \phi$

$\Rightarrow \quad Y \subset \overline{Y} - A,$ since $A \subset \overline{Y}$ .....(6)

$\Rightarrow \quad \overline{Y} \subset \overline{\left(\overline{Y} \sim A\right)}$

$\Rightarrow \quad \overline{Y} \subset \overline{Y} - A \qquad \left(\because\ \overline{Y} - A \text{ is closed in } \overline{Y} \text{ as } A \text{ is open in } \overline{Y} \text{ so } \overline{\left(\overline{Y} \sim A\right)} = \overline{Y} - A\right)$

$\Rightarrow \quad A = \phi$

which contradicts the fact that A is proper.

Case 2. If $G \cap Y = Y$, then by (3),

$A \cap Y = Y$

$\Rightarrow \quad Y \subset A$

$\Rightarrow \quad \overline{Y} \subset \overline{A} = A \qquad \left(A \text{ is closed in } \overline{Y} \Rightarrow \overline{A} = A\right)$

$\Rightarrow \quad \overline{Y} \subset A$

But $A \subset \overline{Y}$, So $A = \overline{Y}$

which contradicts the fact that A is proper. Hence $\overline{Y}$ is not disconnected, that is $\overline{Y}$ is connected.

Theorem 10. *Let G be a connected subset of a topological space* (X, τ). *Let* $G \subset A \cup B$, *where A and B are separated sets, then either* $G \subset A$ *or* $G \subset B$.

Proof : Since $G \subset A \cup B$, so $G \cap (A \cup B) = G$

$$\Rightarrow \quad (G \cap A) \cup (G \cap B) = G \qquad .....(1)$$

Now, to shows that $G \subset A$ or $G \subset B$, we shall show that either

$$G \cap A = \phi$$

or

$$G \cap B = \phi.$$

Let us suppose that $G \cap A \neq \phi$ and $G \cap B \neq \phi$.

Also,

$$(G \cap A) \cap \left(\overline{G \cap B}\right) \subset (G \cap A) \cap \left(\overline{G} \cap \overline{B}\right) \qquad \left(\because \overline{G \cap B} \subset \overline{G} \cap \overline{B}\right)$$

$$\Rightarrow \quad (G \cap A) \cap \left(\overline{G \cap B}\right) \subset \left(G \cap \overline{G}\right) \cap \left(A \cap \overline{B}\right) = \left(G \cap \overline{G}\right) \cap \phi = \phi$$

($\because$ A and B are separated, so $A \cap \overline{B} = \phi$)

$$\Rightarrow \quad (G \cap A) \cap \left(\overline{G \cap B}\right) = \phi \qquad .....(2)$$

Similarly, we can show that

$$\left(\overline{G \cap A}\right) \cap (G \cap B) = \phi \qquad .....(3)$$

From (2) and (3), we can conclude that $G \cap A$ and $G \cap B$ are separated sets. From (1), G can be expressed as the union of two non-empty separated sets. This shows that G is disconnected. Which contradicts the fact that G is connected. So one of the sets $G \cap A$ and $G \cap B$ must be empty.

If $G \cap A = \phi$, (1) gives $G \cap B = G \Rightarrow G \subset B$

and if $G \cap B = \phi$, (1) gives $G \cap A = G \Rightarrow G \subset A$

This completes the proof of the theorem.

Theorem 11. *Let G be a connected subset of a topological space* (X, τ). Let $G \subset A \cup B$, *where A and B are disjoint open (closed) subsets of X, then either* $G \subset A$ *or* $G \subset B$.

Proof : By the theorem 3, A and B are separated if they are disjoint and open. By the theorem 10, $G = A \cup B \Rightarrow$ either $G \subset A$ or $G \subset B$.

Theorem 12. *Let G be a connected subset of a topological space* (X, τ). *H is a subset of X such that* $G \subset H \subset \overline{G}$, *then H is connected.*

Proof : Let, if possible H be disconnected. Then there exist two separated sets A and B such that $H = A \cup B$, then

$$G \subset H \Rightarrow G \subset A \cup B$$

then by theorem 10, we have

$$G \subset A \quad \text{or} \quad G \subset B.$$

Suppose $G \subset A$, then

$$\bar{G} \subset \bar{A} \Rightarrow (\bar{G} \cap B) \subset (\bar{A} \cap B)$$

$$\Rightarrow \bar{G} \cap B \subset \phi \qquad (\because\ A, B \text{ are separated so } \bar{A} \cap B = \phi)$$

$$\Rightarrow \bar{G} \cap B = \phi \qquad (\because\ \phi \subset \bar{G} \cap B) \qquad .....(1)$$

Also $\quad H \subset \bar{G} \Rightarrow A \cup B \subset \bar{G} \qquad (\because\ H = A \cup B)$

$$\Rightarrow B \subset \bar{G}$$

$$\Rightarrow \bar{G} \cap B = B \qquad .....(2)$$

From (1) and (2), $\quad B = \phi$

which is a contradiction as A and B is non-empty being separated sets. Thus our assumption is wrong.

Hence H is connected.

Theorem 13. *Union of arbitrary family of connected subset of a topological space is connected if the family is with non-empty intersection.*

Proof : Let X be a topological space. Let $\{G_\alpha \mid \alpha \in \wedge\}$ be a family of subsets of X such that G_α is connected for all α. Let

$$G = \cup \{G_\alpha \mid \alpha \in \wedge\}.$$

We have to show that G is connected. Let, if possible, G is disconnected. Then there exist two separated sets A and B such that

$$G = A \cup B \qquad .....(1)$$

Since given collection is with non-empty intersection therefore

$$\cap \{G_\alpha \mid \alpha \in \wedge\} \neq \phi.$$

Let $x \in \cap \{G_\alpha \mid \alpha \in \wedge\}$ be arbitrary, then

$$x \in G_\alpha \ \forall\ \alpha \in \wedge$$

and $\quad x \in G = A \cup B \quad$ [by (1)]

$\Rightarrow \quad x \in G_\alpha \ \forall\ \alpha \in \wedge$

and $\quad x \in A$ or $x \in B$

$\Rightarrow \quad x \in G_\alpha \cap A, \ \forall\ \alpha \in \wedge, \ $ if $\ x \in A$

$\Rightarrow \quad G_\alpha \cap A \neq \phi, \ \forall\ \alpha \in \wedge \qquad ....(2)$

Now, since G_α is connected for all α, such that

$$G_\alpha \subset A \cup B$$

where A and B are separated sets, therefore by the theorem 10,

either $\quad G_\alpha \subset A$ or $G_\alpha \subset B \qquad .....(3)$

Now, A and B are separated so they are disjoint, thus

$$A \cap B = \phi \Rightarrow G_\alpha \subset A, \forall \alpha \in \wedge \qquad \text{[by (2)]}$$

$$\Rightarrow \bigcup \{G_\alpha \mid \alpha \in \wedge\} \subset A$$

$$\Rightarrow G \subset A$$

$$\Rightarrow G \subset A \qquad (\because \; A \subset G \text{ as } G = A \cup B)$$

$$\Rightarrow B \subset \phi \qquad (\because \; A \cap B = \phi)$$

which is contradiction as B is non-empty.

Hence G is connected.

Theorem 14. *Union of arbitrary family of connected subset of a topological space is connected if one member of the family intersects every other member of the family.*

Proof : Let $\{G_\alpha \mid \alpha \in \wedge\}$ be a family of connected subsets of a topological space X. Also let G_{α_0} be the fixed member of the family such that

$$G_\alpha \cap G_{\alpha_0} \neq \phi, \; \forall \alpha \in \wedge$$

Now, we have to show that $G = \bigcup_{\alpha \in \wedge} G_\alpha$ is connected. Let $G_\alpha \cup G_{\alpha_0} = H_\alpha$, then H_α is connected for all α as it is union of two connected sets having non-empty intersection (by theorem 13)

Now,
$$\bigcup_{\alpha \in \wedge} H_\alpha = \bigcup_{\alpha \in \wedge} \left(G_\alpha \cup G_{\alpha_0}\right)$$

$$= \left(\bigcup_{\alpha \in \wedge} G_\alpha\right) \cup G_{\alpha_0}$$

$$= \bigcup_{\alpha \in \wedge} G_\alpha \qquad \left(\because \; G_{\alpha_0} \subset \bigcup_{a \in \wedge} G_\alpha\right)$$

$$= G \qquad \text{.....(1)}$$

Also,
$$\bigcap_{\alpha \in \wedge} H_\alpha = \bigcap_{\alpha \in \wedge} \left(G_\alpha \cup G_{\alpha_0}\right)$$

$$= \left(\bigcap_{\alpha \in \wedge} G_\alpha\right) \cup G_{\alpha_0} \neq \phi$$

(G_{α_0} intersects G_α, $\forall \alpha \in \wedge$, so $G_{\alpha_0} \neq \phi$).

Thus

$$\bigcup_{\alpha \in \wedge} H_\alpha$$

is connected by theorem 13, being the union of family of connected sets, where family is with non-empty intersection. Hence by (1) G is connected.

Theorem 15. *A subset of R is connected iff it is an interval.*

Proof : Let A be a connected subset of R. Suppose, if possible, A is not an interval. If A is empty or singleton set then there is nothing to prove, so let A contains more than are point. Let $x, y \in A$ such that $x < y$ and $\exists\, r \in R$ such that $x < r < y$ but $r \notin A$, as A is not an interval.

Now,

$$x < r < y \Rightarrow x \in (-\infty, r),\ y \in (r, \infty)$$

$$\Rightarrow x \in A \cap (-\infty, r),\ y \in A \cap (r, \infty)$$

Now, $\quad [A \cap (-\infty, r)] \cup [A \cap (r, \infty)] = A \quad$ .....(1)

and $\quad [A \cap (-\infty, r)] \cap [A \cap (r, \infty)] = A \cap (-\infty, r) \cap (r, \infty)$

$= \phi \qquad [\because (-\infty, r) \cap (r, \infty) = \phi] \quad$ .....(2)

Also, since $(-\infty, r)$ and (r, ∞) are open in R, therefore $A \cap (-\infty, r)$ and $A \cap (r, \infty)$ are open in A. So, by (1) and (2), we can conclude that A is union of two disjoint open sets, hence by theorem 8, A is disconnected, which is a contradiction. Thus A is an interval.

Conversely, suppose that A is an interval. Let if possible A be not connected that is, A is disconnected. Then there exist two non-empty disjoint sets G and H, both are closed in A such that

$$A = G \cup H.$$

Since $G \cap H = \phi$ and G, H are non-empty therefore we can select two elements $x, y \in R$ such that $x \in G, y \in H$ and $x \neq y$, so $x, y \in A$ also. Without loss of generality, we may assume that $x < y$. Since A is an interval and $x, y \in A$, therefore

$$[x, y] \subset A \Rightarrow [x, y] \subset G \cup H.$$

Let p be supremum of $G \cap [x, y]$, then obviously

$$x \leq p \leq y$$

Since p is supremum of $G \cap [x, y]$, therefore for each $\in > 0$, $\exists\ q \in G \cap [x, y]$ such that

$$p - \in < q \leq p. \qquad \text{(by the definition of supremum)}$$

This shows that every neighbourhood of p contains a point of $G \cap [x, y]$ and hence a point of G. Thus p is an adherent point of G, that is, p is a limit pt. of G or $p \in G$. Since G is closed therefore in both the cases $p \in G$. Since G and H are disjoint therefore $p \notin H$. Now, $y \in H$ so $p \neq y$. Thus from (1), we have $p < y$.

Also, $\qquad \forall \varepsilon > 0,\ p + \varepsilon \in H \qquad (\because p$ is sup. of $G \cap [x, y])$

This shows that every neighbourhood of p contains a point of H other than p, as $p \notin H$. Hence p is a limit point of H. Since H is closed, so $p \in H$ as closed set contains all of its limit point. This is a contradiction as $p \notin H$. So our assumption is wrong. Consequently A is connected.

Corollary : The set R of real numbers is connected.

Proof : Since R is an interval, therefore by the theorem 15, R is connected.

Theorem 16. *Continuous image of a connected space is connected.*

Proof : Let X be a connected topological space and Y be any arbitrary topological space. Let $f : X \to Y$ be a continuous mapping of X into Y. We have to show that $f(X)$ is connected. Suppose, if possible, $f(X)$ is disconnected. Then there exist G_1 and G_2 ; both open sets in Y such that

$$f(X) = [G_1 \cap f(x)] \cup [G_2 \cap f(x)] \qquad ......(1)$$

and $G_1 \cap f(x)$, $G_2 \cap f(x)$ both are non-empty disjoint, open sets in $f(X)$.

Now, $f^{-1}[(G_1 \cap f(X)) \cap (G_2 \cap f(X))] = f^{-1}(\phi)$ $[\because G_1 \cap f(x), G_2 \cap f(X)$ are disjoint$]$

$\Rightarrow \quad f^{-1}[(G_1 \cap G_2) \cap f(X)] = \phi \qquad (\because \; f^{-1}(\phi) = \phi)$

$\Rightarrow \quad f^{-1}(G_1) \cap f^{-1}(G_2) \cap X = \phi$

$\Rightarrow \quad f^{-1}(G_1) \cap f^{-1}(G_2) = \phi \qquad [\because \; f^{-1}(G_1) \cap f^{-1}(G_2)$ is subset of $X]$

and $\quad f^{-1}[(G_1 \cap f(X) \cup (G_2 \cap f(X))] = f^{-1}[f(X)] \qquad$ [by (1)]

$\Rightarrow \quad f^{-1}[(G_1 \cup G_2) \cap f(X)] = X$

$\Rightarrow \quad f^{-1}(G_1 \cup G_2) \cap f^{-1}[f(X)] = X$

$\Rightarrow \quad [f^{-1}(G_1) \cup f^{-1}(G_2)] \cap X = X$

$\Rightarrow \quad f^{-1}(G_1) \cup f^{-1}(G_2) = X \qquad (\because f^{-1}(G_1) \cup f^{-1}(G_2)$ is a subset of $X)$

Since f is continuous, therefore $f^{-1}(G_1)$ and $f^{-1}(G_2)$ are non-empty open sets in X as G_1 and G_2 are non-empty open sets in Y. Thus there exist two non-empty proper open subsets of X which we disjoint and hence X in a disconnected space, which contradicts the fact that X in connected , hence $f(X)$ is connected.

Theorem 17. *If every two points of a subset A of a topological space X are contained in some connected subset of A, then A is connected.*

Proof : Suppose, if possible A is not connected. Then there exist two non-empty subsets G and H of X such that $G \cap \bar{H} = \phi$, $\bar{G} \cap H = \phi$ and $G \cup H = A$. Since $G \neq \phi$, $H \neq \phi$, then $\exists p \in G$, $q \in H$ and p and q is contained in some connected subset B of A.

Now, $\qquad B \subset (G \cup H) \Rightarrow B \subset G \quad \text{or} \quad B \subset H \qquad$ (theorem 11)

$\Rightarrow$ either $p, \; q \in G \quad$ or $\quad p, \; q \in H$

Let $\quad p, q \in G$, but $\; q \in H \Rightarrow G \cap H \neq \phi$

which is a contradiction as $G \cap H = \phi$. Consistently A is connected.

13.4.3. Component :

A maximal connected subspace C of a topological space X is called a component of X. In other words, C is component of X iff it is connected and is not contained in any other connected subspace of X.

Notes :

1. If a topological space is not itself connected then it can be decomposed into a disjoint class of maximal connected subspace.
2. Components are always non-empty, since singleton sub- sets of X are always connected.
3. If X is connected then it has only one component, X itself.
4. Each point in X is contained in exactly are component of X.
5. Every component of X is always closed.

13.4 Locally connected space

Definition : A topological space (X, τ) is said to be **locally connected** at a point $x \in X$ iff for every open neighbourhood G of x, $\exists$ connected open neighborhood H of x such that $H \subset G$, that is iff collection of all connected neighbourhood of x forms a local base at x.

Topological space X is said to be **locally connected** iff it is locally connected at each of its points .

Ex.8. *Every discrete space is locally connected.*

Sol. Let (X, D) be a discrete space. Since Every subsets of a discrete space is open therefore $\{x\}$ is open for every $x \in X$. Also $\{x\}$ is connected being a singleton set. Hence $\{x\}$ is connected open neighbourhood of x, which is contained in every open neighbourhood of x. Thus X is locally connected .But a discrete space containing more than one point is disconnected (example 6.)

Ex.9. *Give an example of a locally connected space which is not connected.*

Sol. A discrete space containing more than one point is not connected but is locally connected (example 8). Let us consider another example. Let (R, U) be the usual topological space.

Let (a, b) and (c, d) be two disjoint open intervals on real line. Let $G = (a, b) \cup (c, d)$ and without loss of generally we may assume that $a < b \leq c < d$.

Now, (a, b) is open in $R \Rightarrow (a, b) \cap G$ is open in G

$\Rightarrow (a, b)$ is open in G $\quad (\because \ (a, b) \subset G)$

Similarly, (c, d) is open in G. Thus G is union of two disjoint non-empty open sets, so G is disconnected. Let $x \in G$ be arbitray and let A be any open neighbourhood of x in G, then $\exists\, \varepsilon > o$ such that

$$(x - \varepsilon, x + \varepsilon) \subset A.$$

We know that any interval in R is connected (Theorem 15). So $(x - \varepsilon, x + \varepsilon)$ connected, and hence connected in G. Thus every open neighourhood of x in G contains an open connected neighbourhood of x in G. Hence G is locally connected as x is arbitrary.

Theorem 18. *The image of a locally connected space under a open continuous mapping is locally connected.*

Proof : Let X be a locally connected topological space. Let f be a open and continuous mapping of X onto an arbitrary topological space Y. We have to show $f(X) = Y$ is locally connected.

Let $y \in Y$ be arbitrary and H be any open neighbourhood of y in Y. Since $y \in Y = f(X)$ therefore $\exists\ x \in X$ such that $y = f(x)$.

Also f is continuous, so H is open in $Y \Rightarrow f^{-1}(H)$ is open in X such that $x \in f^{-1}(H)$.

Thus $f^{-1}(H)$ is a open neighbourhood of x. Since X is locally connected, $\exists$ a connected open neighbourhood G of x such that

$$x \in G \subset f^{-1}(H)$$

$$\Rightarrow \quad f(x) \in f(G) \subset H$$

$$\Rightarrow \quad y \in f(G) \subset H \qquad [\because \quad y = f(x)] \qquad ....(1)$$

Since f is open, $f(G)$ is open set in Y. Also, G is connected in X, so $f(G)$ is connected in Y, being the continuous image of connected set **(by theorem 16)**. Thus, from (1), each open neighborhood H of y contains connected open neighbourhood $f(G)$ of y. Since y is arbitrary element of Y, therefore Y is locally connected.

Self-learning exercise-1

Find true and false statements :

1. Separated sets are always disjoint.

2. Disjoint non-empty sets are always separated.

3. Two disjoint non-empty sets are separated if both are open.

4. A topological space X is disconnected iff it is union of two non-empty disjoint open sets.

5. Every discrete space is always connected.

6. Every indiscrete space is always connected.

7. Closure of a connected set is always connected.

8. (R, U) is disconnected.

9. Locally connected space is always connected.

10. Connected space need not be locally connected.

13.5 Summary

In this unit, we have studied about connectedness of a topological space. Different properties of a connected and disconnected space are studied. Locally connectedness and connectedness are two independent properties, of a topological space, that is, local connectedness neither implies nor is implied by connectedness. Also, both are topological properties.

13.6 Answers to self-learning exercises

1 . true	**2.** false	**3.** true	**4.** true
5. false	**6.** true	**7.** true	**8.** false
9. false	**10.** true.		

13.7 Exercises

1. Show that a cofinite topological space X is connected if X is infinite and disconnected if X is finite.

2. Let A and B be two non-empty subsets of a topological space X, such that both are closed in $A \cup B$. If $A \cap B$ and $A \cup B$ are connected then show that A and B a are connected.

3. Show that any finite subset of (R, U) is disconnected.

4. Show that Q the set of rational numbers is disconnected.

5. Show that a space X is connected iff X con not be expressed as the union of two non-empty disjoint open sets.

6. If f is a continuous mapping of a connected space X onto an arbitrary space Y, then show that Y is connected.

7. If A and B are subsets of a space X such that both are open or both are closed then show that $A - B$ and $B - A$ are separated.

8. Let $X = \{a, b, c, d,\}$ and $\tau = \{ \phi, \{b\}, \{b, c\}, \{b, c, d\}, X\}$. Then show that X is connected.

9. If (X, τ) is connected space and $\tau*$ is a topological on X such that it is coarser than τ, then show that $(X, \tau*)$ is also connected.

□ □ □

UNIT 14 : Product and Quotient Spaces

Structure of the Unit

14.0 Objectives

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14.2 Product space of two spaces

14.2.1 Product pace and product topology

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14.2.4 Product space of finite family of topological spaces

14.3 General product pace

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14.4 Quotient space and quotient topology

14.4.1 Quotient topology

14.4.2 Partition of a set and quotient map

14.4.3 Quotient space

14.5 Summary

14.6 Answers to self-learning exercises

14.7 Exercises

14.0 Objectives

In this unit, we shall study about product space of two topological spaces and product space of arbitrary family of topological spaces, related topologies, base, subbase and their properties through related theorems. We shall also discuss quotient space and quotient topology.

14.1 Introduction

Cartesian product of two sets and properties of this product is well known. In this unit we shall study that how can we construct a topology for the cartesian product of two topological spaces. This product will also be topological space for the constructed topology with the help of open sets of both the spaces. After study of product of two spaces, we will be able to understand the properties of product of finite or countable numbers of topological spaces. After this we shall see that the product of arbitrary family of topological space is again a topological space for the topology having certain subbase. A quotient space and quotient topology need a continuous map called quotient map and an equivalence relation of the given space.

14.2 Product space of two spaces

14.2.1 Product space and product topology :

Let (X, τ) and $(Y, \boldsymbol{V})$ be two topological space. The topology $\boldsymbol{W}$ whose base is the set $\boldsymbol{B} = \{G \times H \mid G \in \tau, H \in \boldsymbol{V}\}$ is called the **product topology** for the cartesian product $X \times Y$ and this product is called **product space** of the space X and Y.

Thus, $(X \times Y, \boldsymbol{W})$ is a product space for the topology $\boldsymbol{W}$ whose **base** $\boldsymbol{B}$ is collecting of cartesian products of τ-open and $\boldsymbol{V}$-open sets.

We can verify that $\boldsymbol{B}$ is a base for some topology as follows :

Theorem 1. *Let (X, τ) and $(Y, \boldsymbol{V})$ be two topological spaces. Then the collection $\boldsymbol{B}$ of cartesian products of τ-open sets and $\boldsymbol{V}$-open sets, that is, $\boldsymbol{B} = \{G \times H \mid G \in \tau, H \in \boldsymbol{V}\}$ is a base for some topology for cartesian product $X \times Y$.*

Proof : In order to show that $\boldsymbol{B}$ is a base for some topology on $X \times Y$, it is sufficient to show that $X \times Y$ is the union of members of $\boldsymbol{B}$ and the intersection of any tow members of $\boldsymbol{B}$ is the union of members of $\boldsymbol{B}$.

(i) Since $X \in \tau$ and $Y \in \boldsymbol{V}$, therefore $X \times Y \in \boldsymbol{B}$ and hence $X \times Y \subset U\ \{G \times H \mid G \times H \in \boldsymbol{B}\}$.

Also, $U\ \{G \times H \mid G \times H \in \boldsymbol{B}\} \in X \times Y$ as $G \times H \subset X \times Y\ \forall\ G \times H \in \boldsymbol{B}$ ($\because\ G \subset X\ \forall\ G \in \tau$ and $H \subset Y,\ \forall\ H \in \boldsymbol{V}$) consequently

$$X \times Y = U\ \{G \times H \mid G \times H \in \boldsymbol{B}\}$$

(ii) Let $G_1 \times H_1$ and $G_2 \times H_2$ be any two members of $\boldsymbol{B}$, then

$$(G_1 \times H_1) \cap (G_2 \times H_2) = (G_1 \cap G_2) \times (H_1 \cap H_2) \in \boldsymbol{B}$$

($\because\ G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau, H_1, H_2 \in \boldsymbol{V} \Rightarrow H_1 \cap H_2 \in \boldsymbol{V}$, τ and $\boldsymbol{V}$ being the topologies)

Thus intersection of any two members of $\boldsymbol{B}$ is again a member of $\boldsymbol{B}$, in other words, intersection of any two members of $\boldsymbol{B}$ is union of members of $\boldsymbol{B}$. Hence $\boldsymbol{B}$ is a base for some topology for $X \times Y$.

Theorem 2. *Let* (X, τ) *and* (Y, V) *be two topological spaces with bases* $\boldsymbol{B}_1$ *and* $\boldsymbol{B}_2$ *respectively. Then*

$$\boldsymbol{B}^* = \{B_1 \times B_2 \mid B_1 \in \boldsymbol{B_1}, B_2 \in \boldsymbol{B_2}\}$$

is a base for the product topology $\boldsymbol{W}$ *for* $X \times Y$.

Proof : Let N be a $\boldsymbol{W}$-neighbourhood of $(x, y) \in X \times Y$, where (x, y) is an arbitrary element of $X \times Y$, then we know that

$$\boldsymbol{B} = \{G \times H \mid G \in \tau, H \in V\}$$

is a base of $\boldsymbol{W}$ (theorem 1), then there exists a member $G \times H \in \boldsymbol{B}$ such that

$$(x, y) \in G \times H \subset N \qquad \text{(by the definition of base)} \qquad .....(1)$$

Since $G \in \tau$ and $\boldsymbol{B}_1$ is a base for τ, therefore there exist some $B_1 \in \boldsymbol{B}_1$ such that

$$x \in B_1 \subset G \qquad .....(2)$$

Again, since $H \in V$ and $\boldsymbol{B}_2$ is a base for V, there exist $B_2 \in \boldsymbol{B}_2$ such that

$$y \in B_2 \subset H \qquad .....(3)$$

From (2) and (3), we have

$$(x, y) \in B_1 \times B_2 \subset G \times H \qquad .....(4)$$

Again, from (1) and (4), we have, for arbitrary $(x, y) \in X \times Y$ and for any $\boldsymbol{W}$-neighborhood N of (x, y), there exists such $B_1 \times B_2 \in \boldsymbol{B}^*$ that

$$(x, y) \in B_1 \times B_2 \subset N$$

This shows that $\boldsymbol{B}^*$ is a base for $\boldsymbol{W}$.

***Ex.*1.** *Let* $X = \{1, 2, 3\}$, $\tau = \{\phi, \{1\}, X\}$

and $Y = \{a, b, c\}$, $V = \{\phi\ \{a\}, \{a, c\}, \boldsymbol{Y}\}$

Find a base for the product topology $\boldsymbol{W}$ *on* $X \times Y$.

Sol. Base $\boldsymbol{B}_1$ for τ and $\boldsymbol{B}_2$ for V will be as follows :

$$\boldsymbol{B}_1 = \{\{1\}, X\}$$

and

$$\boldsymbol{B}_2 = \{\{a\}, \{a, c\}, Y\}$$

Now, the base for $\boldsymbol{W}$ is given by

$$\boldsymbol{B}^* = \{\{1\} \times \{a\},\ \{1\} \times \{a, c\}, \{1\} \times Y, X \times \{a\}, X \times \{a, c\}, X \times Y\}$$
$$= \{\{(1, a)\}, \{(1, a), (1, c)\}, \{(1, a), (1, b), (1, c)\}, \{(1, a), (2, a), (3, a)\},$$
$$\{(1, a), (2, a), (3, a), (1, c), (2, c), (3, c)\}, X \times Y\}$$

Theorem 3. *Let* (X, τ) *and* (Y, V) *be two topological spaces.* $\boldsymbol{C}$ *and* $\boldsymbol{D}$ *be sub-bases for* τ *and* $\boldsymbol{V}$ *respectively. Then the collection* $\boldsymbol{A}$ *of all subsets of* $X \times Y$ *of the form* $C \times Y$, $C \in \boldsymbol{C}$ *and* $X \times D$, $D \in \boldsymbol{D}$ *is a sub-bases for the product topology* $\boldsymbol{W}$ *on* $X \times Y$.

Proof : We shall show that collection $\boldsymbol{C}^*$ of finite intersections of members of $\boldsymbol{A}$ form a base for $\boldsymbol{W}$ on $X \times Y$.

Now, $X \times Y \in \boldsymbol{C}^*$, being empty intersection of members of $\boldsymbol{A}$. Let $C_1, C_2, ..., C_n \in \boldsymbol{C}$ and $D_1, D_2, ..., D_n \in \boldsymbol{D}$, then $C_i \times Y$ $(i = 1, 2, ... n)$ and $X \times D_j$ $(j = 1, 2, ... m)$ are the members of $\boldsymbol{A}$. Also, the finite intersection

$$(C_1 \times Y)(C_2 \times Y) \cap ... \cap (C_n \times Y) \cap (X \times D_1) \cap (X \times D_2) \cap ... \cap (X \times D_m) \in \boldsymbol{C}^*$$

$$\Rightarrow \quad [(C_1 \cap C_2 \cap ... \cap C_n) \times Y] \cap [X \times (D_1 \cap D_2 \cap ... \cap D_m)] \in \boldsymbol{C}^*$$

$$(\because A \times (B \cap C) = (A \times B) \cap (A \times C))$$

$$\Rightarrow \quad [(C_1 \cap C_2 \cap ... \cap C_n) \cap X] \times [Y \cap (D_1 \cap D_2 \cap ... \cap D_m)] \in \boldsymbol{C}^*$$

$$(\because (A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D))$$

$$\Rightarrow \quad \bigcap_{i=1}^{n} C_i \times \bigcap_{j=1}^{m} D_i \in \boldsymbol{C}^* \qquad (\because (C_i \subset X, \forall i, D_j \subset Y, \forall j) \qquad .....(1)$$

Let $\boldsymbol{B_1}$ and $\boldsymbol{B_2}$ be bases for τ and $\boldsymbol{V}$ respectively such that they are generated by the elements of $\boldsymbol{C}$ and $\boldsymbol{D}$ respectively. Since finite intersection of members of subbase is a member of base, therefore

$$\bigcap_{i=1}^{n} C_i \in \boldsymbol{B}_1 \quad \text{and} \quad \bigcap_{j=1}^{m} D_j \in \boldsymbol{B}_2$$

Thus $$\bigcap_{i=1}^{n} C_i \times \bigcap_{j=1}^{m} D_j \in \boldsymbol{B}_1 \times \boldsymbol{B}_2 \qquad .....(2)$$

From (1) and (2), we have

$$\boldsymbol{C}^* = \{B_1 \times B_2 \mid B_1 \in \boldsymbol{B}_1, B_2 \in \boldsymbol{B}_2\}.$$

By the theorem 2, $\boldsymbol{C}^*$ is a base for product topology $\boldsymbol{W}$ on $X \times Y$. Since $\boldsymbol{C}^*$ is obtained from the finite intersections of members of $\boldsymbol{A}$, therefore $\boldsymbol{A}$ is a subbase for $\boldsymbol{W}$.

14.2.2 Projection mappings :

$$\pi_x : X \times Y \to X \quad \text{such that} \quad \pi_x((x, y)) = x, \ \forall (x, y) \in X \times Y$$

and $$\pi_y : X \times Y \to Y \quad \text{such that} \quad \pi_y((x, y)) = y, \ \forall (x, y) \in X \times Y$$

are called **projection mappings** of $X \times Y$ onto X and Y respectively.

Theorem 4. *Let (X, τ) and $(Y, \boldsymbol{V})$ be two topological spaces and $(X \times Y, \boldsymbol{W})$ be the product space of X and Y. Then the projection mappings π_x and π_y are continuous and open mappings.*

Proof : Projection mappings π_x and π_y are given as follows :

$$\pi_x : X \times Y \to X, \ \pi_x((x, y)) = x, \ \forall (x, y) \in X \times Y$$

and $$\pi_y : X \times Y \to Y, \ \pi_y((x, y)) = y, \ \forall (x, y) \in X \times Y.$$

Let G be any τ-open set *in* X, then by definition of π_x, we have

$$\pi_x^{-1}(G) = G \times Y.$$

Since $G \in \tau$, $Y \in \boldsymbol{V}$, therefore $G \times Y \in \boldsymbol{B}$, the basis of $X \times Y$. Thus, G is open in X, $\pi_x^{-1}(G)$ is open in $X \times Y$, hence π_x is $\boldsymbol{W} - \tau$ continuous mapping.

Now let W be any $\boldsymbol{W}$-open set in $X \times Y$. Then W can be expressed as union of members of base $\boldsymbol{B}$ for $\boldsymbol{W}$.

So, $$W = U\ \{G \times H \mid G \in \tau, H \in \boldsymbol{V}, G \times H \in \boldsymbol{B'} \subset \boldsymbol{B}\}$$

$$\Rightarrow \quad \pi_x(W) = \pi_x\,[U\ \{G \times H \mid G \in \tau, H \in \boldsymbol{V}, G \times H \in \boldsymbol{B'} \subset \boldsymbol{B}\}]$$
$$= U\ \{\pi_x\,(G \times H) \mid G \in \tau, H \in \boldsymbol{V}, G \times H \in \boldsymbol{B'} \subset \boldsymbol{B}\}$$
$$= U\ \{G \mid G \in \tau, G \times H \in \boldsymbol{B'} \subset \boldsymbol{B}\} \qquad \text{(by the definition of } \pi_x)$$

Since union of arbitrary family of open sets is open therefore,

$$U\ \{G \mid G \in \tau, G \times H \in \boldsymbol{B'} \subset \boldsymbol{B}\} \in \tau$$

$$\Rightarrow \qquad \pi_x\,(W) \in \tau$$

Thus, W is $\boldsymbol{W}$-open in $X \times Y$, $\pi_x\,(W)$ is τ-open in X. Hence π_x is $\tau - \boldsymbol{W}$ continuous and open mapping.

Similarly, we can prove that π_y is a continuous and open mapping.

Theorem 5. *Product topology $\boldsymbol{W}$ is the weakest (coarsest) topology for which prodjections are continuous.*

Proof : Let $\boldsymbol{W'}$ be any topology for $X \times Y$ for which the projection mappings are continuous and let W be any $\boldsymbol{W}$-open subset of $X \times Y$, then by the definition of base $\boldsymbol{B}$ for $\boldsymbol{W}$, we have

$$W = U\ \{G \times H \mid G \in \tau, H \in \boldsymbol{V} \text{ and } G \times H \in \boldsymbol{B'} \subset \boldsymbol{B}\}$$
$$= U\ \{G \cap X) \times (Y \cap H) \mid G \in \tau, H \in \boldsymbol{V} \text{ and } G \times H \in \boldsymbol{B'} \subset \boldsymbol{B}\}$$
$$(\because\ G \subset X, H \subset Y)$$
$$= U\ \{(G \times Y) \cap (X \times H) \mid G \in \tau, H \in \boldsymbol{V}, G \times H \in \boldsymbol{B'} \subset \boldsymbol{B}\}$$
$$= U\{\pi_x^{-1}\ (G) \cap \pi_y^{-1}\ (H) \mid G \in \tau, H \in \boldsymbol{V} \text{ and } G \times H \in \boldsymbol{B'} \subset \boldsymbol{B}\}$$
$$\text{(by the definition of } \pi_x \text{ and } \pi_y)$$

Now, $\pi_x^{-1}(G) \in \boldsymbol{W'}$, whenever $G \in \tau$, as π_x is $\boldsymbol{W'} - \tau$ continuous. Similarly $\pi_y^{-1}(H) \in \boldsymbol{W'}$. Thus

$$U\ \{\pi_x^{-1}\ (G) \cap \pi_y^{-1}\ (H) \mid G \in \tau, H \in \boldsymbol{V} \text{ and } G \times H \in \boldsymbol{B'} \subset \boldsymbol{B}\}$$

is $\boldsymbol{W'}$-open and hence W is $\boldsymbol{W'}$-open.

So $W \in \boldsymbol{W} \Rightarrow W \in \boldsymbol{W'}$ which shows that $\boldsymbol{W} \subset \boldsymbol{W'}$. Consequently $\boldsymbol{W}$ is the weakest topology for which projections are continuous.

14.2.3 Product invariant properties :

Compactness, countability etc. are product invariant properties for finite products. We shall establish this fact by proving some theorems.

Theorem 6. *The product space* $(X \times Y, \boldsymbol{W})$ *is hausdorff if the space* (X, τ) *and* $(Y, \boldsymbol{V})$ *are Hausdorff.*

Proof : Let the spaces X and Y be Hausdorff. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$ such that $(x_1, y_1) \neq (x_2, y_2)$. Then either $x_1 \neq x_2$ or $y_2 \neq y_2$. Let us suppose that $x_1 \neq x_2$. Since X is Hausdorff and x_1, x_2 are two distinct points of X, therefore there exists two disjoint open neighbourhoods G_1 and G_2 of x_1 and x_2 of respectively. Now, $G_1 \times Y$ and $G_2 \times Y$ are open subsets of $X \times Y$ such that

$$(x_1, y_1) \in G_1 \times Y, \quad (x_2, y_2) \in G_2 \times Y$$

and $$(G_1 \times Y) \cap (G_2 \times Y) = (G_1 \cap G_2) \times Y = \phi \times Y \qquad (\because \; G_1 \cap G_2 = \phi)$$

This shows that $X \times Y$ is a Hausdorff space.

Theorem 7. *The product space* $(X \times Y, \boldsymbol{W})$ *is second countable if* (X, τ) *and* $(Y, \boldsymbol{V})$ *is second countable.*

Or

The product space of two second countable spaces is second countable.

Proof : Let X and Y be two second countable spaces. Let $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ be countable bases for X and Y respectively. Then by theorem 2, set

$$\boldsymbol{B}^* = \{B_1 \times B_2 \mid B_1 \in \boldsymbol{B_1}, B_2 \in \boldsymbol{B_2}\}$$

is a base for product space $X \times Y$. Also, since $\boldsymbol{B_1}$ and $\boldsymbol{B_2}$ are countable, therefore $\boldsymbol{B}^*$ is also countable. Thus $\boldsymbol{B}^*$ is a countable base for product space $X \times Y$. Hence $X \times Y$ is a second countable space.

Theorem 8. *The product space $X \times Y$ is connected if and only if X and Y are connected.*

Proof : First suppose that $X \times Y$ is connected. Mappings

$$\pi_x : X \times Y \to X, \; \pi\,((x, y)) = x, \; \forall\, (x, y) \in X \times Y$$

and $$\pi_y : X \times Y \to Y, \pi\,((x, y)) = y, \; \forall\, (x, y) \in X \times Y$$

are continuous, then X and Y are continuous images of $X \times Y$. Since continuous image of a connected space is connected, therefore X and Y are also connected (Theorem 16, unit-13).

Conversely, suppose X and Y are connected spaces. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$, then $\{x_1\} \times Y$ is homeomorphic to Y and $X \times \{y_2\}$ are continuous images of X and Y respectively, hence $\{x_1\} \times Y$ and $X \times \{y_2\}$ are connected. Also, since

$$(x_1, y_2) \in [\{x_1\} \times Y] \cap [X \times \{y_2\}],$$

therefore $$[\{x_1\} \times Y] \cap [X \times \{y_2\}] \neq \phi$$

so, by the theorem 13 (unit-13)

$$[\{x_1\} \times Y] \cup [X \times \{y_2\}] = E \text{ (say)},$$

is connected. Since (x_1, y_1) and (x_2, y_2) are two arbitrary points of $X \times Y$ and are contained in a connected set $(\{x_1\} \times Y) \cup (X \times \{y_2\})$.

Hence $X \times Y$ is also connected.

Theorem 9. *The product space* $(X \times Y, \boldsymbol{W})$ *is compact if and only if each of the spaces* (X, τ) *and* $(Y, \boldsymbol{V})$ *is compact.*

Proof : First suppose, $X \times Y$ is compact. Since X and Y are continuous images of $X \times Y$ under the projection mappings π_x and π_y respectively, therefore X and Y are compact spaces.

Conversely, suppose that X and Y both are compact. In order to show that $X \times Y$ is compact, we shall show that every basic open cover of $X \times Y$ is reducible to a finite sub-cover. Let

$$\boldsymbol{C} = \{G_\alpha \times H_\alpha \mid \alpha \in \Lambda\}$$

be a basis open cover of $X \times Y$, where $G_\alpha \in \tau$, $H_\alpha \in \boldsymbol{V}$.

Let $x \in X$, then $\{x\} \times Y$ is homeomorphic to Y and hence it is compact. Since $\{x\} \times Y \subset X \times Y$, therefore $\boldsymbol{C}$ is a basic open cover of $\{x\} \times Y$. Since $\{x\} \times Y$ is compact, therefore

$$\{x\} \times Y \subset \bigcup_{i=1}^{n} \left(G_{\alpha_i} \times H_{\alpha_i}\right), \quad \text{for some indices } \alpha_i \in \Lambda$$

$$\Rightarrow \qquad x \in G_{\alpha_i}, \quad \forall i$$

$$\Rightarrow \qquad x \in \bigcap_{i=1}^{n} G_{\alpha_i} = G(x) \text{ (say)},$$

where $G(x)$ is open set being the finite intersection of open sets.

$$\Rightarrow \qquad \{x\} \times Y \subset \bigcup_{i=1}^{n} \left(G(x) \times H_{\alpha_i}\right).$$

Thus for each $x \in X$, we can get open set $G(x)$ such that the collection $\{G(x) \mid x \in X\}$ is an open cover of X. Since X is compact, therefore this cover is also reducible to a finite subcover that is,

$$X \subset \bigcup_{j=1}^{m} G(x_j), \quad \text{for some } x_j \in X. \qquad .....(1)$$

Each $G(x_j)$ is obtained by intersection of τ-open sets, which are members of $\boldsymbol{C}$. Let one of then is $G_{\alpha_{x_j}}$, then

$$G(x_j) \subset G_{\alpha_{x_j}}, \quad \text{for} \quad j = 1, 2, \ldots m.$$

Hence, from (1), $$X \subset \bigcup_{j=1}^{m} G_{\alpha_{x_j}} \qquad .....(2)$$

and for each j ($j = 1, 2,..., m$)

$$G(x_j) \times Y \subset \bigcup_{i=1}^{n} G_{\alpha_{x_j}} \times H_{\alpha_i} \qquad .....(3)$$

From (1), (2) and (3), the collection $\{G_{\alpha_{x_j}} \times H_{\alpha_i} \mid j = 1, 2,... m, i = 1, 2,... n\}$ covers $X \times Y$, which is finite subcover of $\boldsymbol{C}$. Thus $\boldsymbol{C}$ is reducible to a finite subcover. Hence $X \times Y$ is compact.

14.2.4 Product space of finite family of topological spaces :

Let (X_i, τ_i), $i = 1, 2,..., n$ be n topological spaces. Then the collection

$$\boldsymbol{B} = \{G_1 \times G_2 \times G_3 \times ... \times G_n \mid G_i \in \tau_i, i = 1, 2,..., n\}$$

is a base for a topology for $X_1 \times X_2 \times X_3 \times ... \times X_n$. This topology is called product topology and product $X_1 \times X_2 \times X_3 \times ... \times X_n$ is called product space of finite number of topological spaces.

The mapping $\pi_{x_i} : X_1 \times X_2 \times X_3 \times ... \times X_n \to X_i$, such that

$$\pi_{x_i}((x_1, x_2, x_3, ..., x_n)) = x_i, \ \forall (x_1, x_2, x_3, ..., x_n) \in X_1 \times X_2 \times X_3 \times ... \times X_n$$

is called ith projection mapping (or simply ith projection).

Theorems discussed above are also valid for this finite product and projection mappings.

14.3 General product space

Before defining general product topology and space, we shall discuss some notations and definitions related to product of arbitrary collection of sets. Since general product topology is defined in terms of subbase and subbase is defined in terms of inverse image under projection mappings, therefore, here we shall define projection mappings of arbitrary product of topological spaces.

14.3.1 Coordinate :

Let $\{A_\lambda \mid \lambda \in \Lambda\}$ be an arbitrary family of indexed sets, also let product of this family is $\boldsymbol{A}$, then

$$\boldsymbol{A} = \times \{A_\lambda \mid \lambda \in \Lambda\}.$$

An element a of A is a mapping

$$a : \Lambda \to U \{A_\lambda \mid \lambda \in \Lambda\}$$

such that $\quad a(\lambda) = a_\lambda \in A_\lambda, \ \forall \lambda \in \Lambda$.

Here a_λ is called λth coordinate of a and A_λ is called λth coordinate set of the product.

Thus, $\quad A = \{a \mid a : \Lambda \to x, a(\lambda) \in A_\lambda, \ \forall \lambda \in \Lambda\}.$

14.3.2 Projection mapping :

The mapping $\quad \pi_\lambda : A \to A_\lambda$, such that $\pi_\lambda(a) = a_\lambda, \ \forall a \in A$

is called **projection mapping** on A or λth projection of A.

Let $\boldsymbol{B}_\lambda$ be a subset of A_λ then the set $\pi_\lambda^{-1}(\boldsymbol{B}_\lambda)$ is the set of all $a \in A$ whose λth coordinates are the members of $\boldsymbol{B}_\lambda$ and other coordinates are unrestricted. Thus

$$\pi_\lambda^{-1}(\boldsymbol{B}_\lambda) = \{a \mid a \in A, \pi_\lambda(a) = a_\lambda \in B_\lambda\}$$
$$= \times \{Y_\alpha \mid \alpha \in \Lambda\},$$

where $Y_\lambda = \boldsymbol{B}_\lambda$ and $Y_\alpha = A_\alpha \ (\alpha \neq \lambda)$.

14.3.3 Embedding :

A mapping $f : X \to Y$, which defines a homeomorphism of X onto $f(X)$ is said to be an **embedding** of a space X into another space Y.

For example, including map i of subspace X of a space Y to space Y, define as $i(x) = x$, $\forall x \in X$ is an embedding of X into Y.

14.3.4 General product space and Tychonoff topology :

Let $\{\{X_\lambda, \tau_\lambda\} \mid \lambda \in \Lambda\}$ be an arbitrary family of topological spaces. Let $X = \times \{X_\lambda \mid \lambda \in \Lambda\}$, then the topology τ for product X having the subbase

$$\boldsymbol{B}^* = \{\pi_\lambda^{-1}(G_\lambda) \mid, G_\lambda \in \tau_\lambda\}$$

is called **product topology** or **Tychonoff topology** for X and (X, τ) is called general product space or simply product space of the given spaces.

Notes : 1. Now onwards product topology means the topology generated by the collection of all sets of the form $\pi_\lambda^{-1}(G_\lambda)$, $\lambda \in \Lambda$, $G_\lambda \in \tau_\lambda$.

2. If X is a product of countable collection $\{X_1, X_2, X_3, ...\}$ of topological spaces, then

$$X = \times \{X_n \mid n \in N\}, \quad x \in X \Rightarrow x = (x_1, x_2, ...), x_n \in X_n$$

and $$\pi_m^{-1}(G_m) = X_1 \times X_2 \times ... X_{m-1} \times G_m \times X_{m+1} \times ...$$

3. $\pi_\lambda^{-1}(G_\lambda)$ is subbasic open in X. The collection $\boldsymbol{B}$ of all finite intersections of elements of subbase $\boldsymbol{B}_*$ form a base for topology τ. Thus if $B \in \boldsymbol{B}$, then

$$B = \cap \{\pi_\lambda^{-1}(G_\lambda) \mid \lambda \in \Lambda', \ \Lambda' \text{ is finite subset of } \Lambda, G_\lambda \in \tau_\lambda\}$$
$$= \times \{Y_\lambda \mid \lambda \in \Lambda\}, \quad \text{where} \quad Y_\lambda = X_\lambda$$

for all except a finite number of λ' is in Λ, is a basic open set in X.

4. Every τ-open set $\boldsymbol{G}$ in product space X will contain a base member B. So all but a finite number of coordinates of points of G are unrestricted in respective of coordinate spaces.

5. The collection $\times \{G_\lambda \mid \lambda \in \Lambda, G_\lambda \in \tau_\lambda\}$ is also a base for some topology, different from τ. Since τ is more important than the topology obtained from above collection, therefore definition of general product space is not the extension of the definition for product space of two space.

6. Since $G_\lambda \in \tau_\lambda \Rightarrow \pi_\lambda^{-1}(G_\lambda) \in \boldsymbol{B}_* \subset \tau \Rightarrow \pi_{\lambda^{-1}}(G_\lambda) \in \tau$, that is inverse image of τ_λ-open set is τ-open, therefore π_λ is continuous mapping.

Theorem 10. *Let X be a product space of an arbitrary collection* $\{(X_\lambda, \tau_\lambda) \mid \lambda \in \Lambda\}$ *of topological spaces. Then* τ *is the topology for X iff* τ *is the smallest topology for which the projections are continuous.*

Proof : First suppose that τ is the topology for X. We know that each projection map π_λ, $\forall\, \lambda \in \Lambda$ is continuous (Note 6 of 14.3). Let τ^* be any topology on X such that π_λ is $\tau^* - \tau_\lambda$ continuous for each $\lambda \in \Lambda$. Then for every $G_\lambda \in \tau_\lambda$, $\pi_\lambda^{-1}(G_\lambda)$ is τ^*-open, $\forall\, \lambda \in \Lambda$. Thus by the definition of topology, union of finite intersections of members of the collection $\{\pi_\lambda^{-1}(G_\lambda) \mid \lambda \in \Lambda, G_\lambda \in \tau_\lambda\}$ is a member of τ^*. This shows that $\tau \subset \tau^*$ as $\{\pi_\lambda^{-1}(G_\lambda) \mid \lambda \in \Lambda, G_\lambda \in \tau_\lambda\}$ is a subbase of τ. Hence τ is smallest topology for X such that π_λ is continues for each $\lambda \in \Lambda$.

Conversely, suppose that τ is the smallest topology for X for which each π_λ is continuous. Let

$$B_* = \{\pi_\lambda^{-1}(G_\lambda) \mid \lambda \in \Lambda, G_\lambda \in \tau_\lambda\}.$$

By the property of continuous mapping, if τ^* is topology for X, then all the projections π_λ are $\tau^* - \tau_\lambda$ continuous iff $\boldsymbol{B}_* \subset \tau^*$. Hence $\boldsymbol{B}_* \subset \tau$ and since τ is the smallest topology containing $\boldsymbol{B}_*$, therefore τ is generated by $\boldsymbol{B}_*$, that is, B_* is a subbase for τ. Hence by the definition of product topology (14.3.4) , τ is a product topology for X.

Theorem 11. *Let* (X, τ) *be a product space of arbitrary family* $\{(X_\lambda, \tau_\lambda) \mid \lambda \in \Lambda\}$ *of topological spaces. Then projection mapping* π_λ *for* λ *is continuous and open.*

Proof : According to note 6 of (14.3) π_λ is continuous for $\lambda \in \Lambda$. Now, it remains to show that π_λ is open, that is, π_λ is $\tau_\lambda - \tau$ continuous for each $\lambda \in \Lambda$. For this, we shall show that image of every τ-open set in X under π_λ is τ_λ-open in X_λ. Let $\boldsymbol{B}$ be the base for τ and $B \in \boldsymbol{B}$ be arbitrary. Then (by note 3 of 14.3),

$$B = \times \{Y_\lambda \mid \lambda \in \Lambda\}, \quad \text{where } Y_\lambda \in \tau_\lambda \text{ for each } \lambda \in \Lambda$$

and $\qquad Y_\lambda = X_\lambda$ for all except a finite number of λ' s.

Now, $\qquad \pi_\lambda(B) = Y_\lambda \in \tau_\lambda$,

that is image of every basic open set of X under π_λ is τ_λ-open in X_λ. Let G be any τ-open set in X. Then G is union of members of base $\boldsymbol{B}$. Then

$$G = U\,\{B \mid B \in \boldsymbol{B}' \subset \boldsymbol{B}\}$$

$$\Rightarrow \quad \pi_\lambda(G) = \pi_\lambda\,[U\,\{B \mid B \in \boldsymbol{B}' \subset \boldsymbol{B}\}]$$

$$= U\,\{\pi_\lambda(B) \mid B \in \boldsymbol{B}' \subset \boldsymbol{B}\}$$

Since $\pi_\lambda(B)$ is open set in X_λ for each $B \in \boldsymbol{B'}$, therefore $\pi_\lambda(G)$ is open in X_λ, being the union of open sets in X_λ.

Thus $\qquad G \in \tau \Rightarrow \pi_\lambda(G) \in \tau_\lambda, \quad \forall \lambda \in \Lambda.$

Hence π_λ is an open mapping for each $\lambda \in \Lambda$.

Theorem 12. *Let X be a non-empty product space of arbitrary family $\{(X_\lambda, \tau_\lambda) \mid \lambda \in \Lambda\}$ of topological spaces. Then a non-empty product subset $F = \times \{F_\lambda \mid \lambda \in \Lambda\}$ is closed in X iff F_λ is closed in X_λ, for each $\lambda \in \Lambda$.*

Proof : First suppose that F_λ is τ_λ-closed for each $\lambda \in \Lambda$. Then $\pi_\lambda^{-1}(F_\lambda)$ is τ-closed in X, since π_λ is a continuous mapping. Now,

$$\begin{aligned} F &= \times \{F_\lambda \mid \lambda \in \Lambda\} \\ &= \cap \{\pi_\lambda^{-1}(F_\lambda) \mid \lambda \in \Lambda\} \end{aligned}$$

Thus, F is τ-closed being the intersection of τ-closed sets.

Conversely, suppose that $F = \times \{F_\lambda \mid \lambda \in \Lambda\}$ is τ-closed. Let $\alpha \in \Lambda$ be arbitrary and $x_\alpha \in X_\alpha$ be any limit point of F_α.

Let $x \in X$ such that $\pi_\alpha(x) = x_\alpha$ and $\pi_\lambda(x)$ be arbitrary element of F_λ for $\lambda \neq \alpha$. Let $\boldsymbol{B}$ be any basic open set in X such that $x \in B$. Since π_α is open, therefore $\pi_\alpha(B)$ is open in X_α and $x_\alpha \in \pi_\alpha(B)$. Since x_α is a limit point of F_α, $\pi_\alpha(B)$ contains a point $a_\alpha \in F_\alpha$, that

$$\pi_\lambda(z) = \pi_\lambda(x), \quad \text{for } \lambda \neq \alpha \quad \text{and} \quad \pi_\lambda(z) = a_\alpha.$$

Obviously $z \in F$ and z and x differ in αth coordinate, so $x \neq z$. Thus, every basic open set B in X contains a point of F different from x. Hence x is a limit point of F, so $x \in F$ as F is closed. This shows that $\pi_\alpha(x) \in F_\alpha \Rightarrow x_\alpha \in F_\alpha$. Thus F_α contains all of its limit point, hence F_α is closed. Since α was arbitrary, F_λ is closed for each $\lambda \in \Lambda$.

Theorem 13. *The product space $X = \times \{X_\lambda \mid \lambda \in \Lambda\}$ is Hausdorff if and only if each space X_λ is Hausdorff.*

Proof : First suppose that X_λ is Hausdorff for each $\lambda \in \Lambda$. Let $x = \{x_\lambda \mid \lambda \in \Lambda\}$ and $y = \{y_\lambda \mid \lambda \in \Lambda\}$ be two distinct point of the product space X. Since $x \neq y$, therefore $x_\alpha \neq y_\alpha$ for some $\alpha \in \Lambda$ and $x_\alpha, y_\alpha \in X_\alpha$. Since X_α is Hausdorff and $x_\alpha \neq y_\alpha$, therefore, there exists open neighbourhoods G_α and H_α of x_α and y_α respectively such that

$$G_\alpha \cap H_\alpha = \phi, \; x_\alpha \in G_\alpha, \; y_\alpha \in H_\alpha$$

Since π_α is continuous, therefore $\pi_\alpha^{-1}(G_\alpha)$ and $\pi_\alpha^{-1}(H_\alpha)$ are open sets in X such that

$$x \in \pi_\alpha^{-1}(G_\alpha) \quad \text{and} \quad y \in \pi_\alpha^{-1}(H_\alpha) \qquad (\because \pi_\alpha(x) = x_\alpha \text{ and } \pi_\alpha(y) = y_\alpha)$$

and $\qquad \pi_\alpha^{-1}(G_\alpha) \cap \pi_\alpha^{-1}(H_\alpha) = \pi_\alpha^{-1}(G_\alpha \cap H_\alpha) = \phi \qquad (\because G\alpha \cap H_\alpha = \phi)$

Thus for each pair of distinct points $x, y \in X$, there exist two disjoint open neighbourhoods of x and y respectively. Hence X is a Hausdorff space.

Conversely, suppose that X is Hausdorff. Let $\alpha \in \Lambda$ be arbitrary and $x_\alpha, y_\alpha \in X_\alpha$ such that $x_\alpha \neq y_\alpha$. Let $x, y \in X$ such that

$$\pi_\alpha(x) = x_\alpha, \quad \pi_\alpha(y) = y_\alpha \quad \text{and} \quad \pi_\lambda(x) = \pi_\lambda(y),$$

each $\lambda \in \Lambda$ except $\lambda = \alpha$.

Thus x and y differ only in the αth coordinate, and hence $x \neq y$. Since X is Hausdorff, there exist open neighourhoods G and H of x and y respectively such that

$$G \cap H = \phi$$

Let B and C be two basic open sets in X such that

$$x \in B \subset G \quad \text{and} \quad y \in C \subset H \qquad .....(1)$$

and $\quad B = \times \{B_\lambda \mid \lambda \in \Lambda\},\ C = X\{C_\lambda \mid \lambda \in \Lambda\},$

where $\quad B_\lambda, C_\lambda \in \tau_\lambda$ for each λ.

Since $\quad G \cap H = \phi \Rightarrow B \cap C = \phi$

$$\Rightarrow \pi_\alpha(B) \cap \pi_\alpha(C) = \phi$$

$$\Rightarrow B_\alpha \cap C_\alpha = \phi$$

where B_α and C_α are open sets in X_α such that

$$\pi_\alpha(x) = x_\alpha \in B_\alpha, \quad \pi_\alpha(y) = y_\alpha \in C_\alpha \text{ (by (1))}$$

Thus, for two distinct points $x_\alpha, y_\alpha \in X_\alpha$, there exist two open set B_α and C_α in X_α such that

$$x_\alpha \in B_\alpha,\ y_\alpha \in C_\alpha \quad \text{and} \quad B_\alpha \cap C_\alpha = \phi.$$

Hence X_α is Hausdordd. Since α was arbitrary therefore X_α is Hausdorff for each $\lambda \in \Lambda$.

Theorem 14. *The product space $X = \times \{X_\lambda \mid \lambda \in \Lambda\}$ is connected if each space X_λ is connected.*

Proof : First suppose that X_λ is connected for each $\lambda \in \Lambda$. Let a be a fixed point of the product space X, then $a = \{a_\lambda \mid \lambda \in \Lambda\}$, where $a_\lambda \in X_\lambda,\ \forall \lambda \in \Lambda$. Let C be the component of X such that $a \in C$. Let

$$B = \times \{Y_\lambda \mid \lambda \in \Lambda\}$$

be an arbitrary basic open set, where

$$Y_\lambda = G_\lambda \in \tau_\lambda,$$

for each $\lambda \in \Lambda$ and $Y_\lambda = X_\lambda$ for all $\lambda \in \Lambda$ except a finite number of λ's, $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ (say). Let

$$x = \{x_\lambda \mid \lambda \in \Lambda\} \in B.$$

The set $\quad \boldsymbol{A} = A_{\lambda_1} \times A_{\lambda_2} \times \ldots \times A_{\lambda_n}$

is the set of all point $\{p_\lambda \mid \lambda \in \Lambda\}$ such that

$$p_\lambda = a_\lambda \text{ if } \lambda \neq \lambda_1, \lambda_2, ..., \lambda_n,$$

which is homeomorphic to $X_{\lambda_1} \times X_{\lambda_2} \times ... \times X_{\lambda_n}$, which is connected (by theorem 8). Hence its homeomosphic image A is also connected. Since C is maximal connected subset of X, therefore $A \subset C$. Then set C contains the point $\{p_\lambda \mid \lambda \in \Lambda\}$,

$$p_\lambda = a_\lambda \text{ if } \lambda \neq \lambda_1, \lambda_2, ..., \lambda_n,$$

and $$p_\lambda = x_\lambda \text{ for } \lambda \neq \lambda_1, \lambda_2, ..., \lambda_n.$$

This point is in B also. This shows that $x \in \bar{C} = C$, ($C = \bar{C}$ as C is closed being the component). Thus $X \subset C$, but $C \subset X$ so $X = C$. Since C is connected therefore X is connected.

Conversely, let X be connected. Since X_λ is continuous image of X under the project mapping π_λ, therefore X_λ is connected for each $\lambda \in \Lambda$. (Theorem 16, Unit 13).

14.3.5. Finitely short family :

A collection ***C*** of subsets of a topological space (X, τ) is **short** iff ***C*** does not cover X and ***C*** is finitely short iff no finite subfamily of ***C*** covers X. ***C*** is said to be a maximal finitely short iff for each $G \in \tau$, $G \notin$ ***C***, there exists a finite subfamily ***C′*** of ***C*** such that union of members of ***C′*** together with G covers X.

Lemma 1. Let ***M*** be a maximal finitely short family of open subsets of a topological space (X, τ). If some member M of ***M*** is such that ***M*** contains

$$(G_1 \cap G_2 \cap ... \cap G_n), \quad G_i \in \tau \quad \text{for} \quad i = 1, 2, ..., n,$$

then for some i, $G_i \in$ ***M***.

Proof : We shall prove contra positive of the statement. Let $G_1, G_2 \in \tau$ such that $G_1 \notin$ ***M*** and $G_2 \notin$ ***M***.

Then by the definition of maximal finitely short family, for G_1 there exists

$$M_1, M_2, ..., M_r \in \boldsymbol{M}$$

such that $$G_1 \cup M_1 \cup M_2 \cup ... \cup M_r = X \qquad .....(1)$$

and for G_2 there exists $N_1, N_2, ...N_s \in$ ***M*** such that

$$G_2 \cup N_1 \cap N_2 \cap .... \cap N_s = X \qquad ......(2)$$

From (1) and (2), we have

$$(G_1 \cap G_2) \cup (M_1 \cup M_2 \cup .... \cup M_r) \cup (N_1 \cup N_2 \cup .... \cup N_s) = X.$$

Since ***M*** is finitely short, there fore neither $G_1 \cap G_2 \in$ ***M***, nor $(G_1 \cap G_2)$ is contained in any member of ***M. Thus*** we have shown that

"$G_i \notin$ ***M*** for $i = 1, 2 \Rightarrow G_1 \cap G_2$ is not contained in any member of ***M***"

The contra positive of this statement is " $G_1 \cap G_2$ is contained in some member of ***M*** $\Rightarrow$ either $G_1 \in$ ***M*** or $G_2 \in$ ***M***."

Thus lemma is true for $n = 2$ and by finite induction lemma is true for any positive integer n.

Lemma 2. Let $\boldsymbol{F}$ be a finitely short family of open sets of a topological space (X, τ). Then there exists a maximal finitely short subfamily $\boldsymbol{M}$ of τ such that $\boldsymbol{F} \subset \boldsymbol{M}$.

Proof : Let $\boldsymbol{D}$ be the collection of all finitely short subfamilies of τ. Then $\boldsymbol{D}$ is partially ordered set for the relation inclusion. Now, $\boldsymbol{F} \in \boldsymbol{D}$ and $\{\boldsymbol{F}\}$ is an ordered set. By Hausdorff maximal principle, $\exists$ a maximal ordered (chain) subfamily $\boldsymbol{D}'$ of $\boldsymbol{D}$ such that $\{\boldsymbol{F}\} \subset \boldsymbol{D}'$, which shows that $\boldsymbol{F} \in \boldsymbol{D}'$. Let $\cup \boldsymbol{D}' = \boldsymbol{M}$, then $\boldsymbol{M} \subset \tau$. Let $\{M_i \mid i = 1, 2, \ldots, n\}$ be any finite subfamily of $\boldsymbol{M}$ them

$$M_i \in M, \ \forall \ i \Rightarrow M_i \in D_i \text{ for some } \boldsymbol{D}_i \in \boldsymbol{D}', \text{ for } i = 1, 2, \ldots, n. \qquad \ldots..(1)$$

Since D' is a chain, there is one $i = r$ (say) such that

$$\boldsymbol{D}_i \subset \boldsymbol{D}_r, \ \forall \ i. \qquad \ldots...(2)$$

From (1) and (2), we have, $M_i \in \boldsymbol{D}_r, \ \forall \, i$.

Since $\boldsymbol{D}_r \in \boldsymbol{D}' \subset \boldsymbol{D}$, therefore $\boldsymbol{D}_r$ is finitely short family, hence $\{M_i \mid i = 1, 2, \ldots, n\}$ does not cover X, that is, $M_1 \cup M_2 \cup \ldots \cup M_n \neq X$. Thus M is finitely short. Now we shall show that $\boldsymbol{M}$ is maximal. Let M is not maximal, then for some $G \in \tau$, $G \notin \boldsymbol{M}$ and $\boldsymbol{M} \cup \{G\}$ is still finitely short. Since $\cup \boldsymbol{D}' = \boldsymbol{M}$, $\boldsymbol{M}$ contains each member of $\boldsymbol{D}'$, this show that $\boldsymbol{D}' \cup \{\boldsymbol{M} \cup \{G\}\}$ would be simply ordered. Since $G \notin \boldsymbol{M} = \cup \boldsymbol{D}' \Rightarrow \boldsymbol{D}' \cup \{M \cup \{G\}\}$ properly contains $\boldsymbol{D}'$. This is a contradiction as $\boldsymbol{D}'$ is maximal. Thus $\boldsymbol{M}$ is maximal finitely short subfamily of τ such that $\boldsymbol{F} \subset \boldsymbol{M}$.

Theorem 15. (Alexander subbase Lemona) : *A topological space (X, τ) is compact iff every subbasic open cover for X has a finite subcover.*

or

A topological space is compact iff each finitely short subfamily of subbasic open sets is short.

Proof. Let X be compact. Then every open cover for X reducible to a finite subcover, hence every subbasic open cover must be reducible.

Conversely, suppose that every subbasic open cover for X has a finite subcover. We have to show that every open cover for X is reducible. We shall prove contra positive of this statement, that is, each finitely short subfamily of τ is short. Let $\boldsymbol{F}$ be any finitely short subfamily of τ. By lemma 2, $\exists$ a maximal finitely short subfamily of τ. Let it be $\boldsymbol{M}$. Then $\boldsymbol{F} \subset \boldsymbol{M}$. We shall show that $\boldsymbol{M}$ is short.

Let $\boldsymbol{B}_*$ be a subbase for τ. Then $(\boldsymbol{M} \cap \boldsymbol{B}_*) \subset \boldsymbol{M}$ and $\boldsymbol{M} \cap \boldsymbol{B}_*$ is also finitely short as $\boldsymbol{M}$ is finitely short. Here $\boldsymbol{M} \cap \boldsymbol{B}_*$ is the collection of members of $\boldsymbol{M}$ which are subbasic open sets. By the hypothesis $\boldsymbol{M} \cap \boldsymbol{B}_*$ is short, that is,

$$\cup (\boldsymbol{M} \cap \boldsymbol{B}_*) \neq X \qquad \ldots..(1)$$

Let x be any element of $\cup \boldsymbol{M}$, then $x \in M$ for some $M \in \boldsymbol{M}$. Since $M \in \tau$, then $\exists \, B \in \boldsymbol{B}$, the base for τ, such that

$$x \in B \subset M. \qquad \ldots.(2)$$

Since $B \in \boldsymbol{B}$, then E $B_1, B_2, \ldots, B_m \in \boldsymbol{B}_*$ such that

$$B = B_1 \cap B_2 \cap \ldots \cap B_m \qquad .....(3)$$

From (2) and (3), we have

$$x \in (B_1 \cap B_2 \cap \ldots \cap B_m) \subset M \qquad .....(4)$$

Then, by lemma 1, $\quad B_i \in \boldsymbol{M}$, for some i, $\qquad .....(5)$

Now, by (4), $\quad x \in B_i \qquad .....(6)$

Also, $\quad B_i \in \boldsymbol{B}_*, B_i \in \boldsymbol{M}$ (by (5)) $\Rightarrow B_i \in \boldsymbol{M} \cap \boldsymbol{B}_*$

and by (6), $\quad x \in \boldsymbol{M} \cap \boldsymbol{B}_* \Rightarrow x \in \cup (\boldsymbol{M} \cap \boldsymbol{B}_*)$

Thus, $\quad x \in \cup \boldsymbol{M} \Rightarrow x \in \cup (\boldsymbol{M} \cap \boldsymbol{B}_*)$

$$\Rightarrow (\cup \boldsymbol{M}) \subset \cup (\boldsymbol{M} \cap \boldsymbol{B}_*)$$

$$\Rightarrow \cup \boldsymbol{M} \neq X \qquad \text{(by (1))}$$

This shows that $\boldsymbol{M}$ is short and hence $\boldsymbol{F}$ is short as $\boldsymbol{F} \subset \boldsymbol{M}$. Consequently X is compact.

Theorem 16. (Tychonoff Theorem) : *Let (X, t) be a product space of arbitrary family of topological spaces* $\{(X_\lambda, \tau_\lambda) \mid \lambda \in \Lambda\}$. *Then X is compact relative to* τ *iff each* X_λ *is compact relative to* τ_λ.

Proof. First suppose that X is compact. Since each X_λ is a continuous image of the compact space X under the projection mapping π_λ, therefore X_λ is compact for each $\lambda \in \Lambda$.

Conversely, suppose each X_λ is compact. Let

$$\boldsymbol{B}_* = \{\pi_\lambda^{-1}(G_\lambda) \mid \lambda \in \Lambda, G_\lambda \in \tau_\lambda\}.$$

By alexander subbase lemma (theorem 15), to show X compact, it is sufficient to show that each finitely short subfamily of $\boldsymbol{B}_*$ is short. Let $\boldsymbol{C}$ be finitely short family such that $\boldsymbol{C} \subset \boldsymbol{B}_*$. We shall show that $\boldsymbol{C}$ is short. Let $\boldsymbol{C}_\lambda = \{G_\lambda \mid G_\lambda \in \tau_\lambda$ and $\pi_\lambda^{-1}(G_\lambda) \in \boldsymbol{C}\}$.

Since $\boldsymbol{C}$ is finitely short in X, therefore $\boldsymbol{C}_\lambda$ is finitely short in X_λ. Also, since X_λ is compact, $\boldsymbol{C}_\lambda$ is short, that is, $\boldsymbol{C}_\lambda$ does not cover X. Then $\exists\, x_\lambda \in X_\lambda$ such that $x_\lambda \notin G_\lambda$ for any $G_\lambda \in \boldsymbol{C}_\lambda$. Thus, no $x \in X$ will belong to any member of $\boldsymbol{C}$, for which $\pi_\lambda(x) = x_\lambda$. This shows that $x \notin \cup \boldsymbol{C}$, that is, $\boldsymbol{C}$ does not cover X. Hence $\boldsymbol{C}$ is short and consequenly X is compact.

14.4 Quotient space and quotient topology

14.4.1 Quotient topology :

We observed that product topology is the smallest topology for which projection mappings are continuous (theorem 5, theorem 10). Now we shall see that if f is a function from a space (X, τ) onto a set Y, then there exists the largest topology for Y relative to which function f is continuous, that is, Y can be topolized. This topology is called **quotient topology** for Y relative to f and denoted by τ_f. Mapping f is called **quotient map.** Now, we shall show that such a topology always exists.

Theorem 17. *Let (X, τ) be a topological space and Y be any set. Let f be a mapping of X onto Y. Then the collection τ_f of all subsets G of Y such that $f^{-1}(G)$ is open in X is the largest topology for Y such that f is $\tau - \tau_f$ continuous.*

Proof. Let $\tau_f = \{G \subset Y \mid f^{-1}(G) \in \tau\}$. We have to show that τ_f is topology for Y.

(i) $\phi \subset Y$ and $f^{-1}(\phi) = \phi \in \tau \Rightarrow \phi \in \tau_f$

(ii) Let $G_1, G_2 \in \tau_f \Rightarrow G_1, G_2 \subset Y$ and $f^{-1}(G_1), f^{-1}(G_2) \in \tau$.

$\Rightarrow (G_1 \cap G_2) \subset Y$ and $f^{-1}(G_1) \cap f^{-1}(G_2) \in \tau$ ($\because$ τ is topology)

$\Rightarrow (G_1 \cap G_2) \subset Y$ and $f^{-1}(G_1 \cap G_2) \in \tau$

$\Rightarrow G_1 \cap G_2 \in \tau_f$ (by the definition of τ_f)

(iii) Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be any arbitrary collection of members of τ_f. Then $f^{-1}(G_\lambda) \in \tau$ and $G_\lambda \subset Y$ for each $\lambda \in \Lambda$

$$\Rightarrow \quad \bigcup_{\lambda \in \Lambda} f^{-1}(G_\lambda) \in \tau, \left(\bigcup_{\lambda \in \Lambda} G_\lambda\right) \subset Y$$

$$\Rightarrow \quad f^{-1}\left(\bigcup_{\lambda \in \Lambda} G_\lambda\right) \in \tau, \left(\bigcup_{\lambda \in \Lambda} G_\lambda\right) \subset Y$$

$$\Rightarrow \quad \left(\bigcup_{\lambda \in \Lambda} G_\lambda\right) \in \tau_f.$$

Consequently τ_f is a topology for Y. Also, since inverse image of every open set in Y is open in X, hence f is $t - \tau_f$ continuous.

Let τ' be another topology for Y such that f is $\tau - \tau'$ continuous,

so $H \in \tau' \Rightarrow f^{-1}(H) \in \tau$

$\Rightarrow H \in \tau_f$ (by the definition of τ_f)

$\Rightarrow \tau' \subset \tau_f$

Hence τ_f is the largest topology for Y for which f is continuous.

Theorem 18. *A subset A of Y is closed in the quotient topology τ_f, relative to $f : X \to Y$ iff $f^{-1}(A)$ is closed in X.*

Proof. A is τ_f-closed $\Leftrightarrow (Y - A)$ is τ_f-open

$\Leftrightarrow f^{-1}(Y - A)$ is τ-open (by the definition of τ_f)

$\Leftrightarrow f^{-1}(Y) - f^{-1}(A)$ is τ-open

$\Leftrightarrow X - f^{-1}(A)$ is τ-open ($\because$ f is onto)

$\Leftrightarrow f^{-1}(A)$ is τ-closed.

Theorem 19. *A subset G of Y is* τ_f*-open in X iff* $f^{-1}(G)$ *is open in X.*

Proof. Let (X, τ) be a topology space and τ_f is quotient topology for Y relative to $f : X \to Y$.

Let G be τ_f-open in Y, then by the definition of τ_f, $f^{-1}(G)$ is open in X (also by continuity of f).

Conversely, by the definition of τ_f, $G \in \tau_f$ if $f^{-1}(G) \in \tau$, that is, if $f^{-1}(G)$ is open in X, then G is open in Y.

Theorem 20. *Let* (X, τ) *and* (Y, V) *be two topological spaces. Let f be a continuous mapping of X onto Y such that f is either open or closed, then **V** must be quotient topology for Y (that is* $V = \tau_f$*).*

Proof. Case *(i)*. Let f be continuous and open mapping of X onto Y. We know that quotient topology τ_f for Y is the largest topology for which f is continuous, so $V \subset \tau_f$.

Now, let $G \in \tau_f \Rightarrow f^{-1}(G) \in \tau$ (theorem 19)

$$\Rightarrow f[f^{-1}(G)] \in V \quad (\because f \text{ is open})$$
$$\Rightarrow G \in V$$
$$\Rightarrow \tau_f \subset V$$

This shows that $V = \tau_f$, that is, V is the quotient topology for Y.

Case *(ii)*. Let f be continuous and closed mapping of X onto Y. Again, since τ_f is the largest topology for Y for which f is continuous, therefore $V \subset \tau_f$.

Let $G \in \tau_f \Rightarrow f^{-1}(G) \in \tau$ (theorem 19)

$$\Rightarrow X - f^{-1}(G) \text{ is } \tau\text{-closed}$$
$$\Rightarrow f^{-1}(Y) - f^{1}(G) \text{ is } \tau\text{-closed} \quad (\because f \text{ is onto}, f^{-1}(Y) = X)$$
$$\Rightarrow f^{-1}(Y - G) \text{ is } \tau\text{-closed}$$
$$\Rightarrow f[f^{-1}(Y - G)] \text{ is } V\text{-closed} \quad (\because f \text{ is closed})$$
$$\Rightarrow Y - G \text{ is } V\text{-closed}$$
$$\Rightarrow G \text{ is } V\text{-open or } G \in V$$
$$\Rightarrow \tau_f \subset V$$

so, $V = \tau_f$. Hence V is quotient topology for Y.

14.4.2 Partition of a set and quotient map :

Partition *P* (decomposition) of a non-empty set X is the collection of non-empty disjoint subsets of X whose union is the set X.

Let π be a mapping from X onto $\boldsymbol{P}$ such that $\pi(x) = P \in \boldsymbol{P}$ such that $x \in P$, $\forall\ x \in X$. Then π is called the **quotient map** (cannonical map or projection map).

We know that a partition ***P*** on X induces an equivalence relation R on X, such that

$$R = \{(x, y) \mid x, y \in X \text{ belong to same member of } \boldsymbol{P}\}$$
$$= \{(x, y) \mid \pi(x) = \pi(y)\}$$
$$= \cup \{P \times P \mid p \in \boldsymbol{P}\}$$

Conversely, each equivalence relation R on X gives partition ***P*** of X, denoted by X/R, quotient set of X modulo R. Thus,

$$\boldsymbol{P} = X/R = \text{set of all equivalence classes.}$$

14.4.3 Quotient space :

Let (X, τ) be a topological space and R be an **equivalence relation** on X.

Let π be the quotient map of X onto the quotient set X/R of X over R so that $\pi(x) = [x]$, the equivalence class to which x belongs, $\forall\ x \in X$.

Then the X/R with the quotient topology relative to π (that is τ_π) is called **quotient space.**

Notes :

1. According to definition of quotient topology (14.4.1), τ_π is the largest topology for X/R for which π is continuous on X, which consists of all subsets G of X/R such that $\pi^{-1}(G)$ is open in X.
2. Let $A \subset X$, then the set of all points of X which are ***R***-relative of points of A is denoted by $R[A]$ or $[A]$. Hence
$$[A] = R[A] = \{y \in X \mid (x, y) \in R, \text{ for some } x \in A\}$$
$$= \cup \{P \mid P \in X/R \text{ and } P \cap A \neq \phi\}$$
3. If $x \in X$, then $R[x] = [x] = \pi(x)$, where π is the projection of X onto ***P***.
4. If $\boldsymbol{P}' \subset X/R$, then $\pi^{-1}(\boldsymbol{P}') = \cup \{P \mid P \in \boldsymbol{P}'\}$ and $\boldsymbol{P}'$ is τ_π-open (closed) in X/R iff $\cup \{P \mid p \in \boldsymbol{P}'\}$ or $\pi^{-1}(\boldsymbol{P}')$ is τ-open (closed) in X.

Theorem 21. *Let* (X, τ) *be a topological space and X/R be the quotient space of X over R. Let* π *be the quotient mapping of X onto X/R, then the following statements are equivalent* :

(a) π *is an open mapping.*

(b) If G is τ*-open in X, then* $R[G]$ *is* τ*-open.*

(c) If F is a τ*-closed subset of X, then the union of all members of X/R which are subset of F is closed in X.*

Proof. ***(a)*** $\Leftrightarrow$ ***(b)*** **:** First suppose that π is an open and G be any open subset of X. Then $\pi(G)$ is open in X/R, as π is open mapping. Also, since π is continuous, therefore $\pi^{-1}[\pi(G)]$ is open in X and hence $R[G]$ is open in X $\quad$ ($\because\ R[G] = \pi^{-1}[\pi(G)]$.

Conversely, let $R[G]$ be open in X, $\forall\ G \in \tau$, then $\pi^{-1}[\pi(G)]$ is open in X. Thus, $\pi(G)$ is open in X/R (by the definition of quotient topology). This shows that π is open.

***(b)* ⇔ *(c)* :** The union of all those members of X/R which are subsets of F is given by

$$\cup \{P \in X/R \mid P \subset F\} = X - R\,[X - F] \qquad .....(1)$$

First suppose that $R\,[G]$ is open, $\forall\ G \in \tau$ and let F be any τ-closed subset of X, then $X - F$ is open in X, hence by the hypothesis, $R\,[X - F]$ is open in X. So that $X - R\,[X - F]$ is closed in X. Then by (1), the union of all those members of X/R which are subsets of F is closed.

Conversely, suppose that for any closed subset F of X, the union of all those members of X/R, which are subsets of F is closed, that is $X - R\,[X - F]$ is closed (by (1)). Now, let G be any τ-open subset of X, then $X - G$ is closed in X, so that $X - R\,[X - (X - G)]$ is closed (by the hypothesis), that is $X - R\,[G]$ is closed. This shows that $R\,[G]$ is open.

Theorem 22. *Let π be the quotient mapping of the topological space* (X, τ) *onto the quotient space X/R. Then the following statements are equivalent* :

(a) π is a closed mapping.

(b) If G is closed in X, then $R\,[G]$ is closed.

(c) If F is a open subset of X, then the union of all members of X/R which are subset of F is open.

Proof. This is the dual of the theorem 21. The proof of this theorem can be obtained by the words open and closed throughout in the proof of theorem 21.

Theorem 23. *Let X be a topological space and X/R be a quotient space. If X is compact and connected then X/R is also compact and connected.*

Proof. We known that compactness and connectedness are topological invariant properties. Since X/R is continuous image of X, therefore it is compact and connected.

Theorem 24. *Let* (X, τ) *be a topological space such that X/R is Hausdorff quotient space, then R is a closed subset of the product space X × X relative to product topology **V**.*

Proof. We shall show that all the limit points of R belong to R, that is, no point of $(X \times X - R)$ is a limit point of R.

Let $(x, y) \in (X \times X - R) \Rightarrow (x, y) \in R$

$\Rightarrow \pi(x) \neq \pi(y)$

$\Rightarrow \exists$ open sets G, H in X/R

such that $\pi(x) \in G, \pi(y) \in H$ and $G \cap H = \phi$. (∵ X/R is Hausdorff)

$\Rightarrow \pi^{-1}(G)$ and $\pi^{-1}(x)$ are open in X,

and since images of these sets under π are $\pi(\pi^{-1}(G)) = G$ and $\pi(\pi^{-1}(H)) = H$, which are disjoint, so that no member of $\pi^{-1}(G)$ can be R-related to a member of $\pi^{-1}(H)$.

$\Rightarrow \pi^{-1}(G) \times \pi^{-1}(H)$

is an open neighbourhood of (x, y) which does not contain a point of R.

$\Rightarrow (x, y)$ is not a limit point of R

$\Rightarrow$ No point of $X \times X - R$ can be a limit point of R.

$\Rightarrow R$ is closed.

Ex.2. *Give an example to show that the quotient space of a Hausdorff space need not be a Hausdorff.*

Sol. Let (***R***, ***u***) be the usual topological space, we know that R is a Hausdorff space relative to usual topology. Let E be a relation on R such that

$$x \, E \, y \Leftrightarrow x - y \in Q, \text{ the set of rational number.}$$

Obviously E is an equivalence relation. The quotient space R/E will be an indiscrete space, therefore it is not Hausdorff.

Self-learning exercise

State whether the following statements are true or false :

(i) The set of all cartesian products of basic open subsets of space X and space Y forms a base for space $X \times Y$.

(ii) Let (X, τ) and (Y, V) be two topological spaces. Then the set $\{G \times H \mid G \in \tau, H \in V\}$ is a base for some topology for product space X x Y.

(iii) Projection mappings of product space are continuous but not open.

(iv) Connectedness and compactness are topological properties.

(v) Let $(X_\lambda, \tau_\lambda)$ be a topological space for all $\lambda \in \Lambda$, an arbitrary index set. Then $\times \{G_\lambda \in \tau : \lambda \in \Lambda\}$ is a base for product topology for the product space $X = \times \{X_\lambda \mid \lambda \in \Lambda\}$.

(vi) Product topology is the strongest topology for which projections are continuous.

(vii) A space X is compact if each finitely short subfamily of subbasic members is short.

(viii) Let (X, τ) be a topological space. Then the quotient topology τ_f for a set Y is the smallest topology for which function f is continuous.

(ix) Let X/R be a Hausdorff quotient space of a topological space X. Then R is closed in product space $X \times X$.

(x) If X is a Hausdorff space then its quotient space X/R is also Hausdorff.

14.5 Summary

In this unit, we have studied about the product space of finite family of topological spaces as well as product space of an arbitrary family of topological spaces. Product topology of finite product is defined in terms of a base while product topology for arbitrary product is defined in terms of subbase. Topological invariant properties of a product space have been discussed. With the help of an equivalence relation on a topological space we defined the quotient topology and quotient topological space.

14.6 Answers to self-learning exercises

(*i*) true (*ii*) true (*iii*) false (*iv*) true

(*v*) false (*vi*) false (*vii*) true (*viii*) false

(*ix*) true (*x*) false.

14.7 Exercises

1. Show that the product topology on a non-empty set $X \times Y$ is the weak topology for $X \times Y$ determined by the projection mapping π_x and π_y from the topology on X and Y.
2. Let y_0 be a fixed element of Y and $Z = X \times \{y_0\}$. Then the restriction of projection map π_x to Z is a homeomorphism of the subspace Z of $X \times Y$ onto X.
3. Let (X_i, τ_i) be topological spaces for $i = 1, 2, 3$. Show that a mapping $f : X_3 \to X_1 \times X_2$ is continuous if and only if $\pi_{X_1} \circ f : X_3 \to X_1$ and $\pi_{X_2} \circ f : X_3 \to X_2$ are continuous.
4. Show that the product space $X = \times \{X_\lambda \mid \lambda \in \Lambda\}$ is T_0 iff each coordinate space is T_0.
5. Show that the product of each family of locally compact space is locally compact.
6. Show that each coordinate space X_λ of product space $X = \times \{X_\lambda \mid \lambda \in \Lambda\}$ has a quotient topology induced by the projection mapping π_λ.

□ □ □

UNIT 15 : Nets and Filters

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15.5 Summary

15.6 Answers to self-learning exercises

15.7 Exercises

15.0 Objectives

In this unit, we shall study about convergence of a net, related topic like directed set, subnet, cluster point etc. Filter is an important topic in topology. We shall discuss about it also.

15.1 Introduction

Net is a generalized sequence whose domain is a directed set. If directed set is particularly set of natural numbers N, then it is called sequence. Thus net is a general notion while sequence is a particular type of net. Net is also called **generalized sequence** or **Moor-Smith Family.**

15.2 Net and its convergence

15.2.1 Directed set :

A pair $(A, \geq)$ consisting of a non-empty set A and *a* binary relation $\geq$ defined on A such that

(i) $a \in A \Rightarrow a \geq a$ (reflexive)

(ii) $a \geq b,\ b \geq c \Rightarrow a \geq c$ (transitive)

(iii) For any two members a and b of A, $\exists$ a member $c \in A$ such that $c \geq a$ and $c \geq b$, is called **directed set.** We say that relation $\geq$ **directs** A.

Ex.1. *Set of natural numbers N and the set of real numbers R are directed by the relation* $\geq$ (*greater than or equal to*), *that is,* $(N, \geq)$ *and* $(R, \geq)$ *are directed sets in usual sense.*

Ex.2. *Let* $\boldsymbol{P}$ *be the collection of all finite subsets of a set A. Then* $(\boldsymbol{P}, \supset)$ *is a directed set, where "*$X \supset Y$*" denotes "X is superset of Y" or "X contains Y".*

Sol. ***(i)*** Since each set contains itself, therefore $X \in P \Rightarrow X \supset X$.

(ii) By the set theory, $X \supset Y,\ Y \supset Z \Rightarrow X \supset Z$, for $X, Y, Z \in \boldsymbol{P}$.

(iii) For any two finite sets X, $Y \in \boldsymbol{P}$ we have $X \cup Y \in \boldsymbol{P}$ such that $(X \cup Y) \supset X$ and $(X \cup Y) \supset Y$.

Thus the relation $\supset$ directs $\boldsymbol{P}$, hence $(\boldsymbol{P}, \supset)$ is a directed set.

Similar examples : $(\boldsymbol{P}, \subset)$, for the relation 'inclusion' and the collection $\boldsymbol{N}(x)$ of all τ-neighbourhoods of $x \in X$, where (X, τ), is a topological space, for the relation 'inclusion' $(\boldsymbol{N}(x), \subset)$, is a directed sets.

15.2.2 Residual subset :

Let $(A, \geq)$ be a directed set and let $B \subset A$. Then set B is said to be **residual subset** of the set A iff there exist an element $a_0 \in A$ such that $x \geq a_0 \Rightarrow x \in B$.

15.2.3 Cofinal subset :

Let $(A, \geq)$ be a directed set and let $B \subset A$. Then set B is said to be **cofinal subset** of the set A iff for every $a \in A$, there exist an element $b \in \boldsymbol{B}$ such that $b \geq a$.

From the definitions it is clear that every residual subset of A is cofinal subset of A. Also, every cofinal subset of A is directed by $\geq \cdot$

Ex.3. *Let $(N, \geq)$ be a directed set, where $\geq$ is the relation "is greater than or equal to", then the subset $A = \{3, 4, 5, 6, ...\}$ is a residual subset of N because $\exists\ 3 \in N$ such that $x \geq 3 \Rightarrow x \in \boldsymbol{A}$. It is also cofinal subset of N. The set $B = \{2, 4, 6, ...\}$ is cofinal but bot residual subset of N. Set B is directed by $\geq$. Similarly the set of positive even integers is also cofinal and directed by $\geq \cdot$*

15.2.4 Net :

Let $(A, \geq)$ be a directed set and let $f : A \rightarrow X$ be an arbitrary mapping of A into a set X, then f is called ***a*** **net** in the X and we denote it by $(f, X, A, \geq)$ or $\{f(a) : a \in A\}$ or $\{f_a \mid a \in A\}$. If $A = N$ and $\geq$ is the relation "is greater than or equal to", then the net is called sequence. A net is also called **Moor-Smith family** or **generalized sequence.**

15.2.5 Eventually net :

Let $(f, X, A, \geq)$ be a net in X and let Y is subset of X. Then the net f is said to be **eventually** in Y iff $\exists$ a residual subset B of the set A such that $f(B) \subset Y$, that is, iff $\exists\ a_0 \in A$ such that $\forall\ a \in A, a \geq a_0 \Rightarrow f_a \in Y$.

15.2.6 Frequently net :

Let $(f, X, A, \geq)$ be a net in X and let $Y \subset X$. The f is said to be **frequently** in Y iff $\exists$ a cofinal subset B of A such that $f(B) \subset Y$, that is, iff for each $a \in A$, $\exists\ x \in A$ such that $x \geq a$ and $f_x \in Y$.

Note : A Net is frequently in Y iff it is not eventually in Y^c or $X - Y$.

15.2.7 Convergent net :

Let (X, τ) be a topological space and $(A, \geq)$ be a directed set. A net $(f, X, A, \geq)$ in X is said to be **convergent at** $x_0 \in X$ iff it is eventually in every τ-open neighbourhood of x_0. In other words, we can say that f converges to $x_0 \in X$ iff for each τ-open neighbourhood N of x_0 $\exists$ an element $a_0 \in A$ such that $\forall\, a \in A, a \geq a_0 \Rightarrow f_a \in N$.

15.2.8 Cluster point of net :

A point $x_0 \in X$ is said to be a cluster point of a net f in space X iff is frequentlly in every open neighbourhood of x_0.

Ex.4. *Let (X, I) be an indiscrete space, then show that every net $(f, X, A, \geq)$ in X converges to x, $\forall\, x \in X$.*

Sol. The only open neighbourhood of $\forall\, x \in X$ is X and $f_a \in X$, $\forall\, a \in A$, so net is eventually in X. Thus net is convergent at $x \in X$. Since x is arbitrary, therefore every net in X converges to every element of X.

Ex.5. *Let (X, D) be a discrete space and $(f, X, A, \geq)$ be any net in X. Show that f converges to $x_0 \in X$ iff net is eventually in $\{x_0\}$.*

Sol. First suppose that net f converges to $x_0 \in X$, so f is eventually in every D-open neighbourhood of x_0. Since $\{x_0\}$ is D-open neighbourhood of x_0, therefore net f is eventually in it.

Now suppose that the net f is eventually in $\{x_0\}$. Every neighbourhood of x_0 contains the set $\{x_0\}$, so net is eventually in every D-open neighbourhood of x_0.

Theorem 1. *Let Y be subset of topological space (X, τ). The set Y is τ-open iff no net in $(X - Y)$ converges to a point in Y.*

Proof : First suppose that no net in $(X - Y)$ converges to a point in Y. Suppose, if possible, Y is not τ-open, then $\exists\, y_0 \in Y$ such that it is not an interior point of Y, that is each neighbourhood N of y_0 contains atleast one point of $X - Y$, that is,

$$N \cap (X - Y) \neq \phi, \quad \forall\, N \in N(y_0), \qquad .....(1)$$

where $N(y_0)$ be the collection of all neighbourhoods of y_0. Now, $(N(y_0), \subset)$ is a directed set, where $\subset$ is an inclusion relation. We can choose a point $x(N)$ from $N \cap (X - Y)$ for each $N \in N(y_0)$, as $N \cap (X - Y)$ is non-empty for each N (by (1)).

Consider a mapping $f : N(y_0) \to X - Y$ defined by

$$f(N) = x(N), \ \forall\, N \in N(y_0).$$

Since $N(y_0)$ is a directed set, therefore f is a net in $X - Y$. Let G be any open neighbourhood of y_0, then for each $H \in N(y_0)$ such that H is a subset of G, then $H \geq G$ and

$$f(H) = x(H) \in H \cap (X - Y) \subset G, \qquad (\because\ H \subset G)$$

thus $\exists$ a member G of the directed set $N(y_0)$ such that for each

$$H \in N(y_0), H \geq G \Rightarrow f(H) \in G.$$

This shows that net is eventually in every open neighbourhood of y_0 as G is arbitrary. Thus $\exists$ a net in $X - Y$ converging to $y_0 \in Y$. This contradicts the fact that no net in $X - Y$ can converge to a point in Y. Hence Y is open.

Conversely, let Y be open. Suppose if possible, $\exists$ a net in $X - Y$ converging to a point $y_0 \in Y$. Since Y is open therefore it is neighbourhood of y_0 and hence net must be eventually in Y. Thus $Y \cap (X - Y) \neq \phi$ as net is in $X - Y$. This is a contradiction. Thus no net in $X - Y$ can converge to a point in Y.

Theorem 2. *Let Y be subset of topological space* (X, τ). *Then a point* $x_0 \in X$ *is an accumulation point* (*limit point*) *of Y iff* $\exists$ *a net in* $Y - \{x_0\}$ *converging to the point* x_0.

Proof : Let a net $(f, Y - (x_0), A, \geq)$ in $Y - \{x_0\}$ be converging to a point $x_0 \in X$. Let N be any τ-open neighbourhood of x_0. Since net f converges to x_0, therefore net f is eventually in N, so $\exists\ a_0 \in A$ such that $\forall\, a \in A,\ a \geq a_0 \Rightarrow f_a \in N$. Since net is in $Y - \{x_0\}$ therefore $f_a \neq x_0,\ \forall\, a \geq a_0$ and $f_a \in Y - \{x_0\}$, so N contains a point of Y other than x_0. Since N is arbitrary, therefore we can conclude that every τ-open neighbourhood of x_0 contains a point of Y other than x_0. Hence x_0 is an accumulation point of Y.

Conversely, suppose that x_0 is an accumulation point of the set Y. Then every neighbourhood of x_0 contains a point of Y other than x_0, that is

$$N \cap (Y - (x_0)) \neq \phi, \quad \forall\, N \in N(x_0)$$

where $N(x_0)$ is a collection of all τ-neighbourhoods of x_0, which is a directed set for inclusion relation. Since $N \cap (Y - (x_0)) \neq 0$, we may choose a point $x(N) \in N(Y - \{x_0\}),\ \forall\, N \in N(x_0)$. Consider a mapping $f : N(x_0) \to Y - \{x_0\}$ such that

$$f(N) = x(N),\ \forall\, N \in N(y_0).$$

Let G be any open neighbourhood of x_0, then for each $H \in N(x_0)$ such that $H \subset G$, that is, $H \geq G$ we have

$$f(H) = x(H) \in H \cap (Y - \{x_0\}) \subset H \subset G \qquad (\because\ H \subset G),$$

that is, $\exists\ G \in N(x_0)$ such that $H \in N(x_0),\ H \geq G \Rightarrow f(H) \in G$, thus net f in $Y - \{x_0\}$ is eventually in G. Hence net f converges to x_0.

Theorem 3. *Let Y be subset of topological space* (X, τ), *then Y is* τ-*closed iff no net in Y converges to point in* $X - Y$.

Proof : First suppose that no net in Y converges to a point in $X - Y$. Suppose, if possible, Y is not τ-closed. Then $\exists$ an accumulation point x_0 of Y not belonging to Y, that is, $x_0 \in X - Y$. Now, since x_0 is an accumulation point of Y, then by the Theorem 2, there exists a net in $Y - \{x_0\}$ converging to x_0. Since $x_0 \notin Y$, therefore $Y - \{x_0\} = Y$, so we can say that there exists a net in Y converging to $x_0 \in X - Y$. But this is a contradiction as no net in Y converges to a point in $X - Y$. Hence Y is τ-closed.

Conversely, let Y be a τ-closed subset of X. Suppose, if possible there exists a net $(f, Y, A, \geq)$ in Y converging to a point $x_0 \in X - Y$. Since $x_0 \in X - Y$, therefore $x_0 \notin Y$, so f is a net in $Y - \{x_0\}$ also. Then by the Theorem 2, x_0 is an accumulation point of Y. Since Y is τ-closed, therefore all of its limit points must be in Y, that is $x_0 \in Y$. This is a contradiction as $x_0 \notin Y$.

Hence no net in Y can converge to a point in $X - Y$.

Theorem 4. *A topological space is Hausdorff iff every net in the space converge to at most one point.*

Proof : Let (X, τ) be a topological space and let every net in X converge to at most one point. Suppose, if possible, space X is not Hausdorff. Then there exist $x, y \in X$ such that $x \neq y$ and every neighbourhood of x has non-empty intersection with every neighbourhood of Y. Let collection of all neighbourhood of x and y be $N(x)$ and $N(y)$ respectively. Then $(N(x), \subset)$ and $(N(y), \subset)$ both are directed sets for the inclusion relation $\subset$..

Let $N(x) \times N(Y) = M$ be the Cartesian product of $N(x)$ and $N(y)$. Let $(G_1, H_1) = M_1$ (say) and $(G_2, H_2) = M_2$ (say) be two elements of M, where $G_1, G_2 \in N(x)$ and $H_1, H_2 \in N(y)$. Consider a relation $\geq$, defined as

$$M_1 \geq M_2 \Leftrightarrow G_1 \subset G_2, H_1 \subset H_2.$$

Since $N(x)$ and $N(y)$ are directed sets, therefore set M will be directed by the relation $\geq$, defined as above. Hence $(M, \geq)$ is a directed set. Since $G \cap H \neq \phi$, $\forall\, G \in N(x)$, $\forall\, H \in N(y)$, therefore we can choose a point $x(G, H) \in G \cap H$, $\forall\, (G, H) \in M$.

Let $f : M \to X$ be a mapping such that

$$f(G, H) = x(G, H), \quad \forall\, (G, H) \in M$$

Let U and V be any neighbourhoods of x and y respectively. Let $(G, H) \in M$ such that $(G, H) \geq (U, V)$ so that $G \subset U$ and $H \subset V$ so $(G \cap H) \subset (U \cap V)$, then

$$f(G, H) = x(G, H) \in (G \cap H) \subset (U \cap V) \qquad .....(1)$$

Also, $\qquad (U \cap V) \subset U$ and $(U \cap V) \subset V$,

thus, $\qquad f(G, H) \in U$ and $f(G, H) \in V$, $\forall\, (G, H) \in M, (G, H) \geq (U, V)$

This show that net F is eventually in U and V both. Hence F converges to both x and y. This is a contradiction so X must be a Hausdorff space.

Conversely, suppose that the space X is a Haudorff space. Let $x, y \in X$ such that $x \neq y$. Since X is Haudorff, therefore there exist neighbourhoods G and H or x and y respectively such that $G \cap H = \phi$. Any net can not be eventually in G and H both as both are disjoint. Thus no net in X can converge to x and y both. Since x and y are arbitrary elements of X, therefore we can conclude that a net in X can converge at one point.

Note : If every sequence in a space X converges to at most one point then space X need not be Hausdorff as there exists non-Hausdorff space in which every sequence converges to at most one point. For example, co-countable topological space is not Hausdorff but every convergent sequence has a unique limit.

Theorem 5. *Let X and Y be two topological spaces. A mapping $g : X \to Y$ is continuous at $x_0 \in X$ iff whenever a net $(f, X, A, \geq)$ converges to $x_0 \in X$, then the net $(g \, o \, f, X, A, \geq)$ converges to $g\,(x_0) \in Y$.*

Proof : First suppose that mapping g is continuous at $x_0 \in X$ and the net f_a converges to x_0. We shall show that the net $g \, o \, f$ or $g\,(f_a)$ converges to $g\,(x_0) \in Y$. Let H be any open nighbourhood of $g\,(x_0)$ in Y. Since g is continuous at x_0, therefore $g^{-1}\,(H)$ is an open neighbourhood of x_0. Also, the net f_a converges to x_0, so $\exists\, a_0 \in A$ such that for every $a \in A$, $a \geq a_0 \Rightarrow f_a \in g^{-1}\,(H) \Rightarrow g\,(f_a) \in H$. Thus the net $g\,(f_a)$ is eventually in every open neighbourhood of $g\,(x_0)$. Hence the $g \, o \, f$ or $g\,(f_a)$ converges to $g\,(x_0)$ in Y.

Conversely, suppose that whenever a net f_a converges to a point x_0 in X, the net $g\,(f_a)$ converges to $g\,(x_0)$ in Y. We shall show that the mapping g is continuous at the point x_0. Suppose, if possible, g is not continuous at x_0, then there exists a open neighbourhood H of $g\,(x_0)$ in Y, such that for every open neighbourhood N of x_0, $g\,(N)$ is not a subset of H, that is $g\,(N) \not\subset H$, $\forall\, N \in N\,(x_0)$, where $N\,(x_0)$ is collection of all open neighbourhoods of x_0. So, for each $N \in N\,(x_0)$ there exist a point $x_N \in N$ such that $g\,(x_N) \notin H$. Consider a mapping $h : N\,(x_0) \to X$ such that $h\,(N) = x_N$, $\forall\, N \in N\,(x_0)$. Since $N\,(x_0)$ is directed by the inclusion relation $\subset$, therefore h is a net in X. Let $\boldsymbol{N}$ be any open neighbourhood of x_0. Then for every $G \in N\,(x_0)$, such that $G \subset N$, that is $G \geq N$, we have

$$h\,(G) = x_G \in G \subset N,$$

thus, $\exists\, N \in N\,(x_0)$

such that for every $G \in N\,(x_0), \quad G \geq N \Rightarrow h\,(G) \in N,$

so that the net h or $\{x_N \mid N \in N\,(x_0)\}$ is eventually in N, which is arbitrary. Hence, the net $\{x_N \mid N \in N\,(x_0)\}$ converges to $x_0 \in X$.

But the net $\{g\,(x_N) \mid N \in N\,(x_0)\}$ in Y does not converge to $g\,(x_0)$ as H is an open neighbourhood of $g\,(x_0)$ such that $g\,(x_N) \notin H$, $\forall\, N \in N\,(x_0)$ (by the choice of x_N), that is, the net $\{g\,(x_N) \mid N \in N\,(x_0)\}$ is not eventually in H. This is a contradiction. Hence g is continuous at $x_0 \in X$.

Theorem 6. *Let $X = \times \{X_\lambda \mid \lambda \in \Lambda\}$ be a product space. A net $(f, X, A, \geq)$ in X converges to $x_0 \in X$ iff the ne $\{\pi_\lambda\,(f_a) \mid a \in A\}$ converges to $\pi_\lambda\,(x)$ in X_λ for all λ.*

Proof : First suppose that net f_a converges to $x \in X$. Since projection mapping π_λ is continuous for all λ, therefore the net $\pi_\lambda\,(f_a)$ converges to $\pi_\lambda\,(x)$ in X_λ, $\forall\, \lambda \in \Lambda$ (by the theorem 5).

Conversely, suppose that the net $\pi_\lambda(f_a)$ converges to $\pi_\lambda(x)$, $\forall\,\lambda \in \Lambda$.

Let $$\pi_{\lambda_1}^{-1}(G_{\lambda_1}) \cap \pi_{\lambda_2}^{-1}(G_{\lambda_2}) \ldots \cap \pi_{\lambda_n}^{-1}(G_{\lambda_n}) = G \quad \ldots..(1)$$

be any basic neighbourhood of $x \in X$, where $\pi_{\lambda_i}^{-1}(G_{\lambda_i})$ are the members of subbase of product topology X and G_{λ_i} is open neighbourhood of $\pi_{\lambda_i}(x)$ in X_{λ_i} for all i.

Since net $\pi_\lambda(f_a)$ converges to $\pi_\lambda(x)$, $\forall\,\lambda$; therefore net $\pi_{\lambda_i}(f_a)$ converges to $\pi_{\lambda_i}(x)\ \forall\, i$. So for each i, $\exists\, a_i \in A$ such that for all $a \in A$, $a \geq a_i \Rightarrow \pi_{\lambda_i}(f_a) \in G_{\lambda_i}$.

Let $$a_0 \geq a_i, \quad i = 1, 2, \ldots, n, \text{ then}$$

$$\pi_{\lambda_i}(f_a) \in G_{\lambda_i}, \quad \forall\, a \geq a_0,\ a \in A.$$

Thus $\exists\, a_0 \in A$ such that for every $a \in A$, $a \geq a_0$

$$\pi_{\lambda_i}(f_a) \in G_{\lambda_i} \Rightarrow f_a \in \pi_{\lambda_i}^{-1}(G_{\lambda_i}), \quad i = 1, 2, \ldots, n$$

$$\Rightarrow \quad f_a \in \pi_{\lambda_1}^{-1}(G_{\lambda_1}) \cap \pi_{\lambda_2}^{-1}(G_{\lambda_2}) \cap \ldots. \cap \pi_{\lambda_n}^{-1}(G_{\lambda_n})$$

$$\Rightarrow \quad f_a \in G \qquad \text{(by (1))}$$

Hence the net f is eventually in every open neighbourhood of $x \in X$. Consequently net f_a converges to x.

15.3 Ultranet and subnet

15.3.1 Ultranet: *A* net $(f, x, A, \geq)$ in a set X is said to be an **ultranet** or an **universalnet** iff for every subset Y of X, the net f is eventually in Y or in its complement Y^c. From the definition it is clear that if an ultranet is frequently in any subset Y of X, then it must be eventually in Y. If f be an ultranet in a topological space X and x be a cluster point of f, then f will be frequently in every open neighbourhood of x and since f is an ultranet, therefore it must be eventually in every open neighbourhood of x. Hence the net converges to its cluster point x. Thus, every ultranet in a topological space X converges to each of its cluster points.

15.3.2 Subnet :

Let $(f, X, A, \geq)$ and $(g, X, B, \geq^*)$ be two nets in a set X. The net g is said to be a **subnet** of the net f iff there exists a mapping $\phi : B \to A$ defined as

(a) $f \circ \phi = g$

(b) For every $a \in A$, $\exists$ an element $b \in B$ such that $\phi(p) \geq a$, $\forall\ p \geq^* b$ in B.

15.3.3 Isotone mapping :

A mapping ψ of *a* directed set $(A, \geq)$ to another directed set $(B, \geq^*)$ is said to be an isotone mapping iff for $p, q \in A$, $p \geq q \Rightarrow \psi(p) \geq^* \psi(q)$.

Theorem 7. *Let* $(A, \geq)$ *and* $(B, \geq^*)$ *be two directed sets and let* ψ *be an isotone mapping of* B *to* A *such that* $\psi(B)$ *is cofinal in* A. *If* $(f, X, A, \geq)$ *be a net in* X, *then* $f \circ \psi$ *is a subnet of the net* f.

Proof. Since mapping $\psi : B \to A$ and $f : A \to X$, therefore mapping $f \circ \psi : B \to X$. Let $f \circ \psi = g$, then $(g, X, B, \geq^*)$ is a net in X. To show that g is a subnet of f it is sufficient to show that for each $a \in A$, $\exists\, b \in B$ such that $\psi(x) \geq a$, for every $x \geq^* b$ in B.

Since ψ is an isotone mapping, therefore,

$$x \geq^* y \Rightarrow \psi(x) \geq \psi(y),\ x, y \in B \qquad .....(1)$$

Also, since $\psi(B)$ is cofinal in A, therefore, for each $a \in A$, $\exists\, b \in B$ such that

$$\psi(b) \geq a \qquad .....(2)$$

Now, let $x \in B$ such that $x \geq^* b$ then by (1)

$$x \geq^* b \Rightarrow \psi(x) \geq \psi(b) \Rightarrow \psi(x) \geq a \qquad \text{(by (2))}$$

Thus, for each $a \in A$, $\exists\, b \in B$ such that

$$x \geq^* b \text{ in } B \Rightarrow \psi(x) \geq a \text{ in } A$$

consequently $g = f \circ \psi$ is a subnet of the net f.

15.4 Filter and its convergence

15.4.1 Filter : A filter $\boldsymbol{F}$ on a non-empty set X is a non-empty collection of subsets of X satisfying following axioms :

[*F*1] : $\phi \notin \boldsymbol{F}$

[*F*2] : If $A \in \boldsymbol{F}$ and $B \supset A$ then $B \in \boldsymbol{F}$

[*F*3] : If $A, B \in \boldsymbol{F}$ then $A \cap B \in \boldsymbol{F}$

Notes :

1. Form [*F*2] of definition it is clear that $X \in \boldsymbol{F}$ always as it is a superset of every member of $\boldsymbol{F}$.
2. From [*F*3], $A_1, A_2 \in \boldsymbol{F} \Rightarrow A_1 \cap A_2 \in \boldsymbol{F}$. Again if $A_3 \in \boldsymbol{F}$, by [*F*3] $(A_1 \cap A_2) \cap A_3 = A_1 \cap A_2 \cap A_3 \in \boldsymbol{F}$. Thus for $A_1, A_2, A_3, ..., A_n \in \boldsymbol{F} \Rightarrow A_1 \cap A_2 \cap A_3 \cap ... \cap A_n \in \boldsymbol{F}$. By [*F*1] $A_1 \cap A_2 \neq \phi$, $A_1 \cap A_2 \cap A_3 \neq \phi$, and so on. Thus $\boldsymbol{F}$ is with finite intersection property (FIP).
3. Form (2) and form (3), we can conclude that $\boldsymbol{F}$ is closed for finite intersection as empty intersection of members of $\boldsymbol{F}$ is $X \in \boldsymbol{F}$.
4. The power set $P(X)$ of the set X contains ϕ, so it can not be fitter on X. Also, filter can not be empty, so filter $\boldsymbol{F}$ on X is always a proper subset of $P(X)$, that is, $\boldsymbol{F} \neq \phi$, $\boldsymbol{F} \neq P(x)$.
5. Any subset A of X and its complement A^c, both together can not be member of the filter on X. For if $A, A^c \in \boldsymbol{F}$ then by $[\boldsymbol{F}_3]$, $(A \cap A^c) \in \boldsymbol{F}$, That is $\phi \in \boldsymbol{F}$, which is not possible as $\phi \notin \boldsymbol{F}$ $([\boldsymbol{F}_1])$.

6. *A* filter $\boldsymbol{F}$ on X has $F\, l\, P$ (by (2)). It is said to be free iff it has empty intersection, that is iff $\cap \{A] A \in \boldsymbol{F}\} = \phi$ other wise it is said to be fixed.

***Ex*.6.** Let $X = \{a, b, c, d\}$, then

$\boldsymbol{F_1} = \{X\}, \boldsymbol{F}_2 = \{\{a\}, \{a,b\}, \{a, d\}, X\}$

$\boldsymbol{F_3} = \{(a, b), X\}$ are filters on X but

$\boldsymbol{F_4} = \{\phi, X\}$,

$\boldsymbol{F_5} = \{\{a\}, \{b\}, X\}$

are not the filter on X as $\boldsymbol{F_4}$ contains ϕ and in $\boldsymbol{F_5}$, $\{a\} \cap \{b\} = \phi \notin \boldsymbol{F_5}$.

***Ex*.7.** $\boldsymbol{F} = \{X\}$ is a filter if X is non empty. It is called **indiscrete filter.**

***Ex*.8.** Let A_0 be a non-empty subset of X, then show that $\boldsymbol{F} = \{A \mid A \supset A_0\}$ is a filter on X.

Sol. The collection $\boldsymbol{F}$ is non-empty as $A_0 \supset A_0$ so $A_0 \in \boldsymbol{F}$. Also,

[F_1] : Since ϕ does not contain A_0 ($\neq \phi$), so $\phi \notin \boldsymbol{F}$.

[F_2] : Let A be any member of $\boldsymbol{F}$ and $B \supset A$. Since $A \in \boldsymbol{F} \Rightarrow A \supset A_0$ and since $B \supset A \Rightarrow B \supset A_0$, thus $B \in \boldsymbol{F}$.

[F_3] : Let $A, B \in F \Rightarrow A \supset A_0, B \supset A_0 \Rightarrow (A \cap B \supset A_0$, Thus $A \cap B \in \boldsymbol{F}$.

Hence $\boldsymbol{F}$ is a filter

***Ex*.9.** Let X be a topological space and $x \in X$. Let $N(x)$ be the collection of all neighbourhoods of X. Then $N(x)$ is a filter on X. [It is called neighbouhood filter on X.]

Sol. Since X is neighbourhood of x so $X \in N(x)$, hence $N(x)$ is non-empty.

[F_1] : Let $N \in N(x)$ be arbitrary, then N is a neighbourhood of x, so $x \in N$, *that* is, $N \neq \phi$, hence no member of $N(x)$ is empty. Thus $\phi \notin N(x)$.

[F_2] : Let $N \in N(x)$ and $M \supset N$. Since every superset of a neighbourhood is again a neighbourhood of a point in X, hence M is also a neighbourhood of x. So $M \in N(x)$.

[F_3] : Let $M, N \in N(x)$. Since intersection of two neighbourhoods is again a neighbourhood, so $M \supset N \in N(x)$.

Hence $N(x)$ is a filter on X.

***Ex*.10.** Let $x_0 \in X$ and $\boldsymbol{F}$ is the collection of all those subsets of X which contains x_0. Then show that $\boldsymbol{F}$ is a filter on X. [It is called discrete filter].

15.4.2 Finer and coarser filters :

Let $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ be two filters on a set X. Then $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ are said to be comparable if either $\boldsymbol{F_1} \subset \boldsymbol{F_2}$ or $\boldsymbol{F_2} \subset \boldsymbol{F_1}$. If $\boldsymbol{F_1} \subset \boldsymbol{F_2}$, then $\boldsymbol{F_2}$ is said to be finer than $\boldsymbol{F_1}$ or $\boldsymbol{F_1}$ is said to be coarser than $\boldsymbol{F_2}$. Also if $\boldsymbol{F_1} \neq \boldsymbol{F_2}$, then $\boldsymbol{F_2}$ is strictly finer than $\boldsymbol{F_1}$ or $\boldsymbol{F_1}$ is strictly coarser than $\boldsymbol{F_2}$.

Two filters are said to be comparable iff one is finer than another. If we define a relation $\geq$ defined as $\boldsymbol{F_1} \subset \boldsymbol{F_2} \Leftrightarrow F_1, \geq \boldsymbol{F_2}$, then the set of all filters on X is a directed set for the relation defined above.

***Ex.*11.** Let $X = \{1, 2, 3\}$ and

let $\boldsymbol{F_1} = \{X\}$,

$\boldsymbol{F_2} = \{\{1, 2\}, \{X\}$

$\boldsymbol{F_3} = \{\{1\}, \{1, 2\}, X\}$

$\boldsymbol{F_4} = \{\{2\}, \{1, 2\}, X\}$,

then $\boldsymbol{F_1}$ is coarser than $\boldsymbol{F_2}$, $\boldsymbol{F_3}$, $\boldsymbol{F_4}$. $\boldsymbol{F_2}$ is coarser than $\boldsymbol{F_3}$ or $\boldsymbol{F_3}$ is finer than $\boldsymbol{F_2}$. $\boldsymbol{F_3}$ and $\boldsymbol{F_4}$ are not comparable.

15.4.3 Subbase of a filter :

We can construct a filter ***F*** on a non-empty set ***X*** with the help of a non-empty family ***C*** of subsets of X having some certain properties. Then we say that ***F*** is generated by ***C*** and ***C*** is said to be **subbase** of ***F***. Now, we shall discuss that properties of ***C*** and method to construct the filter ***F*** with the help of following theorem.

Theorem 8. *Let X be a non-empty set and ⊆ be any non-empty collection of subsets of X. Then there exists a filter F on X containing ⊆ iff ⊆ has the finite intersection property* (*FIP*).

Proof. First suppose that collection ***C*** has FIP. Let ***B*** be the collection of all possible finite intersections of members of ***C***, that is,

$\boldsymbol{B} = \{B \mid B$ is the intersection of a finite subfamily of $\boldsymbol{C}\}$

and let $\boldsymbol{F} = \{F \mid F \supset B, B \in \boldsymbol{B}\}$,

that is, ***F*** is a collection of supersets of members of ***B***. Now we shall show that ***F*** is a filter on X. Since ***C*** has FIP, therefore members of ***B*** are non-empty, that is, $\phi \in \boldsymbol{B}$. By the definition of ***F*** it is clear that $\boldsymbol{F} \supset \boldsymbol{C}$ and since ***C*** is non-empty, therefore ***F*** is also non-empty.

[F_1] : Since ϕ is not a superset of any member of ***B*** ($\phi \notin \boldsymbol{B}$), so $\phi \notin \boldsymbol{F}$.

[F_2] : Let $F \in \boldsymbol{F}$ and $G \supset F$. Since $F \in \boldsymbol{F}$, therefore it contains a member of ***B*** and hence G must contain that member of ***B***. So that $G \in \boldsymbol{F}$.

[F_3] : Let $F, G \in \boldsymbol{F}$, then $\exists\, A, B \in \boldsymbol{B}$ such that $F \supset A$ and $G \supset B$. Since A and B are members of ***B***, therefore they are finite intersection of members of ***C***, so $A \cap B$ is also a finite intersection of members of ***C*** and hence $A \cap B \in \boldsymbol{B}$. Now,

$(F \cap G) \supset (A \cap B)$ as $F \supset A$ and $G \supset B$

Thus $F \cap G$ contains a member of ***B***. Hence $F \cap G \in \boldsymbol{F}$.

Consequently ***F*** is filter on X containing ***C***.

Conversely, let ***F*** be a filter on X containing ***C***. We shall show that ***C*** has FIP. By the definition of ***F*** it follows that ***F*** contains ***C*** as well as finite intersections of members of ***C*** (that is collection ***B***), as ***F*** is closed for finite intersections of its members. Thus no finite intersection of members of ***C*** can be empty, otherwise $\phi \in \boldsymbol{F}$. Hence ***C*** has FIP.

Now we can define **subbase** as follows :

A non-empty collection ***C*** of subsets of X having FIP can generate a filter ***F*** on X. This filter ***F*** is said to be generated by ***C*** and ***C*** is said to be a subbase of ***F***.

Ex.*12.** Let $X = \{1, 2, 3\}$ and ***C $= \{\{1, \},\{1, 2\}\}$, then construct *a* filter ***F*** on X for which ***C*** is a subbase.

Sol. Since ***C*** is with FIP, therefore it can be a subbase for the filter on X. Let $A = \{A \mid A$ is intersection of a finite subfamily of ***C***$\}$

$$= \{ \{1\}, \{1,2\}, X \},$$

Since X is empty intersection, so $X \in$ ***A***. Now, take all the supersets of members of ***A*** in ***F***, then filter

$$\boldsymbol{F} = \{\{1\}, \{1,2\}, \{1, 3\}, X\}.$$

15.4.4 Filter base :

A **filter base *B*** on a non-empty set X is a non-empty family ***B*** of subsets of X satisfying the following axioms :

$[\boldsymbol{B}_1] : \phi \notin \boldsymbol{B}$

$[\boldsymbol{B}_2]$: if $F, G \in \boldsymbol{B}$, then $\exists\, H \in \boldsymbol{B}$ such that $H \subset (F \cap G)$.

From the definition it follows that ***B*** does not contain empty set and each finite intersection of members of ***B*** contains a members of ***B***, hence we can conclude that ***B*** has FIP.

***Ex.*13.** Show that every filter is a filter base.

Sol. Let ***F*** be the filter. Then $\phi \notin \boldsymbol{F}$ ([***B***1]).

Also, let $F, G \in \boldsymbol{F}$, then $F \cap G \in \boldsymbol{F}$ and since $(F \cap G) \subseteq (F \cap G)$, thus [$B$2] is satisfied. Consequently ***F*** is a filter base.

Theorem 9. Let ***C*** be any non-empty family of subsets of a set X. Then there exists a filter base on X containing ***C*** iff ***C*** has FIP.

Proof. First suppose that ***C*** has FIP and let ***B*** be the collection of all finite intersection of members of ***C***, that is,

$$\boldsymbol{B} = \{B \mid B \text{ is the intersection on finite subfamily of } C\}.$$

We shall show that ***B*** is a filter base on X. Since $\boldsymbol{C} \subset \boldsymbol{B}$ and ***C*** is non-empty, therefore ***B*** is also non-empty.

$[\boldsymbol{B}_1] : \phi \notin \boldsymbol{B}$, since ***C*** has FIP.

$[\boldsymbol{B}_2]$: Let $F, G \in \boldsymbol{B}$, then F and G are finite intersection of members of ***C*** and hence $(F \cap G) \in \boldsymbol{B}$ as $F \cap G$ will also be a finite intersection. Thus for $F, G \in \boldsymbol{B}$, $\exists\, (F \cap G) \in \boldsymbol{B}$ such that $(F \cap G) \subseteq (F \cap G)$.

Hence ***B*** is a filter base containing ***C***.

Conversely, suppose that there exists a filter base $\boldsymbol{B}$ on X containing $\boldsymbol{C}$. Since $\boldsymbol{B}$ is a filter base, therefore no member of $\boldsymbol{B}$ can be empty and by [$\boldsymbol{B}$2], the intersection of every finite subfamily of $\boldsymbol{B}$ contains a member of $\boldsymbol{B}$ which is non-empty. Thus, intersection of every finite subfamily of members of $\boldsymbol{C}$ also contains a member of $\boldsymbol{B}$ which is non-empty. So finite intersection of members of $\boldsymbol{C}$ can not be empty. Consequently $\boldsymbol{C}$ has FIP.

15.4.5 Filter generated by Filter base :

A Filter $\boldsymbol{F}$ on a non-empty set X, consisting of all those subsets of X which contain a member of filter base $\boldsymbol{B}$ is said to be a **filter generated by *B*.**

15.4.6 Base of a filter :

A subfamily $\boldsymbol{B}$ of filter $\boldsymbol{F}$ on X is said to be a **base** of $\boldsymbol{F}$ iff every member of $\boldsymbol{F}$ contains a member of $\boldsymbol{B}$.

Note: From above definitions, if follows that a collection $\boldsymbol{B}$ of subsets of X is a base of some filter on X iff axioms [$\boldsymbol{B}$1] and [$\boldsymbol{B}$2] of definitions of filter base are satisfied or in other words a collection $\boldsymbol{B}$ of subsets of X is a filter base on X iff $\boldsymbol{B}$ is a base of some filter on X.

***Ex.*14.** Let $X = \{1, 2, 3, 4\}$ and $\boldsymbol{C} = \{\{1, 2\},\{1, 3\}\}$, then find base and filter taking $\boldsymbol{C}$ as a subbase.

Sol. Since $\boldsymbol{C}$ has FIP as $\{1, 2\} \cap \{2, 3\} = \{1\} \neq \phi$, therefore it can be a subbase. Taking all finite intersection of members of $\boldsymbol{C}$, we get base $\boldsymbol{B}$ as follows :

$$\boldsymbol{B} = \{\{1\}, \{1,2\}, \{1, 3\}, X\}.$$

It is easy to verify that $\boldsymbol{B}$ satisfy [$\boldsymbol{B}$1] and [$\boldsymbol{B}$2], hence it is a filter base also.

Now, to get filter $\boldsymbol{F}$, take all super sets of members of $\boldsymbol{B}$. Thus

$$\boldsymbol{F} = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 3, 4\}, \{1, 2, 4\}, X\}$$

Note: This process is discussed in theorem 10 and in example 12 also.

15.4.7 Ultrafilter :

A Filter $\boldsymbol{F}$ an a non-empty set X is said to be an **ultrafilter** or a **maximal filter** on X iff there exists no filter on X strictly finer than $\boldsymbol{F}$. Thus if $\boldsymbol{F}$ is an ultrafilter then for every filter $\boldsymbol{F}'$ on X, $\boldsymbol{F} \subset \boldsymbol{F}' \Rightarrow \boldsymbol{F} = \boldsymbol{F}'$.

A filter base of on ultrafilter is said to be an ultrafilter base.

***Ex.*15.** Let $X = \{1, 2, 3\}$, then

$$\boldsymbol{F}_1 = \{\{1\}, \{1, 2\}, \{1, 3\}, X\},$$
$$\boldsymbol{F}_2 = \{\{2\}, \{1, 2\}, \{2, 3\}, X\},$$

and $$\boldsymbol{F}_3 = \{\{3\}, \{1, 3\}, \{2, 3\}, X\},$$

are ultrafilter an X, while

$$\boldsymbol{F}_4 = \{\{1, 2\}, X\}$$

and $$\boldsymbol{F}_5 = \{\{1, 3\}, X\}$$

are not ultrafilter as $\boldsymbol{F}_4 \subset \boldsymbol{F}_1$ and $\boldsymbol{F}_5 \subset \boldsymbol{F}_3$.

Theorem 10. *Every filter* $\boldsymbol{F}$ *on a non-empty set* X *is contained in an ultrafilter on* X.

Proof. Let A be the collection of all those filters on X which contains a filter $\boldsymbol{F}$ on X. Then

$$A = \{M \mid M \text{ is filter on } X \text{ such that } M \supset F\}.$$

Since $\boldsymbol{F} \supset \boldsymbol{F}$, therefore $\boldsymbol{F} \in A$ and hence $A \neq \phi$.

Obviously A is a partially ordered set for the inclusion relation. Let $B = \{F_\alpha : \alpha \in \Lambda\}$ be a linearly ordered sub set of A. Then for $\boldsymbol{F}_1, \boldsymbol{F}_2 \in B$, either $\boldsymbol{F}_1 \subset \boldsymbol{F}_2$ or $\boldsymbol{F}_2 \subset \boldsymbol{F}_1$.

Let $$\boldsymbol{H} = \cup \{\boldsymbol{F}_\lambda \mid \boldsymbol{F}_\lambda \in B\}.$$

We shall show that $\boldsymbol{H}$ is also a filter on X.

[$\boldsymbol{F_1}$] : Since $\phi \notin \boldsymbol{F}_\lambda$, $\forall\, \boldsymbol{F}_\lambda \notin B$ as $\boldsymbol{F}_\lambda$ is a filter, $\forall\, \lambda$, hence $\phi \notin \boldsymbol{H}$, being the union of all $\boldsymbol{F}_\lambda$.

[$\boldsymbol{F_2}$] : Let $H \in \boldsymbol{H}$ and let $G \supset H$, then $H \in \boldsymbol{F}_\lambda$ for some $\alpha \in \Lambda$ and hence $G \in \boldsymbol{F}_\alpha \subset H$ as $\boldsymbol{F}_\alpha$ is a filter. Thus $G \in \boldsymbol{H}$.

[F3] : Let $G, H \in \boldsymbol{H}$. Then $G \in \boldsymbol{F}_\lambda$ and $H \in \boldsymbol{F}_\mu$ for some $\boldsymbol{F}_\lambda, \boldsymbol{F}_\mu \in B \Rightarrow$ either $\boldsymbol{F}_\lambda \subset \boldsymbol{F}_\mu$ or $\boldsymbol{F}_\mu \subset \boldsymbol{F}_\lambda$. Let us suppose that $\boldsymbol{F}_\lambda \subset \boldsymbol{F}_\mu$, then $G, H \in \boldsymbol{F}_\mu$. Since $\boldsymbol{F}_\mu$ is a filter, so $G \cap H \in \boldsymbol{F}_\mu \subset H$. Hence $G \cap H \in \boldsymbol{H}$.

Consequently $\boldsymbol{H}$ is a filter on X. $\boldsymbol{H} \supset \boldsymbol{H}_\lambda$, $\forall\, \boldsymbol{H}_\lambda \in B$ so $\boldsymbol{H}$ is an upper bound of B. Thus A is a poset whose every linearly ordered subset has an upper bound. Hence A contains a maximal element (by Zorn's lemma} which will be an ultrafilter on X containing $\boldsymbol{F}$.

15.4.8 Convergence of a filter :

Let $\boldsymbol{F}$ be a filter on a non-empty set X and let A be a subset of X. Then $\boldsymbol{F}$ is said to be **eventually** in the set A if and only if $A \in \boldsymbol{F}$. Filter $\boldsymbol{F}$ is said to **converge to a point** $a \in X$ iff $\boldsymbol{F}$ is eventually in each open neighborhood of a and a is said to be a **limit point** (or limit) of $\boldsymbol{F}$ and it is written as $F \to a$.

Notes :

1. By the definition it follows that $\boldsymbol{F}$ is eventually in all of its members.
2. If N (a) is the collection of all neighborhoodroods of a then $\boldsymbol{F}$ converges to a iff N (a) $\subset \boldsymbol{F}$.
3. Since each neighbourhood of a contains an open neighbourhood, therefore by [$\boldsymbol{F}_2$], if $\boldsymbol{F}$ is eventually in all open neighbourhoods of a then it must be eventually in all neighbourhood of a. Hence in above definition 'open neighbourhood' can be replaced by 'neighbourhood'.

15.4.9 Frequently filter :

Filter $\boldsymbol{F}$ is said to be **Frequently** in a subset A of X iff A intersects every member of $\boldsymbol{F}$, that is, $A \cap F \neq \phi$, $\forall\ F \in \boldsymbol{F}$.

Note : $\boldsymbol{F}$ is eventually in $A \Rightarrow \boldsymbol{F}$ is frequently in A. But converse is not necessarily true. For example, Let $X = \{1, 2, 3\}$ and $\boldsymbol{F} = \{\{1, 2\}, X\}$ then $\boldsymbol{F}$ is frequently in $\{1\}$ but not eventually in $\{1\}$.

Theorem 11. *A topological space X is haudroff space iff every convergent filter on X has a unique limit.*

Proof. Fist suppose that X is a houdroff space and let $\boldsymbol{F}$ be a filter on X. Let us suppose that $\boldsymbol{F}$ converges to $x, y \in X$ such that $x \neq y$. Since x and y are limit points of $\boldsymbol{F}$, therefore $\boldsymbol{F}$ is eventually in every neighbourhood of x and in every neighbourhood of y. Let $N(x)$ and $N(y)$ be the collection of all neighbourhood of x and y respectively. Then $N(x) \subset \boldsymbol{F}$ and $N(y) \subset \boldsymbol{F}$. Since X is housdorff, therefore $\exists\ N \in N(x)$ and $M \in N(y)$ such that $N \cap M = \phi$. But $N, M \in \boldsymbol{F}$ as $N(x)$ and $N(y)$ both are subsets of $\boldsymbol{F}$ and $N \cap M = \phi$. This is a contradiction. Hence $\boldsymbol{F}$ has a unique limit. Conversely, suppose that every convergent filter on X converges to a unique point. Suppose, if possible X is not hausdorff. Then $\exists$ two distinct points $x, y \in X$ such that

$$N \cap M \neq \phi, \ \forall\ N \in N(x), \ \forall\ M \in N(y).$$

Since $N(x)$ and $N(y)$ are neighbourhood filters on X, therefore $N(x) \cap N(y)$ is also a filter on X. Since every filter is a filter base also and $(N(x) \cap N(Y)) \subset N(x)$, $(N(x) \cap N(y) \subset N(y)$, therefore $N(x) \cap N(y)$ generates a filter $\boldsymbol{F}$ finer than $N(x)$ and $N(y)$ both. Since filter $N(x)$ contains every neighbourhood of x, therefore it is eventually in every neighbourhood of x, thus $N(x)$ converges to x and similarly $N(y)$ converges to Y.

Since $N(x) \subset \boldsymbol{F}$, $N(y) \subset \boldsymbol{F}$, therefore $\boldsymbol{F}$ converges to x and y both. This is a contradiction. Hence X is a Housdorff space.

15.4.10 Limit point of a filter base :

A filter base $\boldsymbol{B}$ on X is said to **converge** to $x \in X$ iff the filter generated by $\boldsymbol{B}$ converges to x and x is called **limit point** of $\boldsymbol{B}$.

15.4.11 Cluster point of a filter :

Let $\boldsymbol{F}$ be a filter an a topological space X. *A* point $x \in X$ is said to be a **cluster point** or an **adherent point** of $\boldsymbol{F}$ iff $\boldsymbol{F}$ is frequently in each open neighbourhood of x.

15.4.12 Cluster point of a filter base :

Let $\boldsymbol{B}$ be a filter base on a topological space X. *A* point $x \in X$ is said to be a **cluster point** or an **adherent pint** of $\boldsymbol{B}$ iff $\boldsymbol{B}$ is frequently in each open neighbourhood of x.

***Ex.*16.** Let $X = \{ 1, 2, 3 \}$ and $\tau = \{\phi, \{1\}, \{1, 2\}, \{1, 3\}, X\}$, then find all cluster points of filter $\boldsymbol{F_1} = \{1, 2\}, X\}$ and $\boldsymbol{F_2} = \{ \{2, 3\}, X\}$

Sol. ***(i)*** Open neighbourhoods of $1 = \{1\}, \{1, 2\}, \{1, 3\}, X$.

Since $\{1\}$ does not intersect each member of $\boldsymbol{F_2}$ so 1 is not a cluster point of $\boldsymbol{F_2}$. Every open neighbourhood intersects every member of $\boldsymbol{F_1}$ so 1 is a cluster point of $\boldsymbol{F_1}$.

(ii) open neighbourhood of $2 = \{1, 2\}, X$. Every open neighbourhood 2 intersects each member of $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ hence 2 is a cluster point of $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ both.

(iii) open neighbourhood of 3 = { 1, 3}, X. Every open neighbourhood of 3 intersects each member of $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$. Hence 3 is a cluster point of $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ both. Thus adh $\boldsymbol{F_1}$ = {1, 2, 3} and adh $\boldsymbol{F_2}$ = {2, 3}.

Self-learning exercise

1. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$, Then find limit points and cluster point of the sequence < a, a, a, a, ...>.
2. Define residual and cofinal subsets of the directed set (A, ≥).
3. Give an example to show that a net can converge to several different points.
4. Give an example to show that the set of all cluster points of a net in a topological space need not be closed.
5. Let X = { a, b, c, d} and let $\boldsymbol{C}$ = {{a, b}, {b, c} then filter whose subbase is $\boldsymbol{C}$.
6. State that whether the following state meuts are true or false.
   (i) A sequence can converge more than are point.
   (ii) A net can converge more than are point.
   (iii) In a housdorff space every convergent net has a unique limit point.
   (iv) A filter contains empty set ϕ.
   (v) A topological space X is hausdorff then every convergent filter in X has a unique limit but converse is not necessarily true.

15.5 Summary

In this unit, we have discussed convergence of a sequence, net and filter. We have observed that net is a generalized sequence. We have studied definitions of limit point, cluster point of net and filter , subnet, ultrafilter, filter base and subbase of a filter.

15.6 Answers to self-learning exercises

1. Cluster points = *a, b, c* and limit points = *a, b, c*.
2. See definitions.
3. See example 4.
4. Let τ be cofinite topology on the set of natural numbers N. The mapping $f : N \to N$ such that $f(n) = 2n - 1$ is a net in N. The set of cluster points of this net <1, 3, 5, ...> is {1, 3, 5,...} which is not closed in N.
5. {{b}, {a, b}, {b, c}, {b, d}, {a, b, c} {b, c, d}, {a, b, d} X}.
6. *(i)* True *(ii)* True *(iii)* True
   (iv) False *(v)* False.

15.7 Exercises

1. Show that every subnet of an ultranet is an ultranet.
2. Show that a mapping g of a space X to a space Y is continuous at $x_0 \in X$ iff every net $(f, X, A, \geq)$ converging to x_0, the composition mapping $g \, o \, f : A \to Y$ converges to $g\,(x_0)$.
3. Let X be a topological space and let Y be a sub set of X. Then Y is τ–open iff no sequence in $X - Y$ converges to a point in Y. Prove it.
4. A subset A of a topological space Y is closed iff no net in A converges to a point in $X - A$.
5. Let $(A, \geq)$ be a directed set and B be a cofinal subset of A so that B is also directed by $\geq$. Let $(f, X, A, \geq)$ be a net. Then show that restriction map of f to B is a subnet of f.
6. Let X be any infinite set. Then show that $F = \{A \subset X \mid X - A$ is finite $\}$ is a filter on X.
7. Let $\{F_\lambda \mid \lambda \in \Lambda\}$ be any non-empty family of filters on a non-empty set X. Then show that intersection of this family that is $\cap \{F_\lambda \mid \lambda \in \Lambda\}$ is also a fitter on X.
8. Let F be a filter on a non-empty set X and let A is a subset of X, then there exists a filter F' finer than F such that $A \in F'$ if and only if $A \cap F \neq \phi$ for every $F \in F'$.
9. Let X be a topological space and let $x \in X$. Then show that local base $\boldsymbol{B}\,(x)$ at x is a filter on X.
10. Show that every filter base on a set X is contained in an ultrafilter on X.

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